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Kvazieuklidovské obory integrity

Katedra algebry

Vedoucí bakalářské práce: Mgr. Jan Šaroch, Ph.D.
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Abstrakt: Tato práce shrnuje některé známé výsledky, týkající se k-stage euklidovských a kvazieuklidovských okruhů a oborů integrity, jistých zobecnění pojmu euklidovského okruhu, a prezentuje nové výsledky. Mezi ně patří zejména zavedení transfinitní konstrukce k-stage euklidovského okruhu, vedoucí k charakterizaci k-stage euklidovských okruhů nevyužívající pojmu normy, a její důsledky. Pozornost je dále věnována tvrzením, dávajícím návod, jak konstruovat nové k-stage euklidovské oruhy z jiných k-stage euklidovských okruhů (a popř. tak, aby se jednalo o obory integrity). Prezentujeme také příklad oboru integrity, který se jeví jako dobrý kandidát na 3-stage euklidovský okruh, který není 2-stage euklidovský.

Klíčová slova: kvazieuklidovský, obor integrity, řetěz dělitelnosti, OIHI

Title: Quasi-Euclidean Domains

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Abstract: In this thesis, we present an overview of some of the known facts about k-stage Euclidean and quasi-Euclidean rings and domains, certain generalisations of the concept of Euclidean ring, as well as some new results. Among the new results, the norm-free characterization of k-stage Euclidean rings based on a transfinite construction of k-stage Euclidean ring is fundamental and has many applications. Statements providing a way to construct new k-stage Euclidean rings from other k-stage Euclidean rings recieve special attention (with the integral domain case in mind). Also, we present an example of a 3-stage Euclidean integral domain which we believe is a good candidate for not being 2-stage Euclidean.

Keywords: quasi-Euclidean, domain, division chain, PID

Contents

List of symbols

ω	set of all natural numbers (including 0)
$\mathbb{Z}$	set (ring) of integers
$\mathbb{Q}$	set (field) of rational numbers
$\mathbb{R}$	set (field) of real numbers
$\mathbb{C}$	set (field) of complex numbers
$\widehat{\mathbb{Z}}_p$	ring of p-adic integers
R/I	quotient ring R modulo I
$R[x_1, x_2, \ldots, x_n]$	ring of polynomials in indeterminates $x_1, x_2, \ldots, x_n$ over ring R
$S^{-1}R$	localization of ring R by multiplicatively closed set S
$R_1 \times R_2$	direct product of rings R_1 and R_2
$\prod_{i\in I} R_i$	direct product of collection $(R_i \mid i \in I)$ of rings
R^A	direct product of collection of copies of R indexed by A
$\overline{K}$	algebraic closure of K
$R[a_1, a_2, \ldots, a_n]$	ring generated by R and $a_1, a_2, \ldots, a_n$
$K(a_1, a_2, \ldots, a_n)$	field obtained by adjoining $a_1, a_2, \ldots, a_n$ to K
$R^\times$	group of units of ring R
aR	ideal of R generated by a
$I + J$	sum of ideals I and J
$\deg f$	degree of polynomial f
$\mathrm{lc}(f)$	leading coefficient of polynomial f
$\lvert z \rvert$	absolute value of (possibly complex) number z
$a \mid b$	a divides b
$\gcd(a, b)$	set of all greatest common divisors of a and b (or greatest common divisor of a and b)
$\left(\begin{array}{c\|c} a & q_1 \ldots q_k \\ b & r_1 \ldots r_k \end{array}\right)$	division chain starting from (a, b) with quotients $q_1, \ldots, q_k$ and remainders $r_1, \ldots, r_k$
$[q_1, q_2, \ldots, q_k]$	continued fraction with coefficients $q_1, q_2, \ldots, q_k$
${}_kR_\alpha$	α-th step of transfinite construction of R as a k-stage Euclidean ring
$(M_i \mid i \in I)$, $(M_i)_{i\in I}$	collection of sets indexed by I
$\mathrm{card}(S)$	cardinality of S
$\varphi : a \mapsto b$	φ maps a to b
$\varphi \circ \psi$	composition of maps ψ and φ

Introduction

Given a Euclidean domain D with a Euclidean norm $\varphi : R \to \omega$, one usually proves that every pair of elements a, b has a greatest common divisor by carrying out a division algorithm which produces a series of equations of the form

$$(*) \qquad \begin{aligned} a &= q_1 b + r_1 \\ b &= q_2 r_1 + r_2 \\ &\vdots \\ r_{k-2} &= q_k r_{k-1} + 0 \end{aligned}$$

and making the observation that r_{k-1} is the greatest common divisor in question. The significance of the Euclidean norm in this process is to ensure that such a series exists — in particular, one chooses the q_i's so that the remainders r_i satisfy $\varphi(r_1) < \varphi(b), \varphi(r_2) < \varphi(r_1), \ldots$, which is possible by definition of Euclidean norm and eventually leads to the last remainder being 0 (since $\varphi(0) < \varphi(r)$ for every nonzero r).

Regarding this purpose, the condition on φ can be, however, significantly relaxed — if φ were to satisfy that for given $a, b \in D$ with $b \neq 0$, there exist q_1, q_2 such that

$$(**) \qquad a = q_1 b + r_1, \quad b = q_2 r_1 + r_2, \quad \varphi(r_2) < \varphi(b),$$

the existence of a series of the form $(*)$ is still ensured (even though the values of $\varphi(r_1), \varphi(r_3), \ldots$ can be arbitrarily large). This observation was made by G. E. Cooke in [7], where a norm satysfying $(**)$ is called 2-stage Euclidean. An obvious modification leads to the concept of k-stage Euclidean norm and further to the concept of ω-stage Euclidean norm, which can be viewed as somewhat 'limit case' of the k-stage Euclidean notion. A domain admitting a k-stage Euclidean norm is called k-stage Euclidean (for $k \leq \omega$).

Another approach was chosen by B. Bogaut in [2], where the concept of Euclidean norm is generalized to the quasi-Euclidean norm, a map $\psi : R^2 \to \omega$ satisfying the condition that for every $(a, b) \in R^2$ with $b \neq 0$, there exists $q \in R$ such that $\psi(b, a - qb) < \psi(a, b)$. Existence of such norm implies the existence of $(*)$ for every pair (a, b) as well. A ring R which can be equipped with such a norm is called quasi-Euclidean. As it turns out, quasi-Euclidean domains are exactly ω-stage Euclidean domains.

This thesis contains an overview of the basic known facts about k-stage Euclidean and quasi-Euclidean rings as well as new results, concerning mostly the k-stage Euclidean rings. Some of the new results provide a way of constructing new k-stage Euclidean rings and domains from other k-stage Euclidean rings. A class of elementary new examples of 3-stage Euclidean domains is constructed and some other (counter-)examples from literature (in particular, [3],[7],[9]) are presented.

Although the main focus is on the case of integral domains, we do not restrict our attention to those entirely — the theory of k-stage Euclidean rings can be developed without greater difficulities even for rings with zero divisors (as shown in [3]). Moreover, k-stage Euclidean rings with zero divisors provide an interesting comparison to the case of integral domains, as well as a new perspective on the theory.

In chapter 1, definitions of k-stage Euclidean and quasi-Euclidean ring are given and basic properties of these rings are established. The focus is on the relation to the existence of greatest common divisors and establishing the concept of the smallest k-stage Euclidean norm on a k-stage Euclidean ring.

In chapter 2, several characterizations of quasi-Euclidean and k-stage Euclidean rings are given. The characterization of k-stage Euclidean rings is a new result and has immedate consequences, which are investigated in this chapter as well.

Chapter 3 contains several examples of k-stage Euclidean and quasi-Euclidean rings which are not Euclidean as well as some counterexamples to possible conjectures about k-stage Euclidean and quasi-Euclidean rings. A special attention recieves the ring of integers of the number field $\mathbb{Q}(\sqrt{-19})$. In [7], Cooke states that the ring is not quasi-Euclidean and refers to the proof of P. M. Cohn in [6] that the ring is not GE_2, a related notion to the quasi-Euclidean notion. We present a more straightforward proof of the fact.

Finally, in Appendix we list some technical lemmas which are used in Chapter 3. These are presented separately for better readability of Chapter 3.

Preliminaries

Throughout this thesis, all rings are commutative rings with unit (and satisfying inequality $0 \neq 1$).

Natural numbers are defined via the standard construction in Zermelo-Fraenkel set theory — that is, natural numbers are considered to be finite ordinals (and thus, 0 is a natural number). The set of all natural numbers is denoted by ω (in particular, $k < \omega$ denotes $k \in \omega$, i.e. k is a natural number).

The following definition of Euclidean ring is used:

Definition. A ring R is said to be *Euclidean* if there exists an ordinal γ and a map $\varphi : R \to \gamma$ satisfying

1) $\varphi(a) = 0$ if and only if $a = 0$, and

2) for every $a \in R$ and every nonzero $b \in R$, there exists $q \in R$ such that $\varphi(a - qb) < \varphi(b)$.

Such a map is further called a *Euclidean norm* on R.

This definition differs from the standard one in the codomain (i.e. the tagret set) of the Euclidean norm — most commonly, the codomain of a Euclidean norm is required to be ω (we call such norms *finite-valued*).

Using this definition, we obtain exactly the same class of rings as studied by P. Samuel in [14]. This class is strictly larger than the one given by the standard definition — in [10], J.-J. Hiblot provided an example of a domain which can be equipped with a Euclidean norm with its range in an ordinal larger than ω but not with a finite-valued Euclidean norm. However, the most important properties of Euclidean rings hold even for Euclidean rings defined in this manner — namely, Euclidean rings are principal and every Euclidean domain is a UFD. Proofs of these statements are analogous to the usual ones.

Consider a ring R and $a, b \in R$. A common divisor c of a and b with the property that for every common divisor d of a and b, $d \mid c$ (that is, d divides c), is called a *greatest common divisor* of a and b. Denote $\gcd(a, b)$ the set of all greatest common divisors of a and b. It follows that for $c_1, c_2 \in \gcd(a, b)$, c_1 and c_2 are associates[1]. Following the usual convention, we write $c = \gcd(a, b)$ instead of $c \in \gcd(a, b)$ (and, sometimes, refer to $\gcd(a, b)$ as an element of R — for example, note that $\gcd(a, b)R$ is a uniquely determined ideal of R provided that there exists a greatest common divisor of a and b) — that is, $c_1 = \gcd(a, b), c_2 = \gcd(a, b)$ does not imply $c_1 = c_2$, however, it does imply that c_1 and c_2 are associates.

Given an integral domain D and its fraction field K, we consider K to be its own fraction field as well. In particular, we someties write (c/d) instead of cd^{-1} for $c,d \in K, d \neq 0$ (which is a fairly common convention if D is the ring of integers and K are the rationals).

We use the following usual abbreviations: PID for principal ideal domain, and UFD for unique factorization domain.

The degree of a zero polynomial is considered to be -1.

[1]For rings with zero divisors, we adopt terminology used by D. D. Anderson and S. Valdes-Leon in [1], that is, c_1 and c_2 are associates if $c_1 \mid c_2$ and $c_2 \mid c_1$, or, equivalently, $c_1R = c_2R$.

1. Definitions and basic properties

1.1 Notions of k-stage Euclidean and quasi-Euclidean ring

Although the notion of k-stage Euclidean ring was originally proposed by Cooke in [7], we adopt a slightly generalised version used by C.-A. Chen and M.-G. Leu in [3] (the generalisation is analogous to the generalisation of Euclidean ring made by Samuel).

Definition 1.1. Let R be a ring and $a, b \in R$.
A series of equations

$$(C) \qquad \begin{aligned} a &= q_1 b + r_1 \\ b &= q_2 r_1 + r_2 \\ r_1 &= q_3 r_2 + r_3 \\ &\vdots \\ r_{k-2} &= q_k r_{k-1} + r_k, \end{aligned}$$

where $q_i, r_i \in R, i = 1, 2, \ldots, k$, is called a *$k$-stage division chain* starting from the pair (a, b). We will also denote such chain (C) as

$$\left(\begin{array}{c|c} a & q_1 \ldots q_k \\ b & r_1 \ldots r_k \end{array} \right).$$

The number k is called the *length* of divison chain (C) (denoted by $l(C)$). A chain (C) is called *terminating* if $r_k = 0$. Given such division chain, b is called 0-th remainder and denoted as r_0.

