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DOCTORAL THESIS

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Model Problems of the Theory of Gravitation

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Dedicated to my mum in memoriam.

I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Modelové problémy teórie gravitace

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Abstrakt: Pohybové rovnice pro obecnou gravitační konexi a ortonormální korepér jsou odvozeny pro Einstein-Cartanovu teorii z Einstein-Hilbertovského typu účinku. Kalibrační volnost plynoucí z obecnosti gravitační konexe je geometricky interpretována. Naše formulace nefixuje ortonormální korepér jako dotykovým k prostorovému řezu a proto umožňuje, aby Lorentzova grupa byla součástí kalibrační volnosti. 3+1 rozklad proměmných zavádí dotykovou Minkowskiho strukturu a Hamilton-Diracův přístup k dynamice pracuje s Lorentzovskou konexí nad prostorovým řezem. Vazby druhého druhu jsou analyzovány a Diracova závorka je zavedena. Fázový prostor je zredukován a popsán kanonickými proměnnými.
Druhá část disertační práce se věnuje kvantové formulace Einstein-Cartanové teorie. Bodová formulace fázového prostoru je zavedena. Základní proměmné, důležité pro kvantovou formulaci, jsou odvozeny pomocí akcí grup na fázovém prostoru a jejich samosdružená reprezentace je sestrojena. Pomocí nekonečného tensorového součinu bodových Hilbertových prostorů je sestrojena reprezentace základních proměnných Einstein-Cartanové teorie.

Klíčová slova: Einstein-Cartanova teorie, Hamiltonovská formulace, Kvantová gravitace

Title: Model Problems of the Theory of Gravitation

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Abstract:
Equations of motion for general gravitational connection and orthonormal coframe from the Einstein-Hilbert type action of the Einstein-Cartan theory are derived. Additional gauge freedom is geometrically interpreted. Our formulation does not fix coframe to be tangential to spatial section hence Lorentz group is still present as part of gauge freedom. 3+1 decomposition introduces tangent Minkowski structures hence Hamilton-Dirac approach to dynamics works with Lorentz connection over spatial section. The second class constraints are analyzed and Dirac bracket is defined.Reduction of phase space is performed and canonical coordinates are introduced. The second part of this thesis is dedicated to quantum formulation of Einstein-Cartan theory. Point version of Einstein-Cartan phase space is introduced. Basic variables, crucial for quantization, are derived via groups acting on the phase space and their selfadjoint representation is found. Representation of basic variables of Einstein-Cartan theory is derived via infinite tensor product of Hilbert spaces.

Keywords: Einstein-Cartan theory, Hamiltonian formulation, Quantum Gravity

Contents

Introduction

One of the open problems in the theory of gravitation is the difficulty with adding the spinors into the theory. There are two physically nonequivalent formulations of a system including gravity and Dirac's field. In the first one the gravitational connection ${}^{(RLC)}\hat{\nabla}$ is strictly geometrical called Riemann-Levi-Civita (RLC), i.e. connection is compatible with metric and its torsion vanishes, and action of our system is a sum of Einstein-Hilbert and Dirac's actions [1]. In Dirac Langrangian the external derivative operator $\hat{\mathrm{d}}$ should be replaced by ${}^{(RLC)}\hat{\nabla}$ in order to have a final theory locally Lorentz invariant. The variation of action is taken with respect to "metric" and Dirac field ψ (metric should be expressed in terms of orthonormal coframe $\mathbf{e}^a$ and ${}^{(RLC)}\hat{\nabla}$ depends on $\mathbf{e}^a$). In the second model the orthonormal coframe $\mathbf{e}^a$ remains unchanged and gravitational connection $\hat{\nabla}$ is now general, i.e. without any restriction like compatibility with metric, etc. These two families of variables represent our configuration space and variations of the action with respect to both of them are independent. These two formulations are equivalent in the case of pure gravity. But if one adds a Dirac field with Lagrangian depending on connection $\hat{\nabla}$ these two theories give different physical results in the region of Planck densities and higher, e.g. bing bang or black hole singularities resolutions occur in the presence of fermionic matter [2]. Another example of Lagrangian which depends on connection is given by Bičák's vector field [3].

We will focus on the second model in this thesis. The motivation for this choice is taken from loop quantum gravity, where Ashtekar connection $\mathbf{A}$ on spatial section Σ is defined by RLC connection of $\mathbf{q}$ ($\mathbf{q}$ is a metric on Σ induced from a 4-dimensional metric $\mathbf{g}$ of spacetime $\mathbf{M}$) and external curvature of 4-dimensional RLC connection. Ashtekar originally began with complex connection $\mathbf{A}$ but problems with reality conditions or hermiticity of inner product of quantum Hilbert space caused that Barbero-Immirzi parameter enters theory and $\mathbf{A}$ becomes real. This parameter plays no role on classical level, but after quantization it causes ambiguity and must be fixed by comparison of Hawking-Bekenstein entropy with entropy computed from loop theory. Fermionic matter was successfully added to loop gravity only on kinematical level and problems of dynamics remain unresolved. And last but not least, problem is that general theory is $\mathbf{SO}(\mathbf{g})$ invariant what is still true in the case of complex Ashtekar connection but the real loop theory broke down this explicit invariance to $\mathbf{SO}(\mathbf{q})$ [5]. If one does not fix coframe to be tangential to Σ in opposite to loop gravity then all degrees of freedom enters the theory which can then be expressed as $\mathbf{SO}(\mathbf{g})$ gauge theory. As is shown in appendix this leads to a theory where torsion appears as the first class constraint in the case of 2+1 dimensional gravity which is good news for 2+1 dimensional theoretical physicists, because one can work with $\mathbf{SO}(\mathbf{g})$ gauge connection instead of 2+1 analogue of Ashtekar connection and problem of vanishing torsion can be solved on quantum level as one wishes. Unfortunately in the case of 3+1 dimensional Einstein-Cartan theory the condition of vanishing torsion is split in two parts where one is the first class and the other is the second class constraints. Therefore new potential problems like introduction of ghosts should be solved on quantum level.

In this thesis, we will focus on the derivation of Hamiltonian-Dirac formulation of our physical system. The work is organized as follows. In section 1.1, Lagrangian formulation of the Einstein-Cartan theory is formulated in the the language of forms valued in the tangent tensor algebra on $\mathbf{M}$. Equations of motion (EOM) are derived and equivalence between theory of General Relativy and Einstein-Cartan theory is also shown in this section. In section 1.2, geometrical interpretation of general solution of gravitational connection given by the equations of motion is done. 3+1 decomposition is performed in section 1.3 and also some useful formulas are evaluated there. In section 1.4, the Hamiltonian of the theory is written and separation of constraints into the first and second class is performed. In section 1.5, Dirac brackets are introduced and coordinates on reduced phase space are defined.

In the second part of this thesis we will focus on the quantum formulation of Einstein-Cartan theory. We will try to construct the kinematical Hilbert space of Einstein-Cartan theory, where selfadjoint representation of certain family of observables can be defined. We will start with point version of the phase space, which can be interpreted as the phase space of the coframe settled in the point of spatial manifold. There are several groups acting on this space. Their existence will be used for correct definition of selfadjoint operators related to classical observables. In the next section basic ideas of von Neumann's [12] construction of tensor product of the infinite family of Hilbert spaces is briefly summarized. Then it is used for final construction of Hilbert space of Einstein-Cartan theory.The final result appears not very satisfactory, since the construction leads to too large family of irreducible representations of the algebra of kinematical variables which makes the quantization procedure rather ambigous at its present stage of construction.

1. Hamiltonian Formulation of Einstein-Cartan Theory

1.1 Lagrangian of Einstein-Cartan Theory

Let ($\mathbf{M} = \mathbb{R}[t]\times\Sigma, \mathbf{g}$) be a spacetime manifold equipped with metric $\mathbf{g}$ (signature($\mathbf{g}$) = $(+,-,-,-)$). Geroch's theorem [6] says that spinor structure over the manifold $\mathbf{M}$ exists iff there exists global orthonormal frame $\mathbf{e}_a$ over $\mathbf{M}$ and $\mathbf{M}$ is orientable. These two conditions restrict possible topological shapes of $\mathbf{M}$ and Σ, e.g. if the spacetime manifold is given by the product $\mathbf{M} = \mathbb{R}\times$"3-dimensional sphere" then Geroch's conditions are not fulfilled and spinor structure can not be defined over such manifold, in other words, if one considers Friedman's models then the closed model violates the Geroch's conditions. We assume Geroch's conditions already now in the case of pure gravity since spinors should be added into the theory later so there is no loss of generality [1]. The first nice simplification is that the coframe $\mathbf{e}^a$ is defined globally and thus every useful geometrical or gravitational variable can be written in a global manner. Let us look at the basic quantities:
metric

$$\mathbf{g} = \eta_{ab}\mathbf{e}^a \otimes \mathbf{e}^b,$$

4-volume form

$$\hat{\mathbf{\Sigma}} = \frac{1}{4!}\boldsymbol{\varepsilon}_{abcd}\mathbf{e}^a \wedge \mathbf{e}^b \wedge \mathbf{e}^c \wedge \mathbf{e}^d,$$

gravitational connection 1-form $\hat{\mathbf{\Gamma}}^b{}_a$

$$\hat{\nabla}_{\mathbf{u}}\mathbf{e}_a = \hat{\mathbf{\Gamma}}^b{}_a(\mathbf{u})\mathbf{e}_b$$

or its curvature 2-form

$$\hat{\mathbf{F}}^a{}_b = \hat{\mathrm{d}}\hat{\mathbf{\Gamma}}^a{}_b + \hat{\mathbf{\Gamma}}^a{}_c \wedge \hat{\mathbf{\Gamma}}^c{}_b.$$

General Relativity sets the connection $\hat{\nabla}$ to be geometrical and the Einstein-Hilbert action of GR is

$$S_{\mathrm{EH}} = \int -\frac{1}{16\pi\kappa}R_{\mathbf{g}}\omega_{\mathbf{g}}$$

where $R_{\mathbf{g}}$ is Ricci scalar related to the RLC connection of metric tensor $\mathbf{g}$, $\omega_{\mathbf{g}} = \sqrt{-\det|g|}\mathrm{d}^4x$ is its volume form and κ is Newton's constant (c=1). Action written in this form explicitly depends on the choice of coordinates and one should overlap few coordinate's neighbourhoods and solve boundary terms if one wants to cover the whole manifold $\mathbf{M}$ in general case. But using our assumption on $\mathbf{e}^a$ one can rewrite the Einstein-Hilbert action into the following geometrical form

$$S_{\mathrm{EH}} = \int -\frac{1}{32\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\eta^{b\bar{b}}\mathbf{R}_{\mathbf{g}}{}^a{}_{\bar{b}} \wedge \mathbf{e}^c \wedge \mathbf{e}^d, \tag{1.1}$$

[1] One may say that we can define spinor structure locally and work with such structure. But there some certain type of phatologic features occur. We will not focus our attention to this problem. Therefore "no loss of generality".

where ${\mathbf{R}_{\mathbf{g}}}^{a}{}_{b}$ is curvature 2-form of RLC connection. The action (1.1) is a functional of basic variables $\mathbf{e}^a = \mathbf{e}^a_\mu \mathrm{d}x^\mu$ and one should make variation of the action with respect to them. The idea of Einstein-Cartan theory is very simple, gravitational connection $\hat{\nabla}$ is no more geometrical. In Einstein-Cartan action being of Einstein-Hilbert type

$$S = \int\limits_{\Omega} -\frac{1}{32\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\eta^{b\bar{b}}\hat{\mathbf{F}}^{a}{}_{\bar{b}} \wedge \mathbf{e}^c \wedge \mathbf{e}^d \tag{1.2}$$

variation should be made independently in both variables $\mathbf{e}^a$ and $\hat{\mathbf{\Gamma}}^{a}{}_{b}$. Ω is a timelike compact set, i.e. $\Omega =< t_i; t_f > \times\Sigma$. For simplicity we assume in this work that Σ is compact manifold, e.g. torus. Let us decompose variable $\hat{\mathbf{\Gamma}}^{a}{}_{b}$ into $\mathbf{O}(\mathbf{g})$-irreducible parts

$$\hat{\mathbf{\Gamma}}^{ab} = \eta^{bc}\hat{\mathbf{\Gamma}}^{a}{}_{c} = \hat{\mathbf{A}}^{ab} + \hat{\mathbf{B}}\eta^{ab} + \hat{\mathbf{C}}^{ab} \tag{1.3}$$

where $\hat{\mathbf{A}}^{ab}$ is antisymmetric and $\hat{\mathbf{C}}^{ab}$ is symmetric and traceless 1-form, respectively. Curvature $\hat{\mathbf{F}}^{a}{}_{b}$ can be expressed as

$$\hat{\mathbf{F}}^{ab} = \eta^{bc}\hat{\mathbf{F}}^{a}{}_{c} = \hat{\mathbf{R}}^{ab} + \mathrm{d}\hat{\mathbf{B}}\eta^{ab} + \hat{\mathcal{D}}\hat{\mathbf{C}}^{ab} + \eta_{cd}\hat{\mathbf{C}}^{ac} \wedge \hat{\mathbf{C}}^{db}$$

where $\hat{\mathcal{D}}$ is metric connection defined by $\hat{\mathcal{D}}\mathbf{u}^a = \mathrm{d}\mathbf{u}^a + \eta_{bc}\hat{\mathbf{A}}^{ab} \wedge \mathbf{u}^c$ and $\hat{\mathbf{R}}^{ab}$ is its curvature. If $\tilde{\mathbf{e}}^a = O^a{}_{\bar{a}}\mathbf{e}^{\bar{a}}$ is a new coframe with $O^a{}_b$ being Lorentz transformation, then $\hat{\mathbf{A}}^{ab}$ transforms as

$$\tilde{\hat{\mathbf{A}}}^{ab} = O^a{}_{\bar{a}}O^b{}_{\bar{b}}\hat{\mathbf{A}}^{\bar{a}\bar{b}} + O^a{}_{\bar{a}}\eta^{\bar{a}\bar{b}}\mathrm{d}O^b{}_{\bar{b}}$$

while $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}^{ab}$ transform like tensors in their indices. The Einstein-Cartan action can be written in new variables $(\mathbf{e}^a, \hat{\mathbf{A}}^{ab}, \hat{\mathbf{B}}, \hat{\mathbf{C}}^{ab})$ as

$$S = \int\limits_{\mathbf{M}} -\frac{1}{32\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\hat{\mathbf{R}}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{e}^d + \int\limits_{\mathbf{M}} -\frac{1}{32\pi\kappa}\eta_{\bar{a}\bar{b}}\boldsymbol{\varepsilon}_{abcd}\hat{\mathbf{C}}^{a\bar{a}} \wedge \hat{\mathbf{C}}^{\bar{b}b} \wedge \mathbf{e}^c \wedge \mathbf{e}^d \tag{1.4}$$

Notice that variable $\hat{\mathbf{B}}$ does not enter the action (1.4). Thus variation of (1.4) with respect to $\hat{\mathbf{B}}$ vanishes identically and no corresponding equation of motion arises, i.e.

$$\delta_{\hat{\mathbf{B}}}S = 0. \tag{1.5}$$

Now if one makes variation with respect to $\hat{\mathbf{C}}^{ab}$ then one gets

$$\delta_{\hat{\mathbf{C}}}S = \int -\frac{1}{16\pi\kappa}\eta_{\bar{a}\bar{b}}\boldsymbol{\varepsilon}_{abcd}\delta\hat{\mathbf{C}}^{a\bar{a}} \wedge \hat{\mathbf{C}}^{b\bar{b}} \wedge \mathbf{e}^c\mathbf{e}^d = 0 \tag{1.6}$$

for $\forall\delta\hat{\mathbf{C}}^{ab}$: $\delta\hat{\mathbf{C}}^{ab} = \delta\hat{\mathbf{C}}^{ba}$ and $\eta_{ab}\delta\hat{\mathbf{C}}^{ab} = 0$. Equation (1.6) is equivalent to

$$\hat{\mathbf{C}}^{ab} = 0. \tag{1.7}$$

If one uses this fact then action (1.4) can be written as

$$S' = \int\limits_{\Omega} -\frac{1}{32\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\hat{\mathbf{R}}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{e}^d, \tag{1.8}$$

its variation (see, e.g., [7]) is

$$\delta_{\hat{\mathbf{A}},\mathbf{e}} S' = \int\limits_{\Omega} \left(\frac{1}{16\pi\kappa} \boldsymbol{\varepsilon}_{abcd} \delta \hat{\mathbf{A}}^{ab} \wedge \mathbf{e}^c \wedge \hat{\mathcal{D}} \mathbf{e}^d - \frac{1}{16\pi\kappa} \boldsymbol{\varepsilon}_{abcd} \delta \mathbf{e}^a \wedge \hat{\mathbf{R}}^{bc} \wedge \mathbf{e}^d \right)$$

and equations of motion are

$$\begin{aligned} 0 &= \frac{1}{8\pi\kappa} \boldsymbol{\varepsilon}_{abcd} \mathbf{e}^c \wedge \hat{\mathcal{D}} \mathbf{e}^d = -\frac{1}{8\pi\kappa} \left(\hat{T}^c_{ab} + \hat{T}^d_{da} \delta^c_b - \hat{T}^d_{db} \delta^c_a \right) \hat{\mathbf{\Sigma}}_c, &(1.9) \\ 0 &= -\frac{1}{16\pi\kappa} \boldsymbol{\varepsilon}_{abcd} \hat{\mathbf{R}}^{bc} \wedge \mathbf{e}^d = -\frac{1}{8\pi\kappa} \hat{G}^c{}_a \hat{\mathbf{\Sigma}}_c, &(1.10) \end{aligned}$$

where the torsion components are given by

$$\hat{\mathcal{D}} \mathbf{e}^a = \hat{\mathbf{T}}^a = \frac{1}{2} \hat{T}^a_{bc} \mathbf{e}^b \wedge \mathbf{e}^c,$$

3-volume forms

$$\hat{\mathbf{\Sigma}}_a = \frac{1}{3!} \boldsymbol{\varepsilon}_{abcd} \mathbf{e}^b \wedge \mathbf{e}^c \wedge \mathbf{e}^d, \tag{1.11}$$

and $\hat{G}^a{}_b$ is the Einstein tensor

$$\begin{aligned} \hat{G}^a{}_b &= \hat{R}^{ca}{}_{cb} - \frac{1}{2} \hat{R}^{cd}{}_{cd} \delta^a_b, \\ \hat{\mathbf{R}}^{ab} &= \frac{1}{2} \hat{R}^{ab}{}_{cd} \mathbf{e}^c \wedge \mathbf{e}^d. \end{aligned}$$

Equation (1.9) implies that connection $\hat{\mathcal{D}}$ is torsion-free and together with metricity of $\hat{\mathcal{D}}$ we have that $\hat{\mathcal{D}}$ is geometrical connection. Equations (1.10) are Einstein equations of General Relativity. Solution for general gravitational connection $\hat{\mathbf{\Gamma}}^{ab}$ is

$$\hat{\mathbf{\Gamma}}^{ab} = \hat{\mathbf{A}}^{ab} + \hat{\mathbf{B}} \eta^{ab}, \tag{1.12}$$

where $\hat{\mathbf{B}}$ is an arbitrary 1-form and $\hat{\mathbf{A}}^{ab}$, $\mathbf{e}^a$ are given by equations (1.9) and (1.10). Connection of type (1.12) is called Cartan connection. Ambiguity of $\hat{\mathbf{\Gamma}}^{ab}$ due to $\hat{\mathbf{B}}$ represents a gauge freedom in $\hat{\mathbf{\Gamma}}^{ab}$ [8]. Spacetime is given by topology of Σ which is established initially and metric $\mathbf{g} = \eta_{ab} \mathbf{e}^a \otimes \mathbf{e}^b$. The metric is given just by knowledge of $\mathbf{e}^a$, hence $\hat{\mathbf{B}}$ does not affect geometry. Thus General Relativity and the Einstein-Cartan Theory are physically equivalent, at least in the case of pure gravity.

1.2 Geometrical interpretation of the gravitational connection

Let V be a four-dimensional real vector space. Two different frames in V are related by a linear transformation $g \in \mathbf{GL}(V)$. The first nontrivial irreducible representations of $\mathbf{GL}(V)$ are given by $\mathbb{T}^1 V = V$ and $\mathbb{T}_1 V = V^*$ (V^* - dual of V). Next candidates for representations are spaces built by tensor products $\mathbb{T}^2 V = V \otimes V$, $\mathbb{T}_2 V = V^* \otimes V^*$ and $\mathbb{T}^1_1 V = V \otimes V^*$ which is isomorphic with $V^* \otimes V$. Anyway, these spaces are not irreducible. In order to see this, let us consider a general element $\mathbf{t} \in \mathbb{T}^2 V$. $\mathbf{t}$ can be written as

$$\mathbf{t} = t^{ab}\mathbf{g}_a \otimes \mathbf{g}_b, \tag{1.13}$$

where $\mathbf{g}_a$ is a base of V. Thus $\mathbf{t}$ can be expressed as a matrix t^{ab}. As we know, any matrix can be written as a sum of symmetric and antisymmetric matrices

$$\mathbf{t} = \frac{1}{2}(t^{ab} - t^{ba})\mathbf{g}_a \otimes \mathbf{g}_b + \frac{1}{2}(t^{ab} + t^{ba})\mathbf{g}_a \otimes \mathbf{g}_b = a^{ab}\mathbf{g}_a \otimes \mathbf{g}_b + s^{ab}\mathbf{g}_a \otimes \mathbf{g}_b \tag{1.14}$$

and since $g^* a^{ab} = -g^* a^{ba}$ and $g^* s^{ab} = g^* s^{ba}$ we can see that $\mathbb{T}^2 V$ is reducible. g^* means an action of $g \in \mathbf{GL}(V)$. $\mathbb{T}^2 V$ can be decomposed as $\mathbb{T}^2 V = \mathbb{A}^2 V \oplus \mathbb{S}^2 V$, where $\mathbb{A}^2 V$ or $\mathbb{S}^2 V$ means antisymmetric or symmetric part of $\mathbb{T}^2 V$. Of course this is not a proof of irreducibility of $\mathbb{X}^2 V$ ($\mathbb{X} \in \{\mathbb{A}, \mathbb{S}\}$) but these facts about $\mathbf{GL}(V)$ are well known and we will not go further into details. Similar analysis can be done on $\mathbb{T}_2 V$ and we can also write $\mathbb{T}_2 V = \mathbb{A}_2 V \oplus \mathbb{S}_2 V$. Now, let us focus on $\mathbb{T}^1_1 V$. General element $\mathbf{t} \in \mathbb{T}^1_1 V$ is $\mathbf{t} = t^a_b \mathbf{g}_a \otimes \mathbf{g}^b$ where $\mathbf{g}^a$ is dual base and t^a_b can be expressed as sum of trace and traceless parts

$$\mathbf{t} = \frac{1}{4}\delta^a_b \delta^d_c\, t^c_d\, \mathbf{g}_a \otimes \mathbf{g}^b + \left(\delta^a_c \delta^d_b - \frac{1}{4}\delta^a_b \delta^d_c\right) t^c_d\, \mathbf{g}_a \otimes \mathbf{g}^b. \tag{1.15}$$

Trace part transforms like scalar while traceless part gives us another representation of $\mathbf{GL}(V)$.

Let $\mathbf{M}$ be a four-dimensional orientable manifold. We also suppose that $\mathbf{LM} = \mathbf{M} \times \mathbf{GL}(\mathbf{M})$, where $\mathbf{LM}$ is a frame bundle over $\mathbf{M}$ (see, e.g.,[7]) and $\mathbf{GL}(\mathbf{M}) \equiv \mathbf{GL}(\mathbb{T}^1\mathbf{M})$. This is a nontrivial assumption. In the case of the metric manifolds $(\mathbf{M}, \mathbf{g})$, it is equivalent to the first Geroch's condition of a global section of a bundle $\mathbf{OM}$ of all orthonormal frames over $(\mathbf{M}, \mathbf{g})$. Together with the orientability of $\mathbf{M}$ we have a generalized version of both Geroch's conditions . Thanks to these assumptions we can represent $\mathbb{T}^1\mathbf{M}$ as $\times^4_{i=1} F(\mathbf{M})$, where $F(\mathbf{M})$ is a space of functions over $\mathbf{M}$. Let us denote this representation as $\hat{\mathbb{T}}^1\mathbf{M}$ and let $\hat{\mathbf{g}}_a$ be a base coresponding with $\mathbf{g}_a$, etc. Thus we have a representation $\hat{\mathbb{T}}\mathbf{M}$ of tensor algebra $\mathbb{T}\mathbf{M}$. We can define an algebra $\Lambda\mathbb{T}\mathbf{M} = \Lambda\mathbf{M} \otimes \hat{\mathbb{T}}\mathbf{M}$ where $\Lambda\mathbf{M}$ is Cartan algebra of forms over $\mathbf{M}$. A product $\wedge$ on $\Lambda\mathbb{T}\mathbf{M}$ is defined via formula

$$\hat{\mathbf{a}} \wedge \hat{\mathbf{b}} = (\mathbf{a}^{a_1 \dots a_n}{}_{b_1 \dots b_m} \wedge \mathbf{b}^{c_1 \dots c_{n'}}{}_{d_1 \dots d_{m'}})\hat{\mathbf{g}}_{a_1} \otimes \cdots \otimes \hat{\mathbf{g}}_{c_{n'}} \otimes \hat{\mathbf{g}}^{b_1} \otimes \cdots \otimes \hat{\mathbf{g}}^{d_{m'}} \tag{1.16}$$

and an exterior derivative operator $\hat{\mathrm{d}}$ on $\Lambda\mathbb{T}\mathbf{M}$ is given by $\hat{\mathrm{d}} = \hat{\mathrm{d}} \otimes \mathrm{id}$, where $\hat{\mathrm{d}}$ on the left-hand side is operator on $\Lambda\mathbb{T}\mathbf{M}$ while $\hat{\mathrm{d}}$ on the right-hand side is the usual exterior derivative operator on $\Lambda\mathbf{M}$. One can also define a covariant exterior derivative operator for some general connection $\hat{\nabla}$ in similar way on $\Lambda\mathbb{T}\mathbf{M}$. From now we will omit basis (co)vectors and will write just indexed forms instead of whole expressions.

