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Fractional Cointegration of Daily High and Low Stock Prices

Master Thesis

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ACADEMIC YEAR: **2012/2013**

Declaration of authorship

I hereby declare that I wrote this master thesis on my own under the guidance of my supervisor and that all resources and literature I have used are referenced herein.

I hereby declare that this thesis was not used to obtain another or equal degree.

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Prague, May 16, 2013 Sylvie Dvořáková

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Abstract

In this thesis we provide unique empirical support for the fractional cointegration of daily high and low stock prices. The recently formalized fractionally cointegrated VAR model by Johansen and Nielsen (2012) is used due to its ability to capture both the cointegration between daily high and low stock prices and the long memory of their linear combination, the range. Daily high and low stock prices are of particular interest as they provide valuable information about range-based volatility, which is considered a highly efficient and robust estimator of volatility. We provide a comparison of the Czech PX 50 index with the developed market indices (DAX, FTSE 100, S&P 500 and NIKKEI 225) during the 2003-2012 period as well as before and after the crisis. We find that the range of all indices displays long memory and is mostly in the non-stationary region (except for the ranges of the PX 50 and NIKKEI 225 indices in the pre-crisis period). These findings provide evidence that volatility may not be a stationary process. No common pattern is detected among all five market indices and different behaviour is also observed in the pre-crisis and post-crisis periods.

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Abstrakt

Tato práce přináší jedinečnou empirickou analýzu podporující frakční kointegraci denních maximálních a minimálních cen akcií. Používáme zde frakčně kointegrovaný VAR model, který byl pouze nedávno zobecněn Johansenem a Nielsenem (2012). Tento model je schopen popsat jak frakční kointegraci mezi denními maximálními a minimálními cenami akcií, tak i dlouhou paměť jejich lineární kombinace, tzv. rozpětí (range). Denní extrémní ceny jsou obzvláště zajímavé, protože poskytují cenné informace o odhadu volatility pomocí rozpětí (range-based volatility). Tento odhad je v literatuře považován za vysoce eficientní a robustní odhad volatility. V práci přinášíme srovnání českého indexu PX 50 s vyspělými tržními indexy (DAX, FTSE 100, S&P 500 a NIKKEI 225) v období 2003-2012. Tyto indexy také zkoumáme v období před a po krizi. Zjistili jsme, že rozpětí (range) všech indexů vykazuje dlouhou paměť a je většinou nestacionární (s výjimkou rozpětí indexů PX 50 a NIKKEI 225 v období před krizí). Tyto poznatky naznačují, že volatilita nemusí být stacionární proces. Nenašli jsme žádné společné vlastnosti napříč všemi indexy, odlišné chování je také vypozorováno v obdobích před a po krizi.

Klasifikace	C32, C58, G15
Klíčová slova	Frakční kointegrace, dlouhá paměť, FCVAR model, denní maximální a minimální ceny
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List of Acronyms

ACF	Autocorrelation Function
ADF	Augmented Dickey-Fuller test
ARCH	Autoregressive Conditional Heteroscedasticity
ARFIMA	Autoregressive Fractionally Integrated Moving Average
ARMA	Autoregressive Moving Average
BIC	Bayesian Information Criterion
CVAR	Cointegrated Vector Autoregression
EGARCH	Exponential Generalized Autoregressive Conditional Heteroscedasticity
ELW	Exact Local Whittle
FIGARCH	Fractionally Integrated Generalized Autoregressive Conditional Heteroscedasticity
FCVAR	Fractionally Cointegrated Vector Autoregression
GARCH	Generalized Autoregressive Conditional Heteroscedasticity
GPH	Geweke and Porter-Hudak
IV	Initial Values
LR	Likelihood Ratio
LW	Local Whittle
NBER	National Bureau of Economic Research
VAR	Vector Autoregression
VECM	Vector Error Correction Model

Master Thesis Proposal

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Proposed Topic:

Fractional Cointegration of Daily High and Low Stock Prices

Topic Characteristics:

The purpose of the thesis is to estimate the fractional cointegration of daily high and low prices of stock indices. Daily highs and lows provide valuable information, which is not included in the open/close prices. Moreover, the difference between high and low prices (known as range) can be used as an estimation of volatility. Parkinson (1980) shows that the price range-based estimation of volatility is a highly efficient and robust estimator.

The stock prices in developed markets tend to exhibit stochastic trends. If this is the case, then each series can drift, but, their difference should not diverge. In most studies, they are shown to be *I(1)* processes (see e.g. Cheung, 2007). However, as Caporin et al. (2011) show, the decision between *I(0)* and *I(1)* processes can be too restrictive and the degree of integration of daily highs and lows should be estimated in a fractional or long-memory framework. Thus, the series are supposed to be cointegrated of order *d*. In order to determine the relationship between highs and lows we will implement a fractional vector autoregressive model with error correction (FVECM) proposed by Johansen (2008). The motivation for using this methodology stems from the assumed cointegration between highs and lows and from the belief that the range should display a long memory.

To support this idea, we consider four world main stock indices, namely American Dow Jones Industrial index, German DAX, Japanese NIKKEI 225 and UK's FTSE 100 Index over 2002-2012 period available at Yahoo! Finance and we compare the results with Czech PX index over the same period. The provided period enables us to compare both the cointegration and the estimated volatility before and during the recent financial crisis.

Hypotheses:

1. Daily highs and lows are fractionally cointegrated.
2. The level of cointegration differs among indices.
3. The level of cointegration is different before and during the crisis.
4. The range displays long memory.

Methodology:

As already mentioned above, we suppose that highs and lows are cointegrated and that their difference displays a long memory. This leads us to applying the FVECM framework, which captures both features very well. The cointegration of stock prices has been studied profoundly. However, little has been done in the field of fractional cointegration, which is not limited to the discussion between *I(0)* or *I(1)* process and instead postulates that the series are cointegrated of order *d*. Moreover, the presence of memory property of high-low range is not consistent with the traditional tests of cointegration.

First, we will apply the univariate Lagrange Multiplier test of Breitung and Hassler (2002) to test the null hypothesis of unit roots against fractional alternatives. Then, we will estimate the fractional degree of persistence of daily highs and lows by Whittle estimator of Shimotsu and Philips (2005). We will also use the approach proposed in Nielsen and Shimotsu (2007) which tests the equality of integration orders. The modification of general VEC form to account for fractional cointegration (and the long-memory) was proposed by Johansen (2008), which makes the distinction between cofractional relations and common trends possible. The model will be then estimated by Maximum Likelihood Estimator (MLE).

Outline:

1. Introduction
2. Evolution of stock markets in selected countries over 2002-2012
    2.1. Pre-crisis period
    2.2. Crisis period
3. Preliminary analysis of the cointegration
    3.1. Data description
    3.2. Test for fractional cointegration
4. Empirical model – modeling Daily Highs and Lows
    4.1. Model of fractional cointegration
    4.2. Augmented models
5. Concluding remarks
6. References

Core Bibliography:

Baillie, R. T. (1996): "Long memory processes and fractional integration in econometrics," Journal of Econometrics, 73(1), 5–59.

Breitung, J., and U. Hassler (2002): "Inference on the cointegration rank in fractionally integrated processes," Journal of Econometrics, 110(2), 167–185.

Caporin, M., Ranaldo, A., and Santucci de Magistris, P. (2011): "On the Predictability of Stock Prices: A Case for High and Low Prices," "Marco Fanno" Working Papers 0136, Dipartimento di Scienze Economiche "Marco Fanno".

Cheung, Y. (2007): "An empirical model of daily highs and lows," International Journal of Finance and Economics, 12, 1–20.

Engle, R., and C. Granger (1987): "Cointegration and error correction: representation estimation, and testing," Econometrica, 55, 251276.

Fama, E. F. (1970): "Efficient Capital Markets: A Review of Theory and Empirical Work," Journal of Finance, 25(2), 383–417.

Fama, E. F. (1991): "Efficient Capital Markets: II," Journal of Finance, 46(5), 1575–617.

Johansen, S. (1991): Estimation and hypothesis testing of cointegration vectors in Gaussian vector autoregressive models. Econometrica 59: 1551–1581.

Johansen, S, and Juselius, K. (1990). Maximum likelihood estimation and inference on cointegration}with applications to the demand for money. Oxford Bulletin of Economics and Statistics 52: 169–210.

Johansen, S. (2008): "A representation theory for a class of vector autoregressive models for fractional processes," Econometric Theory, Vol 24, 3, 651–676.

Nielsen, M. Ø. (2005): "Multivariate Lagrange Multiplier Tests for Fractional Integration," Journal of Financial Econometrics, 3(3), 372–398.

Nielsen, M. Ø., and K. Shimotsu (2007): "Determining the cointegration rank in nonstationary
fractional system by the exact local Whittle approach," Journal of Econometrics, 141, 574–596.

Parkinson, M. (1980): "The Extreme Value Method for Estimating the Variance of the Rate of Return," The Journal of Business, 53, 61–65.

Shimotsu, K., and P. Phillips (2005): "Exact local Whittle estimation of fractional integration.,"
Annals of Statistics, 33, 1890–1933.

Author

Supervisor

1 Introduction

Daily high and low stock prices provide valuable information about range-based volatility, which is not included in the open and close prices. Parkinson (1980) shows that the daily price range-based estimator of volatility, defined as the difference between daily high and low prices, is a highly efficient and robust estimator of volatility.

The stock prices in developed markets tend to exhibit stochastic trends, and in most studies they are shown to be $I(1)$ processes (see e.g. Cheung, 2007). However, as Caporin et al. (2011) show, the choice between $I(0)$ and $I(1)$ processes can be too restrictive and the degree of integration of daily highs and lows should be estimated in a fractional or long-memory framework. Thus, the series are assumed to be integrated of order d and cointegrated of order $(d-b)$, where $0 < b \leq d$. In order to determine the relationship between highs and lows we implement a fractionally cointegrated vector autoregressive model (FCVAR) as proposed by Johansen (2008) and further analysed by Johansen and Nielsen (2010, 2012). The motivation for using this methodology is twofold. First, daily highs and lows are assumed to be cointegrated, i.e. in the short term they may diverge, but in the long term they have an embedded convergence path. Second, their linear combination, the range, is assumed to display a long memory. This property of range is well captured by the fractional autoregressive technique and the range is, in fact, the error correction term in the cointegrating relation between high and low log-prices.

In the thesis, the above-described methodology is applied in the following way. The high and low log-prices and range are first tested for the presence of a unit root based on the Augmented Dickey-Fuller test (ADF). The results imply stationarity of the range, which is, however, not consistent with its slowly decaying autocorrelation function. This is an indication of a potential long-memory nature of the range process, which we estimate through the Exact Local Whittle and Geweke Porter-Hudak estimators. Substantial evidence of the presence of long memory has been demonstrated in the literature not only in the volatility of asset prices, but also in the interest rate differentials, inflation rates, forward premiums, or exchange rates (Baillie, 1996). This concept has been widely examined in the literature and there is more or less a consensus that volatility is a long-memory process. However, not many studies report that volatility is a non-stationary process (e.g. when the order of integration of range is bigger than 0.5). Kellard et al. (2010) were one of the first who reported volatility as a non-stationary process.

When long memory is present in the range, the general well-known tests for cointegration are not suitable. We thus use two cointegration estimators which account for the long memory. Namely, we apply the methodology proposed by Nielsen and Shimotsu (2007) and the cointegration rank test which is part of the FCVAR methodology described by Johansen and Nielsen (2012). From the very first concept of cointegration analysis proposed by Granger (1986) or Engle and Granger (1987), cointegration has been defined for a general case where the series are integrated of order d but are cointegrated of order $(d - b)$, where $0 < b \leq d$. However, a complete estimation method has been developed only for the case when the series are $I(1)$ and cointegrated of order 0 (e.g. when $d = b = 1$). This method has been predominantly used in the literature for the analysis of cointegrated series. Only recently, Johansen and Nielsen (2012) formalized the statistical methodology for the general case. Hence, its application in this thesis is a valuable contribution to current research.

The analysis is performed on four of the global stock market's principal indices, namely, the American Standard and Poor's 500, the German DAX, the Japanese NIKKEI 225, and the UK's FTSE 100 index over 2003-2012 period. These results are compared to the Czech PX index over the same period. For all five indices, logarithmic transformation of daily high and low stock prices is considered. The period of interest is divided into two sub-periods with December 2007 as the breakpoint, which enables us to compare both cointegration and estimated volatility before and during the recent financial crisis.

Our findings are as follows. We find significant evidence of long memory in daily range. In most cases, the range-based volatility estimate is in the non-stationary region. There are only two exceptions: the ranges of the PX 50 index and the NIKKEI 225 index in the 2003-2007 period, which are in the stationary region. The estimates of the long memory parameters are, however, quite sensitive to the selected methodology. We also show that the first period (2003-2007) is rather calm, as the estimates of long memory are generally the lowest, while the second period, 2008-2012, is more turbulent. Overall, the PX 50 index displays the lowest estimates of the order of price range integration, and its behaviour is very similar to that of the NIKKEI 225. The ranges of S&P 500, FTSE 100 and DAX indices display, on the other hand, relatively higher orders of integration. Furthermore, we show that the unrestricted FCVAR performs better in detecting the stationarity of range than the FCVAR specification with restrictions on the cointegrating vector.

The remainder of the study is organised as follows. Chapter 2 contains motivation for using daily high and low prices and data description. Chapter 3 deals with the regular CVAR framework, describes the unit root and cointegration tests, and defines the CVAR model. Chapter 4 discusses long-memory processes as well as the formal tests for long memory and the test for the equality of integration orders. Chapter 5 describes fractional cointegration and presents the FCVAR model and the results of our analysis. Chapter 6 concludes.

2 Data description

Stock prices have been widely examined in different fields of economic research. Most studies employ the close-to-close return analysis (as standard econometric tools require stationarity of the data) and frequently ignore the information stemming from daily high and low prices. By the high price we understand the maximum price observed during the day, and the low price is the minimum price achieved during that day. These prices contain additional information about the change of direction of excess demand (Cheung, 2007). Other reasons why daily high and low prices can be of great importance are summarised by Caporin et al. (2011). First, daily high and low prices can have the function of a reference level. Market agents use these reference levels to make assumptions and predictions about future developments and use daily highs and lows as reference values. Second, daily highs and lows might work as a stop-loss indicator and may contain information about liquidity provision and the price discovery process. Third, high and low prices are more probable to correspond to ask and bid quotes, respectively, implying they may be influenced by transaction costs and other frictions (e.g. price discreteness, stale prices, or tick size). Moreover, daily high and low prices tend to react to unanticipated public announcements or other unexpected shocks.

Daily high and low stock prices are mainly valuable as a measure of dispersion. Parkinson (1980) shows that a variance estimator based on close-to-close returns is a far less efficient volatility estimator than the price range (defined as a difference between daily high and low log-prices). Alizadeh et al. (2002) also show that the range-based estimator of volatility is highly statistically efficient and robust with respect to many microstructure frictions. Also, they state that range-based volatility proxies are much less contaminated by measurement error and they explain not only the autocorrelation of volatility but also the volatility of volatility. Furthermore, Corwin and Schultz (2012) argue that since daily high and low prices are mostly buy and sell trades, respectively, the price range thus represents a fundamental volatility as it reflects both the stock's variance and its bid-ask spread. Alizadeh et al. (2002) note the range as a volatility proxy has a "long and colourful" history in finance (e.g. Garman and Klass, 1980; Parkinson, 1980 or Andersen and Bollerslev, 1998). More recently, Caporin et al. (2011) find evidence of long memory in the range of all 30 components of the DJIA index during the 2003-2010 period.

This thesis contains analysis of daily high and low log-prices of four major global indices over the 2003-2012 period. This sample period is quite representative, since the 10 years cover the calm and liquid markets of the last boom period as well as the recent financial crisis. We consider 5 world indices: the American S&P 500, the German DAX, the Japanese NIKKEI 225, and the UK's FTSE 100, available from Yahoo! Finance, which are examined and compared to the Czech PX stock market index (available at stocktrading.cz). Each index is examined during the whole 10-year period from January 2003 to December 2012 as well as during two sub-periods. The first sub-period covers the pre-crisis years from January 2003 till December 2007. This break has been chosen based on the statement of the National Bureau of Economic Research (NBER), which has identified December 2007 as the peak of the pre-crisis economic activity. As rationale the NBER states that the subsequent decline in economic activity was large enough to be qualified as a recession. The second sub-period (form January 2008 till December 2012) thus covers the crisis and post-crisis periods. According to the NBER, the recession that started at the end of 2007 ended in June 2009, which is the last trough of the economic activity as well as the last business cycle reference date. This breakpoint is, however, not considered in our analysis. There were no more breakpoints announced by the NBER during our study sample.

For the United States we analyse the Standard and Poor's 500 index (S&P 500). This index is one of the most closely watched American indices, along with the DJIA or the NASDAQ Composite (NASDAQ). It is quoted in USD and contains 500 leading companies publicly traded in the US. As it covers stocks from all major American exchanges, this index is considered by many to be the best representation of the US market. FTSE 100, quoted in GBP, is a share index of the 100 largest companies by market capitalization listed on the London Stock Exchange. DAX is a German stock market index composed of 30 major companies traded on the Frankfurt Stock Exchange and is quoted in EUR. NIKKEI 225 is a stock market index for the Tokyo Stock Exchange quoted in JPY and is the most extensively quoted average of Japanese equities. And finally, PX 50 is the official index of the major stocks trading on the Prague Stock Exchange and is quoted in CZK.

