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Penzijní modely

Katedra pravděpodobnosti a matematické statistiky

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Abstrakt: Diplomová práce se zabývá problémem udržitelné míry spotřeby během důchodového věku, což je podstatný kvantitativní problém v rámci penzí. Postupně je vybudován model, který intuitivně spojuje tři klíčové faktory týkající se finančního plánování důchodu: náhodnou délku lidského života, náhodné výnosy z investic a rychlost spotřeby úspor. V rámci vybudovaného modelu odvodíme pomocí momentové metody aproximaci pro pravděpodobnost ruinování jedince. Přesnost odvozené aproximace analyzujeme pomocí rozsáhlých Monte Carlo simulací. Odvozená metoda je aplikována na datech pro Českou republiku. Výsledky obsahují hodnoty pravděpodobnosti ruinování a maximální udržitelné míry spotřeby pro různé kombinace rychlosti spotřeby úspor a parametrů investičního portfolia.

Klíčová slova: životní anuity, důchodové plánování, udržitelnost penzí

Title: Pension Models

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Abstract: The thesis is concerned with the problem of sustainable spending towards the end of the human life cycle, which is a substantial quantitative problem in the pension framework. We gradually build a model, which coherently links the three key factors of retirement planning: uncertain length of human life, uncertain investment returns and spending rates. Within the framework of our intuitive model, we apply the method of moment matching to derive an approximation for the probability of individual's retirement ruin. The accuracy of presented approximation is analysed via extensive Monte Carlo simulations. A numerical case study using Czech data is provided, including calculated values for the probability of ruin and maximal sustainable spending rate under various combinations of wealth-to-spending ratios and investment portfolio characteristics.

Keywords: life annuities, retirement planning, sustainable pensions

Contents

List of Abbreviations

AAO	Arithmetic Asian option
AM	arithmetic mean
CDF	cumulative distribution function
CZK	Czech crown
CZSO	Czech Statistical Office
EPoR	eventual probability of ruin
GBM	geometric Brownian motion
GM	geometric mean
IFM	instantaneous force of mortality
LPoR	lifetime probability of ruin
LSDE	linear stochastic differential equation
MC	Monte Carlo
NWP	net wealth process
ODE	ordinary differential equation
PDF	probability density function
PoR	probability of ruin
PSE	Prague Stock Exchange
PVSP	present value of a stochastic perpetuity
r.v.	random variable
SBM	standard Brownian motion
SDE	stochastic differential equation
SPV	stochastic present value
SSR	sustainable spending rate
WLLN	Weak Law of Large Numbers

List of Symbols

$\mathbb{R}$	set of real numbers
$\mathsf{P}[\cdot]$	probability
$\mathsf{E}[\cdot]$	mean
$\mathsf{var}[\cdot]$	variance
$\mathsf{M}[\cdot]$	median
$M_X(\cdot)$	moment-generating function of random variable X
$m(\cdot)$	sample mean
$s^2(\cdot)$	sample variation
T_x	remaining lifetime random variable
$F_x(\cdot)$	cumulative distribution function of T_x
$f_x(\cdot)$	probability density function of T_x
${}_tp_x$	conditional probability of survivor to age $x+t$
${}_tq_x$	conditional probability of dying before age $x+t$
$\lambda(x+t)$	instantaneous force of mortality evaluated at age $x+t$
ω	ultimate age of the life table
$\Gamma(a,c)$	upper incomplete Gamma function with parameters a and c
m	location parameter of the Gompertz IFM curve
b	dispersion parameter of the Gompertz IFM curve
i	effective annual interest rate
r	continuously compounded interest rate
$\tilde{g}$	annualized growth rate of a risky investment (r.v.)
S_t	value of a risky investment at time t
B_t	standard one-dimensional Brownian motion
$B_t^{(\nu,\sigma)}$	Brownian motion with a drift of ν and a volatility of σ
$\mathrm{Gamma}(\alpha,\beta)$	Gamma distribution with parameters α and β
$\mathrm{LN}(a,b^2)$	log-normal distribution with parameters a and $b^2>0$
$\mathrm{N}(\mu,\sigma^2)$	normal distribution with a mean of μ and a variance of σ^2
$\mathrm{RG}(\alpha,\beta)$	reciprocal Gamma distribution with parameters α and β
$A(\xi\|m,b,x)$	Gompertz price of a life annuity under c.c. interest rate ξ
$\tilde{M}_1$	first moment of the SPV random variable
$\tilde{M}_2$	second (non-central) moment of the SPV random variable
$\hat{\alpha}$, $\hat{\beta}$	fitted parameters of the reciprocal gamma approximation
W_t	value of the net wealth process at time t

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Introduction

Due to the population ageing phenomenon, which occurs almost everywhere around the world, the social insurance systems in many economically developed countries are predicted to face serious sustainability problems in a short time. Most likely, the future retirees will have to (at least partially) rely on their own lifetime savings in the issue of providing a sufficient income stream to finance their retirement consumption.

In essence, there are two distinct ways to cope with the problem of transforming one's retirement savings into a periodic income stream. The most natural strategy is to buy a life annuity and thus secure a life-contingent cash flow. However, as the relatively low worldwide demand for annuities indicates, many individuals actually decide for an alternative strategy, namely to replicate a synthetic life annuity. We refer to the later approach as the *self-annuitization*. This strategy comprises a discretionary management of pension assets and periodic withdrawals for the purposes of consumption. The retirees who decide for the self-annuitization strategy will likely share a common dilemma: How large portion of one's retirement savings can annually be spent without an excessive exposure to the risk that the savings run out prior to death of the individual? To answer this question, one needs to link the three essential factors of retirement planning: mortality considerations, investment returns and spending rates.

In order to simulate financial life-cycles, many financial planners have resorted to Monte Carlo simulations. However, the Monte-Carlo-based studies have their own drawbacks. They provide only little insight into the dynamics of the trade-off between risk and return during retirement and, most importantly, the number of simulations required for convergence tends to be very large, which makes these and similar methods quite time-consuming.

In this text, we approach the problem of sustainable retirement spending from a different perspective. By deriving an analytic relationship between the three essential variables of retirement planning, we present an intuitive and consistent model. Since both the length of human life and the future investment returns are stochastic, the event of the retirement savings complete depletion prior to one's death can be viewed as a realization of random variable. Similarly to the classical ruin theory in insurance, we are interested in the *probability of ruin* of the individual, i.e. we investigate the probability that the retirement savings run out prior to death of the individual.

In order to solve this problem, we introduce the concept of *stochastic present value* and demonstrate that the probability of ruin can be represented as the probability that the stochastic present value of future consumption is greater than the initial capital (retirement savings) of the individual. Our approach is strictly analytic, without depending on Monte Carlo simulations or historical studies.

In the limit case of perpetual consumption it turns out that, under certain assumptions on investment returns and mortality dynamics, the stochastic present value follows the reciprocal gamma distribution, as was first shown by Milevsky (1997). Unfortunately, in the case of (finite) random remaining lifetime, there is no closed-form density function for the stochastic present value. However, as shown by Milevsky & Robinson (2000), the true probability density function can be approximated by matching the first two moments of the stochastic present value to the first two moments of the reciprocal gamma distribution. We refer to this approach as the *reciprocal gamma approximation*.

In this thesis, we start from scratch and gradually build the model for the two key stochastic variables of our analysis: length of human life and investment returns. We continue with the main analytic part, i.e. the derivation of the reciprocal gamma approximation for the probability of ruin. Afterwards, using extensive Monte Carlo simulations, the accuracy of the reciprocal gamma approximation is verified. The discrepancy between the reciprocal gamma approximation results and the Monte Carlo values is analysed in dependency on the investment volatility, for various values of retiring age, spending rate and expected rate of return. Last but not least, we provide a numeric case study using Czech data. In particular, we present values for the lifetime probability of ruin and *sustainable spending rate* under various combinations of wealth-to-consumption ratios and investment portfolio characteristics.

The contribution of this thesis can be viewed from several different perspectives. In the main analytic part of this text, we draw extensively from the aforementioned articles by M. A. Milevsky. More specifically, all the concepts regarding the reciprocal gamma approximation originate in his work. In Milevsky (2006) and Milevsky & Robinson (2000), the author derived the reciprocal gamma approximation formula under the assumption of the *exponential law of mortality* and the *Gompertz law of mortality*, respectively. We extend his theoretical result by deriving the corresponding reciprocal gamma formula also for the *Gompertz-Makeham law of mortality.* As another part of our analytic contribution we provide a non-trivial step-by-step derivation of the second moment of the stochastic present value, which, to our best knowledge, has not been published yet.

The quantitative contribution of our work can be understood in the enclosed results for the probability of ruin and the sustainable spending rate in the Czech environment, as well as in the verification of the reciprocal gamma approximation accuracy. We believe our approach to be valuable in the sense that the potential reader will not only find numeric results and recommendations suitable for his individual circumstances, but he will also get a notion of how precise the corresponding values are.

Finally, we see the qualitative contribution of this text in a comprehensive presentation and explanation of all the concepts required to understand the main theoretic result, which is the reciprocal gamma formula. The reader does not necessarily need to be familiar with the results of actuarial mathematics or stochastic calculus prior to reading the text.

The remainder of the text is organized as follows.

In *Chapter 1* we start with a demonstration of the population ageing phenomenon size and a discussion of its consequences. The key concepts of self-annuitization and ruin in retirement are introduced. We also discuss the benefits and drawbacks of the classical annuitization approach and review the theoretic result by Yaari (1965), which concerns the demand for life annuities.

Chapter 2 presents the actuarial models for description of human mortality, including the remaining lifetime random variable, the instantaneous force of mortality and life tables. Two various analytic mortality laws are introduced and their respective characteristics are discussed.

Chapter 3 deals with the problem of modelling a risky financial investment with a stochastic rate of return. The key concept of this chapter is the geometric Brownian motion for the portfolio price with the underlying log-normal distribution of returns.

In *Chapter 4* we derive the theoretic results for the probability of retirement ruin under the assumptions of the Gompertz-Makeham law of mortality and the market prices driven by the geometric Brownian motion. First, we present the exact result of Milevsky (1997) for the eventual probability of ruin. Afterwards, we derive the reciprocal gamma approximation for the lifetime probability of ruin.

In *Chapter 5* we analyse the accuracy of the reciprocal gamma approximation via comparison with the results obtained from Monte Carlo simulations.

Chapter 6 provides the numerical case study. First, we fit the Gompertz law of mortality to the Czech life table and calibrate the financial model. Afterwards, the results for the lifetime probability of ruin and the sustainable spending rate are presented and discussed.

Chapter 1

The Retirement Planning Problem

To motivate the reader and demonstrate the importance of the topic discussed in the following text, we start this chapter with a short description of the late demographic development across chosen European countries. In particular, we focus on the ageing of European population. To illustrate the size of the problem, we present data compiled by Eurostat[1]. Naturally, the demographic situation in the Czech Republic is of the highest importance for us. However, we see fit to describe the problem in the European context, because the analysis we carry out later can be (with a slight change of assumptions) applied universally across different countries.

Later we explain how the population ageing problem influences the social insurance system and thus concerns the pension planning of future retirees. We also introduce the fundamental concepts of self-annuitization, sustainable spending rate and probability of ruin.

1.1 The Demographic Picture

For the entire recorded human history the world population has never been as old as nowadays. Over the last couple of years the ageing of the population has become an often discussed world phenomenon. According to the *World Population Ageing: 1950-2050*[2] record elaborated by the United Nations, the rate of global ageing in the 21st century is predicted to be larger than in the previous century. The ageing of the population is caused mainly by two demographic effects: *increasing longevity* and *declining fertility*.

Thanks to the progress of medical care, the average length of human life has been steadily on the rise during the last centuries. A common demographic indicator used among other things for longevity measurement is the *life expectancy at birth*[3], defined as the mean number of years that a newborn child can expect to live. Table 6.5 in the Appendix shows an overview of the life expectancy

[1] Data available for download at *http://epp.eurostat.ec.europa.eu/portal/page/portal/eurostat/home*.

[2] The record is available at `www.un.org`.

[3] See Section 2.1 for an exact definition of the life expectancy at birth in the context of actuarial mathematics.

development in selected European countries. Observe that in every country the life expectancy at birth has increased over the last ten years. In the Czech Republic, for instance, the life expectancy has grown by 2.7 years for males and by 2.5 years for females. Similarly, the average increase over all countries (males and females combined) is approximately 2.5 years, which is a substantial number, given the period over which the change is measured.

However important the increasing longevity nowadays might appear, according to the aforementioned United Nations report, the contribution of the low fertility to the ageing of world population is even larger. A substantial decline in fertility rate in the most developed countries has been evident especially in the second half of the 20th century. Table 6.6 in the Appendix shows the *total fertility rate*[4], i.e. number of children per woman, across the same group of European countries. For example, the fertility rate in the Czech Republic started at 1.86 children per woman in 1991, then dropped over ten years to the critical value of 1.14 in 2001 and rose again to 1.43 until 2011. In general, the decrease in fertility is not as evident as the increase in life expectancy shown in the previous table. The values vary significantly amongst different countries and in many cases the total fertility rate has even grown during the last twenty years, since the stage of the demographic transition towards lower fertility and mortality is different for every country. Still, most of the recent figures are far bellow the *replacement level fertility rate*[5], which is considered to be 2.1 in highly developed countries.

Without taking the migration into account, a long-term sub-replacement fertility rate generally leads to a gradual population decline in the given region and thus (possibly combined with the increasing longevity effect) the population is ageing, since the number of born babies is declining with time. An illustrative indicator of the population ageing suitable for our purpose is the *old-age dependency ratio*. This indicator is defined as the ratio between the total number of elderly persons of an age when they are generally economically inactive (aged 65 and over) and the number of persons of working age (from 15 to 64). Table 1.1 shows the development of old-age dependency ratio in selected European countries (the same countries as in the previous case), along with estimated future values up to the year 2060. While the average ratio hovered somewhere around 20% in 1991 (19.1% for the Czech Republic), it has jumped over twenty years to an average value of 25% in 2011 (22.3% in the Czech Republic) and the average value in the year 2060 is estimated to be about 50% (55% in the Czech Republic). Stated differently, on average, the European population is estimated to age so fast that by the year 2060 there will be only two persons of working age for every retiree. Observe that despite the variations, the old-age dependency ratio is estimated to increase significantly in every country.

[4]The mean number of children that would be born alive to a woman during her lifetime if she were to pass through her childbearing years conforming to the fertility rates by age of a given year. This rate is therefore the completed fertility of a hypothetical generation, computed by adding the fertility rates by age for women in a given year (the number of women at each age is assumed to be the same).

[5]Replacement fertility is the total fertility rate at which newborn girls would have an average of exactly one daughter over their lifetimes. That is, women have just enough female babies to replace themselves (or, equivalently, adults have just enough total babies to replace themselves).

	Year					
Country	**1991**	**2001**	**2011**	**2020**	**2040**	**2060**
Austria	22.2	22.8	26.0	29.8	46.8	50.7
Belgium	22.5	25.7	26.0	30.3	41.0	43.8
Czech Republic	19.1	19.8	22.3	30.4	40.1	55.0
Denmark	23.1	22.2	25.7	31.4	41.9	43.5
Finland	20.0	22.4	26.5	36.2	43.5	47.4
France	21.2	24.5	25.8	32.7	44.4	46.6
Germany	21.7	24.5	31.2	35.8	56.4	59.9
Greece	20.6	24.7	29.0	32.6	47.8	56.7
Iceland	16.6	17.8	18.4	25.1	34.5	33.5
Ireland	18.5	16.6	17.2	22.8	33.1	36.7
Italy	22.0	27.4	30.9	34.8	51.7	56.7
Lithuania	16.6	21.3	26.6	26.6	41.8	56.7
Netherlands	18.7	20.1	23.3	30.8	47.3	47.5
Norway	25.2	23.2	22.8	27.4	38.5	43.0
Poland	15.7	18.0	18.9	26.9	39.9	64.6
Romania	15.9	19.6	21.3	25.7	40.7	64.8
Spain	20.7	24.7	25.2	28.9	46.7	56.4
Sweden	27.7	26.8	28.4	33.5	40.5	46.2
Switzerland	21.3	22.9	24.9	29.5	45.7	54.4
United Kingdom	24.2	24.3	25.3	29.6	38.9	42.1

Note: Figures for years 2020, 2040 and 2060 are projections.

Source: Eurostat.

Table 1.1: Old-age-dependency ratio [in %] across selected European countries.

1.2 Unsustainable Social Security System

In the previous part we have shown the potential magnitude of the European population ageing problem. In this section we aim to discuss one of the most serious socio-economic consequences of this demographic effect: an unsustainable social security system.

The *social insurance systems* in most countries (including the Czech Republic) are based on the *pay-as-you-go* principle, meaning in essence that the current generation of working people pays to meet the living costs of those who are currently in retirement. This makes the older people within the society directly dependent on the current younger generation in terms of financial support.

The increasing longevity leads to an extended retirement period of human life[6], while the length of the active labour period remains unchanged. Along with the decreasing fertility, this leads to a higher replacement ratio and thus the burden for the younger generation becomes larger. Consequently, the social security systems in many countries (including the Czech Republic) begin to experience sustainability problems.

In the last decade, many countries have adopted policies to strengthen the financial sustainability of their pension systems, but the pace of population ageing is so fast that the problem usually remains more or less unresolved. The result for most of the European future retirees remains the same: most likely they will have to generate a substantial part of their retirement income from their own pension savings, as the income stream generated by the social insurance system will not be sufficient to maintain a dignified standard of living.

1.3 Self-Annuitization

Clearly, the aforementioned problem should motivate people to put aside money during their active labour years for the purpose of creating a sufficiently large nest egg (initial corpus of investment). Once an individual retires, this capital should be used to generate an income stream, which should, along with the benefits from the social insurance system, provide enough money to maintain the desired standard of living in retirement.

So far the financial services industry has focused mainly on the problems connected with the phase of savings accumulation. However, in this text we are not interested in answering the question of how much money should be put aside every year, how large the nest egg at the age of retirement should be or how the funds should be invested. We aim to analyse the second part of the problem, regarding the strategy of creating a suitable income stream from the nest egg once an individual retires.

Assume an individual just entered the retiring age and so far he has accumulated certain amount of savings. There are two reasonable choices regarding the transformation of his pension capital into a retirement income stream: *classical annuitization* and *self-annuitization*.

[6] See Table 6.7 in the Appendix for a demonstration of the change in *life expectancy at age 65* over the last twenty years. This value might be considered as a rough estimate of the average time spent in retirement by a single retiree in given country.

By the classical annuitization we mean buying a life annuity from an insurance company. An obvious advantage of this choice is the fact that such an income stream cannot be outlived. On the other hand, by buying an annuity the retiree gives up the liquidity of his pension funds. Moreover, he implicitly foregoes the chance of leaving a bequest in case of an early death, since the unconsumed capital remains with the insurance company.

Being the second possible choice, the self-annuitization strategy means a discretionary management of pension funds with systematic withdrawals for consumption purposes. The benefits and drawbacks of the self-annuitization approach are the exact opposite to those accompanying the purchase of a life annuity. The consumer preserves both the flexibility and the possibility of leaving a bequest, but on the expense of a substantial risk that his pension capital might run out while he is still alive.

In general, life annuities offered by insurance companies are priced to cover operating costs and costs of adverse selection[7] along with the basic longevity risk. Consequently, the self-annuitization might be a favourable choice, assuming one can reasonably quantify the probability that the money will run out prior to death of the individual, also known as the *probability of ruin (PoR)*.

1.4 Ruin in Retirement

Suppose that a retiring individual with an initial capital (nest egg) w wants to consume an amount of c real (inflation-free) Czech crowns (CZK) per annum for the rest of his life. Let's say that he lives in a fully deterministic world, where the time of his death T and the fixed real future interest rate r are known. In this simple case, the value of his future consumption would be

$$c\,\bar{a}_{\overline{T}|} = c \int_0^T e^{-rt}\,dt. \tag{1.1}$$

In this situation, there is no need for quantifying the probability of ruin. More precisely, the PoR is trivially either one or zero, depending on the size of w relative to $c\,\bar{a}_{\overline{T}|}$. If the initial wealth w is lower than the expression in Equation (1.1), the individual does not have enough capital to finance the desired consumption stream and the probability that the ruin occurs is equal to one.

Now, we want to extend this concept to a stochastic world where both the rate of investment return and the time of death are random. We need to link both these sources of uncertainty and provide a quantification of the individual probability of retirement ruin[8]. Stated differently, given an initial nest egg and other individual circumstances, we want to find a *sustainable spending rate*, under which the PoR is reasonably small.

[7] Generally, the people buying life annuities tend to have above-average life expectancy. This forces the insurance providers to set higher prices, which makes annuities expensive for people with average or below-average life expectancy.

[8] It might be worth pointing out that our problem of quantifying the PoR is tied to the classical *ruin theory* in insurance, which was first introduced in 1903 by the Swedish actuary Filip Lundberg. However, in contrast to the classical model, we are interested in a ruin probability from a personal perspective. Specifically, in our case deterministic premiums are substituted by the stochastic investment returns and random claims are replaced by the deterministic consumption stream.

In order to proceed with the analysis we need to make some assumptions concerning the two random factors: *human mortality* and *random investment returns*. We therefore devote the next two chapters of this text to the description of these factors and in both cases we suggest a possible way to cope with the underlying uncertainty.

1.5 Further Reading

Let us close this chapter with a reference to a couple of interesting articles dealing with a problem linked to ours.

In the first and classical one, Yaari (1965) takes a more theoretic approach to the general problem of finding an optimal consumption plan over a random time horizon. In particular, the author shows that, under the assumptions of actuarially fairly priced annuities and intertemporally separable utility function, a consumer without bequest motives always annuitizes (in the classical sense) all his savings. More specifically, Yaari takes the *chance-constrained programming* approach to the problem of maximizing the expected value of the Fisher utility function, subject to the probabilistic constraint that the consumer's net assets at time of death must almost surely be non-negative. The fact that the consumer will always annuitize all his savings follows from the argument that fairly priced actuarial notes always bring a higher return than regular notes, as long as the consumer lives. The only drawback tied to the annuitization of all assets is that the consumer's bequest becomes automatically equal to zero, independently on the time of death. But since there are no bequest motives, the consumer is not concerned with his posthumous asset position, as long as the bequest remains non-negative, so that the constraint is fulfilled. Thus, the consumer will hold all his assets in the form of a life annuity.