Let $\mathbf{t}^{ab} \in \Lambda\mathbf{M} \times \hat{\mathbb{T}}^2\mathbf{M} \equiv \Lambda\mathbb{T}^2\mathbf{M} \subset \Lambda\mathbb{T}\mathbf{M}$ and $\mathbf{a}^{ab}$ or $\mathbf{s}^{ab}$ be its antisymmetric or symmetric parts, respectively. We have immediately from the linearity of $\hat{\nabla}$ that $\hat{\nabla}\mathbf{a}^{ab} \in \Lambda\mathbb{A}^2\mathbf{M}$ and $\hat{\nabla}\mathbf{s}^{ab} \in \Lambda\mathbb{S}^2\mathbf{M}$. Similar results can be obtained for $\Lambda\mathbb{T}_2\mathbf{M}$ and its antisymmetric and symmetric parts. Since contraction of indices and $\hat{\nabla}$ commute, the covariant derivative operator $\hat{\nabla}$ respects decomposition of the spaces $\Lambda\mathbb{T}^2\mathbf{M}$, $\Lambda\mathbb{T}_2\mathbf{M}$ and $\Lambda\mathbb{T}^1_1\mathbf{M}$ into the irreducible subspaces of $\mathbf{O}(\mathbf{g})$.

Now we are going to explore what happens if we equip **M** with a metric. As before we start with the real four-dimensional vector space V and $\mathbf{g}$ is the metric with signature $(+,-,-,-)$ which can be written as $\mathbf{g} = g_{ab}\mathbf{g}^a \otimes \mathbf{g}^b$. There exists canonical way how to pick up a certain subgroup called orthonormal group $\mathbf{O}(\mathbf{g}) \subset \mathbf{GL}(V)$ given by

$$\mathbf{O}(\mathbf{g}) = \{g \in \mathbf{GL}(V) : g^* g_{ab} = g_{ab}\}\,. \tag{1.17}$$

Because V is equipped with the metric then there exists a canonical isomorphism between V and V^* given by $\mathbf{g} : V \to V^*$. This map can be easily extended into isomorphisms between tensor spaces of the same rank. Examples for rank=2 are given by maps to $\mathbb{T}^2V$, let $\mathbf{t}^* \in \mathbb{T}_2V$ and $\bar{\mathbf{t}} \in \mathbb{T}^1_1V$ and $\mathbf{g}^*$ be an action of such isomorphism

$$\begin{aligned} \mathbf{g}^*\mathbf{t}^* &= \quad g^{ac}g^{bd}\,t^*_{cd} \quad \mathbf{g}_a \otimes \mathbf{g}_b, & (1.18)\\ \mathbf{g}^*\bar{\mathbf{t}} &= \qquad g^{ac}\,\bar{t}^b_c \quad \mathbf{g}_a \otimes \mathbf{g}_b. & (1.19) \end{aligned}$$

We already know that $\mathbb{T}^2V$ and $\mathbb{T}_2V$ can be split into the symmetric and antisymmetric parts while the decomposition of $\mathbb{T}^1_1V$ is given by trace and traceless parts. These spaces generate irreducible representations of $\mathbf{GL}(V)$. Since there exist isomorphisms between the tensor spaces of rank=2 and $\mathbf{O}(\mathbf{g})$ is subgroup of $\mathbf{GL}(V)$ there should exist some common decompositions of spaces $\mathbb{T}^2\mathbf{M}$, $\mathbb{T}_2\mathbf{M}$ and $\mathbb{T}^1_1\mathbf{M}$ into the irreducible representations of $\mathbf{O}(V)$. Indeed, the space $\mathbb{T}^2V$ can be decomposed into the three subspaces $\mathbb{A}^2V$, $\mathbb{B}^2V$ and $\mathbb{C}^2V$ by the following projections

$$\begin{aligned} P^{\mathbb{A}^2V\,ab}{}_{cd}\;\; t^{cd} &= \frac{1}{2}(\delta^a_c\delta^b_d - \delta^a_d\delta^b_c)\;\; t^{cd},\\ P^{\mathbb{B}^2V\,ab}{}_{cd}\;\; t^{cd} &= \frac{1}{4}g^{ab}g_{cd}\;\; t^{cd},\\ P^{\mathbb{C}^2V\,ab}{}_{cd}\;\; t^{cd} &= \left(\frac{1}{2}(\delta^a_c\delta^b_d + \delta^a_d\delta^b_c) - \frac{1}{4}g^{ab}g_{cd}\right)\;\; t^{cd}. \end{aligned} \tag{1.20}$$

$\mathbb{A}^2V$ is our well known antisymmetric subspace, $\mathbb{B}^2V$ is trace part and $\mathbb{C}^2V$ is symmetric traceless subspace of $\mathbb{T}^2V$. Similar projections work on $\mathbb{T}_2V$

$$\begin{aligned} P^{\mathbb{A}_2V\,cd}{}_{ab}\;\; t_{cd} &= \frac{1}{2}(\delta^c_a\delta^d_b - \delta^d_a\delta^c_b)\;\; t_{cd},\\ P^{\mathbb{B}_2V\,cd}{}_{ab}\;\; t_{cd} &= \frac{1}{4}g_{ab}g^{cd}\;\; t_{cd},\\ P^{\mathbb{C}_2V\,cd}{}_{ab}\;\; t_{cd} &= \left(\frac{1}{2}(\delta^c_a\delta^d_b + \delta^d_a\delta^c_b) - \frac{1}{4}g_{ab}g^{cd}\right)\;\; t_{cd} \end{aligned} \tag{1.21}$$

and on $\mathbb{T}^1_1V$

$$\begin{aligned} P^{\mathbb{A}^1_1V\,ad}{}_{bc}\;\; t^c_d &= \frac{1}{2}(\delta^a_c\delta^d_b - g^{ad}g_{bc})\;\; t^c_d,\\ P^{\mathbb{B}^1_1V\,ad}{}_{bc}\;\; t^c_d &= \frac{1}{4}\delta^a_b\delta^d_c\;\; t^c_d,\\ P^{\mathbb{C}^1_1V\,ad}{}_{bc}\;\; t^c_d &= \left(\frac{1}{2}(\delta^a_c\delta^d_b + g^{ad}g_{bc}) - \frac{1}{4}\delta^a_b\delta^d_c\right)\;\; t^c_d. \end{aligned} \tag{1.22}$$

These spaces $\mathbb{X}V$ ($\mathbb{X} \in \{\mathbb{A}^2, \mathbb{B}^2, \mathbb{C}^2, \mathbb{A}^1_1, \ldots, \mathbb{C}_2\}$) are irreducible [2] representations of the group $\mathbf{O}(\mathbf{g})$. $\mathbb{B}$-spaces are equivalent to trivial $\mathbb{R}$ but the rest of $\mathbb{X}$ are representations

[2] Recall that we are working with real representations of $\mathbf{O}(\mathbf{g})$. If we work with complex tensors and $\mathbf{SO}(\mathbf{g})$ then $\mathbb{A}V$ representation is reducibile into self and antiselfdual antisymmetric matrices. Thus in terms of (n,m) counting of complex irreducible representation of $\mathbf{SO}(\mathbf{g})$ $\mathbb{A}V$ is $(1,0) \oplus (0,1)$ while $\mathbb{C}V$ is $(1,1)$.

of higher degree than V. Now we are in a point where our vector space preparation is over and we can finally turn our attention to the metric manifold $(\mathbf{M}, \mathbf{g})$.

We have seen few lines above that general connection $\hat{\nabla}$ preserves structure of irreducible representations of $\mathbf{GL}(V)$ on tensor spaces of rank=2. But the situation is different in the case of group $\mathbf{O(g)}$. Since generally $\hat{\nabla} P^{\mathbb{X}\mathbf{M}}{}_{bc}^{ad} \neq 0$ then e.g. $\hat{\nabla}\mathbb{A}_1^1 \not\subseteq \mathbb{A}_1^1$. The question is how does general connection preserving irreducible structure of $\mathbf{O}(g)$ look like? Necessary conditions for such connection are given by $\hat{\nabla} P^{\mathbb{X}\mathbf{M}}{}_{bc}^{ad} = \gamma^{\mathbb{X}} P^{\mathbb{X}\mathbf{M}}{}_{bc}^{ad}$ (no summation over $\mathbb{X}$!). Thanks to (1.20)-(1.22) we have that these equations are equivalent to

$$\hat{\nabla}(g^{ab} g_{cd}) = \gamma g^{ab} g_{cd}, \tag{1.23}$$

where $\gamma^{\mathbb{X}} = \alpha^{\mathbb{X}} \gamma$ and $\alpha^{\mathbb{X}}$ are constants while γ is arbitrary function. Let us fix the frame $\mathbf{g}_a$ to be an orthonormal $\mathbf{e}_a$ then the metric is $\mathbf{g} = \eta_{ab} \mathbf{e}^a \otimes \mathbf{e}^b$. We can use the decomposition (1.3) and obtain

$$\hat{\nabla}(\eta^{ab}\eta_{cd}) = 2\hat{\mathbf{C}}^{ab}\eta_{cd} - 2\eta^{ab}\eta_{c\bar{c}}\eta_{d\bar{d}}\hat{\mathbf{C}}^{\bar{c}\bar{d}} = \gamma\eta^{ab}\eta_{cd}. \tag{1.24}$$

Necessary condition for existence of solution of (1.24) is given by

$$\eta_{ab}\hat{\nabla}(\eta^{ab}\eta_{cd}) = -8\eta_{c\bar{c}}\eta_{d\bar{d}}\hat{\mathbf{C}}^{\bar{c}\bar{d}} = 4\gamma\eta_{cd}. \tag{1.25}$$

This equation has solution only if $\gamma = 0$ and then the solution is $\hat{\mathbf{C}}^{ab} = 0$ which is also the solution of (1.24) if $\gamma = 0$. Thus general shape of connection preserving irreducible structure of $\mathbf{O(g)}$ on tensor spaces with rank=2 is given by $\hat{\mathbf{\Gamma}}^{ab} = \hat{\mathbf{A}}^{ab} + \hat{\mathbf{B}}\eta^{ab}$ and considering (1.12) we figure out that this is exactly the same type like Cartan connection given by solution of equations of motion in Einstein-Cartan theory. From now until the end of this section $\hat{\nabla}$ is Cartan connection.

Another possible interpretation of Cartan connection is based on notion of symmetrization and antisymmetrization. Space $\mathbb{T}^p_q\mathbf{M}$ is $\mathbf{g}^*$-isomophic to $\mathbb{T}^0_{p+q}\mathbf{M}$. Let $\mathbf{t} \in \mathbb{T}^p_q\mathbf{M}$ then $\mathbf{g}^*\mathbf{t} \in \mathbb{T}^0_{p+q}\mathbf{M}$. (Anti)symmetric projections on $\mathrm{T}^p_q\mathbf{M}$ are defined by

$$\Pi_{\mathbb{A}}\mathbf{t} = (\mathbf{g}^*)^{-1}\hat{\Pi}_{\mathbb{A}}\mathbf{g}^*\mathbf{t}, \tag{1.26}$$
$$\Pi_{\mathbb{S}}\,\mathbf{t} = (\mathbf{g}^*)^{-1}\hat{\Pi}_{\mathbb{S}}\mathbf{g}^*\mathbf{t}, \tag{1.27}$$

where

$$\begin{aligned} \hat{\Pi}_{\mathbb{A}}\mathbf{g}^*\mathbf{t}(u_1, \ldots, u_{p+q}) &= \frac{1}{(p+q)!}\sum_{\sigma} sgn(\sigma)\mathbf{g}^*\mathbf{t}(\sigma(u_1, \ldots, u_{p+q})), \\ \hat{\Pi}_{\mathbb{S}}\mathbf{g}^*\mathbf{t}(u_1, \ldots, u_{p+q}) &= \frac{1}{(p+q)!}\sum_{\sigma} \mathbf{g}^*\mathbf{t}(\sigma(u_1, \ldots, u_{p+q})), \end{aligned}$$

σ means permutation, $sgn(\sigma) = 1$ if σ is even and $sgn(\sigma) = -1$ if σ is odd. Since $\hat{\nabla}\hat{\Pi}_{\mathbb{X}} = 0$ ($\mathbb{X} \in \{\mathbb{A}, \mathbb{S}\}$) because $\hat{\Pi}_{\mathbb{X}}$ are linear combinations of δ-s and $(\mathbf{g}^*)^{-1}\hat{\Pi}_{\mathbb{X}}\mathbf{g}^*\mathbf{t}$ contains p-times multiplied expressions of type $\eta^{ab}\eta_{cd}$ we have immediately that $\hat{\nabla}\Pi_{\mathbb{X}} = 0$.

It should be noted that since in general $\hat{\nabla}\mathbf{g}^* \neq \mathbf{g}^*\hat{\nabla}$ for metric isomorphism between two tensor spaces of the same rank, there is no physical reason for such a feature, the $\mathbb{B}$-part of $\hat{\mathbf{\Gamma}}^{ab}$ should represent a gauge degree of freedom and physical connection should be compatible with metric $\mathbf{g}$ which reflects a well known fact from General Relativity.

1.3 3+1 Decomposition

We have already assumed that the spacetime $\mathbf{M}$ is given by the product $\mathbb{R} \times \Sigma$. This assumption is equivalent to the existence of a global Cauchy surface and hence solution of equations (1.9) and (1.10) can be evolved from initial data on Σ uniquely upto gauge transformation[3]. Our basic variables $\mathbf{e}^a$, $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}^{ab}$ belong to the algebra $\Lambda\mathbb{T}\mathbf{M}$ while $\hat{\mathbf{A}}^{ab}$ are connection forms on $\mathbf{M}$, so it will be useful to preserve this structure even in Hamiltonian formulation. Since we assume that Geroch's conditions are valid, there exists a global orthonormal frame $\mathbf{e}_a$. Let $x \in \Sigma$ then $\mathscr{M}_x = \mathrm{Span}\left\{\mathbf{e}_a\big|_x\right\}$ together with the metric $\mathbf{g}\big|_x$ define a tangent Minkowski space settled at the point x. Since x is the arbitrary point of Σ then space $\mathscr{M} = \cup_{x\in\Sigma}\mathscr{M}_x$ plays analogous role as $\mathbb{T}^1\Sigma$ but it is little bit bigger since $\mathscr{M}$ contains even non tangential vectors. Important thing is that $\mathscr{M}$ can be represented as $\hat{\mathscr{M}} = \times_{\mathrm{dim}\mathbf{M}}F(\Sigma)$ and it is also equipped with Minkowski metric η_{ab}. Hat over $\mathscr{M}$ will be omitted from now and space $\mathscr{M}$ and its representation will be identified. $\mathscr{M}$ is vector space and we can define its tensor algebra $\mathbb{T}\mathscr{M}$ and algebra of forms on Σ valued in this space $\Lambda\mathbb{T}\mathscr{M} = \Lambda\Sigma \times \mathbb{T}\mathscr{M}$. Let $\mathbf{e}_a$ and $\tilde{\mathbf{e}}_a$ be two orthonormal frames in $\mathscr{M}$. Then due to Geroch's conditions there exists just one $g \in \mathbf{O}(\mathbf{g}) \times \Sigma$ such that $\tilde{\mathbf{e}_a} = g^*\mathbf{e}_a$. Thus, we can see that there exists a trivial principal bundle $\mathbf{O}(\mathbf{g})\mathscr{M} = \Sigma \times \mathbf{O}(\mathbf{g})$ over Σ, where part of gauge freedom is given by Lorentz group $\mathbf{O}(\mathbf{g})$. Now we can start detail analysis of 3+1 decomposition of our variables.

Let $\hat{\mathbf{T}} \in \Lambda\mathbb{T}\mathbf{M}$ be a p-form valued in $\hat{\mathbb{T}}\mathbf{M}$, then $\hat{\mathbf{T}}$ can be uniquely decomposed into pure spatial $(p-1)$-form $\check{\mathbf{T}}$ and p-form $\mathbf{T}$ valued in $\mathscr{M}$

$$\hat{\mathbf{T}} = \check{\mathbf{T}} \wedge \mathrm{d}t + \mathbf{T}.$$

Another important geometric object is an external derivative operator. Let us denote by $\hat{\mathrm{d}}$ external derivative on $\mathbf{M}$ while we keep d for Σ. Anyway we still write $\mathrm{d}t$ with the hope that this will not cause any problem. Let us apply $\hat{\mathrm{d}}$ on $\hat{\mathbf{T}}$ to obtain

$$\hat{\mathrm{d}}\hat{\mathbf{T}} = \mathrm{d}\check{\mathbf{T}} \wedge \mathrm{d}t + \mathrm{d}t \wedge \dot{\mathbf{T}} + \mathrm{d}\mathbf{T}$$

where dot means action of Lie derivate along ∂_t which is just simple time derivative of components, e.g. for spatial 1-form $\dot{\mathbf{T}} = \partial_t \mathrm{T}_\alpha \mathrm{d}x^\alpha$, etc. So one can project spacetime p-form onto pure spatial p-form and (p-1)-form on Σ and even $3+1$ dimensional external derivative is also writen in language of spatial forms and their time and spatial derivatives.

Let us explore what happens with orthonormal coframe $\mathbf{e}^a$. We can write

$$\mathbf{e}^a = \lambda^a \mathrm{d}t + \mathbf{E}^a = \lambda^a \mathrm{d}t + E^a_\alpha \mathrm{d}x^\alpha, \tag{1.28}$$

where $\alpha, \beta, \gamma, \ldots = 1, 2, 3$ are spatial coordinate indices while $a, b, c, \ldots = 0, 1, 2, 3$ are reserved for tensors on $\mathscr{M}$. It is useful for our purposes to decompose even frame $\mathbf{e}_a$ into tangential and time parts

$$\mathbf{e}_a = \lambda_a \partial_t + \mathbf{E}_a = \lambda_a \partial_t + E^\alpha_a \partial_\alpha. \tag{1.29}$$

It should be noted that $\lambda_a \neq \eta_{ab}\lambda^a$. We hope that this notation is not confusing since if we need to in/de-crease indices then it will be explicitly written using metric tensor.

[3]One equation is still missing as we will see at the end of this section. But this equation is conservation of constraints given by (1.9) and (1.10).

We have $\mathbf{e}^a(\mathbf{e}_b) = \delta^a_b$ what is

$$\begin{pmatrix}\lambda^a & E^a_\alpha\end{pmatrix}\begin{pmatrix}\lambda_b \\ E^\alpha_b\end{pmatrix} = \lambda^a\lambda_b + E^a_\alpha E^\alpha_b = \delta^a_b, \tag{1.30}$$

thus matrices (λ^a, E^a_α) and $(\lambda_a, E^\alpha_a)^{\mathrm{T}}$ are mutually inverse and since they are finite dimensional we also have

$$\begin{pmatrix}\lambda_a \\ E^\alpha_a\end{pmatrix}\begin{pmatrix}\lambda^a & E^a_\beta\end{pmatrix} = \begin{pmatrix}1 & 0 \\ 0 & \delta^\alpha_\beta\end{pmatrix}, \tag{1.31}$$

or

$$\begin{aligned}\lambda_a\lambda^a &= 1, \qquad & \lambda_a E^a_\alpha &= 0,\\ E^\alpha_a\lambda^a &= 0, \qquad & E^\alpha_a E^a_\beta &= \delta^\alpha_\beta.\end{aligned} \tag{1.32}$$

As we expected, variables λ^a, λ_a, $\mathbf{E}^a$ and $\mathbf{E}_a$ are not independent and we can express vector coefficients using covectors via well known formula for inverse matrix

$$e\lambda_a = \frac{\partial e}{\partial\lambda^a}, \tag{1.33}$$

$$eE^\alpha_a = \frac{\partial e}{\partial E^a_\alpha}, \tag{1.34}$$

where

$$e = \frac{1}{3!}\boldsymbol{\varepsilon}_{abcd}\bar{\varepsilon}^{\alpha\beta\gamma}\lambda^a E^b_\alpha E^c_\beta E^d_\gamma \tag{1.35}$$

is determinant of matrix (λ^a, E^a_α). Coordinates' (co)vectors can be written with the help of previous formulas as

$$\begin{aligned}\mathrm{d}t &= \lambda_a\mathbf{e}^a \qquad & \mathrm{d}x^\alpha &= E^\alpha_a\mathbf{e}^a\\ \partial_t &= \lambda^a\mathbf{e}_a \qquad & \partial_\alpha &= E^a_\alpha\mathbf{e}_a\end{aligned} \tag{1.36}$$

thus we see that vector $\partial_t \in \mathbb{T}^1\mathbf{M}$ is represented by vector $\lambda^a \in \mathscr{M}$ and similar for $\mathrm{d}t \in \mathbb{T}_1\mathbf{M}$ we have $\lambda_a \in \mathbb{T}_1\mathscr{M}$.

Since $\mathscr{M}$ is isomorphic to $\mathbb{T}^1\mathbf{M}$ and there exists a natural decomposition of $\mathbb{T}^1\mathbf{M}$ into subspaces collinear with embedding of Σ and ∂_t there should also exist similar structure on space $\mathscr{M}$. We have immediately from relation $(\lambda^a\lambda_c)(\lambda^c\lambda_b) = \lambda^a\lambda_b$ that $\lambda^a\lambda_b$ is projection on $\mathscr{M}$. We can rearrange equation (1.30) as

$$\mathbf{E}^a_b = \mathbf{E}^a(\mathbf{E}_b) = E^a_\alpha E^\alpha_b = \delta^a_b - \lambda^a\lambda_b \tag{1.37}$$

and another supplemental projection $\mathbf{E}^a_b$ on $\mathscr{M}$ appears. It is clear from (1.36) that $\lambda^a\lambda_b$ maps a general vector $v^a \in \mathscr{M}$ on that part of v^a which is proportional to ∂_t and $\mathbf{E}^a_b$ on that tangent to Σ.

We were working with a general orthonormal frame until now. From this moment $\mathbf{e}^a$ is supposed to be righthanded and future oriented. This assumption restricts our variables λ^a, $\mathbf{E}^a$ and following conditions should be fulfilled

$$\lambda^0 > 0 \tag{1.38}$$

$$\eta_{ab}\lambda^a\lambda^b > 0, \tag{1.39}$$

$$e > 0, \tag{1.40}$$

$$\mathbf{q} = \eta_{ab}\mathbf{E}^a \otimes \mathbf{E}^b < 0, \tag{1.41}$$

where $\mathbf{q}$ is spatial metric and $\mathbf{q} < 0$ means that this tensor on Σ is strictly negative, i.e $\forall \mathbf{v} \neq 0 \in \mathbb{T}^1\Sigma : \mathbf{q}(\mathbf{v}, \mathbf{v}) < 0$. Let $\mathbf{SO}(\mathbf{g})$ be a subgroup of $\mathbf{O}(g)$ preserving conditions (1.38)-(1.41). If one wants to work with the whole $\mathbf{O}(\mathbf{g})$ then configuration manifold splits into four disjoint parts given by future/past and right/left hand orientation and this discrete structure should be taken into account on quantum level, but this is far at the moment.