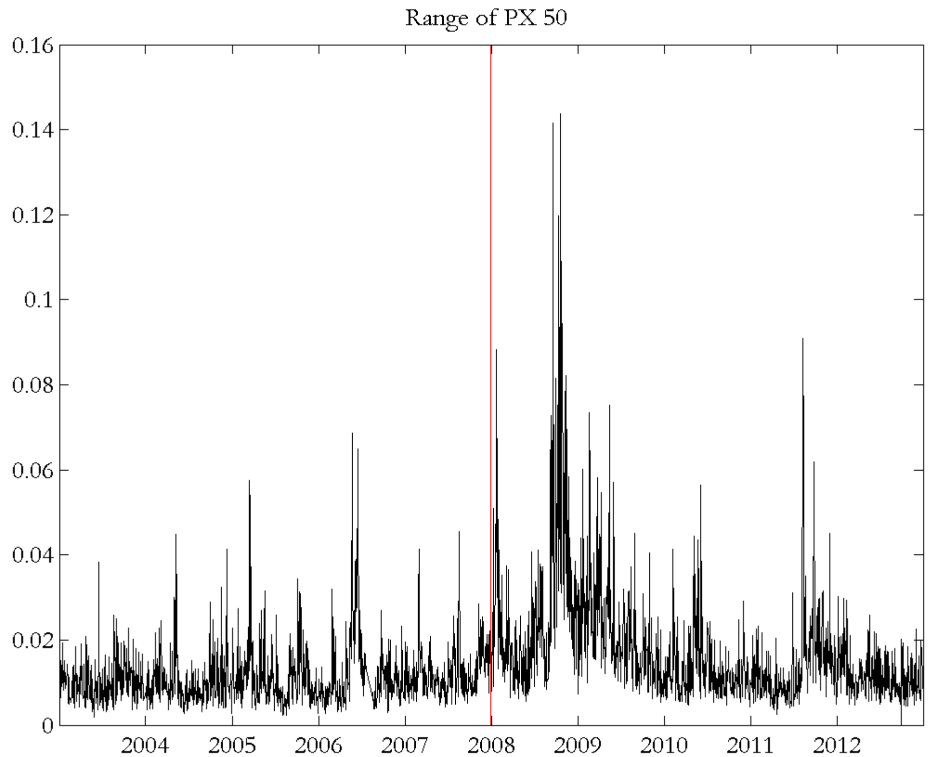
Daily data on high stock prices (P_t^H) and low stock prices (P_t^L) is available for the five indices of interest. Following Cheung (2007) or Caporin et al. (2011), we consider the logarithmic transformation, $p_t^H = \log(P_t^H)$ and $p_t^L = \log(P_t^L)$. In the two following figures (Figure 1 and Figure 2) we can see the development of daily high and low log-prices of the PX index and their difference – the range. The red line marks the NBER peak in December 2007. From Figure 1 we can see that the PX 50 index was experiencing steep growth in the first period, but at the end of 2007 it was severely hit by the crisis. After a stable period from mid-2009 through mid-2011, another drop followed; however it was less dramatic than the one seen at the end of 2007. In Figure 2 we can see that range-based volatility (measured as the difference between daily high and low log-prices) is significantly higher after the outbreak of the crisis and reaches its maximum at the end of 2008; then it gradually returns to its pre-crisis values.

Figure 1: High and Low Log-Prices of PX 50

Similar figures for all remaining series can be found in the Appendix (see Figure 6 and Figure 7 for the DAX, Figure 8 and Figure 9 for the FTSE 100, Figure 10 and Figure 11 for the NIKKEI 225 and finally, Figure 12 and Figure 13 for the S&P 500). With the DAX we can see a slightly different behaviour, mainly after the crisis, when it has grown steadily since the drop in 2007, with the exception of a short period of decline at the end of 2011. This was also the case with the S&P 500. However, the FTSE 100 displayed only a very slight or no growth after the crisis, and the NIKKEI 225 has actually declined.

As for volatility, the DAX started the period with relatively high levels of volatility, which then slightly decreased during the first period, after which two peaks followed, one at the end of 2008 and another at the end of 2011. The ranges of FTSE 100 and S&P 500 exhibit a similar pattern. The range of NIKKEI 225 is quite stable in the first period, then a peak follows at the end of 2008, similarly to the other indices. However, there is another high peak at the beginning of 2011 caused by a drop in the daily low log-price.

Figure 2: Range of PX 50

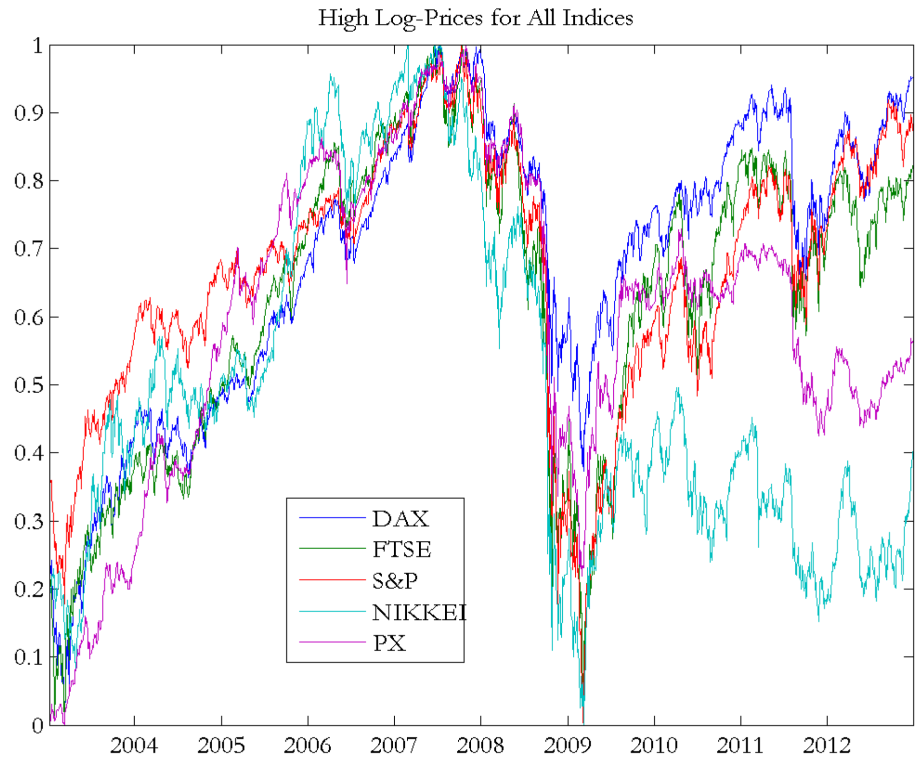


Comparison of high log-prices for all five indices is displayed in Figure 3. We present only the development of high log-prices since the development of the low log-prices is very similar, as can be seen in the individual plots of high and low log-prices of the indices. In this figure, all high log-prices are standardised to make comparison possible. We apply the following transformation:

$$\frac{p_t^H - \min(p_t^H)}{\max(p_t^H) - \min(p_t^H)}.$$

We only include dates which are present in all series in order to maintain constant data length. Because each index comes from a different country, some observations might otherwise be missing for some of the indices, for example due to different public holidays creating gaps in the time-series of each index. To avoid inconsistencies, even if only one index is missing the observation on a given day, the whole day is excluded from all five series.

Figure 3: Comparison of High Log-Prices for All Indices



All indices seem to be influenced by similar common factors; however, the reaction to the crisis is quite different for each of them. All indices experienced a rapid growth during the 2003-2007 period. At the end of 2007, a steep downturn followed. This figure shows that the DAX and PX 50 indices were the two indices least hit by the crisis. The DAX was also the first index in our sample to recover from the crisis and remains the best-performing one. On the other hand, the PX 50 index was the second-least hit by the crisis, but is the second worst-performing index as of the end of 2012. Until mid-2009 it was closely following the DAX, but then it lost pace and fell at the end of 2011. The other indices also experienced this fall, but contrary to the PX 50 and NIKKEI 225 indices they were able to resume their previous growth. Even though NIKKEI 225 outperformed the remaining indices slightly before the crisis, this was no longer true after the crisis, when its performance has been the worst; it never quite recuperated from the crisis. Since the beginning of 2012 the S&P 500 has been catching up with DAX, and the

FTSE 100 follows close behind. Overall, we can see that before the crisis, all indices were growing together, but the reactions to the crisis quite varied.

3 Cointegrated VAR framework

Following the intuition that daily high and low log-prices are non-stationary and share a common trend, we may use the original cointegrated VAR as it is used for example in Cheung (2007). However, later in the study we show why the CVAR is not the most appropriate model for our analysis. First, the stationarity of the data is checked by the Augmented Dickey-Fuller test. Then the possible cointegrating relation is discussed and the CVAR is defined.

3.1 Unit root tests

There is a common belief that stock market prices are non-stationary. The assumption that the best estimate of future price is today's price leads to the hypothesis of prices following a random walk. Having this in mind, we first test the data for stationarity.

Daily data on high stock prices (P_t^H) and low stock prices (P_t^L) is available. We employ the logarithmic transformation $p_t^H = \log(P_t^H)$ and $p_t^L = \log(P_t^L)$ following Cheung (2007) or Caporin et al. (2011). We define the following variable $X_t \equiv (p_t^H, p_t^L)'$, which will be used throughout the remainder of the study. It is a (2×1) vector; however, the methodology is developed for a general case, where X_t is an $(n \times 1)$ vector. Based on the simple visualization of the daily high and low log-prices (see Figure 3 in the previous chapter), a strong non-stationarity is apparent.

For preliminary unit-root testing, we apply the Augmented Dickey-Fuller (ADF) test with the null hypothesis that each component of the series X_t has a unit root (i.e. is an $I(1)$ process) against the alternative that each component of X_t is a stationary $I(0)$ process, with the assumption that the data contains an ARMA structure. This test is influenced by the presence, but not by the value, of the deterministic terms in the regression, meaning that different critical values should be applied if a constant, a trend, or both are present.

The ADF test can be formally written in the following way (e.g. Cipra, 2008):

$$\Delta X_t = c + \delta t + \Pi X_{t-1} + \sum_{j=1}^{p} \Psi_j \Delta X_{t-j} + \varepsilon_t,$$

$$\mathcal{H}_0 \colon \Pi = 0, c = 0, \delta = 0,$$

$$\mathcal{H}_1 : \Pi \neq 0, c \neq 0, \delta \neq 0$$

where c is an n-dimensional vector of constants, δ is an n-dimensional vector of trend parameters, t denotes the trend, Π and Ψ_j are diagonal matrices of parameters, ε_t is an n-dimensional vector of random shocks and ΔX_{t-j} captures the remaining serial correlation in the residuals, where $j = 1, 2, \dots, p$. The value of p is selected to minimize the Bayesian information criteria (BIC). For most indices and sub-periods we consider only the inclusion of a constant as the sole deterministic element; the trend is not included as it is found insignificant in the regressions.

The hypotheses then are in the following form:

$$\mathcal{H}_0 \colon \Delta X_t = \sum_{j=1}^{p} \Psi_j \Delta X_{t-j} + \varepsilon_t,$$

$$\mathcal{H}_1 \colon \Delta X_t = c + \Pi X_{t-1} \sum_{j=1}^{p} \Psi_j \Delta X_{t-j} + \varepsilon_t,$$

The only exceptions are the tests of unit root for the daily high and low log-prices of DAX, FTSE 100 and S&P 500 in the first period, where we include both a constant and a trend. It is readily apparent from the visualisation of the series that a trend is present in these series in the first period. The trend is not, however, included when testing the stationarity of daily highs and lows of PX 50 and NIKKEI 225 in the first period, as the trend is found insignificant in the regressions.

The hypotheses for these three particular cases (DAX, FTSE 100, S&P 500 in the first period) are in the following form:

$$\mathcal{H}_0\colon \Delta X_t = \sum_{j=1}^{p} \Psi_j \Delta X_{t-j} + \varepsilon_t,$$

$$\mathcal{H}_1\colon \Delta X_t = c + \delta t + \Pi X_{t-1} \sum_{j=1}^{p} \Psi_j \Delta X_{t-j} + \varepsilon_t.$$

The results of these tests are presented below in Table 1.

Table 1: P-values for an ADF test for levels and first-differences

		ADF_H		ADF_L		ADF_R
		level	Δ	level	Δ	level
S&P 500						
c	2003-2012	0.2947	0.001	0.2697	0.001	0.001
c, t	2003-2007	0.6369	0.001	0.2830	0.001	0.001
c	2008-2012	0.5718	0.001	0.5333	0.001	0.001
FTSE 100						
c	2003-2012	0.2683	0.001	0.1467	0.001	0.001
c, t	2003-2007	0.0225	-	0.0180	-	0.001
	2008-2012	0.2863	0.001	0.1917	0.001	0.001
DAX						
c	2003-2012	0.3666	0.001	0.3380	0.001	0.001
c, t	2003-2007	0.0419	-	0.0329	-	0.001
c	2008-2012	0.3246	0.001	0.1799	0.001	0.001
NIKKEI 225						
c	2003-2012	0.4489	0.001	0.3779	0.001	0.001
c	2003-2007	0.4578	0.001	0.4349	0.001	0.001
c	2008-2012	0.1224	0.001	0.0979	0.001	0.001
PX 50						
c	2003-2012	0.1107	0.001	0.1162	0.001	0.001
c	2003-2007	0.2578	0.001	0.3114	0.001	0.001
c	2008-2012	0.2488	0.001	0.1570	0.001	0.001

Note: The letter *"c"* denotes the inclusion of a constant only, letter *"t"* denotes additional inclusion of a trend.

Table 1 presents the p-values of the ADF test for daily high (ADF_H) and low (ADF_L) log-prices as well as for their difference, range (ADF_R). The stationarity of range is tested because we believe that a stationary linear combination of daily high and low stock prices exists. The ADF test is first applied on levels and if they are not found stationary, the test is then applied on first-differences. Thus we decide whether a time series is stationary (an

$I(0)$ process) or difference-stationary (when only first differencing is needed we call it an $I(1)$ process).

What we can see from Table 1 is that the range is always stationary and thus an $I(0)$ process and that the daily highs and lows are mostly $I(1)$ processes. There are, however, two exceptions. When we include the trend into the regression of DAX and FTSE 100 we find that the daily high and low log-prices in the first period are stationary on the 5% level of significance. This means that they are trend-stationary. This can be a slight problem for further application of the fractional methodology since it is a possibility that the long memory is caused by the trend. If we detrend these series, then based on the ADF test, we get a stationary process; however, the ACFs of the detrended series decline at a very slow rate, which is definitely not consistent with a stationary $I(0)$ process. The plots of the ACF of both detrended series in the first period are in the Appendix (see Figure 14 and Figure 15).

The shocks to a stationary $I(0)$ process decay at an exponential rate, while no dissipation of shocks occurs in a non-stationary $I(1)$ process. However, it may happen that based on the standard unit-root test the series is stationary, but the autocorrelation function displays a slow (non-exponential) decay, which is not consistent with a stationary process. In such a case we can assume the presence of long memory and consider the series to be fractionally integrated of order d, meaning that the d'th difference is a stationary $I(0)$ process, where d can be any real number. These fractionally integrated processes allow for more flexibility than the extreme assumption of a unit root and the corresponding implications for a shock persistence in the system.

To illustrate the possibility of the presence of long memory in range we examine its autocorrelation functions (ACF). In Figure 4 we show the ACF of DAX range in all periods. We can see that despite the stationarity of range in all periods being implied by the ADF test, the ACF of the range is in no period declining exponentially, but rather displays a high degree of persistence. This can be an indication of long memory. On the other hand, the range of the PX 50 index in the first period, reported in the second panel of Figure 5 does not display a long memory, as its ACF decays exponentially. All remaining ACFs are reported in the Appendix (see Figure 16, Figure 17, and Figure 18);

and all ranges display long memory. All autocorrelations are significant even after 40 lags, except for the PX 50 index in the first period, displayed below in Figure 5.

The overall conclusion from the dependence in the autocorrelation functions is that the highest dependence is present either in the full study period of 2003-2012 or in the second period of 2008-2012. The first period, 2003-2012, displays the lowest degree of persistence, especially in the case of NIKKEI 225 and PX 50, where we might even say that long memory is not present. On the other hand, the behaviour of DAX is quite an exception. The highest degree of persistence can be seen in the first period and the lowest in the second, which is the opposite of the other series' behaviour.