Definition 1.2. Let R be a ring and γ be an ordinal. A map $\varphi : R \to \gamma$ is called a *norm* on R if for every $r \in R$, $\varphi(r) = 0$ if and only if $r = 0$.
Let k be a positive integer. We say that a norm $\varphi : R \to \gamma$ is *k-stage Euclidean* if for every pair $(a, b) \in R^2$ with $b \neq 0$, there exists $n \leq k$ and an n-stage division chain $\left(\begin{array}{c|c} a & q_1 \ldots q_n \\ b & r_1 \ldots r_n \end{array} \right)$ starting from (a, b) such that $\varphi(r_n) < \varphi(b)$.

We say that a norm $\varphi : R \to \gamma$ is *ω-stage Euclidean* if for every pair $(a, b) \in R^2$ with $b \neq 0$, there exists a positive integer n and an n-stage division chain $\left(\begin{array}{c|c} a & q_1 \ldots q_n \\ b & r_1 \ldots r_n \end{array} \right)$ starting from (a, b) such that $\varphi(r_n) < \varphi(b)$.
We say that R is *k-stage Euclidean* if there exists a k-stage Euclidean norm on R. We say that R is *quasi-Euclidean* (or *ω-stage Euclidean*[1]) if there exists an ω-stage Euclidean norm on R.

Remark 1.3. Clearly, the definition of 1-stage Euclidean norm is equivalent to the definition of Euclidean norm (given in Preliminaries). Therefore, 1-stage Euclidean domains are exactly Euclidean domains.

It is also obvious that for $k \geq n$, every n-stage Euclidean norm is also k-stage Euclidean and also ω-stage Euclidean. Hence every n-stage Euclidean ring is k-stage Euclidean for every $k \geq n$ and also quasi-Euclidean.

[1] This notation is used in formulations such as 'k-stage Euclidean ring for $k \leq \omega$'.

Remark 1.4. Two division chains $\left(\begin{array}{c|c} a & q_1 \dots q_k \\ b & r_1 \dots r_k \end{array} \right)$ and $\left(\begin{array}{c|c} a' & q'_1 \dots q'_k \\ b' & r'_1 \dots r'_l \end{array} \right)$ are said to be *equivalent* if $a = a', b = b'$ and $r_k = r'_l$.

Let R be a ring and $a, b, q, r \in R$ elements such that $a = qb + r$. Then the following pair of equalities holds:

$$a = (q+1)b + (r-b),$$
$$b = (-1)(r-b) + r.$$

As a consequence (by induction on k), we infer that every n-stage division chain is equivalent to some k-stage division chain for every $k \geq n$.

This leads to an alternative definition of k-stage Euclidean norm:
A norm $\varphi : R \to \gamma$ is k-stage Euclidean iff for every $(a, b) \in R^2$ with $b \neq 0$, there exists a k-stage division chain $\left(\begin{array}{c|c} a & q_1 \dots q_k \\ b & r_1 \dots r_k \end{array} \right)$ such that $\varphi(r_k) < \varphi(b)$.

This observation simplifies formulation of most proofs and definitions concerning k-stage Euclidean rings and therefore is used throughout this thesis without further mentioning.

1.2 Relation to divisibility in a domain

The basic motivation behind the concept of k-stage Euclidean and quasi-Euclidean ring is to obtain a class of rings with the property that for every pair of elements, there exists its greatest common divisor and, moreover, it can be (at least theoretically) obtained via a division algorithm similar to the classical Euclidean algorithm. Although in this section the main focus is on the case of integral domains, some results are useful in the case of rings with zero divisors as well and are therefore stated more generally.

Proposition 1.5 ([7]). *Let R be a ring and let $\left(\begin{array}{c|c} a & q_1 \dots q_{n-1}\, q_n \\ b & r_1 \dots r_{n-1}\, 0 \end{array} \right)$ be a terminating division chain. Then $r_{n-1} = \gcd(a, b)$.*

Proof. For $s, t \in R$, denote $\operatorname{cd}(s, t)$ the set of all common divisors of s and t (note that $\gcd(s, t)$ depends only on $\operatorname{cd}(s, t)$). Consider the equation $c = qd + r$. Then $\operatorname{cd}(c, d) = \operatorname{cd}(d, r)$. Clearly, $(x \mid c \;\&\; x \mid d)$ implies $(x \mid d \;\&\; x \mid r)$, since $r = c - qd$. On the other hand, if $y \mid d$ and $y \mid r$, then $y \mid c$, since $c = qd + r$.

By induction on $k \leq n$, we infer that

$$\operatorname{cd}(a, b) = \operatorname{cd}(b, r_1) = \cdots = \operatorname{cd}(r_i, r_{i+1}) = \cdots = \operatorname{cd}(r_{k-1}, r_k)\,.$$

It follows that $\gcd(a, b) = \gcd(r_{n-1}, 0) = r_{n-1}$. □

Proposition 1.6 ([7]). *Let R be a quasi-Euclidean ring. Then for every pair $(a, b) \in R^2$ with $b \neq 0$, there exists a terminating divison chain starting from (a, b).*

Proof. Let φ be an ω-stage Euclidean norm on R and consider $(a, b) \in R^2$ with $b \neq 0$. Recursively define $(a_n, b_n) \in R^2$. Set $(a_0, b_0) = (a, b)$. If (a_{n-1}, b_{n-1}) has been defined and $b_{n-1} \neq 0$, then there exists $k_n < \omega$ and a k_n-stage division chain

$$\left(\begin{array}{c|c} a_n & q_1^n \dots q_{k_n}^n \\ b_n & r_1^n \dots r_{k_n}^n \end{array} \right)$$

with $\varphi(r^n_{k_n}) < \varphi(b_{n-1})$. In this case, set $(a_n, b_n) = (r^n_{k_n-1}, r^n_{k_n})$.

We claim that this process stops at some $N < \omega$. Assume the contrary, i.e. $((a_n, b_n) \mid n < \omega)$ is well-defined and for every $n < \omega$, $\varphi(b_n) > 0$. Then $\{\varphi(b_n) \mid n < \omega\}$ is a nonempty subset of γ and thus it contains its smallest element α. Let k be an integer such that $0 < \varphi(b_k) = \alpha$. Then $\varphi(b_{k+1}) < \varphi(b_k) = \alpha$, a contradiction. Therefore there exists $N < \omega$ such that $b_N = 0$.

Thus, by concatenation of the considered division chains, a terminating division chain

$$\left(\begin{array}{c|c} a & q^1_1 \dots q^1_{k_1}\, q^2_1 \dots q^2_{k_2} \dots \quad \dots q^N_{k_N} \\ b & r^1_1 \dots r^1_{k_1}\, r^2_1 \dots r^2_{k_2} \dots \quad \dots\ 0 \end{array}\right)$$

is obtained. □

Corollary 1.7. *Every quasi-Euclidean domain R is a GCD domain. Moreover, every quasi-Euclidean ring is Bézout (that is, every finitely generated ideal is principal).*

Proof. It is enough to show that for $a, b \in R$, $aR + bR = (\gcd(a, b))R$. Since the case $b = 0$ is trivial, we can assume $b \neq 0$. Then there exist a positive integer k and a k-stage division chain $\left(\begin{array}{c|c} a & q_1 \dots q_{k-1}\, q_k \\ b & r_1 \dots r_{k-1}\ 0 \end{array}\right)$ where, by Proposition 1.5, $r_{k-1} = \gcd(a, b)$.

By induction on $n \leq k$, we show that there exists $\alpha, \beta \in R$ satisfying the equation

$$\alpha a + \beta b = r_{n-1}.$$

Case $n = 1$ is trivial, since $r_0 = b$, and the case $n = 2$ follows directly from the equation $a = q_1 b + r_1$. Consider $2 < n \leq k$ and assume that there exists $\alpha', \beta', \alpha'', \beta'' \in R$ such that

$$\begin{aligned} \alpha'' a + \beta'' b &= r_{n-3}, \\ \alpha' a + \beta' b &= r_{n-2}. \end{aligned}$$

Then from $r_{n-3} = q_{n-1} r_{n-2} + r_{n-1}$ we get

$$\begin{aligned} r_{n-1} = r_{n-3} - q_{n-1} r_{n-2} &= \alpha'' a + \beta'' b - q_{n-1}(\alpha' a + \beta' b) \\ &= (\alpha'' - q_{n-1}\alpha')a + (\beta'' - q_{n-1}\beta')b. \end{aligned}$$

For $n = k$, this gives us $\alpha, \beta \in R$ such that $\alpha a + \beta b = \gcd(a,b)$, which implies $aR + bR \supseteq (\gcd(a, b))R$. The other inclusion is trivial. □

A domain D is called a *uniqe factorization domain* or a *UFD* if every nonzero nonunit element $x \in D$ can be written as $x = p_1 p_2 \dots p_k$, where p_i are irreducible elements, and this decomposition is unique up to order and associated elements. Recall that D is a UFD if and only if D is a GCD domain and every nonzero nonunit element can be written as a product of irreducible elements. The following statement is a direct consequence of this fact.

Corollary 1.8 ([7])**.** *Let D be a quasi-Euclidean domain such that every nonzero nonunit element can be written as a product of irreducible elements. Then D is a UFD.* □

1.3 Further properties

Consider a k-stage Euclidean ring R (where $k \leq \omega$) and a nonempty family $(\varphi_i : R \to \gamma_i \mid i \in I)$ of k-stage Euclidean norms on R. Then by setting $\gamma = \sup_{i \in I} \gamma_i$, we can treat the norms φ_i as $\varphi_i : R \to \gamma$ (since γ_i is an initial segment of γ). This allows us to partially order $(\varphi_i \mid i \in I)$ pointwise — that is, $\varphi_i < \varphi_j$ iff for every $a \in R$, $\varphi_i(a) \leq \varphi_j(a)$ and there exists $b \in R$ such that $\varphi_i(b) < \varphi_j(b)$. This raises the natural question whether there exists *the smallest k-stage Euclidean norm on R* — that is, a k-stage Euclidean norm τ such that given an arbitrary k-stage Euclidean norm φ on R, $\tau \leq \varphi$.

Lemma 1.9 ([3]). *Let R be a ring and k a positive integer or $k = \omega$. Let $(\varphi_i : R \to \gamma \mid i \in I)$ be a nonempty collection of k-stage Euclidean norms on R. Define $\varphi(a) = \min_{i \in I} \varphi_i(a)$ for every $a \in R$. Then $\varphi : R \to \gamma$ is also a k-stage Euclidean norm.*

Proof. Clearly, $\varphi(r) = 0$ iff there exists $i \in I$ such that $\varphi_i(r) = 0$. Since φ_i is a k-stage Euclidean norm, this occurs if and only if $r = 0$.

Consider $(a, b) \in R^2$ with $b \neq 0$ and $i \in I$ such that $\varphi_i(b) = \varphi(b)$. Then there exists $n \leq k$ (or $n < \omega$, if $k = \omega$) and a division chain $\left(\begin{array}{c|c} a & q_1 \dots q_n \\ b & r_1 \dots r_n \end{array} \right)$ with $\varphi_i(r_n) < \varphi_i(b)$. Then $\varphi(r_n) \leq \varphi_i(r_n) < \varphi_i(b) = \varphi(b)$. $\square$

Proposition 1.10. *Given a k-stage Euclidean (resp. quasi-Euclidean) ring R, there exists the smallest k-stage Euclidean (resp. ω-stage Euclidean) norm on R.*

Proof. Let $k \leq \omega$ and R be k-stage Euclidean. Consider a cardinal κ such that $\mathrm{card}(R) < \kappa$. By Lemma 1.9, there exists the smallest k-stage Euclidean norm τ among the collection of all k-stage Euclidean norms on R with range contained in κ. Consider an arbitrary k-stage Euclidean norm $\varphi : R \to \gamma$. Then the image $\varphi(R)$ of φ is a well-ordered set with $\mathrm{card}(\varphi(R)) < \kappa$, hence its order type α is an ordinal satisfying $\alpha < \kappa$. If we denote $\theta : \varphi(R) \to \alpha$ the order isomorphism, then the composition $\theta \circ \varphi : R \to \alpha$ is clearly a k-stage Euclidean norm on R such that $\tau \leq \theta \circ \varphi \leq \varphi$. $\square$

The following two propositions show that for given $k < \omega$, the class of k-stage Euclidean rings are closed under certain constructions — namely, under finite direct products and under taking the quotient ring [2].