Decomposition of variables $\hat{\mathbf{B}}^{ab}$, $\hat{\mathbf{C}}^{ab}$ is given by

$$\hat{\mathbf{B}}^{ab} = \mathcal{B}^{ab}\mathrm{d}t + \mathbf{B}^{ab}, \tag{1.42}$$

$$\hat{\mathbf{C}}^{ab} = C^{ab}\mathrm{d}t + \mathbf{C}^{ab} \tag{1.43}$$

and we can now focus on the metric connection variable $\hat{\mathbf{A}}^{ab}$. We can write

$$\hat{\mathbf{A}}^{ab} = \Lambda^{ab}\mathrm{d}t + \mathbf{A}^{ab}. \tag{1.44}$$

It should be noted that Λ^{ab} transforms like tensor under $g \in \mathbf{SO}(\mathbf{g}) \times \Sigma$. Let $\tilde{\hat{\mathbf{e}}}^a = g^*\hat{\mathbf{e}}^a = O^a{}_b\hat{\mathbf{e}}^b$ be a new coframe[4] on $\mathbb{T}_1\mathscr{M}$ then transformation law for $\mathbf{A}^{ab}$ is given by formula

$$\tilde{\mathbf{A}}^{ab} = O^a{}_{\bar{a}}O^b{}_{\bar{b}}\mathbf{A}^{\bar{a}\bar{b}} + O^a{}_{\bar{a}}\eta^{\bar{a}\bar{b}}\mathrm{d}O^b{}_{\bar{b}}.$$

Let $\hat{\mathbf{v}}^a = \check{\mathbf{v}}^a \wedge \mathrm{d}t + \mathbf{v}^a \in \Lambda\mathbb{T}\mathbf{M}$ then $\hat{\mathcal{D}}\hat{\mathbf{v}}^a$ can be written as

$$\hat{\mathcal{D}}\hat{\mathbf{v}}^a = \mathcal{D}\check{\mathbf{v}}^a \wedge \mathrm{d}t + \mathrm{d}t \wedge \dot{\mathcal{D}}\mathbf{v}^a + \mathcal{D}\mathbf{v}^a, \tag{1.45}$$

where $\mathcal{D}$ is spatial covariant external derivative operator on $\mathbf{SO}(\mathbf{g})\mathscr{M}$ given by

$$\mathcal{D}\mathbf{v}^a = \mathrm{d}\mathbf{v}^a + \eta_{bc}\mathbf{A}^{ab} \wedge \mathbf{v}^c \tag{1.46}$$

and $\dot{\mathcal{D}}$ is covariant time derivative

$$\dot{\mathcal{D}}\mathbf{v}^a = \dot{\mathbf{v}}^a + \eta_{bc}\Lambda^{ab}\mathbf{v}^c. \tag{1.47}$$

Since Λ^{ab} and $\mathbf{A}^{ab}$ are antisymmetric in their indices we have immediately that

$$\mathcal{D}\eta_{ab} = 0 \tag{1.48}$$

and

$$\dot{\mathcal{D}}\eta_{ab} = 0. \tag{1.49}$$

Thus operators $\mathcal{D}$ and $\dot{\mathcal{D}}$ are compatible with metric η_{ab} on $\mathscr{M}$.

Let us summarize our situation. We started with connection $\hat{\mathcal{D}}$ on $\mathbf{M}$ with gauge group $\mathbf{SO}(\mathbf{g})$. $3 + 1$ decomposition of space $\Lambda\mathbb{T}\mathbf{M}$ leads us to pure spatial connection $\mathcal{D}$ on Σ with the same group $\mathbf{SO}(\mathbf{g})$ which is good news for us. Since as we wanted or expected the $\mathbf{SO}(\mathbf{g})$ structure is preserved even in the language of spatial forms on Σ. This is to be contrasted with standard ADM/real Loop formulation[5] where gauge group is only $\mathbf{SO}(\mathbf{q})$. So far we are still working with real variables which is again in contrast with complex Loop theory where gauge group is $\mathbf{SO}(\mathbf{g})$ but the prize paid for

[4] $\mathbf{e}^a$ is coframe on $\mathbb{T}_1\mathbf{M}$, $\hat{\mathbf{e}}^a$ is its representation on $\mathbb{T}_1\mathscr{M}$

[5] Of course ADM formalism works with spatial metric $\mathbf{q}$ and therefore there are no coframe variables. For example in the Loop gravity Hamiltonian formulation starts with ADM, then orthonormal coframe $\mathbf{e}^i$ on Σ is introduced and metric is expressed by orthonormality of this coframe, i.e. $\mathbf{q}$ $(i, j = 1, 2, 3)$.

that is the loss of reality of variables.

In general theory of gauge connections a notion of a curvature is well known. Vanishing of the curvature expresses the condition that a horizontal subspace in a fibre bundle over given manifold is integrable. In usual words this means that parallel transport along closed path of a given object (the object should be valued in nontrivial representantion space of the gauge group) is given by identity (see details in ,e.g., [7]). That is why the curvature plays an important role even for the general gauge group G (recall $\hat{\mathbf{F}} = \hat{\mathrm{d}}\hat{\mathbf{A}}$ in Maxwell theory or more complicated objects in Standard Model). For our purposes it is sufficient to write down an explicit formula which is

$$\mathbf{R}^{ab} = \mathcal{D}\mathbf{A}^{ab} = \mathrm{d}\mathbf{A}^{ab} + \eta_{cd}\mathbf{A}^{ac} \wedge \mathbf{A}^{db}$$

for our $\mathbf{SO(g)}$ connection $\mathbf{A}^{ab}$ on $\Lambda\mathbb{T}\mathscr{M}$. Spacetime curvature $\hat{\mathbf{R}}^{ab}$ can be decomposed as

$$\hat{\mathbf{R}}^{ab} = \mathbf{R}^{ab} + \mathrm{d}t \wedge \dot{\mathbf{A}}^{ab} + \mathcal{D}\Lambda^{ab} \wedge \mathrm{d}t. \tag{1.50}$$

Next geometrical object on $\mathbf{M}$ which plays important role in the Einstein-Cartan Theory is torsion $\hat{\mathbf{T}}^a = \hat{\mathcal{D}}\mathbf{e}^a$. How does its spatial counterpart look like? Coframe $\mathbf{e}^a$ is not object from $\Lambda\mathbb{T}\mathscr{M}$ because it contains $\mathrm{d}t$. We can project $\mathbf{e}^a$ with $\mathbf{E}^a_b$ and have $\mathbf{E}^a = \mathbf{E}^a_b\mathbf{e}^b$ what is already the object from $\Lambda\mathbb{T}\mathscr{M}$. Thus, let us define $\mathbf{SO(g)}$-torsion by formula

$$\mathbf{T}^a = \mathcal{D}\mathbf{E}^a. \tag{1.51}$$

Since we are not and will not be working with the 3-dimensional $\mathbf{SO(q)}$-connection let us call for simplicity $\mathbf{T}^a$ as torsion on places where no confusion can arise. Another motivation for its name appears if we write spacetime torsion $\hat{\mathbf{T}}^a$ in 3 + 1 manner

$$\hat{\mathcal{D}}\mathbf{e}^a = \mathcal{D}\mathbf{E}^a + \mathcal{D}\lambda^a \wedge \mathrm{d}t + \mathrm{d}t \wedge \dot{\mathcal{D}}\mathbf{E}^a. \tag{1.52}$$

As we can see, spatial part of spacetime torsion $\hat{\mathbf{T}}^a$ is just $\mathbf{SO(g)}$-torsion $\mathbf{T}^a$.

It will be useful in a while and also in next sections to have derived few formulas. In order to do this, let us consider 2-form $\mathbf{P}_{ab}$ which is antisymmetric in its indices ab, i.e.

$$\mathbf{P}_{ab} = \frac{1}{2}\tilde{P}^{\alpha}_{ab}\varepsilon_{\alpha\beta\gamma}\mathrm{d}x^{\beta} \wedge \mathrm{d}x^{\gamma}. \tag{1.53}$$

$\mathbf{P}_{ab}$ can be decomposed in its tensor indices into tangential and time parallel parts as

$$\mathbf{P}_{ab} = 2\overset{\perp}{\mathbf{P}}_{[a}\lambda_{b]} + \hat{\mathbf{P}}_{ab}, \tag{1.54}$$

where

$$\overset{\perp}{\mathbf{P}}_a = \mathbf{P}_{ab}\lambda^b, \tag{1.55}$$

note that $\overset{\perp}{\mathbf{P}}_a\lambda^a = 0$, and

$$\hat{\mathbf{P}}_{ab} = \mathbf{E}^{\bar{a}}_a\mathbf{E}^{\bar{b}}_b\mathbf{P}_{\bar{a}\bar{b}}. \tag{1.56}$$

Let us focus on the tangential part $\hat{\mathbf{P}}_{ab}$. We can multiply it by $\mathbf{E}^a$

$$\mathbf{K}^c_{ab} = \hat{\mathbf{P}}_{ab} \wedge \mathbf{E}^c \tag{1.57}$$

It is easy to show that there is a one to one correspondence between $\hat{\mathbf{P}}_{ab}$ and $\mathbf{K}^c_{ab}$ iff $\lambda_c\mathbf{K}^c_{ab} = 0$, $\mathbf{K}^c_{ab} = -\mathbf{K}^c_{ba}$ and $\lambda^a\mathbf{K}^c_{ab} = 0$. Let $\mathbf{K}^c_{ab} = \tilde{K}^c_{ab}\mathrm{d}^3x$, then $\tilde{K}^c_{ab} = \hat{\tilde{P}}^\alpha_{ab}E^c_\alpha$ and due to $\lambda_c\mathbf{K}^c_{ab} = 0$ we can express $\hat{\tilde{P}}^\alpha_{ab} = \tilde{K}^c_{ab}E^\alpha_c$. Equation (1.57) can be rearranged without any loss of information by multiplying it with $\bar{\boldsymbol{\varepsilon}}^{a\bar{b}\bar{c}\bar{d}}\lambda_{\bar{b}}$, since bottom indices are spatial and antisymmetric, into the 3-form

$$\mathbf{K}^{ab} = \frac{1}{2}\bar{\boldsymbol{\varepsilon}}^{a\bar{b}\bar{c}\bar{d}}\lambda_{\bar{b}}\mathbf{K}^b_{\bar{c}\bar{d}} = \frac{1}{2}\bar{\boldsymbol{\varepsilon}}^{a\bar{b}\bar{c}\bar{d}}\lambda_{\bar{b}}\mathbf{P}_{\bar{c}\bar{d}} \wedge \mathbf{E}^b, \tag{1.58}$$

which can be written as a sum of symmetric and antisymmetric parts

$$\mathbf{K}^{ab} = \mathbf{K}^{(ab)} + \mathbf{K}^{[ab]}. \tag{1.59}$$

Antisymmetric part can be rewritten as

$$\begin{aligned} \mathbf{P}^{\parallel}_a = \boldsymbol{\varepsilon}_{bcda}\lambda^b\mathbf{K}^{[cd]} &= -\frac{1}{2}\boldsymbol{\varepsilon}_{abcd}\bar{\boldsymbol{\varepsilon}}^{c\bar{b}\bar{c}\bar{d}}\lambda^b\lambda_{\bar{b}}\hat{\mathbf{P}}_{\bar{c}\bar{b}} \wedge \mathbf{E}^d = \ldots \\ \mathbf{P}^{\parallel}_a &= \mathbf{E}^b_a\mathbf{P}_{bc} \wedge \mathbf{E}^c. \end{aligned} \tag{1.60}$$

Thus whole information about $\mathbf{P}_{ab}$ is encoded in three independent components
$\mathbf{P}^{\perp}_a$ – 2-form spatial covector,
$\mathbf{P}^{\parallel}_a$ – 3-form spatial covector,
$\boldsymbol{\sigma}^{ab}$ – spatial symmetric 3-form,
where (sign and 2 is just convention)

$$\boldsymbol{\sigma}^{ab} = -2\mathbf{K}^{(ab)} = \frac{1}{2}\mathbf{P}_{\bar{a}\bar{b}}\lambda_{\bar{c}} \wedge \left(\bar{\boldsymbol{\varepsilon}}^{\bar{a}\bar{b}\bar{c}a}\mathbf{E}^b + \bar{\boldsymbol{\varepsilon}}^{\bar{a}\bar{b}\bar{c}b}\mathbf{E}^a\right).$$

Let us consider linear map of $\mathbf{P}_{ab}$ given by the integral

$$\mathbf{P}(\mathbf{B}) = \int_\Sigma \frac{1}{2}\mathbf{P}_{ab} \wedge \mathbf{B}^{ab}, \tag{1.61}$$

where $\mathbf{B}^{ab}$ is 1-form antisymmetric in its indices. Since we can decompose $\mathbf{P}_{ab}$ into three parts we expect that similar decomposition works for its dual $\mathbf{B}^{ab}$. We can write

$$\frac{1}{2}\mathbf{P}_{ab} \wedge \mathbf{B}^{\perp ab} = \mathbf{P}^{\perp}_a \wedge \mathbf{B}^a = \frac{1}{2}\mathbf{P}_{ab} \wedge 2\mathbf{B}^{[a}\lambda^{b]}, \tag{1.62}$$

thus $\mathbf{B}^{\perp ab} = 2\mathbf{B}^{[a}\lambda^{b]}$

$$\frac{1}{2}\mathbf{P}_{ab} \wedge \mathbf{B}^{\parallel ab} = \mathbf{P}^{\parallel}_a\mathcal{B}^a = \frac{1}{2}\mathbf{P}_{ab} \wedge 2\mathcal{B}^{\bar{a}}\mathbf{E}^{[a}_{\bar{a}}\mathbf{E}^{b]}, \tag{1.63}$$

thus $\mathbf{B}^{\parallel ab} = 2\mathcal{B}^{\bar{a}}\mathbf{E}^{[a}_{\bar{a}}\mathbf{E}^{b]}$ and

$$\frac{1}{2}\mathbf{P}_{ab}\mathbf{B}^{M ab} = \boldsymbol{\sigma}^{ab}M_{ab} = \frac{1}{2}\mathbf{P}_{ab}\lambda_{\bar{a}}\bar{\boldsymbol{\varepsilon}}^{ab\bar{a}\bar{b}} \wedge \mathbf{E}^{\bar{c}}M_{\bar{b}\bar{c}}, \tag{1.64}$$

thus $\mathbf{B}^{M ab} = \bar{\boldsymbol{\varepsilon}}^{ab\bar{a}\bar{b}}\mathbf{E}^{\bar{c}}\lambda_{\bar{a}}M_{\bar{b}\bar{c}}$. In other words we can decompose dual to $\mathbf{P}_{ab}$ as

$$\mathbf{B}^{ab} = 2\mathbf{B}^{[a}\lambda^{b]} + 2\mathcal{B}^{\bar{a}}\mathbf{E}^{[a}_{\bar{a}}\mathbf{E}^{b]} + \bar{\boldsymbol{\varepsilon}}^{ab\bar{a}\bar{b}}\mathbf{E}^{\bar{c}}\lambda_{\bar{a}}M_{\bar{b}\bar{c}}, \tag{1.65}$$

where $\mathbf{B}^a$ is arbitrary 1-form vector, $\mathscr{B}^a$ is 0-form vector and M_{ab} is symmetric matrix.

We already derived equations of motion of the Einstein-Cartan theory from Lagrangian in section 1.1 and now it is the right time to explore them in details. Anyway, we present here only brief description and leave the rest to the next chapters where Hamiltonian-Dirac formalism is explored in full details. Recall that torsion equation (1.9) sets connection to be just geometrical; in other words $\hat{\mathbf{A}}^{ab}$, can be written as function(al) of the metric $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$ ($\mu, \nu = t, \alpha$ are spacetime coordinate indices) and initial value formulation for Einstein equations (1.10) written using $g_{\mu\nu}$ is well known and understood problem (see, e.g. [10]). If we follow ideas of Einstein-Cartan theory and work with our variables $\mathbf{A}^{ab}$, $\mathbf{E}^a$, etc. then the set of equations given by (1.9) and (1.10) is not complete. Missing equations should be derived from the condition preserving the constraints given by parts of equation (1.9) and (1.10). Let us look what happens here. Decomposition of (1.9) leads to

$$0 = \frac{1}{8\pi\kappa} \boldsymbol{\varepsilon}_{abcd} \mathbf{E}^c \wedge \mathcal{D}\mathbf{E}^d, \tag{1.66}$$

$$0 = \frac{1}{8\pi\kappa} \boldsymbol{\varepsilon}_{abcd} \left(\lambda^c \mathcal{D}\mathbf{E}^d + \mathbf{E}^c \wedge \mathcal{D}\lambda^d - \mathbf{E}^c \wedge \dot{\mathcal{D}}\mathbf{E}^d \right). \tag{1.67}$$

Equations (1.10) can be rewritten similarly as

$$0 = -\frac{1}{16\pi\kappa} \boldsymbol{\varepsilon}_{abcd} \mathbf{R}^{bc} \wedge \mathbf{E}^d, \tag{1.68}$$

$$0 = -\frac{1}{16\pi\kappa} \boldsymbol{\varepsilon}_{abcd} \left(\mathbf{R}^{bc} \lambda^d + \dot{\mathbf{A}}^{bc} \wedge \mathbf{E}^d - \mathcal{D}\Lambda^{bc}. \wedge \mathbf{E}^d \right) \tag{1.69}$$

The expression on the right-hand side of (1.67) is a 2-form with antisymmetric indices and we can use decompositon (1.65). We obtain an evolution equation and a constraint

$$0 = \dot{\mathcal{D}}\mathbf{E}^a - \mathcal{D}\lambda^a, \tag{1.70}$$

$$0 = \mathbf{E}^{(a} \mathbf{E}^{b)}_c \wedge \mathcal{D}\mathbf{E}^c. \tag{1.71}$$

Here is no problem with ambiguity. The equation (1.69) is a 2-form with one tensor index that expresses $4 \times 3 = 12$ conditions for $\dot{\mathbf{A}}^{ab}$ with $6 \times 3 = 18$ degrees of freedom. We see that we are not able to determine connection velocities and some equation(s) is(are) still missing. We will see later that conditions (1.68) and (1.66) represent the first class contraints while equation (1.71) is the constraint of the second class. The missing equation can be obtained by applying the time derivative on (1.71). Since (1.68) and (1.66) are the first class constraints no new conditions appear and we have closed system of equations determining $\mathbf{E}^a$ and $\mathbf{A}^{ab}$. The variables λ^a and Λ^{ab} are arbitrary. The missing equation is

$$0 = \mathbf{E}^{(a} \mathbf{E}^{b)}_c \wedge (\mathbf{R}^{c\bar{a}} \eta_{\bar{a}\bar{b}} \lambda^{\bar{b}} + \mathbf{H}^{c\bar{a}} \eta_{\bar{a}\bar{b}} \wedge \mathbf{E}^{\bar{b}}), \tag{1.72}$$

where $\mathbf{H}^{ab} = \dot{\mathbf{A}}^{ab} - \mathcal{D}\Lambda^{ab}$. Now we can determine $\mathbf{H}^{ab}$ as a certain function(al) of λ^a, $\mathbf{E}^a$, $\mathbf{A}^{ab}$ but we will not do that because we do not need it anywhere. It is enough for our purposes to know that our set of equations determines uniquely, up to gauge transformation, evolution of our system.

1.4 Hamiltonian

In the section 1.1 we have introduced the Lagrangian of the Einstein-Cartan theory. The next step towards its quantum formulation should be done by its conversion into

Table 1.1: Basic variables

Variables	Momenta	Velocities
λ^a	$\boldsymbol{\pi}_a = \tilde{\pi}_a \mathrm{d}^3x$ where $\tilde{\pi}_a = \partial\mathscr{L}/\partial\dot{\lambda}^a$	$\nu^a = \dot{\lambda}^a$
$\mathbf{E}^a = E^a_\alpha \mathrm{d}x^\alpha$	$\mathbf{p}_a = \frac{1}{2}\tilde{p}^\alpha_a \varepsilon_{\alpha\beta\gamma}\mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma$ where $\tilde{p}^\alpha_a = \partial\mathscr{L}/\partial\dot{E}^a_\alpha$	$\mathbf{b}^a = \dot{\mathbf{E}}^a$
Λ^{ab}	$\mathbf{\Pi}_{ab} = \tilde{\Pi}_{ab}\mathrm{d}^3x$ where $\tilde{\Pi}_{ab} = \partial\mathscr{L}/\partial\dot{\Lambda}^{ab}$	$\Gamma^{ab} = \dot{\Lambda}^{ab}$
$\mathbf{A}^{ab} = A^{ab}_\alpha \mathrm{d}x^\alpha$	$\mathbf{p}_{ab} = \frac{1}{2}\tilde{p}^\alpha_{ab}\varepsilon_{\alpha\beta\gamma}\mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma$ where $\tilde{p}^\alpha_{ab} = \partial\mathscr{L}/\partial\dot{A}^{ab}_\alpha$	$\mathbf{B}^{ab} = \dot{\mathbf{A}}^{ab}$
$\mathcal{B}$	$\boldsymbol{\varphi} = \tilde{\varphi}\mathrm{d}^3x$ where $\tilde{\varphi} = \partial\mathscr{L}/\partial\dot{\mathcal{B}}$	$\mathcal{Y} = \dot{\mathcal{B}}$
$\mathbf{B} = B_\alpha \mathrm{d}x^\alpha$	$\mathbf{u} = \frac{1}{2}\tilde{u}^\alpha\varepsilon_{\alpha\beta\gamma}\mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma$ where $\tilde{u}^\alpha = \partial\mathscr{L}/\partial\dot{B}_\alpha$	$\mathbf{Y} = \dot{\mathbf{B}}$
$\mathcal{C}^{ab}$	$\mathbf{\Phi}_{ab} = \tilde{\Phi}_{ab}\mathrm{d}^3x$ where $\tilde{\Phi}_{ab} = \partial\mathscr{L}/\partial\dot{\mathcal{C}}^{ab}$	$\mathcal{X}^{ab} = \dot{\mathcal{C}}^{ab}$
$\mathbf{C}^{ab} = C^{ab}_\alpha \mathrm{d}x^\alpha$	$\mathbf{U}_{ab} = \frac{1}{2}\tilde{U}^\alpha_{ab}\varepsilon_{\alpha\beta\gamma}\mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma$ where $\tilde{U}^\alpha_{ab} = \partial\mathscr{L}/\partial\dot{C}^{ab}_\alpha$	$\mathbf{X}^{ab} = \dot{\mathbf{C}}^{ab}$

the canonical form. Since our system contains velocities of basic variables at best linearly, standard Hamilton procedure can not be used. Therefore we must use Dirac procedure for constrained dynamic [9]. In the standard and even in the Dirac approach to dynamics the notion of momentum for variable q^A is introduced by $p_A = \frac{\partial L}{\partial \dot{q}^A}$, where L is the Lagrangian of a system. Since the action is $S = \int \mathrm{d}tL$ we can see that action and Lagrangian for field theory can be written within 4-form $\mathbf{L}$ called Lagrangian form as $S = \int_\Omega \mathbf{L}$ and $L = \int_\Sigma i_{\partial_t}\mathbf{L}$, where $\mathbf{L} = \mathscr{L}\mathrm{d}^4x$ and $\mathscr{L}$ is Lagrangian density. If we suppose that configuration space is built just by generalized n-forms $\mathbf{Q}^A = \frac{1}{n!}Q^A_{\alpha\ldots\beta}\mathrm{d}x^\alpha \wedge\cdots\wedge \mathrm{d}x^\beta$, e.g. $\mathbf{E}^a$, $\mathbf{A}^{ab}$ in our system, all variables in Standard Model, etc., then we can see that their momenta $\tilde{p}_A^{\alpha\ldots\beta} = \frac{\delta L}{\delta \dot{Q}^A_{\alpha\ldots\beta}} = \frac{\partial\mathscr{L}}{\partial \dot{Q}^A_{\alpha\ldots\beta}}$ transform like densities under coordinate transformation and therefore objects $\mathbf{p}_A = \frac{1}{n!(3-n)!}\tilde{p}_A^{\alpha\ldots\beta}\varepsilon_{\alpha\ldots\beta\gamma\ldots\delta}\mathrm{d}x^\gamma \wedge\cdots\wedge \mathrm{d}x^\gamma$ are $(3-n)$-forms and even more $\mathbf{p}_A \wedge \dot{\mathbf{Q}}^A = \frac{1}{n!}\tilde{p}_A^{\alpha\ldots\beta}\dot{Q}^A_{\alpha\ldots\beta}\mathrm{d}^3x$ what is exactly the first term in the definition of Hamiltonian $H = \int_\Sigma \mathbf{p}_A \wedge \dot{\mathbf{Q}}^A - L$. Recall that $Q^A_{\alpha\ldots\beta}$ and $\tilde{p}_A^{\alpha\ldots\beta}$ are antisymmetric in their coordinate indices therefore every term in $\tilde{p}_A^{\alpha\ldots\beta}\dot{Q}^A_{\alpha\ldots\beta}$ is $n!$-times repeated while every velocity should enter the Hamiltonian just once. Our configuration space is described by variables $\lambda^a, \ldots, \mathbf{C}^{ab}$ and its velocities (see table 1.1 for details). Variables $\mathcal{B}, \ldots, \mathbf{C}^{ab}$ enters the Lagrangian (1.75) in a certain special way. We can decompose it as sum of two Lagrangians $\mathbf{L} = \mathbf{L}^{(\mathrm{EC})} + \mathbf{L}^{(\mathrm{Rest})}$ where

$$\mathbf{L}^{(\mathrm{Rest})} = -\mathrm{d}t \wedge \frac{1}{16\pi\kappa}\eta_{\bar{a}\bar{b}}\boldsymbol{\varepsilon}_{abcd}(\mathcal{C}^{a\bar{a}}\mathbf{C}^{b\bar{b}} \wedge \mathbf{E}^c \wedge \mathbf{E}^d + \mathbf{C}^{a\bar{a}} \wedge \mathbf{C}^{b\bar{b}} \wedge \lambda^c\mathbf{E}^d) \tag{1.73}$$

and $\mathbf{L}^{(\mathrm{EC})}$ does not depend on $\mathcal{C}^{ab}$, $\mathbf{C}^{ab}$ while as we already know, the whole Lagrangian $\mathbf{L}$ does not depend on $\mathcal{B}$, $\mathbf{B}$. Thus we can consider this subsystem independently. Hamiltonian $\mathbf{H}^{(\mathrm{Rest})}$ is given by

$$\begin{aligned}\mathbf{H}^{(\mathrm{Rest})} = \quad & \boldsymbol{\varphi} \wedge \mathcal{Y} + \mathbf{u} \wedge \mathbf{Y} + \frac{1}{2}\mathbf{\Phi}_{ab} \wedge \mathcal{X}^{ab} + \frac{1}{2}\mathbf{U}_{ab} \wedge \mathbf{X}^{ab} + \\ & + \frac{1}{16\pi\kappa}\eta_{\bar{a}\bar{b}}\boldsymbol{\varepsilon}_{abcd}(\mathcal{C}^{a\bar{a}}\mathbf{C}^{b\bar{b}} \wedge \mathbf{E}^c \wedge \mathbf{E}^d + \mathbf{C}^{a\bar{a}} \wedge \mathbf{C}^{b\bar{b}} \wedge \lambda^c\mathbf{E}^d) \end{aligned} \tag{1.74}$$

with primary constraints $\boldsymbol{\varphi} = \mathbf{u} = \mathbf{\Phi}_{ab} = \mathbf{U}_{ab} = 0$. Secondary constraints are $\mathcal{C}^{ab} = \mathbf{C}^{ab} = 0$. Since $\mathbf{\Phi}_{ab}$, $\mathbf{U}_{ab}$ and $\mathcal{C}^{ab}$, $\mathbf{C}^{ab}$ are canonical variables their Poisson bracket is an identity. They are thus the second class constraints and we must use the Dirac procedure. Dirac bracket for this subsystem is just Poisson bracket on canonical variables $\mathscr{B}$, $\mathbf{B}$ and its momenta $\boldsymbol{\varphi}$, $\mathbf{u}$ while its reduced Hamiltonian is $\mathbf{H}^{(\mathrm{Rest})} = \boldsymbol{\varphi} \wedge \mathscr{Y} + \mathbf{u} \wedge \mathbf{Y}$. Hence we can focus ourselves for a while just on $\mathbf{L}^{(\mathrm{EC})}$ and its hamiltonization. Final Hamiltonian will be obtained by sum $\mathbf{H} = \mathbf{H}^{(\mathrm{EC})} + \mathbf{H}^{(\mathrm{Rest})}$.