Figure 4: ACF of DAX Range in All Periods

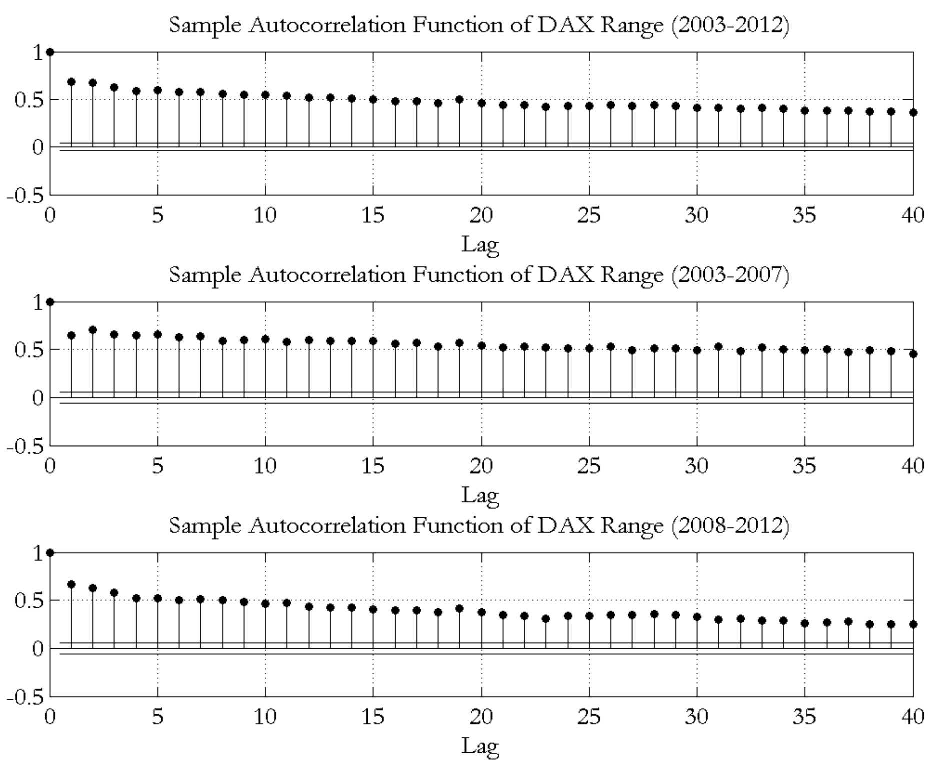
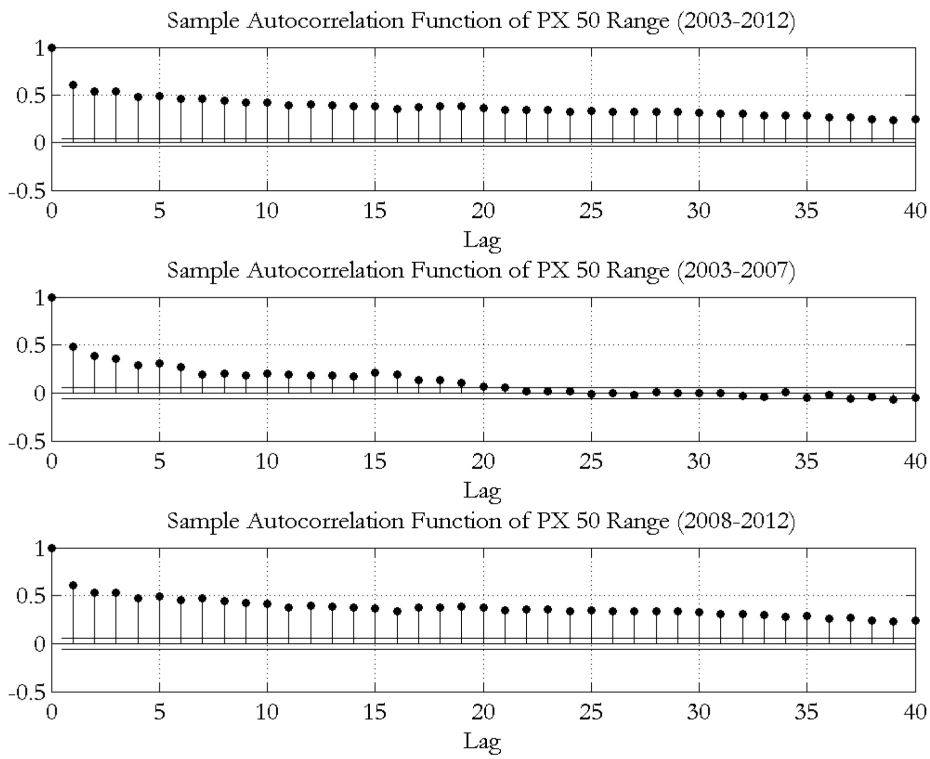


Figure 5: ACF of PX 50 in All Periods



3.2 Cointegration tests

Based on the ADF test results and the inspection of the autocorrelation functions of the detrended DAX and FTSE 100 in the first period (which were identified as trend-stationary by the ADF test) we can conclude that all series are non-stationary. Engle and Granger (1987) state that a linear combination of two or more non-stationary series can be stationary. If this is the case, we say that the non-stationary series are cointegrated and the stationary linear combination is called a cointegrating equation. This equation is then interpreted as a long-run equilibrium relationship among the variables. If daily highs and lows are cointegrated, it means that in the short term they may diverge, but in the long term they have an embedded convergence path. In the previous section we have tested the stationarity of range, which is a linear combination of two non-stationary series. The stationarity of range was confirmed and thus we can postulate that daily high and low log-prices are cointegrated.

In order to be able to apply a proper model which takes the cointegration into account, we first define a Vector Autoregression (VAR) model. This model requires the series to be stationary. If the series are non-stationary, the usual method of dealing with this issue is to use first differences. Even though this transition is statistically correct, it omits the long-run relationship between the original (non-differenced) variables if the variables are cointegrated. In the presence of cointegration the basic VAR is transformed into the cointegrated VAR (CVAR) model, which enables us to describe the short-term relationships between the growth components and also corrects for short-term deviations from the long-run equilibrium.

Consider the VAR model of order p, (see e.g. Cipra, 2008)

$$X_t = \mu + \sum_{i=1}^{p} \Pi_i X_{t-i} + \varepsilon_t, \qquad t = 1, \dots T,$$

where X_t is an $(n \times 1)$ vector, in our case it is containing only the series of daily high and low logarithmic stock prices, μ is an n-dimensional vector of constants, Π_i is an $(n \times p)$ matrix of autoregressive coefficients, $i = 1, 2, \dots, p$, and ε_t is an n-dimensional $i.i.d.(0, \Omega)$ vector of random shocks, where Ω is a positive-definite variance matrix. The VAR model is frequently used by econometricians, as it provides a simple tool for the analysis of multivariate time series. It is a natural extension of the univariate autoregressive model. Among other advantages, all variables are considered endogenous, so there is no need to specify which variables are to be considered endogenous and which exogenous. Each variable depends on its own lagged values, white noise, and on the lagged values of the other variables included in the model.

We can rewrite this VAR model as

$$\Delta X_t = \mu + \Pi X_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta X_{t-i} + \varepsilon_t, \qquad t = 1, \dots T,$$

where $\Pi = \sum_{i=1}^{p-1} \Pi_i - \mathrm{I}$ and $\Gamma_i = -\sum_{j=i+1}^{p} \Pi_j$ for $i = 1, 2, \dots, p-1$.

If the coefficient matrix Π is of reduced rank $< n$, then there exist matrices α and β, both of rank r, such that $\Pi = \alpha\beta'$ and $\beta' X_t$ is a stationary $I(0)$ process; r is then the cointegration rank representing the number of cointegrating relations. Furthermore, each of the columns of β is a cointegrating vector and α is a vector of adjustment parameters.

In our particular case where $n = 2$, we are mostly interested in the situation $r = 1$. In this case, out of the two eigenvalues of matrix Π exactly one eigenvalue is nonzero, meaning that there is exactly one unit root in the system of two time series.

To formally test for cointegration the Johansen procedure is commonly used. This procedure consists of estimating the Π matrix by unrestricted VAR and testing whether we can reject the restrictions implied by the reduced rank of Π.

The Johansen procedure (taken from Cipra, 2008) tests for significant canonical correlations measuring the partial dependency between the n-dimensional vectors ΔX_t and X_{t-1} when $\Delta X_{t-1}, \ldots, \Delta X_{t-p+1}$ are fixed. These canonical correlations are the squared roots of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of a positive semi-definite matrix closely related to Π. In particular, the number of positive values among $\lambda_1, \ldots, \lambda_r$ is equal to the rank r of the matrix Π. There are two tests which can be applied as a preliminary analysis of cointegration (for more details see Johansen, 1991): the Johansen Trace Test and the Johansen Maximum Eigenvalue Test. These tests are essentially based on likelihood ratios (LR), whose critical values are simulated, and test the null of λ being zero.

Consider the Johansen Trace Test with the test statistic:

$$\lambda_{trace}(r) = -T \sum_{i=r+1}^{n} \ln(1 - \hat{\lambda}_i),$$

which has the joint null hypothesis of no more than r cointegrating relations, against the alternative of more than r cointegrating relations. The test is performed successively for $r = 0,1, \ldots n - 1$. (In our case where $n = 2$, it would only be performed for $r = 0,1$.)

The Johansen Maximum Eigenvalue Test has the test statistic:

$$\lambda_{max}(r) = -T \ln\left(1 - \hat{\lambda}_{r+1}\right),$$

with the null hypothesis of r cointegrating relations against the alternative of $(r+1)$ cointegrating relations. The test is conducted in the same manner as the Johansen Trace Test.

We do not provide the results of the cointegration tests here, because these tests are not applicable for fractionally cointegrated variables. The definition of fractional cointegration is provided later in this study. However, the presence of cointegration is, in most cases, obvious since the daily high and low log-prices are driven by common trends and are the prices of one underlying index. Moreover, based on the ADF tests of range we can see that there exists a combination of two non-stationary variables (the daily high and low stock prices) which is stationary. There are two exceptions: In the first period DAX and FTSE 100 seem to be trend-stationary; however, based on the visualisation of the autocorrelation functions of the detrended series we can see that they remain non-stationary even after detrending. We will, however, keep these results in mind and address this later in the study. In the preliminary analysis, the presence of cointegration was confirmed and the cointegrating vector was mostly $(1,-1)$. These findings are, however, only valid as long as we do not consider the presence of long memory (e.g. the presence of fractional integration/cointegration). Thus a more general approach, which takes long memory into account, shall be discussed later and the results of this generalised test will be provided.

3.3 Empirical model

When the variables are non-stationary and cointegrated, one can apply the cointegrated VAR model (or the vector error correction model – VECM), in which information about the long-run equilibrium is not lost. For an n-dimensional non-stationary time series X_t we define the CVAR in the following form (see e.g. Johansen and Nielsen, 2010):

$$\Delta X_t = \alpha(\beta' X_{t-1} + \rho') + \sum_{i=1}^{p} \Gamma_i \Delta X_{t-1} + \varepsilon_t, \qquad t = 1, \dots, T,$$

where $\Delta X_{t-i} = X_{t-i} - X_{t-i-1}$ and where ε_t is an n-dimensional vector of $i.i.d.(0, \Omega)$ random shocks, where Ω is a positive-definite variance matrix. The model is used to describe the long-run economic relations given by the stationary combinations $\beta' X_t$. The parameters $\Gamma = (\Gamma_1, \dots, \Gamma_p)$ govern the short-run dynamics. The parameter ρ is the restricted constant term (since the constant term in the model is restricted to be of the form $\mu = \alpha\rho'$) and is interpreted as the mean level of the long-run equilibrium.

We do not estimate the cointegrated VAR, because this methodology is not suitable for fractionally cointegrated processes with an error correction term which is potentially non-stationary. This model needs some generalization in order to be able to take into account the fractional cointegration of daily high and low log-prices and the potential long memory of their linear combination. This generalization and the definition of the long memory process are described in the following chapters.

4 Long memory processes

In this chapter we analyse the so called long memory processes, for whose description the fractional models are very useful. So far, we have assumed that a series can be integrated of order 0 or 1; it is, however, possible that the series is instead integrated of order d, where d can be any real number. When $d < 0$, we say that the process is anti-persistent. When $d > 0$ we refer to the process as having long memory. The process is said to be stationary when $d < 1/2$ and non-stationary when $d > 1/2$.

Long-memory models have been used by the natural sciences (specifically, hydrology and climatology) since 1950's.They drew the attention of econometricians around 1980 when Granger (1980) and Granger and Joyeux (1980) developed the autoregressive fractionally integrated moving average (ARFIMA) and Geweke and Porter-Hudak (1983) proposed a technique for estimating the d parameter. Later on, the fractionally integrated GARCH(p, d, q) (FIGARCH) was proposed by Baillie (1996). The long memory processes are characterized by slowly decaying autocorrelation functions, with the decay rate being hyperbolic. This decay of the ACF is, for example, slower than the exponential decay associated with ARMA models. The persistence in the ACF is consistent neither with an $I(1)$ process nor with an $I(0)$ process. If the series displays long memory, the

ACF may appear to be non-stationary while at the same time the differenced series appears over-differenced.

Substantial evidence of the presence of long memory has been demonstrated on financial data such as the volatility of asset prices, interest rate differentials, inflation rates, forward premiums, or exchange rates (Baillie, 1996). Evidence of long-range dependence in asset price volatility was also found for example by Ding, Granger and Engel (1993), Andersen and Bollerslev (1997), Breidt et al. (1998), Kellard et al. (2010) or Garvey and Gallagher (2012). Breidt et al. (1998) show that this type of persistence cannot be appropriately modelled by the classic autoregressive conditional heteroskedastic models (ARCH) and their commonly used variations the generalized ARCH (GARCH) or exponential generalized ARCH (EGARCH). These models are short-memory models and the only way to incorporate long-memory persistence is by unit root. Both ARFIMA and FIGARCH thus allow d to take any value between 0 and 1. In this study, we are dealing with a multivariate case, which is why we later on describe the fractionally cointegrated vector autoregressive (FCVAR) model, which is capable of capturing both the long memory and the fact that the series are cointegrated.

The literature also mentions some challenges to the long-memory findings. Lobato and Savin (1997) note that the presence of long memory can be spurious due to the effect of aggregation. This means that the individual stocks may not exhibit long memory, but the index constructed from them does due to aggregation. Granger and Hyung (2004) propose structural breaks as another challenge to the presence of long memory. They show that a structural break in volatility can reproduce a slowly decaying ACF and other properties of long-memory processes.

We can define a long-memory process as in Brockwell and Davis (1991): A weakly stationary process has long memory if its autocorrelation function ACF $\rho(\cdot)$ has a hyperbolic decay,

$$\rho(h) \sim C \cdot h^{2d-1} \text{ as } h \to \infty,$$

where $C \neq 0, d < 0.5$.

In contrast, a weakly stationary process has short memory if its autocorrelation function ACF $\rho(\cdot)$ is geometrically bounded.

$$|\rho(h)| \leq C \cdot r^{|h|} \text{ for some } C > 0, 0 < r < 1.$$

The evidence of long-range dependence in asset price volatility means that a shock affecting the evolution of volatility will persist for a long time.

We have already learned that daily range is not an $I(1)$ process and should be considered an $I(0)$ process based on the ADF results. This implies that daily high and low stock prices are cointegrated, as the stationary range is their linear combination. However, the high dependency in most of the ACF functions of range for the five indices is not compatible with the $I(0)$ assumption derived from the ADF test. This leads us to characterize range by a fractional degree of integration. Caporin et al. (2007) note that the traditional notion of cointegration is not consistent with the existence of long memory, so they analyse the degree of integration of daily high and low stock prices and of their difference (range) in a fractional or long-memory framework.

Classical cointegration is considered in the "$I(1)/I(0)$" context. We mentioned that an $(n \text{ x } 1)$ vector X_t with $I(1)$ elements is cointegrated if there exists a linear combination $\beta' X_t$ which is an $I(0)$ process. Robinson and Yajima (2002) note that the possibility of existence of a long-run, stable relationship among non-stationary series X_t does not depend on whether or not the series are $I(1)$. There can exist a cointegration with $\beta' X_t$ which is stationary but not necessarily $I(0)$. This need for a flexible approach is solved by considering an $I(d)$ series with a real-valued d.

According to Robinson and Yajima (2002), for $d < \frac{1}{2}$, we say that X_t is $I(d)$ if

$$u_t = (1 - L)^d X_t$$

is $I(0)$, where L is the lag operator and

$$(1 - L)^d = \sum_{i=0}^{\infty} \frac{\Gamma(i - d)}{\Gamma(-d)\Gamma(i + 1)} L^i,$$

where $\Gamma(.)$ is the *gamma* function defined as $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx$ for $a > 0$, while for $a = -n$, $n = 0, 1, \ldots$, $\Gamma(a)$ has simple poles with residues $\frac{(-1)^n}{n}$, and for other $a < 0$, $\Gamma(a)$ can be obtained by the recursion $\Gamma(a) = \frac{\Gamma(a+1)}{a}$. We then say that X_t is fractionally integrated of order $d < 1/2$ and is covariance-stationary.

For $d \geq \frac{1}{2}$ we define a non-stationary $I(d)$ series $X_t = (1-L)^{-d}u_t I\{t \geq 1\}, t = 0, \pm 1, \pm 2, \ldots$, where $\mathrm{I}(.)$ is the indicator function.

Shimotsu and Phillips (2005) show that the representation can also be rewritten in the following form, giving a valid linear representation of X_t:

$$u_t I\{t \geq 1\} = (1-L)^d X_t = \sum_{k=0}^{t-1} \frac{(-d)_k}{k!} X_{t-k},$$

so

$$X_t = (1-L)^{-d} u_t I\{t \geq 1\} = \sum_{k=0}^{t-1} \frac{(d)_k}{k!} u_{t-k},$$

where $(d)_k = \frac{\Gamma(d+k)}{\Gamma(d)} = d(d+1)\ldots(d+k-1)$ is the forward factorial function.

If $d > 0$ we say that the process has long memory, and if $d < 0$ we say that the process is anti-persistent. One can easily see that if $d = 1$ then the process represents a random walk and if $d = 0$ the process is stationary. The parameter d is called the fractional differencing parameter, fractional degree of persistence or fractional order of integration and describes the memory properties of X_t (Robinson and Yajima, 2002).

There are several methods of estimating the fractional degree of persistence d. It can be estimated within the FCVAR framework as it is described in Sub-section 5.2.1, as well as separately. Here we present two semi-parametric methods, a univariate exact local Whittle estimator (ELW) proposed by Shimotsu and Phillips (2005) and a procedure proposed by Geweke and Porter-Hudak (GPH) (1983). Since we are estimating the order of integration for two series (daily high and low log-prices) we also describe and conduct a test for the equality of integration orders proposed by Nielsen and Shimotsu (2007), as this is a condition for further FCVAR analysis.

4.1 Exact local Whittle estimator

The exact local Whittle estimator proposed by Shimotsu and Phillips (2005) makes no assumptions about the presence of cointegration and is consistent both when cointegration is present and when it is absent. This estimator is also applicable to both the stationary and non-stationary case, which removes the limitation of the original tests proposed by Robinson and Yajima (2002) which only allow stationary data. It also improves the original local Whittle (LW) estimator whose asymptotic theory was discontinuous for $d = \frac{3}{4}$ and again for $d = 1$.

We consider the fractional process X_t generated by the model

$$(1 - L)^d X_t = u_t, \qquad t = 1, 2, \dots T,$$

where u_t is stationary with zero mean and spectral density $f_u(\lambda)$.

Next we define the discrete Fourier transform of a time series x_t evaluated at frequency λ as

$$\omega_x(\lambda_j) = (2\pi T)^{-\frac{1}{2}} \sum_{t=1}^{n} x_t e^{it\lambda},$$

and the periodogram of a time series x_t computed at the Fourier frequency λ_j as

$$I_x(\lambda_j) = \left|\omega_x(\lambda_j)\right|^2, \qquad \lambda_j = \frac{2\pi j}{T} \ (j = 1, \dots, m < T).$$

Following Shimotsu and Phillips (2005), we minimize the (negative) Whittle likelihood function of stationary innovations u_t based on frequencies up to λ_m and up to scale multiplication. By applying the local approximation of spectral density, $f_u(\lambda) \sim G$ for $\lambda \to 0$, and an algebraic transformation, the likelihood function becomes data-dependent:

$$Q_m(G,d) = \frac{1}{m}\sum_{j=1}^{m}\left[\log\left(G\lambda_j^{-2d}\right) + \frac{1}{G}I_{\Delta^{\mathrm{d}}x}(\lambda_j)\right],$$

where $I_{\Delta^{\mathrm{d}}x}(\lambda_j)$ is the periodogram of $\Delta^{\mathrm{d}}X_t = (1-L)^d X_t$ at the Fourier frequency λ_j of the fractionally differenced series u_t. The number $m = m(T)$ denotes a bandwidth parameter determining the number of periodogram ordinates used in the equation. In our estimation we use two bandwidth parameter specifications; $m = T^{0.5}$ and $m = T^{0.6}$, which are the same bandwidth parameters as those chosen by Nielsen and Shimotsu (2007) for their empirical application.