However, in stark contrast to Yaari's result, the actual worldwide demand for life annuities is rather low. The discrepancy between the theoretical result and the observed state of annuity market is often referred to as the *annuity puzzle*. For example, Cipra (2012) offers several explanations for this phenomenon (e.g., uneven distribution of retirement consumption, impact of social security benefits, adverse selection, etc.). The author also presents a graphic explanation of Yaari's result (see Cipra, 2012, Section 11.1).

In addition to the literature dealing with the demand for life annuities, we also like to refer to Schmeiser & Post (2005), where the authors suggest an interesting "family strategy" the retiree and his heirs might adopt in order to reduce the risk connected with self-annuitization.

Finally, in another master's thesis dealing with the problem of pension planning for Czech retirees, Langová (2011) describes the so-called "do-it-yourself-and-then-switch" strategy, which is a third possible solution to the retirement wealth annuitization decision problem. Being introduced by Milevsky (1998), this approach is based on the idea that the retiring individual can defer the annuitization of his pension wealth until the mortality adjusted rate of return from life annuity outgrows the benefits of self-annuitization.

Chapter 2

Models of Human Mortality

As mentioned in the introductory section, there are two sources of uncertainty the retirement consumption planners should be concerned with, one of them being the random length of human life. The main goal of this chapter is therefore to introduce the basic concepts of the human mortality modelling. We start with the idea of the remaining lifetime random variable and its connection to mortality tables. Later we introduce two different models for the instantaneous force of mortality, known as the exponential- and Gompertz(-Makeham) laws of mortality.

Since it is impossible here to cover all the relevant aspects of actuarial mortality models, this chapter should perhaps be perceived rather as an overview of actuarial formulas and notation, which shall be used through the rest of the text. For more information on the topic and more detailed explanation of the underlying mathematics, we refer an interested reader to a classical (and comprehensive) actuarial textbook Bowers et al. (1997).

2.1 Remaining Lifetime Random Variable

When trying to formulate the uncertainty of death in probability concepts, we believe it is a good idea to start off with the *remaining lifetime* random variable. Let us consider an individual aged x and suppose we want to describe the probability distribution governing the time till the event of his ore her death. In what follows we will denote T_x a (continuous-type) random variable representing the *remaining lifetime* of an individual aged x. We shall write F_x for the cumulative distribution function (CDF) of T_x, i.e.

$$F_x(t) = \mathsf{P}[T_x \le t], \qquad t \ge 0.$$

We will assume that there exists a corresponding probability density function (PDF) such that

$$F_x(t) = \int_0^t f_x(s)ds.$$

In order to receive further results, we shall assume that for $t \ge 0$ the probability distribution of random variable T_{x+t} is the same as the conditional distribution of $T_x - t \mid T_x > t$. The interpretation of this assumption is, that the additional information about the life of an individual available at age $x + t$ does not change the remaining lifetime distribution we would otherwise a person of age x expected to have after surviving t more years. Note that this distribution equality can be written as

$$\mathsf{P}[T_{x+t} \leq z] = \mathsf{P}[T_x - t \leq z \mid T_x > t], \qquad t \geq 0. \tag{2.1}$$

Now, since we will try to follow the standard actuarial notation[1] whenever possible in the rest of the text, let us introduce symbols ${}_tp_x$ and ${}_tq_x$ as the *conditional probability of survival* and *death*, respectively. That is: a living individual aged (exactly) x has the probability ${}_tp_x$ of surviving t more years and, on the other hand, the probability that he will die prior to reaching age $x + t$ is the complementary ${}_tq_x := 1 - {}_tp_x$. In the context of our remaining lifetime random variable this means nothing else than that for $t \geq 0$:

$${}_tq_x = \mathsf{P}[T_x \leq t] = F_x(t) \tag{2.2}$$

and

$${}_tp_x = \mathsf{P}[T_x > t] = 1 - F_x(t). \tag{2.3}$$

In order to be conform with the notation of other authors we shall write p_x and q_x instead of ${}_1p_x$ and ${}_1q_x$.

An overview of estimated values q_x (along with other characteristics) for the age spectrum $x = 0, \ldots, \omega$ is called the *life table*. The *ultimate age* ω of a mortality table is defined as the lowest age x for which $q_x = 1$ occurs. For instance, in the life table for the Czech Republic (annually published by the Czech Statistical Office (CZSO)) the ultimate age is set to be $\omega = 105$ (since 2010). In general, mortality tables play a fundamental part in an evaluation of every life-contingent asset. See Table 6.8 in the Appendix for an example of a life table. There are many different types of mortality tables in practice and once again we refer to the aforementioned literature for more details.

The last concept that remains to be introduced in this section is the *life expectancy* for given age x. It is defined simply as the mean of T_x. Recall from probability theory that the expected value of a non-negative continuous random variable T_x with PDF and CDF denoted by $f_x(t)$ and $F_x(t)$, respectively, can be expressed as

$$\begin{aligned} \mathsf{E}[T_x] &= \int_0^\infty t f_x(t)\, dt && (2.4) \\ &= \int_0^\infty (1 - F_x(t))\, dt. && (2.5) \end{aligned}$$

In addition, due to (2.3), in the case of remaining lifetime r.v. the last formula can be rewritten as

$$\mathsf{E}[T_x] = \int_0^\infty {}_tp_x \, dt. \tag{2.6}$$

It is worth mentioning that an estimate of the value $\mathsf{E}[T_x]$ for the complete spectrum of ages x is one of the characteristics that are usually presented in mortality tables. In particular, the estimate of $\mathsf{E}[T_0]$ is a very useful demographic development indicator of the underlying population.

[1] See Bowers et al. (1997, Appendix 4) for an overview of general rules for symbols of actuarial functions.

2.2 Instantaneous Force of Mortality

Let us now for a moment consider an individual aged $x+t$ and the probability of his or her death in a very near future. In terms of our model this is represented by the expression $\mathsf{P}[0 < T_{x+t} \leq \Delta t]$, where $\Delta t \to 0^+$. Using standard rules for conditional probability and assumption (2.1), we obtain

$$\mathsf{P}[0 < T_{x+t} \leq \Delta t[= \mathsf{P}[t < T_x \leq t + \Delta t \mid T_x > t] = \frac{F_x(t+\Delta t) - F_x(t)}{1 - F_x(t)}.$$

Now, denoting $\lambda(x+t) := \lim_{\Delta t \to 0^+} \frac{\mathsf{P}[t<T_x \leq t+\Delta t | T_x > t]}{\Delta t}$ and assuming $F'_x(t^+) = F'_x(t)$, we can write:

$$\lambda(x+t) = \frac{F'_x(t)}{1 - F_x(t)}, \qquad t \geq 0. \tag{2.7}$$

The function $\lambda(x+t)$ is known as the *instantaneous force of mortality* (IFM) and it can be thought of as the instantaneous rate of death at age $x+t$. Observe that, thanks to the assumed existence of $f_x(t)$, the IFM can be also represented as

$$\lambda(x+t) = \frac{f_x(t)}{1 - F_x(t)} \tag{2.8}$$

$$= -\frac{d}{dt} \ln(1 - F_x(t)). \tag{2.9}$$

A couple of things should be noticed. Firstly, examining (2.8) and taking properties of $f_x(t)$ and $F_x(t)$ into account yields $\lambda(x+t) \geq 0$, which means that the instantaneous rate of death at any age is non-negative. In addition, by substituting (2.3) into (2.8) we receive a useful equation

$$f_x(t) = {}_tp_x \, \lambda(x+t). \tag{2.10}$$

Last but not least, integrating both sides of (2.9) and applying once again (2.3) leads to

$${}_tp_x = \exp\left\{ -\int_0^t \lambda(x+s)\, ds \right\}, \tag{2.11}$$

or, by a simple change of variables

$${}_tp_x = \exp\left\{ -\int_x^{x+t} \lambda(s)\, ds \right\}. \tag{2.12}$$

Altogether, the relationships (2.8), (2.10) and (2.11) allow us to switch comfortably from $\lambda(x+t)$ to $F_x(t)$ or $f_x(t)$ and back without having to use complicated calculus.

Now, let us step back for a moment from general results and introduce two particular ways to model the behaviour of human mortality.

2.3 Exponential Law of Mortality

The easiest case is when the assumption is made, that the IFM curve is constant over all ages, i.e. $\lambda(x+t) \equiv \lambda$. Notice that this means that the instantaneous probability of death is the same at every age. Of course, this assumption is rather unrealistic, at least for modelling human mortality[2], but it is very convenient to work with and in some cases the obtained results are remarkably similar to the outcomes derived from more complicated models (see e.g. Milevsky, 2006, Chapter 9).

Using the relationship (2.12), the distribution function of T_x under constant IFM curve can easily be derived, since

$$F_x(t) = 1 - {}_tp_x = 1 - \exp\left\{ -\int_x^{x+t} \lambda \, ds \right\} = 1 - e^{-\lambda t}. \tag{2.13}$$

Thus, under the assumption $\lambda(x+t) \equiv \lambda$, the remaining lifetime random variable T_x follows an exponential distribution[3] with the rate parameter λ. We therefore say that such a model follows the *exponential law of mortality*.

2.4 Gompertz-Makeham Law of Mortality

The second human mortality model we would like to introduce overcomes the problem of too unrealistic constant IFM. Similarly to the exponential one, it can be defined by a particular shape of the IFM curve, namely

$$\lambda(x) = \lambda + \frac{1}{b} \exp\left\{ \frac{x-m}{b} \right\}, \qquad x \geq 0, \tag{2.14}$$

where $\lambda \geq 0$, $m > 0$ and $b > 0$ are non-negative parameters. The value $m > 0$ can be interpreted as the modal length of human life, while $b > 0$ is the dispersion coefficient. We will show later why this is true. For now, observe that the IFM in (2.14) is a sum of an age-independent component λ and an exponential term increasing with age. The constant λ represents the component of the death rate that is attributable to accidental deaths, whereas the exponential curve captures the natural death causes. We will refer to model (2.14) as to the *Gompertz-Makeham law* when $\lambda > 0$ and call it simply the *Gompertz law*[4] when $\lambda = 0$. Since the value of λ tends to be negligible in practice, assuming $\lambda = 0$ and thus working only with the Gompertz version does not cause a large inaccuracy.

Since the Gompertz-Makeham law of mortality will be essential to our later analysis, let us investigate some of its properties. What does, for instance, the conditional probability of survival look like? Using the previously derived formula

---

[2]However, it is known that, for example, the mortality of lobsters follows a model with constant IFM curve.

[3]Recall that the *exponential distribution with a rate parameter* λ corresponds to the CDF $F(t) = 1 - e^{-\lambda t}$, $t \geq 0$. This leads to the PDF $f(t) = \lambda e^{-\lambda t}$, $t \geq 0$ and the expected value is then $1/\lambda$.

[4]This model was first introduced by a British mathematician Benjamin Gompertz (see Gompertz, 1825), although he used a different parametrization

$$\lambda(x) = B\, c^x, \qquad x \geq 0,\ B > 0,\ c \geq 0.$$

Equivalence of this parametrization with (2.14) for $\lambda = 0$ can be easily verified by setting $c = e^{1/b}$ and $B = b^{-1} e^{-m/b}$.

(2.12) and integrating by substitution, we find that

$$
\begin{aligned}
{}_tp_x &= \exp\left\{-\int_x^{x+t} \lambda + \frac{1}{b}e^{(s-m)/b}\,ds\right\} \\
&= \exp\left\{-\lambda t + e^{(x-m)/b}\left(1 - e^{t/b}\right)\right\}. \qquad (2.15)
\end{aligned}
$$

Of course, the conditional probability of survival under the Gompertz law can be easily obtained from the last formula by setting $\lambda = 0$.

Plugging the result for ${}_tp_x$ into (2.10) leaves us with

$$
f_x(t) = \exp\left\{-\lambda t + e^{(x-m)/b}\left(1 - e^{t/b}\right)\right\}\left(\lambda + \frac{1}{b}\exp\left\{\frac{x-m}{b}\right\}\right) \qquad (2.16)
$$

as the PDF of the remaining lifetime T_x under the Gompertz-Makeham law. In the case of $\lambda = 0$ the density simplifies to

$$
f_x(t) = \frac{1}{b}\exp\left\{\frac{x-m+t}{b} + e^{(x-m)/b} - e^{(x-m+t)/b}\right\}. \qquad (2.17)
$$

We are now in a position to demonstrate the aforementioned interpretation of parameters $m > 0$ and $b > 0$ in the Gompertz IFM expression (2.14). The following analysis draws from Carriere (1992) and assumes $\lambda = 0$.

The first of both Gompertz parameters represents the modal length of human life because it is the mode of the density $f_0(t)$. Under the Gompertz law we have

$$
f_0(t) = \frac{1}{b}\exp\left\{\frac{t-m}{b} + e^{-m/b} - e^{(t-m)/b}\right\}. \qquad (2.18)
$$

Using the inequality $e^y \geq 1 + y$, $\forall y \in \mathbb{R}$, one can show that

$$
e^{-m/b} - 1 \geq e^{-m/b} - e^{(t-m)/b} + \frac{t-m}{b}
$$

and therefore conclude

$$
f_0(m) \geq f_0(t), \qquad \forall t > 0,
$$

which demonstrates that m is the mode.

Moreover, observe that for an arbitrary $\varepsilon > 0$ it holds

$$
\lim_{b\to 0^+}\left[{}_{m-\varepsilon}p_0 - {}_{m+\varepsilon}p_0\right] = \lim_{b\to 0^+}\exp\left\{e^{-m/b}\right\}\left[\exp\left\{-e^{-\varepsilon/b}\right\} - \exp\left\{-e^{\varepsilon/b}\right\}\right] = 1.
$$

Since the last result can be rewritten as

$$
\lim_{b\to 0^+}\left[F_0(m+\varepsilon) - F_0(m-\varepsilon)\right] = 1, \qquad \forall \varepsilon > 0,
$$

this means that all the probability mass concentrates around m when b is small. Thus we have just shown that b measures the dispersion of $f_0(t)$ around the location parameter m.

Now, let us remark that since the Gompertz law of mortality is a relatively simple model, it usually does not fit the mortality table well when the complete age spectrum data is used. However, it is well known that the Gompertz model is an excellent description of the mortality pattern at the older ages. This means that one usually applies the Gompertz curve to describe the mortality dynamics only for ages $x = x_0, \ldots, \omega$, where $x_0 \geq 40$.

Suppose we are given a mortality table with crude probabilities $\hat{q}_x$, $x = x_0, \ldots, \omega$ and the total amount of death claims D_x associated with each of the crude rates. We want to find such (positive) parameter values $\hat{m}$ and $\hat{b}$ that the corresponding Gompertz curve

$$q_x(\hat{m}, \hat{b}) := 1 - \exp\left\{ e^{(x-\hat{m})/\hat{b}} \left(1 - e^{1/\hat{b}}\right)\right\} \tag{2.19}$$

"fits" the table when evaluated at $x = x_0, \ldots, \omega$. A useful and simple method of the Gompertz parameters estimation applicable in this situation was presented in Carriere (1994). We will now describe this estimation technique and apply it later in our numerical study (see Chapter 6). The idea of this technique is to define the robust loss function

$$L\,(m, b) := \sum_{x=x_0}^{\omega} \sqrt{D_x} \left| 1 - \frac{q_x(m, b)}{\hat{q}_x} \right| \tag{2.20}$$

and set

$$(\hat{m}, \hat{b}) := \underset{m>0,\, b>0}{\arg\min}\, L\,(m, b)\,. \tag{2.21}$$

Observe that the loss function is a sum of weighted relative differences between the crude rate $\hat{q}_x$ and the Gompertz curve evaluated at age x. Every element of the sum is weighted by a square root of the total amount of death claims associated with the corresponding crude rate and thus lower weights are assigned to the more volatile rates $\hat{q}_x$ based on a small number of observations.

Carriere (1994) applied this estimation method using the male and female ultimate mortality rates from the *1975–80 Basic Tables of the Society of Actuaries* for ages $x = 40, \ldots, 100$. Using the male data, he found values $\widehat{m_M} = 82,153$ and $\widehat{b_M} = 10,304$, while the results for the female data were $\widehat{m_F} = 87,281$ and $\widehat{b_F} = 10,478$. Observe that while the values of scaling parameters are quite similar, the modal lifetimes of men and women differ considerably. The author concludes that the Gompertz model fits the data well.

An interesting justification for the Gompertz law as a model of the human mortality was presented in Brillinger (1961). Suppose that the human body can be represented as a series system of independent[5] identically distributed components where the whole system will fail with the first failure of one of its components. Due to the *Fisher-Tippett theorem* from the extreme value theory (see Haan & Ferreira, 2006, Theorem 1.1.3), it holds that the time of failure of such a system attains in the limit an *extreme value distribution*, namely one of the three different classes of extreme value distributions: *Fréchet*, *Gumbel* or *Weibull*. Since the Gompertz distribution is equivalent to the truncated Gumbel distribution for minima, the Gompertz law is a suitable tool for modelling the time to failure of the described system.

Finally, to demonstrate another key property of the Gompertz-Makeham model – its connection to the incomplete Gamma function – let us conclude

[5] As Brillinger (1961) states, the condition of independence can be relaxed in a sense that only components "sufficiently far apart" must be independent.

this section with a derivation of the formula for life expectancy. According to (2.6) we have

$$
\begin{aligned}
\mathsf{E}[T_x] &= \int_0^\infty \exp\left\{-\lambda t + e^{(x-m)/b}\left(1 - e^{t/b}\right)\right\}\, dt \\
&= \exp\left\{e^{(x-m)/b}\right\} \int_0^\infty \exp\left\{-\lambda t - e^{(x-m+t)/b}\right\}\, dt \\
&= b\exp\left\{e^{(x-m)/b} + \lambda(x-m)\right\} \int_{\exp\left\{\frac{x-m}{b}\right\}}^\infty e^{-u} u^{-\lambda b - 1}\, du \\
&= b\exp\left\{e^{(x-m)/b} + \lambda(x-m)\right\} \Gamma\left(-\lambda b, e^{(x-m)/b}\right), \qquad (2.22)
\end{aligned}
$$

where

$$
\Gamma(a, c) = \int_c^\infty e^{-t} t^{a-1}\, dt \qquad (2.23)
$$

is the *(upper) incomplete Gamma function* mentioned above. In general, it turns out that under the Gompertz-Makeham law many actuarial functions can be expressed in terms of this function. Once again, we refer to Carriere (1994) for more information about the Gompertz law of mortality and also for a method of numerical approximation to the incomplete Gamma function.

Chapter 3

Models of Risky Financial Investments

Recall that we aim to link three key factors of retirement planning: mortality, investment returns and spending rates towards the end of the life cycle. While in the previous chapter we described a possible way to deal with the uncertainty accompanying the length of human life, this part of the text is devoted to the models of random investment returns.

The chapter is organized as follows. The first section presents the continuously compounded investment growth rate and an important assumption concerning its distribution. In Section 3.2 we define the standard Brownian motion and summarize its basic properties. Section 3.3 introduces the geometric Brownian motion model of market prices and in the last section we conclude the chapter with the stochastic differential equation representation of our model.

3.1 Continuously Compounded Growth Rate

Since we want to develop our financial model in continuous time, we will use the continuously compounded interest. Recall that the idea of continuous compounding is based on the equality

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r,$$

which can be easily proved with the help of l'Hôspital's rule from calculus. Hence, the relationship between the *continuously compounded interest rate* r and the *effective annual rate* i is

$$1+i=e^r.$$

Our focus in the later analysis will be on investments in a stock portfolio. Due to the relatively higher volatility of stock markets, such an investment is more risky than investing in, say, bonds with fixed yield to maturity i. To emphasize this difference, we introduce a different notation for a portfolio growth rate. Suppose the value of our portfolio with the initial value S_0 (at time 0) has grown to

$$S_t = S_0 e^{\tilde{g}t} \tag{3.1}$$

at time t. In this case we say that the realized (continuously compounded) *annualized growth rate* of the portfolio over the past t years is $\tilde{g}$. It follows from

(3.1) that the growth rate $\tilde{g}$ can be expressed as a time-scaled logarithmic ratio of portfolio prices S_t and S_0 (LogRatio).

It is worth mentioning that the annualized growth rate $\tilde{g}$ over t years computed from (3.1) is always less than or equal to the average annual return (computed as an arithmetic mean). This is a consequence of the *AM-GM inequality*[1], which states that the *arithmetic mean* (AM) of a series of non-negative real numbers is always greater than or equal to the *geometric mean* (GM) of the same series. We will encounter another consequences of this inequality later.

In our context, we will approach the future growth rate $\tilde{g}$ of a portfolio as a random variable because we obviously never know its realized value in advance. Different examples from the stock market history show that the possible range for realized values of $\tilde{g}$ is large towards both positive and negative values, which is, of course, what makes the investment risky.

Now, in order to quantify the risk, we have to accept some assumptions about the probability distribution of $\tilde{g}$. We will adopt a relatively convenient approach and assume that $\tilde{g}$ is normally distributed with a mean of ν and a variance of σ^2/t, i.e.

$$\tilde{g} \sim \mathrm{N}\left(\nu, \frac{\sigma^2}{t}\right), \tag{3.2}$$

where t denotes the length of period (in years) over which the growth rate is being analysed, ν is the *expected growth rate* and $\sigma > 0$ is *volatility*.

Put differently, we can write

$$\tilde{g}t \sim \mathrm{N}(\nu t, \sigma^2 t). \tag{3.3}$$

Note that while the variance of the annualized growth rate $\tilde{g}$ decreases with the length of the investment horizon, the variance of the *cumulative growth rate* $\tilde{g}t$ increases. To model the development of the cumulative growth rate $\tilde{g}t$ in time, one needs to define a suitable stochastic process. In the next section we therefore introduce a crucial concept of this chapter - the Brownian motion.

3.2 Brownian Motion

The *standard Brownian motion* (also called *Wiener process*) is one of the best known stochastic processes and besides probability theory it plays a fundamental role also in some areas of physics and biology.

Formally, we say that a continuous-time stochastic process $\{B_t,\ t \geq 0\}$ is a *standard Brownian motion* (SBM) if the following conditions are met:

(i) $\mathsf{P}[B_0 = 0] = 1$;

(ii) the function $t \mapsto B_t$ is almost surely everywhere continuous;

(iii) for arbitrary $0 \leq t_0 < t_1 < \cdots < t_n$ the increments

$$B_{t_1} - B_{t_0},\ B_{t_2} - B_{t_1},\ \ldots,\ B_{t_n} - B_{t_{n-1}}$$

are independent;

[1]See Steele (2004, Problem 2.1) for the proof and additional information on the AM-GM inequality.