Let us substitute the decomposition of variables $\mathbf{e}^a$, $\hat{\mathbf{A}}^{ab}$ into Langrangian $\mathbf{L}^{(\mathrm{EC})}$

$$
\begin{aligned}
i_{\partial_t}\mathbf{L}^{(\mathrm{EC})} &= -\frac{1}{16\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\lambda^a\mathbf{R}^{bc} \wedge \mathbf{E}^d + \frac{1}{32\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\mathscr{D}\Lambda^{ab} \wedge \mathbf{E}^c \wedge \mathbf{E}^d \\
&\quad -\frac{1}{32\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\dot{\mathbf{A}}^{ab} \wedge \mathbf{E}^c \wedge \mathbf{E}^d. \qquad (1.75)
\end{aligned}
$$

We use this in definition of Hamiltonian. Our procedure then yields the following result

$$
\mathsf{H}^{(\mathrm{EC})} = \int_\Sigma \mathbf{H}^{(\mathrm{EC})} = \boldsymbol{\pi}(\nu) + \boldsymbol{\Pi}(\Gamma) + \mathbf{p}(\mathbf{b}) + \mathbf{P}(\mathbf{B}) + \mathbf{R}(\lambda) + \mathbf{T}(\Lambda), \qquad (1.76)
$$

where

$$
\begin{aligned}
\boldsymbol{\pi}(\nu) &= \int_\Sigma \boldsymbol{\pi}_a \wedge \nu^a, \\
\mathbf{p}(\mathbf{b}) &= \int_\Sigma \mathbf{p}_a \wedge \mathbf{b}^a, \\
\boldsymbol{\Pi}(\Gamma) &= \int_\Sigma \frac{1}{2}\boldsymbol{\Pi}_{ab} \wedge \Gamma^{ab}, \\
\mathbf{P}(\mathbf{B}) &= \int_\Sigma \frac{1}{2}\left(\mathbf{p}_{ab} + \frac{1}{16\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\mathbf{E}^c \wedge \mathbf{E}^d\right) \wedge \mathbf{B}^{ab} = \int_\Sigma \frac{1}{2}\mathbf{P}_{ab} \wedge \mathbf{B}^{ab}, \\
\mathbf{R}(\lambda) &= \int_\Sigma \frac{1}{16\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\lambda^a\mathbf{R}^{bc} \wedge \mathbf{E}^d = \int_\Sigma \lambda^a\mathbf{R}_a, \\
\mathbf{T}(\Lambda) &= \int_\Sigma -\frac{1}{32\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\mathscr{D}\Lambda^{ab} \wedge \mathbf{E}^c \wedge \mathbf{E}^d = \int_\Sigma -\frac{1}{16\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\Lambda^{ab} \wedge \mathbf{E}^c \wedge \mathscr{D}\mathbf{E}^d = \int_\Sigma \frac{1}{2}\Lambda^{ab}\mathbf{T}_{ab},
\end{aligned}
$$

where $\mathbf{P}_{ab} = \mathbf{p}_{ab} + \frac{1}{16\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\mathbf{E}^c \wedge \mathbf{E}^d$, $\mathbf{R}_a = \frac{1}{16\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\mathbf{R}^{bc} \wedge \mathbf{E}^d$ and $\mathbf{T}_{ab} = -\frac{1}{8\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\mathbf{E}^c \wedge \mathscr{D}\mathbf{E}^d$. The existence of the primary constraints represents the fact that we are working with a degenerated Lagrangian and therefore we are not able to express velocities as function(al)s of momenta (they are given by conditions $\frac{\partial\mathscr{L}}{\partial\dot{Q}^A} = 0$). Our system is degenerated and primary contraints are given by

$$
\begin{aligned}
\boldsymbol{\pi}(\nu) = 0 &\quad \forall\nu^a \in \Lambda_0\mathbb{T}\mathscr{M} \quad \Leftrightarrow \quad \boldsymbol{\pi}_a = 0, \\
\mathbf{p}(\mathbf{b}) = 0 &\quad \forall\mathbf{b}^a \in \Lambda_1\mathbb{T}\mathscr{M} \quad \Leftrightarrow \quad \mathbf{p}_a = 0, \\
\boldsymbol{\Pi}(\Gamma) = 0 &\quad \forall\Gamma^{ab} \in \Lambda_0\mathbb{T}\mathscr{M} \quad \Leftrightarrow \quad \boldsymbol{\Pi}_{ab} = 0, \\
\mathbf{P}(\mathbf{B}) = 0 &\quad \forall\mathbf{B}^{ab} \in \Lambda_1\mathbb{T}\mathscr{M} \quad \Leftrightarrow \quad \mathbf{P}_{ab} = \mathbf{p}_{ab} + \frac{1}{16\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\mathbf{E}^c \wedge \mathbf{E}^d = 0.
\end{aligned}
$$

Since these constraints should be valid through the whole time evolution of our physical system their time derivatives should vanish too and this implies further conditions which should be fulfilled[6],

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{\pi}(\tilde{v})}{\mathrm{d}t} &= \left\{\boldsymbol{\pi}(\tilde{v});\mathsf{H}^{(\mathrm{EC})}\right\} = -\mathbf{R}(\tilde{v}) = 0, && (1.77)\\
\frac{\mathrm{d}\boldsymbol{\Pi}(\tilde{\Gamma})}{\mathrm{d}t} &= \left\{\boldsymbol{\Pi}(\tilde{\Gamma});\mathsf{H}^{(\mathrm{EC})}\right\} = -\mathbf{T}(\tilde{\Gamma}) = 0, && (1.78)\\
\frac{\mathrm{d}\mathbf{p}(\tilde{\mathbf{b}})}{\mathrm{d}t} &= \left\{\mathbf{p}(\tilde{\mathbf{b}});\mathsf{H}^{(\mathrm{EC})}\right\} = \\
&= \int \frac{1}{16\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\tilde{\mathbf{b}}^a \wedge \left(\mathbf{B}^{bc}\wedge\mathbf{E}^d + \lambda^b\mathbf{R}^{cd} - \mathcal{D}\Lambda^{bc}\wedge\mathbf{E}^d\right) = 0, && (1.79)\\
\frac{\mathrm{d}\mathbf{P}(\tilde{\mathbf{B}})}{\mathrm{d}t} &= \left\{\mathbf{P}(\tilde{\mathbf{B}});\mathsf{H}^{(\mathrm{EC})}\right\} = \\
&= \int \frac{1}{16\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\tilde{\mathbf{B}}^{ab} \wedge \left(\mathbf{b}^c\wedge\mathbf{E}^d + \eta_{\bar{a}\bar{b}}\Lambda^{c\bar{a}}\mathbf{E}^{\bar{b}}\wedge\mathbf{E}^d - \mathcal{D}(\lambda^c\mathbf{E}^d)\right) = 0. && (1.80)
\end{aligned}
$$

The first two of them are secondary constraints. It is clear that (1.79) is equal to (1.69), while (1.80) is connected with (1.67); they determine Lagrange multipliers $\mathbf{b}^a$, $\mathbf{B}^{ab}$. As we have already promised in previous section we will show how to do this now. Since these equations are same one can also use the same procedure there (recall that $\mathbf{b}^a = \dot{\mathbf{E}}^a$ and $\mathbf{B}^{ab} = \dot{\mathbf{A}}^{ab}$). We can express equations (1.79), (1.80) as:

$$
\begin{aligned}
0 &= \frac{1}{16\pi\kappa}\boldsymbol{\varepsilon}_{abcd}(\mathbf{H}^{bc}\wedge\mathbf{E}^d + \mathbf{R}^{bc}\lambda^d), && (1.81)\\
0 &= \frac{1}{8\pi\kappa}\boldsymbol{\varepsilon}_{abcd}(\mathbf{h}^c\wedge\mathbf{E}^d - \lambda^c\mathcal{D}\mathbf{E}^d), && (1.82)
\end{aligned}
$$

where $\mathbf{H}^{ab} = \mathbf{B}^{ab} - \mathcal{D}\Lambda^{ab}$ and $\mathbf{h}^a = \mathbf{b}^a + \Lambda^{a\bar{a}}\eta_{\bar{a}\bar{b}}\mathbf{E}^{\bar{b}} - \mathcal{D}\lambda^a$. Let us focus on the second equation (1.82). We can multiply it again by general 1-form $\tilde{\mathbf{B}}^{ab}$ and since it is antisymmetric in its indices we can decompose it as (1.65)

$$\frac{1}{8\pi\kappa}(\tilde{\mathbf{B}}^a\lambda^b + \tilde{\mathcal{B}}^{\bar{a}}\mathbf{E}^a_{\bar{a}}\mathbf{E}^b + \frac{1}{2}\bar{\boldsymbol{\varepsilon}}^{ab\bar{a}\bar{b}}\mathbf{E}^{\bar{c}}\lambda_{\bar{a}}\tilde{M}_{\bar{b}\bar{c}})\wedge\boldsymbol{\varepsilon}_{abcd}(\mathbf{h}^c\wedge\mathbf{E}^d - \lambda^c\mathcal{D}\mathbf{E}^d) = 0. \quad (1.83)$$

This expression can be split into three independend equations

$$
\begin{aligned}
\frac{1}{8\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\lambda^b\mathbf{h}^c\wedge\mathbf{E}^d &= 0, && (1.84)\\
\frac{1}{8\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\mathbf{E}^a_{\bar{a}}\mathbf{E}^b\wedge(\mathbf{h}^c\wedge\mathbf{E}^d - \lambda^c\mathcal{D}\mathbf{E}^d) &= 0, && (1.85)\\
-\frac{1}{8\pi\kappa}\mathbf{E}^{(a}\mathbf{E}^{b)}_c\wedge\mathcal{D}\mathbf{E}^c &= 0. && (1.86)
\end{aligned}
$$

We can use constraint $\mathbf{T}_{ab} = 0$ in the second equation which together with the first one implies that $\mathbf{h}^a = 0$, while the third equation is another secondary constraints,

$$\mathbf{S}(M) = \int_\Sigma \frac{1}{8\pi\kappa}M_{ab}\mathbf{E}^a\mathbf{E}^b_c\wedge\mathcal{D}\mathbf{E}^c = \int_\Sigma M_{ab}\mathbf{S}^{ab} = 0, \quad (1.87)$$

where $\mathbf{S}^{ab} = \frac{1}{8\pi\kappa}\mathbf{E}^{(a}\mathbf{E}^{b)}_c\wedge\mathcal{D}\mathbf{E}^c$ and M_{ab} is arbitrary function symmetric in its indices. Let us substitute the decomposition

$$\mathbf{H}^{ab} = 2\mathbf{H}^{[a}\lambda^{b]} + 2\mathcal{H}^{\bar{a}}\mathbf{E}^{[a}_{\bar{a}}\mathbf{E}^{b]} + \bar{\boldsymbol{\varepsilon}}^{ab\bar{a}\bar{b}}\mathbf{E}^{\bar{c}}\lambda_{\bar{a}}\gamma_{\bar{b}\bar{c}} \quad (1.88)$$

[6]We omitted writing of details like $\forall\tilde{v}^a\ldots$ in constraint expressions.

into equation (1.81). We obtain

$$\frac{1}{16\pi\kappa}\boldsymbol{\varepsilon}_{abcd}(2\mathbf{H}^b\lambda^c\wedge\mathbf{E}^d+2\mathcal{H}^{\bar{b}}\mathbf{E}^b_{\bar{b}}\mathbf{E}^c\wedge\mathbf{E}^d+\mathbf{R}^{bc}\lambda^d)=0. \tag{1.89}$$

If we multiply it by λ^a then we have immediately that $\mathbf{E}^a_b\mathcal{H}^b=0$ while $\lambda_a\mathcal{H}^a$ is arbitrary but we do not need it since it does not enter $\mathbf{H}^{ab}$. Hence this equation is reduced as

$$\frac{1}{16\pi\kappa}\boldsymbol{\varepsilon}_{abcd}(2\mathbf{H}^b\lambda^c\wedge\mathbf{E}^d+\mathbf{R}^{bc}\lambda^d)=0, \tag{1.90}$$

which can be rewritten after some algebraic manipulations as

$$2H^{[a}_{d}\lambda^{b]}+2H^c_c\delta^{[a}_d\lambda^{b]}=-2R^{c[a}{}_{cd}\lambda^{b]}, \tag{1.91}$$

where $R^{ab}{}_{cd}=i_{\mathbf{E}_d}i_{\mathbf{E}_c}\mathbf{R}^{ab}$ and $H^a_b=i_{\mathbf{E}_b}\mathbf{H}^a$. Constraint $\mathbf{R}_a\lambda^a=0$ is equivalent to $R^{ab}{}_{ab}=0$ and if we sum the previous equation over $a=d$ then $H^a_a=0$ and we finally have

$$2\mathbf{H}^{[a}\lambda^{b]}=-2i_{\mathbf{E}_c}\mathbf{R}^{c[a}\lambda^{b]} \tag{1.92}$$

or

$$\mathbf{H}^{ab}=-2i_{\mathbf{E}_c}\mathbf{R}^{c[a}\lambda^{b]}+\bar{\boldsymbol{\varepsilon}}^{ab\bar{a}\bar{b}}\mathbf{E}^{\bar{c}}\lambda_{\bar{a}}\gamma_{\bar{b}\bar{c}}, \tag{1.93}$$

where $\gamma_{ab}=\gamma_{ba}$ is not determined yet. But there is no need to worry since our analysis is not over. We have just finished the first level of the Dirac procedure, however conservation of the secondary constraints should be analyzed too and there will appear the missing equation for γ_{ab}. In order to do this let us compute time derivatives of secondary constraints (1.77), (1.78) and (1.87)

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{R}(\mu)}{\mathrm{d}t} &= \left\{\mathbf{R}(\mu),\mathsf{H}^{(\mathrm{EC})}\right\}= \\
&= \int_\Sigma \frac{1}{16\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\,\mu^a(\mathbf{R}^{bc}\wedge\mathbf{b}^d+\mathcal{D}\mathbf{B}^{bc}\wedge\mathbf{E}^d)=0, &(1.94)\\
\frac{\mathrm{d}\mathbf{T}(\Theta)}{\mathrm{d}t} &= \left\{\mathbf{T}(\Theta),\mathsf{H}^{(\mathrm{EC})}\right\}= \\
&= \int_\Sigma -\frac{1}{16\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\Theta^{ab}(\mathbf{E}^c\wedge\mathcal{D}\mathbf{b}^d+\mathbf{E}^c\wedge\mathbf{B}^{d\bar{a}}\eta_{\bar{a}\bar{b}}\wedge\mathbf{E}^{\bar{b}})=0, &(1.95)\\
\frac{\mathrm{d}\mathbf{S}(M)}{\mathrm{d}t} &= \left\{\mathbf{S}(M),\mathsf{H}^{(\mathrm{EC})}\right\}= \\
&= \int_\Sigma \frac{1}{8\pi\kappa}M_{ab}\left(\mathbf{E}^a\mathbf{E}^b_c\wedge\mathcal{D}\mathbf{b}^c+\mathbf{E}^a\mathbf{E}^b_c\wedge\mathbf{B}^{c\bar{a}}\eta_{\bar{a}\bar{b}}\wedge\mathbf{E}^{\bar{b}}\right)=0, &(1.96)
\end{aligned}$$

where the terms obviously proportional to the constraints are omitted. We can substitute the expression for $\mathbf{b}^a$ from $\mathbf{h}^a=\mathbf{b}^a+\Lambda^{a\bar{a}}\eta_{\bar{a}\bar{b}}\mathbf{E}^{\bar{b}}=0$ into (1.94) and thanks to generalized Bianchi $\mathcal{D}\mathbf{R}^{ab}=0$ and Ricci $\mathcal{D}\mathcal{D}\Lambda^{ab}=\mathbf{R}^{a\bar{a}}\eta_{\bar{a}\bar{b}}\Lambda^{\bar{a}b}+\mathbf{R}^{b\bar{a}}\eta_{\bar{a}\bar{b}}\Lambda^{a\bar{b}}$ identities we have immediately

$$\mathcal{D}\left(\frac{\boldsymbol{\varepsilon}_{abcd}}{16\pi\kappa}(\mathbf{R}^{bc}\lambda^d+\mathbf{H}^{bc}\wedge\mathbf{E}^d)\right)-\frac{\boldsymbol{\varepsilon}_{abcd}}{16\pi\kappa}\mathbf{R}^{bc}\Lambda^{d\bar{a}}\eta_{\bar{a}\bar{b}}\mathbf{E}^{\bar{b}}+\frac{\boldsymbol{\varepsilon}_{abcd}}{8\pi\kappa}\mathbf{R}^{b\bar{a}}\eta_{\bar{a}\bar{b}}\Lambda^{\bar{b}c}\wedge\mathbf{E}^d=0.$$

The first term vanishes due to (1.81). The last term can be transformed with the help of identity $\mathbf{R}^{ab} = \frac{1}{4}\bar{\boldsymbol{\varepsilon}}^{ab\bar{a}\bar{b}}\boldsymbol{\varepsilon}_{\bar{a}\bar{b}\bar{c}\bar{d}}\mathbf{R}^{\bar{c}\bar{d}}$ into expression

$$\frac{\boldsymbol{\varepsilon}_{abcd}}{8\pi\kappa}\mathbf{R}^{b\bar{a}}\eta_{\bar{a}\bar{b}}\Lambda^{\bar{b}c}\wedge\mathbf{E}^{d} = \frac{\boldsymbol{\varepsilon}_{abcd}}{16\pi\kappa}\mathbf{R}^{bc}\Lambda^{d\bar{a}}\eta_{\bar{a}\bar{b}}\mathbf{E}^{\bar{b}} - \frac{1}{16\pi\kappa}\eta_{ab}\Lambda^{b\bar{a}}\boldsymbol{\varepsilon}_{\bar{a}\bar{b}\bar{c}\bar{d}}\mathbf{R}^{\bar{b}\bar{c}}\mathbf{E}^{\bar{d}}.$$

Hence no new condition appears from equation (1.94) since last term is proportional to $\mathbf{R}_a = 0$.

Equation (1.95) can be rewritten with the help $\mathbf{h}^a = 0$ and due to the fact that constraints $\mathbf{T}_{ab} = \mathbf{S}^{ab} = 0$ imply $\mathcal{D}\mathbf{E}^a = 0$ as

$$\frac{1}{32\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\mathbf{R}^{cd}\wedge\mathbf{E}^{\bar{a}}\eta_{\bar{a}\bar{b}}\lambda^{\bar{b}} + \frac{1}{8\pi\kappa}\eta_{\bar{a}[a}\boldsymbol{\varepsilon}_{b]c\bar{b}\bar{c}}\mathbf{E}^{\bar{a}}\wedge i_{\mathbf{E}_{\bar{d}}}\mathbf{R}^{\bar{d}\bar{b}}\lambda^{\bar{c}}\wedge\mathbf{E}^{c} = 0, \tag{1.97}$$

where (1.93) has been already substituted. Since any 4-form on the three-dimensional manifold vanishes identically we have that $\mathbf{E}^a\wedge\mathbf{R}^{bc}\wedge\mathbf{E}^d = 0$. We can apply interior product on it $i_{\mathbf{E}_b}(\mathbf{E}^a\wedge\mathbf{R}^{bc}\wedge\mathbf{E}^d) = \mathbf{E}^a_b\mathbf{R}^{bc}\wedge\mathbf{E}^d - \mathbf{E}^a\wedge i_{\mathbf{E}_b}\mathbf{R}^{bc}\wedge\mathbf{E}^d - \mathbf{E}^a\wedge\mathbf{R}^{bc}\mathbf{E}^d_b = 0$ and now we can express from this the term proportional to $i_{\mathbf{E}_b}\mathbf{R}^{bc}$ and substitute it into previous equation. If we use again $\mathbf{R}^{ab} = \frac{1}{4}\bar{\boldsymbol{\varepsilon}}^{ab\bar{a}\bar{b}}\boldsymbol{\varepsilon}_{\bar{a}\bar{b}\bar{c}\bar{d}}\mathbf{R}^{\bar{c}\bar{d}}$ then we finally find out that (1.97) is proportional to $\mathcal{D}\mathbf{E}^a$. Hence again no new constraint appears from (1.95).

Equation (1.96) can be rewritten as

$$\frac{1}{8\pi\kappa}\mathbf{E}^{(a}\mathbf{E}^{b)}_c\wedge(\mathbf{R}^{c\bar{a}}\eta_{\bar{a}\bar{b}}\lambda^{\bar{b}} + \mathbf{H}^{c\bar{a}}\eta_{\bar{a}\bar{b}}\wedge\mathbf{E}^{\bar{b}}) = 0. \tag{1.98}$$

This is the equation which determines γ_{ab} entering (1.93). However, we do not need explicit expression. For our purposes it is sufficient to show that this equation determines γ_{ab} uniquely. In order to see this we should substitute the expression (1.93) instead of $\mathbf{H}^{ab}$ into this equation. Since (1.98) is linear in $\mathbf{H}^{ab}$ it is also linear in γ_{ab}, i.e. $c_A + Q^B_A\gamma_B$=0, where $A, B = (ab)$, and hence it is sufficient to show that Q^A_B is invertible. The first observation is that (1.98) actually represents 6 equations for 6 pieces $\mathbf{E}^a_{\bar{a}}\mathbf{E}^b_{\bar{b}}\gamma_{ab}$ hence we can consider only the term proportional to γ_{ab} which is $\lambda_b\lambda_{\bar{b}}\eta_{c\bar{c}}\bar{\boldsymbol{\varepsilon}}^{dcb(a}\bar{\boldsymbol{\varepsilon}}^{\bar{a})\bar{b}\bar{c}\bar{d}}\gamma_{d\bar{d}} = \tilde{G}^{a\bar{a}b\bar{b}}\gamma_{b\bar{b}}$ and as we will see in the next section the expression $\tilde{G}^{a\bar{a}b\bar{b}}$ standing in front of $\gamma_{b\bar{b}}$ is invertible on spatial subspace.