Proposition 1.11. *Let R be a k-stage Euclidean ring, where k is a positive integer, and $I \subseteq R$ its proper ideal. Then R/I is k-stage Euclidean.*

Proof. Denote $\varphi : R \to \gamma$ a k-stage Euclidean norm on R. For $a + I \in R/I$, set

$$\overline{\varphi}(a + I) = \min\{\varphi(a') \mid a' + I = a + I\}.$$

Then $\overline{\varphi} : R/I \to \gamma$ is k-stage Euclidean norm. Indeed, $\overline{\varphi}(a + I) = 0$ if and only if $a + I = 0 + I$. Let $a + I, b + I$ be arbitrary cosets with $b + I \neq 0 + I$. Choose

[2]Proposition 1.11 was independently proven by Chen and Leu in [4], which was published while this thesis was being written.

the representant b such that $\varphi(b) = \overline{\varphi}(b+I)$. Then there exist a k-stage division chain

$$\left(\begin{array}{c|c} a & q_1 \dots q_k \\ b & r_1 \dots r_k \end{array} \right)$$

in R with $\varphi(r_k) < \varphi(b)$. This division chain induces a k-stage division chain

$$\left(\begin{array}{c|c} a+I & q_1+I \dots q_k+I \\ b+I & r_1+I \dots r_k+I \end{array} \right)$$

in R/I with $\overline{\varphi}(r_k+I) \leq \varphi(r_k) < \varphi(b) = \overline{\varphi}(b+I)$. □

Proposition 1.12. *Let R_1 and R_2 be rings and k be a positive integer, $k \geq 2$. Then $R_1 \times R_2$ is k-stage Euclidean if and only if both R_1 and R_2 are k-stage Euclidean.*

Proof. Both $R_1 \times \{0\}$ and $\{0\} \times R_2$ are ideals in $R_1 \times R_2$. Thus, by Proposition 1.11, if $R_1 \times R_2$ is k-stage Euclidean, so are $R_1 \simeq (R_1 \times R_2)/(\{0\} \times R_2)$ and $R_2 \simeq (R_1 \times R_2)/(R_1 \times \{0\})$.

Conversely, consider k-stage Euclidean norms $\varphi_i : R_i \to \gamma,\ i = 1, 2$ on R_1, R_2 respectively and for $(a^1, a^2) \in R_1 \times R_2$, set

$$\psi(a^1, a^2) = \max\{\varphi_1(a^1), \varphi_2(a^2)\}\,.$$

Then $\psi : R_1 \times R_2 \to \gamma$ is a k-stage Euclidean norm. Clearly $\psi(a^1, a^2) = 0$ iff $\varphi_1(a^1) = 0 = \varphi_2(a^2)$, which is equivalent to $(a^1, a^2) = (0, 0)$. Consider $(a^1, a^2), (b^1, b^2) \in R_1 \times R_2$ such that $(b^1, b^2) \neq (0, 0)$.

If both b_1 and b_2 are not 0, consider k-stage division chains

$$\left(\begin{array}{c|c} a^1 & q_1^1 \dots q_k^1 \\ b^1 & r_1^1 \dots r_k^1 \end{array} \right), \quad \left(\begin{array}{c|c} a^2 & q_1^2 \dots q_k^2 \\ b^2 & r_1^2 \dots r_k^2 \end{array} \right)$$

in R_1, R_2 respectively, such that $\varphi_i(r_k^i) < \varphi_i(b^i)$, $i = 1, 2$.

Suppose that $b^1 \neq 0, b^2 = 0$. Then there exists a k-stage division chain

$$\left(\begin{array}{c|c} a^1 & q_1^1 \dots q_k^1 \\ b^1 & r_1^1 \dots r_k^1 \end{array} \right)$$

and a 2-stage division chain

$$\begin{aligned} a^2 &= 0 \cdot 0 + a^2 \\ 0 &= 0a^2 + 0 \ . \end{aligned}$$

As stated in Remark 1.4, there exists an equivalent k-stage division chain

$$\left(\begin{array}{c|c} a^2 & q_1^2 \dots q_{k-1}^2\, q_k^2 \\ b^2 & r_1^2 \dots r_{k-1}^2\ 0 \end{array} \right).$$

The case $b^1 = 0, b^2 \neq 0$ can be treated similarly.

In either case, by combining the considered k-stage division chains in R_1 and R_2, we obtain a k-stage division chain in $R_1 \times R_2$

$$\left(\begin{array}{c|c} (a^1, a^2) & (q_1^1, q_1^2) \dots (q_k^1, q_k^2) \\ (b^1, b^2) & (r_1^1, r_1^2) \dots (r_k^1, r_k^2) \end{array} \right)$$

such that $\psi(r_k^1, r_k^2) = \max\{\varphi_1(r_k^1), \varphi_2(r_k^2)\} < \max\{\varphi_1(b^1), \varphi_2(b^2)\} = \psi(b^1, b^2)$, which completes the proof. $\square$

The condition $k \geq 2$ is not essential for the statement. In [3], a different proof of Proposition 1.12, involving the case $k = 1$, is given (alternatively, the case $k = 1$ can be found in [14]). On the other hand, the above proof shows that for $k \geq 2$, the codomain of k-stage Euclidean norm on $R_1 \times R_2$ does not need to exceed maximum of the codomains of k-stage Euclidean norms on R_1, R_2 respectively. As a special case it follows that the product of two rings equipped with finite-valued k-stage Euclidean norms can be again equipped with a finite-valued k-stage Euclidean norm for $k \geq 2$. This, however, does not hold for $k = 1$. In [14], Samuel shows that the ring $\mathbb{Z} \times \mathbb{Z}$ cannot be equipped with finite-valued Euclidean norm, although the ring $\mathbb{Z}$ can.

Note that the assertions of Proposition 1.12 and Proposition 1.11 hold in the case of quasi-Euclidean rings (that is, in the case $k = \omega$) as well and they can be proved using the same proof with slight differences in notation (namely, instead of k-stage division chains one should consider n-stage division chains for some $n < \omega$; in the case of Proposition 1.12, Remark 1.4 is used to modify the considered pair of division chains to respectively equivalent pair of division chains of the same length).

Finally, by induction and Proposition 1.12 it easily follows that the class of k-stage Euclidean rings is closed under finite direct products (where $k \leq \omega$), with an obvious generalisation of the observation about finite-valued norms for $k \geq 2$.

2. Characterizations

The goal of this chapter is to provide conditions equivalent to the k-stage Euclidean and quasi-Euclidean conditions, respectively, with the additional requirement that the conditions do not use the concept of a norm. These characterizations allow one to work with k-stage Euclidean and quasi-Euclidean rings more effectively and eventually lead to new results (which are summarized in Section 2.4).

We start with the quasi-Euclidean case, where this is achieved quite easily.

2.1 Alternative definitions of quasi-Euclidean ring

Proposition 2.1 ([2],[7],[9]). *Let R be a ring. Then the following conditions are equivalent:*

(i) R is quasi-Euclidean.

(ii) For every pair $(a,b) \in R^2$ with $b \neq 0$, there exists a terminating division chain starting from (a,b).

(iii) There exists a map $\psi : R^2 \to \omega$ with the property that for every $(a,b) \in R^2$ with $b \neq 0$, there exists $q \in R$ such that $\psi(b, a-qb) < \psi(a,b)$.

(iv) There exists a partial order $\leq$ on R^2 satisfying the descending chain condition such that for every $(a,b) \in R^2$ with $b \neq 0$, there exists $q \in R$ with $(b, a-qb) < (a,b)$.

Proof. *(i)→(ii)* is proved in Proposition 1.6.

(ii)→(i): Let φ be an arbitrary norm on R. For $(a,b) \in R^2$ with $b \neq 0$, consider a terminating division chain $\left(\begin{array}{c|c} a & q_1 \ldots q_{k-1}\, q_k \\ b & r_1 \ldots r_{k-1}\, 0 \end{array}\right)$. Then $\varphi(0) = 0 < \varphi(b)$. Hence φ is ω-stage Euclidean.

(ii)→(iii): Define ψ as follows: For $(a,b) \in R^2$ with $b \neq 0$, consider the set $TC(a,b)$ of all terminating divison chains starting from (a,b). Then put

$$\psi(a,b) = \begin{cases} \min\{l(C) \mid (C) \in TC(a,b)\} & \text{if } b \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $(a,b) \in R^2$ with $b \neq 0$, denote $k = \psi(a,b)$ (clearly, $k > 0$). By definition of ψ, there exists a terminating k-stage division chain $\left(\begin{array}{c|c} a & q_1 \ldots q_{k-1}\, q_k \\ b & r_1 \ldots r_{k-1}\, 0 \end{array}\right)$. Then $\psi(b, a-q_1b) < \psi(a,b)$: Clearly $r_1 = a - q_1 b$. If $r_1 = 0$, then $\psi(b, a-q_1b) = 0$ and $\psi(a,b) = 1 > 0$. Otherwise $\left(\begin{array}{c|c} b & q_2 \ldots q_{k-1}\, q_k \\ r_1 & r_2 \ldots r_{k-1}\, 0 \end{array}\right)$ is a terminating division chain of length $k-1$. Therefore, $\psi(b, a-q_1b) \leq k-1 < k = \psi(a,b)$.

(iii)$\to$*(iv)*: Set

$$(a,b) \leq (c,d) \overset{\text{def}}{\leftrightarrow} (\,\psi(a,b) < \psi(c,d) \text{ or } (a,b) = (c,d)\,).$$

It is obvious that $\leq$ is a partial order on R^2 with desired properties.

(iv)$\to$*(ii)*: Consider $(a,b) \in R^2$ with $b \neq 0$. We define $(a_n, b_n) \in R^2$ recursively. Set $(a_0, b_0) = (a,b)$. Assume that (a_{n-1}, b_{n-1}) is defined and $b_{n-1} \neq 0$. This implies the existence of $q_n \in R$ such that $(b_{n-1}, a_{n-1} - q_n b_{n-1}) < (a_{n-1}, b_{n-1})$. Then set $(a_n, b_n) = (b_{n-1}, a_{n-1} - q_n b_{n-1})$.

Since $\leq$ satisfies the descending chain condition, this process clearly stops at some $N < \omega$, that is, $b_N = 0$. Thus, we obtain a terminating division chain

$$\left(\begin{array}{c|c} a & q_1 \ldots q_{N-1}\, q_N \\ b & r_1 \ldots r_{N-1}\; 0 \end{array} \right).$$

□

The condition *(iii)* in the above proposition is the original definition of quasi-Euclidean ring used by Bogaut in [2]. The definition used in this thesis is due to Cooke and was originally referred to as ω-stage Euclidean ring.

Note that replacing ω by a possibly larger ordinal γ in *(iii)* would not be an essential generalisation, since such condition implies *(ii)* (using the same proof), which is equivalent to *(iii)*.

From the proof of the implication *(ii)*$\to$*(i)* it follows that given a quasi-Euclidean ring R, *any* norm on R is ω-stage Euclidean, which, in a way, makes the concept of ω-stage Euclidean norm superfluous[1].

2.2 Relation to continued fractions

In the case of integral domains, there is a natural way of connecting the quasi-Euclidean condition to the relation between the domain and its fraction field. This is done via continued fractions, which are closely related to division chains.

Definition 2.2. Let D be a domain and K its fraction field. A *continued fraction with coefficients in D* is an element of K denoted by $[q_1, q_2, \ldots, q_k]$, where $q_i \in D,\ i = 1, 2, \ldots k$, and defined recursively as follows:

1) For $q_1 \in D$, set $[q_1] = q_1$.

2) Suppose that $[q_2, \ldots, q_k]$ is defined and nonzero. Then set

$$[q_1, q_2, \ldots, q_k] = q_1 + \frac{1}{[q_2, \ldots, q_k]}.$$

The following lemma makes the declared connection precise[2].

[1]However, it is important to emphasize that this is not the case of k-stage Euclidean norms for $k < \omega$.

[2]The relation of quasi-Euclidean domain to the continued fractions in its fraction field was described by Cooke. However, the case of possible division by zero was not treated and therefore we state the result in a different, more precise way and using a different proof. See [7] for comparison.

Lemma 2.3. *Consider a domain D and its elements $a, b \in D, b \neq 0$. Denote K the fraction field of D.*

(1) Suppose that there exists a terminating division chain $\left(\begin{array}{c|c} a & q_1 \dots q_{k-1}\, q_k \\ b & r_1 \dots r_{k-1}\, 0 \end{array} \right)$ *in D such that $r_i \neq 0,\ i = 1, 2, \dots, k-1$. Then $(a/b) = [q_1, \dots, q_k]$ in K.*

(2) Conversely, if $(a/b) = [q_1, \dots, q_k]$ in K, where $q_i \in D$, $i = 1, 2, \dots, k$, then there exists a terminating division chain $\left(\begin{array}{c|c} a & q_1 \dots q_{k-1}\, q_k \\ b & r_1 \dots r_{k-1}\, 0 \end{array} \right)$ *in D such that $r_i \neq 0$, $i = 1, 2, \dots, k-1$.*

Proof. We prove both assertions by induction on k.