(iv) for arbitrary $0 \leq s < t$:

$$B_t - B_s \sim \mathrm{N}(0, t - s).$$

Since we are looking for a proper stochastic process to model the cumulative growth rate of an investment, the condition $B_t \sim N(0, t)$ seems to be too restrictive for our purpose. A more complex version of the SBM is the non-standard Brownian motion with a *drift rate* of ν and a *volatility* of $\sigma > 0$, which is defined by

$$B_t^{(\nu,\sigma)} = \nu t + \sigma B_t. \tag{3.4}$$

This means that the non-standard Brownian motion is just the SMB scaled with parameter $\sigma > 0$ and added to a deterministic linear trend. The SBM is obviously a special case of $B_t^{(\nu,\sigma)}$ with $\nu = 0$ and $\sigma = 1$.

It follows from (3.4) that $\mathsf{E}[B_t^{(\nu,\sigma)}] = \nu t$ and $\mathsf{var}[B_t^{(\nu,\sigma)}] = \sigma^2 t$. In addition, since a linear function of a normally distributed random variable remains normal, we may conclude

$$B_t^{(\nu,\sigma)} \sim \mathrm{N}\left(\nu t, \sigma^2 t\right). \tag{3.5}$$

This makes $\{B_t^{(\nu,\sigma)},\ t \geq 0\}$ a suitable candidate for modelling the development of the investment cumulative growth rate $\tilde{g}t$ from (3.3) because its assumed distribution is exactly the same.

The Brownian motion has many remarkable properties, most of which go, however, beyond the scope of this thesis. Fortunately, due to the importance of this concept across different fields of mathematical science, a plenty of literature has been written on this topic. See for instance Karatzas & Shreve (1991, Section 2.9) for the derivation of some sample path properties of the SBM, which are scale invariance, quadratic variation, non-differentiability and others. Finally, let us remark that the Brownian motion can be also intuitively obtained as the limit of a scaled random walk (see Shreve, 2004, Chapter 3).

3.3 Geometric Brownian Motion

When the cumulative growth rate of a stock portfolio in time is governed by the (non-standard) Brownian motion $B_t^{(\nu,\sigma)}$, the according portfolio price follows a related stochastic process - the *geometric Brownian motion*[2] (GBM). Denoting the GBM by $\{S_t,\ t \geq 0\}$, we can define it as

$$S_t = S_0 e^{\nu t + \sigma B_t}, \tag{3.6}$$

where $\{B_t,\ t \geq 0\}$ is the standard Brownian motion introduced in the previous section.

[2]The geometric Brownian motion (also referred to as the *exponential Wiener process*) was first implemented in the late 1960s by an American Nobel laureate in Economics Robert C. Merton. See Merton (1990) for an extensive collection of his work on continuous-time finance.

Note that, since $\nu t + \sigma B_t \sim \mathrm{N}(\nu t, \sigma^2 t)$, the price ratio S_t/S_0 follows a *log-normal distribution*[3], i.e.

$$\frac{S_t}{S_0} \sim \mathrm{LN}(\nu t, \sigma^2 t). \tag{3.7}$$

Recall that the PDF corresponding to the log-normal distribution $\mathrm{LN}(a, b^2)$ is

$$f(z) = \frac{1}{z\sqrt{2\pi}b} e^{-(\ln z - a)^2/2b^2}, \qquad z > 0.$$

It follows from (3.7) that the conditional distribution of the portfolio price at time t is

$$S_t \mid S_0 \sim \mathrm{LN}(\ln S_0 + \nu t, \sigma^2 t).$$

Let us now point out an important property of the log-normal distribution concerning the difference between its median and expected value.

First, observe that the *(conditional) median value* of the portfolio price at time t is

$$\mathsf{M}[S_t \mid S_0] = S_0 e^{\nu t}. \tag{3.8}$$

To verify this, note that for an arbitrary $u > 0$ we can write

$$\begin{aligned} \mathsf{P}[S_t \le u \mid S_0] &= \mathsf{P}\left[\ln S_0 + \nu t + \sigma B_t \le \ln u \mid S_0\right] \\ &= \mathsf{P}\left[\frac{B_t}{\sqrt{t}} \le \frac{\ln(u/S_0) - \nu t}{\sigma\sqrt{t}} \,\middle|\, S_0\right] \\ &= \int_{-\infty}^{w(u)} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz, \end{aligned} \tag{3.9}$$

where

$$w(u) := \frac{\ln(u/S_0) - \nu t}{\sigma\sqrt{t}}. \tag{3.10}$$

In (3.9) we have used the fact that B_t is normally distributed. Recall that the CDF of a normal distribution with zero mean is point symmetric. Consequently, a value $u_m > 0$ for which $w(u_m) = 0$ holds[4], also satisfies $\mathsf{P}[S_t \le u_m \mid S_0] = 1/2$ and therefore $\mathsf{M}[S_t \mid S_0] = u_m$. The result then follows directly from (3.10).

In contrast, the *(conditional) expected value* of S_t is

$$\begin{aligned} \mathsf{E}[S_t \mid S_0] &= \int_{-\infty}^{\infty} S_0 e^{\nu t + \sigma z} \frac{1}{\sqrt{2\pi t}} e^{-z^2/2t}\, dz \\ &= S_0 e^{(\nu + \sigma^2/2)t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-(z-\sigma t)^2/2t}\, dz \\ &= S_0 e^{\mu t}, \end{aligned} \tag{3.11}$$

where we define

$$\mu := \nu + \frac{\sigma^2}{2}. \tag{3.12}$$

---

[3]Log-normal distribution $\mathrm{LN}(a, b^2)$ is defined as the distribution of the random variable $Z = e^X$, where $X \sim \mathrm{N}(a, b^2)$. The support of a log-normally distributed r.v. is $(0, \infty)$. On a logarithmic scale, a and $b > 0$ can be interpreted as a *location parameter* and a *scale parameter*, respectively.

[4]Since we condition on S_0, the function $w(\cdot)$ is deterministic.

Altogether, we have obtained the relationship between the median and the expected value of the log-normally distributed portfolio price:

$$\mathsf{M}[S_t \mid S_0] = e^{-t\sigma^2/2}\,\mathsf{E}[S_t \mid S_0].$$

The factor $e^{-t\sigma^2/2}$ can be interpreted as a *convexity correction*[5] and is tied to the difference between the arithmetic and the geometric mean. We shall therefore carefully distinguish between the arithmetic and the geometric average of annual returns, denoted by μ and ν, respectively. Note also that (3.12) corresponds with the aforementioned AM-GM inequality.

3.4 The SDE Representation

In this section we introduce a more general representation of the geometric Brownian motion. In general, the stochastic differential equation (SDE) approach requires a theoretical background from *stochastic calculus*. However, for purposes of the following text we only need certain basic results of this discipline, namely Ito's lemma (see Theorem 1). Still, we shall omit some technical details and we refer to the aforementioned literature (e.g. Shreve, 2004) for more detailed information.

To begin with, let us introduce a general type of stochastic processes, which can be defined as a solution to a specific stochastic differential equation. We say that $\{Y_t,\, t \geq 0\}$ is a *diffusion process (Itô process)* with a *drift* of $\mu(Y_t, t)$ and a *diffusion* of $\sigma(Y_t, t)$, if it is a solution of the SDE

$$dY_t = \mu(Y_t, t)\,dt + \sigma(Y_t, t)\,dB_t, \qquad t \geq 0, \tag{3.13}$$

where $\{B_t,\, t \geq 0\}$ is the SBM.

Note that, in general, both the drift and the diffusion in (3.13) depend on time t and value of Y_t. A special example of a diffusion process is the (non-standard) Brownian motion $B_t^{(\nu,\sigma)}$. Besides the formula (3.4), the Brownian motion can be alternatively characterized as a diffusion process with a constant drift of ν and a constant diffusion (volatility) of σ. That is, $B_t^{(\nu,\sigma)}$ satisfies the SDE

$$dY_t = \nu\,dt + \sigma\,dB_t, \qquad t \geq 0, \tag{3.14}$$

with the initial condition $Y_0 = 0$.

Indeed, integrating both sides of (3.14) yields

$$Y_t - Y_0 = \nu t + \sigma(B_t - B_0)$$

and due to the initial condition we receive

$$Y_t = \nu t + \sigma B_t.$$

[5] Recall that the log-normal distribution is defined as an exponential transformation of the normal distribution. In general, the median of a normally distributed r.v. is equal to its expected value. However, as we have just shown, this is not true for the log-normal distribution. Because the exponential transformation is non-linear, the term $e^{-t\sigma^2/2}$ may be interpreted as the convexity correction.

Now, which is the SDE representation of the geometric Brownian motion defined in (3.6)? In order to find this out, we need a theoretical result from the stochastic calculus called the *Ito's lemma*, which concerns the concept of a stochastic differentiation.

Theorem 1 (Ito's lemma)**.** *Let* $\{Y_t,\, t \geq 0\}$ *be a diffusion process defined by Equation* (3.13) *and let* $f(t, y)$ *be a function for which the partial derivatives* $\partial f/\partial t$, $\partial f/\partial y$ *and* $\partial^2 f/\partial y^2$ *are defined and continuous. Then*

$$df(t, Y_t) = \left(\frac{\partial f}{\partial t} + \mu(Y_t, t)\frac{\partial f}{\partial y} + \frac{1}{2}\sigma^2(Y_t, t)\frac{\partial^2 f}{\partial y^2} \right) dt + \sigma(Y_t, t)\frac{\partial f}{\partial y}\, dB_t, \qquad t \geq 0.$$

Proof. The proof is described in Shreve (2004, Section 4.4.2). ❑

In case of the GBM, let us take $f(t, y) := S_0 e^y$ so that we have $S_t = f(t, B_t^{(\nu,\sigma)})$. Then the partial derivatives are $\partial f/\partial t = 0$, $\partial f/\partial y = f(t, y)$ and $\partial^2 f/\partial y^2 = f(t, y)$ and a straightforward application of Theorem 1 leads to

$$dS_t = \mu S_t\, dt + \sigma S_t\, dB_t, \qquad t \geq 0. \tag{3.15}$$

The differential formula for the GBM we have just derived, is very suitable for our purposes and we will therefore use it in the following chapters rather than the original relationship (3.6). Note that the arithmetic average of returns μ is used instead of the geometric average ν used in (3.14).

Chapter 4

Computing the Probability of Ruin

Suppose a retiring individual at a given age, with a certain initial capital (nest egg) at his disposal, decides to finance his constant retirement consumption stream via the self-annuitization approach. What is the probability of retirement ruin in his case? In this chapter, we try to give an answer to this problem...

We proceed to derive an analytic formula and demonstrate that the probability of ruin can be expressed as the probability that the stochastic present value (SPV) of future consumption is greater than the ratio of the initial wealth and the desired annual consumption. In the special (limiting) case of a perpetual consumption, we provide an exact analytic solution because the stochastic present value is known to obey a reciprocal gamma distribution. In the case of a lifetime consumption, we use the technique of moment matching to approximate the true probability distribution of the SPV under various types of mortality dynamics.

Most parts of the analysis carried out in this chapter come from Milevsky & Robinson (2000). The first of the two authors originally applied these concepts in the context of option pricing (see Milevsky & Posner, 1998).

4.1 Model for Mortality and Investment Returns

In order to receive further results, we need to make assumptions about both random elements concerning the individual probability of ruin: *investment returns* and *human mortality*. Using the concepts developed in the previous two chapters, from now on we shall assume that:

- The length of human life follows the Gompertz-Makeham law of mortality (described in Section 2.4), i.e. the instantaneous force of mortality is

$$\lambda(x) = \lambda + \frac{1}{b} \exp\left\{\frac{x-m}{b}\right\}, \qquad x \geq 0, \tag{4.1}$$

  where the parameters $m > 0$ and $b > 0$ are the modal length of human life and the dispersion coefficient, respectively.
- The market prices are driven by the geometric Brownian motion (introduced in Section 3.3), i.e. the real (inflation-free) price of the investment portfolio obeys the stochastic differential equation

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t, \qquad S_0 = 1, \tag{4.2}$$

where $\{B_t, t \geq 0\}$ is the standard Brownian motion (defined in Section 3.2).

- The average real rate of return μ and the volatility σ in (4.2) satisfy the portfolio growth restriction[1]

$$\mu - \frac{\sigma^2}{2} > 0 \tag{4.3}$$

and also

$$\mu - 2\sigma^2 \neq 0. \tag{4.4}$$

- The remaining lifetime random variable T_x is independent of the standard Brownian motion driving the portfolio price S_t from Equation (4.2).

4.2 Net Wealth Process

Let's say that a retiree at the age of x invests his initial wealth w to a stock portfolio and plans to withdraw a constant stream of k real CZK per annum for the rest of his life.

Due to the assumptions about the portfolio returns made in the previous section, the *net wealth process (NWP)* of the retiree, denoted by $\{W_t,\ t \geq 0\}$, must obey the SDE

$$dW_t = (\mu W_t - k)\ dt + \sigma W_t\, dB_t, \qquad W_0 = w. \tag{4.5}$$

Note that, while the diffusion of the net wealth process remains the same as the portfolio volatility σ, the drift has changed to $\mu W_t - k$. Consequently, depending on the size of k relative to μW_t, the drift might become negative over time, hence the value of the (real) net wealth W_t might eventually attain zero.

In the next step we want to solve the *linear stochastic differential equation (LSDE)* (4.5) and thus receive an explicit formula for the NWP. We will use the stochastic analogy to the method of variation of coefficients, which we present in the form of a lemma.

Lemma 2 (Karatzas & Shreve, 1991, formula (6.32)). *Consider a one-dimensional LSDE*

$$dX_t = [A(t)X_t + a(t)]\ dt + [S(t)X_t + \sigma(t)]\ dB_t, \tag{4.6}$$

where $\{B_t, t \geq 0\}$ is a one-dimensional standard Brownian motion with a filtration $\{\mathcal{F}_t\}$ and the coefficients A, a, S, σ are measurable, $\{\mathcal{F}_t\}$-adapted [2]*, almost surely locally bounded processes. Define*

$$Z_t := \exp\left\{\int_0^t A(u)\, du + \int_0^t S(u)\, dB_u - \frac{1}{2}\int_0^t S^2(u)\right\} du, \qquad t \geq 0.$$

Then there exists a unique solution to (4.6) *and this solution is given by the formula*

$$X_t = Z_t\left[X_0 + \int_0^t \frac{1}{Z_u}\{a(u) - S(u)\sigma(u)\}\ du + \int_0^t \frac{\sigma(u)}{Z_u}\, dB_u\right], \qquad t \geq 0.$$

[1] Recall the difference between arithmetic and geometric mean mentioned in Section 3.3. Using Equation (3.12), the inequality (4.3) can be rewritten as $\nu > 0$. Thus, appealing to the implicitly made assumption (3.2), the imposed parameter restriction (4.3) actually says that the expected value of the real annualized portfolio growth rate $\tilde{g}$ has to be positive.

[2] See e.g. Karatzas & Shreve (1991, Chapter 1) for the definition of a *filtration* and an *adapted process*.

Thanks to this theoretical result, we may formulate the following theorem:

Theorem 3 (Milevsky & Robinson, 2000, Theorem 1)**.** *The net wealth process* $\{W_t,\, t \geq 0\}$ *defined by* (4.5) *can be solved explicitly to yield*

$$W_t = S_t \left[w - k \int_0^t \frac{1}{S_u}\, du \right], \qquad t \geq 0, \tag{4.7}$$

where

$$S_s = \exp\left\{ \left(\mu - \frac{\sigma^2}{2} \right) s + \sigma B_s \right\}, \qquad s \geq 0$$

is the GBM solution to (4.2)*.*

Proof. Using the notation from the previous lemma, we set $X_t := W_t$, $A(t) := \mu$, $a(t) := -k$, $S(t) := \sigma$, $\sigma(t) := 0$. After applying the stochastic method of variation of coefficients, we get

$$Z_t = \exp\left\{ \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma B_t \right\}$$

and thus

$$\begin{aligned} W_t &= Z_t \left[W_0 + \int_0^t \frac{1}{Z_u} \{-k\}\, du \right] \\ &= \exp\left\{ \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma B_t \right\} \left[w - k \int_0^t \frac{1}{S_u}\, du \right]. \end{aligned}$$

❑

Note that another way to prove the previous theorem is applying Ito's Lemma (Theorem 1) to (4.7) and then arguing by the uniqueness of the solution of LSDE from Lemma 2.

Observe that by the definition of our problem, the ruin of the retiree occurs when the net wealth process hits or breaches a value of zero at some point prior to the (stochastic) time of death T_x. This means the probability of ruin can be expressed as

$$\text{PoR}\,(w) = \mathsf{P}\left[\inf_{0 \leq t \leq T_x} W_t \leq 0 \middle| W_0 = w \right]. \tag{4.8}$$

The PoR is an explicit function of the initial capital size w and an implicit function of the NWP parameters μ, σ, k as well as the parameters x, m, b governing mortality.

We can now use the explicit NWP formula (4.7) to prove an important property of the net wealth process: it never breaches zero more than once. In other words, once the NWP enters the negative region, it stays there.

Theorem 4 (Milevsky & Robinson, 2000, Theorem 2)**.** *For all* $T \in (0, \infty]$ *it holds that*

$$\mathsf{P}\left[\inf_{0 \leq t \leq T} W_t \leq 0 \middle| W_0 = w \right] = \mathsf{P}\left[W_T \leq 0 | W_0 = w \right]. \tag{4.9}$$

Proof. The NWP in Equation (4.7) consists of an exponential function S_t multiplied by the term in brackets. Since an exponential function is always positive, the process $\{W_t,\ t \geq 0\}$ attains a non-positive value at time t if and only if

$$\frac{w}{k} \leq \int_0^t \frac{1}{S_u}\, du.$$

Note that the integral on the right-hand side is monotonically increasing in the upper bound t of integration. Thus once the value of the integral exceeds w/k, it stays greater than w/k. Consequently, both probabilities in (4.9) are equal. ❑

4.3 The Eventual Probability of Ruin

Let us now for a while assume that the death of the individual never occurs. Then we would be interested in the so-called *eventual probability of ruin (EPoR)* defined as

$$\mathrm{EPoR}\,(w) := \mathsf{P}\left[\inf_{0\leq t\leq\infty} W_t \leq 0 \middle| W_0 = w\right]. \tag{4.10}$$

Although taking $T_x = \infty$ might seem like a step aside from our initial goal because obviously no one can live forever, we will see that this special case leads to a very important analytical result, which will be very useful to us later.

First of all, note that Theorem 4 also covers the eventual case, since it was formulated for all $T \in (0, \infty]$. The EPoR in Equation (4.10) can be therefore expressed as

$$\begin{aligned} \mathrm{EPoR}\,(w) &= \mathsf{P}\left[W_\infty \leq 0 | W_0 = w\right] \\ &= \mathsf{P}\left[\frac{w}{k} \leq \int_0^\infty \frac{1}{S_u}\, du\right] \\ &= \mathsf{P}\left[\frac{w}{k} \leq \int_0^\infty e^{-\left(\mu-\sigma^2/2\right)u-\sigma B_u}\, du\right]. \end{aligned} \tag{4.11}$$

The integral in the last equation can be interpreted as the *present value of a stochastic perpetuity (PVSP)*. This perpetuity pays continuously one real CZK per annum and is subjected to a stochastic Brownian real rate of interest. Equation (4.11) says that the eventual probability of ruin is equal to the probability that the PVSP is greater than or equal to the ratio w/k. This means that we have reduced our problem to finding a probability distribution for the PVSP random variable

$$Z := \int_0^\infty \frac{1}{S_u}\, du. \tag{4.12}$$

Before we proceed, let us briefly review an important probability distribution that will play a critical role in the rest of this chapter. First, recall that a random variable X follows the *gamma distribution* with a shape parameter $\alpha > 0$ and a rate parameter[3] $\beta > 0$, denoted by $X \sim \mathrm{Gamma}(\alpha, \beta)$, if the corresponding PDF

[3] A *rate parameter* of a probability distribution is defined simply as the reciprocal of its scale parameter. Another common parametrization of the gamma distribution is the *shape-scale* parametrization, which uses the scale parameter $\theta = 1/\beta$.

is

$$g(x;\ \alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \qquad x \geq 0.$$

Now in addition, we say that a random variable Y obeys the *reciprocal gamma (RG) distribution*[4] with a shape parameter $\alpha > 0$ and a scale parameter $\beta > 0$, denoted by $Y \sim \mathrm{RG}(\alpha,\beta)$, if

$$Y = \frac{1}{X},$$

where $X \sim \mathrm{Gamma}(\alpha,\beta)$. Consequently, the PDF of an RG random variable must satisfy

$$\begin{aligned} g_R(y;\ \alpha,\beta) &= \frac{g(1/y;\ \alpha,\beta)}{y^2} \\ &= \exp\left\{-\frac{1}{y\beta}\right\} \frac{y^{-(\alpha+1)}}{\Gamma(\alpha)\beta^\alpha}, \qquad y > 0. \end{aligned}$$

Note that we changed the role of β and it now represents the scale parameter. Writing $G(x;\ \alpha,\beta)$ and $G_R(x;\ \alpha,\beta)$ for the CDF of the gamma distribution and RG distribution respectively, we have for any $x > 0$:

$$G(x;\ \alpha,\beta) = \mathsf{P}[X \leq x] = \mathsf{P}[Y \geq 1/x] = 1 - G_R(1/x;\ \alpha,\beta).$$

The first two (non-central) moments of the RG distribution are

$$\mathsf{E}[Y] = \frac{1}{\beta(\alpha-1)}, \tag{4.13}$$

$$\mathsf{E}[Y^2] = \frac{1}{\beta^2(\alpha-1)(\alpha-2)}. \tag{4.14}$$

A picture of the probability density functions of an RG random variable under various parameter values is provided by Figure 6.6 and Figure 6.7 in the Appendix.

We are now in a position to present the main result for the eventual probability of ruin.