Let us summarize this section. We have built the Hamiltonian formulation of Einstein-Cartan theory. The Hamiltonian is given by the sum of two Hamiltonians

$$\begin{aligned}\mathsf{H} = \mathsf{H}^{(\mathrm{EC})}+\mathsf{H}^{(\mathrm{Rest})} &= \boldsymbol{\pi}(\nu)+\boldsymbol{\Pi}(\Gamma)+\mathbf{p}(\mathbf{b})+\mathbf{P}(\mathbf{B})+\mathbf{R}(\mu)+\mathbf{T}(\Theta)+\mathbf{S}(M)\\ &\quad+\boldsymbol{\varphi}(\mathcal{Y})+\mathbf{u}(\mathbf{Y}).\end{aligned} \tag{1.99}$$

Constraints given by $\boldsymbol{\pi}(\nu)$, $\boldsymbol{\Pi}(\Gamma)$, $\mathbf{R}(\mu)$, $\mathbf{T}(\Theta)$, $\boldsymbol{\varphi}(\mathcal{Y})$ and $\mathbf{u}(\mathbf{Y})$ do not determine any Lagrange multipliers, therefore they are the first class constraints. The remaining constraints $\mathbf{p}(\mathbf{b})$, $\mathbf{P}(\mathbf{B})$ and $\mathbf{S}(M)$ are of the second class. Lagrange multipliers $\mathbf{b}^a$ and $\mathbf{B}^{ab}$ are

$$\mathbf{b}^a = \mathcal{D}\lambda^a - \Lambda^{a\bar{a}}\eta_{\bar{a}\bar{b}}\mathbf{E}^{\bar{b}}, \tag{1.100}$$

$$\mathbf{B}^{ab} = \mathcal{D}\Lambda^{ab} + \mathbf{H}^{ab}, \tag{1.101}$$

where $\mathbf{H}^{ab}$ does not depend on Λ^{ab} and it is the solution of (1.81) and (1.98). We will continue with Dirac analysis in the next section where we will introduce Dirac brackets and consider the reduced phase space of our physical system.

1.5 Dirac Brackets

The first level of the Hamilton-Dirac approach to the dynamics has been completed in previous section. In the case when physical system possesses the second class constraints C_A standard Poisson brackets can not be quantized by the usual rule

$$i\hbar\varrho\left(\{A,B\}\right)|\psi\rangle = \left[\varrho(A),\varrho(B)\right]|\psi\rangle,$$

where ϱ is a representation of basic variables, since in the case when A, B are the constraints C_A then there is zero vector $\left(\varrho(C_A)\varrho(C_B)-\varrho(C_B)\varrho(C_A)\right)|\psi\rangle$ on the right-hand side while the operator on the left-hand side $\varrho\left(\{C_A,C_B\}\right)$ is invertible. Hence there exists only one possibility for all physical states solving quantum analogue of classical constraints represented by quantum equation $\varrho(C_A)|\psi\rangle = 0$ given by $|\psi\rangle = 0$. Dirac solved this problem by introducing new brackets and quantization is formulated by the representation of the Dirac instead of the Poisson algebra (See details in [9]). Let C_A be the second class contraints and so $\{C_A,C_B\} = U_{AB}$ is invertible; then Dirac brackets are defined by

$$\{A,B\}^* = \{A,B\} - \{A,C_A\}U^{AB}\{C_B,B\}, \tag{1.102}$$

where $U_{AB}U^{BC} = \delta^C_A$. We divide our job in two parts. In the first part we define certain simple brackets $\{\,,\,\}'$ and then we use these partial brackets in the definition of the final Dirac brackets $\{\,,\,\}^*$.

Let us define weak equivalence before we start our analysis of constraints. We say that two variables A, A' are weakly equivalent, $A\hat{=}A'$, if their difference is proportional to the second class constraints. The second class constraints for our system are ($\mathbf{b}^a$, $\mathbf{B}^{ab}$, M_{ab} are arbitrary)

$$\begin{aligned}
\mathbf{p}(\mathbf{b}) &= \int_\Sigma \mathbf{p}_a\wedge\mathbf{b}^a = \int_\Sigma \tilde{p}^\alpha_a b^a_\alpha \mathrm{d}^3x,\\
\mathbf{P}(\mathbf{B}) &= \int_\Sigma \frac{1}{2}\left(\mathbf{p}_{ab} + \frac{1}{16\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\mathbf{E}^c\wedge\mathbf{E}^d\right)\wedge\mathbf{B}^{ab} = \int_\Sigma \frac{1}{2}\mathbf{P}_{ab}\wedge\mathbf{B}^{ab} = \int_\Sigma \frac{1}{2}\tilde{P}^\alpha_{ab}B^{ab}_\alpha \mathrm{d}^3x,\\
\mathbf{S}(M) &= \int_\Sigma \frac{1}{8\pi\kappa}M_{ab}\mathbf{E}^a\mathbf{E}^b_c\wedge\mathcal{D}\mathbf{E}^c = \int_\Sigma M_{ab}\mathbf{S}^{ab} = \int_\Sigma M_{ab}\tilde{S}^{ab}\mathrm{d}^3x.
\end{aligned}$$

We start the analysis by their decompositions

$$\begin{array}{lcl}
\mathbf{p}^{\perp} = \mathbf{p}_a\lambda^a & \longleftrightarrow & \tilde{p}^{\perp\alpha} = \tilde{p}^\alpha_a\lambda^a,\\
\mathbf{p}^{\parallel}_a = \mathbf{E}^{\bar{a}}_a\mathbf{p}_{\bar{a}} & \longleftrightarrow & \tilde{p}^{\parallel\alpha}_a = \mathbf{E}^{\bar{a}}_a\tilde{p}^\alpha_{\bar{a}},\\
\mathbf{P}^{\perp}_a = \mathbf{P}_{ab}\lambda^b & \longleftrightarrow & \tilde{P}^{\perp\alpha}_a = \tilde{P}^\alpha_{ab}\lambda^b,\\
\mathbf{P}^{\parallel}_a = \mathbf{E}^{\bar{a}}_a\mathbf{P}_{\bar{a}b}\wedge\mathbf{E}^b & \longleftrightarrow & \tilde{P}^{\parallel}_a = \mathbf{E}^{\bar{a}}_a\tilde{P}^\alpha_{\bar{a}b}E^b_\alpha,\\
\boldsymbol{\sigma}^{ab} = \mathbf{P}_{\bar{a}\bar{b}}\lambda_{\bar{c}}\wedge\bar{\boldsymbol{\varepsilon}}^{\bar{a}\bar{b}\bar{c}(a}\mathbf{E}^{b)} & \longleftrightarrow & \tilde{\sigma}^{ab} = \tilde{P}^\alpha_{\bar{a}\bar{b}}\lambda_{\bar{c}}\bar{\boldsymbol{\varepsilon}}^{\bar{a}\bar{b}\bar{c}(a}E^{b)}_\alpha.
\end{array}$$

Now we are going to eliminate constraints $\mathbf{p}^{\perp}$, $\mathbf{p}^{\parallel}_a$ and their "canonical friends" $\mathbf{P}^{\perp}_a$, $\mathbf{P}^{\parallel}_a$ by introducing "partial Dirac bracket" $\{\,,\,\}'$. This bracket plays important role even in the context of full Dirac brackets. In order to introduce them we need the following expressions

$$\begin{aligned}
\{\mathbf{P}(\mathbf{B}), \mathbf{p}(\mathbf{b})\} &= \int_\Sigma \frac{1}{16\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\mathbf{B}^{ab}\wedge\mathbf{b}^c\wedge\mathbf{E}^d \\
&\updownarrow \\
\{\tilde{P}^\alpha_{ab}(\mathbf{x}), \tilde{p}^\beta_c(\mathbf{y})\} &= \frac{1}{8\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\bar{\varepsilon}^{\alpha\beta\gamma}E^d_\gamma\delta_{\mathbf{xy}}.
\end{aligned}$$

Hence nontrivial Poisson brackets are

$$\begin{aligned}
\{\mathbf{P}(\mathbf{B}^\perp), \mathbf{p}(\mathbf{b}^\parallel)\} &\hat{=} \int_\Sigma -\frac{1}{8\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\mathbf{B}^a\wedge\mathbf{b}^b\lambda^c\wedge\mathbf{E}^d \\
&\updownarrow \\
\{\tilde{P}^{\perp\alpha}_{\ a}(\mathbf{x}), \tilde{p}^{\parallel\beta}_{\ b}(\mathbf{y})\} &\hat{=} -\frac{1}{8\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\bar{\varepsilon}^{\alpha\beta\gamma}\lambda^c E^d_\gamma\delta_{\mathbf{xy}} = U^{\alpha\beta}_{ab}\delta_{\mathbf{xy}},
\end{aligned}$$

$$\begin{aligned}
\{\mathbf{P}(\mathbf{B}^\parallel), \mathbf{p}(\mathbf{b}^\perp)\} &\hat{=} \int_\Sigma \frac{1}{8\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\mathcal{B}^a\mathbf{b}\wedge\lambda^b\mathbf{E}^c\wedge\mathbf{E}^d \\
&\updownarrow \\
\{\tilde{P}^{\parallel}_{\ a}(\mathbf{x}), \tilde{p}^{\perp\alpha}(\mathbf{y})\} &\hat{=} \frac{1}{8\pi\kappa}\boldsymbol{\varepsilon}_{abcd}\bar{\varepsilon}^{\alpha\beta\gamma}\lambda^b E^c_\beta E^d_\gamma\delta_{\mathbf{xy}} = U^\alpha_a\delta_{\mathbf{xy}}.
\end{aligned}$$

It is easy to find that the matrix U^a_α (inverse to U^α_a) is

$$U^a_\alpha = -\frac{4\pi\kappa}{e}E^a_\alpha, \qquad \text{where} \qquad U^a_\alpha U^\alpha_b = \mathbf{E}^a_b \qquad \text{and} \qquad U^a_\alpha U^\beta_a = \delta^\beta_\alpha. \tag{1.103}$$

Next step is to look for the inverse matrix to $U^{\alpha\beta}_{ab}$. We can use ansatz $U^{ab}_{\alpha\beta} = AE^a_\alpha E^b_\beta + BE^a_\beta E^b_\alpha$ and the result is given by the expression

$$U^{ab}_{\alpha\beta} = -\frac{4\pi\kappa}{e}(E^a_\alpha E^b_\beta - 2E^a_\beta E^b_\alpha), \qquad \text{where} \qquad U^{ab}_{\alpha\beta}U^{\beta\gamma}_{bc} = \mathbf{E}^a_c\delta^\gamma_\alpha. \tag{1.104}$$

Now we have prepared everything what we need in order to define the partial Dirac bracket as follows

$$\begin{aligned}
\{A,B\}' = \{A,B\} &+ \int_\Sigma d^3x\{A, \tilde{P}^{\perp\alpha}_{\ a}(\mathbf{x})\}U^{ab}_{\alpha\beta}(\mathbf{x})\{\tilde{p}^{\parallel\beta}_{\ b}(\mathbf{x}), B\} \\
&- \int_\Sigma d^3x\{B, \tilde{P}^{\perp\alpha}_{\ a}(\mathbf{x})\}U^{ab}_{\alpha\beta}(\mathbf{x})\{\tilde{p}^{\parallel\beta}_{\ b}(\mathbf{x}), A\} \\
&+ \int_\Sigma d^3x\{A, \tilde{P}^{\parallel}_{\ a}(\mathbf{x})\}U^a_\alpha(\mathbf{x})\{\tilde{p}^{\perp\alpha}(\mathbf{x}), B\} \\
&- \int_\Sigma d^3x\{B, \tilde{P}^{\parallel}_{\ a}(\mathbf{x})\}U^a_\alpha(\mathbf{x})\{\tilde{p}^{\perp\alpha}(\mathbf{x}), A\}.
\end{aligned}$$

The final Dirac brackets are going to be introduced within partial brackets and remaing constraints $\boldsymbol{\sigma}^{ab}$, $\mathbf{S}^{ab}$. First of all we should mention the following property of the partial bracket. Let A be an arbitrary variable on full phase space; then

$$\{\boldsymbol{\sigma}(m), A\}' \hat{=} \{\boldsymbol{\sigma}(m), A\},$$

since $\{\boldsymbol{\sigma}(m), \mathbf{p}(\mathbf{b})\} = \{\mathbf{P}(\mathbf{B}^m), \mathbf{p}(\mathbf{b})\} \hat{=} \int -\frac{1}{4\pi\kappa}\delta^{\bar{a}\bar{b}}_{ab}\mathbf{b}^a\lambda_{\bar{a}}m_{\bar{b}\bar{c}} \wedge \mathbf{E}^{\bar{c}} \wedge \mathbf{E}^b = 0$ and we have also $\{\boldsymbol{\sigma}(m), \mathbf{P}(\mathbf{B})\} = \{\mathbf{P}(\mathbf{B}^m), \mathbf{P}(\mathbf{B})\} = 0$. Hence we have as a consequence

$$\{\boldsymbol{\sigma}(m), \boldsymbol{\sigma}(m')\}' \hat{=} 0 \quad \longleftrightarrow \quad \{\tilde{\sigma}^{ab}(\mathbf{x}), \tilde{\sigma}^{cd}(\mathbf{y})\} \hat{=} 0.$$

The next important classical commutator is

$$\{\boldsymbol{\sigma}(m), \mathbf{S}(M)\}' \hat{=} \{\boldsymbol{\sigma}(m), \mathbf{S}(M)\} = \int_\Sigma -\frac{1}{8\pi\kappa} m_{a\bar{a}} M_{b\bar{b}} \eta_{c\bar{c}} \lambda_d \lambda_{\bar{d}} \bar{\boldsymbol{\varepsilon}}^{abcd} \bar{\boldsymbol{\varepsilon}}^{\bar{a}\bar{b}\bar{c}\bar{d}} \boldsymbol{\omega}, \tag{1.105}$$

where $\boldsymbol{\omega} = \frac{1}{3!}\boldsymbol{\varepsilon}_{abcd}\lambda^a \mathbf{E}^b \wedge \mathbf{E}^c \wedge \mathbf{E}^d = e\, \mathrm{d}^3x$. Now it is time to pay debt from the previous section where we have stated that $\tilde{G}^{a\bar{a}b\bar{b}}$ is invertible. We are going to do even more. We are going to calculate inverse of $U^{a\bar{a}b\bar{b}} = \frac{e}{8\pi\kappa}\tilde{G}^{a\bar{a}b\bar{b}}$. We can write

$$\begin{aligned} \{\boldsymbol{\sigma}(m), \mathbf{S}(M)\} &= \int_\Sigma \frac{1}{8\pi\kappa} m_{a\bar{a}} M_{b\bar{b}} \tilde{G}^{a\bar{a}b\bar{b}} \boldsymbol{\omega} \\ &\updownarrow \\ \{\tilde{\sigma}^{a\bar{a}}(\mathbf{x}), \tilde{S}^{b\bar{b}}(\mathbf{y})\} &= \frac{e}{8\pi\kappa}\tilde{G}^{a\bar{a}b\bar{b}}\delta_{\mathbf{xy}} = U^{a\bar{a}b\bar{b}}\delta_{\mathbf{xy}} \end{aligned}$$

and

$$\tilde{G}^{a\bar{a}b\bar{b}} = -\frac{1}{2}\eta_{c\bar{c}}\lambda_d\lambda_{\bar{d}}\left(\bar{\boldsymbol{\varepsilon}}^{abcd}\bar{\boldsymbol{\varepsilon}}^{\bar{a}\bar{b}\bar{c}\bar{d}} + \bar{\boldsymbol{\varepsilon}}^{\bar{a}bcd}\bar{\boldsymbol{\varepsilon}}^{a\bar{b}\bar{c}\bar{d}}\right). \tag{1.106}$$

Let us transform $U^{a\bar{a}b\bar{b}}$ into more suitable form. In order to do so we need to use the spatial metric tensor which is due to our choice of signature strictly negative

$$\mathbf{q} = \eta_{ab}\mathbf{E}^a \otimes \mathbf{E}^b = \mathbf{q}_{ab}\mathbf{E}^a \otimes \mathbf{E}^b = q_{\alpha\beta}\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta, \tag{1.107}$$

where $\mathbf{q}_{ab} = \eta_{\bar{a}\bar{b}}\mathbf{E}^{\bar{a}}_a\mathbf{E}^{\bar{b}}_b$, its inverse matrix is $q^{\alpha\beta}q_{\beta\gamma} = \delta^\alpha_\beta$ or $\mathbf{q}^{ab}\mathbf{q}_{bc} = \mathbf{E}^a_c$ and determinant

$$q\varepsilon_{\alpha\beta\gamma} = q_{\alpha\bar{\alpha}}q_{\beta\bar{\beta}}q_{\gamma\bar{\gamma}}\bar{\varepsilon}^{\bar{\alpha}\bar{\beta}\bar{\gamma}}.$$

It should be noted that $\mathbf{q}^{ab} \neq \mathbf{E}^a_{\bar{a}}\mathbf{E}^b_{\bar{b}}\eta^{\bar{a}\bar{b}}$. Now we can write

$$U^{a\bar{a}b\bar{b}} = \frac{\overset{*}{\lambda}{}^2}{16\pi\kappa}(2\mathbf{q}^{a\bar{a}}\mathbf{q}^{b\bar{b}} - \mathbf{q}^{ab}\mathbf{q}^{\bar{a}\bar{b}} - \mathbf{q}^{a\bar{b}}\mathbf{q}^{\bar{a}b}), \tag{1.108}$$

where we have used formula $q = -e^2\overset{*}{\lambda}{}^2$ and $\overset{*}{\lambda}{}^2 = \eta^{ab}\lambda_a\lambda_b$. Now we are looking for inverse matrix to $U^{a\bar{a}b\bar{b}}$ in the form $U_{a\bar{a}b\bar{b}} = A\mathbf{q}_{a\bar{a}}\mathbf{q}_{b\bar{b}} + B(\mathbf{q}_{ab}\mathbf{q}_{\bar{a}\bar{b}} + \mathbf{q}_{a\bar{b}}\mathbf{q}_{\bar{a}b})$ and the result is given by the expression

$$U_{a\bar{a}b\bar{b}} = \frac{4\pi\kappa}{\overset{*}{\lambda}{}^2 e}(\mathbf{q}_{a\bar{a}}\mathbf{q}_{b\bar{b}} - \mathbf{q}_{ab}\mathbf{q}_{\bar{a}\bar{b}} - \mathbf{q}_{a\bar{b}}\mathbf{q}_{\bar{a}b}), \quad \text{where} \quad U^{a\bar{a}b\bar{b}}U_{b\bar{b}c\bar{c}} = \mathbf{E}^{(a}_c\mathbf{E}^{\bar{a})}_{\bar{c}}. \tag{1.109}$$

Finally we can define the full Dirac bracket as

$$\begin{aligned} \{A, B\}^* = \; & \{A, B\}' + \int \mathrm{d}^3x \{A, \tilde{\sigma}^{a\bar{a}}(\mathbf{x})\}' U_{a\bar{a}b\bar{b}}(\mathbf{x}) \{\tilde{S}^{b\bar{b}}(\mathbf{x}), B\}' - \\ & - \int \mathrm{d}^3x \{B, \tilde{\sigma}^{a\bar{a}}(\mathbf{x})\}' U_{a\bar{a}b\bar{b}}(\mathbf{x}) \{\tilde{S}^{b\bar{b}}(\mathbf{x}), A\}' - \\ & - \int \mathrm{d}^3x\mathrm{d}^3y \{A, \tilde{\sigma}^{a\bar{a}}(\mathbf{x})\}' U_{a\bar{a}b\bar{b}}(\mathbf{x}) \{\tilde{S}^{b\bar{b}}(\mathbf{x}), \tilde{S}^{c\bar{c}}(\mathbf{y})\}' U_{c\bar{c}d\bar{d}}(\mathbf{y}) \{\tilde{\sigma}^{d\bar{d}}(\mathbf{y}), B\}'. \end{aligned}$$

In order to finish the phase space reduction we need to describe a reduced manifold. Let us start with the full phase space $\tilde{\Gamma}$ described by canonical variables λ^a, $\boldsymbol{\pi}_a$, $\ldots$, U_{ab} (check table 1.1). As we have seen in section 1.4 the first reduction is given by $C^{ab} = \mathbf{C}^{ab} = \boldsymbol{\Phi}_{ab} = \mathrm{U}_{ab} = 0$ while conditions $\boldsymbol{\varphi} = \mathbf{u} = 0$ are the first class contraints. These contraints mean that $\mathcal{B}$, $\mathbf{B}$ are arbitrary and physics does not depend on them. Hence we can write $\tilde{\Gamma}\big|_{\mathrm{red}} = \hat{\Gamma} \times \Lambda\Sigma$, where $\Lambda\Sigma$ is Cartan algebra of all forms on Σ of varibles $\mathcal{B}$, $\ldots$, $\mathbf{u}$ and $\hat{\Gamma}$ is described by variables λ^a, $\ldots$, $\mathbf{p}_{ab}$. Whole dynamics takes place in $\hat{\Gamma}$. We totally ignore topological properties at this moment, but if one wishes then one can imagine that all variables are, for example, differentiable functions with their standard topology. Let us consider a set[7]

$$\mathfrak{Conf} = \{\times_{\mathbf{x}\in\Sigma}(\lambda^a(\mathbf{x}), \mathbf{E}^a(\mathbf{x}));\ \forall \mathbf{x} \in \Sigma :\ e > 0,\ \eta_{ab}\lambda^a\lambda^b > 0,\ \lambda^0 > 0,\ \mathbf{q} < 0\}.$$

Hence due to condition $e > 0$ we have $\mathfrak{Conf} \subset (\mathbf{GL}^+(\mathscr{M}))^\Sigma = \times_{\mathbf{x}\in\Sigma}\mathbf{GL}^+(\mathscr{M}_\mathbf{x})$. However $\mathfrak{Conf}$ is not a group. Nevertheless for every sufficiently small change $(\Delta\lambda^a, \Delta\mathbf{E}^a)$ the new element is again from $\mathfrak{Conf}$, i.e. $(\lambda^a + \Delta\lambda^a, \mathbf{E}^a + \Delta\mathbf{E}^a) \in \mathfrak{Conf}$; in other words $\mathfrak{Conf}$ is a manifold. Hence we can construct canonically its cotangent bundle $\mathbb{T}^*\mathfrak{Conf} = \mathbb{T}_1\mathfrak{Conf}$ with symplectic structure $\omega_{\mathfrak{Conf}}$ on it. $\mathbb{T}^*\mathfrak{Conf}$ is described by canonical coordinates (λ^a, $\mathbf{E}^a$, $\boldsymbol{\pi}_a$, $\mathbf{p}_a$). Another structure of $\hat{\Gamma}$ is given by space

$$\mathfrak{G} = (\Lambda_0\mathbb{A}^2\Sigma \times \Lambda_3\mathbb{A}_2\Sigma) \times (\Lambda_1\mathbb{A}^2\Sigma \times \Lambda_2\mathbb{A}_2\Sigma) \tag{1.110}$$

described by variables (Λ^{ab}, $\boldsymbol{\Pi}_{ab}$; $\mathbf{A}^{ab}$, $\mathbf{p}_{ab}$). Hence $\hat{\Gamma} = \mathbb{T}^*\mathfrak{Conf} \times \mathfrak{G}$.