(1) Consider $a, b \in D, b \neq 0$ and suppose there exists a terminating division chain

$$\left(\begin{array}{c|c} a & q_1 \dots q_{k-1}\, q_k \\ b & r_1 \dots r_{k-1}\, 0 \end{array} \right)$$

with $r_1 \neq 0,\ i = 1, 2, \dots, k-1$. If $k = 1$, the statement is trivial. Assume that $k > 1$ and therefore $r_1 \neq 0$. Since $b \neq 0$, the equation

$$a = q_1 b + r_1$$

implies

$$\frac{a}{b} = q_1 + \frac{r_1}{b}.$$

Since $\left(\begin{array}{c|c} b & q_2 \dots q_{k-1}\, q_k \\ r_1 & r_2 \dots r_{k-1}\, 0 \end{array} \right)$ is a terminating division chain of the length $k-1$ starting from (b, r_1) with $r_1 \neq 0,\ i = 2, \dots, k-1$, by induction hypothesis we have

$$\frac{b}{r_1} = [q_2, \dots, q_k].$$

Hence

$$\frac{a}{b} = q_1 + \frac{r_1}{b} = q_1 + \frac{1}{\frac{b}{r_1}} = q_1 + \frac{1}{[q_2, \dots, q_k]} = [q_1, \dots, q_k].$$

(2) Assume $(a/b) = [q_1, \dots, q_k]$, where $q_i \in D, i = 1, 2, \dots, k$.

The case $k = 1$ is trivial, since $(a/b) = [q_1]$ implies $a = q_1 b$. Suppose that $(a/b) = [q_1, \dots, q_k]$ and $k > 1$. Then from the equation

$$\frac{a}{b} = [q_1, \dots, q_k] = q_1 + \frac{1}{[q_2, \dots, q_k]}$$

we obtain

$$a = q_1 b + \frac{b}{[q_2, \dots, q_k]},$$

where

$$r_1 = \frac{b}{[q_2, \dots, q_k]} = a - q_1 b$$

is a nonzero element of D. Moreover, $(b/r_1) = [q_2, \dots, q_k]$ and thus, by induction hypothesis, there exists a terminating division chain

$$\left(\begin{array}{c|c} b & q_2 \dots q_{k-1}\, q_k \\ r_1 & r_2 \dots r_{k-1}\, 0 \end{array} \right)$$

with $r_i \neq 0,\ i = 2, 3, \dots, k-1$. It follows that $\left(\begin{array}{c|c} a & q_1 \dots q_{k-1}\, q_k \\ b & r_1 \dots r_{k-1}\ 0 \end{array} \right)$ is a terminating division chain with $r_i \neq 0,\ i = 1, 2, \dots, k-1$. ☐

Proposition 2.4. *An integral domain D is quasi-Euclidean if and only if every element of its fraction field can be expressed as a continued fraction with coefficients in D.*

Proof. Using Lemma 2.3, we infer that the condition is sufficient. Conversely, assume that D is quasi-Euclidean and consider $(a, b) \in D^2$ with $b \neq 0$. By Proposition 2.1, there exists a terminating division chain starting form (a, b). Choose one of the terminating division chains $\left(\begin{array}{c|c} a & q_1 \dots q_{k-1}\, q_k \\ b & r_1 \dots r_{k-1}\ 0 \end{array} \right)$ such that the length k is minimal. Then the condition $r_i \neq 0,\ i = 1, 2, \dots, k-1$ is staisfied and thus, $(a/b) = [q_1, \dots, q_k]$ by Lemma 2.3. ☐

2.3 Transfinite construction of k-stage Euclidean ring

In the case of k-stage Euclidean rings, where k is a positive integer, finding a norm-independent characterization is more complicated. The basic idea is to use the fact that every k-stage Euclidean ring R admits its smallest k-stage Euclidean norm $\tau : R \to \gamma$. Description of the level sets $(\{b \in R \mid \tau(b) \leq \alpha\})_{\alpha<\gamma}$ by the means of transfinite recursion leads to a construction which can be performed in a general ring and, in a sense, measures 'how far is the ring from being k-stage Euclidean'[3].

Throughout the rest of the chapter, k denotes a fixed positive integer, if not specified otherwise.

Definition 2.5 (transfinite construction of k-stage Euclidean ring)**.** Let R be a ring and γ be a sufficiently large ordinal. For a positive integer k, we define a sequence $({}_kR_\alpha)_{\alpha<\gamma}$ of subsets of R as follows:

1) We set ${}_kR_0 = \{0\}$.

2) Assume that $\alpha < \gamma$ and for every $\beta < \alpha$, ${}_kR_\beta$ is defined. Set

$$ {}_kR'_\alpha = \bigcup_{\beta<\alpha} {}_kR_\beta \,. $$

For $b \in R$, denote

$$ {}_kR^b_\alpha = \{r_1 \in R \mid \exists r_k \in {}_kR'_\alpha\ \exists r_2, r_3, \dots, r_{k-1} \in R : \\ r_1 \mid (b - r_2),\ r_2 \mid (r_1 - r_3),\ \dots,\ r_{k-1} \mid (r_{k-2} - r_k)\}. $$

Then we set

$$ {}_kR_\alpha = \{b \in R \mid \pi_b\left({}_kR^b_\alpha\right) = R/bR\}, $$

where $\pi_b : R \to R/bR$ is the canonical projection.

[3]This construction is a generalization of transfinite construction of Euclidean ring introduced by Samuel in [14] as well as transfinite construction of 2-stage Euclidean ring by Chen and Leu in [3].

Remark 2.6.

(1) It follows that the sequence $({}_kR_\alpha)_{\alpha<\gamma}$ is strictly increasing (in particular, ${}_kR_\beta \setminus {}_kR'_\beta \neq \emptyset$) until the first ordinal α such that

$$\forall \beta, \alpha \leq \beta < \gamma : {}_kR_\beta = {}_kR_\alpha.$$

Consider an ordinal γ such that $\mathrm{card}(\gamma) > \mathrm{card}(R)$. Then γ is sufficiently large in the sense that there exists $\alpha < \gamma$ such that the increase of the above sequence stops at α. This is the intended meaning of „sufficiently large" in the definition above and it is further used in this manner.

(2) Note that the condition in the definition of ${}_kR^b_\alpha$ can be equivalently restated as follows:

There exists a $(k-1)$-stage division chain $\left(\begin{array}{c|c} b & q_2 \ldots q_k \\ r_1 & r_2 \ldots r_k \end{array}\right)$ *with* $r_k \in {}_kR'_\alpha$.

It is further clear that $\pi_b(r_1) = a + bR$ if and only if $a = q_1 b + r_1$ for some $q_1 \in R$ (since, by definition, $\pi_b(r_1) = r_1 + bR$). Using these facts, we infer that $b \in {}_kR_\alpha$ if and only if for every $a \in R$, there exists a k-stage division chain $\left(\begin{array}{c|c} a & q_1 \ldots q_k \\ b & r_1 \ldots r_k \end{array}\right)$ with $r_k \in {}_kR'_\alpha$.

The following theorem shows that the transfinite construction defined in this manner corresponds to the level sets as discussed above.

Theorem 2.7. *Let R be a k-stage Euclidean ring with the smallest k-stage Euclidean norm τ. Let γ be a sufficiently large ordinal. For $\alpha < \gamma$, set*

$${}_k\overline{R}_\alpha = \{b \in R \mid \tau(b) \leq \alpha\}, \ \ {}_k\overline{R'}_\alpha = \{b \in R \mid \tau(b) < \alpha\} \ .$$

Then

$$\forall \alpha < \gamma : {}_k\overline{R}_\alpha = {}_kR_\alpha \ .$$

Proof. Clearly, ${}_k\overline{R}_0 = {}_kR_0$.

Assume that $0 < \alpha < \gamma$ and for every $\beta < \alpha$, ${}_k\overline{R}_\beta = {}_kR_\beta$. Then we have

$${}_k\overline{R'}_\alpha = \bigcup_{\beta<\alpha} {}_k\overline{R}_\beta = \bigcup_{\beta<\alpha} {}_kR_\beta = {}_kR'_\alpha \ .$$

Consider $b \in {}_k\overline{R}_\alpha$, i.e. $\tau(b) \leq \alpha$, and let a be an arbitrary element of R. Then there exists a k-stage division chain $\left(\begin{array}{c|c} a & q_1 \ldots q_k \\ b & r_1 \ldots r_k \end{array}\right)$ such that $\tau(r_k) < \tau(b) \leq \alpha$. This means that $r_k \in {}_k\overline{R'}_\alpha = {}_kR'_\alpha$, which by Remark 2.6 (2) implies that $b \in {}_kR_\alpha$. Thus, ${}_k\overline{R}_\alpha \subseteq {}_kR_\alpha$.

Suppose that there exists $c \in {}_kR_\alpha \setminus {}_k\overline{R}_\alpha$. Then we set

$$\tilde{\tau}(r) = \begin{cases} \tau(r), & r \neq c \ , \\ \alpha, & r = c \ . \end{cases}$$

We claim that $\tilde{\tau}$ is a k-stage Euclidean norm. Clearly, $\tilde{\tau}(r) = 0$ iff $\tau(r) = 0$, which is equivalent to $r = 0$, since τ is a norm. Consider $(a, b) \in R^2$ with $b \neq 0$.

We distinguish two cases:

(a) $\underline{b \neq c}$: Then, since τ is a k-stage Euclidean norm, there exists a k-stage division chain $\left(\begin{array}{c|c} a & q_1 \dots q_k \\ b & r_1 \dots r_k \end{array} \right)$ with $\tau(r_k) < \tau(b)$.
Then $\tilde{\tau}(r_k) \leq \tau(r_k) < \tau(b) = \tilde{\tau}(b)$.

(b) $\underline{b = c}$: Then $b \in {}_kR_\alpha$, therefore (by Remark 2.6 (2)) there exists a k-stage division chain $\left(\begin{array}{c|c} a & q_1 \dots q_k \\ b & r_1 \dots r_k \end{array} \right)$ with $r_k \in {}_kR'_\alpha = {}_k\overline{R'}_\alpha$, that is, $\tau(r_k) < \alpha$.
Then $\tilde{\tau}(r_k) = \tau(r_k) < \alpha = \tilde{\tau}(b)$.

Since $\tilde{\tau} < \tau$, this is a contradiction to the fact that τ is the smallest k-stage Euclidean norm on R. Therefore, ${}_k\overline{R}_\alpha = {}_kR_\alpha$. ☐

Finally, we obtain a norm-free characterization of k-stage Euclidean rings, as states the following theorem.

Theorem 2.8. *A ring R is k-stage Euclidean if and only if $\bigcup_{\alpha<\gamma} {}_kR_\alpha = R$, where γ is a sufficiently large ordinal.*

Proof. Let R be k-stage Euclidean and $\tau : R \to \gamma$ its smallest k-stage Euclidean norm. Then by Theorem 2.7 we have $R = \bigcup_{\alpha<\gamma} {}_k\overline{R}_\alpha = \bigcup_{\alpha<\gamma} {}_kR_\alpha$.

Assume that $\bigcup_{\alpha<\gamma} {}_kR_\alpha = R$. Then we define $\tau : R \to \gamma$ as follows:

$$\tau(r) = \alpha \overset{\text{def}}{\leftrightarrow} r \in {}_kR_\alpha \setminus {}_kR'_\alpha .$$

Then τ is a k-stage Euclidean norm. It is obvious that $\tau(r) = 0$ iff $r = 0$. Consider $(a, b) \in R^2$ with $b \neq 0$ and denote $\alpha = \tau(b)$. Then there exists $r_1 \in {}_kR^b_\alpha$ such that $r_1 + bR = a + bR$ (i.e. $a = r_1 + q_1 b$ for some $q_1 \in R$) and $r_2, \dots, r_k \in R$ with $r_k \in {}_kR'_\alpha$ such that

$$b - r_2 = q_2 r_1, r_1 - r_3 = q_3 r_2, \ \dots \ , r_{k-2} - r_k = q_k r_{k-1}.$$

It follows that $\left(\begin{array}{c|c} a & q_1 \dots q_k \\ b & r_1 \dots r_k \end{array} \right)$ is a k-stage division chain with $\tau(r_k) < \alpha$. ☐

2.4 Consequences of norm-free characterizations

In this section, we list some applications of the investigated characterizations. The main focus is on the case of k-stage Euclidean rings. The advantage of the characterization using the transfinite construction is that it introduces transfinite induction as a useful tool for proving statements concerning the k-stage Euclidean rings.