Theorem 5 (Milevsky & Robinson, 2000, Theorem 3)**.** *The PVSP random variable*

$$Z = \int_0^\infty e^{-\left(\mu-\sigma^2/2\right)u-\sigma B_u}\, du$$

obeys a reciprocal gamma distribution with parameters $\alpha = \frac{2\mu}{\sigma^2} - 1$ *and* $\beta = \frac{\sigma^2}{2}$.

Proof. The proof relies on non-trivial martingale techniques from the theory of stochastic calculus and goes beyond the scope of this text. We refer the interested reader to Milevsky (1997) where the detailed steps are described. ❏

[4] Sometimes also referred to as the *inverse gamma distribution*.

Thus, using the formula (4.11) together with the last theorem, we receive our final expression for the eventual probability of ruin:

$$\begin{aligned}\text{EPoR}\,(w) &= \mathsf{P}\left[\frac{w}{k} \leq Z\right] \\ &= 1 - G_R\left(\frac{w}{k};\, \frac{2\mu}{\sigma^2} - 1, \frac{\sigma^2}{2}\right) = G\left(\frac{k}{w};\, \frac{2\mu}{\sigma^2} - 1, \frac{\sigma^2}{2}\right). \end{aligned} \tag{4.15}$$

To sum up, we have shown that the eventual probability of ruin is equal to the probability that the present value of a stochastic perpetuity is greater than or equal to the ratio of the initial capital w and the desired annual consumption k. Since the PVSP follows a reciprocal gamma distribution with parameters depending on the characteristics μ and σ of the underlying Brownian motion, we conclude that the EPoR can be computed as the CDF of the corresponding gamma distribution evaluated at k/w.

4.4 The Lifetime Probability of Ruin

In order to separate our main problem from the special case discussed in Section 4.3, let us introduce a new notation for the *lifetime probability of ruin (LPoR)*, where the random time of death T_x (i.e. the remaining lifetime of an individual aged x) takes only finite values. In accordance with Equation (4.8), we define the lifetime probability of ruin as

$$\text{LPoR}\,(w) = \mathsf{P}\left[\inf_{0 \leq t \leq T_x} W_t \leq 0 \middle| W_0 = w\right]. \tag{4.16}$$

Similarly as in the previous section, we can apply Theorem 4 to receive

$$\begin{aligned}\text{LPoR}\,(w) &= \mathsf{P}\left[W_{T_x} \leq 0 | W_0 = w\right] \\ &= \mathsf{P}\left[\frac{w}{k} \leq \int_0^{T_x} \frac{1}{S_u}\, du\right] &(4.17)\\ &= \mathsf{P}\left[\frac{w}{k} \leq \int_0^{T_x} e^{-\left(\mu - \sigma^2/2\right)u - \sigma B_u}\, du\right]. &(4.18)\end{aligned}$$

In other words, the lifetime probability of ruin can be expressed as the probability that the *stochastic present value (SPV)* of a life annuity paying continuously one CZK per annum is greater than or equal to the ratio of initial capital w and the desired (real) annual consumption k. Hence, denoting the SPV for a fixed maturity by

$$Z_t := \int_0^t e^{-(\mu - \sigma^2/2)s - \sigma B_s}\, ds, \tag{4.19}$$

we have transformed our problem to the one to find a suitable probability distribution for Z_{T_x}, which is the SPV of lifetime consumption.

Unfortunately, according to Milevsky & Robinson (2000), there is no probability density function for Z_{T_x}, which could be expressed in a closed form. However, several approximation techniques have been proposed in this case.

For instance, the authors of the aforementioned article have noticed a remarkable connection between the SPV random variable and a financial derivative called the *Arithmetic Asian option (AAO)*[5]. Recall that the payoff from a plain vanilla call option is $\max[\hat{S}_T - K, 0]$, where K is the strike price and $\hat{S}_T$ denotes the price of the underlying at the time of maturity T. On the other hand, the payoff from the Arithmetic Asian call option is

$$\max\left[\frac{1}{n}\sum_{i=1}^{n}\hat{S}_{t_i} - K, 0\right], \tag{4.20}$$

where $\{t_1, \ldots, t_n\}$ is the set of predetermined observation dates. In addition, in the limiting case for a large number of observation dates compared to the lifetime of the option we can define the payoff from a "continuous" AAO as

$$\max\left[\frac{1}{T}\int_0^T \hat{S}_t\,dt - K, 0\right],$$

where $\{\hat{S}_t,\ 0 \le t \le T\}$ is the price process of the underlying. Consequently, the (real-world) probability that an AAO expires in-the-money can be expressed as

$$\mathsf{P}\left[\int_0^T \hat{S}_t\,dt \ge TK\right].$$

Therefore, if we define a new price process $\hat{S}_t := 1/S_t$, we can conclude that the lifetime probability of ruin (see (4.17)) is equal to the (real-world) probability that a certain Arithmetic Asian option will expire in-the-money.

The last result led the authors of Milevsky & Robinson (2000) to the idea of approximating the distribution of the lifetime SPV random variable Z_{T_x} via techniques that had been successfully used in the AAO pricing literature[6] to approximate the distribution of $\int_0^T \hat{S}_t\,dt$.

4.5 Reciprocal Gamma Approximation

In this section we will present a convenient method of the SPV random variable distribution approximation, which was also used e.g. by Milevsky & Posner (1998) in the Arithmetic Asian option pricing context.

First, recall the definition of the lifetime SPV r.v.

$$Z_{T_x} := \int_0^{T_x} e^{-(\mu-\sigma^2/2)s-\sigma B_s}\,ds. \tag{4.21}$$

In Section 4.3 we have shown that the present value of a stochastic perpetuity follows the reciprocal gamma distribution. In some sense, the PVSP random variable Z is the limiting case for the SPV r.v. defined by Equation (4.21). The

[5]The AAOs are one of the basic forms of exotic options. Unlike in the case of a plain vanilla option, the payoff from the AAO is path-dependant. More specifically, the value of AAO is determined by the arithmetic average of prices of the underlying asset, measured over some predetermined set of observation dates. In general, Asian options are typically cheaper than (European or American) vanilla options, since the averaging feature reduces the volatility inherent in the option.

[6]See for example Milevsky & Posner (1998).

RG distribution therefore becomes a natural candidate for our approximation of Z_{T_x}. In particular, we will use the moment matching technique. Since the RG distribution has two degrees of freedom, we match the first two moments.

We will now proceed to obtain the first moment of Z_{T_x}. First of all, recall that for a random variable X the moment-generating function evaluated at $t \in \mathbb{R}$ is defined as $M_X(t) := \mathsf{E}[e^{tX}]$, providing that this expectation exists. In particular, for a normally distributed r.v. $Y \sim \mathrm{N}(\mu, \sigma^2)$ we receive

$$M_Y(t) = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}, \qquad t \in \mathbb{R}. \tag{4.22}$$

Consequently, since the standard Brownian motion fulfils $B_s \sim \mathrm{N}(0, s)$, we may write

$$M_{B_s}(-\sigma) = \mathsf{E}\left[e^{-\sigma B_s}\right] = e^{s\sigma^2/2}. \tag{4.23}$$

Thus, the first moment of the SPV r.v. can be calculated as

$$\begin{aligned}
\mathsf{E}\left[Z_{T_x}\right] &= \mathsf{E}\left[\mathsf{E}\left[\int_0^t e^{-(\mu-\sigma^2/2)s-\sigma B_s}\, ds \middle| T_x = t\right]\right] && (4.24)\\
&= \mathsf{E}\left[\int_0^t e^{-(\mu-\sigma^2/2)s}\, \mathsf{E}\left[e^{-\sigma B_s}\right]\, ds \middle| T_x = t\right] \\
&= \mathsf{E}\left[\int_0^t e^{-(\mu-\sigma^2)s}\, ds \middle| T_x = t\right], && (4.25)
\end{aligned}$$

where in (4.24) we used the law of iterated expectations, i.e.

$$\mathsf{E}[Z_{T_x}] = \mathsf{E}[\mathsf{E}[Z_{T_x} | T_x = t]].$$

Denoting the PDF of the remaining lifetime r.v. T_x by $f_x(t)$, we can change the order of integration in (4.25) to obtain

$$\begin{aligned}
\mathsf{E}\left[Z_{T_x}\right] &= \int_0^\infty \int_0^t e^{-(\mu-\sigma^2)s}\, ds f_x(t)\, dt \\
&= \int_0^\infty e^{-(\mu-\sigma^2)s} \int_s^\infty f_x(t)\, dt\, ds \\
&= \int_0^\infty e^{-(\mu-\sigma^2)s}\, {}_s p_x\, ds. && (4.26)
\end{aligned}$$

The last equality in (4.26) follows from formula (2.3).

Let us now, for the sake of convenience, define the following present value operator:

$$A\,(\xi|\cdot) := \int_0^\infty \exp\{-\xi s\}\, {}_s p_x\, ds. \tag{4.27}$$

This expression can be interpreted as the price of a continuous life annuity under a continuously compounded interest rate ξ. Rewriting the result (4.26), we can now conclude that the *first moment of the SPV random variable* is

$$\tilde{M}_1 = \mathsf{E}\left[Z_{T_x}\right] = A(\mu - \sigma^2|\cdot). \tag{4.28}$$

Note that we have derived this result without using any specific information about the remaining lifetime random variable T_x (except for the assumption that the PDF of T_x exists). The last formula is therefore universally applicable for an arbitrary mortality law. The imposed mortality dynamics then determine the shape of $A(\xi|\cdot)$ with the corresponding parameters instead of the dot.

Let us now use our assumption that T_x follows the Gompertz-Makeham law of mortality. In Section 2.4 we have shown that in this case the conditional probability of survival is

$$ {}_tp_x = \exp\left\{-\lambda t + e^{(x-m)/b}\left(1 - e^{t/b}\right)\right\}. \tag{4.29} $$

Substituting (4.29) into (4.27) and performing the change of variables $z := e^{(x-m+s)/b}$ then gives

$$ \begin{aligned} A(\xi|\lambda, m, b, x) &= \exp\left\{e^{(x-m)/b}\right\}\int_0^\infty \exp\left\{-(\xi+\lambda)s - e^{(x-m+s)/b}\right\} ds \\ &= b\exp\left\{e^{(x-m)/b} + (\xi+\lambda)(x-m)\right\}\int_{\exp\{\frac{x-m}{b}\}}^\infty z^{-(\xi+\lambda)b-1}e^{-z}\, dz \\ &= b\exp\{e^{(x-m)/b} + (\xi+\lambda)(x-m)\}\Gamma(-(\xi+\lambda)b, e^{(x-m)/b}). \end{aligned} \tag{4.30} $$

Recall that the last expression in (4.30) stands for the (upper) incomplete Gamma function defined by (2.23). Also observe that if we assumed T_x to follow just the *Gompertz* law of mortality instead of the Gompertz-Makeham model, it would suffice to set $\lambda = 0$ in (4.30) to yield

$$ A(\xi|0, m, b, x) = b\exp\{e^{(x-m)/b} + \xi(x-m)\}\Gamma(-\xi b, e^{(x-m)/b}). \tag{4.31} $$

We will now proceed to obtain the second moment of the SPV random variable. The technique is similar to the previous case. We have

$$ \begin{aligned} \mathsf{E}\left[Z_{T_x}^2\right] &= \mathsf{E}\left[\mathsf{E}\left[Z_t^2\middle|T_x = t\right]\right] \\ &= \mathsf{E}\left[\mathsf{E}\left[\left(\int_0^t \frac{1}{S_s}\,ds\right)^2\middle|T_x = t\right]\right] \\ &= \mathsf{E}\left[\mathsf{E}\left[2\int_0^t\int_0^s \frac{1}{S_s}\frac{1}{S_z}\,dz\,ds\middle|T_x = t\right]\right] && (4.32) \\ &= \mathsf{E}\left[2\int_0^t\int_0^s e^{-(\mu-\sigma^2/2)(s+z)}\,\mathsf{E}\left[e^{-\sigma B_s}e^{-\sigma B_z}\right]\,dz\,ds\middle|T_x = t\right]. && (4.33) \end{aligned} $$

In (4.32) we have used a generally applicable formula

$$ \left(\int_0^t h(s)\,ds\right)^2 = 2\int_0^t\int_0^s h(s)h(z)\,dz\,ds, $$

which can be validated via integration by parts (providing that the integrals on the both sides exist).

We now require the covariance term $\mathsf{E}[e^{-\sigma B_s}e^{-\sigma B_z}]$, where $z < s$. Recall the definition of the standard Brownian motion from Section 3.2. Using the properties (iii) and (iv) together with the moment-generating function for a normal random variable, we have

$$ \begin{aligned} \mathsf{E}\left[e^{-\sigma B_s}e^{-\sigma B_z}\right] &= \mathsf{E}\left[e^{-\sigma(B_s - B_z)}\right]\mathsf{E}\left[e^{-2\sigma B_z}\right] \\ &= e^{(s-z)\sigma^2/2}e^{2z\sigma^2} = e^{(s+3z)\sigma^2/2}. \end{aligned} \tag{4.34} $$

Thanks to the assumption (4.4), substituting (4.34) into (4.33) now yields

$$
\begin{aligned}
\mathsf{E}\left[Z_{T_x}^2\right] &= \mathsf{E}\left[2\int_0^t e^{-(\mu-\sigma^2)s}\int_0^s e^{-(\mu-2\sigma^2)z}\,dz\,ds\middle|T_x=t\right] \\
&= \frac{2}{\mu-2\sigma^2}\,\mathsf{E}\left[\int_0^t e^{-(\mu-\sigma^2)s}-e^{-(2\mu-3\sigma^2)s}\,ds\middle|T_x=t\right] \\
&= \frac{2}{\mu-2\sigma^2}\int_0^\infty\int_0^t\left(e^{-(\mu-\sigma^2)s}-e^{-(2\mu-3\sigma^2)s}\right)ds f_x(t)\,dt \\
&= \frac{2}{\mu-2\sigma^2}\int_0^\infty\left(e^{-(\mu-\sigma^2)s}-e^{-(2\mu-3\sigma^2)s}\right)\int_s^\infty f_x(t)\,dt\,ds \\
&= \frac{2}{\mu-2\sigma^2}\int_0^\infty\left(e^{-(\mu-\sigma^2)s}-e^{-(2\mu-3\sigma^2)s}\right){}_sp_x\,ds. \qquad (4.35)
\end{aligned}
$$

Finally, using the notation introduced in Equation (4.27), we conclude that the *second moment of the SPV random variable* is

$$
\tilde{M}_2 = \mathsf{E}\left[Z_{T_x}^2\right] = \frac{2}{\mu-2\sigma^2}\left[A(\mu-\sigma^2|\cdot)-A(2\mu-3\sigma^2|\cdot)\right]. \qquad (4.36)
$$

Now, recall that according to (4.13) and (4.14) the first two moments of the RG distribution with (positive) parameters α and β are

$$
M_1 = \frac{1}{\beta(\alpha-1)} \qquad (4.37)
$$

$$
M_2 = \frac{1}{\beta^2(\alpha-1)(\alpha-2)}. \qquad (4.38)
$$

The last two equations can be easily solved for α and β to yield

$$
\alpha = \frac{2M_2-M_1^2}{M_2-M_1^2}, \qquad \beta = \frac{M_2-M_1^2}{M_2M_1}.
$$

Hence, the fitted parameter values are

$$
\hat{\alpha} = \frac{2\tilde{M}_2-\tilde{M}_1^2}{\tilde{M}_2-\tilde{M}_1^2}, \qquad \hat{\beta} = \frac{\tilde{M}_2-\tilde{M}_1^2}{\tilde{M}_2\tilde{M}_1}, \qquad (4.39)
$$

where $\tilde{M}_1$ and $\tilde{M}_2$ are defined by (4.28) and (4.36), respectively. We conclude that the approximate distribution of the SPV random variable is

$$
Z_{T_x} \stackrel{\sim}{\sim} \mathrm{RG}(\hat{\alpha},\hat{\beta}). \qquad (4.40)
$$

Thus, going all the way back to the formula (4.18), we can summarize that

$$
\begin{aligned}
\mathrm{LPoR}(w) &= \mathsf{P}\left[\frac{w}{k}\le Z_{T_x}\right] \\
&\cong 1-G_R\left(\frac{w}{k};\,\hat{\alpha},\hat{\beta}\right) = G\left(\frac{k}{w};\,\hat{\alpha},\hat{\beta}\right), \qquad (4.41)
\end{aligned}
$$

where $\hat{\alpha}$ and $\hat{\beta}$ are defined by (4.39). In other words, the lifetime probability of ruin is approximately equal to the CDF of a suitable gamma distribution evaluated at the ratio of the desired consumption k to the initial wealth w.

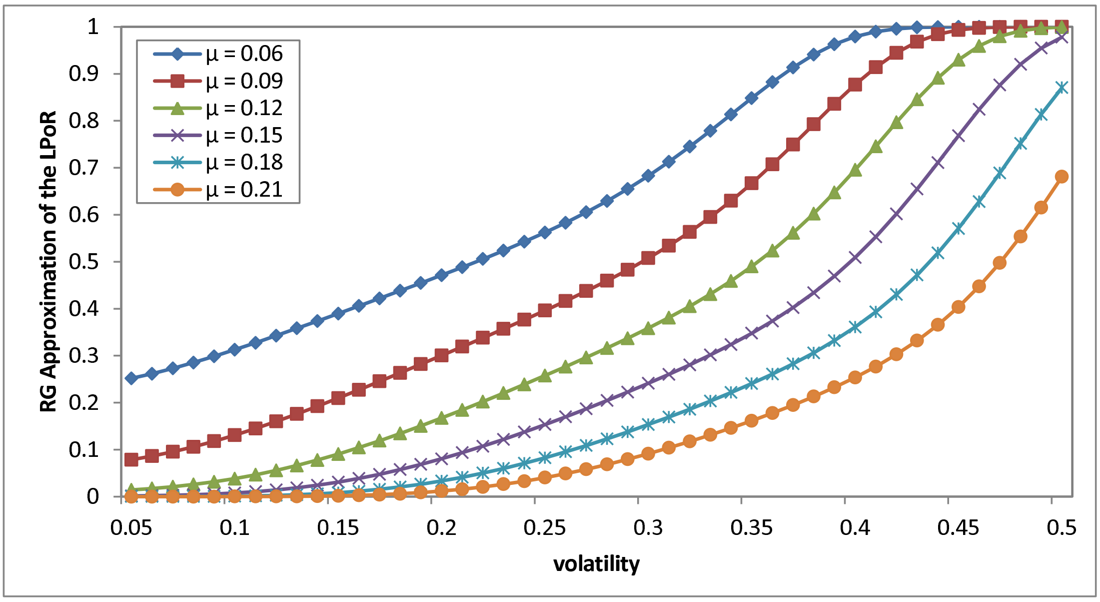


Figure 4.1: The RG approximation of the LPoR as a function of the investment volatility σ, for various values of the expected real rate of return μ. A 65-year-old individual with an initial wealth $w = 12$ CZK, who plans to consume $k = 1$ real CZK per year was assumed. The parameters of the Gompertz distribution were set to $m = 85$ and $b = 9$.

Figure 4.1 shows the LPoR approximated by the RG formula (4.41). In this case the mortality was assumed to follow the Gompertz law. The LPoR is displayed as a function of the investment volatility σ for various values of the expected real rate of return μ. The retiree was assumed to be aged $x = 65$. Note that although the initial capital and the annual consumption rate were set to $w = 12$ CZK and $k = 1$ CZK, respectively, only the ratio of k and w matters. In different words, as long as the ratio k/w does not change, the both values can be scaled arbitrarily without affecting the LPoR. Figure 4.1 evidently corresponds with the intuitive idea that (all other things being equal) the LPoR should be decreasing in μ and increasing in σ. Thus, a natural investment strategy minimizing the LPoR would be to invest the initial capital to a portfolio with the highest possible μ and the lowest possible σ. However, in an efficient financial market only a shift of the both μ and σ in the same direction can be accomplished, making the portfolio either riskier (higher values of μ and σ) or more conservative (lower values of μ and σ). The difference in the LPoR corresponding to diversely conservative portfolios will be examined in Section 6.5.

Some additional facts regarding the reciprocal gamma approximation should be mentioned. First of all, let us once again point out that the RG approximation procedure is quite universal with respect to the mortality assumptions. If we had imposed different mortality dynamics, our results would remain the same except for the shape of $A(\xi|\cdot)$. In (4.30) we have derived the shape of this expression under the Gompertz-Makeham law. What would the result, for instance, in the case of the Exponential law of mortality look like? Recall from Section 2.3 that in this case the IFM curve is constant across all ages, i.e. $\lambda(x) \equiv \lambda$. The corresponding conditional probability of survival is then ${}_tp_x = e^{-\lambda t}$, as we have shown in (2.13).

Substituting into (4.27) then gives[7]

$$A(\xi|\lambda) = \int_0^\infty e^{-\xi s} e^{-\lambda s}\, ds = \frac{1}{\xi + \lambda}. \tag{4.42}$$

Observe how simple the result looks compared to the formula for $A(\xi|\lambda, m, b, x)$ we have received under the Gompertz-Makeham law. In fact, in this case it makes sense to substitute $A(\xi|\lambda)$ into (4.28) and (4.36) to obtain

$$\tilde{M}_{1,E} = \frac{1}{\mu - \sigma^2 + \lambda}, \qquad \tilde{M}_{2,E} = \frac{2}{(\mu - \sigma^2 + \lambda)(2\mu - 3\sigma^2 + \lambda)}.$$

Plugging the exponential SPV moments into (4.39) leads to

$$\hat{\alpha}_E = \frac{2\mu + 4\lambda}{\sigma^2 + \lambda} - 1, \qquad \hat{\beta}_E = \frac{\sigma^2 + \lambda}{2}. \tag{4.43}$$

Since the gamma distribution is usually available in spreadsheet calculators, this actually means that, if one is willing to simplify the situation somewhat by assuming the Exponential law of mortality, the approximate lifetime probability of ruin can be easily calculated, for example, in MS Excel. It is also worth noticing that the results obtained under the Exponential law are in line with the statement we made about the perpetual case. Indeed, after setting $\lambda = 0$ in (4.43), which means that the mortality decrement is ignored, we arrive at the parameter values presented in Theorem 5.

Finally, let us remark that there is a number of ways in which the RG formula can be manipulated. Perhaps the most useful one is calculating the so-called *sustainable spending rate (SSR)* by inverting (4.41) to yield

$$k \cong w \cdot G^{-1}\left(\mathrm{LPoR}(w);\ \hat{\alpha}, \hat{\beta}\right). \tag{4.44}$$

This means, one can use (4.44) to calculate the maximum annual spending rate for a fixed initial capital w, so that the corresponding consumption plan results in a given tolerated lifetime probability of ruin. A numerical application of (4.44) is presented in Section 6.4.