We obtain the estimates of d and G by minimizing $Q_m(G,d)$, hence,

$$\left(\hat{G},\hat{d}\right) = \underset{G\in(0,\infty);d\in[\Delta_1,\Delta_2]}{\arg\min}\ Q_m(G,d),$$

where Δ_1 and Δ_2 are the lower and upper bounds, respectively, of the admissible values of d such that $-\infty < \Delta_1 < \Delta_2 < \infty$.

Furthermore, if we concentrate $Q_m(G,d)$ with respect to G, the Exact Local Whittle estimator of d reduces to

$$\hat{d}_i = \underset{d\in[\Delta_1,\Delta_2]}{\arg\min}\ R_i(d),$$

where

$$R_i(d) = \log\hat{G}_{ii}(d) - 2d\frac{1}{m}\sum_{j=1}^{m}\log\lambda_j \text{ and } \hat{G}_{ii}(d) = \frac{1}{m}I_{\Delta^{\mathrm{d}}\mathrm{x}_i}(\lambda_j).$$

In our case, $i = H, L$, $i = R$, respectively. Shimotsu and Phillips (2005) show that this exact local Whittle estimator is consistent and has a normal limit distribution $N(0,\frac{1}{4})$ for all values of d if the optimization covers an interval of a width lesser than $\frac{9}{2}$ and if the initial values of the process are assumed to be known. The word "exact" in the estimator's name is used to distinguish this estimator from the original LW estimator. It is "exact" because the estimator relies on an exact algebraic manipulation, rather than on an approximation as the conventional LW estimator does.

In Table 2 we present the estimates of the long memory parameter d based on the 2-step ELW estimator. For the trend-stationary series (DAX in the first period and FTSE 100 in the first period) we also include the results of the 2-step ELW estimator with prior detrending. The section "Bandwidths" contains three bandwidth figures: the first is the total number of observations (T) of each index in each period, and the two remaining figures are the bandwidth parameters used ($T^{0.5}$ and $T^{0.6}$). Then, we report the estimates of three long memory parameters for each bandwidth; the first one is for the series of high log-prices $(\hat{d}_H)$, the second one is for the series of low log-prices $(\hat{d}_L)$, and the third one is for their difference, the range $(\hat{d}_R)$. The estimates of the long memory parameters are provided for both bandwidths considered in order to show that the choice of bandwidth may, to a certain extent, influence the results. It is also possible to calculate the standard errors of the estimates, which are defined as $(4m)^{-1/2}$, where m is equal to either $T^{0.5}$ or $T^{0.6}$, depending on the bandwidth chosen. This means that for each bandwidth the standard errors are the same for all three integration orders ($\hat{d}_H$, $\hat{d}_L$, $\hat{d}_R$), as it only depends on the number of observations used for the calculation of the estimates.

The results in Table 2 support two our initial hypotheses. First, that daily highs and lows are not stationary. The order of integration is generally close to 1; however, in the full period and in the second sub-period unitary integration is substantially exceeded. Only in 25 cases out of 60 is the integration order of daily highs and lows smaller than 1. The lowest values of integration of daily highs and lows are achieved in the first period. In the case of the DAX and the FTSE 100, which, based on the ADF test, are assumed to be trend-stationary in the first period, we can see that even if the series are detrended first, they remain non-stationary as their order of integration is greater than 0.5. For the smaller bandwidth ($m = T^{0.5}$), the level of integration of FTSE 100 is closer to the marginal value of 0.5, but when the higher bandwidth is used the order of integration is around 0.8. In the case of DAX, both bandwidth specifications suggest non-stationarity of the detrended series. We thus conclude that based on the ELW estimates both series are non-stationary in all periods rather than trend-stationary.

Our second hypothesis was that the difference between high and low log-prices (range) is not stationary and displays long memory. We can see that $\hat{d}_R$ is significantly different from zero and in most cases it is in the region of non-stationarity (i.e., $d_R > 0.5$). Moreover, the order of integration of range is higher in the second period than in the first one, implying that the second period is more non-stationary. This supports our initial findings from the visualisation of the autocorrelations of range (except for the DAX).

Table 2: Exact local Whittle estimator of the fractional degree of integration

	Bandwidths			$ELW_{m=T^{0.5}}$				$ELW_{m=T^{0.6}}$			
	T	$T^{0.5}$	$T^{0.6}$	$\hat{d}_H$	$\hat{d}_L$	$\hat{\boldsymbol{d}}_{\boldsymbol{R}}$	s.e	$\hat{d}_H$	$\hat{d}_L$	$\hat{\boldsymbol{d}}_{\boldsymbol{R}}$	s.e.
S&P 500											
2003-2012	2517	50	109	1.0972	1.0709	**0.7626**	0.0707	1.0442	1.0036	**0.6216**	0.0479
2003-2007	1258	35	72	0.9094	0.8547	**0.5870**	0.0845	0.9265	0.8932	**0.5740**	0.0589
2008-2012	1259	35	72	1.1428	1.0853	**0.6643**	0.0845	1.0512	1.0130	**0.6867**	0.0589
FTSE 100											
2003-2012	2526	50	110	0.9990	0.9763	**0.6575**	0.0707	0.9673	0.9369	**0.6324**	0.0477
2003-2007	1264	35	72	0.8518	0.8182	**0.6220**	0.0845	0.8539	0.8357	**0.6028**	0.0589
detrending				0.5765	0.5877			0.8138	0.8011		
2008-2012	1262	35	72	1.0377	1.0026	**0.6278**	0.0845	0.9799	0.9568	**0.6169**	0.0589
DAX											
2003-2012	2556	50	110	1.0221	1.0055	**0.5884**	0.0707	1.0434	1.0058	**0.5887**	0.0477
2003-2007	1274	35	72	0.9711	0.9158	**0.6142**	0.0845	0.9817	0.9461	**0.5968**	0.0589
detrending				0.9076	0.8451			0.9630	0.9289		
2008-2012	1282	35	73	1.0913	1.0766	**0.7001**	0.0845	1.0237	1.0057	**0.6229**	0.0585
NIKKEI 225											
2003-2012	2543	49	108	1.0572	1.0372	**0.4833**	0.0714	1.0564	1.0428	**0.6493**	0.0481
2003-2007	1229	35	71	0.9909	0.9630	**0.4955**	0.0845	0.9755	0.9447	**0.3916**	0.0593
2008-2012	1224	34	71	1.1082	1.1018	**0.6131**	0.0857	1.0697	1.0544	**0.6044**	0.0593
PX 50											
2003-2012	2494	49	109	1.0990	1.0876	**0.5158**	0.0714	1.1659	1.1367	**0.5197**	0.0479
2003-2007	1235	35	71	0.8911	0.8601	**0.2811**	0.0845	1.0174	0.9659	**0.4455**	0.0593
2008-2012	1259	35	72	1.2337	1.2189	**0.5723**	0.0845	1.1216	1.0986	**0.5165**	0.0589

The PX 50 index is quite different from the other indices. In the first period, the range is in the stationary region; this is also true for the Japanese NIKKEI 225, but not for the other indices, as their range is not stationary in any of the three periods. We have also noted this weak dependence in the range of the PX 50 and NIKKEI 225 indices in the visualisation of the ACF functions in Section 3.1. Moreover, the range of NIKKEI 225 displays the highest variation in the estimates of the order of integration. For example, when the $m = T^{0.5}$ bandwidth is used the range in the full period is in the stationary region, but when the $m = T^{0.6}$ bandwidth is used the range is in the non-stationary region. The only conclusion drawn from the ELW estimator which differs from the conclusions drawn from the autocorrelation functions is in the case of the DAX index. The ELW estimator suggests that the highest dependence is present in the second period, but based on the ACF function of range this period should display the lowest dependence.

4.2 GPH estimator

The above-discussed procedure by Shimotsu and Phillips (2005) provides estimators for the fractional degree of persistence of daily high $(\hat{d}_H)$ and low $(\hat{d}_L)$ log-prices and their difference – range $(\hat{d}_R)$. Similar procedure used to obtain estimates of the long-memory parameters is proposed by Geweke and Porter-Hudak (GPH) (1983). They prove that provided a large enough sample (more than 50 observations), the asymptotic theory is reliable.

GPH (1983) again consider the fractional process $(1-L)^d X_t = u_t$, where u_t is a stationary linear process with spectral density $f_u(\lambda)$. The spectral density function is considered to be finite, bounded away from zero and continuous on the interval $\langle -\pi, \pi \rangle$. They also define the spectral density function of $\{X_t\}$ in the following form: $f(\lambda) = \frac{\sigma^2}{2T}\{4\, sin^2(\lambda)\}^{-d} f_u(\lambda)$.

The estimator is based on the following linear regression (taken from Garvey and Gallagher, 2012) of the log periodogram on a deterministic regressor:

$$\ln I(\lambda_j) = c - d\ln(4\left(\sin^2\left(\frac{\lambda_j}{2}\right)\right) + \epsilon(\lambda_j),$$

where $I_x(\lambda_j) = \left|\omega_x(\lambda_j)\right|^2$ is the periodogram of the data and $\lambda_j,\ j = 0, \dots, m < T$ is the Fourier frequency close to zero, c is a constant term and ϵ denotes the residuals. This estimator is semi-parametric, only assuming near the zero frequencies and treating spectral density non-parametrically when away from the origin.

The least-squares estimator of the slope parameter in the regression ($\hat{d}$) formed by only the lowest frequency ordinates of the log periodogram is then the estimator of our long memory parameter. The purpose of this estimator is to capture any fractional structure in the lower frequencies, i.e. it is looking for the correlation structure that is $I(d)$, where d is to be estimated. As noted by Kanzler (1998), the lower frequencies are covered exclusively, so no short-memory process specification is needed. The estimation only makes use of a relatively small number of the lowest frequencies; thus, only a restricted number of observations is included into the regression. If too many frequencies are considered, high-frequency cycles can be introduced, which would disrupt the analysis. On the other hand, if we do not include enough frequencies, the reliability of the OLS regression's results may become suspect. In the method proposed by Kanzler (1998), the first T^{-1} frequencies are excluded from the regression and all the remaining lower frequencies are included, up to the $(m = T^{0.5})$-th frequency. Kanzler (1998) argues that 0.5 is the standard value used in the literature. In our analysis we consider two values of this bandwidth parameter, $m = T^{0.5}$ and $m = T^{0.6}$, in the same manner as in the ELW estimation. The distribution of the long memory parameter is asymptotically normal with standard errors $\frac{\pi}{(6T)^{-\frac{1}{2}}}$ provided certain conditions are fulfilled (for more details see GPH, 1983, pp. 226-227). GPH (1983) also demonstrate that this estimator of the long memory parameter is biased and inefficient when the regression residuals are substantially autocorrelated.

In Table 3 we present the results obtained with the GPH estimator. We again list the bandwidths used for the estimation alongside the estimated degrees of fractional integration of daily high ($\hat{d}_H$) and low ($\hat{d}_L$) log-prices and their difference ($\hat{d}_R$) (range). Standard errors are displayed in the last column. This time the standard errors do not depend on the bandwidth parameter but only on the total number of observations (T), thus only one standard error is provided for all the estimates of each index in each period. In the case of the two trend-stationary series (DAX and FTSE 100 in the first period), as implied by the ADF test, we detrend the series first, then we apply the GPH estimator and we find that they remain non-stationary. The order of integration for FTSE 100 is in both specifications around 0.65 and in the case of DAX it is even higher. This supports the results from the ELW estimator. Thus, for the remainder of the study we consider these two series non-stationary and use them in their original form (with no detrending).

Table 3: GPH estimator of the fractional degree of integration

	Bandwidths			$GPH_{m=T^{0.5}}$			$GPH_{m=T^{0.6}}$			
	T	$T^{0.5}$	$T^{0.6}$	$\hat{d}_H$	$\hat{d}_L$	$\boldsymbol{\hat{d}_R}$	$\hat{d}_H$	$\hat{d}_L$	$\boldsymbol{\hat{d}_R}$	s.e.
S&P 500										
2003-2012	2517	50	109	1.0261	1.0202	**0.7646**	1.0819	1.0593	**0.6742**	0.0256
2003-2007	1258	35	72	1.0921	1.0819	**0.5204**	1.0914	1.0945	**0.6778**	0.0362
2008-2012	1259	35	72	1.2206	1.2483	**0.7526**	1.0390	1.0398	**0.7211**	0.0361
FTSE 100										
2003-2012	2526	50	110	1.0187	1.0095	**0.7095**	1.0524	1.0306	**0.7123**	0.0255
2003-2007	1264	35	72	1.0299	1.0157	**0.5145**	1.1522	1.1606	**0.5960**	0.0361
detrending				0.6543	0.6425		0.6674	0.6732		
2008-2012	1262	35	72	1.1200	1.0774	**0.7387**	0.9274	0.9292	**0.7010**	0.0361
DAX										
2003-2012	2556	50	110	1.0756	1.0647	**0.6518**	1.1018	1.1023	**0.7059**	0.0254
2003-2007	1274	35	72	1.0695	1.0623	**0.8031**	1.0768	1.0819	**0.8264**	0.0359
detrending				0.7396	0.7131		0.8819	0.8758		
2008-2012	1282	35	73	1.1732	1.1692	**0.7099**	0.9912	0.9828	**0.7384**	0.0358
NIKKEI 225										
2003-2012	2543	49	108	1.0696	1.0841	**0.4989**	1.0753	1.0869	**0.6589**	0.0254
2003-2007	1229	35	71	1.0403	1.0310	**0.4627**	1.0703	1.0590	**0.4013**	0.0366
2008-2012	1224	34	71	1.2168	1.1852	**0.7009**	1.0606	1.0344	**0.7841**	0.0367
PX 50										
2003-2012	2494	49	109	1.0653	1.0673	**0.5765**	1.0871	1.0859	**0.6167**	0.0257
2003-2007	1235	35	71	1.0166	1.0159	**0.2106**	1.0158	1.0128	**0.3155**	0.0365
2008-2012	1259	35	72	0.8935	0.9182	**0.7603**	0.8478	0.8336	**0.5805**	0.0361

The results presented in Table 3 generally support the results of estimation using ELW in two ways. First, the integration orders of daily highs and lows are close to 1 and second, range is not integrated of order 0 and displays long memory. However, we can see some differences between the two estimation procedures. The GPH estimates of the integration orders of daily highs and lows using each bandwidth are much closer to each other than the ELW estimates. However, if we look at the first (calmer) period, ELW estimates the integration orders of daily highs and lows below 1, but GPH estimates are mostly greater than 1 (overall, in only 8 cases out of 60 are the GPH integration orders of daily highs and lows smaller than 1). Surprisingly, from the GPH results with $m = T^{0.6}$

bandwidth one would believe that the log-prices in the second (after-crisis) period are less non-stationary than in the first (pre-crisis) period. If we compare this result with results using the $m = T^{0.5}$ bandwidth we arrive at opposite conclusion.

The estimates of integration orders of range vary substantially between the two bandwidth specifications. Similarly to the ELW estimator, in the case of NIKKEI 225 in the full period the choice of bandwidth changes the conclusion about the stationarity of range. In 24 cases out of 30 the GPH estimator provides higher estimates of the long memory of the range than the ELW estimator. Among indices, the integration orders of range vary substantially, from 0.21 for the PX 50 index in the first period up to 0.83 for DAX in the same period. The range of the PX 50 index and of NIKKEI 225 is definitely stationary in the first period and non-stationary in the others. The range of the other indices is non-stationary in all periods. One can also see the large difference between the ELW and GPH estimates of the integration order of the range of DAX in the first period; the GPH estimate is more than 30% higher than the ELW estimate with the same bandwidth specification. The GPH estimate is more in line with the findings from the inspection of the DAX autocorrelation function then the ELW estimate.

Overall, one can see that the results are sensitive to both the selected estimator (ELW or GPH) and to the bandwidth parameter. However, it is clear that the daily high and low log-prices are not stationary and display long memory. Range also displays long memory, but in some cases it is in the stationary region.

In the following section we describe and provide a test statistic for testing the equality of integration orders of daily high and low log-prices, which is a necessary condition for the further FCVAR analysis.

4.3 Testing the equality of integration orders

Nielsen and Shimotsu (2007) present the possibility of testing the equality of integration orders. This approach is based on the fact that the presence or absence of cointegration is not known when we estimate the fractional integration orders. They propose a test statistic which takes this into account.

They design two test hypotheses: one is the pairwise equality of the integration orders; the other is the equality of all integration orders. For each hypothesis a test statistic is defined. However, in our bivariate case, these two possible hypotheses collapse into one, allowing us to only focus on the hypothesis of equality of all integration orders:

$$\mathcal{H}_0: d_H = d_L = d_*.$$

The test statistic is the following:

$$\hat{T}_0 = m(S\hat{d})' \left(S\frac{1}{4}\hat{D}^{-1}(\hat{G} \odot \hat{G})\hat{D}^{-1}S' + h(T)^2\right)^{-1} (S\hat{d}),$$

where $\odot$ denotes the Hadamard product, $\hat{d} = (\hat{d}_H, \hat{d}_L)$, $D = diag(G_{11}, G_{22})$, $S = (1, -1)'$ and $h(T) = \log(T)^{-k}$ for $k > 0$. Under the null hypothesis $(\mathcal{H}_0)$ Nielsen and Shimotsu (2007) prove that if the variables are cointegrated (i.e., $r = 1$) then the test statistic $\hat{T}_0$ should converge in probability to 0 (i.e., $\hat{T}_0 \rightarrow_p 0$). However, if they are not cointegrated ($r = 0$) then under the null hypothesis the test statistic $\hat{T}_0$ should converge in distribution to the χ_1^2 distribution ($\hat{T}_0 \rightarrow_d \chi_1^2$). This means that if the value of the test statistic $\hat{T}_0$ is significantly large with respect to the χ_1^2 density, we can take this as evidence that the null of the equality of integration orders is rejected.