Proposition 2.9. *Let R be a k-stage Euclidean ring such that for every nonzero $b \in R$, R/bR is finite. Then the smallest k-stage Euclidean norm τ on R is finite-valued.*

Proof. We show that ${}_kR_\omega \setminus {}_kR'_\omega = \emptyset$. Then clearly ${}_kR'_\omega = R$ and τ is finite-valued (see Remark 2.6 (1) and Theorem 2.7).

Consider a nonzero element $b \in {}_kR_\omega$ and choose $a^1, a^2, \ldots, a^n \in R$ such that $\{a^1 + bR, a^2 + bR, \ldots, a^n + bR\} = R/bR$ (this is possible since R/bR is finite). Then there exist k-stage division chains

$$\left(\begin{array}{c|c} a^1 & q_1^1 \ldots q_k^1 \\ b & r_1^1 \ldots r_k^1 \end{array} \right), \left(\begin{array}{c|c} a^2 & q_1^2 \ldots q_k^2 \\ b & r_1^2 \ldots r_k^2 \end{array} \right), \ldots, \left(\begin{array}{c|c} a^n & q_1^n \ldots q_k^n \\ b & r_1^n \ldots r_k^n \end{array} \right)$$

such that $r_k^i \in {}_kR'_\omega,\ i = 1, 2, \ldots, n$. If we denote $\beta_i = \tau(r_k^i)$, then, using Theorem 2.7, it is easily seen that $r_k^i \in {}_kR_{\beta_i}$ and $\beta_i < \omega,\ i = 1, 2, \ldots, n$.

Finally, by putting $\beta = \max\{\beta_1, \beta_2, \ldots, \beta_n\}$, we obtain a natural number such that $\pi_b\left({}_kR^b_{\beta+1}\right) = R/bR$, which implies $b \in {}_kR_{\beta+1} \subseteq {}_kR'_\omega$. This shows that ${}_kR_\omega \subseteq {}_kR'_\omega$ which is equivalent to ${}_kR_\omega \setminus {}_kR'_\omega = \emptyset$. ☐

Recall that given a number field (i.e. an extension of $\mathbb{Q}$ of finite degree) K, the ring $\mathcal{O}_K$ of all algebraic integers contained in K has the property that for every nonzero $b \in \mathcal{O}_K$, the ring $\mathcal{O}_K/b\mathcal{O}_K$ is finite[4]. Thus, by Proposition 2.9, if $\mathcal{O}_K$ is k-stage Euclidean, then it admits a finite-valued k-stage Euclidean norm[5].

Proposition 2.10. *Let D be a quasi-Euclidean domain and $S \subseteq D$ be a multiplicatively closed subset (containing 1 and not containing 0). Then $S^{-1}D$ is quasi-Euclidean.*

Proof. This is immediate from Proposition 2.4 and the fact that the fraction field of $S^{-1}D$ is the fraction field of D (that is, the map $(a/b)/(c/d) \mapsto (ad/bc)$ between fraction fields of $S^{-1}D$ and D respectively is a field isomorphism). ☐

Proposition 2.10 shows that the class of quasi-Euclidean domains is closed under localizations. Similar assertion holds for the class of k-stage Euclidean domains, where k is a positive integer[6].

Proposition 2.11. *Let D be a k-stage Euclidean domain and $S \subseteq D$ a multiplicatively closed subset (containing 1 and not containing 0). Then $R = S^{-1}D$ is k-stage Euclidean.*

Proof. Denote τ the smallest k-stage Euclidean norm on D. It suffices to show that for $b \in {}_kD_\alpha$ and $t \in S$, $(b/t) \in {}_kR_\alpha$. Proceed by transfinite induction on α.

For $\alpha = 0$, the statement is trivial, since ${}_kD_\alpha = \{0\}, {}_kR_\alpha = \{0\}$.

Assume that $\alpha > 0$ and for every $\beta < \alpha$, the statement holds. Consider $b \in {}_kD_\alpha$, $t \in S$ and $(a/s) \in R$ an arbitrary element of R (that is, $a \in D$ and $s \in S$). Then there exists a k-stage division chain

$$(C) = \left(\begin{array}{c|c} a & q_1 \ldots q_k \\ b & r_1 \ldots r_k \end{array} \right)$$

[4]In particular, $\mathrm{card}(\mathcal{O}_K/b\mathcal{O}_K) = |N_{K/\mathbb{Q}}(b)|$, where $N_{K/\mathbb{Q}}$ denotes the field norm. For further details, see [8, p. 98, Theorem 15].

[5]Similar result, concerning the rings of integers, was provided by Cooke. In [7], Proposition 13 states that, assuming that $\mathcal{O}_K$ is quasi-Euclidean, it is k-stage Euclidean for some $k < \omega$. Since Cooke's definition of k-stage Euclidean domain requires a k-stage Euclidean norm to be finite-valued, the assertion for k follows as a consequence (however, the statement does not provide a way of controlling the size of k).

[6]Proposition 2.11 was proved independently by Chen and Leu. See [4] for a proof that does not use the transfinite construction.

such that $\tau(r_k) < \tau(b) \leq \alpha$, i.e. $r_k \in {}_kD_\beta \subseteq {}_kD'_\alpha$, where $\beta = \tau(r_k)$ (see Theorem 2.7). For k odd, denote $v = t$, $w = s$ and for k even, denote $v = s$, $w = t$. From the division chain (C), a k-stage division chain

$$\left(\begin{array}{c|cccc} \frac{a}{s} & \frac{q_1 t}{s} & \frac{q_2 s}{t} & \frac{q_3 t}{s} \cdots & \frac{q_k v}{w} \\ \frac{b}{t} & \frac{r_1}{s} & \frac{r_2}{t} & \frac{r_3}{s} \cdots & \frac{r_k}{w} \end{array} \right)$$

in R can be derived. Using the induction hypothesis on $r_k \in {}_kD_\beta$, we infer that $(r_k/w) \in {}_kR_\beta \subseteq {}_kR'_\alpha$. Hence $(b/t) \in {}_kR_\alpha$. ☐

Throughout the literature, it is still fairly common for the definition of Euclidean norm to involve the additional condition

(m) For every $a, b \in R$ with $ab \neq 0$, $\varphi(a) \leq \varphi(ab)$.

It is a well-known fact that *(m)* is not essential for the properties of Euclidean rings — for instance, in [14], Samuel shows that any Euclidean ring (defined without the requirement *(m)* on Euclidean norm) can be equipped with a norm satisfying *(m)*. In particular, the smallest Euclidean norm always satisfies *(m)*. Note that the condition *(m)* implies that associated elements are indistinguishable by the norm. The next proposition is a generalisation of this statement to the smallest k-stage Euclidean norm on a k-stage Euclidean ring [7].

Proposition 2.12. *Let R be a k-stage Euclidean ring with its smallest k-stage Euclidean norm τ. Then for every $u \in R^\times$ and every $b \in R$, $\tau(ub) = \tau(b)$.*

Proof. By transfinite induction on α, we show that for $b \in {}_kR_\alpha$ and $u \in R$, $ub \in {}_kR_\alpha$.

The case $\alpha = 0$ is trivial. Assume that $\alpha > 0$ and that the assertion holds for every $\beta < \alpha$. For $b \in {}_kR_\alpha$, a unit u and an arbitrary $a \in R$ there exists a k-stage division chain $\left(\begin{array}{c|c} au^{-1} & q_1 \ldots q_k \\ b & r_1 \ldots r_k \end{array} \right)$ with $\tau(r_k) < \tau(b) \leq \alpha$, i.e. $r_k \in {}_kR'_\alpha$. By considering the division chain

$$\left(\begin{array}{c|c} a & q_1 \ \ldots \ q_k \\ ub & ur_1 \ldots ur_k \end{array} \right)$$

and using the induction hypothesis on ur_k, we infer that $ur_k \in {}_kR'_\alpha$, hence $ub \in {}_kR_\alpha$.

In conclusion, we have that $\tau(ub) \leq \tau(b)$ for every $u \in R^\times$ and every $b \in R$. Applying this on $u^{-1} \in R^\times$ and $ub \in R$, we infer that also $\tau(b) \leq \tau(ub)$, and hence, the equality $\tau(b) = \tau(ub)$ holds. ☐

In the case of integral domains, even more can be said. The following proposition proves that the condition *(m)* holds for the smallest k-stage Euclidean norm on arbitrary k-stage Euclidean domain.

[7] More precisely, following the terminology introduced in [1], we prove this only for *strongly associated* elements.

Proposition 2.13. *Let D be a k-stage Euclidean domain with the smallest k-stage Euclidean norm $\tau : D \to \gamma$. Then for all nonzero $b, c \in D$, $\tau(b) \leq \tau(bc)$.*

Proof. We use transfinite induction on $\alpha < \gamma$ to show that for nonzero $b, c \in D$, $bc \in {}_kD_\alpha$ implies $b \in {}_kD_\alpha$.

The case $\alpha = 0$ is trivial, since D has no zero divisors.

Consider $\alpha > 0$ and suppose that for every $\beta < \alpha$, the statement holds. Let b, c be nonzero elements of D such that $bc \in {}_kD_\alpha$. For $a \in D$, there exists a k-stage division chain

$$\left(\begin{array}{c|c} ac & q_1 \ldots q_k \\ bc & r_1 \ldots r_k \end{array} \right)$$

with $r_k \in {}_kD'_\alpha$. By induction on $i \leq k$, we can see that $c \mid r_i$, i.e. there exists r'_i such that $r_i = r'_i c$. From $ac = q_1 bc + r_1$ we have

$$r_1 = ac - q_1 bc = (a - q_1 b)c\,.$$

Suppose that $r_i = r'_i c$, $r_{i-1} = r'_{i-1}c$ and consider the equation $r_{i-1} = q_{i+1}r_i + r_{i+1}$. Then

$$r_{i+1} = r_{i-1} - q_{i+1}r_i = r'_{i-1}c - q_{i+1}r'_i c = (r'_{i-1} - q_{i+1}r'_i)c\,.$$

As a conseqence, the considered division chain is of the form

$$\left(\begin{array}{c|c} ac & q_1 \;\ldots\; q_k \\ bc & r'_1 c \ldots r'_k c \end{array} \right).$$

But since D is an integral domain and $c \neq 0$, the equation $r'_{i-2}c = q_i r'_{i-1}c + r'_i c$ implies that $r'_{i-2} = q_i r'_{i-1} + r'_i$ for $i \in \{2, \ldots, k\}$, and, similarly, we have that $a = q_1 b + r'_1$. Thus, we obtain a division chain

$$\left(\begin{array}{c|c} a & q_1 \ldots q_k \\ b & r'_1 \ldots r'_k \end{array} \right).$$

If $r_k = 0$, then $0 = r_k = r'_k c$ and $r'_k = 0$ (since $c \neq 0$ and D is a domain). If $r_k \neq 0$, we have $r'_k c = r_k \in {}_kD'_\alpha$, i.e. there exists $\beta < \alpha$ such that $r'_k c \in {}_kD_\beta$ and both r'_k, c are nonzero elements. Using the induction hypothesis it follows that $r'_k \in {}_kD_\beta$. In either case, $r'_k \in {}_kD'_\alpha$. Thus, $b \in {}_kD_\alpha$, which completes the proof. ∎

3. Examples and counterexamples

3.1 Elementary examples

Example 1. Consider a field K and the ring $R = K[x, y]$, where x, y are indeterminates. Then it is a well-known fact that R is a UFD. On the other hand, R is not Bézout, hence (according to Corollary 1.7), R is not quasi-Euclidean. By Proposition 2.4, this implies that its fraction field $K(x, y)$ contains an element which cannot be expressed as a continued fraction with coefficients in R. From Lemma 2.3 and the given proof of Corollary 1.7 it is clear that (x/y) is such an element, since the finitely generated ideal $xR + yR$ is not principal (otherwise it would necessarily contain 1, which is a greatest common divisor of x and y) and thus, by the proof of Corollary 1.7, no terminating division chain starting from (x, y) exists.

In the following example, we use the fact that given arbitrary $n < \omega$, there exists a pair of (nonzero) elements $(a, b) \in \mathbb{Z}^2$ such that every terminating division chain starting from the pair (a, b) is of the length greater or equal to n. Proof of this uses a lemma given by P. Glivický and J. Šaroch in [9] and can be found in Appendix (see Corollary 2 in Appendix).