Since $\mathbf{A}^{ab}$ is antisymmetric matrix 1-form we can decompose it as

$$\mathbf{A}^{ab} = 2\mathbf{A}^{[a}\lambda^{b]} + 2\mathcal{A}^{\bar{a}}\mathbf{E}^{[a}_{\bar{a}}\mathbf{E}^{b]} + \bar{\boldsymbol{\varepsilon}}^{ab\bar{a}\bar{b}}\mathbf{E}^{\bar{c}}\lambda_{\bar{a}}\alpha_{\bar{b}\bar{c}}. \tag{1.111}$$

Relevant information about $\mathbf{A}^a$ and $\mathcal{A}^a$ is encoded in the variable

$$\mathbf{F}_a = \frac{1}{2}\boldsymbol{\varepsilon}_{abcd}\mathbf{A}^{bc} \wedge \mathbf{E}^d \quad \longleftrightarrow \quad \mathbf{F}(\mathbf{K}) = \int_\Sigma \frac{1}{2}\boldsymbol{\varepsilon}_{abcd}\mathbf{K}^a \wedge \mathbf{A}^{bc} \wedge \mathbf{E}^d, \tag{1.112}$$

while α_{ab} does not enter $\mathbf{F}_a$. Since $\{\boldsymbol{\sigma}(m), \mathbf{F}(\mathbf{K})\}' \hat{=} 0$ and $\{\boldsymbol{\sigma}(m), \mathbf{E}(\mathbf{Q})\}' \hat{=} 0$, where $\mathbf{E}(\mathbf{Q}) = \int_\Sigma \mathbf{Q}_a \wedge \mathbf{E}^a$ we have that

$$\{\mathbf{E}(\mathbf{Q}), \mathbf{F}(\mathbf{K})\}^* \hat{=} \{\mathbf{E}(\mathbf{Q}), \mathbf{F}(\mathbf{K})\}' = -8\pi\kappa \int_\Sigma \mathbf{Q}_a \wedge \mathbf{K}^a. \tag{1.113}$$

Analogously, we obtain the rest of Dirac brackets for our variables on $\hat{\Gamma}$. The nontrivial results are

$$\{\lambda^a, \boldsymbol{\pi}(\mu)\}^* \ \hat{=}\ \mu^a, \tag{1.114}$$
$$\{\Lambda^{ab}, \boldsymbol{\Pi}(\Gamma)\}^* \ \hat{=}\ \Gamma^{ab}. \tag{1.115}$$

The reduction of $\hat{\Gamma}$ is almost finished. We can express α_{ab} from the condition $\mathbf{S}^{ab} = 0$ as function(al) of λ^a, $\mathbf{E}^a$ and $\mathbf{F}_a$. The remaining second class contraints are trivially

[7] If one wants to work with C^1 forms then one must replace Cartesian product $\times$ with "C^1-Cartesian product" $\times^{C^1}$, i.e. one must change in definition of Cartesian product "all functions" by "all C^1-functions". Anyway all familiar theorems about Cartesian product are not valid anymore, hence whole theory about C^1-Cartesian product has to be built from beginning. Similarly for any C^ω, where $\omega = 1, \ldots, \infty$, analytic.

soluble. Since variables $\mathcal{B}, \ldots, \mathbf{u}$ do not describe any dynamics we can cast them away by additional fixation $\mathcal{B} = 0$ and $\mathbf{B} = 0$. Similar, we can proceed with Λ^{ab} but for different reason. Hence we have the final reduced phase space

$$\Gamma = \mathbb{T}^*\mathfrak{Conf} \tag{1.116}$$

described by variables $(\lambda^a, \mathbf{E}^a, \boldsymbol{\pi}_a, \mathbf{F}_a)$ with symplectic structure defined by (1.113) and (1.114). The reason for excluding Λ^{ab} is very simple. Since $\mathbf{\Pi}_{ab} = 0$, we have that Λ^{ab} plays the role of a Lagrange multiplier which in our notation is given by Γ^{ab}.

2. Kinematical Hilbert Space for Einstein-Cartan Theory

2.1 Preliminaries

We have successfully constructed the phase space $\mathbb{T}^*\mathfrak{Conf}$ of Einstein-Cartan theory in the previous part of this thesis. Now it is time to build a quantum algebra of the basic variables. Before we start let us focus our attention to the following simple excersice well known from the quantum mechanics of the particle moving on the half line. Canonical variables of this system are x and p, where x is a position of the particle on the half line $x > 0$ and p is its canonical momentum. We can naively represent them on $\mathscr{H} = L^2(\mathbb{R}^+, \mathrm{d}x)$ as $\varrho(x) = x$, $\varrho(p) = -\mathrm{i}\partial_x$. The operators $\varrho(x)$ and $\varrho(p)$ are symmetric but $\varrho(p)$ can not be extended into the selfadjoint operator on $\mathscr{H}$. In order to see this let us compute its deficiency indices n_ε, where $\varepsilon = \pm 1$ (See details about extensions of the symmetric operators in [13]). Equations

$$-\mathrm{i}\partial_x\psi^{(\varepsilon)} - \mathrm{i}\varepsilon\psi^{(\varepsilon)} = 0$$

have solutions

$$\psi^{(\varepsilon)} = A^{(\varepsilon)}\mathrm{e}^{-\varepsilon x}.$$

Solution $\psi^{(+1)}$ belongs to the space $L^2(\mathbb{R}^+, \mathrm{d}x)$ while $\psi^{(-1)}$ is not square integrable function on $\mathbb{R}^+$. Since $n_+ = 1$ and $n_- = 0$ we have $n_+ \neq n_-$. Thus we can not construct the selfadjoint extenstion of the operator $-i\partial_x$. Hence if one wants to describe the quantum particle on the half line then one has to choose different set of basic variables. The first observation is that $\mathbb{R}^+$ is a group $\mathbf{GL}^+(\mathbb{R})$. Invariant measure on $\mathbf{GL}^+(\mathbb{R})$ is $\omega_{\mathbf{GL}^+(\mathbb{R})} = \frac{\mathrm{d}x}{x}$ hence the good candidate for the "momentum" operator is given by $\varrho(xp) = -\mathrm{i}x\partial_x$. Indeed, the operator $\varrho(xp)$ is symmetric on $L^2(\mathbb{R}^+, \frac{\mathrm{d}x}{x})$.

$$\langle\psi_2|\varrho(xp)\psi_1\rangle = \int_{\mathbb{R}^+} \frac{\mathrm{d}x}{x}\overline{(\psi_2)}(-\mathrm{i}x\partial_x\psi_1) = \int_{\mathbb{R}^+} \frac{\mathrm{d}x}{x}\overline{(-\mathrm{i}x\partial_x\psi_2)}\psi_1 = \langle\varrho(xp)\psi_2|\psi_1\rangle$$

and its deficiency indices are determined by the following equations

$$-\mathrm{i}x\partial_x\psi^{(\varepsilon)} - \mathrm{i}\varepsilon\psi^{(\varepsilon)} = 0$$

with solutions

$$\psi^{(\varepsilon)} = A^{(\varepsilon)}x^{-\varepsilon},$$

which do not belong to $L^2(\mathbb{R}^+, \frac{\mathrm{d}x}{x})$. Hence $n_+ = n_- = 0$ and the operator $\varrho(xp)$ is essentially selfadjoint. The algebra of the basic variables is a space spanned on operators $\varrho(x), \varrho(xp)$ with nontrivial commutator

$$[\varrho(x), \varrho(xp)] = \mathrm{i}\varrho(\{x, xp\}) = \mathrm{i}\varrho(x).$$

As we have seen on this simple exercise the choice of the basic variables plays the crucial role in the context of quantization. In the next section we will try to understand a point version of Einstein-Cartan phase space. We will find there a reprepresentation of the basic variables, which seperate points in the phase space. The third section of this part is dedicated to a brief summary of von Neumann construction of tensor product of infinite sequence of Hilbert spaces. In the last section of this part we will find a representation of basic variables of Einstein-Cartan theory. Anyway this representation is highly reducible and cannot be used in quantum formulation of Einstein-Cartan theory for reasons explained in conclusion.

2.2 Point Algebra of Basic Variables

We will focus in this section on the introduction of a Hilbert space $\mathscr{H}_{\mathbf{x}}$ associated with an arbitrary point $\mathbf{x}$ in the spatial section Σ. We will define a point representation of the basic variables related to the canonical coordinates on the phase space $\mathbb{T}^*\mathfrak{Conf}$. Let us mention that all canonical variables $\lambda^a(\mathbf{x})$, $\mathbf{E}^a(\mathbf{x})$, $\boldsymbol{\pi}_a(\mathbf{x})$, $\mathbf{G}_a(\mathbf{x}) = -\frac{1}{8\pi\kappa}\mathbf{F}_a(\mathbf{x})$ are local functions of the point $\mathbf{x}$. No derivatives, no complicated integrals or any kind of dislocation are presented, hence we can explore them in the single point $\mathbf{x}$. Before we start, we will introduce spacetime notation[1]

$$
\begin{aligned}
e^a_\mu &= (e^a_t = \lambda^a; e^a_\alpha = E^a_\alpha),\\
p^\mu_a &= (p^t_a = \tilde{\pi}_a; p^\alpha_a = \tilde{G}^\alpha_a).
\end{aligned}
$$

Since we are working with the point variables, their canonical relations are given by

$$
\{e^a_\mu, p^\nu_b\} = \delta^a_b \delta^\nu_\mu
$$

and the phase space is defined in accordance to the Einstein-Cartan phase space as $\mathbb{T}^*\mathfrak{conf}$, where

$$
\mathfrak{conf} = \{(e^a_\mu); e = \det(e^a_\mu) > 0,\ \eta_{ab} e^a_t e^b_t > 0,\ e^0_t > 0,\ \eta_{ab} e^a_\alpha e^b_\beta < 0\}.
$$

Thanks to the positivity of the determinant e we can see that $\mathfrak{conf} \subset \mathbf{GL}^+(\mathbb{R}^4) \equiv \mathbf{GL}^+$, anyway the subset $\mathfrak{conf}$ is not a group. Now we will try to construct a representation of the basic variables. Let us define a Hilbert space $\mathscr{H}_{\mathbf{x}} \equiv \mathscr{H}$ as a space of square integrable functions over $\mathfrak{conf}$

$$
\mathscr{H} = \mathsf{L}^2\left(\mathfrak{conf}, \frac{\mathrm{d}e}{e^4}\right), \tag{2.1}
$$

where $\frac{\mathrm{d}e}{e^4}$ is left/right-invariant[2] Haar's measure on the $\mathbf{GL}^+$, which is unique up to the multiplicative constant. $\mathrm{d}e = \mathrm{d}e^0_t \mathrm{d}e^0_x \dots \mathrm{d}e^3_y \mathrm{d}e^3_z$ is Lebesgue measure on the coordinates $(e^a_\mu) \in \mathbb{R}^{16}$ of the space $\mathfrak{conf}$. The representation ϱ of e^a_μ is given by trivial multiplication

$$
\varrho(e^a_\mu)\psi(e^a_\mu) = e^a_\mu \psi(e^a_\mu).
$$

It is well known fact that such operators can be extentended into the selfadjoint operators. The problems occure with variables p^a_μ, since the action of $\varrho(p^a_\mu) = -\mathrm{i}\partial_{e^a_\mu}$ given by the "unitary" transformation

$$
\mathrm{e}^{\mathrm{i}\vartheta^a_\mu \varrho(p^\mu_a)}\psi(e^a_\mu) = \psi(e^a_\mu + \vartheta^a_\mu)
$$

[1] Explicit writing of the point $\mathbf{x}$ is omitted till the end of this section.

[2] In the case of the general noncompact group it may happen that left and right invariant measures are not equal.

maps vectors from $\mathscr{H}$ out of this space, therefore the operators $\varrho(p^\mu_a)$ are not selfadjoint (they are neither symmetric). What we can do with that? We know, thanks to the Stone's theorem, that every one-parametric strongly continuous unitary group is related to the selfadjoint operator and vice versa. This implies that if we wish to find the selfadjoint operators for the momenta or their functions, we need to find certain groups acting on the space $\mathfrak{conf}$. Indeed, a following statement is valid.

Statement 1. *Let* $\mathfrak{X} \subset \mathbb{R}^n$ *and* $\mathrm{d}x$ *be the Lebesgue measure on* $\mathbb{R}^n$*. If* $U(t)$ *is one-parametric unitary group acting on the Hilbert space* $\mathscr{H} = \mathsf{L}^2(\mathfrak{X}, g\mathrm{d}x)$*, where* $g \geq 0$ *is locally integrable function on* $\mathfrak{X}$*, and if* Φ_t *is a continuous flow on* $\mathfrak{X}$ *associated with* $U(t)$*, then* $U(t)$ *is strongly continuous.*

A proof of the statement is based on the fact that function $I(t) : \mathbb{R} \to \mathbb{R}$, defined as

$$I(t) = \int\limits_{\Phi^*_t(K)} f\mathrm{d}x,$$

is continuous, where $\Phi_t : \mathfrak{X} \times \mathbb{R} \to \mathfrak{X}$ is continous mapping, K is compact subset of $\mathfrak{X}$ and f is locally integrable function. It is sufficient to prove that $\|(1 - U(t))\psi\|$ is continuous in $t = 0$ for all $\psi \in \mathfrak{D}$, where $\mathfrak{D}$ is some dense subset in $\mathsf{L}^2(\mathfrak{X}, g\mathrm{d}x)$, since for any convergent sequence $\psi_n \in \mathfrak{D} \to \psi_0 \in \mathsf{L}^2(\mathfrak{X}, g\mathrm{d}x)$ we have

$$\|(1 - U(t))\psi_0\| \leq \|(1 - U(t))(\psi_0 - \psi_n)\| + \|(1 - U(t))\psi_n\| \leq 2\|\psi_0 - \psi_n\| + \|(1 - U(t))\psi_n\|.$$

The set of simple functions is dense in $\mathsf{L}^2(\mathfrak{X}, g\mathrm{d}x)$, hence for the general simple function

$$f = \sum_{i=1}^{m} f_i \chi_{K_i},$$

where $m \in \mathbb{N}$, f_i are complex constants, $K_i \subset \mathfrak{X}$ are compacts and $K_i^o = K_i \setminus \partial K_i$ are mutually disjoint, we have

$$\begin{aligned} \|(1 - U(t))f\|^2 &= \sum_{i,j=1}^{m} \int g\,\mathrm{d}x\, \bar{f}_i f_j \left(\chi_{K_i}\chi_{K_j} + \chi_{\Phi^*_t(K_i)}\chi_{\Phi^*_t(K_j)} - \chi_{\Phi^*_t(K_i)}\chi_{K_j} - \chi_{K_i}\chi_{\Phi^*_t(K_j)}\right) \\ &= \sum_{i=1}^{m} \int g\,\mathrm{d}x\, |f_i|^2 \left(\chi_{K_i} + \chi_{\Phi^*_t(K_i)}\right) - \sum_{i,j=1}^{n} \int g\,\mathrm{d}x\, \bar{f}_i f_j \left(\chi_{\Phi^*_t(K_i)\cap K_j} + \chi_{K_i\cap\Phi^*_t(K_j)}\right), \end{aligned}$$

what is continuous in t. Hence $U(t)$ is strongly continuous.

Now we can try to find group(s) acting on the space $\mathfrak{conf}$. The positive linear group $\mathbf{GL}^+$ is not a good candidate, since, as before in the case of p^μ_a, there exists transformation g from $\mathbf{GL}^+$ which does not preserve the space $\mathfrak{conf}$, e.g. rotation in a plane spaned on e^0_t, e^1_t maps $e^0_t \to -e^0_t$ and $e^1_t \to -e^1_t$. The problem is caused by the fact that group $\mathbf{GL}^+$ ignores a metric η_{ab}. Indeed, if we consider a Lorentz group acting on e^a_μ via

$$e^a_\mu \to \left(\mathrm{e}^{\Lambda\eta}\right)^a_b e^b_\mu, \tag{2.2}$$

where $(\Lambda\eta)^a_b = \Lambda^{ac}\eta_{cb}$ and $\Lambda^{ab} = -\Lambda^{ba}$, then we have that $\mathrm{e}^{\Lambda\eta}(\mathfrak{conf}) \subset \mathfrak{conf}$ and even more the transformation (2.2) is continuous. We can define an operator

$$\mathsf{U}^{\mathrm{L}}(\Lambda^{ab})\psi(e^a_\mu) = \psi\left(\left(\mathrm{e}^{\Lambda\eta}\right)^a_b e^b_\mu\right), \tag{2.3}$$

which is, thanks to the invariance of the measure $\frac{de}{e^4}$, unitary. Let Λ^{ab} be arbitrary, but fixed, then

$$U_\Lambda(t) = \mathsf{U}^{\mathrm{L}}(t\Lambda^{ab})$$

is the one-parametric strongly continuous unitary group and, due to the Stone's theorem, we have that its generator is a selfadjoint operator. We have fixed arbitrary Λ^{ab}, hence we have for every Λ^{ab} its own generator. Λ^{ab} has six degrees of freedom, thus there are six independent generators L_{ab} and we can write

$$\mathsf{U}^{\mathrm{L}}(\Lambda^{ab}) = \mathrm{e}^{\mathrm{i}\frac{1}{2}\Lambda^{ab}\mathsf{L}ab}.$$

Let $\psi(e^a_\mu) \in C^\infty_{\mathrm{C}}(\mathfrak{conf}) \subset \mathscr{H}$, where $C^\infty_{\mathrm{C}}(\mathfrak{conf})$ is the set of all ∞-times differentiable functions with compact support on $\mathfrak{conf}$, which is dense in $\mathscr{H}$, then we can use Taylor expansion

$$\begin{aligned}\mathsf{U}^{\mathrm{L}}(t\Lambda^{ab})\psi(e^a_\mu) &= \psi\left(\left(\mathrm{e}^{t\Lambda\eta}\right)^a_b e^b_\mu\right) = \psi\left(e^a_\mu + \left(\left(\mathrm{e}^{t\Lambda\eta}\right)^a_b e^b_\mu - e^a_\mu\right)\right) = \\ &= \psi(e^a_\mu) + t\Lambda^{ac}\eta_{cb}e^b_\mu\partial_{e^a_\mu}\psi(e^a_\mu) + t^2 o(t, e^a_\mu) \end{aligned} \tag{2.4}$$

where $o(t, e^a_\mu)$ is some C^∞-function on $\mathbb{R} \times \mathfrak{conf}$ with compact support on $\mathfrak{conf}$ for every t given by Taylor's expansion remainder. The remainder $o(t, e^a_\mu)$ can be restricted for $|t| < \delta$ as $|o(t, e^a_\mu)| \leq M\chi_{\bar{K}_\delta}$, where

$$K_\delta = \cup_{|t|<\delta} K_t,$$

K_t is a support of $o(t, e^a_\mu)$ in $\mathfrak{conf}$ for given t. Since the closure of $\cup_{|t|<\delta}\{t\}\times K_t$ is compact in $\mathbb{R} \times \mathfrak{conf}$ we have that closure $\bar{K}_\delta$ is also compact in $\mathfrak{conf}$. Now we can compute the generator $\mathsf{L}(\Lambda^{ab}) = \frac{1}{2}\Lambda^{ab}\mathsf{L}_{ab}$ as a limit $t \to 0$

$$\mathrm{i}\mathsf{L}(\Lambda^{ab})\psi = \lim_{t\to 0} \frac{\mathsf{U}^{\mathrm{L}}(t\Lambda^{ab}) - 1}{t}\psi.$$

If we use expansion (2.4), then we have

$$\begin{aligned}\frac{1}{t}\left\|\left(\mathsf{U}^{\mathrm{L}}(t\Lambda^{ab}) - 1\right)\psi - \mathrm{i}t\mathsf{L}(\Lambda^{ab})\psi\right\|^2 &= \frac{1}{t}\int\limits_{\mathfrak{conf}} \left|t\Lambda^{ac}\eta_{cb}e^b_\mu\partial_{e^a_\mu}\psi + t^2 o(t, e^a_\mu) - \mathrm{i}t\mathsf{L}(\Lambda^{ab})\psi\right|^2 \frac{\mathrm{d}e}{e^4} \leq \\ &\leq tM^2 \int\limits_{K_\delta} \frac{\mathrm{d}e}{e^4},\end{aligned}$$

iff

$$\mathsf{L}(\Lambda^{ab}) = \frac{1}{2}\Lambda^{ab}\mathsf{L}_{ab} = -\mathrm{i}\Lambda^{ab}\eta_{bc}e^c_\mu\partial_{e^a_\mu} = -\mathrm{i}\Lambda^{ab}\eta_{bc}\lambda^c\partial_{\lambda^a} - \mathrm{i}\Lambda^{ab}\eta_{bc}E^c_\alpha\partial_{E^a_\alpha}. \tag{2.5}$$

Thus we have as a final conclusion that the operator $\mathsf{L}(\Lambda^{ab})$, given by previous expression, with domain $\mathfrak{D}(\mathsf{L}(\Lambda^{ab})) = C^\infty_{\mathrm{C}}(\mathfrak{conf})$ is essentially selfadjoint for every Λ^{ab}.

This is not everything what the Lorentz group can show us. There exists another transformation of Lorentz group acting on coordinate indices μ. The metric $\mathbf{g}$ can be written as

$$\mathbf{g} = \eta_{ab}e^a_\mu e^b_\nu \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu, \ \ (\mathrm{d}x^\mu = \mathrm{d}t, \mathrm{d}x^\alpha)$$

and its inverse matrix $g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu_\nu$ as $g^{\mu\lambda} = \eta^{ab}e^\mu e^\nu$. The transformation prescribed as

$$e^a_\mu \to \left(e^{\Gamma g^{-1}}\right)^\nu_\mu e^a_\nu; \quad \left(\Gamma g^{-1}\right)^\nu_\mu = \Gamma_{\mu\lambda}g^{\lambda\nu} \quad \text{and} \quad \Gamma_{\mu\lambda} = -\Gamma_{\lambda\mu}, \tag{2.6}$$

preserves inverse metric $g^{\mu\nu}$, while $e^{g^{-1}\Gamma}$ preserves $g_{\mu\nu}$. Therefore $e^{\Gamma g^{-1}}\mathfrak{conf} \subset \mathfrak{conf}$. Thanks to the similar arguments as in the previous case, we have that the operators

$$\mathsf{U}^{\mathsf{Q}}(\Gamma_{\mu\nu})\psi(e^a_\mu) = e^{i\frac{1}{2}\Gamma_{\mu\nu}\mathsf{Q}^{\mu\nu}}\psi(e^a_\mu) = \psi\left(\left(e^{\Gamma g^{-1}}\right)^\nu_\mu e^a_\nu\right) \tag{2.7}$$

are unitary with selfadjoint generators

$$\mathsf{Q}(\Gamma_{\mu\nu}) = \frac{1}{2}\Gamma_{\mu\nu}\mathsf{Q}^{\mu\nu} = -i\Gamma_{\mu\nu}\eta^{ab}e^\nu_b\partial_{e^a_\mu}. \tag{2.8}$$

We are not finished yet with the Lorenz Group. Let us use again 3+1 decomposition $e^a_\mu = (\lambda^a, E^a_\alpha)$. As we already know λ^a are components of vector ∂_t in the frame $\mathbf{e}_a$. Since the time vector can be choosen arbitrary there is no reason to have tied variables λ^a, E^a_α together. Hence we can work with λ^a, E^a_α independently. Let us consider Lorentz group acting on λ^a, then the generators of this action are given by

$$\mathsf{L}^{(\lambda)}_{ab} = -i\eta_{bc}\lambda^c\partial_{\lambda^a} + i\eta_{ac}\lambda^c\partial_{\lambda^b}.$$

We obtain similar result for the Lorentz action on E^a_α

$$\mathsf{L}^{(\mathbf{E})}_{ab} = -i\eta_{bc}E^c_\alpha\partial_{E^a_\alpha} + i\eta_{ac}E^c_\alpha\partial_{E^b_\alpha}.$$

Let us compare this results with (2.5), we can see that

$$\mathsf{L}_{ab} = \mathsf{L}^{(\lambda)}_{ab} + \mathsf{L}^{(\mathbf{E})}_{ab}$$

as one expected. Generators $\mathsf{L}^{(\lambda)}_{ab}, \mathsf{L}^{(\mathbf{E})}_{ab}$ play an important role, since, as we will see in a while, their classical analogues can be used as coordinates on the phase space.
Lorentz group does not change lengths of the vectors, while ∂_t can be arbitrary long. We need to cover this featur of ∂_t. Let us define a following transformation

$$\lambda^a \to e^N\lambda^a, \qquad E^a_\alpha \to E^a_\alpha.$$

Let $\mathsf{U}^\pi(N)$ be its unitary operator defined via

$$\mathsf{U}^\pi(N)\psi(\lambda^a, E^a_\alpha) = \psi(e^N\lambda^a, E^a_\alpha)$$

and its selfadjoint generator is

$$\pi = -i\lambda^a\partial_{\lambda^a}. \tag{2.9}$$

A final transformation acting on the space $\mathfrak{conf}$ is given by group $\mathbf{GL}^+(\mathbb{R}^3) \equiv \mathbf{GL}^{+3}$ acting on the spatial indices α. Let θ^α_β be an arbitrary real matrix, then the transformation given by

$$\lambda^a \to \lambda^a, \qquad E^a_\alpha \to \left(e^\theta\right)^\beta_\alpha E^a_\beta \tag{2.10}$$

represents the change of spatial frame $\partial_\alpha \to (\mathrm{e}^\theta)^\beta_\alpha \partial_\beta$. Since the transformation does not change a signature of $q_{\alpha\beta} = \eta_{ab} E^a_\alpha E^b_\beta$, we have that $\mathrm{e}^\theta \mathfrak{conf} \subset \mathfrak{conf}$ and operators

$$\mathsf{U}^\Delta(\theta)\psi(\lambda^a, E^a_\alpha) = \psi\left(\lambda^a, \left(\mathrm{e}^\theta\right)^\beta_\alpha E^a_\beta\right)$$

are unitary and their selfadjoint generators are

$$\Delta^\alpha_\beta = -\mathrm{i} E^a_\beta \partial_{E^a_\alpha}.$$

Let us summarize our situation. We have constructed family of unitary transformation with action in the space $\mathfrak{conf}$. Now it is a time to find classical variables associated with their generator. We can suppose that formal relation $\varrho(p^\mu_a) = -\mathrm{i}\partial_{e^a_\mu}$ will lead us to the final representation. Let us focus on the last four families of the generators. We have

$$\begin{aligned} \mathbf{L}^{(\lambda)}(\Lambda) &= \Lambda^{ab}\eta_{bc}\lambda^c \boldsymbol{\pi}_a, \\ \boldsymbol{\pi}(N) &= N\lambda^a \boldsymbol{\pi}_a, \\ \mathbf{L}^{(\mathbf{E})}(\Lambda) &= \Lambda^{ab}\eta_{bc}\mathbf{E}^c \wedge \mathbf{G}_a, \\ \boldsymbol{\Delta}(\theta) &= \theta(\mathbf{E}^a) \wedge \mathbf{G}_a. \end{aligned}$$