[7] Observe that in this case, since both values add up in the denominator, additive changes in ξ and λ have exactly the same effect on $A(\xi|\lambda)$ in (4.42). Taking the aforementioned interpretation of $A(\xi|\cdot)$ into account, this means that for the value of a lifetime annuity under the Exponential law of mortality only the sum of the interest rate ξ and the IFM λ is relevant.

Chapter 5

Accuracy of Reciprocal Gamma Approximation

In this chapter we verify the accuracy of the reciprocal gamma approximation for the lifetime probability of ruin. Being the key theoretical result of this text, the RG approximation formula (4.41) was obtained via matching of the first two moments of the stochastic present value random variable. Since the exact value of the LPoR is not known, we compare the results of the RG approximation with the results obtained by the *Monte Carlo (MC) simulations* approximation of the true LPoR.

5.1 The Monte Carlo Simulation Technique

The term *Monte Carlo methods (Monte Carlo experiments, Monte Carlo simulations)* refers to a broad class of computational algorithms sharing a common feature: they rely on a repeated random sampling to obtain numerical results. Although several attempts were made, there is no general consensus on how exactly the Monte Carlo should be defined. In this text, we use this term to refer to our technique of the LPoR approximation based on stochastic simulations.

Relying on the assumptions made in Section 4.1, one can simulate the event of retirement ruin by generating the time of death (i.e. sampling from the Gompertz distribution with the PDF (2.16)) and then simulating the path of the net wealth process (independently on the time of death). In a given trial, the ruin of the retiree occurs if the NWP path crosses zero prior to the simulated time of death. Thinking of the result of a given trial as a Bernoulli distributed r.v., the LPoR can be viewed as the probability of success, and hence also as the mean of the corresponding Bernoulli distribution. Thanks to the *Weak Law of Large Numbers (WLLN)*, one can claim that, after repeating this trial many times (under the same conditions but independently), the LPoR can be approximated by the ratio of observed ruins and the total number of trials.

The number of trials one needs to carry out to obtain a reasonably precise MC approximation of the LPoR depends on the speed of convergence (assured by the WLLN) to the true expected value. Figure 5.1 displays the MC LPoR curve as a function of the investment volatility σ for various number of MC simulations. Obviously, the more trials carried out, the closer approaches the MC approximation to the true LPoR, and thus the MC LPoR curve in the figure

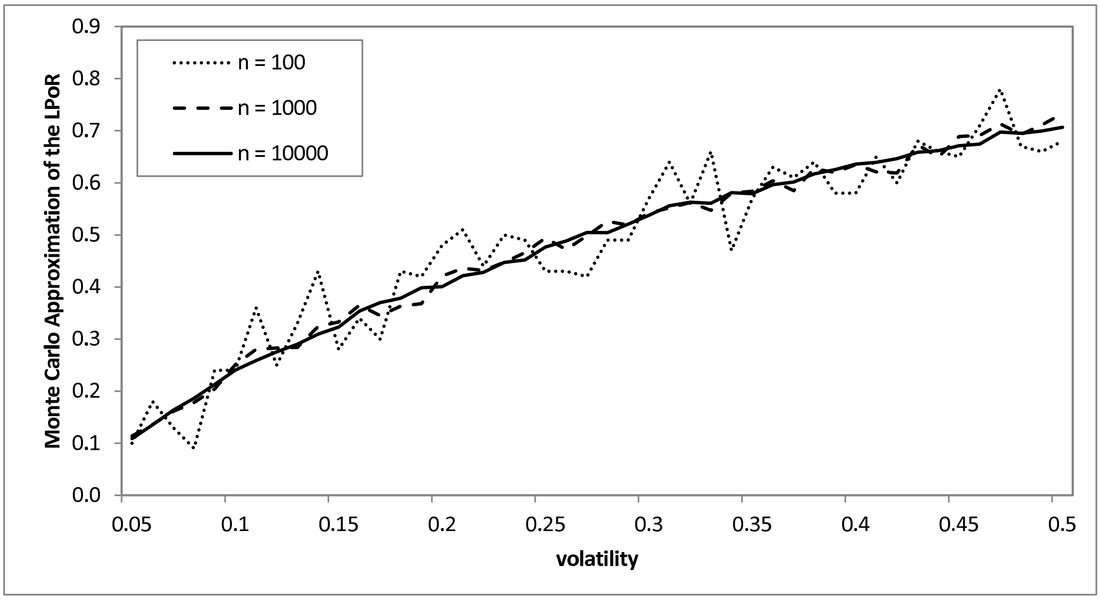


Figure 5.1: The speed of convergence of the Monte Carlo LPoR approximation. The figure displays the MC approximation of the LPoR as a function of the investment volatility σ, for various number of simulations n. A 65-year-old individual with an initial wealth $w = 10$ CZK, who plans to consume $k = 1$ real CZK per year was assumed. The parameters of the Gompertz distribution were set to $m = 85$ and $b = 9$. The assumed expected real rate of investment return is $\mu = 0.1$.

becomes smoother. On the other hand, since the amount of (pseudo)random numbers to be generated for every simulation is not negligible[1] and the time complexity of the MC approximation is increasing linearly with the number of simulations n, the possible amount of trials is significantly limited. Based on a graphical analysis of the MC LPoR results, we have decided to set $n = 10\,000$ as a reasonable compromise solution to this trade-off between time complexity and precision. The following section provides a comparison of the RG LPoR with the results of the MC LPoR approximation obtained for $n = 10\,000$.

5.2 Measuring the RG LPoR Accuracy

Recall Figure 4.1, which showed the reciprocal gamma approximation of the LPoR as a function of the investment volatility σ, for various values of the expected real rate of return μ. In the same manner, Figure 5.2 displays the results of the Monte Carlo LPoR approximation. The assumed parameter values are the same as in Figure 4.1. Observe that, similarly to the RG LPoR, the MC LPoR is increasing in σ and decreasing in μ. However, for high levels of volatility the MC values are significantly lower than the RG values in Figure 4.1. Actually, since the LPoR curves in Figure 5.2 seem fairly smooth, one can assume that the MC values are not far from the true LPoR. Thus, in this situation, one can use

[1] The simulation of the net wealth process path was constructed by discretizing time in increments of 1/250 years (one increment per trading day). This means that one (random) increment of the NWP has to be generated for every trading day of the individual's remaining lifetime.

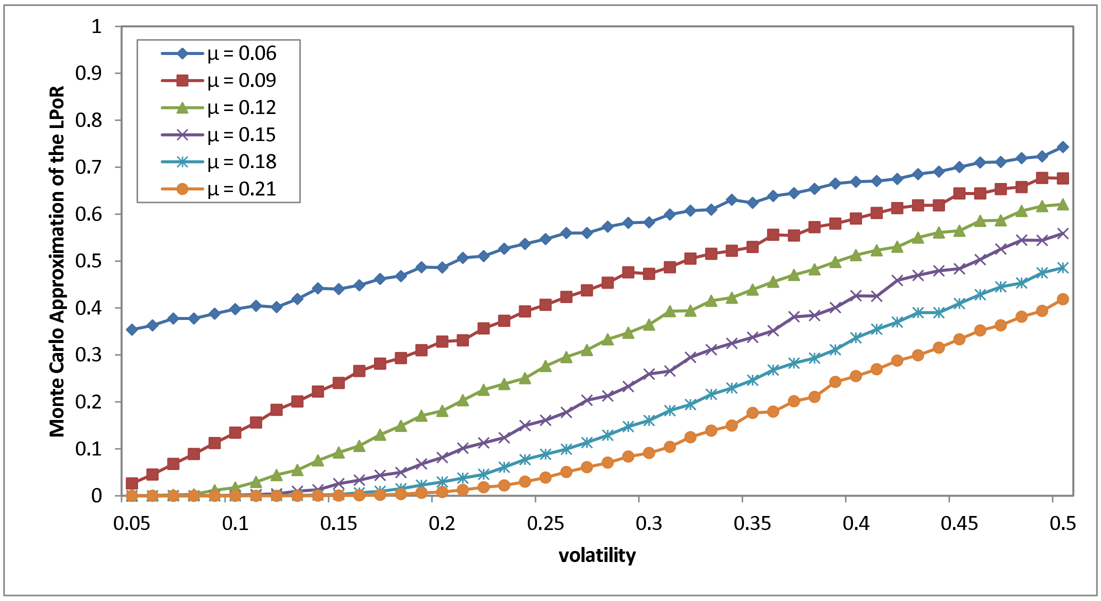


Figure 5.2: The MC approximation of the LPoR as a function of the investment volatility σ, for various values of the expected real rate of return μ. A 65-year-old individual with an initial wealth $w = 12$ CZK, who plans to consume $k = 1$ real CZK per year was assumed. The parameters of the Gompertz distribution were set to $m = 85$ and $b = 9$.

the difference between the RG LPoR and the MC LPoR as a measure of the RG approximation accuracy.

Figure 5.3 displays the discrepancy between the RG LPoR and the MC LPoR as a function of the investment volatility σ in the same setting as in Figure 4.1 and Figure 5.2. Observe that in all cases the discrepancy remains reasonably small as long as σ does not increase beyond a certain level at which the RG approximation starts to overestimate the LPoR significantly. The higher the expected rate of return μ, the later (in terms of σ) starts the RG approximation to deteriorate. To explain this effect, recall the formula (4.35) for the second moment of the stochastic present value random variable:

$$\mathsf{E}\left[Z_{T_x}^2\right] = \frac{2}{\mu - 2\sigma^2} \int_0^\infty \left(e^{-(\mu-\sigma^2)s} - e^{-(2\mu-3\sigma^2)s}\right) {}_s p_x \, ds. \tag{5.1}$$

First, note that the integrand in (5.1) is a sum of two exponentials multiplied by a probability of survival. Regardless of μ and σ, the equality

$${}_s p_x = 0, \qquad s > \omega - x$$

ensures that the value of the integral always remains finite. Thus, thanks to the assumption (4.4)

$$\mu - 2\sigma^2 \neq 0,$$

the second moment of the SPV always exists. The relationship between μ and σ, in particular the sign of $\mu - 2\sigma^2$, determines which of the exponentials in the integrand outweighs the other for $s \to \infty$. In the case of high volatility relative to the expected rate of return, the second moment (5.1) becomes very large due to the second exponential term in the integrand and hence the moment

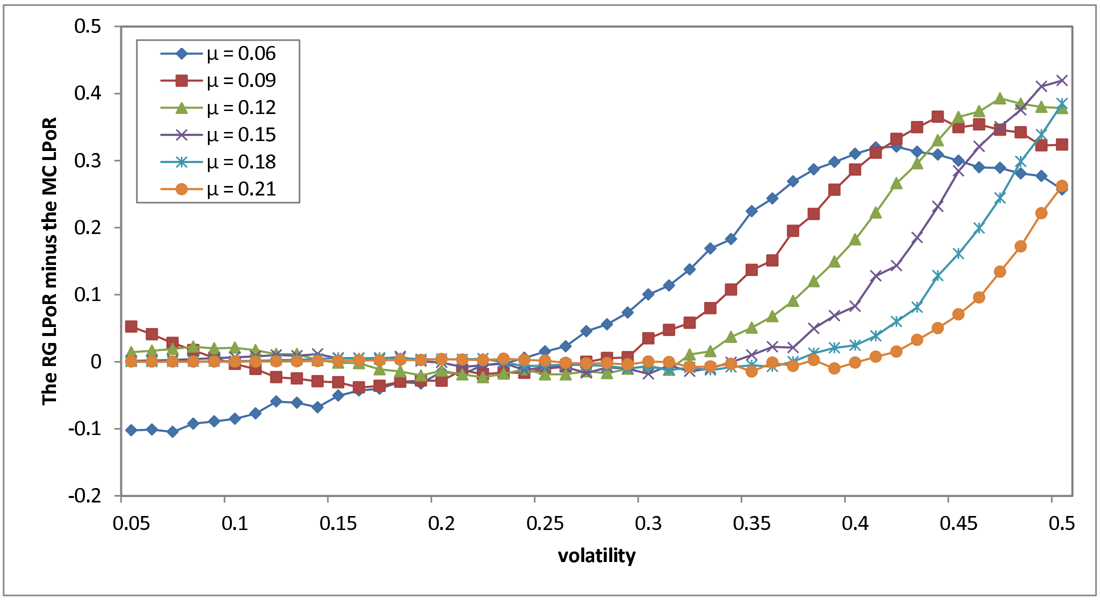


Figure 5.3: The discrepancy between the RG and the MC approximations of the LPoR as a function of the investment volatility σ, for various values of the expected real rate of return μ. A 65-year-old individual with an initial wealth $w = 12$ CZK, who plans to consume $k = 1$ real CZK per year was assumed. The parameters of the Gompertz distribution were set to $m = 85$ and $b = 9$.

matching approximation deteriorates. The discrepancy in Figure 5.3 starts to grow significantly approximately when σ increases beyond $\sqrt{2\mu/3}$, which is when the sign of the second exponent in (5.1) becomes positive. Due to the ${}_sp_x$ factor in the integrand, the exact threshold at which the RG approximation noticeably deteriorates also depends on the Gompertz parameters m and b, as well as on the initial age x of the individual.

So far, we have illustrated the behaviour of the both LPoR approximations in dependency on the expected real rate of investment return μ. In the same manner, one can analyse the behaviour of the LPoR in dependency on the ratio of the initial capital w and the real annual spending rate k. Figure 6.8 and Figure 6.9 in the Appendix show the RG LPoR and the MC LPoR, respectively, as a functions of the investment volatility σ, for various amounts of the initial capital w. The assumed spending rate is set to $k = 1$ real CZK per year. The both figures confirm the intuitive idea that the LPoR decreases in w when the consumption rate k and all other parameters are fixed.

The discrepancy between the RG LPoR and the MC LPoR is displayed in Figure 5.4. Similarly to the previous case, the RG approximation overestimates the LPoR at higher levels of volatility. However, observe that unlike in Figure 5.3, the threshold at which the RG approximation starts to deteriorate appears to be roughly the same for every curve, which corresponds with the fact that the second moment of the SPV does not depend on w.

Since for a given population the Gompertz parameters m and b are usually fixed, the last factor influencing the accuracy of the RG LPoR approximation we are interested in, is the initial age x of the retiree. Similarly to the previous two cases, the RG LPoR and the MC LPoR for various values of x are shown in Figure 6.10 and Figure 6.11 in the Appendix. It is apparent from the figures that

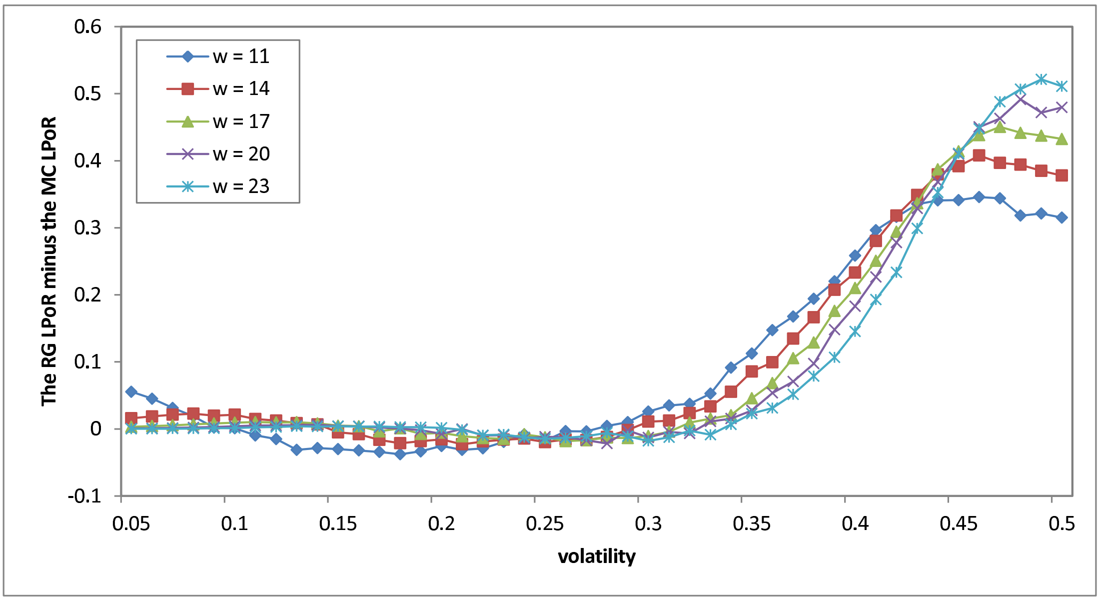


Figure 5.4: The discrepancy between the RG and the MC approximations of the LPoR as a function of the investment volatility σ, for various values of the initial wealth w. A 65-year-old individual, who plans to consume $k = 1$ real CZK per year was assumed. The parameters of the Gompertz distribution were set to $m = 85$ and $b = 9$. The expected real rate of investment return was set to $\mu = 0.1$.

both the RG LPoR and MC LPoR are decreasing in x when all other variables are held constant. The later the individual retires, the smaller is the probability that a given consumption plan will lead to the retirement ruin.

Figure 5.5 displays the discrepancy between the RG LPoR and the MC LPoR as a function of σ. Again, at first the RG values seem to stay reasonably close to the MC LPoR, but with higher levels of volatility they overgrow the MC values significantly. Since the value of (5.1) depends on x via the conditional probability of survival ${}_sp_x$, the level of volatility at which the RG approximation begins to deteriorate is different for every x. Namely, the higher the initial age x, the sooner attains the ${}_sp_x$ factor in (5.1) zero, keeping the second moment of the SPV smaller. Thus, with all else held constant, the higher the initial age x, the higher is the volatility level at which the RG approximation starts to deteriorate noticeably, as confirmed by Figure 5.5.

The fact that at high levels of volatility the RG approximation becomes more precise with an older initial age actually might appear rather peculiar at first. Recall that, in the case of the eventual probability of ruin, the PVSP random variable (4.12) follows exactly the reciprocal gamma distribution, as stated in Theorem 5. Being the limit case for the LPoR as the death rate decreases towards zero, the EPoR is the upper bound for the true LPoR, and the younger the individual is, the closer the true LPoR and the EPoR are. Hence, one would expect that for younger ages it should be possible to approximate the true LPoR more precisely by the RG procedure. Nevertheless, this fact does not affect the problem of the SPV second moment growth when σ increases, the consequences of which become noticeable already at a slightly lower level of volatility for younger ages.

To sum up, as shown by the previous figures, the reciprocal gamma LPoR

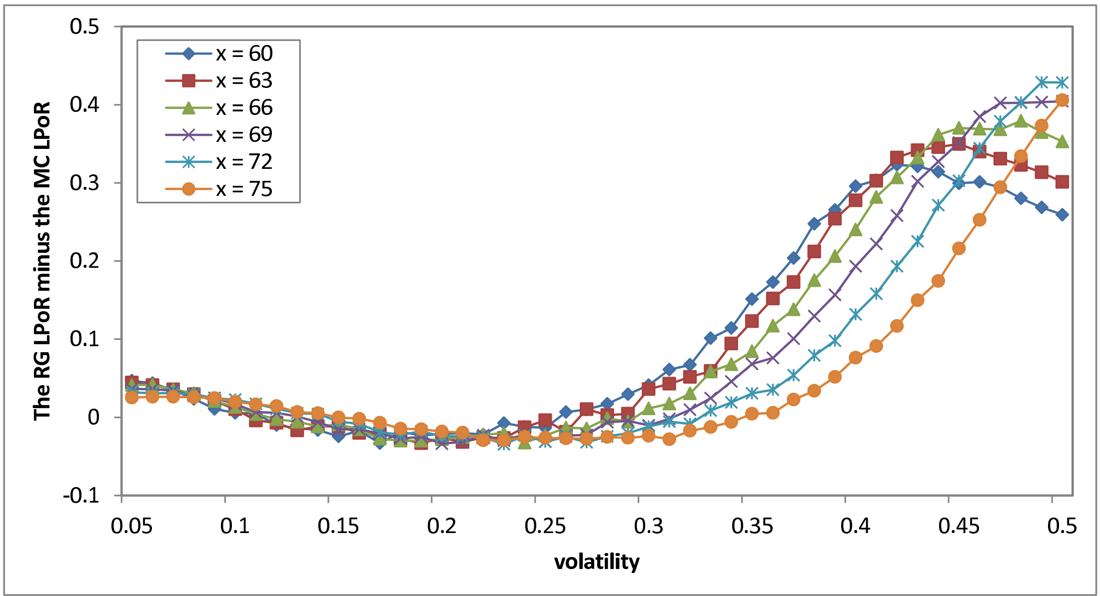


Figure 5.5: The discrepancy between the RG and the MC approximations of the LPoR as a function of the investment volatility σ, for various values of the age x of the retiree. An individual with the initial wealth of $w = 12$ CZK, who plans to consume $k = 1$ real CZK per year was assumed. The parameters of the Gompertz distribution were set to $m = 85$ and $b = 9$. The expected real rate of investment return was set to $\mu = 0.1$.

approximation usually provides values that are reasonably close to the Monte Carlo LPoR values. However, at high levels of volatility the RG approximation significantly overestimates the real LPoR and thus becomes unusable. The exact level of volatility at which the RG approximation starts to deteriorate depends primarily on the expected real rate of return, but also on the initial age of the retiree and on the used mortality dynamics (Gompertz parameters).

5.3 Benefits of the RG Approximation

Bearing the results from the previous section in mind, one might wonder what the benefits of the reciprocal gamma LPoR approximation are, given that the Monte Carlo procedure usually provides a more precise value and most importantly, it does not break at high levels volatility. In this section we therefore briefly discuss the possible advantages of the RG approach over the MC method.

There are two reasons why the RG approximation might be preferred over the MC method in some situations. Firstly, while the RG LPoR can be easily computed in any spreadsheet calculator (including for example MS Excel), one needs to use a relatively advanced (and ergo expensive) software tool[2] for a successful implementation of the MC procedure. Thus the RG formula (4.41) implemented in a simple spreadsheet calculator might be the right tool for an interested retirement planner or a beginning financial advisor.