Example 2. Consider the ring $R = \mathbb{Z}^\omega$, i.e. the countable product of copies of the ring $\mathbb{Z}$.

Then R is a GCD ring, moreover, R is Bézout. To see that, consider arbitrary elements $(a_n)_{n<\omega}, (b_n)_{n<\omega} \in R$. For $n < \omega$, denote $g_n = \gcd(a_n, b_n)$ (with respect to the ring $\mathbb{Z}$). Then it is easy to see that

$$(g_n)_{n<\omega} = \gcd\left((a_n)_{n<\omega}, (b_n)_{n<\omega}\right) .$$

Since $\mathbb{Z}$ is Euclidean, it is a PID and therefore Bézout. This implies that for every $n < \omega$, there exist $\alpha_n, \beta_n \in \mathbb{Z}$ such that $\alpha_n a_n + \beta_n b_n = g_n$. It follows that

$$(\alpha_n)_{n<\omega} \cdot (a_n)_{n<\omega} + (\beta_n)_{n<\omega} \cdot (b_n)_{n<\omega} = (g_n)_{n<\omega},$$

hence R is Bézout.

Suppose now that for every $n < \omega$, (c_n, d_n) is a pair of integers with the property that every terminating divison chain starting form (c_n, d_n) is of the length greater or equal to n (and, moreover, $d_n \neq 0$). Consider the elements $(c_n)_{n<\omega}, (d_n)_{n<\omega}$. Assume that there exists a terminating division chain

$$\left(\begin{array}{c|c} (c_n)_{n<\omega} & (q_n^1)_{n<\omega} \dots (q_n^{k-1})_{n<\omega} \; (q_n^k)_{n<\omega} \\ (d_n)_{n<\omega} & (r_n^1)_{n<\omega} \dots (r_n^{k-1})_{n<\omega} \quad 0 \end{array} \right).$$

Choose arbitrary $N > k$. Then we obtain an induced terminating chain

$$\left(\begin{array}{c|c} c_N & q_N^1 \dots q_N^{k-1} \; q_N^k \\ d_N & r_N^1 \dots r_N^{k-1} \; 0 \end{array} \right)$$

of the length $k < N$, a contradiction. Thus, no terminating division chain starting from $((c_n)_{n<\omega}, (d_n)_{n<\omega})$ exists.

This is an elementary example of Bézout ring which is not quasi-Euclidean. Moreover, since the ring $\mathbb{Z}$ is Euclidean and hence k-stage Euclidean for every $k \leq \omega$, this example shows that the assertion of Proposition 1.12 cannot be extended to infinite direct products.

The following example is motivated by the effort to find a 3-stage Euclidean domain which is not 2-stage Euclidean[1].

Example 3. Let D be a Euclidean domain such that for every $a, b \in D$ with $b \neq 0$, there exists a terminating 2-stage division chain in D (in particular, this holds for any discrete valuation ring). Denote K the fraction field of D and $\varphi : D \to \gamma$ the Euclidean norm on D. Consider an order isomorphism $\theta : (\omega \times \gamma, <) \to \gamma \cdot \omega$, where $\cdot$ denotes the ordinal multiplication. Then the ring

$$R = D + xK[x] = \{f \in K[x] \mid f(0) \in D\},$$

a subring of $K[x]$, is a 3-stage Euclidean ring with respect to the norm $\psi = \theta \circ \rho$, where $\rho : D \to \omega \times \gamma$ is defined by

$$\rho(f) = (\deg f + 1, \varphi(f(0))).$$

Proof. Clearly, $\rho(f) = (0,0)$ if and only if $f = 0$.

Consider $f, g \in R$ with $g \neq 0$. We can write $f = f(0) + xf_1$, $g = g(0) + xg_1$, $f_1, g_1 \in K[x]$. We distinguish several cases:

(1) $\underline{g \text{ is constant, i.e. } g_1 = 0}$: Since D is Euclidean and $g(0) \neq 0$, there exist $q_1, r_1 \in D$ such that $f(0) = q_1 g(0) + r_1$ and $\varphi(r_1) < \varphi(g(0))$. Then

$$f = f(0) + xf_1 = q_1 g + r_1 + xf_1 = \left(q_1 + \frac{xf_1}{g}\right) g + r_1,$$

where $\left(q_1 + \frac{xf_1}{g}\right) \in R$ and $\rho(r_1) \leq (1, \varphi(r_1)) < (1, \varphi(g(0))) = \rho(g)$.

(2) $\underline{\deg g > 0}$: Denote n the degree of g. Consider the following subcases:

(a) $\underline{\deg f < n}$: Then $f = 0g + f$ with $\rho(f) < \rho(g)$.

(b) $\underline{\deg f = n}$: Consider $c \in D$ such that $cf, cg \in D[x]$. Denote $a = \mathrm{lc}(cf)$, $b = \mathrm{lc}(cg)$. Then there exist $q_1, q_2, r_1 \in D$ such that

$$\begin{aligned} a &= q_1 b + r_1, \\ b &= q_2 r_1 + 0. \end{aligned}$$

It follows that

$$\begin{aligned} f &= q_1 g + \frac{r_1}{c} x^n + h_1, \\ g &= q_2 \left(\frac{r_1}{c} x^n + h_1\right) + h_2, \end{aligned}$$

where $h_1, h_2 \in R$ are suitable polynomials of degree less than n. In particular, $\rho(h_2) < \rho(g)$.

[1] And thus, to resolve an open question raised by Glivický and Šaroch in [9].

(c) $\underline{\deg f > n}$: Since $K[x]$ is Euclidean with respect to the norm $\deg(-)+1$, there exist $q, r \in K[x]$ such that

$$f = qg + r, \quad \deg r < n.$$

Write $q = q(0) + xq_1$. Then

$$f = xq_1 g + (q(0)g + r),$$

where $h = (q(0)g + r) \in R$, since its constant term is equal to $f(0)$. If $\deg h < n$, then $\rho(h) < \rho(g)$. If $\deg h = n$, using steps in (b) on the pair (g, h) we obtain a 3-stage division chain such that the degree of the last remainder is less than n. □

Notice that, assuming D is not a field, the ring $R = D + xK[x]$ is not Noetherian. Given a nonzero nonunit element $d \in D$, we obtain an infinite chain

$$xR \subsetneq (x/d)R \subsetneq (x/d^2)R \subsetneq \cdots \subsetneq (x/d^k)R \subsetneq \dots$$

of ideals in R. Thus, we obtain our first example of a non-PID domain which is k-stage Euclidean for some $k > 1$ (regardless of whether it is 2-stage Euclidean or not).

3.2 The ring of all algebraic integers

Example 4. Denote $\mathbb{A}$ the ring of all algebraic integers — that is, an integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$, where $\overline{\mathbb{Q}}$ denotes the field of all algebraic numbers, i.e. the algebraic closure of $\mathbb{Q}$. Recall that $\mathbb{A}$ is a Bézout domain[2] with the fraction field $\overline{\mathbb{Q}}$.

The following proposition was proved by T. van Aardenne-Ehrenfest and H. W. Lenstra Jr. in [15]. For reader's convenience, we present the proof here as well.

Proposition 3.1. *Given an arbitrary algebraic number z, there exist algebraic integers a, b such that*

$$z = a + \frac{1}{b}.$$

Proof. From the fact that $\overline{\mathbb{Q}}$ is the fraction field of $\mathbb{A}$ and the fact that $\mathbb{A}$ is a GCD domain it follows that we can write

$$z = \frac{c}{d},$$

where c, d are coprime algebraic integers with $d \neq 0$. Since $\mathbb{A}$ is Bézout, there exist algebraic integers r, s such that

$$rc + sd = 1. \tag{1}$$

[2]For proof of this statement, see [12, p. 72].

Consider the field $K = \mathbb{Q}(c, d, r, s)$ and denote $R = \mathcal{O}_K$ the ring of algebraic integers contained in K. From (1) we see that $c + dR$ is a unit in the ring R/dR, since

$$(r + dR)(c + dR) = rc + dR = 1 + dR.$$

Since the ring R/dR is finite, its group of units is finite as well and thus, there exists a positive integer n such that $(c + dR)^n = 1 + dR$. That is, there exists an algebraic integer (contained in K) e such that

$$c^n + de = 1.$$

Using the facts that $\mathbb{A}$ is a GCD domain and that $\gcd(c, d) = 1$, we infer that $\gcd(c^{n-1}, d^{n-1}) = 1$. This implies that there exist algebraic integers u', v' such that $u'c^{n-1} + v'd^{n-1} = 1$. Then $u = u'e,\ v = v'e$ are algebraic integers satisfying

$$uc^{n-1} + vd^{n-1} = e.$$

Define $a \in \mathbb{A}$ as a solution of the equation

$$x^n + ux^{n-1} + v = 0. \tag{2}$$

We immediately see that $a \neq z$, since

$$\begin{aligned} z^n + uz^{n-1} + v &= \left(\frac{c}{d}\right)^n + u\left(\frac{c}{d}\right)^{n-1} + v = \frac{1}{d^n}(c^n + d(uc^{n-1} + vd^{n-1})) \\ &= \frac{1}{d^n}(c^n + de) = \frac{1}{d^n} \neq 0. \end{aligned}$$

Finally, set $b = 1/(z - a)$. Then the desired identity $z = a + (1/b)$ holds and it remains only to show that b is an algebraic integer. Since

$$a = z - \frac{1}{b} = \frac{c}{d} - \frac{1}{b},$$

by substitution into (2) we have

$$\left(\frac{c}{d} - \frac{1}{b}\right)^n + u\left(\frac{c}{d} - \frac{1}{b}\right)^{n-1} + v = 0,$$

which implies

$$(cb - d)^n + udb(cb - d)^{n-1} + vd^nb^n = 0.$$

That is, b is a root of the polynomial

$$f(x) = (cx - d)^n + udx(cx - d)^{n-1} + vd^nx^n$$

defined over $D = \mathbb{Z}[c, d, u, v]$, an integral extension of $\mathbb{Z}$. Since the leading coefficient of f is

$$c^n + udc^{n-1} + vd^n = c^n + d(uc^{n-1} + vd^{n-1}) = c^n + de = 1,$$

b is integral over D. It follows that b is integral over $\mathbb{Z}$, i.e. $b \in \mathbb{A}$. $\square$

Having the correspondence between terminating division chains and continued fractions as stated in Lemma 2.3 in mind, from Proposition 3.1 we immediately see that every pair of algebraic integers a, b (with $b \neq 0$) has a terminating 2-stage division chain. That is, ${}_2\mathbb{A}_1 = \mathbb{A}$, in particular, $\mathbb{A}$ is a 2-stage Euclidean domain. However, $\mathbb{A}$ is easily seen to be non-Noetherian and thus non-principal, since

$$2\mathbb{A} \subsetneq \sqrt{2}\mathbb{A} \subsetneq \sqrt[4]{2}\mathbb{A} \subsetneq \cdots \subsetneq \sqrt[2^k]{2}\mathbb{A} \subsetneq \ldots$$

is a strictly increasing chain of ideals in $\mathbb{A}$. For completeness' sake, we comment on this further. The number $\sqrt[2^k]{2}$ is an algebraic integer for every $0 < k < \omega$, since it is a root of the monic polynomial

$$x^{2^k} - 2$$

(which is the minimal polynomial of $\sqrt[2^k]{2}$ over $\mathbb{Q}$, since it is irreducible by Eisenstein's criterion). The strictness of the inclusions follows from the fact that the minimal polynomial of $1/(\sqrt[2^k]{2})$ over $\mathbb{Q}$ is

$$x^{2^k} - \frac{1}{2}$$

(irreducibility of which follows from the irreducibility of $x^{2^k} - 2$ via the substitution $x \to \frac{1}{y}$). Since it does not have coefficients in $\mathbb{Z}$, the element $1/(\sqrt[2^k]{2})$ is not an algebraic integer for any $1 < k < \omega$.

Moreover, every algebraic integer d can be written as $d = \sqrt{d}\sqrt{d}$ in $\overline{\mathbb{Q}}$, and similar arguments show that $\sqrt{d}$ is an algebraic integer. Thus, the ring $\mathbb{A}$ has no irreducible elements (since $\sqrt{d}$ is a unit iff d is a unit).