Table 4 presents the test statistics for testing the equality of integration orders of daily high and low log-prices. For each index and time period we present two test statistics; the first is estimated using the $T^{0.5}$ bandwidth, and the second using the $T^{0.6}$ bandwidth. In our case we use $k = 1$, and thus $h(T) = 1/\log(T)$. Since the maximum test statistic is 1.3361 and the lowest critical value of the χ_1^2 distribution is 2.71 (for the 90% confidence interval), we cannot reject the null hypothesis of equality of the integration orders. This implies that we can perform the FCVAR estimation, which requires the integration orders to be the same (i.e., $d_H = d_L$).

Table 4: Test statistics for the equality of integration orders

	Bandwidths			$\hat{T}_0$ *statistics*	
	T	$T^{0.5}$	$T^{0.6}$	$\hat{T}_0(m = T^{0.5})$	$\hat{T}_0(m = T^{0.6})$
S&P 500					
2003-2012	2517	50	109	0.2636	1.3361
2003-2007	1258	35	72	0.7113	0.5400
2008-2012	1259	35	72	0.8033	0.7043
FTSE 100					
2003-2012	2526	50	110	0.1979	0.7689
2003-2007	1264	35	72	0.2716	0.1596
2008-2012	1262	35	72	0.3005	0.2635
DAX					
2003-2012	2556	50	110	0.1041	1.1610
2003-2007	1274	35	72	0.7085	0.6007
2008-2012	1282	35	73	0.0525	0.1604
NIKKEI 225					
2003-2012	2543	49	108	0.1493	0.1517
2003-2007	1229	35	71	0.1919	0.4660
2008-2012	1224	34	71	0.0096	0.1129
PX 50					
2003-2012	2494	49	109	0.0488	0.6988
2003-2007	1235	35	71	0.2329	1.2668
2008-2012	1259	35	72	0.0538	0.2520

5 Fractionally cointegrated VAR

Having confirmed that range displays long memory and that the integration orders of daily high and low log-prices are the same, we can now define the fractionally cointegrated VAR. The fractionally cointegrated vector error correction model (FVECM) or fractionally cointegrated VAR (FCVAR) was first proposed by Johansen (2008) based on an idea of Granger (1986), and further analysed by Johansen and Nielsen (2010, 2012). In this chapter we introduce the fractionally cointegrated VAR model and show how it can be estimated. We also describe the cointegration rank tests which should be applied in the fractional framework.

5.1 Derivation of FCVAR

The cointegrated VAR was already described in Section 3.3; however, in order to explain the generalization to the fractional form we recall the definition once more. The cointegrated VAR for an n-dimensional non-stationary time series X_t is

$$\Delta X_t = \alpha(\beta' X_{t-1} + \rho') + \sum_{i=1}^{p} \Gamma_i \Delta X_{t-1} + \varepsilon_t, \qquad t = 1, \dots T, \tag{1}$$

where $\Delta X_{t-i} = X_{t-i} - X_{t-i-1}$ and where ε_t is an n-dimensional $i.i.d.(0, \Omega)$ vector of random shocks, Ω is a positive-definite variance matrix.

We want to arrive at a VAR model which is able to capture fractional processes characterized by slowly decaying autocorrelation functions. The generalised model should also allow X_t to be fractional of order d and cofractional of order $d - b$; that is, $\beta' X_t$ should be fractional of order $d - b \geq 0$, thereby extending the model (1) to fractional processes. In other words, fractional cointegration assumes the existence of a common stochastic trend which is integrated of order d, and the short-term departures from the long-run equilibrium being integrated of order $d - b$ (Caporin et al., 2011).

The following methodology is taken from Johansen and Nielsen (2012) and Nielsen and Morin (2012).

We derive the model in two steps. First, the usual lag operator $L = 1 - \Delta$ and the difference operator Δ are replaced by the fractional lag operator and fractional difference operator, $L_b = 1 - \Delta^b$ and $\Delta^b = (1 - L)^b$. The fractional difference operator is defined by the binomial expansion $\Delta^b Z_t = \sum_{n=0}^{\infty} (-1)^n \binom{b}{n} Z_{t-n}$. Second, the resulting model is applied to $Z_t = \Delta^{d-b} X_t$. Thus we define the fractionally cointegrated $\text{VAR}_{\text{d,b}}(\text{p})$ by

$$\Delta^d X_t = \Delta^{d-b} L_b \alpha \beta' X_t + \sum_{i=1}^{p} \Gamma_i \Delta^d L_b^i X_t + \varepsilon_t, \qquad t = 1, \dots T, \tag{2}$$

where ε_t is an n-dimensional $i.i.d.(0, \Omega)$, Ω is a positive-definite variance matrix and α and β are $(n \times r)$ matrices, $0 \leq r \leq n, d \geq b > 0$.

If we want to model data with a non-zero mean, e.g. $Y_t = \mu + X_t$ where X_t is given by (2), then $\Delta^a Y_t = \Delta^a(\mu + X_t) = \Delta^a X_t$ because $\Delta^a 1 = 0$ for $a > 0$. Y_t thus satisfies the same equations as $Y_t = \mu + X_t$. This means that the model with $d > b$ is invariant to inclusion of a restricted constant term ρ, when this term is included into the model in a similar way as in (1). Therefore we consider the inclusion of a constant term only in the model with $d = b$:

$$\Delta^d X_t = L_d \alpha(\beta' X_t + \rho') + \sum_{i=1}^{p} \Gamma_i \Delta^d L_d^i X_t + \varepsilon_t, \qquad t = 1, \dots T, \tag{3}$$

Both models include the original cointegrated VAR as the special case $d = b = 1$. The interpretation is similar as before. The model also describes cointegration and adjustment towards equilibrium, but it is more general, as it incorporates fractional integration and cointegration. X_t are integrated of order d, and b is the strength of the cointegrating relations (a higher b means less persistence in the cointegrating relations; b can also be called the cointegration gap). Moreover, if $d - b < 1/2$ then $\beta' X_t$ in (2) is asymptotically a zero-mean stationary process. If we write $\Pi = \alpha\beta'$, where the $(n \times r)$ matrices α and β with $r \leq n$ are assumed to have full column rank r, the columns of β are then the r cointegrating (cofractional) relations determining the long-run equilibria. The rank r is called the cointegration or cofractional rank. The parameter α determines the speed of adjustment towards the equilibria. The parameters $\Gamma = (\Gamma_1, \dots, \Gamma_p)$ govern the short-run dynamics. The parameter ρ is the restricted constant term (since the constant term in the model is restricted to be of the form $\mu = \alpha\rho'$) and is interpreted as the mean level of the long-run equilibria. In the special case when $d = b$, $(\beta' X_t + \rho')$ is a zero-mean process of fractional order zero. Having defined the fractional cointegrated VAR, we shall now take a look at its estimation.

5.2 FCVAR estimation

5.2.1 Maximum likelihood estimation

According to Johansen and Nielsen (2012), the model (2) is estimated by conditional likelihood (given initial values) by maximising the function

$$\log\, L_t(\lambda) = -\frac{T}{2}\log\det(T^{-1}\sum_{t=1}^{T}\epsilon_t(\lambda)\,\epsilon_t(\lambda)')\,,$$

where

$$\epsilon_t(\lambda) = \Delta^d X_t - \Delta^{d-b} L_b \alpha\beta' X_t - \sum_{i=1}^{p}\Gamma_i \Delta^d L_b^i X_t\,, \qquad \lambda = (d, b, \alpha, \beta, \Gamma),$$

for model (2) and

$$\epsilon_t(\lambda) = \Delta^d X_t - L_d \alpha(\beta' X_t + \rho') - \sum_{i=1}^{p}\Gamma_i \Delta^d L_d^i X_t\,, \qquad \lambda = (d, b, \alpha, \beta, \rho, \Gamma),$$

for model (3). Johansen and Nielsen (2012) state that the model is relatively easy to estimate, since for fixed (d, b) the model is estimated by reduced rank regression, thus reducing the numerical problem to the optimization of a function of two variables. The parameters $(\alpha, \beta, \rho, \Gamma)$ can be concentrated out of the likelihood function and the profile likelihood function is optimized over the two fractional parameters d and b using numerical optimization. The existence, uniqueness and consistency of the maximum likelihood estimator are also shown in Johansen and Nielsen (2012), which allows the application of standard likelihood theory.

Johansen and Nielsen (2012) also examine the asymptotic theory. They prove that for i.i.d. errors with suitable moments conditions, the estimated cointegrating vectors $(\hat{\beta}, \hat{\rho})$ are locally asymptotically mixed normal when $b > 1/2$. The remaining conditional maximum likelihood parameter estimates $(\hat{d}, \hat{b}, \hat{\alpha}, \hat{\Gamma}_1, \dots, \hat{\Gamma}_p)$ are asymptotically Gaussian.

For this case of "strong cointegration" the asymptotic theory is non-standard and requires type-II fractional Brownian motion.

When $b < 1/2$ the asymptotic theory is standard and all conditional maximum likelihood parameter estimates $(\hat{d}, \hat{b}, \hat{\alpha}, \hat{\beta}, \hat{\rho}, \hat{\Gamma}_1, \dots, \hat{\Gamma}_p)$ are asymptotically Gaussian. When b is close to $1/2$, a large number of observations is needed to achieve the asymptotic results. Having in mind these results, asymptotically standard (χ^2) inference on all model parameters, including the orders of fractionality and the cointegrating relations, is achieved through the application of quasi-likelihood ratio tests.

5.2.2 Imposing restrictions on parameters

As Nielsen and Morin (2012) describe, it is possible to impose various restrictions, the most interesting being those on α and β. We can impose either linear restrictions or restrictions to known values. This can be achieved through partitioning α and β into sets of columns, $\alpha = [\alpha_1, \alpha_2, \dots]$ and $\beta = [\beta_1, \beta_2, \dots]$, such that each α_i and β_i are $n \times r_i$ matrices with $\sum_i r_i = r$. For each α_i and β_i we can specify a restriction to known values a_i and b_i. Imposing linear restrictions is achieved through specifying matrices $A_i(n \times s_i)$ and $H_i\ (n \times s_i)$ so that $\alpha_i = A_i\psi_i$ and $\beta_i = H_i\phi_i$ for conforming matrices ψ_i and ϕ_i (for more details see Johansen, 1996). The switching algorithm proposed in Johansen (1996) is used for estimation. For each pair of submatrices α_i and β_i the parameters are estimated by taking other submatrices as fixed. The estimation is repeated trough the sequence of submatrices until convergence is achieved.

When a restricted constant term is present, we also have to impose restrictions on this constant term, because the two parameters β and ρ are estimated jointly trough $\beta^* = (\beta', \rho')'$ of dimension $n1 \times r$, where $n1 = n + 1$. If we want to specify some columns of β as known, we have to do the same for the corresponding columns of ρ. For linear restrictions, we make the modification $\beta_i^* = H_i^*\phi_i$, where H_i^* is $(n1\ \times s_i)$. We can impose any linear restrictions we wish on the fractional parameters d and b. For the matrix of short term adjustments Γ_i, only exclusion restrictions are allowed in the Matlab software package proposed by Nielsen and Morin (2012). If no restrictions are specified, the model is estimated unrestricted.

5.2.3 Initial values

Another issue one has to address when estimating the FCVAR is specifying the appropriate number of observations reserved for initial values that the model will be conditioned upon in the estimation. Johansen and Nielsen (2012) propose a generalization regarding initial values. The initial values are neither modelled nor assumed to be zero as it was done in the previous work, but are instead assumed to be bounded. The likelihood function depends on $\Delta^a X_t$ for different values of $a > 0$, and since we do not know all the initial values of X_t we assume we have observations of $X_t, t = -N_0 + 1, \dots, T$ and define initial values for the calculation as $\tilde{X}_{-n} = X_{-n}, n = 0, \dots, N_0 - 1$ and $\tilde{X}_{-n} = 0, n \geq N_0$. Thus N_0 observations are set aside for initial values, which are assumed to be uniformly bounded (Johansen and Nielsen, 2012). The choice of N_0 carries a bias/efficiency trade-off: a smaller number of initial values introduces more bias, but leaves a greater number of observations for the estimation. This was the case in Nielsen and Morin (2012), who together with the Matlab software package for estimation of FCVAR provide an example in which they have only 55 observations, so they use only five observations as the initial values. In our case, we have more than a thousand observations, so the number of initial values is chosen so as to optimize the performance and achieve convergence. The exact number of initial values used for estimation is listed together with the results from the cointegration rank test by Johansen and Nielsen (2012) in Sub-section 0.

5.3 Cointegration rank tests

As we have already noted, the traditional tests for cointegration proposed by Johansen (1991) are not applicable in the presence of long memory. Instead, more recent tests allowing for fractional cointegration should be applied. Fractional cointegration is defined as follows: X_t is said to be fractionally cointegrated $CI(d,b)$ if X_t has $I(d)$ elements and for some $b > 0$ there exists β such that $\beta' X_t$ is integrated of order $(d - b)$. Originally, we assumed that $d = b = 1$; however, whenever either d or b (or both) are different from 1 we are dealing with fractional cointegration.

We define here two cointegration rank tests, one proposed by Nielsen and Shimotsu (2007) and the other by Johansen and Nielsen (2012).

5.3.1 Cointegration rank test by Nielsen and Shimotsu (2007)

The cointegration rank determination procedure proposed by Nielsen and Shimotsu (2007) extends the one found in Robinson and Yajima (2002) since it allows for both stationary and non-stationary fractionally integrated processes. Robinson and Yajima (2002) allow only stationary data, which is quite limiting for financial data. The ability to consider any value of the fractional differencing parameter d follows from the application of the exact local Whittle analysis of Shimotsu and Phillips (2005). The exact local Whittle estimate of d is then used to examine the rank of the spectral density matrix of the d'th differenced process around the origin in order to provide a consistent estimate of the cointegration rank. This semi-parametric method only requires information about the behaviour of the spectral density matrix around the origin, but it also relies on the choice of the bandwidth and threshold parameters. On the other hand, this approach does not require the cointegrating vectors to be estimated in order to determine the cointegration rank.

The estimate of the cointegration rank r of X_t is achieved through estimating the matrix G and its eigenvalues. For simplicity, Nielsen and Shimotsu (2007) assume the equality of integration orders, in our case $d_H = d_L = d_*$. In this study we do not have to assume the equality, because we have provided a formal test for the equality of integration orders.

Nielsen and Shimotsu (2007) define

$$\hat{G}(d_*) = \frac{1}{m_1} \sum_{j=1}^{m_1} \mathrm{Re}\left[I_{\Delta(L,d_*,d_*)x}(\lambda_j)\right], \tag{4}$$

where $I_{\Delta(L,d_*,d_*)x}(\lambda_j)$ is in our bivariate case the periodogram of $(\Delta^{d_*} p_t^H, \Delta^{d_*} p_t^L)'$, and $\mathrm{Re}(.)$ denotes the real part of the periodogram. The estimator $\hat{G}(d_*)$ depends on a new bandwidth parameter $m_1(T)$ as some complications may arise when d_* is estimated. Since

d_* is unknown, it has to be substituted with an estimate, but as Nielsen and Shimotsu (2007) state, we cannot use the multivariate version of the exact local Whittle estimator to obtain d_*, because when p_t^H and p_t^L are cointegrated, the matrix G is not of full rank. That is why the matrix G is estimated by (4) based on m_1 periodogram ordinates and each d_i is estimated by $\hat{d}_i = \arg\min_{d\in[\Delta_1,\Delta_2]} R_i(d)$ using m ordinates such that $\frac{m}{m_1} \to 0$.

We consider two combinations of m and m_1: first, $m = T^{0.6}$ and $m_1 = T^{0.55}$, and second, $m = T^{0.5}$ and $m_1 = T^{0.45}$. However, since both specifications yield similar results, for the sake of brevity, we only present here the first combination.

Denoting $\hat{\delta}_i$ the i'th eigenvalue of $\hat{G}$, it is possible to apply a model selection procedure to determine the cointegration rank r.

$$\hat{r} = \arg\min_{u=0,1} L(u),$$

where the loss function is defined as

$$L(u) = v(T)(2-u) - \sum_{i=1}^{2-u} \hat{\delta}_i \,,$$ [1]

for a $v(T) > 0$ such that $v(T) + \frac{1}{m_1^{1/2}\, v(T)} \to 0$.

Table 5 summarizes the results from the cointegration rank estimates proposed by Nielsen and Shimotsu (2007). We only provide here the results for the combination of $m = T^{0.6}$ and $m_1 = T^{0.55}$, but the same analysis was also conducted for $m_1 = T^{0.5}$ and $m = T^{0.45}$. Also, for the rank estimates we only show the results for the smallest $v(T) = m_1^{-0.45}$ and for the largest $v(T) = m_1^{-0.05}$, but we have also considered all of the following values of $v(T)$: $m_1^{-0.40}$, $m_1^{-0.35}$, $\ldots m_1^{-0.10}$. These all yielded the same result of one cointegrating vector. The first column of Table 5 represents the common integration order $\bar{d}_*$, which is used in the fractional cointegration analysis and is simply computed as an average of the estimated integration orders of daily high and low log-prices from the

[1] In the original definition there is $(p-u)$ instead of $(2-u)$, but in their notation p is the dimension of X_t, which in our notation is $n = 2$.

ELW estimator based on a given bandwidth. The other two columns stand for the estimates of the eigenvalues of matrix $\hat{G}(\bar{d}_*)$ from (4). Their values indicate that one eigenvalue of $\hat{G}(\bar{d}_*)$ could be zero. This implies that there could be one cointegrating relation. $L(0)$ and $L(1)$ are the values of the loss function evaluated with regard to a cointegration rank of 0 or 1. The cointegration rank $\hat{r}$ is then determined by $\arg\min$ of $L(u)$. In all cases the loss function $L(1)$ is smaller than $L(0)$, implying that there is exactly one cointegrating relation between daily high and low log-prices.