The second advantage of the RG approximation over the MC method is the lower time complexity. While evaluating the CDF of the gamma distribution

[2]The MC procedure for the LPoR approximation was implemented in *Wolfram Mathematica 8.0*. See `http://www.wolfram.com/mathematica/`.

is a relatively fast and straightforward procedure, sampling from the Gompertz distribution and, above all, generating a path of the net wealth process, takes some time. Moreover, we have mentioned that the MC simulation has to be repeated at least ten thousand times to yield a reasonably precise approximation of the LPoR.

To illustrate the difference in the time complexity of the both procedures, let us once again take a look at Figure 6.10 and Figure 6.11 in the Appendix. For example, the evaluation of the RG LPoR for one set of parameter values (i.e. the calculation of a single point in Figure 6.10) took[3] less than 7 seconds. On the other hand, for a similar procedure in the case of the MC method (i.e. the evaluation of a single point in Figure 6.11) almost 8 minutes were needed. Put differently, while the evaluation of a single curve in Figure 6.10 takes no more than 5 minutes, the same calculation requires almost 6 hours using the MC method.

We conclude that although the inaccuracy at higher levels of volatility is a significant drawback, thanks to the aforementioned advantages over the MC method, the reciprocal gamma approximation can certainly be useful in practice.

[3]To ensure comparability of results, both the RG LPoR approximation and the MC LPoR procedure (with $n = 10\,000$) were implemented in *Wolfram Mathematica 8.0* and the results were calculated on the same computer.

Chapter 6

Sustainable Spending Rate: The Czech Case

This chapter provides a numerical case study for the lifetime probability of ruin using the Czech life table and historical investment return estimates. First, we describe the both mortality dynamics and capital market parameters estimation techniques and discuss their possible shortcomings. We continue with the presentation of numerical results for the LPoR and the sustainable spending rate. The chapter ends with a numerical analysis of an investment strategy influence on the LPoR.

6.1 Estimating the Gompertz Parameters for the Czech Population

Recall the assumption from Chapter 4 that the length of human life follows the Gompertz-Makeham law of mortality, which was defined in Section 2.4 by the IFM curve

$$\lambda(x) = \lambda + \frac{1}{b} \exp\left\{\frac{x-m}{b}\right\}, \qquad x \geq 0. \tag{6.1}$$

We have shown that the parameters $m > 0$ and $b > 0$ represent the modal length of human life and the dispersion coefficient, respectively. The parameter $\lambda \in \mathbb{R}$ stands for the component of the death rate that is attributable to accidental deaths. Since the value of λ tends to be negligible in practice, assuming $\lambda = 0$ and thus working only with the Gompertz law does not lead to a substantial inaccuracy.

We now use the technique presented in Section 2.4 to fit the Gompertz curve to the Czech life table (2011) prepared by the CZSO (see Table 6.8 in the Appendix). Since we are interested only in description of the retirement ages mortality pattern, we set the lower border age for the estimation to $x_0 = 60$. We estimate the both male and female Gompertz parameters separately because the male and female mortality differ significantly from each other.

Running the minimization procedure (2.21) for $x = 60, \dots, 105$ yielded the male Gompertz parameters $\hat{m}_M = 82.51$ and $\hat{b}_M = 10.54$. Similarly, the female parameters were estimated as $\hat{m}_F = 87.87$ and $\hat{b}_F = 7.64$.

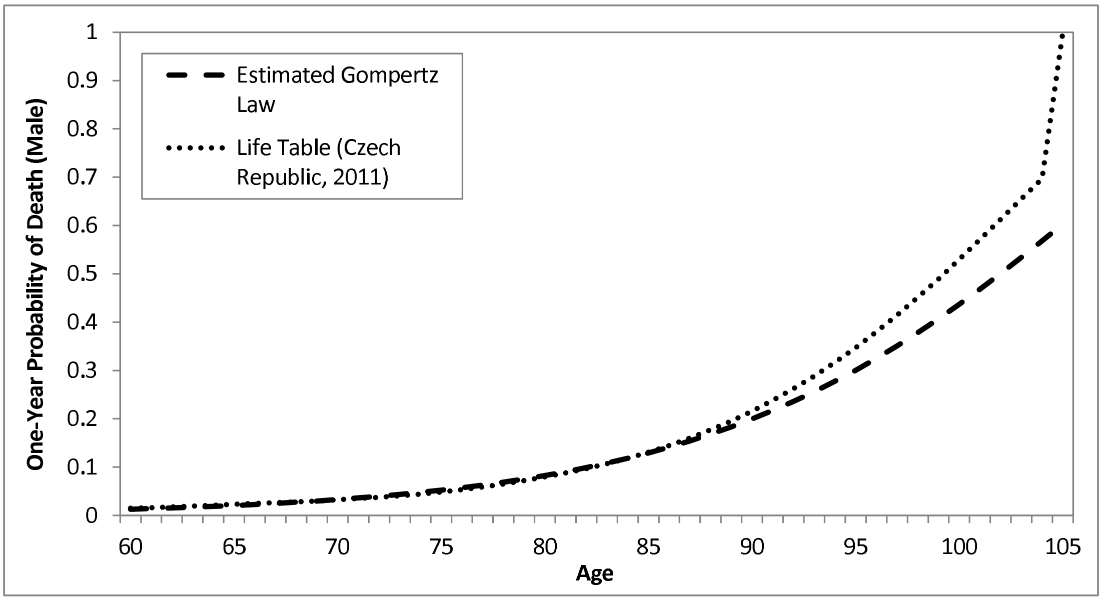


Figure 6.1: An illustration of the Gompertz law fit to the male life table (Czech Republic, 2011). The estimated parameter values are $\hat{m}_M = 82.51$ and $\hat{b}_M = 10.54$.

Figure 6.1 illustrates the achieved fit of the Gompertz curve (2.19) with parameters $\hat{m}_M$ and $\hat{b}_M$ to the Czech male life table. Observe that our model fits the male life table rather accurately[1] at the "lower" ages, while starting at $x = 89$ it begins to underestimate the actual death rate. This goodness of fit imbalance is caused by the fact that the estimation technique (2.21) implicitly (by assigning higher weights) gives preference to the achievement of a tight fit at the ages, at which the most of people tend to die. Stated differently, the (Czech male life table) conditional probability of survival ${}_{89}p_{60} = 0.149$ indicates that, if all males retired exactly at the age of 60, the slight imprecision in our model would affect only 15% of them. Hence, one does not have to be much concerned about the deviation from the actual life table values at the older ages. Moreover, this underestimation can be accepted as longevity risk margin by pension providers.

Similarly, the estimated Gompertz curve for the Czech female population is shown by Figure 6.2. Observe that the result resembles the male case. However, due to the higher female modal age of death, the distinguishable underestimation of the actual death rate does not start until $x = 95$. Thus, the distribution of the fit accuracy is more even. For the sake of completeness, the female (table) conditional probability of survival is ${}_{95}p_{60} = 0.053$. The fact that the death rate underestimation at the older ages is not as significant as in the male case will actually lead to a minor curiosity in a part of our results. We attend to this problem in Section 6.4.

[1] The tightness of the fit is actually not that surprising, since a part of the crude death rates smoothing procedure used by the Czech Statistical Office involves smoothing by the Gompertz-Makeham curve. More information on the CZSO life table preparation methodology available at `http://www.czso.cz/csu/redakce.nsf/i/umrtnostni_tabulky_metodika`.

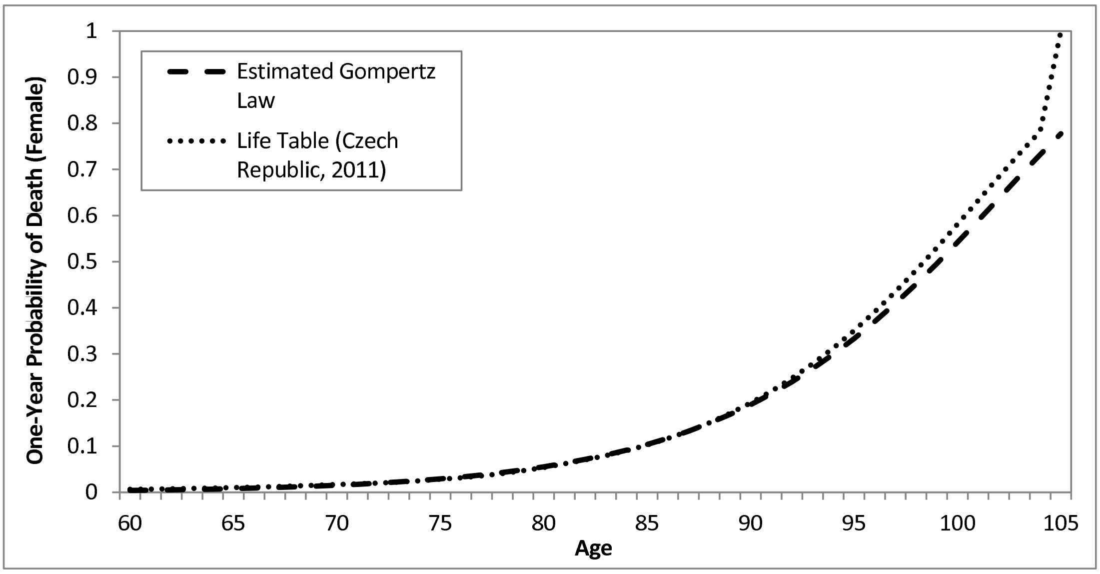


Figure 6.2: An illustration of the Gompertz law fit to the female life table (Czech Republic, 2011). The estimated parameter values are $\hat{m}_F = 87.87$ and $\hat{b}_F = 7.64$.

6.2 Calibrating the Financial Model

The second key assumption we made in Chapter 4 is that the market prices are driven by the geometric Brownian motion, i.e. the real price of the investment portfolio follows the equation

$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma B_t}, \qquad t \geq 0. \tag{6.2}$$

For the purpose of our numerical study, we have chosen the *PX index*[2] traded at the *Prague Stock Exchange (PSE)* as a representative of a simple stock portfolio easily available in the Czech financial environment. From now on, we shall assume that at the first day of retirement the individual invests his or her initial capital w to the PX index. The desired yearly income stream of k real CZK is to be created via a systematic withdrawal plan, which regularly[3] sells off the required number of the PX index units[4].

To calibrate our financial model, we now need to estimate the GBM parameters μ and σ. We will use a technique based on historical prices presented in Cipra (2008). Suppose we have the historical index prices $\{P_{t_i}\}_{i=0}^{n}$ at our disposal, where the set of observation dates is $\{t_i = i\Delta,\, i = 0, \ldots, n\}$, so that $\Delta > 0$ is the time difference between any two adjacent price observations. For instance, in the case of daily index prices, we would have $\Delta = 1/250$ since we measure time in years and there is approximately 250 trading days in a year.

[2]The official index of the Prague Stock Exchange. The PX index was first calculated in March, 2006, when it replaced the PX 50 and PX-D indices. The PX index took over the historical values of the PX 50 and continues in its development. The PX index is a price index and dividend yields are not considered in its calculation.

[3]In the real world it is obviously not possible to make continuous withdrawals from a portfolio. We can only approach the limiting continuous case by making regular discrete withdrawals on e.g. monthly, weekly or daily basis.

[4]All numbers are prior to any income taxes.

Recall from Section 3.1 that we assume the corresponding (annualized) rates of return

$$\tilde{g}_{t_i} = \frac{1}{\Delta} \ln \frac{P_{t_i}}{P_{t_{i-1}}}, \qquad i = 1, \dots, n$$

to be normally distributed:

$$\tilde{g}_{t_i} \sim \mathrm{N}\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{\Delta}\right), \qquad i = 1, \dots, n.$$

Hence, for every $i = 1, \dots, n$ we have

$$\mathsf{E}[\tilde{g}_{t_i}] = \mu - \frac{\sigma^2}{2}, \qquad \mathsf{var}[\tilde{g}_{t_i}] = \frac{\sigma^2}{\Delta}. \tag{6.3}$$

On the other hand, the *sample mean* and the *sample variance* of the observed rates of return are

$$m(\tilde{g}) = \frac{1}{n} \sum_{i=1}^{n} \tilde{g}_{t_i}, \qquad s^2(\tilde{g}) = \frac{1}{n-1} \sum_{i=1}^{n} \left(\tilde{g}_{t_i} - m(\tilde{g})\right)^2. \tag{6.4}$$

Putting the empirical moments (6.4) equal to the theoretical moments (6.3) and solving the system of two equations for μ and σ then yields the GBM parameters estimators:

$$\hat{\mu} = m(\tilde{g}) + \frac{\Delta s^2(\tilde{g})}{2}, \qquad \hat{\sigma} = \sqrt{\Delta} s(\tilde{g}). \tag{6.5}$$

It follows from the properties of sample mean and sample variance that the both estimators in (6.5) are consistent and (asymptotically) unbiased.

Going back to our situation, we have decided to apply this estimation technique on the PX index historical daily prices observed in the last 10 years[5]. Since we are interested in the real (inflation-free) price parameters, we have reconstructed the real PX index historical prices by subtracting the inflation rate[6] from the nominal prices. The development of the both nominal and real price of the PX index over the last ten years is shown by Figure 6.3.

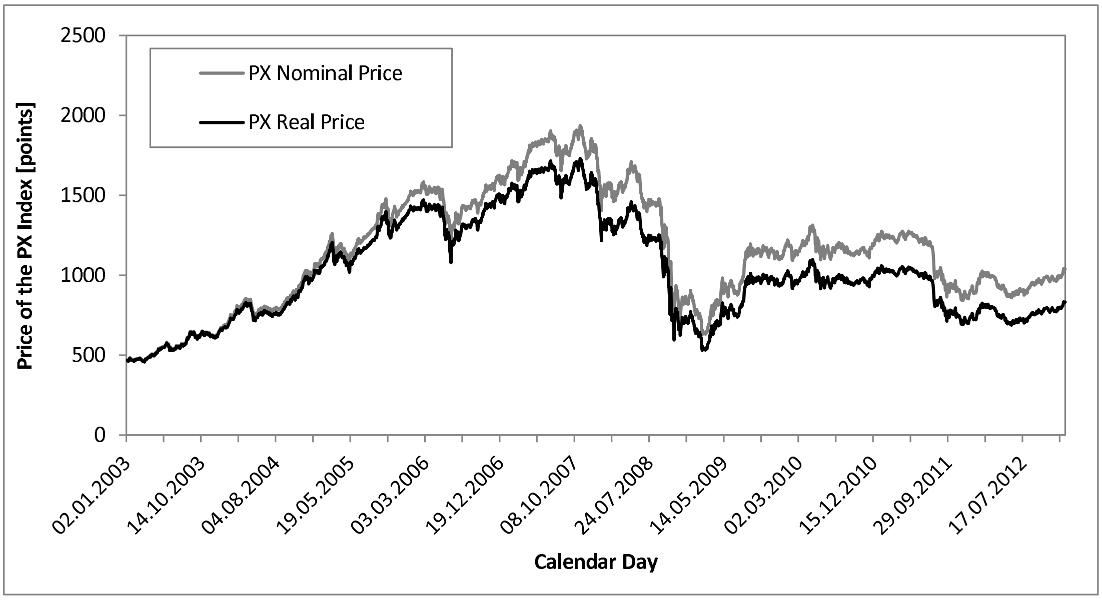
In conclusion, using the real daily PX index prices observed between 2.1.2003 and 28.12.2012, setting $\Delta = 1/250$ and performing the aforementioned estimation technique yielded the values $\hat{\mu}_{PX} = 0.087867$ and $\hat{\sigma}_{PX} = 0.244746$. Thus, we assume the real PX index price to follow the GBM equation

$$P_t = P_0 e^{(\hat{\mu}_{PX} - \hat{\sigma}^2_{PX}/2)t + \hat{\sigma}_{PX} B_t}, \qquad t \geq 0.$$

Being equipped with the estimations of the both GBM and Gompertz parameters, we are now in a position to introduce the numerical results for the lifetime

[5]The choice of the length of the historical prices reference period is an important decision. Obviously, we need to use a relatively large number of observations to receive reasonable estimates of μ and σ. On the other hand, too old historical prices might be irrelevant for our purposes, for instance, because of possible changes in the market factors driving the index price over time. After several trials we settled for a 10 years long reference period as a reasonable compromise. Thus, the total number of used price observations was $n = 2515$. Altogether, employing more complex techniques for estimation (or prediction) of μ and σ is an area for a possible further research on this topic.

[6]Table 6.9 in the Appendix displays the estimates of the monthly inflation rate in the Czech Republic. The estimates were prepared by the CZSO.



Source of the data: PSE (`www.bcpp.cz`), CZSO (`www.czso.cz`).

Figure 6.3: Development of the daily PX index price over the last ten years (2.1.2003 - 28.12.2012). The real price was calculated by subtracting the (monthly) inflation from the nominal price.

probability of ruin obtained via the reciprocal gamma formula (4.41). Before we do so in the next section, let us briefly attend to the expected precision of our results. Recall from Chapter 5 that the main factor influencing the accuracy of the RG LPoR approximation is the relationship between the investment parameters μ and σ. Since we have $\hat{\mu}_{PX} \doteq 0.09$ and $\hat{\sigma}_{PX} \doteq 0.24$, our situation *roughly* corresponds to the $\sigma = 0.24$ point of the $\mu = 0.09$ curve in Figure 5.3. Judging by the fact that the discrepancy for this combination of the GBM parameters values is rather small, we can expect our numerical results to be relatively precise with respect to the Monte Carlo LPoR.

6.3 The Czech Lifetime Probability of Ruin

In this section we present numerical results for the LPoR obtained via the RG approximation method. All the results were calculated using the Gompertz parameters estimated in Section 6.1 and the GBM parameters estimated in Section 6.2. Since the implementation of the RG formula in a spreadsheet calculator is not difficult at all, the range of presented results is intentionally not as extensive as it could be. The purpose of the following tables is rather to provide an insight on the impact of various parameter changes (such as increase in the consumption rate or decrease in the retirement age) on the LPoR in the Czech environment.

Table 6.1 displays a brief overview of the Czech male lifetime probability of ruin results using the both reciprocal gamma and Monte Carlo approximation methods. The first column of the table displays the retirement age x and the second column provides the corresponding assumed expected age at death, calculated as $x + \mathsf{E}[T_x]$, using the formula (2.22) with the parameter values $\hat{m}_M$, $\hat{b}_M$

Retire-ment age	Expected age at death		Spending rate (per 100 CZK) 2 CZK	4 CZK	6 CZK	8 CZK	10 CZK
		RG	2.0%	11.1%	25.7%	41.9%	56.9%
60	79.9	MC	2.2%	11.8%	27.9%	42.8%	56.0%
		D	-0.2%	-0.7%	-2.2%	-0.9%	0.9%
		RG	1.2%	7.5%	18.8%	32.5%	46.4%
65	81.1	MC	1.3%	8.5%	20.9%	34.0%	47.9%
		D	-0.1%	-1.0%	-2.1%	-1.5%	-1.5%
		RG	0.7%	4.6%	12.4%	23.0%	34.7%
70	82.8	MC	0.6%	5.2%	14.6%	26.9%	37.5%
		D	0.1%	-0.6%	-2.2%	-3.9%	-2.8%
		RG	0.3%	2.5%	7.3%	14.4%	23.1%
75	84.8	MC	0.3%	2.9%	9.0%	17.5%	27.0%
		D	0.1%	-0.3%	-1.7%	-3.1%	-3.9%
		RG	0.2%	1.2%	3.8%	7.9%	13.5%
80	87.4	MC	0.1%	1.3%	4.5%	10.6%	17.8%
		D	0.1%	-0.1%	-0.7%	-2.6%	-4.3%

Note: RG = reciprocal gamma, MC = Monte Carlo, D = RG - MC.

Table 6.1: Overview of the lifetime probability of retirement ruin for Czech males.

and $\lambda = 0$. Every entry in the table then shows the LPoR of a Czech male, who retires at age x, invests his initial capital of the size $w = 100$ CZK to the PX index and decides to withdraw the given amount of k real CZK per year for the rest of his life. Once again, we stress that the probabilities in the table are universally applicable, since only the ratio of k and w matters. Thus, for instance, the RG LPoR 12.4% from Table 6.1 holds as well for a retiree aged 70 years with the initial capital of $w = 1$ million CZK and the real annual spending rate $k = 60$ thousand CZK. Also observe that, as predicted by Figure 5.3, the differences between the RG LPoR and the MC LPoR are relatively small, which inspires additional confidence in the RG approximation procedure.

A similar overview of the RG and the MC LPoR values, this time for Czech females, is provided by Table 6.10 in the Appendix. Observe that in this case the both RG and MC LPoR values are slightly higher, which corresponds with the fact that females live on average longer than males and hence they need a more conservative withdrawal plan to reach the same level of sustainability. Also note that the differences between the RG and the MC female LPoR values are more or less similar to those in Table 6.1, meaning that the small change in the Gompertz parameter values from $\hat{m}_M$ and $\hat{b}_M$ to $\hat{m}_F$ and $\hat{b}_F$, respectively, does not have a substantial influence on the discrepancy between the both results.

Age at retirement	1% M	1% F	5% M	5% F	10% M	10% F	20% M	20% F
	Tolerated lifetime probability of ruin							
60	1.56	1.35	2.87	2.45	3.82	3.26	5.27	4.48
61	1.62	1.40	2.96	2.53	3.95	3.36	5.43	4.60
62	1.68	1.45	3.06	2.61	4.07	3.46	5.60	4.74
63	1.74	1.50	3.16	2.70	4.21	3.57	5.79	4.88
64	1.80	1.56	3.28	2.79	4.36	3.69	5.98	5.03
65	1.87	1.62	3.40	2.90	4.51	3.82	6.19	5.20
66	1.95	1.69	3.53	3.00	4.68	3.96	6.41	5.38
67	2.03	1.76	3.66	3.12	4.86	4.11	6.65	5.58
68	2.11	1.84	3.81	3.25	5.05	4.27	6.91	5.79
69	2.20	1.92	3.97	3.39	5.25	4.44	7.18	6.02
70	2.30	2.01	4.13	3.53	5.47	4.63	7.47	6.27
71	2.40	2.11	4.31	3.69	5.70	4.84	7.79	6.54
72	2.51	2.21	4.50	3.87	5.95	5.06	8.13	6.83
73	2.63	2.32	4.71	4.06	6.22	5.30	8.49	7.15
74	2.75	2.44	4.93	4.26	6.51	5.57	8.89	7.50
75	2.89	2.57	5.17	4.49	6.83	5.86	9.31	7.89
76	3.03	2.72	5.42	4.73	7.16	6.17	9.77	8.31
77	3.18	2.87	5.70	5.00	7.53	6.52	10.26	8.77
78	3.35	3.04	6.00	5.29	7.92	6.89	10.79	9.28
79	3.53	3.22	6.32	5.61	8.34	7.31	11.37	9.84

Note: M = male, F = female. The values are in CZK.