Quantum commutators and their classical analogues are

$$\begin{aligned} [\lambda(\mathbf{k}), \pi(N)] = \mathrm{i}\lambda(N\mathbf{k}) &\leftrightarrow \{\lambda(\mathbf{k}), \boldsymbol{\pi}(N)\} = \lambda(N\mathbf{k}) \\ \left[\lambda(\mathbf{k}), \mathsf{L}^{(\lambda)}(\Lambda)\right] = \mathrm{i}\lambda(\mathbf{k}\Lambda\eta) &\leftrightarrow \left\{\lambda(\mathbf{k}), \mathbf{L}^{(\lambda)}(\Lambda)\right\} = \lambda(\mathbf{k}\Lambda\eta) \\ \left[\mathsf{L}^{(\lambda)}(\Lambda), \mathsf{L}^{(\lambda)}(\Lambda')\right] = -\mathrm{i}\mathsf{L}^{(\lambda)}(\Lambda\eta\Lambda' - \Lambda'\eta\Lambda) &\leftrightarrow \{\mathbf{L}^{(\lambda)}(\Lambda), \mathbf{L}^{(\lambda)}(\Lambda')\} = -\mathbf{L}^{(\lambda)}(\Lambda\eta\Lambda' - \Lambda'\eta\Lambda) \\ [\mathbf{E}(\mathbf{h}), \Delta(\theta)] = \mathrm{i}\mathbf{E}(\theta(\mathbf{h})) &\leftrightarrow \{\mathbf{E}(\mathbf{h}), \boldsymbol{\Delta}(\theta)\} = \mathbf{E}(\theta(\mathbf{h})) \\ \left[\mathbf{E}(\mathbf{h}), \mathsf{L}^{(\mathbf{E})}(\Lambda)\right] = \mathrm{i}\mathbf{E}(\Lambda\eta\mathbf{h}) &\leftrightarrow \{\mathbf{E}(\mathbf{h}), \mathbf{L}^{(\mathbf{E})}(\Lambda)\} = \mathbf{E}(\Lambda\eta\mathbf{h}) \\ \left[\mathsf{L}^{(\mathbf{E})}(\Lambda), \mathsf{L}^{(\mathbf{E})}(\Lambda')\right] = \mathrm{i}\mathsf{L}^{(\mathbf{E})}(\Lambda\eta\Lambda' - \Lambda'\eta\Lambda) &\leftrightarrow \{\mathbf{L}^{(\mathbf{E})}(\Lambda), \mathbf{L}^{(\mathbf{E})}(\Lambda')\} = -\mathbf{L}^{(\mathbf{E})}(\Lambda\eta\Lambda' - \Lambda'\eta\Lambda) \end{aligned}$$

As we can see we have constructed a selfadjoint representation of the variables on the space $\mathscr{H} = \mathsf{L}^2\left(\mathfrak{conf}, \frac{\mathrm{d}e}{e^4}\right)$. The question is whether these variables seperate points of the phase space. Now, we will show that the answer is affirmative. The variables λ^a, $\mathbf{E}^a$ are clear, so let us turn our attention on $\mathbf{L}^{(\lambda)}_{ab}$, $\boldsymbol{\pi}$, $\mathbf{L}^{(\mathbf{E})}_{ab}$, $\boldsymbol{\Delta}^\alpha_\beta$. We have

$$\begin{aligned} \tilde{L}^{(\lambda)}_{ab}\lambda^a E^b_\alpha &= -(\lambda)^2\tilde{\pi}_a E^a_\alpha + \lambda^a\tilde{\pi}_a\eta_{bc}\lambda^c E^c_\alpha, \\ \tilde{\pi} &= \tilde{\pi}_a\lambda^a, \\ \tilde{L}^{(\mathbf{E})}_{ab}\lambda^a E^b_\alpha &= q_{\alpha\beta}\tilde{G}^\beta_a\lambda^a - \eta_{ab}\lambda^a E^b_\beta \tilde{G}^\beta_c E^b_\alpha, \\ \Delta^\alpha_\beta &= E^a_\beta\tilde{G}^\alpha_a, \end{aligned}$$

where $(\lambda)^2 = \eta_{ab}\lambda^a\lambda^b$, $\tilde{L}^{(\lambda)}_{ab}\mathrm{d}^3x = \mathbf{L}^{(\lambda)}_{ab}$, $\tilde{L}^{(\mathbf{E})}_{ab}\mathrm{d}^3x = \mathbf{L}^{(\mathbf{E})}_{ab}$. As we can see, we can invert these equations and we can express canonical momenta $\boldsymbol{\pi}_a$, $\mathbf{G}_a$ as functions of new variables. The projected variables $\mathbf{L}^{(\lambda)}_{\bar{a}\bar{b}}\mathbf{E}^{\bar{a}}_a\mathbf{E}^{\bar{b}}_b$, $\mathbf{L}^{(\mathbf{E})}_{\bar{a}\bar{b}}\mathbf{E}^{\bar{a}}_a\mathbf{E}^{\bar{b}}_b$ are not independent. They play similar roles like angular momenta in quantum mechanincs. So, we have found representation of algebra of new variables.

2.3 Tensor Product Hilbert Space

In the previous section we have constructed the Hilbert space $\mathscr{H}_\mathbf{x}$ associated with the point $\mathbf{x} \in \Sigma$ as $\mathscr{H}_\mathbf{x} = \mathsf{L}^2(\mathfrak{conf}_\mathbf{x}, \mathfrak{e}_\mathbf{x})$, where $\mathfrak{e} = \frac{\mathrm{d}e}{e^4}$ and $\mathbf{x}$ means that it is taken at the point $\mathbf{x}$. A main goal of this section is to briefly summarize ideas of von Neumann's article on tensor product of family of Hilbert spaces labeled by index set of arbitrary cardinality (details can be found in [12]). In our case we can formally write

$$\mathscr{H}_\Sigma = \otimes_{\mathbf{x}\in\Sigma}\mathscr{H}_{\mathbf{x}}.$$

We have a set $\{\mathscr{H}_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma}$ of Hilbert spaces's labeled by points of Σ. A sequence of the states $\{\psi_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma}$ belongs to the Cartesian product $\mathscr{H}_\Sigma^\times = \times_{\mathbf{x}\in\Sigma}\mathscr{H}_{\mathbf{x}}$, but this space is too large, we need to pick up a certain subset of $\mathscr{H}_\Sigma^\times$. Let us call $\{\psi_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma}$ a C-sequence iff a product

$$||\{\psi_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma}|| = \prod_{\mathbf{x}\in\Sigma} ||\psi_{\mathbf{x}}||_{\mathbf{x}} \tag{2.11}$$

converges. Let $C_\Sigma = \{\{\psi_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma}\colon C\text{-sequence}\}$ be a set of all C-sequences. A value of the product limit (2.11) can be positive or zero. We need some criteria for convergence of such limits. They can be found in ([12]).

Citation(α - index and I is an index set with arbitrary cardinality):

Lemma 2.4.1.(p.13):

If all z_α are real and ≥ 0, then
(I) $\prod_{\alpha\in I} z_\alpha$ converges if and only if either $\sum_{\alpha\in I}\mathrm{Max}(z_\alpha - 1, 0)$ converges, or some $z_\alpha = 0$
(II) $\prod_{\alpha\in I} z_\alpha$ converges and is $\neq 0$ if and only if $\sum_{\alpha\in I}|z_\alpha - 1|$ converges and all $z_\alpha \neq 0$.

Lemma 2.4.2.(p.15):

If the z_α are arbitrary complex numbers, then $\prod z_\alpha$ converges if and only if
(I) either $\prod_{\alpha\in I}|z_\alpha|$ converges and its value is 0,
(II) or $\prod_{\alpha\in I}|z_\alpha|$ converges and its value is $\neq 0$, and $\sum_{\alpha\in I}|\mathrm{arcus}\, z_\alpha|$ converges[3]

Definition 2.5.1.(p.18):

$\prod_{\alpha\in I} z_\alpha$ is quasi-convergent if and only if $\prod_{\alpha\in I}|z_\alpha|$ is convergent. Its value is
(I) the value of $\prod_{\alpha\in I} z_\alpha$ if it is even convergent
(II) 0, if it is not convergent.

End of citation.

The reason why we need a notion of quasi-convergence is that if $\{\psi_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma}, \{\phi_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma} \in C_\Sigma$ then product $\prod_{\mathbf{x}\in\Sigma}\langle\psi_{\mathbf{x}}|\pi_{\mathbf{x}}\rangle_{\mathbf{x}}$ is only quasi-convergent in general.
Now we can define a functional ψ_Σ associated with $\{\psi_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma}$ on the set C_Σ of all C-sequences as

$$\psi_\Sigma(\{\phi_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma}) = \prod_{\mathbf{x}\in\Sigma}\langle\phi_{\mathbf{x}}|\psi_{\mathbf{x}}\rangle_{\mathbf{x}},$$

[3] Is $z \neq 0$, $z = |z|e^{\theta}$ with $-\pi < \theta \leq \pi$, then $\mathrm{arcus}\, z = \theta$

where $\{\phi_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma} \in C_\Sigma$ and product is taken in the sence of quasi-convergence. It should be noted that ψ_σ does not imply that $\{\psi_{\mathbf{x}} = 0\}_{\mathbf{x}\in\Sigma}$, e.g. for C-sequence $\{\psi_{\mathbf{x}_0} = 0, \{\psi_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma\setminus\{\mathbf{x}_0\}}\}$ its associated functional vanishes on whole C_Σ. Let us define a complex linear space $\mathscr{H}^0_\Sigma$ of such functionals, where

$$(a\psi_\Sigma + b\phi_\Sigma)(\{\omega_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma}) = a\psi_\Sigma(\{\omega_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma}) + b\phi_\Sigma(\{\omega_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma}).$$

We can define an inner product on $\mathscr{H}^0_\Sigma$ as follows

$$\langle\psi_\Sigma|\phi_\Sigma\rangle = \prod_{\mathbf{x}\in\Sigma}\langle\psi_{\mathbf{x}}|\phi_{\mathbf{x}}\rangle_{\mathbf{x}}. \tag{2.12}$$

The closure $\mathscr{H}_\Sigma = \overline{\mathscr{H}^0_\Sigma}$ in the topology defined via inner product (2.12) is a Hilbert space and we call it as a tensor product of the sequence $\{\mathscr{H}_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma}$

$$\mathscr{H}_\Sigma = \otimes_{\mathbf{x}\in\Sigma}\mathscr{H}_{\mathbf{x}}. \tag{2.13}$$

We wish to characterize the space $\mathscr{H}_\Sigma$ is some way. In order to do so we need to introduce a notion of C_0-sequence and classes of equivalence on them. A sequence $\{\psi_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma}$ is a C_0-sequence iff $\sum_{\mathbf{x}\in\Sigma}\left|\|\psi_{\mathbf{x}}\|_{\mathbf{x}} - 1\right|$ converges. Every C_0-sequence is a C-sequence and every C-sequence $\{\psi_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma}$ is a C_0-sequence iff its functional $\psi_\Sigma \neq 0$. We will say that two C_0-sequences are equivalent $\{\psi_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma} \sim \{\phi_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma}$ iff $\sum_{\mathbf{x}\in\Sigma}\left|\langle\psi_{\mathbf{x}}|\phi_{\mathbf{x}}\rangle_{\mathbf{x}} - 1\right|$ converges, what is equivalent to the mutual convergence of both series $\sum_{\mathbf{x}\in\Sigma}\|\psi_{\mathbf{x}} - \phi_{\mathbf{x}}\|^2$, $\sum_{\mathbf{x}\in\Sigma}\left|\mathfrak{I}(\langle\psi_{\mathbf{x}}|\phi_{\mathbf{x}}\rangle_{\mathbf{x}})\right|$, where $\mathfrak{I}(z)$ is the imaginary part of z. Hence we see immediately that if $\{\psi_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma}$, $\{\phi_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma}$ differ in finite number of points of Σ then they are equivalent. Let us label equivalence classes by γ and a set of all equivalence classes on $\mathscr{H}_\Sigma$ by $\mathfrak{C}(\mathscr{H}_\Sigma)$.

Now we can finish this bries summary of [12] with the following statement. If two C_0-sequences $\{\psi_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma}$, $\{\phi_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma}$ or their functional ψ_Σ, ϕ_Σ belong to two equivalence classes $\gamma(\psi_\Sigma) \neq \gamma(\phi_\Sigma)$, then $\langle\psi_\Sigma|\phi_\Sigma\rangle = 0$. If $\gamma(\psi_\Sigma) = \gamma(\phi_\Sigma)$ and $\langle\psi_\Sigma|\phi_\Sigma\rangle = 0$ then there exists $\mathbf{x}_0$ where $\langle\psi_{\mathbf{x}_0}|\phi_{\mathbf{x}_0}\rangle_{\mathbf{x}_0} = 0$. Hence we see that $\mathscr{H}_\Sigma$ can be decomposed as

$$\mathscr{H}_\Sigma = \oplus_{\gamma\in\mathfrak{C}(\mathscr{H}_\Sigma)}\mathscr{H}_\gamma, \tag{2.14}$$

where $\mathscr{H}_\gamma$ is a Hilbert space associated with γ.

We will use a following example later. Let $K_\Sigma = \{K_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma}$ be sequence of compact sets where $K_{\mathbf{x}} \subset \mathfrak{conf}_{\mathbf{x}}$. K_Σ can be identified with Cartesian product $\times_{\mathbf{x}\in\Sigma}K_{\mathbf{x}}$. Let us define a sets of all sequences of compact sets with unit measure as

$$J^1(\mathfrak{Conf}) = \left\{K_\Sigma = \{K_{\mathbf{x}}\}_{\mathbf{x}\in\Sigma} : \forall\mathbf{x}\in\Sigma;\quad \mathfrak{e}_{\mathbf{x}}(K_{\mathbf{x}}) = 1\right\}$$

We can associate with $K_\Sigma \in J^1(\mathfrak{Conf})$ an element in $\mathscr{H}_\Sigma$ via

$$\chi_{K_\Sigma} = \{\chi_{K_{\mathbf{x}}}\}_{\mathbf{x}\in\Sigma}. \tag{2.15}$$

Let $K_\Sigma, K'_\Sigma \in J^1(\mathfrak{Conf})$ and $\sigma \subset \Sigma$ be a set of all $\mathbf{x}$ where $K_{\mathbf{x}} \neq K'_{\mathbf{x}}$. We will use a notation $\mathbf{e} = (e^a_\mu)$, $\mathbf{e}_{\mathbf{x}} = (e^a_\mu\big|_{\mathbf{x}})$. Let $\mathbf{e} \in K_\Sigma \setminus K'_\Sigma$. If we suppose that for $\forall\mathbf{x}\in\sigma$ exists an open neighbourhood of $\mathbf{e}_{\mathbf{x}} \in U_{\mathbf{x}}$ with property $\overline{U_{\mathbf{x}}} \subset K_{\mathbf{x}} \setminus K'_{\mathbf{x}}$ and $\mathfrak{e}_{\mathbf{x}}(U_{\mathbf{x}}) > \delta \in (0,1)$ and σ is not a finite set then

$$\langle\chi_{K_\Sigma}|\chi_{K'_\Sigma}\rangle = \prod_{\mathbf{x}\in\Sigma}\mathfrak{e}_{\mathbf{x}}(K_{\mathbf{x}} \cap K'_{\mathbf{x}}) = 0,$$

since $1 > 1 - \delta > 1 - \mathfrak{e}_{\mathbf{x}}(K_{\mathbf{x}} \setminus K'_{\mathbf{x}}) = \mathfrak{e}_{\mathbf{x}}(K_{\mathbf{x}} \cap K'_{\mathbf{x}})$.

2.4 Quantum Algebra of Basic Variables

Now it is time to construct a representation of the basic variables of the Einstein-Cartan theory. Inspired by the point version of the phase space we will not work with canonical variables, but we will construct a representation of the following variables

$$
\begin{aligned}
\lambda_{\mathbf{x}}(\mathbf{k}) &= \mathbf{k}_a \lambda^a\Big|_{\mathbf{x}}, \\
\mathbf{L}^{(\lambda)}(\Lambda) &= \int\limits_{\Sigma} \Lambda^{ab}\eta_{bc}\lambda^c \boldsymbol{\pi}_a, \\
\boldsymbol{\pi}(N) &= \int\limits_{\Sigma} N\lambda^a \boldsymbol{\pi}_a, \\
\mathbf{E}_{\mathbf{x}}(\mathbf{h}) &= \mathbf{h}_a \wedge \mathbf{E}^a\Big|_{\mathbf{x}}, \\
\mathbf{L}^{(\mathbf{E})}(\Lambda) &= \int\limits_{\Sigma} \Lambda^{ab}\eta_{bc}\mathbf{E}^c \wedge \mathbf{G}_a, \ (\text{where } \mathbf{G}_a = -\frac{1}{8\pi\kappa}\mathbf{F}_a) \\
\boldsymbol{\Delta}(\theta) &= \int\limits_{\Sigma} \theta(\mathbf{E}^a) \wedge \mathbf{G}_a.
\end{aligned}
$$

with similar algebra as in the point version (trivial brackets are not written)

$$
\begin{aligned}
\left\{\lambda_{\mathbf{x}}(\mathbf{k}),\, \boldsymbol{\pi}(N)\right\}^* &= \lambda_{\mathbf{x}}(N\mathbf{k}), \\
\left\{\lambda_{\mathbf{x}}(\mathbf{k}), \mathbf{L}^{(\lambda)}(\Lambda)\right\}^* &= \lambda_{\mathbf{x}}(\mathbf{k}\Lambda\eta), \\
\left\{\mathbf{L}^{(\lambda)}(\Lambda), \mathbf{L}^{(\lambda)}(\Lambda')\right\}^* &= -\mathbf{L}^{(\lambda)}(\Lambda\eta\Lambda' - \Lambda'\eta\Lambda), \\
\left\{\mathbf{E}_{\mathbf{x}}(\mathbf{h}), \boldsymbol{\Delta}(\theta)\right\}^* &= \mathbf{E}_{\mathbf{x}}(\theta(\mathbf{h})), \\
\left\{\mathbf{E}_{\mathbf{x}}(\mathbf{h}), \mathbf{L}^{(\mathbf{E})}(\Lambda)\right\}^* &= \mathbf{E}_{\mathbf{x}}(\Lambda\eta\mathbf{h}), \\
\left\{\mathbf{L}^{(\mathbf{E})}(\Lambda), \mathbf{L}^{(\mathbf{E})}(\Lambda')\right\}^* &= -\mathbf{L}^{(\mathbf{E})}(\Lambda\eta\Lambda' - \Lambda'\eta\Lambda),
\end{aligned}
$$

Before we start to costruct a representation of this algebra, we need to discuss properties of a certain family of operators. Let $\mathsf{A}_{\mathbf{x}}$ be a selfadjoint operator with action on $\mathscr{H}_{\mathbf{x}}$ with dense domain $\mathfrak{D}(\mathsf{A}_{\mathbf{x}})$. We wish to represent it on the space $\mathscr{H}_\Sigma$. Since $\mathscr{H}_\Sigma \simeq \mathscr{H}_{\mathbf{x}} \otimes \mathscr{H}_{\Sigma\setminus\{\mathbf{x}\}}$ we can use theory of finite tensor product of bounded operator and we see that expression

$$\mathsf{U}_\Sigma(t)\psi_\Sigma = \{\mathsf{U}_{\mathbf{x}}(t)\psi_{\mathbf{x}}; \{\psi_{\mathbf{y}}\}_{\mathbf{y}\neq\mathbf{x}}\}, \tag{2.16}$$

where ψ_Σ is C-sequence, defines an unitary operator on whole $\mathscr{H}_\Sigma$, which is strongly continuous at t. $\mathsf{U}_\Sigma(t)\psi_{\mathbf{x}}$ determines a generator A_Σ associated with it and $\mathfrak{D}(\mathsf{A}_\Sigma) \supset \mathfrak{D}_o(\mathsf{A}_{\mathbf{x}}) = \mathrm{Span}\{\psi_{\mathbf{x}} \otimes \psi_{\Sigma\setminus\{\mathbf{x}\}}; \psi_{\mathbf{x}} \in \mathfrak{D}(\mathsf{A}_{\mathbf{x}}), \psi_{\Sigma\setminus\{\mathbf{x}\}} \in \mathscr{H}^0_{\Sigma\setminus\{\mathbf{x}\}}\}$. Restricted operator $\mathsf{A}_\Sigma\big|_{\mathfrak{D}_o(\mathsf{A}_{\mathbf{x}})}$ is essentially selfadjoint and acts on C-sequences $\psi_\Sigma \in \mathfrak{D}_o(\mathsf{A}_{\mathbf{x}})$ as

$$\mathsf{A}_\Sigma\big|_{\mathfrak{D}_o(\mathsf{A}_{\mathbf{x}})}\psi_\Sigma = \{\mathsf{A}_{\mathbf{x}}\psi_{\mathbf{x}}, \{\psi_{\mathbf{y}}\}_{\Sigma\setminus\{\mathbf{x}\}}\}.$$

Let us start with variables $\lambda_{\mathbf{x}}(\mathbf{k})$, $\mathbf{E_x}(\mathbf{h})$. Both of them are acting on the space $\mathscr{H}_{\mathbf{x}}$, hence we can represent them via previous construction on the space $\mathscr{H}_\Sigma$ by formula for C-sequence $\psi_\Sigma \in C_\Sigma$

$$\varrho\Big(\lambda_{\mathbf{x}}(\mathbf{k})\Big)\psi_\Sigma = \Big\{\lambda_{\mathbf{x}}(\mathbf{k})\psi_{\mathbf{x}}(\mathbf{e}); \{\psi_{\mathbf{y}}\}_{\mathbf{y}\in\Sigma\setminus\{\mathbf{x}\}}\Big\},$$
$$\varrho\Big(\mathbf{E_x}(\mathbf{h})\Big)\psi_\Sigma = \Big\{\mathbf{E_x}(\mathbf{h})\psi_{\mathbf{x}}(\mathbf{e}); \{\psi_{\mathbf{y}}\}_{\mathbf{y}\in\Sigma\setminus\{\mathbf{x}\}}\Big\}.$$

We have used the actions of the groups $\mathbf{SO}^+(\eta)$ for λ^a, $\mathbf{SO}^+(\eta)$ for $\mathbf{E}^a$, $\mathbb{R}^+$ and $\mathbf{GL}^{+3}$ on the space $\mathfrak{conf}$. Now, we wish to generalize this idea to Einstein-Cartan theory. Let $\mathbf{G_x}$ be one, same for all $\mathbf{x}$, of the previous groups acting on the space $\mathfrak{conf}_{\mathbf{x}}$ and let $\Phi^{\mathbf{x}}_t$ be flow associated with some one parametric subgroup of $\mathbf{G_x}$. Then we have a group $\mathbf{G}^\Sigma = \times_{\mathbf{x}\in\Sigma}\mathbf{G_x}$ acting on the space $\mathfrak{Conf} = \times_{\mathbf{x}\in\Sigma}\mathfrak{conf}_{\mathbf{x}}$ by the flow $\Phi^\Sigma_t(\mathbf{e}) = \{\Phi^{\mathbf{x}}_t(\mathbf{e_x})\}_{\mathbf{x}\in\Sigma}$ [4]. Let ψ_Σ be a C-sequence, then an operator defined for any $\Psi \in \mathscr{H}^0_\Sigma$

$$\mathsf{U}^\Sigma(t)\Psi = \sum_{j=1}^{m\in\mathbb{N}} c_j \mathsf{U}^\Sigma(t)\psi^j_\Sigma = \sum_{j=1}^{m\in\mathbb{N}} c_j\Big\{\psi^j_{\mathbf{x}}(\Phi^{\mathbf{x}}_t\mathbf{e_x})\Big\}_{\mathbf{x}\in\Sigma},$$

where $\Psi = \sum_{j=1}^{m\in\mathbb{N}} c_j\psi^j_\Sigma$ and ψ^j_Σ are C-sequences, can be extended to the one-parametric unitary grup acting on whole $\mathscr{H}_\Sigma$. We know nothing about its continuity at the moment.