Table 5: Cointegration rank test by Whittle and Shimotsu (2007)

		Eigenvalues of $\hat{G}(\bar{d}_*)$		Rank estimates					
				$v(T) = m_1^{-0.45}$			$v(T) = m_1^{-0.05}$		
	$\bar{d}_*$	δ_1	δ_2	$L(0)$	$L(1)$	$\hat{r}$	$L(0)$	$L(1)$	$\hat{r}$
S&P 500									
2003-2012	1.0239	0.3169	0.0010	-1.7117	-1.8491	1	-0.3872	-1.1868	1
2003-2007	0.9099	0.2151	0.0009	-1.6561	-1.8197	1	-0.3553	-1.1694	1
2008-2012	1.0321	0.5076	0.0022	-1.6561	-1.8189	1	-0.3553	-1.1686	1
FTSE 100									
2003-2012	0.9521	0.4360	0.0010	-1.7117	-1.8511	1	-0.3872	-1.1889	1
2003-2007	0.8448	0.3286	0.0014	-1.6561	-1.8194	1	-0.3553	-1.1691	1
2008-2012	0.9684	0.6313	0.0018	-1.6561	-1.8222	1	-0.3553	-1.1718	1
DAX									
2003-2012	1.0246	0.5136	0.0017	-1.7117	-1.8491	1	-0.3872	-1.1869	1
2003-2007	0.9639	0.4063	0.0025	-1.6591	-1.8172	1	-0.3569	-1.1662	1
2008-2012	1.0147	0.7520	0.0029	-1.6591	-1.8217	1	-0.3569	-1.1706	1
NIKKEI 225									
2003-2012	1.0496	0.4847	0.0011	-1.7099	-1.8504	1	-0.3861	-1.1885	1
2003-2007	0.9601	0.4551	0.0010	-1.6561	-1.8235	1	-0.3553	-1.1731	1
2008-2012	1.0621	0.6491	0.0021	-1.6529	-1.8199	1	-0.3537	-1.1703	1
PX 50									
2003-2012	1.1513	0.3563	0.0009	-1.7099	-1.8496	1	-0.3861	-1.1877	1
2003-2007	0.9917	0.4132	0.0016	-1.6561	-1.8200	1	-0.3553	-1.1696	1
2008-2012	1.1101	0.5811	0.0035	-1.6561	-1.8159	1	-0.3553	-1.1656	1

5.3.2 Cointegration rank test by Johansen and Nielsen (2012)

In this sub-section, we return to the methodology proposed by Johansen and Nielsen (2012) as it was summarised by Nielsen and Morin (2012) and we describe another approach to estimating the cointegration rank r. We again consider the model

$$\Delta^d X_t = \Delta^{d-b} L_b \Pi X_t + \sum_{i=1}^{p} \Gamma_i \Delta^d L_b^i X_t + \varepsilon_t, \qquad t = 1, \dots T,$$

and we want to test the hypothesis $\mathcal{H}_r : rank(\Pi) = r$ against $\mathcal{H}_n : rank(\Pi) = n$. Let $L(d, b, r)$ be the profile likelihood function given a rank r, where (α, β, Γ) have been concentrated out by regression and reduced rank regression (see Johansen and Nielsen, 2012, p. 23). In the case of the model with a constant we test $\mathcal{H}_r : rank(\Pi, \mu) = r$ against $\mathcal{H}_n : rank(\Pi, \mu) = n$ and the profile likelihood function given rank r is then $L(d, r)$, where again the parameters $(\alpha, \beta, \rho, \Gamma)$ have been concentrated out.[2]

We maximize the profile likelihood function under both hypotheses $\mathcal{H}_r$ and $\mathcal{H}_n$; the likelihood ratio (LR) statistic is $LR_T(q) = 2\log(L(\hat{d}_n, \hat{b}_n, n)/L(\hat{d}_r, \hat{b}_r, r))$, where $q = n - r$ and

$$L(\hat{d}_n, \hat{b}_n, n) = \max_{d,b} L(d, b, n), \qquad L(\hat{d}_r, \hat{b}_r, r) = \max_{d,b} L(d, b, r),$$

similarly for the model with a constant. The asymptotic distribution of $LR_T(q)$ depends both qualitatively and quantitatively on the parameter b and on $q = n - r$. This dependence on the unknown parameter b makes the empirical analysis more complicated; however, a computer program exists, made available by MacKinnon and Nielsen (2012), which, given the dimension of the problem, $q = n - r$ (the number of variables of interest less the cointegration rank), and the value of b, computes the asymptotic critical values and asymptotic p-values for the LR rank test in this fractional framework.

[2] The model with an inclusion of a constant is considered only when $d = b$ and that is why the profile likelihood function depends only on the given rank r and the parameter d.

MacKinnon and Nielsen (2012) deal with the model's uncertainty in the specification of response surface regressions by using model averaging instead of model selection.

As we have already discussed, in the case of "weak cointegration", when $0 < b < 1/2$, $LR_T(q)$ has a standard asymptotic distribution, $LR_T(q) \overset{D}{\rightarrow} \chi^2(q^2)$. This means that we can use the standard critical values of the χ^2 distribution to determine the statistical significance of the cointegration rank. The situation is, however, different when $1/2 < b \leq d$. Then the asymptotic theory is nonstandard and

$$LR_T(q) \overset{D}{\rightarrow} \mathrm{Tr}\left\{\int_0^1 dW(s)F'(s)\left(\int_0^1 F(s)F'(s)ds\right)^{-1}\int_0^1 F(s)dW'(s)\right\},$$

where the vector process dW is the increment of ordinary vector Brownian motion of dimension $q = n - r$. The vector F depends on deterministics in a similar way as the CVAR model in Johansen (1996). If we do not include any deterministic terms into the model, then $F(u) = W_b(u)$. If the restricted constant term is included into the model, then $F(u) = (W_b'(u), 1)'$, where $W_b(u) = \Gamma(b)^{-1}\int_0^u (u-s)^{b-1}dW(s)$ is vector fractional type-II Brownian motion.

If we are in the situation where $b > 1/2$, we use the computer program proposed by MacKinnon and Nielsen (2012) to determine the appropriate critical value given the parameters q and b, and based on this critical value we decide on the significance of the cointegration rank. In particular, values of b in the interval $1/2 < b \leq 2$ are of interest. However, the lowest value of b, which the procedure permits, is 0.51 rather than 0.5, because there is no theory for the case when $b = 1/2$.

The results of this cointegration test are presented in the following Table 6. The first column, IV, stands for the number of initial values used in the estimation. Then, for each rank $r = 0, 1, 2$ we present the estimates of the parameter of the fractional order of integration ($\hat{d}$), the parameter of the cointegration gap ($\hat{b}$), and the corresponding likelihood ratio statistic (LR) and its critical value at 5% level of significance. When b is smaller than 0.5, it follows the $\chi^2(q^2)$ distribution; this means that for cointegration rank $r = 0$, $q = 2$ and $\chi^2_{0.95}(4) = 9.49$. When $r = 1$, then $q = 1$ and $\chi^2_{0.95}(1) = 3.84$. If b is greater than 0.51, we calculate the corresponding critical value using the programme provided by MacKinnon and Nielsen (2012).

Table 6: Cointegration rank test by Johansen and Nielsen (2012)

		$rank = 0$				$rank = 1$				$rank = 2$	
	IV	$\hat{d}$	$\hat{b}$	LR	$CV_{5\%}$	$\hat{d}$	$\hat{b}$	LR	$CV_{5\%}$	$\hat{d}$	$\hat{b}$
S&P 500											
2003-2012	70	0.681	0.372	271.232	9.490	**0.998**	**0.302**	0.058	3.840	1.002	0.295
2003-2007	80	0.517	0.517	117.535	9.362	**0.999**	**0.467**	1.338	3.840	0.984	0.481
2008-2012	60	0.719	0.329	136.629	9.490	**0.981**	**0.334**	1.699	3.840	1.033	0.233
FTSE 100											
2003-2012	80	0.623	0.416	254.914	9.490	**0.970**	**0.357**	2.368	3.840	0.954	0.387
2003-2007	70	0.497	0.497	118.642	9.490	**0.984**	**0.380**	2.510	3.840	0.968	0.424
2008-2012	30	0.667	0.361	120.148	9.490	**0.987**	**0.463**	2.035	3.840	1.009	0.416
DAX											
2003-2012	60	0.609	0.439	233.689	9.490	**0.987**	**0.388**	3.808	3.840	0.966	0.408
2003-2007	50	0.561	0.524	38.776	9.359	**1.042**	**0.476**	0.323	3.840	1.055	0.480
2008-2012	80	0.638	0.405	133.836	9.490	**0.967**	**0.336**	0.302	3.840	0.955	0.361
NIKKEI 225											
2003-2012	80	0.591	0.476	138.368	9.490	**1.004**	**0.517**	0.254	3.636	0.997	0.515
2003-2007	60	0.513	0.513	80.636	9.365	**1.019**	**0.635**	0.003	3.587	1.018	0.635
2008-2012	60	0.708	0.370	79.292	9.490	**0.996**	**0.010**	28.181	3.840	0.978	0.547
PX 50											
2003-2012	80	0.520	0.520	109.031	9.360	**1.007**	**0.475**	1.289	3.840	0.988	0.472
2003-2007	60	0.532	0.532	65.024	9.360	**1.021**	**0.680**	5.991	6.373*	0.977	0.666
2008-2012	60	0.511	0.511	53.246	9.367	**0.982**	**0.454**	1.092	3.840	1.010	0.451

Note: Asterisk (*) denotes the 1% critical value rather than the 5% critical value. The cointegrating relation is not significant at the 5%, but it is at the 1% level of significance.

We can see from Table 6 that we find one significant cointegrating relation, except for the NIKKEI 225 in the second period, where no cointegrating vector is found. In the case of the PX 50 index in the first period, the LR statistic for one cointegrating vector ($rank = 1$) is significant only at the 1% level and not at the 5% level of significance, which we mark by an asterisk. When $r = 0$, the likelihood ratio (LR) statistic is significantly larger than the corresponding critical value, meaning that we reject the null hypothesis of zero cointegrating relations. When $r = 1$ the LR statistic is significantly smaller than the corresponding critical value and thus we do not reject the null of one cointegrating relation. This is not true for the NIKKEI 225 in the second period, where we do not find a significant cointegrating vector even though its presence was hinted at by the results of the cointegration rank test proposed by Nielsen and Shimotsu (2007). One of the possible explanations may be that the optimization procedure did not converge for NIKKEI 225 in the second period and the lowest possible value for b, $b = 0.01$ for one cointegrating relation.

5.4 FCVAR estimation results

The estimation of the cointegration rank presented above is performed together with the estimation of the FCVAR model using the Matlab software package proposed by Nielsen and Morin (2012). Having in mind the result that in each specification there is one significant cointegrating vector (except for the NIKKEI 225 in the second period) we use this information for the FCVAR estimation. We also specify the lag length of the short-term deviations to be $p = 1$. We use only one lag as that is sufficient to capture the autocorrelation of residuals. MacKinnon and Nielsen (2012) state that in the fractional model a single lag is usually sufficient, which is in contrast with the standard cointegrated VAR, where several more lags are needed to capture the serial correlation in residuals. We can also specify the number of initial values used for the estimation, which we have discussed in Sub-section 5.2.3. The number of initial values used for the estimation of the FCVAR model is the same number as the number of initial values used for the estimation of cointegration rank in Table 6. You can see that the number of initial values varies from 30 to 80. We have estimated several models with various numbers of initial values and present here only those which have performed the best (in terms of low standard errors, finding a significant cointegrating vector, stationarity, and convergence). We may also want to impose restrictions, as we discussed in Sub-section 5.2.2. We are particularly interested in the order of integration of range. Since range is defined as the difference between the maximum and minimum daily log-prices, i.e. $(p_t^H - p_t^L)$, we would like the cointegrating vector to be $(1, -1)$. If the cointegrating vector is different from $(1, -1)$, we cannot interpret the difference $(d - b)$ as the order of integration of range. That is why we first estimate the model without any restrictions imposed to see whether the model yields a significant cointegrating vector and significant estimates of $\hat{d}$ and $\hat{b}$, and then we impose the $(1, -1)$ restriction on the cointegrating vector. However, when a restriction is imposed, the standard errors are not provided, thus we cannot make any inference on the significance of the estimated parameters. We estimate the model for the case when $d \neq b$; however, the procedure is capable of detecting whether d and b are close to equality; if they are, we would re-estimate the model with the restriction $d = b$. This situation did, however, not occur; that is in line with our previous findings, as the equality of the d and b parameters would imply that the order of integration of range is 0, which we have rejected based on both the ELW and GPH estimator results.

We first present the results of the FCVAR estimation without any restrictions imposed, in Table 7. The model is estimated in the following form (it is derived from model (2), here with $p = 1$)

$$\Delta^d X_t = \Delta^{d-b} L_b \alpha \beta' X_t + \Gamma_1 \Delta^d L_b X_t + \varepsilon_t, \qquad t = 1, \ldots T,$$

The first two columns of Table 7 contain the estimates of the two parameters of interest ($\hat{d}$ and $\hat{b}$). These estimates are the same as the results from the cointegration rank test presented in Table 6 for one cointegrating vector (i.e., when $rank = 1$). Next we present the estimated cointegrating vector ($\hat{\beta}$) and the remaining six columns list the FCVAR parameters for the speed of adjustment, $\alpha = (\alpha_H, \alpha_L)'$, and the estimates of the matrix of short-run dynamics, $\Gamma_1 = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}$.

In all specifications, the parameters of interest ($\hat{d}$ and $\hat{b}$) are significant (based on small standard errors) and different from each other (thus the specification $d \neq b$ is correct). Also, the estimates of the cointegrating vector are very close to the desired vector of $(1, -1)$. All this suggests that a linear combination of the daily high and low log-prices (possibly the range) is integrated of a non-zero order. However, since the estimate of the cointegrating vector is not exactly $(1, -1)$, we cannot interpret the difference $(d - b)$ as the order of integration of range.

Daily high and low log-prices are integrated of an order close to 1. Surprisingly, in the first period, which we consider the calmer period, daily log-prices are more non-stationary than in the second period or in the full period. Also, the orders of integration of daily log-prices are smaller than unity in 9 out of 14 cases.

Table 7: FCVAR estimation results (no restrictions)

		$\hat{d}$	$\hat{b}$	$\hat{\beta}$	α_H	α_L	γ_{11}	γ_{12}	γ_{21}	γ_{22}
S&P 500										
u	2003-2012	0.998	0.302	(1,-1.007)	-0.195	3.986	-0.773	1.050	-1.614	2.619
		(0.014)	(0.028)		(0.302)	(0.945)	(0.310)	(0.316)	(0.727)	(0.776)
	2003-2007	0.999	0.467	(1,-1.002)	-1.072	1.680	0.153	0.076	-0.282	0.924
		(0.009)	(0.038)		(0.254)	(0.363)	(0.178)	(0.193)	(0.258)	(0.283)
u	2008-2012	0.981	0.334	(1,-1.008)	-0.013	3.081	-0.755	1.049	-0.935	1.890
		(0.015)	(0.035)		(0.348)	(0.922)	(0.363)	(0.370)	(0.696)	(0.752)
FTSE 100										
	2003-2012	0.970	0.357	(1,-1.005)	-0.657	2.308	-0.263	0.569	-0.620	1.505
		(0.011)	(0.024)		(0.218)	(0.466)	(0.190)	(0.204)	(0.358)	(0.386)
u	2003-2007	0.984	0.380	(1,-1.003)	-1.344	2.339	0.245	-0.009	-0.994	1.727
		(0.009)	(0.027)		(0.331)	(0.513)	(0.252)	(0.280)	(0.426)	(0.442)
	2008-2012	0.987	0.463	(1,-1.004)	-0.429	1.344	-0.309	0.524	-0.049	0.669
		(0.009)	(0.031)		(0.189)	(0.305)	(0.162)	(0.171)	(0.231)	(0.247)
DAX										
	2003-2012	0.987	0.388	(1,-1.004)	-0.436	1.891	-0.332	0.571	-0.433	1.101
		(0.011)	(0.033)		(0.209)	(0.440)	(0.195)	(0.202)	(0.327)	(0.354)
	2003-2007	1.042	0.476	(1,-1.001)	-0.551	1.520	-0.145	0.319	-0.169	0.526
		(0.013)	(0.064)		(0.259)	(0.429)	(0.194)	(0.203)	(0.279)	(0.305)
	2008-2012	0.967	0.336	(1,-1.007)	-0.016	2.651	-0.695	0.980	-0.910	1.718
		(0.016)	(0.035)		(0.393)	(0.846)	(0.413)	(0.415)	(0.675)	(0.711)
NIKKEI 225										
	2003-2012	1.004	0.517	(1,-1.002)	-0.175	1.318	-0.295	0.535	-0.066	0.543
		(0.012)	(0.043)		(0.134)	(0.271)	(0.124)	(0.133)	(0.183)	(0.205)
	2003-2007	1.019	0.635	(1,-1.001)	-0.361	1.144	-0.154	0.309	-0.171	0.522
		(0.010)	(0.053)		(0.151)	(0.242)	(0.120)	(0.125)	(0.168)	(0.186)
	2008-2012	-	-	-	-	-	-	-	-	-
PX 50										
	2003-2012	1.007	0.475	(1,-1.003)	-0.682	1.281	-0.172	0.340	-0.447	0.861
		(0.017)	(0.045)		(0.168)	(0.310)	(0.127)	(0.136)	(0.236)	(0.258)
	2003-2007	1.021	0.680	(1,-1.002)	-0.410	0.612	0.021	0.204	0.123	0.243
		(0.015)	(0.065)		(0.119)	(0.168)	(0.095)	(0.096)	(0.120)	(0.130)
	2008-2012	0.982	0.454	(1,-1.003)	-0.732	1.346	-0.272	0.453	-0.600	1.031
		(0.017)	(0.056)		(0.253)	(0.445)	(0.194)	(0.211)	(0.355)	(0.389)

Note: Standard errors in brackets, *"u"* denotes that model is unstable (some roots of the characteristic polynomial lie outside the unit root circle).