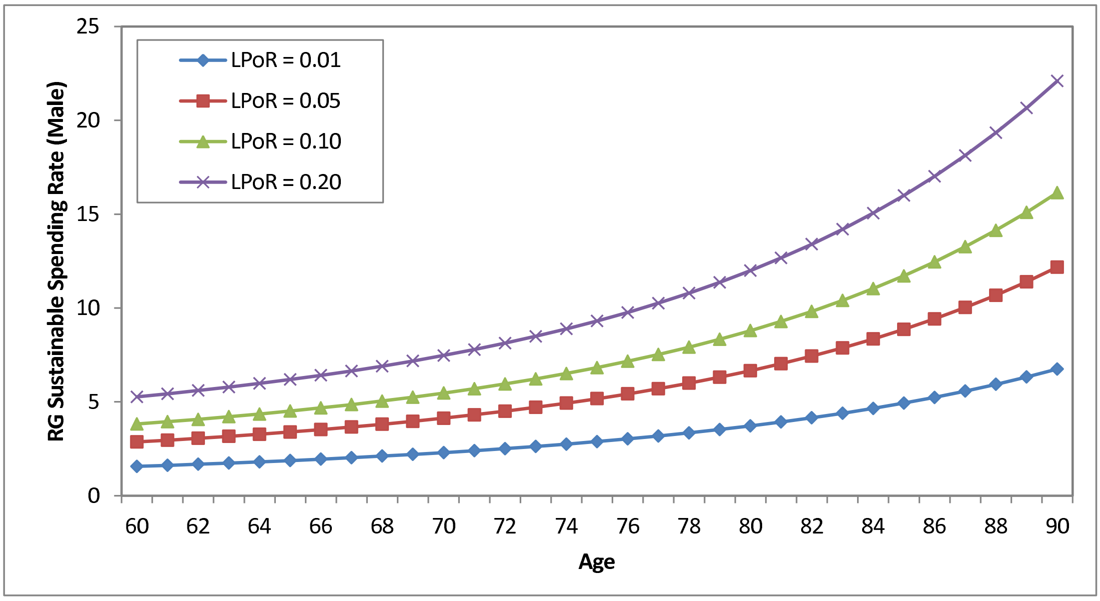
Table 6.2: The sustainable spending rate in the Czech Republic. The table displays the maximal real annual spending (k) [in CZK] per 100 CZK of initial wealth (w) for a given tolerated lifetime probability of ruin.

6.4 The Czech Sustainable Spending Rate

Recall that by inverting the RG formula, one can calculate the (maximal) sustainable spending rate (4.44) for a given tolerated lifetime probability of ruin. Let's say that a Czech retiree with a certain initial pension capital is willing to tolerate a probability of 5% that his or her funds will run out before he or she dies. In other words, the retiree desires a consumption plan with a 95% chance of sustainability. What is the maximal affordable spending rate, so that the requirement is met? How would the affordable spending rate change if the retiree was willing to tolerate only a 1% probability of ruin?

Table 6.2 displays the sustainable spending rate (SSR) for a Czech retiree who invests his or her pension funds to the PX index. The values in the table represent the SSR in real CZK per 100 CZK of initial capital. Similarly to the previous table, the values can be multiplied arbitrarily, provided that the ratio of k and w remains unchanged.

A number of things are apparent from Table 6.2. The numbers fully confirm the intuition that the higher the retirement age, the more can the individual consume. Naturally, the sustainable consumption rate also increases with greater tolerance for probability of ruin. For example, a 65-year-old male who desires



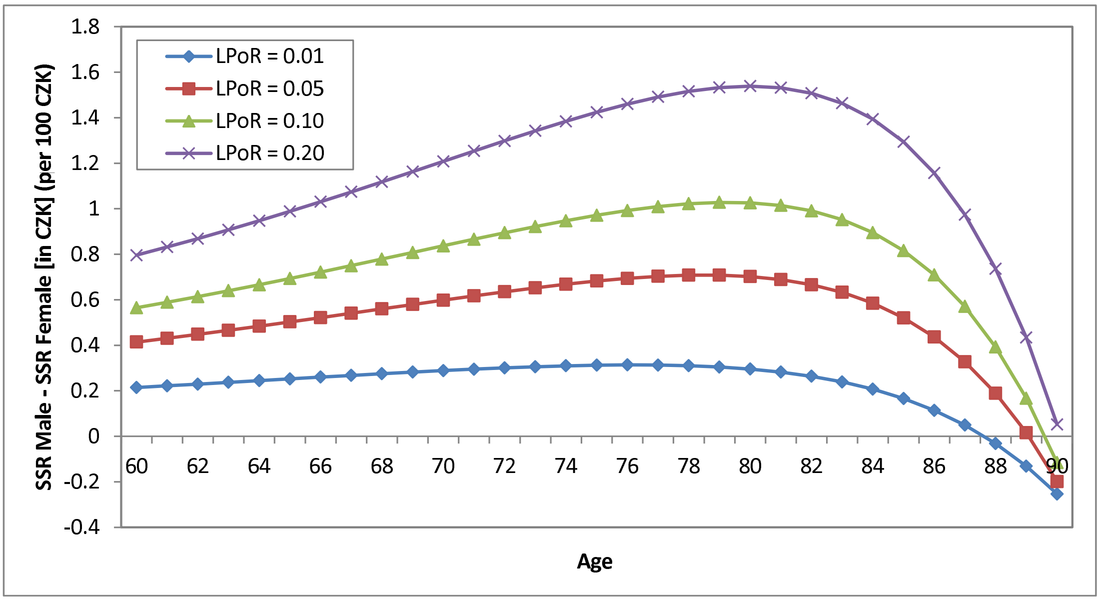
Note: The values are in real CZK, per 100 CZK of initial capital.

Figure 6.4: The sustainable spending rate for Czech males as a function of retirement age x, for various lifetime probability of ruin tolerance. The SSR values are in real CZK per 100 CZK of initial capital.

a 95% chance of sustainability can afford to consume 3.40 real CZK per 100 CZK of his initial capital every year. Doubling his LPoR tolerance to 10% would allow him to spend 4.51 real CZK (per 100 CZK of his initial capital) every year. Approximately the same amount (4.50 real CZK) could be consumed in case he would defer retirement to age 72, provided the (real) size of his initial capital would remain unchanged. Of course, it also holds that the male SSR is higher than the female SSR because, on average, in the female case the savings will have to last longer.

For various values of the tolerated LPoR, the sustainable spending rate for Czech males in dependency on the retirement age x is graphically illustrated by Figure 6.4. Observe that all the curves are convex, meaning that, with all else held constant, the higher the retirement age x, the greater is the increase in the SSR caused by deferring the retirement to the next year.

Figure 6.5 shows the difference between the male SSR (shown in Figure 6.4) and the female SSR. Observe that an interesting fact is revealed, since the figure indicates that the male SSR at very old ages is lower than the female SSR. Indeed, according to Table 6.8 in the Appendix, the female death rate at ages older than 94 is higher than the male one. The exact age at which the female SSR overgrows the male SSR depends on the tolerated LPoR. The lower the tolerated probability of ruin, the sooner exceeds the female SSR the male one. The negative difference in SSR shown in the right bottom part of Figure 6.5 is amplified by the fact that the Gompertz model estimated in Section 6.1 underestimates the actual male death rate at very old ages stronger than it underestimates the female one (see Figure 6.1 and Figure 6.2). However, as indicated by the death rates in Table 6.8, the female SSR would eventually exceed the male SSR anyway, even without this minor flaw of our model. Nonetheless, since almost no one retires later than at the age of 85 and only a small part of Czech retirees lives to their nineties, this



Note: The values are in real CZK, per 100 CZK of initial capital.

Figure 6.5: Difference between the male and the female sustainable spending rate in the Czech Republic as a function of retirement age x, for various lifetime probability of ruin tolerance. The values are in real CZK per 100 CZK of initial capital.

curiosity is of little importance.

6.5 Impact of an Investment Strategy

So far we have considered only the case when the individual invests the whole initial capital to a stock index. However, in practice, most retirees would probably choose to diversify their portfolio and thus adjust the investment parameters to their liking. We already know that, with all else held constant, the LPoR decreases with a higher expected return or a lower volatility. However, in an efficient financial market we rather expect both μ and σ to change in the same direction. A more conservative investor preferring low volatility must deal with the disadvantage of low expected returns. On the other hand, an investor with a low risk aversion prefers a more volatile portfolio, which naturally brings the benefit of higher expected returns.

In this section we provide numerical results for the impact of a simple portfolio diversification strategy on the lifetime probability of ruin approximated by the RG formula (4.41). In particular, we focus on two different asset classes: *stock index* and *government bonds*. We assume that the individual can allocate his or her initial capital investment arbitrarily between these two assets. While a stock index is usually a relatively risky investment, the government bonds represent the conservative component of the portfolio. Hence, by mixing these two components, a relatively wide range of resulting portfolio characteristics can be achieved. An important assumption we are making, is that the real rate of return on the bonds is not correlated with the real rate of return on the stock index.

Recall that in Section 6.2 we have estimated the GBM parameters of the PX index as $\hat{\mu}_{PX} = 0.087867$ and $\hat{\sigma}_{PX} = 0.244746$. Our task remains to find a suitable

Allocation to bonds	μ **(drift)**	σ **(volatility)**
0%	0.08787	0.24475
20%	0.07244	0.19580
40%	0.05702	0.14685
60%	0.04159	0.09790
80%	0.02616	0.04895
100%	0.01074	0.00000

Note: The residual is allocated to the PX index.

Table 6.3: Expected real rate of return and volatility of portfolios corresponding to various investment allocations between the PX index and the retail government savings bonds.

representative for the "bond class" of our financial portfolio. The *Ministry of Finance of the Czech Republic* regularly issues saving government bonds. The latest tranche of the *Retail Savings Bonds*[7] was issued in May, 2013. Since we aim for a long-term investment, we have chosen the *Reinvestment Savings Government Bond of the Czech Republic*[8] with a maturity of 5 years as a representative of the bonds asset class of our portfolio. In order to motivate the investor to hold the bond to maturity, the annual returns of this bond (paid at the end of each respective year) were increasing: 0.5%, 1%, 3%, 4% and 7%. This is equivalent to a fixed annual yield of 3.074%. According to the *Czech National Bank*[9], the long-term expectation of the annual inflation rate in the Czech Republic is 2%. Subtracting the inflation rate leads to the real rate of return $\mu_B = 1.074\%$. Note that – just as in the case of the stock index returns – we do not take the income tax[10] into account. The reason for not taking the income tax into consideration is that the investment income taxation rules in the Czech Republic are currently subject to changes. Thus, imposing an income taxation rule would expose the results to the risk of becoming outdated in a short time.

Table 6.3 displays the portfolio parameters μ and σ for various investment allocations between the PX index and the Czech government savings bonds. The investment allocations are varied by increments of 20%. For example, a 40% allocation to the government bonds and a 60% allocation to the PX index results in the expected real rate of return of $\mu = 0.05702$ and in the volatility of $\sigma = 0.14685$. Naturally, the portfolio with a 100% allocation to the PX index corresponds to $\hat{\mu}_{PX}$ and $\hat{\sigma}_{PX}$, while the portfolio with a 100% allocation to the government bonds results in μ_B and $\sigma_B = 0$. Once again, we stress the key assumption that the real rates of return on the government bond and the PX index are uncorrelated.

Now, consider a Czech male retiring at the age of 65. Table 6.4 displays the RG approximation of the LPoR in this case, for various spending rates and invest-

[7] More information available at `http://www.sporicidluhopisycr.cz/`.

[8] ISIN Number: CZ0001004006. Issue Number/Tranche: 82/1.

[9] See `http://www.cnb.cz/en/monetary_policy/forecast/index.html`.

[10] The yields of the retail saving bonds in the Czech Republic are currently subject to a taxation of 15%. The stock sell-off taxation in the Czech Republic currently depends on how long the stock was held by the investor.

Spending rate per 100 CZK [in CZK]	Initial portfolio allocation to bonds (the residual is allocated to the stock index) 0%	20%	40%	60%	80%	100%
2.00	1.20%	0.59%	0.28%	0.15%	0.14%	0.32%
3.00	3.63%	2.27%	1.43%	1.04%	1.12%	2.17%
4.00	7.50%	5.47%	4.11%	3.54%	4.13%	7.09%
5.00	12.65%	10.24%	8.66%	8.29%	10.01%	15.66%
6.00	18.76%	16.34%	15.00%	15.37%	18.79%	27.15%
7.00	25.49%	23.42%	22.75%	24.33%	29.68%	40.07%
8.00	32.52%	31.07%	31.38%	34.41%	41.50%	52.88%
9.00	39.57%	38.91%	40.34%	44.79%	53.12%	64.43%
10.00	46.44%	46.61%	49.14%	54.77%	63.70%	74.10%

Note: The spending rate is in real CZK per 100 CZK of initial capital.

Table 6.4: The RG LPoR for a 65-year-old Czech male corresponding to various (real) consumption rates and investment allocations. The top of the table shows the proportion allocated to the retail government savings bonds with expected real rate of return μ_B, while the residual of the portfolio is allocated to the PX index with parameters $\hat{\mu}_{PX}$ and $\hat{\sigma}_{PX}$.

ment allocations. The top of the table shows the proportion of initial investment allocated to the reinvestment savings government bond, while the remainder of the investment is allocated to the PX index. Down the vertical axis the assumed annual spending rate (per 100 CZK of initial capital) is shown.

The results in Table 6.4 are quite remarkable. Note that the investment allocation at which the lowest LPoR occurs depends strongly on the desired spending rate. In general, for low spending rates a conservative portfolio composed mainly of the government bonds seems to be optimal. However, the differences in the LPoR are negligible in this case, at least relatively to the imprecision of the RG approximation. On the other hand, with increasing spending rate, the ruin probability tends to attain minimal values with a higher-risk/higher-return portfolio, where the majority of the investment is allocated to the PX index. Nevertheless, if the retiree desires a spending rate higher than 8 CZK even the 100% PX index portfolio does not provide a satisfactory low value of the LPoR. In general, assuming that most retirees are willing to tolerate a LPoR around 1% - 5%, a well balanced portfolio with roughly a 60% allocation to bonds can be advised.

Last of all, let us remark that generating the same table as Table 6.4, this time for females, yields uniformly higher ruin probabilities for all spending rates and investment allocations. The behaviour of the female LPoR is identical to the male case and hence roughly the same portfolio composition is advised.

Conclusion

Thanks to the late demographic development, the issue of maintaining an acceptable standard of living towards the end of the human life cycle has developed a new urgency. More and more future retirees are becoming aware of the fact that they will need their own retirement savings to complete the payouts from the social security system. However, as the low demand for life annuities indicates, a substantial part of retirees will not be willing to annuitize their pension assets, probably because of reluctance to give up the benefits of liquidity and the possibility of leaving a bequest in case of early death. The retirees who decide for discretionary management of their pension assets will need a guidance on how much money can annually be spent without an excessive exposure to the risk of retirement ruin.

In this thesis we have presented a useful method for sustainability testing of retirement plans with a constant consumption stream. Our model was built on the assumptions of log-normal investment returns and Gompertz-Makeham law of mortality. We have shown that the individual's probability of retirement ruin can be expressed as the probability that the stochastic present value of his or her future consumption is greater than the initial size of retirement savings. Since there is no closed-form density function, the stochastic present value has been approximated via a moment matching technique by the reciprocal gamma distribution.

We have tested the accuracy of the reciprocal gamma approximation for the probability of ruin by comparing with the results obtained from extensive Monte Carlo simulations. The discrepancy between the results of the both methods has been analysed in dependency on the investment volatility and for various values of spending rate, retiring age and expected rate of return. We have found out that the main factor driving the accuracy of the reciprocal gamma approximation is the relationship between expected rate of return and volatility of the investment portfolio. As a rule of thumb, the reciprocal gamma approximation seems to be reasonably accurate when $\sigma < \sqrt{2\mu/3}$.

In the numerical case study we have calculated values for the probability of ruin in the Czech environment. We have also calculated the maximal sustainable spending rates under various combinations of wealth-to-consumption ratios and investment portfolio characteristics. Based on these results, we can conclude e.g. that a 65-year-old Czech retiree might want to allocate roughly a 60% of his retirement savings to the government savings bonds, while the rest of the savings should be invested to the PX index.

We have given arguments that despite the significant inaccuracy for high levels of volatility, the reciprocal gamma formula can be useful in practice. Although the key role in the field of retirement planning rightfully belongs to Monte Carlo based studies, the reciprocal gamma formula can at least serve as a calibration point for more complicated simulations. It also helps to explain the connection between the three fundamental factors affecting retirement planning: uncertain investment returns, uncertain length of human life and spending rates. Similarly to the Black-Scholes formula in the context of option prizing, although the underlying assumptions may be questionable, it can coexist with more sophisticated models based on simulations. It also enables a deeper understanding of the risk and return trade-offs.

There are several directions for a possible further research on the problem at hand. Firstly, all the numeric results presented in Chapter 6 are strongly dependent on the estimated values of expected return and volatility. However, we have used only a relatively simple estimation technique based on empiric moments of the historical prices observations. Thus, employing more complex techniques for forecasting long-term investment returns and volatility can significantly enhance the accuracy of our results.

In addition, according to Milevsky & Robinson (2000), apart from the reciprocal gamma distribution there is a second possible candidate for approximation of the stochastic present value - the *Type II Johnson distribution*. The Johnson family of distributions was first described in Johnson (1949). Since the Type II Johnson distribution has four degrees of freedom, one will have to match the first four moments of the stochastic present value (4.19).

References

Bowers, N. L., Gerber, H. U., Hickman, J. C., Jones, D. A., & Nesbitt, C. J. (1997). *Actuarial Mathematics* (Vol. 2). Society of Actuaries Schaumburg, IL.

Brillinger, D. R. (1961). A justification of some common laws of mortality. *Transactions of the Society of Actuaries*, *13*, 116–119.

Carriere, J. F. (1992). Parametric models for life tables. *Transactions of the Society of Actuaries*, *44*, 77–99.

Carriere, J. F. (1994). An investigation of the gompertz law of mortality. *Actuarial Research Clearing House*, *2*, 161–177.

Cipra, T. (2008). *Finanční ekonometrie.* Ekopress.

Cipra, T. (2012). *Penze: kvantitativní přístup.* Ekopress.

Gompertz, B. (1825). On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies. *Philosophical transactions of the Royal Society of London*, *115*, 513–583.

Haan, L. de, & Ferreira, A. (2006). *Extreme value theory: An introduction.* Springer.

Johnson, N. L. (1949). Systems of frequency curves generated by methods of translation. *Biometrika*, *36*(1/2), 149–176.

Karatzas, I., & Shreve, S. (1991). *Brownian motion and stochastic calculus (graduate texts in mathematics).* Springer.

Langová, N. (2011). *Plánování důchodu.* Unpublished master's thesis, Univerzita Karlova v Praze, Matematicko-fyzikální fakulta.

Merton, R. C. (1990). *Continuous-time finance.* Blackwell.

Milevsky, M. A. (1997). The present value of a stochastic perpetuity and the gamma distribution. *Insurance: Mathematics and Economics*, *20*(3), 243–250.

Milevsky, M. A. (1998). Optimal asset allocation towards the end of the life cycle: to annuitize or not to annuitize? *Journal of Risk and Insurance*, 401–426.

Milevsky, M. A. (2006). *The calculus of retirement income: Financial models for pension annuities and life insurance.* Cambridge University Press.

Milevsky, M. A., & Posner, S. E. (1998). Asian options, the sum of lognormals, and the reciprocal gamma distribution. *Journal of financial and quantitative analysis*, *33*(3).

Milevsky, M. A., & Robinson, C. (2000). Self-annuitization and ruin in retirement. *North American Actuarial Journal*, *4*(4), 112–124.

Schmeiser, H., & Post, T. (2005). Life annuity insurance versus self-annuitization: An analysis from the perspective of the family. *Risk Management and Insurance Review*, *8*(2), 239–255.

Shreve, S. (2004). *Stochastic calculus for finance ii: Continuous-time models (springer finance)*. Springer.

Steele, J. M. (2004). *The cauchy-schwarz master class: An introduction to the art of mathematical inequalities*. The Mathematical Association of America.

Yaari, M. E. (1965). Uncertain lifetime, life insurance, and the theory of the consumer. *The Review of Economic Studies*, *32*(2), 137–150.

Appendix

Country	Year					
	2001		2006		2011	
	Male	Female	Male	Female	Male	Female
Austria	75.6	81.7	77.1	82.8	78.3	83.9
Belgium	74.9	81.2	76.6	82.3	77.8	83.2
Czech Republic	72.1	78.6	73.5	79.9	74.8	81.1
Denmark	74.7	79.3	76.1	80.7	77.8	81.9
Finland	74.6	81.7	75.9	83.1	77.3	83.8
France	75.5	83.0	77.3	84.5	78.7	85.7
Germany	75.6	81.4	77.2	82.4	78.4	83.2
Greece	75.9	81.0	77.2	81.9	78.5	83.1
Iceland	78.3	83.2	79.5	82.9	80.7	84.1
Ireland	74.5	79.9	77.3	82.1	78.3	82.8
Italy	77.1	83.1	78.5	84.2	80.1	85.3
Lithuania	65.9	77.6	65.3	77.0	68.1	79.3
Netherlands	75.8	80.8	77.7	82.0	79.4	83.1
Norway	76.2	81.6	78.2	82.9	79.1	83.6
Poland	70.0	78.4	70.9	79.7	72.6	81.1
Romania	67.5	74.9	69.2	76.2	71.0	78.2
Spain	76.4	83.3	77.9	84.5	79.4	85.4
Sweden	77.6	82.2	78.8	83.1	79.9	83.8
Switzerland	77.5	83.2	79.2	84.2	80.5	85.0
United Kingdom	75.8	80.5	77.3	81.7	79.1	83.1

Source: Eurostat.

Table 6.5: Life expectancy at birth across selected European countries.