Let $K_\Sigma \in J^1(\mathfrak{Conf})$ be a constant sequence of compact sets, i.e. $\forall\mathbf{x}\ K_{\mathbf{x}} = K$, and let $\Phi^{\mathbf{x}}_t = \Phi_t$ for $\forall\mathbf{x}\in\sigma\subset\Sigma$ and $\Phi^{\mathbf{x}}_t = \mathrm{id}$ for $\forall\mathbf{x}\in\Sigma\setminus\sigma$. Let us explore an expression

$$u(t) = \left\|\left(1 - \mathsf{U}^\Sigma_t\right)\chi_{K_\Sigma}\right\|^2.$$

It is clear by definition, that $u(0) = 0$. Let $t \neq 0$, then we can write

$$u(t) = \left\langle\chi_{K_\Sigma}\middle|\left(1-\mathsf{U}^\Sigma_{-t}\right)\left(1-\mathsf{U}^\Sigma_t\right)\chi_{K_\Sigma}\right\rangle = 2 - \left\langle\chi_{K_\Sigma}\middle|\mathsf{U}^\Sigma_{-t}\chi_{K_\Sigma}\right\rangle - \left\langle\chi_{K_\Sigma}\middle|\mathsf{U}^\Sigma_t\chi_{K_\Sigma}\right\rangle.$$

The last two terms are zero in the case when σ is not finite due to the arguments from the end of the previous section. Hence we have, as a consequence, that operator $\mathsf{U}^\Sigma(t)$ is not strongly continuous in the general case. Therefore there does not exist selfadjoint generator of $\mathsf{U}^\Sigma(t)$ in the general case.

What we can do is to explore the case when the group $\mathbf{G}^\Sigma$ acts on $\mathfrak{Conf}$ nontrivially only on some finite subset $\sigma\subset\Sigma$. Let us start with $\sigma = \{\mathbf{x}\}$. This case were explored few rows above and point generators $\mathsf{T}_{\mathbf{x}}$ of such action were found in section 2.2. Generalization to the case when $\sigma = \{\mathbf{x}_1,\dots,\mathbf{x}_n\}$ is clear and the resulting generator is $\mathsf{T}_\sigma = \sum_{\mathbf{x}\in\Sigma}\mathsf{T}_{\mathbf{x}}$.

Now we can write explicitly the generators of our groups acting on the $\mathfrak{Conf}$. They are

$$\begin{aligned}
\pi(N) &= \sum_{\mathbf{x}\in\Sigma} -\mathrm{i}N(\mathbf{x})\lambda^a(\mathbf{x})\partial_{\lambda^a(\mathbf{x})},\\
\mathsf{L}^{(\lambda)}(\Lambda) &= \sum_{\mathbf{x}\in\Sigma} -\mathrm{i}\Lambda^{ab}(\mathbf{x})\eta_{bc}\lambda^c(\mathbf{x})\partial_{\lambda^a(\mathbf{x})},\\
\Delta(\theta) &= \sum_{\mathbf{x}\in\Sigma} -\mathrm{i}\theta^\beta_\alpha(\mathbf{x})E^a_\beta(\mathbf{x})\partial_{E^a_\alpha(\mathbf{x})},\\
\mathsf{L}^{(\mathbf{E})}(\Lambda) &= \sum_{\mathbf{x}\in\Sigma} -\mathrm{i}\Lambda^{ab}(\mathbf{x})\eta_{bc}E^b_\alpha(\mathbf{x})\partial_{E^a_\alpha(\mathbf{x})},
\end{aligned}$$

[4] No summation over $\mathbf{x}$! $\mathbf{e_x}$ is a point in the manifold $\mathfrak{conf}_{\mathbf{x}}$.

where $N(\mathbf{x})$, $\Lambda^{ab}(\mathbf{x})$, $\theta_\alpha^\beta(\mathbf{x})$ has support on a finite set. Commutator algebra of basic quantum observables is generated by

$$\begin{aligned}
\Big[\varrho\Big(\lambda_{\mathbf{x}}(\mathbf{k})\Big)\,,\,\pi(N)\,\Big] &= \mathrm{i}\varrho\Big(\lambda_{\mathbf{x}}(N\mathbf{k})\Big),\\
\Big[\varrho\Big(\lambda_{\mathbf{x}}(\mathbf{k})\Big),\mathsf{L}^{(\lambda)}(\Lambda)\Big] &= \mathrm{i}\varrho\Big(\lambda_{\mathbf{x}}(\mathbf{k}\Lambda\eta)\Big),\\
\Big[\mathsf{L}^{(\lambda)}(\Lambda),\mathsf{L}^{(\lambda)}(\Lambda')\Big] &= -\mathrm{i}\mathsf{L}^{(\lambda)}(\Lambda\eta\Lambda'-\Lambda'\eta\Lambda),\\
\Big[\varrho\Big(\mathbf{E}_{\mathbf{x}}(\mathbf{h})\Big),\Delta(\theta)\Big] &= \mathrm{i}\varrho\Big(\mathbf{E}_{\mathbf{x}}(\theta(\mathbf{h}))\Big),\\
\Big[\varrho\Big(\mathbf{E}_{\mathbf{x}}(\mathbf{h})\Big),\mathsf{L}^{(\mathbf{E})}(\Lambda)\Big] &= \mathrm{i}\varrho\Big(\mathbf{E}_{\mathbf{x}}(\Lambda\eta\mathbf{h})\Big),\\
\Big[\mathsf{L}^{(\mathbf{E})}(\Lambda),\mathsf{L}^{(\mathbf{E})}(\Lambda')\Big] &= -\mathrm{i}\mathsf{L}^{(\mathbf{E})}(\Lambda\eta\Lambda'-\Lambda'\eta\Lambda).
\end{aligned}$$

Hence we see that we found representation of classical variables of Einstein-Cartan theory.

Now, let us explore a reducibility of this representation. As we already know, space $\mathscr{H}_\Sigma$ can be decomposed into the mutually orthogonal subspaces labeled by class of equivalences of C_0-sequences $\mathfrak{C}(\mathscr{H}_\Sigma)$. Our representation does not mix this decomposition hence it is reducible. Number of irreducible representation in $\mathscr{H}$ is equal to the number of equivalence classes on $\mathscr{H}_\Sigma$, what is "huge" infinite, e.g. for every element of $\mathsf{L}^2\left(\mathfrak{conf},\frac{de}{e^4}\right)$ there exists its own equivalence class, etc. One may partially save the situation by using unitary version of basic variables and represents operators $\mathsf{U}^{\Sigma}_{\mathbf{G}_\Sigma}(\xi)$, where $\xi = N$ for $\mathbb{R}^+$, etc., instead of its generators $\mathsf{T}(\xi)$, with action on whole $\mathfrak{Conf}$ which mix orthogonal decomposition of $\mathscr{H}_\Sigma$. Anyway for $K^1_\Sigma, K^2_\Sigma \in J^1(\mathfrak{Conf})$, where K^1_Σ is built by simple connected sets and K^2_Σ is built by union of two simple connected sets, there is no element of $\mathbf{G}_\Sigma$ which mixes their equivalence classes and reducibility of unitary representation is still to huge. Hence some additional superselection rules should be used if one wants to quantize Einstein-Cartan theory with this representation.

Conclusion

In section 1.1 we have started with the orthonormal coframe $\mathbf{e}^a$ and general gravitational connection $\hat{\nabla}$ described by its forms $\hat{\boldsymbol{\Gamma}}^{ab} = \eta^{b\bar{b}}\hat{\boldsymbol{\Gamma}}^a{}_{\bar{b}}$. We have derived the equations of motion which have fixed $\hat{\boldsymbol{\Gamma}}^{ab} = \hat{\mathbf{A}}^{ab} + \hat{\mathbf{B}}\eta^{ab}$ where $\hat{\mathbf{A}}^{ab}$ is related to the metric connection $\hat{\mathscr{D}}$ and $\hat{\mathbf{B}}$ is arbitrary 1-form. Torsion of $\hat{\mathscr{D}}$ vanishes as a consequence of EOM, hence $\hat{\mathbf{A}}^{ab}$ can be expressed as a functional of coframe $\mathbf{e}^a$ which is given by the solution of Einstein equations. Algebraic interpretation of such kind of connection, as described in section 1.2, is given by the condition that operator $\hat{\nabla}$ preserves (anti)symmetric structures on $\Lambda\mathbb{T}\mathbf{M}$ which in special case $\Lambda\mathbb{T}^p_q\mathbf{M}$, where $p+q=2$, means that $\hat{\nabla}$ does not mix irreducible structures $\Lambda\mathbb{A}^p_q\mathbf{M}$, $\Lambda\mathbb{B}^p_q\mathbf{M}$ and $\Lambda\mathbb{C}^p_q\mathbf{M}$ on $\Lambda\mathbb{T}^p_q\mathbf{M}$. The author have not found any reference in the literature about such interpretation of the Cartan connection. We have induced the geometrical structure on the spatial section Σ inherited from spacetime $\mathbf{M}$ and hence $\mathbf{SO}(\mathbf{g})$ is still (part of) gauge freedom which is opposite to the standard loop formulation of gravity where the orthonormal coframe $\mathbf{e}^a$ is fixed to be tangential to Σ and its time vector is normal to Σ. Then we have used the $\mathbf{SO}(\mathbf{g})$ structure in the Hamilton-Dirac formulation of the Einstein-Cartan theory. Since our system is degenerated and it contains both classes of constraints the Dirac bracket has been introduced. The Dirac procedure has been finished by introducing the reduced phase space described by coordinates $(\lambda^a, \mathbf{E}^a, \boldsymbol{\pi}_a, \mathbf{F}_a)$.

The loop theory (LQG) is successful theory of quantum gravity. But there exist some unresolved problems in this theory. One of them is Barbero-Immirzi parameter which causes ambiguity in the LQG and this parameter should be fixed by Hawking-Bekenstein(HB) entropy. Honestly, we do not know yet whether HB entropy is in accordance with nature or not. Also this procedure resembles derivation of the Stefan-Boltzmann law of the black body radiation from classical thermodynamics where the Stefan-Boltzmann constant appears like an integration constant and should be fixed by experiment. Only the Planck derivation of this law based on quantum theory predicts this constant from the first principles. In fact, observables like entropy of a black hole have to be predicted by full quantum theory of gravitation. Our approach does not contain such parameter, but there is another problem, one may say a huge problem, caused by the high degree of reducibility of the representation of basic variables constructed in 2.4. Its origin lies in the kernel of the method of construction of $\mathscr{H}_\Sigma$ used here. Similar thing happens if one wants to represent basic variables Φ, Π of scalar fied Φ on the space $\mathscr{H}^\Phi_\Sigma$ given by infinite tensor product of spaces $\mathscr{H}^\Phi_\mathbf{x}$. Hence it seems that this problem of huge ambiguity is not caused by the choise of the kinematical variables on the phase space, but by the choice of method of construction of $\mathscr{H}_\Sigma$. This problem should be solved in the future. A one possible solution of this problem is represented by the following idea. If $\mathbf{G}$ is a topological group, then there always exists its Bohr compactification $\overline{\mathbf{G}}$ based on the notion of almost periodical functions over $\mathbf{G}$. Since $\overline{\mathbf{G}}$ is a compact group there always exists unique left/right invariant[5] measure $\omega^{\overline{\mathbf{G}}}_{\mathrm{L/R}}$ on $\overline{\mathbf{G}}$ with propery $\omega^{\overline{\mathbf{G}}}_{\mathrm{L/R}}(\overline{\mathbf{G}}) = 1$. Of course the space $\mathfrak{Conf}$ is not a group, but it is a subset of $\mathbf{GL}^+$ and there exist actions of the groups $\mathbf{SO}(\eta)$, $\mathbb{R}^+$ and $\mathbf{GL}^{+3}$ on $\mathfrak{Conf}$. Hence the idea is to construct (if it possible, unique, etc.) the space of $\mathscr{H}_\Sigma$ by analogous compactification of the space $\mathfrak{Conf}$ by using the almost periodicity defined via groups $\mathbf{SO}(\eta)$,

[5] Since $\overline{\mathbf{G}}$ is not a Lie group, left and right invariant measures may differ in general.

$\mathbb{R}^+$ and $\mathbf{GL}^{+3}$ acting on $\mathfrak{Conf}$. But this must be explored in detail in future.

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Appendix

2+1 Dimensional Einstein-Cartan Theory

If we already start with metric[6] connection then Lagrangian for 2+1 dimensional Einstein-Cartan theory can be written as

$$\mathbf{L} = \frac{1}{2}\boldsymbol{\varepsilon}_{abc}\hat{\mathbf{R}}^{ab} \wedge \mathbf{e}^c. \tag{2.17}$$

EOM:

$$\begin{aligned} \hat{\mathbf{R}}^{ab} &= 0, && (2.18)\\ \hat{\mathbf{T}}^{a} &= 0. && (2.19) \end{aligned}$$

Using 2+1 decomposition

$$\begin{aligned} \mathbf{e}^a &= \lambda^a \mathrm{d}t + \mathbf{E}^a,\\ \hat{\mathbf{A}}^{ab} &= \Lambda^{ab}\mathrm{d}t + \mathbf{A}^{ab} \end{aligned}$$

leads to Hamiltonian:

$$\mathsf{H} = \boldsymbol{\pi}(\nu) + \boldsymbol{\Pi}(\Gamma) + \mathbf{p}(\mathbf{b}) + \mathbf{P}(\mathbf{B}) + \mathbf{R}(\lambda) + \mathbf{T}(\Lambda), \tag{2.20}$$

where

$$\boldsymbol{\pi}(\nu) = \int_\Sigma \nu^a \wedge \boldsymbol{\pi}_a, \tag{2.21}$$

$$\boldsymbol{\Pi}(\Gamma) = \int_\Sigma \frac{1}{2}\Gamma^{ab} \wedge \boldsymbol{\Pi}_{ab}, \tag{2.22}$$

$$\mathbf{p}(\mathbf{b}) = \int_\Sigma \mathbf{b}^a \wedge \mathbf{p}_a, \tag{2.23}$$

$$\mathbf{P}(\mathbf{B}) = \int_\Sigma \frac{1}{2}\mathbf{B}^{ab} \wedge (\mathbf{p}_{ab} - \boldsymbol{\varepsilon}_{abc}\mathbf{E}^c), \tag{2.24}$$

$$\mathbf{R}(\lambda) = \int_\Sigma -\frac{1}{2}\boldsymbol{\varepsilon}_{abc}\lambda^a\hat{\mathbf{R}}^{bc}, \tag{2.25}$$

$$\mathbf{T}(\Lambda) = \int_\Sigma -\frac{1}{2}\boldsymbol{\varepsilon}_{abc}\Lambda^{ab}\mathcal{D}\mathbf{E}^c. \tag{2.26}$$

Momenta and velocities variables are given by table 2.1.

Primary constraints are

$$\begin{aligned} \boldsymbol{\pi}(\nu) &= 0,\\ \mathbf{p}(\mathbf{b}) &= 0,\\ \boldsymbol{\Pi}(\Gamma) &= 0,\\ \mathbf{P}(\mathbf{B}) &= 0. \end{aligned}$$

Table 2.1: Table of basic variables

Variables	Momentum	Velocities
λ^a	$\boldsymbol{\pi}_a = \tilde{\pi}_a \mathrm{d}^2 x$ where $\tilde{\pi}_a = \partial\mathscr{L}/\partial\dot{\lambda}^a$	$\nu^a = \dot{\lambda}^a$
$\mathbf{E}^a = E^a_\alpha \mathrm{d}x^\alpha$	$\mathbf{p}_a = \tilde{p}^\alpha_a \varepsilon_{\alpha\beta}\mathrm{d}x^\beta$ where $\tilde{p}^\alpha_a = \partial\mathscr{L}/\partial\dot{E}^a_\alpha$	$\mathbf{b}^a = \dot{\mathbf{E}}^a$
Λ^{ab}	$\boldsymbol{\Pi}_{ab} = \tilde{\Pi}_{ab}\mathrm{d}^2 x$ where $\tilde{\Pi}_{ab} = \partial\mathscr{L}/\partial\dot{\Lambda}^{ab}$	$\Gamma^{ab} = \dot{\Lambda}^{ab}$
$\mathbf{A}^{ab} = A^{ab}_\alpha \mathrm{d}x^\alpha$	$\mathbf{p}_{ab} = \tilde{p}^\alpha_{ab}\varepsilon_{\alpha\beta}\mathrm{d}x^\beta$ where $\tilde{p}^\alpha_{ab} = \partial\mathscr{L}/\partial\dot{A}^{ab}_\alpha$	$\mathbf{B}^{ab} = \dot{\mathbf{A}}^{ab}$

Poisson brackets between Hamiltonian and $\mathbf{p}(\mathbf{b})$ or $\mathbf{P}(\mathbf{B})$ lead to Lagrange multipleirs

$$\begin{aligned}\mathbf{B}^{ab} &= \mathcal{D}\Lambda^{ab},\\ \mathbf{b}^a &= \mathcal{D}\lambda^a - \eta_{\bar{a}\bar{b}}\Lambda^{a\bar{a}}\mathbf{E}^{\bar{b}},\end{aligned}$$

while $\boldsymbol{\pi}(\nu)$ and $\boldsymbol{\Pi}(\Gamma)$ give new constraints

$$\begin{aligned}\mathbf{R}(\nu) &= \int_\Sigma -\frac{1}{2}\boldsymbol{\varepsilon}_{abc}\nu^a\hat{\mathbf{R}}^{bc},\\ \mathbf{T}(\Gamma) &= \int_\Sigma -\frac{1}{2}\boldsymbol{\varepsilon}_{abc}\Gamma^{ab}\mathcal{D}\mathbf{E}^c.\end{aligned}$$

No other new constraints appear and $\mathbf{p}$, $\mathbf{P}$ are the second class constrains. Next step is the definition of Dirac bracket thus we need evaluate

$$\left\{\mathbf{P}(\tilde{\mathbf{B}}), \mathbf{p}(\tilde{\mathbf{b}})\right\} = \int_\Sigma -\frac{1}{2}\boldsymbol{\varepsilon}_{abc}\tilde{\mathbf{B}}^{ab}\wedge\tilde{\mathbf{b}}^c,$$

what is equal with

$$\left\{\tilde{P}^\alpha_{ab}(x), \tilde{p}^\beta_c(y)\right\} = -\boldsymbol{\varepsilon}_{abc}\bar{\varepsilon}^{\alpha\beta}\delta_{xy}.$$

Dirac bracket is defined as

$$\begin{aligned}\{A,B\}^* = \{A,B\} &+ \int \frac{\mathrm{d}x}{2}\{A,\tilde{P}^\alpha_{ab}\}\bar{\boldsymbol{\varepsilon}}^{abc}\varepsilon_{\alpha\beta}\{\tilde{p}^\beta_c, B\}\\ &- \int \frac{\mathrm{d}x}{2}\{B,\tilde{P}^\alpha_{ab}\}\bar{\boldsymbol{\varepsilon}}^{abc}\varepsilon_{\alpha\beta}\{\tilde{p}^\beta_c, A\}\end{aligned}\tag{2.27}$$

and constraints algebra is given by commutators

$$\{\mathbf{R}(\mu),\mathbf{R}(\nu)\}^* = 0,\tag{2.28}$$
$$\{\mathbf{R}(\mu),\mathbf{T}(\Lambda)\}^* = -\mathbf{R}(\Lambda\eta\mu),\tag{2.29}$$
$$\{\mathbf{T}(\Lambda),\mathbf{T}(\Gamma)\}^* = \mathbf{T}(\tilde{\Lambda}),\tag{2.30}$$

where $\tilde{\Lambda}^{ab} = 2\delta^{ab}_{\bar{a}\bar{b}}\Lambda^{\bar{a}\bar{c}}\eta_{\bar{c}\bar{d}}\Gamma^{\bar{d}\bar{b}}$ and $(\Lambda\eta\mu)^a = \Lambda^{ab}\eta_{bc}\mu^c$. We see that constraints algebra in 2+1 dimensional Einstein-Cartan theory is Poincaré algebra.

[6]Similar analysis of general connection can be done as in 3+1 case, but for simplicity we fix connection to be compatible with metric already now.

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List of Abbreviations

Manifold structure and indices:

$\mathbf{M}$ - spacetime, Σ - spatial section of $\mathbf{M} = \mathbb{R} \times \Sigma$

$a, b, \cdots = 0, 1, 2, 3$ - frame indices

$\alpha, \beta, \cdots = 1, 2, 3$ - spatial coordinates indices

η_{ab} - Minkowski metric with signature $(+, -, -, -)$

Groups:

$\mathbf{GL}(V)$ - general linear group over (real) vector space V

$\mathbf{O}(\mathbf{g})$ - orthonormal group over metric vector space $(V, \mathbf{g})$ or manifold $(\mathbf{M}, \mathbf{g})$

$\overline{\mathbf{SO}}(\mathbf{g}) \subset \mathbf{O}(\mathbf{g})$ - special orthonormal group over vector space $(V, \mathbf{g})$ or manifold $(\mathbf{M}, \mathbf{g})$

$\mathbf{SO}(\mathbf{g}) \subset \overline{\mathbf{SO}}(\mathbf{g})$ - proper Lorentz group over vector space $(V, \mathbf{g})$ or manifold $(\mathbf{M}, \mathbf{g})$ preserving righthand and future time orientation

(Anti)symmetrization:

$A^{[ab]} = \frac{1}{2}(A^{ab} - A^{ba})$

$S^{(ab)} = \frac{1}{2}(S^{ab} + S^{ba})$

etc.

Antisymmetric delta and Levi-Civita symbol:

$\delta^{a...b}_{c...d} = \delta^{[a}_{\ c} \ldots \delta^{b]}_{d} = \delta_{[c}^{\ a} \ldots \delta^{b}_{d]} = \delta^{[a}_{[c} \ldots \delta^{b]}_{d]}$

$\boldsymbol{\varepsilon}_{abcd} = \boldsymbol{\varepsilon}_{[abcd]}$, $\bar{\boldsymbol{\varepsilon}}^{abcd} = \bar{\boldsymbol{\varepsilon}}^{[abcd]}$ and $\boldsymbol{\varepsilon}_{0123} = \bar{\boldsymbol{\varepsilon}}^{0123} = 1$

$\varepsilon_{\alpha\beta\gamma} = \varepsilon_{[\alpha\beta\gamma]}$, $\bar{\varepsilon}^{\alpha\beta\gamma} = \bar{\varepsilon}^{[\alpha\beta\gamma]}$ and $\varepsilon_{123} = \bar{\varepsilon}^{123} = 1$

Cartan algebra and exterior product:

$(\Lambda\mathbf{M}, \wedge)$ - Cartan algebra of all spacetime forms. $\Lambda_p\mathbf{M}$ - space of spacetime p-forms.

$(\Lambda\Sigma, \wedge)$ - Cartan algebra of all spatial forms. $\Lambda_p\Sigma$ - space of spatial p-forms.

If $\alpha, \beta \in \Lambda_1\mathbf{M}$ or $\Lambda_1\Sigma$ then $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$

Interior product:

$(i_v\alpha)(u_1, ..., u_{p-1}) = \alpha(v, u_1, ..., u_{p-1})$ $\forall \alpha \in \Lambda_p\mathbf{M}$ or $\Lambda_p\Sigma$

Derivative operators:

$\hat{\mathrm{d}}$ - exterior derivative operator on spacetime $\mathbf{M}$. Anyway we write $\mathrm{d}t = \hat{\mathrm{d}}t$

d - spatial exterior derivative operator on Σ

$\hat{\nabla}$ - general covariant exterior derivative operator on $\mathbf{M}$, or general connection associated with $\hat{\mathbf{\Gamma}}^a{}_b$

$\hat{\mathcal{D}}$ - $\mathbf{SO}(\mathbf{g})$- covariant exterior derivative operator on $\mathbf{M}$, or $\mathbf{SO}(\mathbf{g})$ connection associated with $\hat{\mathbf{A}}^a{}_b = \eta_{bc}\hat{\mathbf{A}}^{ac}$

$\mathcal{D}$ - spatial $\mathbf{SO}(\mathbf{g})$- covariant exterior derivative operator on Σ, or spatial $\mathbf{SO}(\mathbf{g})$ connection associated with $\mathbf{A}^a{}_b = \eta_{bc}\mathbf{A}^{ac}$