The adjustment coefficients α_H and α_L capture the speed of adjustment of p_t^H and p_t^L towards equilibrium. These coefficients are mostly significant, non-zero, and with expected signs (α_H is negative and α_L is positive) meaning that they move in opposite directions to restore equilibrium after a shock to the system occurs. We can note that the absolute values of the estimates of α_H are much smaller than of α_L, which suggests that the correction in the equation for daily lows overshoots the long-run equilibrium. Caporin et al. (2011) obtained similar results when analysing the DJIA index. When interpreting the short-run dynamics parameters (γ_{11}, …, γ_{22}) we may notice that the coefficients of lagged daily highs are mostly negative, whereas those of lagged daily lows are mostly positive. Cheung (2007) states that negative coefficients imply a regressive behaviour, whereas positive coefficients are an indication of spill-over effects. He argues that higher daily highs tend to fall to a lower level, lower daily highs tend to drift up to a higher level, and higher daily lows lead to higher daily highs.

In one case, namely the second period of NIKKEI 225, we were unable to estimate the FCVAR model without restrictions, the model did not converge for any specified number of initial values. Also, we have to note that in 3 cases (S&P 500 in the full period and in the second sub-period and FTSE 100 in the first sub-period) the model was not stable. We say that a model is stable when the roots of the characteristic polynomial are smaller than unity. In these three situations, the roots exceeding unity are 1.365, 1.374, and 1.101, respectively, and thus the model should be interpreted with caution in these cases. In Table 7 we mark this situation by the letter *"u"* set before the period specification of the affected periods. We also test the residuals for the remaining autocorrelation and heteroscedasticity. Based on the Ljung-Box Q-test we reject in most cases the null of no autocorrelation; however, the value of the statistic is rather marginal. Based on the visualization of the autocorrelation functions the dependency is quite weak and it disappears after the second lag. Also, based on the visualization of the autocorrelation function of squared residuals we can detect some heteroscedasticity; however, it is again very weak. Neither of these findings impacts the quality of our estimates: they are still unbiased and consistent; however, the reliability of inference may be slightly affected, as standard errors may be biased.[3]

[3] For the sake of brevity, this residual diagnostic is not presented here, but is available upon request.

Since the estimates of the cointegrating vector from the unrestricted model are very close to the desired vector of $(1, -1)$, we impose the restriction on the vector to be exactly $(1, -1)$ in order to be able to interpret the cointegrating relation as range. Table 8 contains the results of this estimation. In three specifications we had to moreover impose the restriction $d = 1$ in order to achieve convergence of the model (we mark this restriction by the letter *"d"* set before the period specification of the affected periods in Table 8). The imposition of this restriction solved the instability of the model for FTSE 100 in the first period. However, this restriction was of no use in the case of S&P 500. In the case of S&P 500, the models in the full period and in the second sub-period remain unstable (which we mark by *"u"*). The roots of the characteristic polynomial exceeding unity are 1.205 and 1.237, respectively.

We can see that the order of integration of daily log-prices is in all cases greater than 1 (which is different from the estimation without the restriction, where the order of integration was mostly below 1). The estimates of the cointegration gap $\hat{b}$ are quite similar to the unrestricted specification, apart from the case of NIKKEI 225 and PX 50 in the first period. In these two situations the difference $(d - b)$ changed from 0.384 to 0.539 and from 0.341 to 0.533, respectively (see Table 9). The estimates of the order of integration of range have thus moved from the stationary region into the non-stationary region. We have already discussed these two indices when analysing the ACF functions, where we note that in the first period the presence of long memory is arguable; this was further supported by the results from the GPH and ELW estimators, where range was found stationary (but not integrated of order 0). In all other cases the maximum change in the estimate of the difference $(d - b)$ was 0.05 and the implication for stationarity or non-stationarity remained unchanged. In the cases of NIKKEI 225 and PX 50 in the first period we can also note the highest differences among the α and γ parameters in the estimation with and without the restriction. In the other specifications, these six parameters vary slightly, but nowhere near as much as in these two cases. The adjustment coefficients α_H are again negative, and the adjustment coefficients α_L remain positive. When interpreting these signs, we can make use of the fact that the cointegrating vector is now the range. An increase in daily range is reduced in the next day by decreasing the high log-price and boosting the low log-price for that day. The short-run dynamics parameters (γ_{11}, …, γ_{22}) are again mostly negative for the lagged daily highs and mostly positive for the lagged daily lows; the interpretation remains the same as in the previous specification. We can conclude that the imposition of the $(1,-1)$ restriction on the cointegrating vector may have impaired the results, even though the original estimates of the cointegrating vector were fairly close to the desired $(1,-1)$.

Table 8: FCVAR estimation results (with restrictions)

	$\hat{d}$	$\hat{b}$	$\hat{\beta}$	α_H	α_L	γ_{11}	γ_{12}	γ_{21}	γ_{22}
S&P 500									
u 2003-2012	1.035	0.339	(1,-1)	-0.258	3.483	-0.692	0.846	-1.362	2.139
d 2003-2007	1.000	0.426	(1,-1)	-1.114	1.834	0.145	0.129	-0.396	0.997
u 2008-2012	1.024	0.366	(1,-1)	-0.154	2.788	-0.646	0.819	-0.827	1.583
FTSE 100									
2003-2012	1.013	0.390	(1,-1)	-0.623	2.135	-0.303	0.487	-0.595	1.286
d 2003-2007	1.000	0.419	(1,-1)	-1.055	1.890	0.078	0.140	-0.743	1.295
2008-2012	1.018	0.468	(1,-1)	-0.593	1.215	-0.224	0.369	0.021	0.515
DAX									
2003-2012	1.018	0.409	(1,-1)	-0.398	1.876	-0.392	0.543	-0.495	1.039
2003-2007	1.054	0.446	(1,-1)	-0.685	1.670	-0.103	0.253	-0.252	0.591
2008-2012	1.013	0.373	(1,-1)	-0.098	2.343	-0.647	0.795	-0.790	1.385
NIKKEI 225									
2003-2012	1.028	0.489	(1,-1)	-0.210	1.511	-0.342	0.539	-0.202	0.645
2003-2007	1.046	0.507	(1,-1)	-0.590	1.631	-0.138	0.261	-0.468	0.828
2008-2012	1.001	0.516	(1,-1)	-0.079	1.123	-0.295	0.542	0.188	0.297
PX 50									
2003-2012	1.039	0.455	(1,-1)	-0.753	1.510	-0.201	0.306	-0.642	0.993
2003-2007	1.053	0.520	(1,-1)	-0.624	1.049	0.041	0.160	-0.121	0.493
d 2008-2012	1.000	0.427	(1,-1)	-0.863	1.493	-0.252	0.399	-0.727	1.122

Note: "u" denotes that model is unstable (some roots of characteristic polynomial lie outside the unit root circle), *"d"* denotes that the restriction $d = 1$ had to be imposed in order to achieve model convergence.

The residual diagnostics are very similar to the previous case. According to the Ljung-Box Q-test, there is some marginal autocorrelation in the residuals, which may bias the standard errors; also, there is some small heteroscedasticity in the residuals based on the visualization of the ACF of squared residuals; however, as this model does not provide any standard errors, neither of these shortcomings changes any of our inferences.[4]

[4] For the sake of brevity, this residual diagnostic is not presented here, but is available upon request.

Finally, having all the estimates we can make the final comparison of the level of integration of range, which was our prior interest. We have first examined the stationarity of range based on the ADF test. In all cases the range was stationary, which was not consistent with our inspection of the autocorrelation function of range. Except for the PX 50 index in the first period, and possibly the NIKKEI 225 in the first period, in all other specifications we could see a substantial amount of dependence in the autocorrelation function, which is an indication of the possibility of long memory in range. Formal tests for long memory were applied, namely the ELW and GPH estimators. Both estimators confirmed our hypothesis of long memory in range. These estimators have also confirmed the stationarity of range for both the NIKKEI 225 and PX 50 indices in the first period and the non-stationarity of the remaining indices in all periods. Finally, we have estimated the long memory of range using the fractionally cointegrated VAR model, after imposing a restriction to the values $(1, -1)$ on the cointegrating vector. Due to this restriction, the order of integration of range was represented by the difference between the order of integration of daily log-prices and the cointegration gap, i.e. the integration order of range was the difference $(d - b)$. All these results are summarised in Table 9. In the first four columns of Table 9, we present the integration orders of range from GPH and ELW estimators with both bandwidth specifications we considered earlier in the study. The integration orders of range from the FCVAR framework (both with and without the imposed restriction) are displayed in the last two columns of Table 9.

In Table 9 we can note that the estimates of long memory in range are quite sensitive to both the chosen methodology and the chosen bandwidth parameter. In the case of S&P 500, even though there were some problems with the stability of the FCVAR model, we can see that the FCVAR estimates of long memory in range are in line with both the GPH and ELW estimates (with the smaller bandwidth parameter). For the FTSE 100 index in the second period, the FCVAR estimates of long memory appear to be quite underestimated, as the values would imply that the second period is less non-stationary than the first one, which contrasts all other results (ELW and GPH estimates and inspection of ACF). In the case of DAX, we can note that each estimator chooses different period to be the most and the least non-stationary, despite the fact that the estimates are in magnitude close to each other.

Table 9: Comparison of integration orders of range

	GPH		ELW		FCVAR	
	$m = T^{0.5}$	$m = T^{0.6}$	$m = T^{0.5}$	$m = T^{0.6}$	R	NR
S&P 500						
2003-2012	0.765	0.674	0.763	0.622	0.696	0.696
2003-2007	0.520	0.678	0.587	0.574	0.574	0.532
2008-2012	0.753	0.721	0.664	0.687	0.658	0.647
FTSE 100						
2003-2012	0.710	0.712	0.658	0.632	0.623	0.613
2003-2007	0.515	0.596	0.622	0.603	0.581	0.604
2008-2012	0.739	0.701	0.628	0.617	0.550	0.524
DAX						
2003-2012	0.652	0.706	0.588	0.589	0.609	0.599
2003-2007	0.803	0.826	0.614	0.597	0.608	0.566
2008-2012	0.710	0.738	0.700	0.623	0.640	0.631
NIKKEI 225						
2003-2012	0.499	0.659	0.483	0.649	0.539	0.487
2003-2007	0.463	0.401	0.496	0.392	0.539	0.384
2008-2012	0.701	0.784	0.613	0.604	0.485	-
PX 50						
2003-2012	0.577	0.617	0.516	0.520	0.584	0.532
2003-2007	0.211	0.316	0.281	0.446	0.533	0.341
2008-2012	0.760	0.581	0.572	0.517	0.573	0.528

Note: "R" denotes model with restriction on the cointegrating vector; "NR" denotes model without restriction.

The FCVAR results for NIKKEI 225 are quite poor. Contrary to the ELW and GPH estimates, the FCVAR with restrictions does not confirm the stationarity of range in the first period, and it overestimates the order of integration. Moreover, in the second period the dependence is significantly underestimated. This may be caused by the fact that the FCVAR model does not work very well for NIKKEI 225 in the second period (the model does not converge when no restrictions are imposed). We can see that the FCVAR model also fails to confirm the stationarity of the range of the PX 50 index in the first period, and overestimates the dependence as well. The FCVAR results in the two remaining periods are quite similar to both the ELW and GPH estimates (with the bigger bandwidth parameter).

Overall we can see that the FCVAR model with restrictions fails to detect the lower orders of integration of range and suggests that the range is in the non-stationary region, when it should be stationary according to the results from other applied tests. However, we should note that when the restriction on the cointegrating vector was not imposed, the FCVAR model did detect the stationarity of the "range". However, without the restriction, interpreting the error correction term in the model as range is incorrect, even though the cointegrating vector was fairly close to the value required for the interpretation to be valid even in the unrestricted model. Also, when no restrictions are imposed, we obtain the standard errors of the estimated parameters, whereas no standard errors are provided with the restricted model, which makes it impossible to draw an inference about the significance of the estimates.

The most unanimous conclusion is that apart from the ranges of PX 50 and NIKKEI 225 in the first period, which are in the stationary region, the remaining ranges are non-stationary and display long memory. The best results can be seen in the case of S&P 500, where all four different methods for examining long memory yield results closest to each other, despite the instability of the FCVAR model for this index.

6 Conclusion

In this thesis, we provide unique empirical support for fractionally cointegrated daily high and low stock prices. These maximum and minimum daily prices are examined because they provide valuable information about range-based volatility. The range, defined as a difference between daily high and low log-prices, is considered a highly efficient and robust estimator of volatility (Parkinson, 1980).

We apply the fractionally cointegrated VAR methodology formalized only recently by Johansen and Nielsen (2012) to model the high and low prices. This framework is able to capture both the cointegration between daily high and low log-prices and the long memory of their linear combination, the range. In this concept, the range is, in fact, the error correction term in the FCVAR model and is allowed to fall into a non-stationary region.

Since the original definition of cointegration by Granger (1986) and Engle and Granger (1987), the case when series are integrated of order 1 and their linear combination is integrated of order 0 have been predominantly considered in the literature. However, this may be too restrictive and that is why we apply here the very general case when the series can be integrated of order d and cointegrated of order $(d-b)$, where $0<b\leq d$. If the cointegrating relation has an interpretation of range, the order of the fractional cointegration between daily high and low log-prices determines the stationarity and long memory of range. If $(d-b)>0$, we then say that range displays long memory, and if $(d-b)>0.5$, we say that range is non-stationary.

The analysis is performed on four major global indices, namely, the American S&P 500, the German DAX, the Japanese NIKKEI 225 and the UK's FTSE 100, and the results are compared to the Czech PX 50 index. We consider three periods, the base period being 2003-2012, and its division into two sub-periods, with year 2007 as the breakpoint. The first sub-period captures the relatively calm time before the crisis, and the second period covers the outbreak of the crisis and the post-crisis turbulences.

Our analysis is conducted as follows. We first test the stationarity of daily high and low log-prices and of price range. The ADF test implies that range is stationary, which contradicts findings from the inspection of autocorrelation functions, where a significant dependency is visible. We thus employ the Exact Local Whittle and Geweke Porter-Hudak estimators of long memory and confirm the existence of long memory in range. The evidence of long memory in asset price volatility was also found for example by Ding, Granger and Engel (1993), Andersen and Bollerslev (1997), Breidt et al. (1998), Kellard et al. (2010) or Garvey and Gallagher (2012).

Since the general well-known tests for cointegration proposed by Johansen (1991) are not suitable when series display long memory, the cointegration of daily high and low log-prices is tested in the fractional framework. We employ the cointegration rank test proposed by Nielsen and Shimotsu (2007) with the result of one cointegrating vector. This finding is also confirmed by the cointegration rank test, which is part of the FCVAR estimation procedure. Two FCVAR specifications are considered, one without any restrictions imposed and one with a restriction on the cointegrating vector in order to get an interpretation of range.

We find significant evidence that range-based volatility estimate displays a long memory. Moreover, in most cases range is in the non-stationary region (i.e. is integrated of order greater than 0.5). Only the ranges of the PX 50 and NIKKEI 225 indices in the first (2003-2007) period are in the stationary region. Overall, the estimates of the long memory are quite sensitive to the selected methodology. In general, the first period is rather the calmer period, in which the estimates of the long memory parameters are mostly the lowest. The second period (2008-2012) is more turbulent. We also show that in the FCVAR framework with restrictions imposed the results are slightly inferior to the original unrestricted FCVAR model, mainly in the situation when range should be in the stationary region based on other applied tests. Furthermore, the results for the PX 50 index are very similar to the results for the NIKKEI 225 as their ranges display the lowest estimates of integration orders. Integration orders of ranges of S&P 500, FTSE 100 and DAX indices are, on the other hand, relatively higher.

7 References

Alizadeh, S., M. W. Brandt, and F. X. Diebold (2002)
Range–Based Estimation of Stochastic Volatility Models, The Journal of Finance, 57(3), 1047–1091.

Andersen, T. G., and T. Bollerslev (1997)
Heterogeneous Information Arrivals and Return Volatility Dynamics: Uncovering the Long–Run in High Frequency Returns, The Journal of Finance, 52(3), 975–1005.

Andersen, T., and T. Bollerslev (1998)
Answering the Skeptics: Yes, Standard Volatility Models Do Provide Accurate Forecasts, International Economic Review, 39(4), 885–905.

Baillie, R. T. (1996)
Long Memory Processes and Fractional Integration in Econometrics, Journal of Econometrics, 73(1), 5–59.

Breidt, F. J., N. Crato, and P. de Lima (1998)
The detection and estimation of long memory in stochastic volatility, Journal of Econometrics, 83 (1–2), 325–348.

Brockwell, P.J., and R.A. Davis (1991)
Time Series: Theory and Methods, 2nd edition, Springer–Verlag, New York.

Caporin, M., Ranaldo, A., and P. Santucci de Magistris (2011)
On the Predictability of Stock Prices: A Case for High and Low Prices, "Marco Fanno" Working Papers 0136, Dipartimento di Scienze Economiche "Marco Fanno".

Cheung, Y. (2007)
An empirical model of daily highs and lows, International Journal of Finance and Economics, 12(1), 1–20.

Cipra, T. (2008)
Finanční ekonometrie, (Financial Econometrics), first edition, Ekopress 2008.

Corwin, S. A., and P. H. Schultz (2012)
A Simple Way to Estimate Bid–Ask Spreads from Daily High and Low Prices, The Journal of Finance, 67(2), 719–760.

Ding, Z., Granger, C.W.J., and R.T. Engle (1993)
A long memory property of stock market returns and a new model, Journal Empirical Finance 1(1), 83–106.

Engle, R., and C.W.J. Granger (1987)
Cointegration and error correction: representation estimation, and testing, Econometrica, 55(2), 251–276.

Garman, M. B., and M. J. Klass (1980)
On the Estimation of Security Price Volatilities from Historical Data, The Journal of Business, 53(1), 67–78.

Garvey, J. F., and L.A. Gallagher (2012)
The Realised–Implied Volatility Relationship: Recent Empirical Evidence from FTSE–100 Stocks. J. Forecast., 31(7), 639–660.

Geweke, J., and S. Porter–Hudak (1983)
The estimation and application of long memory time series models, Journal of Time Series Analysis 4(4), 221–238.

Granger, C.W.J. (1980)
Long memory relationships and the aggregation of dynamic models, Journal of Econometrics, 14(2), 227–238.

Granger, C.W. J. (1986)
Developments in the study of cointegrated economic variables, Oxford Bulletin of Economics and Statistics, 48(3), 213-228.

Granger, C.W.J., and N. Hyung (2004)
Occasional structural breaks and long memory with an application to the S&P 500 absolute stock returns, Journal of Empirical Finance, 11(3), 399–421.