Country	Year 1991	2001	2011
Austria	1.51	1.33	1.42
Belgium	1.66	1.67	1.81
Czech Republic	1.86	1.14	1.43
Denmark	1.68	1.74	1.75
Finland	1.79	1.73	1.83
France	*	1.90	2.01
Germany	*	1.35	1.36
Greece	1.38	1.25	1.42
Iceland	2.18	1.95	2.02
Ireland	2.08	1.94	2.05
Italy	1.30	1.25	1.40
Lithuania	2.01	1.30	1.76
Netherlands	1.61	1.71	1.76
Norway	1.92	1.78	1.88
Poland	2.07	1.31	1.30
Romania	1.59	1.27	1.25
Spain	1.33	1.24	1.36
Sweden	2.11	1.57	1.90
Switzerland	1.58	1.38	1.52
United Kingdom	1.82	1.63	1.96

* - value not available.

Source: Eurostat.

Table 6.6: Total fertility rate (number of children per woman) across selected European countries.

Country	1991 Male	1991 Female	2001 Male	2001 Female	2011 Male	2011 Female
Austria	14.5	18.1	16.3	20.0	18.1	21.7
Belgium	14.5	18.8	15.9	19.9	17.8	21.5
Czech Republic	12.0	15.7	14.0	17.3	15.6	19.2
Denmark	14.3	18.1	15.2	18.3	17.3	20.1
Finland	14.0	18.2	15.7	19.8	17.7	21.7
France	*	*	17.0	21.5	19.3	23.8
Germany	14.2	17.9	16.1	19.8	18.2	21.2
Greece	15.8	17.9	16.5	18.7	18.5	20.6
Iceland	15.5	19.9	17.5	21.2	18.9	21.5
Ireland	13.5	17.0	15.0	18.5	17.9	20.7
Italy	15.2	19.1	16.9	21.0	18.8	22.6
Lithuania	13.5	17.2	13.5	17.9	14.0	19.2
Netherlands	14.5	19.2	15.6	19.4	18.1	21.2
Norway	14.9	19.0	16.2	19.9	18.2	21.4
Poland	12.3	16.0	13.7	17.7	15.4	19.9
Romania	13.0	15.2	13.3	16.0	14.3	17.5
Spain	15.7	19.4	16.9	21.1	18.7	22.8
Sweden	15.5	19.4	16.9	20.2	18.5	21.3
Switzerland	15.6	20.1	17.3	21.3	19.2	22.6
United Kingdom	*	*	16.1	19.2	18.6	21.2

* - value not available.

Source: Eurostat.

Table 6.7: Life expectancy at age 65 across selected European countries.

	Male		Female	
Age	**Dx**	**qx**	**Dx**	**qx**
60	1088	0.014795	485	0.006394
61	1144	0.016130	546	0.006857
62	1207	0.017528	576	0.007472
63	1430	0.019345	649	0.008341
64	1471	0.021072	731	0.009246
65	1432	0.022911	754	0.010244
66	1329	0.024569	725	0.011122
67	1450	0.026522	780	0.012234
68	1400	0.027894	794	0.013662
69	1329	0.029905	855	0.015261
70	1305	0.032429	901	0.016829
71	1347	0.035249	874	0.018358
72	1296	0.037506	882	0.019757
73	1231	0.040798	922	0.022046
74	1194	0.043760	981	0.025073
75	1347	0.048429	1132	0.028289
76	1368	0.053291	1233	0.031497
77	1496	0.059087	1337	0.035690
78	1523	0.064875	1499	0.040594
79	1693	0.072086	1795	0.046779
80	1696	0.079714	1974	0.053862
81	1727	0.087811	2147	0.061754
82	1643	0.096681	2253	0.070191
83	1529	0.107070	2374	0.079582
84	1526	0.118332	2402	0.090250
85	1494	0.130875	2424	0.102550
86	1346	0.144746	2483	0.116635
87	1195	0.160059	2461	0.132579
88	1087	0.176928	2340	0.150582
89	917	0.195466	2153	0.170847
90	728	0.215786	1803	0.193584
91	594	0.237992	1429	0.218991
92	295	0.262178	727	0.247256
93	150	0.288424	517	0.278536
94	121	0.316784	406	0.312946
95	110	0.347285	397	0.350538
96	133	0.379915	434	0.391282
97	110	0.414615	327	0.435039
98	59	0.451271	219	0.481541
99	27	0.489704	132	0.530362
100	17	0.529661	87	0.580909
101	15	0.570812	60	0.632409
102	5	0.612745	28	0.683919
103	2	0.654969	11	0.734353
104	4	0.696920	13	0.782539
105	1	1.000000	10	1.000000

Source: Czech Statistical Office.

Table 6.8: Life table (Czech Republic, 2011) for ages $x = 60, \ldots, 105$.

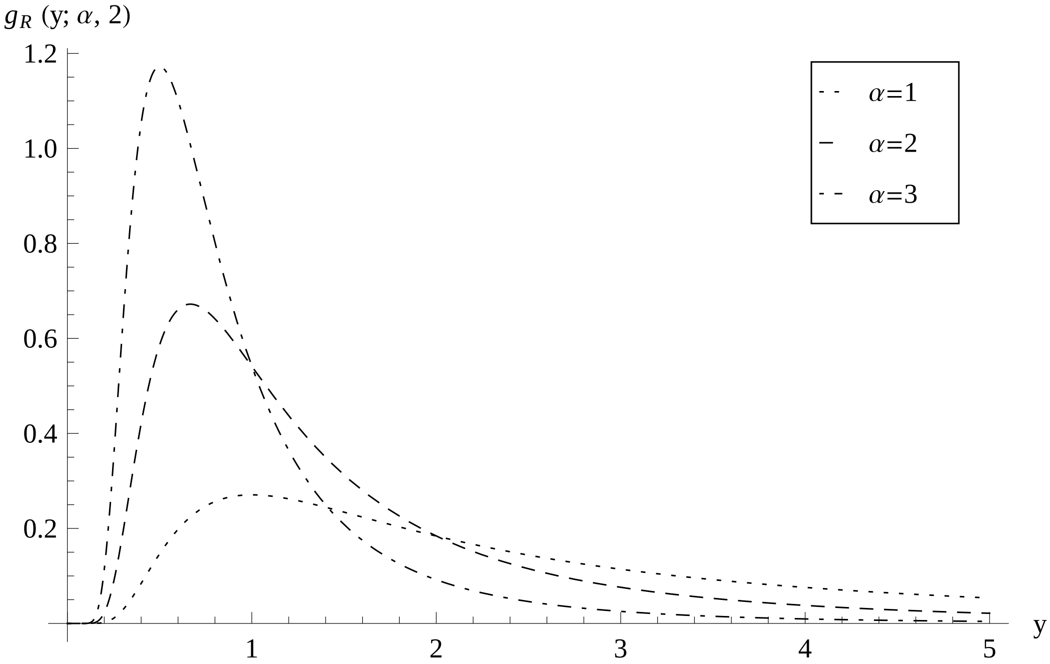


Figure 6.6: The PDF of a reciprocal Gamma random variable with a scale parameter $\beta = 2$ and various values of the shape parameter α.

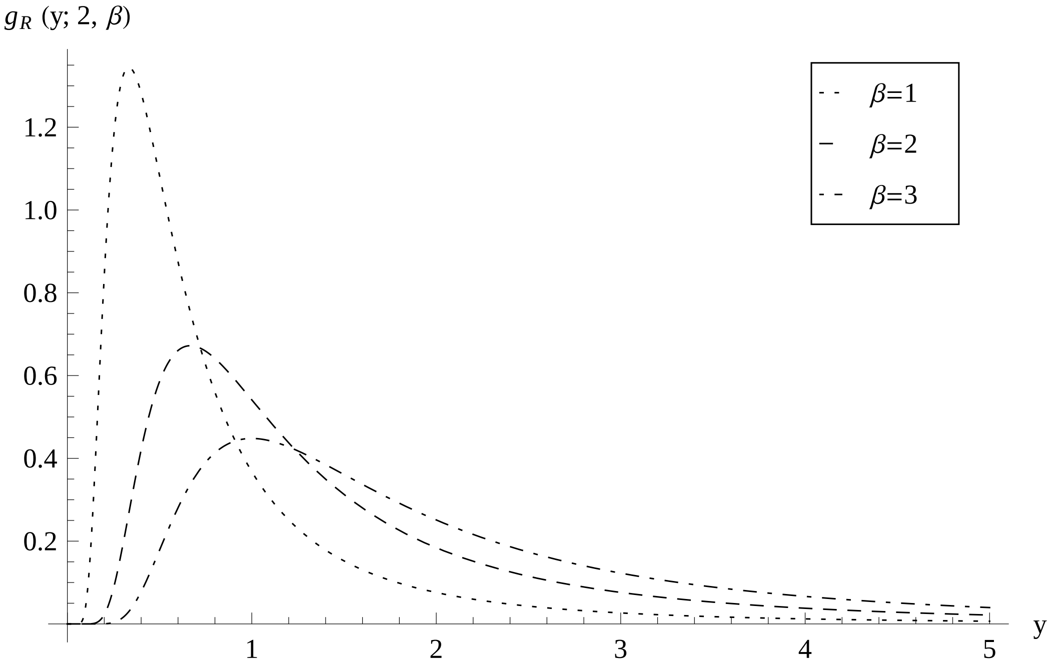


Figure 6.7: The PDF of a reciprocal Gamma random variable with a shape parameter $\alpha = 2$ and various values of the scale parameter β.

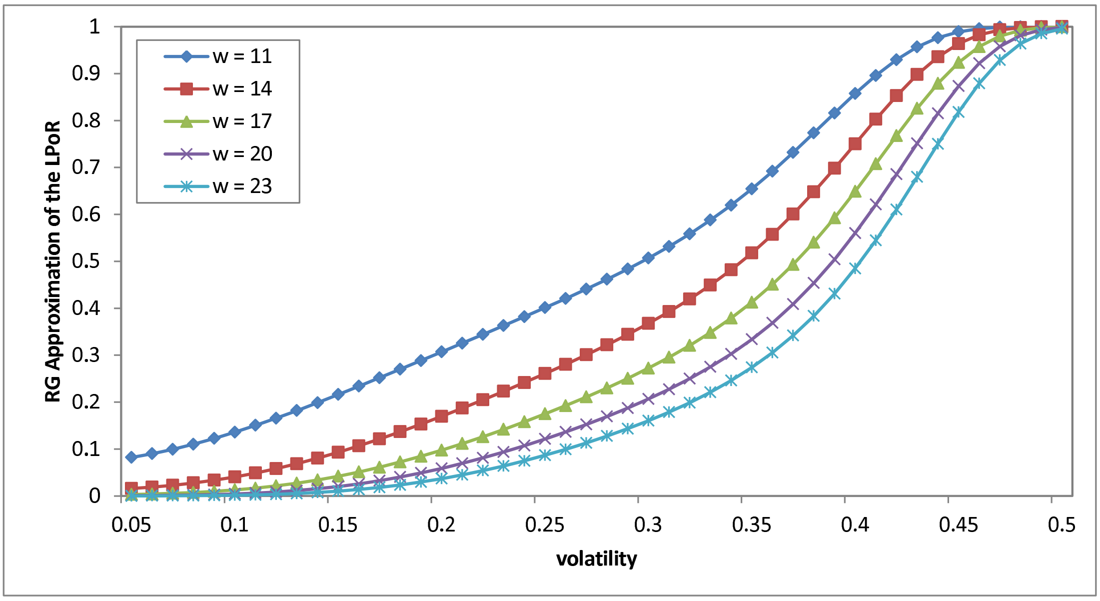


Figure 6.8: The RG approximation of the LPoR as a function of the investment volatility σ, for various values of the initial wealth w. A 65-year-old individual, who plans to consume $k = 1$ real CZK per year was assumed. The parameters of the Gompertz distribution were set to $m = 85$ and $b = 9$. The expected real rate of investment return was set to $\mu = 0.1$.

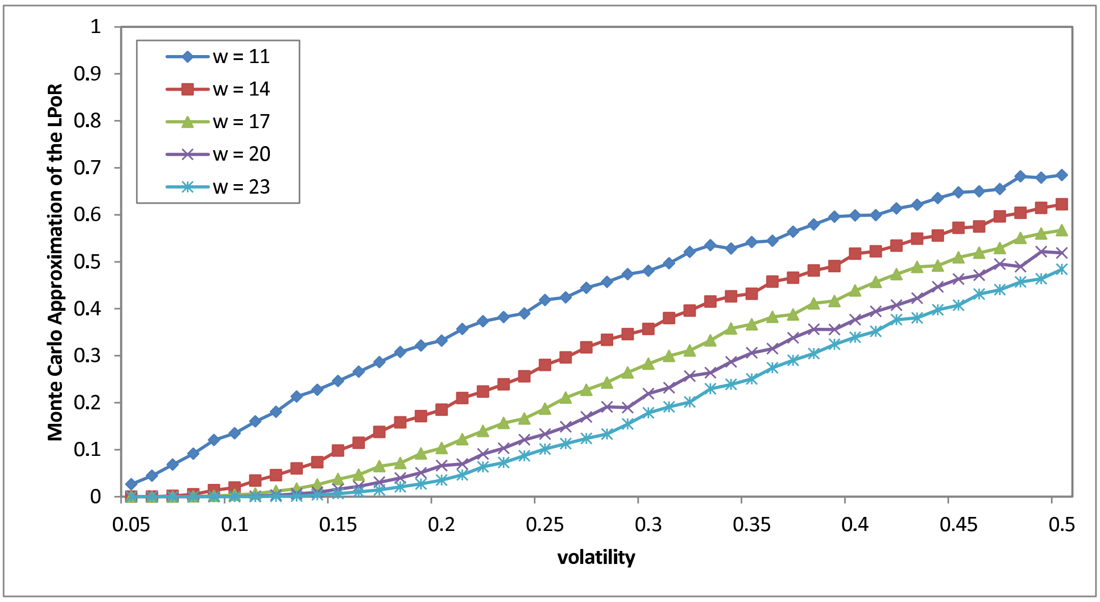


Figure 6.9: The MC approximation of the LPoR as a function of the investment volatility σ, for various values of the initial wealth w. A 65-year-old individual, who plans to consume $k = 1$ real CZK per year was assumed. The parameters of the Gompertz distribution were set to $m = 85$ and $b = 9$. The expected real rate of investment return was set to $\mu = 0.1$.

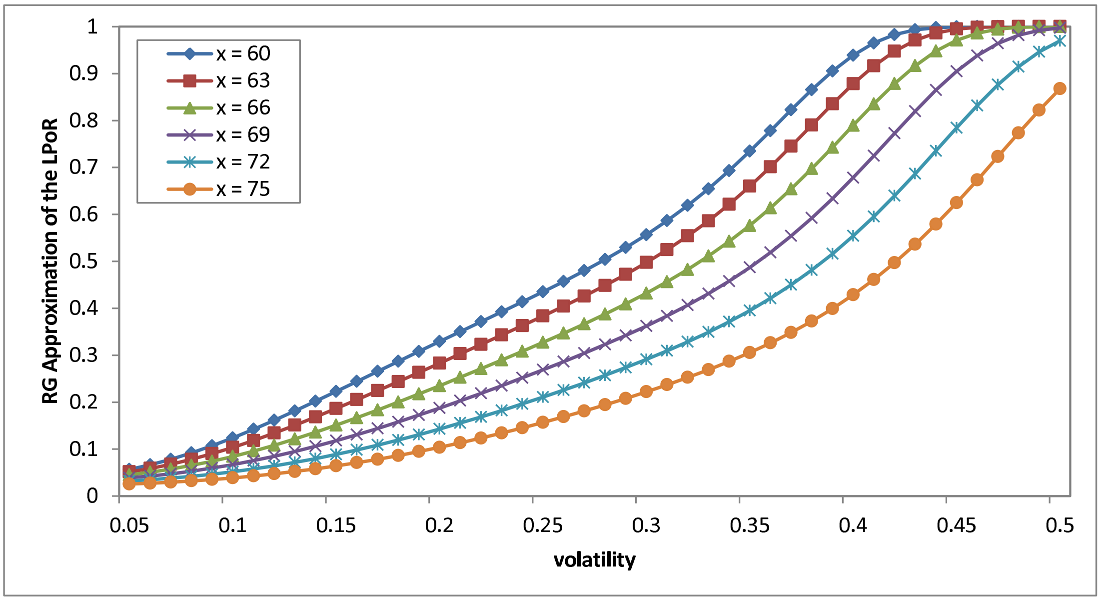


Figure 6.10: The RG approximation of the LPoR as a function of the investment volatility σ, for various values of the age x of the retiree. An individual with the initial wealth of $w = 12$ CZK, who plans to consume $k = 1$ real CZK per year was assumed. The parameters of the Gompertz distribution were set to $m = 85$ and $b = 9$. The expected real rate of investment return was set to $\mu = 0.1$.

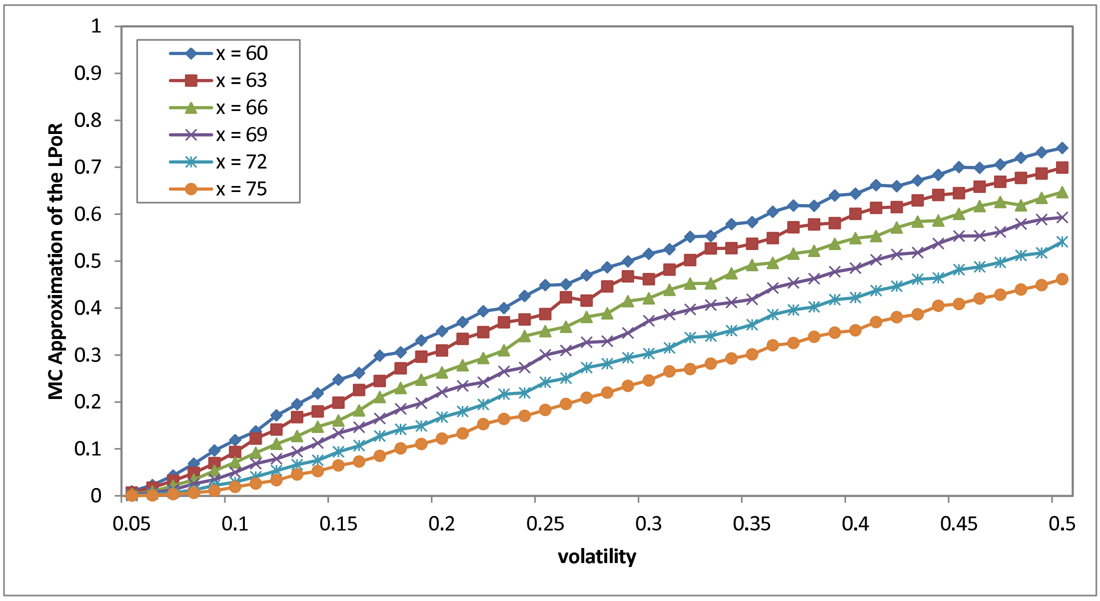


Figure 6.11: The MC approximation of the LPoR as a function of the investment volatility σ, for various values of the age x of the retiree. An individual with the initial wealth of $w = 12$ CZK, who plans to consume $k = 1$ real CZK per year was assumed. The parameters of the Gompertz distribution were set to $m = 85$ and $b = 9$. The expected real rate of investment return was set to $\mu = 0.1$.

Year	Month											
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
2003	0.6	0.2	-0.1	0.2	0.0	0.0	0.1	-0.2	-0.5	0.1	0.5	0.2
2004	1.8	0.2	0.1	0.0	0.4	0.2	0.4	0.0	-0.8	0.5	-0.1	0.1
2005	0.7	0.2	-0.1	0.1	0.2	0.6	0.3	0.0	-0.3	0.9	-0.3	-0.1
2006	1.4	0.1	-0.1	0.1	0.5	0.3	0.4	0.2	-0.7	-0.5	-0.1	0.2
2007	1.0	0.3	0.3	0.7	0.4	0.3	0.4	0.3	-0.3	0.6	0.9	0.5
2008	3.0	0.3	-0.1	0.4	0.5	0.2	0.5	-0.1	-0.2	0.0	-0.5	-0.3
2009	1.5	0.1	0.2	-0.1	0.0	0.0	-0.4	-0.2	-0.4	-0.2	0.2	0.2
2010	1.2	0.0	0.3	0.3	0.1	0.0	0.3	-0.3	-0.3	-0.2	0.2	0.5
2011	0.7	0.1	0.1	0.3	0.5	-0.2	0.3	-0.3	-0.2	0.3	0.4	0.4
2012	1.8	0.2	0.2	0.0	0.2	0.2	-0.1	-0.1	-0.1	0.2	-0.2	0.1

Source: Czech Statistical Office.

Table 6.9: Inflation rate [in %] in the Czech Republic. Every value represents the relative percentage increase in the Consumer Price Index (CPI) between the reference month and the preceding month.

Retirement age	Expected age at death		Spending rate (per 100 CZK)				
			2 CZK	**4 CZK**	**6 CZK**	**8 CZK**	**10 CZK**
		RG	3.0%	15.8%	34.5%	53.2%	68.5%
60	84.3	MC	3.2%	16.7%	35.0%	52.3%	64.8%
		D	-0.2%	-0.9%	-0.4%	0.9%	3.7%
		RG	1.8%	11.2%	26.6%	43.8%	59.4%
65	84.8	MC	2.4%	12.3%	28.8%	45.0%	58.4%
		D	-0.6%	-1.1%	-2.1%	-1.3%	1.0%
		RG	1.0%	6.9%	18.2%	32.4%	46.9%
70	85.6	MC	1.1%	7.4%	20.4%	33.2%	47.9%
		D	-0.1%	-0.5%	-2.1%	-0.8%	-0.9%
		RG	0.5%	3.6%	10.6%	20.6%	32.2%
75	86.8	MC	0.5%	4.2%	12.4%	23.5%	35.9%
		D	0.0%	-0.5%	-1.7%	-2.8%	-3.7%
		RG	0.2%	1.6%	5.1%	10.8%	18.1%
80	88.5	MC	0.1%	1.5%	5.8%	13.5%	21.3%
		D	0.1%	0.1%	-0.7%	-2.8%	-3.2%

Note: RG = reciprocal gamma, MC = Monte Carlo, D = RG - MC.

Table 6.10: Overview of the lifetime probability of retirement ruin for Czech females.