3.3 The ring of integers of $\mathbb{Q}(\sqrt{-19})$

For the purposes of the next example, we define the norm N on the field $\mathbb{C}$ as follows. For $z = a + ib, a, b \in \mathbb{R}$, we set $N(z) = a^2 + b^2$. Note that since $N(z) = |z|^2$, N is subadditive, that is, $N(z_1 + z_2) \leq N(z_1) + N(z_2)$ for every $z_1, z_2 \in \mathbb{C}$. Also, observe that N is multiplicative, i.e. $N(z_1 z_2) = N(z_1)N(z_2)$ for every $z_1, z_2 \in \mathbb{C}$, and thus, since $N(1) = 1$, $N(z_1/z_2) = N(z_1)/N(z_2)$ for every $z_1, z_2 \in \mathbb{C}$ with $z_2 \neq 0$. We denote the restriction of this norm to a subring of $\mathbb{C}$ by N as well.

Example 5. Consider the algebraic number field $K = \mathbb{Q}(\sqrt{-19})$ and let R be the ring of algebraic integers contained in K. It is a well-known fact that $R = \mathbb{Z}[\vartheta]$, where $\vartheta = (1 + i\sqrt{19})/2$ (that is, if we consider $\mathbb{Q}(\sqrt{-19})$ to be the subring $\mathbb{Q}(i\sqrt{19})$ of $\mathbb{C}$)[3].

It is a matter of straightforward verification that given an arbitrary element $a = r + s\vartheta,\ r, s \in \mathbb{Z}$ of R, $N(a) = r^2 + rs + 5s^2$, in particular, it is a natural number. As a consequence, any nonempty subset $M \subseteq R$ contains an element $b \in M$ such that $N(b) = \min\{N(a) \mid a \in M\}$. It is also useful to observe that the fact that $N(a)$ is natural number for any $a \in R$ together with multiplicativity of the norm N imply that ± 1 are the only units in R (and for any nonzero nonunit element b, $N(b) \geq 2$ holds).

[3]See [8, p. 55] for proof.

Lemma 3.2. *An element $a \in R$ is divisible by 2 in R if and only if $N(a)$ is divisible by 2 in $\mathbb{Z}$.*

Proof. If $2 \mid a$, that is, $a = 2a'$ for $a' \in R$, then $2 \mid N(a)$ in $\mathbb{Z}$, since $N(a) = N(2a') = N(2)N(a') = 4N(a')$, where $N(a')$ is a natural number.

Conversely, consider an element $a = r + s\vartheta$, $r, s \in \mathbb{Z}$ such that $2 \mid N(a)$. Since $N(a) = r^2 + rs + 5s^2$ is an even number, by analysis of all four possible combinations of parities of r, s it is clear that both r and s are even. Thus, $a = 2r' + 2s'\vartheta = 2(r' + s'\vartheta)$. □

Proposition 3.3. *R is a principal ideal domain.*

Our proof of the above statement follows the arguments given by O. A. Cámpoli in [5] and is influenced by a modification of these made by R. A. Wilson in [16].

Proof. Let I be a nonzero ideal of R. Choose an element $a \in I$ with minimal nonzero norm (i.e. $N(a) = \min\{N(b) \mid b \in I \setminus \{0\}\}$). Suppose that there exists $b \in I \setminus aR$. Then for every $\alpha, \beta \in R$, $\alpha a + \beta b$ is an element of I, hence $N(\alpha a + \beta b) \geq N(a)$ or $\alpha a + \beta b = 0$.

Denote $b/a = r + is$, where $r,s \in \mathbb{R}$. Consider $n \in \mathbb{Z}$ such that

$$|s + n\sqrt{19}/2| \leq \sqrt{19}/4$$

and $k \in \mathbb{Z}$ such that

$$|r + n/2 + k| \leq 1/2.$$

Set $t = (n\vartheta + k)$. Then

$$\frac{b + at}{a} = \frac{b}{a} + t = \frac{b}{a} + \frac{n(1 + i\sqrt{19})}{2} + k = \left(r + \frac{n}{2} + k\right) + i\left(s + \frac{n\sqrt{19}}{2}\right).$$

Consider the following cases:

(1) $\underline{|s + n\sqrt{19}/2| < \sqrt{3}/2}$:
Then we have

$$N\left(\frac{b + at}{a}\right) = \left(r + \frac{n}{2} + k\right)^2 + \left(s + \frac{n\sqrt{19}}{2}\right)^2 < \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1,$$

that is, $N(b + at) < N(a)$. This implies that $b + at = 0$ and therefore $b \in aR$, a contradiction.

(2) $\underline{\sqrt{3}/2 \leq |s + n\sqrt{19}/2| \leq \sqrt{19}/4}$:
Without loss of generality, we may assume $\sqrt{3}/2 \leq s + n\sqrt{19}/2 \leq \sqrt{19}/4$, i.e. the imaginary part of the considered fraction is positive. If $s + n\sqrt{19}/2 < 0$, by taking $\tilde{a} = -a$, we obtain again an element of I of minimal nonzero norm (since $N(\tilde{a}) = N(a)$). Moreover, $b \notin \tilde{a}R = aR$, and by taking $\tilde{t} = -t$, we obtain a similar fraction

$$\frac{b + \tilde{a}\tilde{t}}{\tilde{a}} = -\frac{b + at}{a},$$

that is, the estimates for absolute values of real and imaginary part of the fraction still hold and the imaginary part is positive.

Consider $k' \in \mathbb{Z}$ such that

$$|2r + n + 2k - 1/2 + k'| \leq 1/2$$

and set $t' = 2t + k' - \vartheta$. Then

$$\begin{aligned}
\frac{2b + at'}{a} &= \frac{2(b + at)}{a} + k' - \frac{1 + i\sqrt{19}}{2} \\
&= \left(2r + n + 2k - \frac{1}{2} + k'\right) + i\left(2s + \frac{(2n-1)\sqrt{19}}{2}\right).
\end{aligned}$$

Since the following estimates

$$\left|2r + n + 2k - \frac{1}{2} + k'\right| \leq \frac{1}{2},$$

$$-\frac{\sqrt{3}}{2} < \sqrt{3} - \frac{\sqrt{19}}{2} \leq \left(2s + \frac{(2n-1)\sqrt{19}}{2}\right) \leq 0$$

hold, we conclude (similarly to the case (1)) that

$$N\left(\frac{2b + at'}{a}\right) < 1,$$

i.e. $N(2b + at') < N(a)$. Thus, $2b + at' = 0$ and from $t' = 2t - \vartheta + k'$, we have

$$\begin{aligned}
0 = 2b + at' &= 2b + a\left(2t - \vartheta + k'\right), \\
2(b + at) &= a\left(\vartheta - k'\right).
\end{aligned}$$

By comparing the norms of the elements on the left-hand and right-hand side of the last equation, it is easily seen that $2 \mid a$ in R, since

$$N(2(b + at)) = N(2)N(b + at) = 4N(b + at)$$

is an even number ($N(b + at)$ is a natural number, since $b + at \in R$), and

$$N\left(\vartheta - k'\right) = {k'}^2 - k' + 5$$

is an odd number (for both possible parities of k'). Therefore $N(a)$ is even and the rest follows from Lemma 3.2. It follows that $a = 2a'$ with $a' \in R$ and

$$b + at = a'\left(\vartheta - k'\right).$$

By multiplying both sides of the equation by $(1 - k' - \vartheta)$ (i.e. the conjugate element of $(\vartheta - k')$) we get

$$(b + at)\left(1 - k' - \vartheta\right) = a'({k'}^2 - k' + 5).$$

From the expression on the left-hand side of the last equation it follows that $a'({k'}^2 - k' + 5) \in I$. Since $2a' = a \in I$ and $({k'}^2 - k' + 5)$ is odd, we infer that $a' \in I$ ($a' = ({k'}^2 - k' + 5)a' + ma$ for suitable $m \in \mathbb{Z}$). But then

$$N(a) = N(2a') = N(2)N(a') = 4N(a')$$

and since $N(a) > 0$, it follows that $N(a') < N(a)$. This contradicts the fact that a is the element of I of minimal nonzero norm.

The assumption $b \in I \setminus aR$ leads to contradiction, therefore $I = aR$. $\square$

The considered ring R is probably the most well-known example of a PID which is not Euclidean (see, for example, [5]). We aim to show that R is not even quasi-Euclidean. To prove this, a rather technical lemma is needed.

Lemma. *Let R be a ring and $a, b \in R$, $b \neq 0$. Suppose that the pair (a, b) has a terminating division chain in R. Then there exists a terminating division chain*

$$\left(\begin{array}{c|c} a & q_1 \dots q_{k-1}\, q_k \\ b & r_1 \dots r_{k-1}\, 0 \end{array}\right)$$

such that for every $i > 1$, q_i is a nonzero nonunit element.

Proof of this lemma can be found in Appendix (where it is listed as Lemma 4).

Proposition 3.4. *R is not quasi-Euclidean. More specifically, the pair of elements $3 - \vartheta, 2 + \vartheta$ does not have a terminating division chain.*

The following proof is inspired by the proof of a related assertion by Cohn[4].

Proof. Firstly observe that $3 - \vartheta$ is not a divisor of $2 + \vartheta$ and vice versa (this is easily seen from multiplicativity of N, since $N(3 - \vartheta) = N(2 + \vartheta) = 11$ and the only elements of norm 1 in R are ± 1). Suppose that there exists a terminating division chain

$$(C) = \left(\begin{array}{c|c} 3-\vartheta & q_1\, q_2 \dots q_{k-1}\, q_k \\ 2+\vartheta & r_1\, r_2 \dots r_{k-1}\, 0 \end{array}\right).$$

It is clear that $k \geq 2$ and by the lemma above, we may further assume that $q_i \notin \{0,1,-1\}$ for $i > 1$.

If $q_1 \notin \{0,1,-1\}$, set $s_{-1} = 3 - \vartheta, s_0 = 2 + \vartheta$. Otherwise put $s_{-1} = 2 + \vartheta$ and $s_0 = r_1$. By analysis of possible values of s_{-1}, s_0 it is easily verified that $0 < N(s_{-1}) \leq N(s_0)$. Moreover, there exists a terminating division chain

$$(D) = \left(\begin{array}{c|c} s_{-1} & t_1 \dots t_{n-1}\, t_n \\ s_0 & s_1 \dots s_{n-1}\, 0 \end{array}\right)$$

such that $n \geq 1$ and $t_i \notin \{0,1,-1\}$, $i = 1, \dots, n$, that is, $N(t_i) \geq 2$ for every i (if $q_1 \notin \{0,1,-1\}$, the chain (D) is the same as (C); in the other case, (D) is formed by all the equations used in (C) except the first one). Denote the last remainder (that is, 0) by s_n.

Consider arbitrary $i \in \{1,2,\dots,n\}$. Then, using subadditivity and multplicativity of N, we have that

$$\begin{aligned} N(t_i s_{i-1}) &= N(t_i s_{i-1} - s_{i-2} + s_{i-2}) \leq \\ &\leq N(t_i s_{i-1} - s_{i-2}) + N(s_{i-2}) = \\ &= N(-s_i) + N(s_{i-2}) = N(s_i) + N(s_{i-2}), \end{aligned}$$

and thus,

$$\begin{aligned} 2N(s_{i-1}) \leq N(t_i)N(s_{i-1}) = N(t_i s_{i-1}) &\leq N(s_i) + N(s_{i-2}), \\ N(s_{i-1}) - N(s_{i-2}) &\leq N(s_i) - N(s_{i-1}). \end{aligned}$$

[4] Cohn proves that R is not GE_2, that is, the group $GL_2(R)$ is not generated by matrices of elementary transformations. See [6, § 6] for the proof and [7, § 2] for establishing the relationship between the concept of GE_2 ring and the quasi-Euclidean condition.

In particular, we have that

$$0 \leq N(s_0) - N(s_{-1}) \leq N(s_1) - N(s_0) \leq \cdots \leq N(s_n) - N(s_{n-1}),$$

and therefore

$$\forall i \in \{0,1,\dots n\} : N(s_i) - N(s_{i-1}) \geq 0, \text{ i.e. } N(s_i) \geq N(s_{i-1}).$$

It follows that

$$0 < N(s_{-1}) \leq N(s_0) \leq N(s_1) \leq \cdots \leq N(s_{n-1}) \leq N(s_n) = N(0) = 0.$$

This is a contradiction. Hence, no terminating division chain starting from $(3 - \vartheta, 2 + \vartheta)$ exists. ☐

Example 6. We close this section with a comment that for many quadratic number fields, the respective rings of algebraic integers are 2-stage Euclidean. As was proved by Cooke in [7], the ring of algebraic integers contained in the field $\mathbb{Q}(\sqrt{d})$ is 2-stage Euclidean for the following values of d [5]:

$$14,\ 22,\ 31,\ 43,\ 46,\ 53,\ 61,\ 69,\ 77.$$

3.4 Quasi-Euclidean domains which are not k-stage Euclidean

Example 7. It was proved by Chen and Leu that the subring

$$\mathbb{Z} + x\mathbb{Q}[x] = \{f \in \mathbb{Q}[x] \mid f(0) \in \mathbb{Z}\}$$

of $\mathbb{Q}[x]$ is a quasi-Euclidean ring which is not 2-stage Euclidean.

More generally, in [3] it is proved that given a quasi-Euclidean domain D with finitely many units equipped with a function $\psi : D \to \gamma$ satisfying the property *(m)* (see Section 2.4) and an additional property that for every element $a \in D$, there exists an irreducible element p with $\psi(p) > \psi(a)$, the ring $D + xK[x]$, where K denotes the fraction field of D, is quasi-Euclidean domain which is not 2-stage Euclidean.

Note that any such ring is non-Noetherian (by the same argument as used in Example 3).

Example 8. Let $\prod = \prod_{p \in \mathbb{P}} \widehat{\mathbb{Z}}_p$, that is, the product of rings of p-adic integers, where p runs over all primes. In [9], Glivický and Šaroch provided a set of subrings $\mathcal{S} = \{R_\lambda \mid \lambda \in \prod\}$ of $\mathbb{Q}[x]$ such that for every $\lambda \in \prod$, R_λ is a quasi-Euclidean domain which is not k-stage Euclidean for any $k < \omega$ [6].

Moreover, $\lambda_1 \neq \lambda_2$ implies $R_{\lambda_1} \neq R_{\lambda_2}$ and thus, $\operatorname{card}(\mathcal{S}) = \operatorname{card}(\prod) = 2^\omega$ (it is further proved that 2^ω of the constructed domains are principal and 2^ω are non-Noetherian).

We note that the case $\lambda = 0$ produces the ring $\mathbb{Z} + x\mathbb{Q}[x]$, the ring described in Example 7. Therefore, the ring $\mathbb{Z} + x\mathbb{Q}[x]$ cannot serve as an example of a k-stage Euclidean domain which is not 2-stage Euclidean.

[5] Some of these, i.e. the cases $d = 14, 69$, are actually Euclidean.

[6] The definition of k-stage Euclidean norm used in [9] allows only finite-valued norms. However, the proof that R_λ is not k-stage Euclidean works even when norms with possibly larger codomains are considered, since the only aspect of ω used in the proof is the descending chain condition of its order.

4. Appendix

In this section we list some additional statements about division chains, which are rather technical.

The first of these lemmas is a version of Lemma 3.5 given by Glivický and Šaroch in [9] for the ring $\mathbb{Z}$ instead of rings discussed in the article. The proof of the statement, however, can be used without any modifications and can be found in [9]. As can be seen in a proof of Corollary 2, we use this lemma to relate the continued fractions with coefficients in $\mathbb{Z}$ as defined in Chapter 2 to the widely studied simple continued fractions.

Lemma 1 ([9]). *Consider $a, b \in \mathbb{Z}$, $a, b > 0$ and a division chain $\left(\begin{array}{c|c} a & q_1 \dots q_k \\ b & r_1 \dots r_k \end{array}\right)$ in $\mathbb{Z}$ (where $k \geq 1$). Then there exists a division chain $\left(\begin{array}{c|c} a & q'_1 \dots q'_n \\ b & r'_1 \dots r'_n \end{array}\right)$ such that for every $i \geq 2$, $q_i > 0$, $|r_k| = |r'_n|$ and $n \leq 2k - 1$.*

Corollary 2. *Given $n < \omega$, there exist $a, b \in \mathbb{Z}$ such that the smallest possible length of a terminatig division chain starting on (a, b) is at least n.*

It is a well-known fact that every rational number can be expressed as a continued fraction with positive integer coefficients (except the first one) — that is, as a *simple continued fraction* — in exactly two ways and the lengths (i.e. numbers of coefficients used) of these continued fraction expressions differ by one[1]. This is a fact used in the following proof.

Proof. Consider the rational number q with the following continued fraction expression

$$q = [\underbrace{1, 1, \dots, 1}_{(2n)\times}]$$

and choose $a, b \in \mathbb{Z}$ such that $a,b > 0$, $(a/b) = q$ and, for unambiguity's sake, such that a, b are coprime[2]. Suppose there exists a terminating division chain

$$\left(\begin{array}{c|c} a & q_1 \dots q_{k-1}\, q_k \\ b & r_1 \dots r_{k-1}\, 0 \end{array}\right)$$

with $k \leq n - 1$. Then by Lemma 1, there exists a terminating division chain

$$\left(\begin{array}{c|c} a & q'_1 \dots q'_{m-1}\, q'_m \\ b & r'_1 \dots r'_{m-1}\, 0 \end{array}\right)$$

such that $q'_i > 0$ for $i > 1$ and $m \leq 2k - 1$. But then, using Lemma 2.3, we see that

$$[\underbrace{1, 1, \dots, 1}_{(2n)\times}] = q = \frac{a}{b} = [q'_1, \dots, q'_{m-1}, q'_m],$$

[1] See [13, p. 14, Theorem 1.1] and the following discussion.

[2] Then it is easy to check that a and b are two consecutive Fibonacci numbers.

where the lengths of the continued fractions on the opposite sides differs by at least 3, since

$$m \leq 2k-1 \leq 2(n-1)-1 = 2n-3.$$

This is a contradiction and hence, the assumption $k \leq n-1$ does not hold. $\square$

Lemma 3. *Let R be a ring. Consider $a,b \in R$, $b \neq 0$, and $u,v \in R^\times$. Given a division chain*

$$\left(\begin{array}{c|c} a & q_1\, q_2 \dots q_{k-1}\, q_k \\ b & r_1\, r_2 \dots r_{k-1}\, r_k \end{array} \right),$$

there exists a division chain

$$\left(\begin{array}{c|ccccc} ua & uv^{-1}q_1 & u^{-1}vq_2 & \dots & u^{\varepsilon}v^{-\varepsilon}q_{k-1} & u^{-\varepsilon}v^{\varepsilon}q_k \\ vb & ur_1 & vr_2 & \dots & wr_{k-1} & tr_k \end{array} \right),$$

where $\varepsilon = (-1)^k$ and $w = u$, $t = v$ for k even and $w = v$, $t = u$ for k odd.

Proof. This is proved by induction on k, using straightforward computation. $\square$

Note that as a consequence, if the chain starting from (a,b) is terminating, so is the derived one for the pair (ua, vb) and, moreover, both division chains are of the same length. This fact is used in the given proof of the following lemma.

Lemma 4. *Let R be a ring and $a, b \in R$, $b \neq 0$. Suppose that the pair (a,b) has a terminating division chain in R. Then there exists a terminating division chain*

$$\left(\begin{array}{c|c} a & q_1 \dots q_{k-1}\, q_k \\ b & r_1 \dots r_{k-1}\, 0 \end{array} \right)$$

such that for every $i > 1$, q_i is a nonzero nonunit element.

Proof. First we make the following observations. Consider a series of equations

$$\begin{aligned} c &= t_1 d + s_1, \\ d &= t_2 s_1 + s_2, \\ s_1 &= t_3 s_2 + s_3. \end{aligned}$$

(1) Suppose that $t_2 = 0$. It follows that $s_2 = d$ and thus, we can write

$$c = t_1 d + s_1 = t_1 d + (t_3 d + s_3) = (t_1 + t_3) d + s_3,$$

omitting the intermediate steps.

(2) Assume t_2 is a unit. Then from the middle equation we have

$$-t_2^{-1} s_2 = s_1 - t_2^{-1} d,$$

and thus, we obtain

$$\begin{aligned} c &= t_1 d + s_1 = (t_1 + t_2^{-1}) d + s_1 - t_2^{-1} d = (t_1 + t_2^{-1}) d - t_2^{-1} s_2, \\ d &= t_2 s_1 + s_2 = t_2(t_3 s_2 + s_3) + s_2 = (t_2 t_3 + 1) s_2 + t_2 s_3 \\ &= (-t_2)(t_2 t_3 + 1)(-t_2^{-1} s_2) + t_2 s_3, \end{aligned}$$

that is, from the considered 3-stage chain we obtain a 2-stage chain at the cost that any continuation of the chain needs to start from the pair $(-t_2^{-1} s_2, t_2 s_3)$ instead of (s_2, s_3) (where $-t_2^{-1}, t_2$ are units).

(3) Suppose that the chain is terminating, i.e. $s_3 = 0$, and that $t_3 = 0$ as well. Then the last two steps are clearly superfluous and we can simply write

$$c = t_1 d + 0$$

instead.

(4) Finally, suppose that $s_3 = 0$ and t_3 is a unit. Then it follows that $s_2 = t_3^{-1} s_1$ and thus,

$$\begin{aligned} c &= t_1 d + s_1, \\ d &= (t_2 + t_3^{-1}) s_1 + 0. \end{aligned}$$

Suppose that a terminating division chain

$$\left(\begin{array}{c|c} a & q_1 \dots q_{k-1}\, q_k \\ b & r_1 \dots r_{k-1}\, 0 \end{array} \right)$$

does not meet the condition of the proposition, that is, $k \geq 2$ and there exist $i > 1$ such that $q_i \in R^\times \cup \{0\}$. If $k \geq 3$, using one of the transformations described in (1)–(4) we can modify the chain to obtain a strictly shorter terminating division chain starting from (a, b) (if we need to use the step (2), we further replace the continuation of the chain by a terminating chain of the same length as described in Lemma 3). If $k = 2$, the case $q_2 = 0$ implies $b = 0$, a contradiction, and the case $q_2 \in R^\times$ can be treated as in (4).

Thus, by considering a terminating division chain (C) of the minimal length, we infer that the condition of the proposition for (C) holds. □

Conclusion

The concepts of k-stage Euclidean and quasi-Euclidean domains were introduced by G. E. Cooke in [7] in 1976 and were motivated by their applications to algebraic number theory. In the same article, Cooke admits that he does not know an example of quasi-Euclidean domain which is not 2-stage Euclidean.

Such examples had not been known until recently. In [3], Chen an Leu provided an example of quasi-Euclidean domain which is not 2-stage Euclidean, and in [9], Glivický and Šaroch described a set of examples of quasi-Euclidean domains which are even not k-stage Euclidean for any positive integer k.

There seems to be a lot of potential to generalise known results concerning the Euclidean domains to the k-stage Euclidean case, where k is an integer. In particular, the Hiblot's article [11] can be interesting in this regard. Since many of the statements in the article work with the transfinite construction of Euclidean ring as described in [14], the transfinite constructions presented in this thesis could be used toward such purpose.

We conclude this thesis by a short list of open questions. First of these is a slightly generalized question raised by Glivický and Šaroch in [9].

Question 1. Does there exist $k \geq 2$ and a $(k+1)$-stage Euclidean ring which is not k-stage Euclidean?

It is the author's conjecture that the Example 3 given in Chapter 3 is an example of 3-stage Euclidean ring which is not 2-stage Euclidean (which would answer this question even for integral domains). However, no proof (or disproof) has been found so far.

Question 2. For a given positive integer k, does there exist a k-stage Euclidean ring R such that the smallest norm on R is not finite-valued?

In the case $k = 1$, Samuel proved that the Euclidean ring $\mathbb{Z} \times \mathbb{Z}$ admits no finite-valued Euclidean norm in [14]. A more sophisticated example, answering the question for $k = 1$ and the case of integral domains, provided Hiblot in [10]. However, the given proof of Proposition 1.12 shows that the Samuel's approach does not provide a counterexample for any $k \geq 2$. Also note that in the case $k = \omega$ no such example exists, since for every quasi-Euclidean ring, the range of the smallest ω-stage Euclidean norm is $\{0, 1\}$.

Question 3. Given a positive integer k and k-stage Euclidean ring R, does there exist a k-stage Euclidean norm $\varphi : R \to \gamma$ such that for every $a, b \in R$ with $ab \neq 0$, $\varphi(a) \leq \varphi(ab)$ (that is, thet the condition *(m)* is satisfied)? In particular, does the condition hold for the smallest k-stage Euclidean norm on R?

In Chapter 2, Proposition 2.12 shows that strongly associated elements are indistiguishable by the smallest k-stage Euclidean norm. This question is motivated by the effort to show that the associated elements are indistiguishable by the norm as well (see [1] for the distinction). The question is answered in the affirmative for $k = 1$, which can be found in [14], and for arbitrary positive integer k in the case that R is an integral domain, which is the statement of Propostition 2.13 of this thesis.

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