Granger, C. W. J., and R. Joyeux, (1980)
An introduction to long memory time series models and fractional differencing, Journal of Time Series Analysis, 1(1), 15–29.

Johansen, S. (1991)
Estimation and hypothesis testing of cointegration vectors in Gaussian vector autoregressive models, Econometrica 59(6), 1551–1581.

Johansen, S. (1996)
Likelihood–Based Inference in Cointegrated Vector Autoregressive Models, 2nd edition, Oxford University Press, Oxford.

Johansen, S. (2008)
A representation theory for a class of vector autoregressive models for fractional processes, Econometric Theory, 24(3), 651–676.

Johansen, S., and M. O. Nielsen (2010)
Likelihood inference for a nonstationary fractional autoregressive model, Journal of Econometrics 158(1), 51–66.

Johansen, S., and M. O. Nielsen (2012)
Likelihood inference for a fractionally cointegrated vector autoregressive model, Econometrica, 80(6), 2667–2732.

Kanzler, L. (1998)
GPH: MATLAB module to calculate Geweke–Porter–Hudak long memory statistic, Statistical Software Components.

Kellard, N., Dunis, C., and N. Sarantis (2010)
Foreign exchange, fractional cointegration and the implied–realized volatility relation, Journal of Banking & Finance, 34(4), 882–891.

Lobato, I.N., and N.E. Savin (1997)
Real and spurious long–memory properties of stock–market data, Journal of Business and Economic Statistics, 16(3), 261–268.

MacKinnon, J. G., and M. Ø. Nielsen (2012)
Numerical distribution functions of fractional unit root and cointegration tests, Journal of Applied Econometrics.

NBER
US Business Cycle Expansions and Contractions, available at: http://www.nber.org/cycles/, last accessed on 15. 5. 2012.

Nielsen, M., and L. Morin (2012)
FCVARmodel.m: A Matlab software package for estimation and testing in the fractionally cointegrated VAR model, QED working paper 1273, Queen's University.

Nielsen, M. Ø., and K. Shimotsu (2007)
Determining the cointegration rank in nonstationary fractional system by the exact local Whittle approach, Journal of Econometrics, 141(2), 574–596.

Parkinson, M. (1980)
The Extreme Value Method for Estimating the Variance of the Rate of Return, The Journal of Business, 53(1), 61–65.

Robinson, P. M., and Y. Yajima (2002)
Determination of cointegrating rank in fractional systems, Journal of Econometrics, 106(2), 217–241.

Shimotsu, K., and P. Phillips (2005)
Exact local Whittle estimation of fractional integration, Annals of Statistics, 33(4), 1890–1933.

8 Appendix

Figure 6: High and Low Log-Prices of DAX

Figure 7: Range of DAX

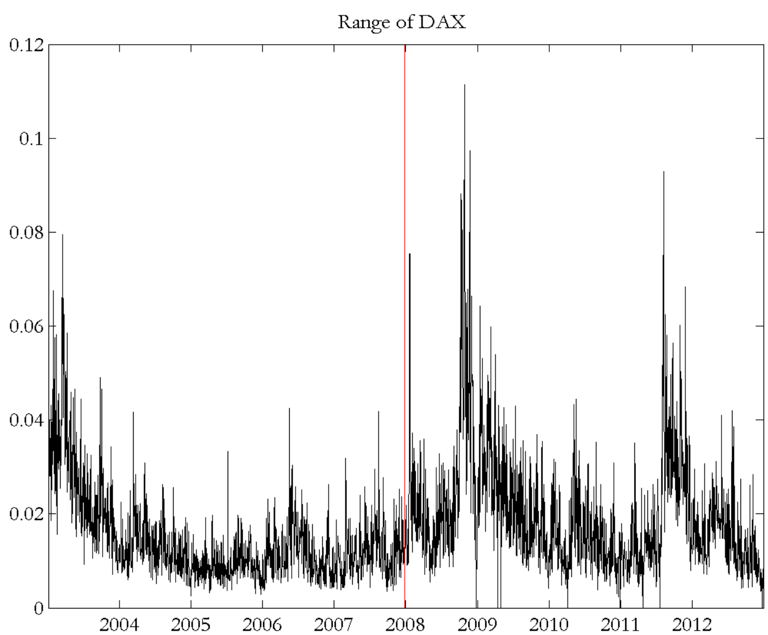


Figure 8: High and Low Log-Prices of FTSE 100

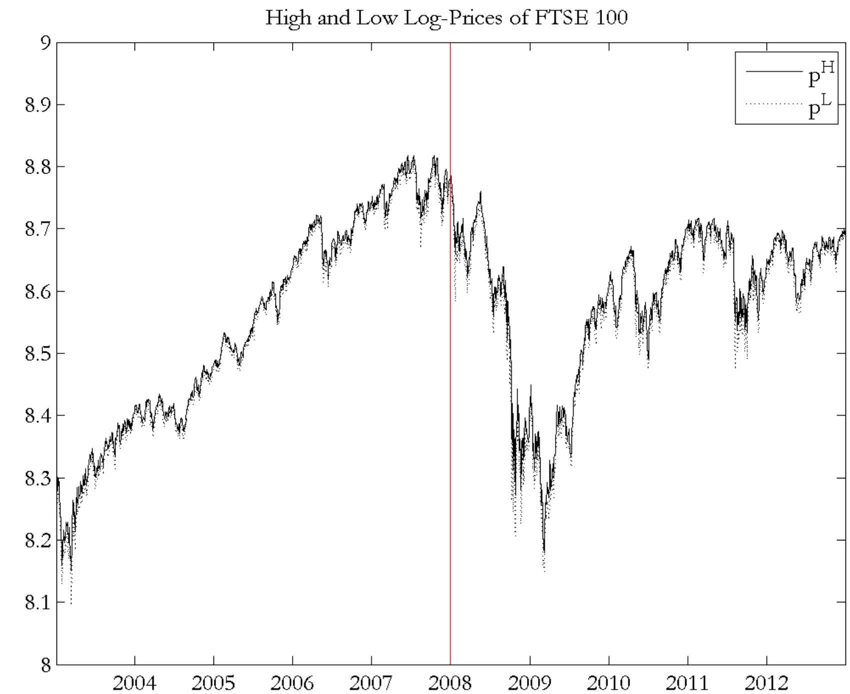


Figure 9: Range of FTSE 100

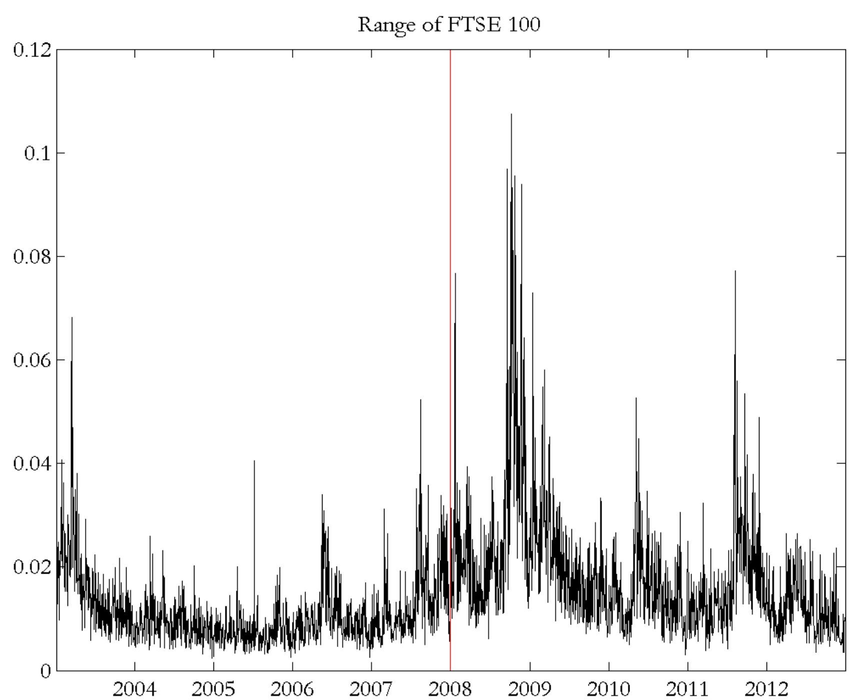


Figure 10: High and Low Log-Prices of NIKKEI 225

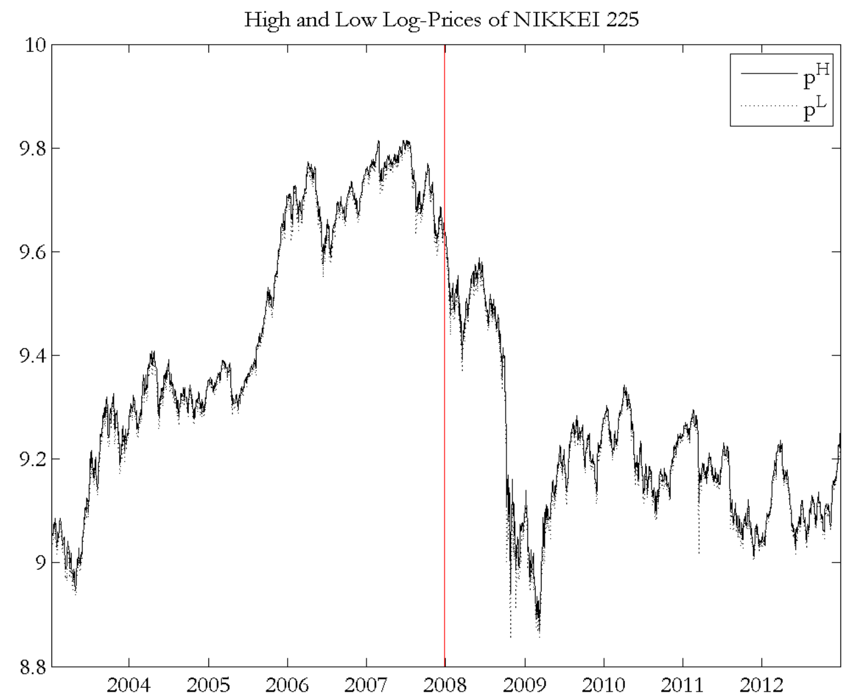


Figure 11: Range of NIKKEI 225

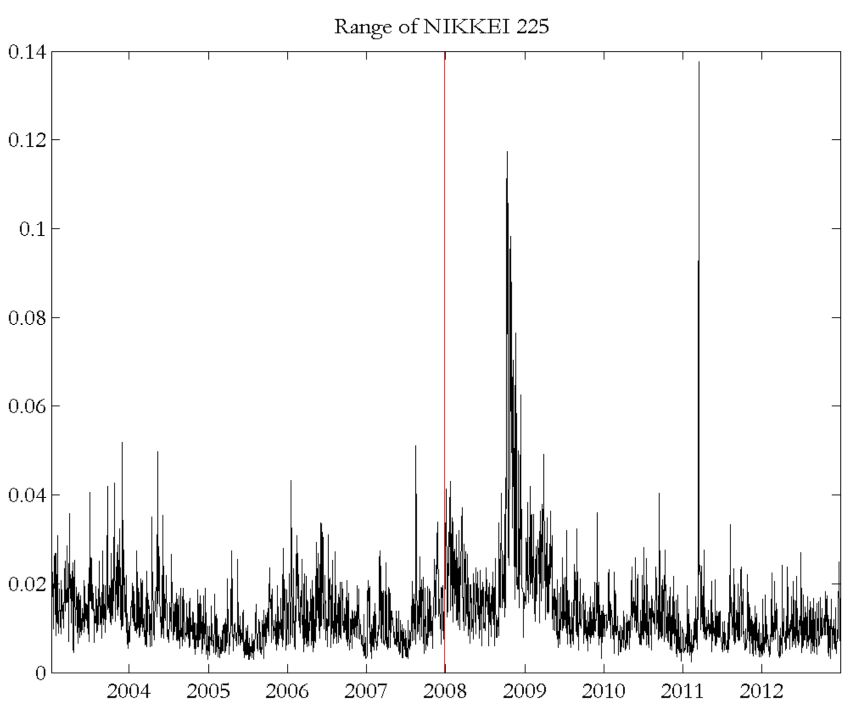


Figure 12: High and Low Log-Prices of S&P 500

Figure 13: Range of S&P 500

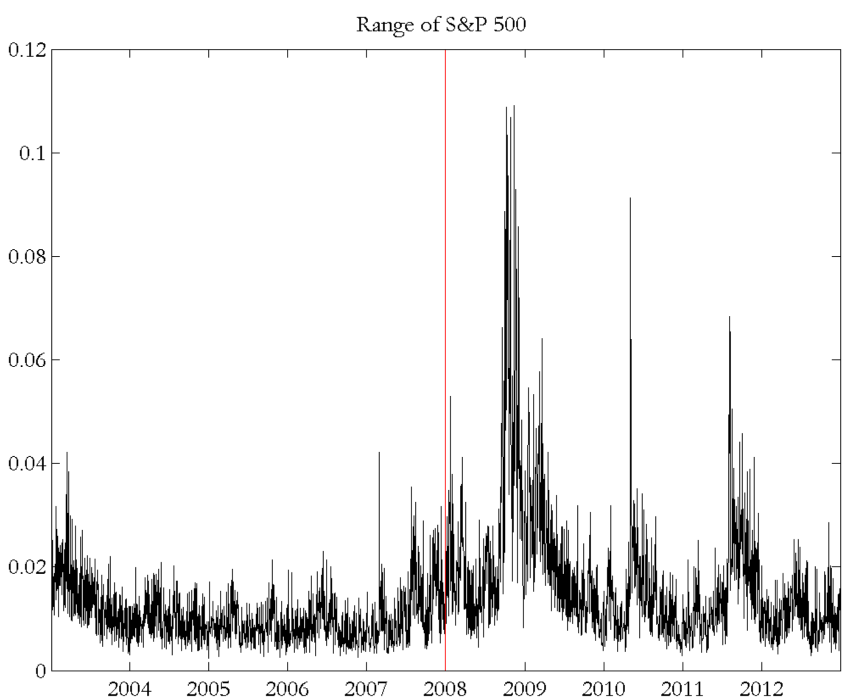


Figure 14: ACF of Detrended High and Low Log-Prices of DAX (2003-2007)

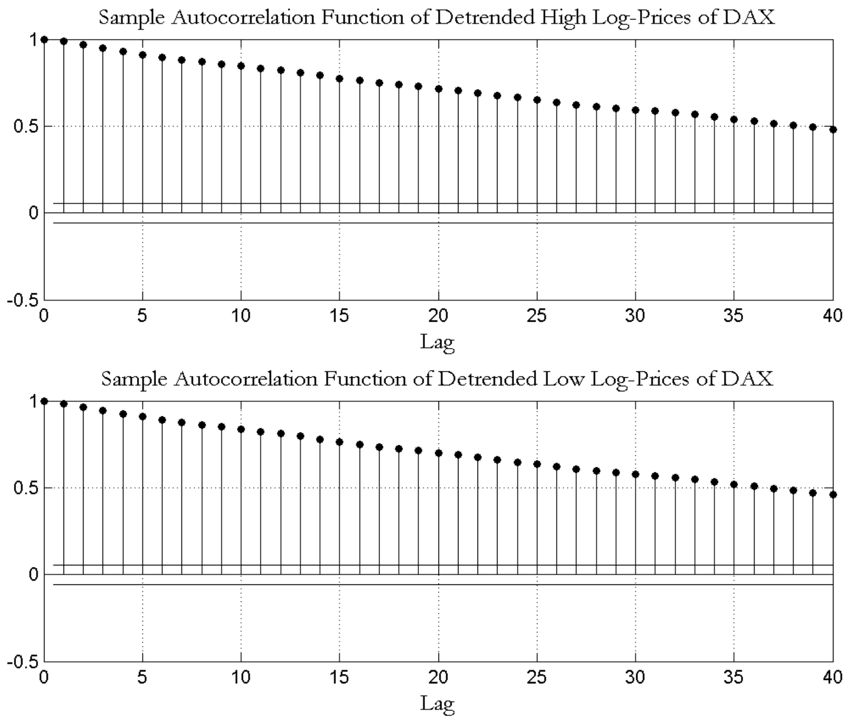


Figure 15: ACF of Detrended High and Low Log-Prices of FTSE 100 (2003-2007)

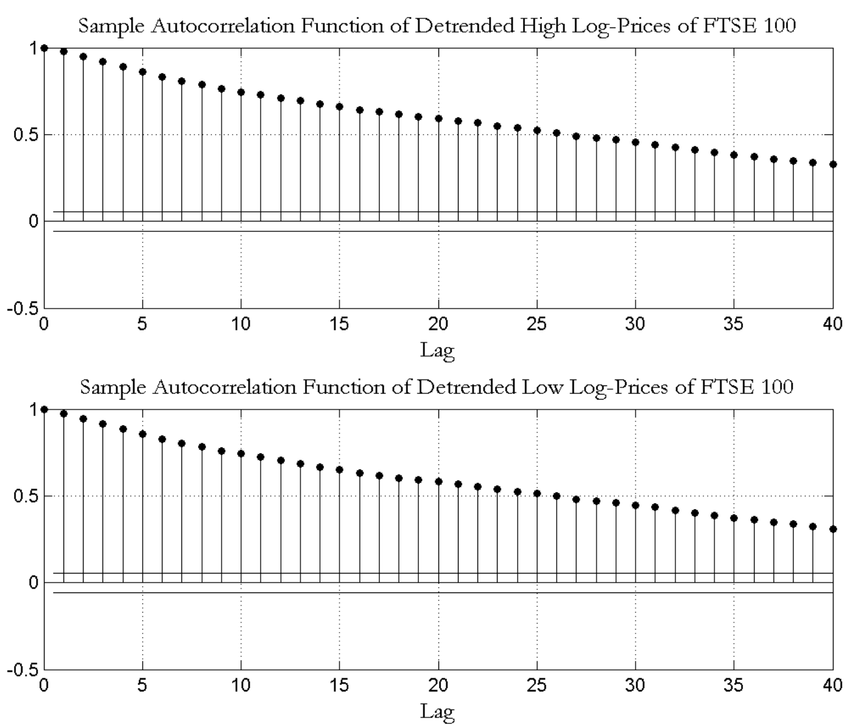


Figure 16: ACF of FTSE 100 Range in All Periods

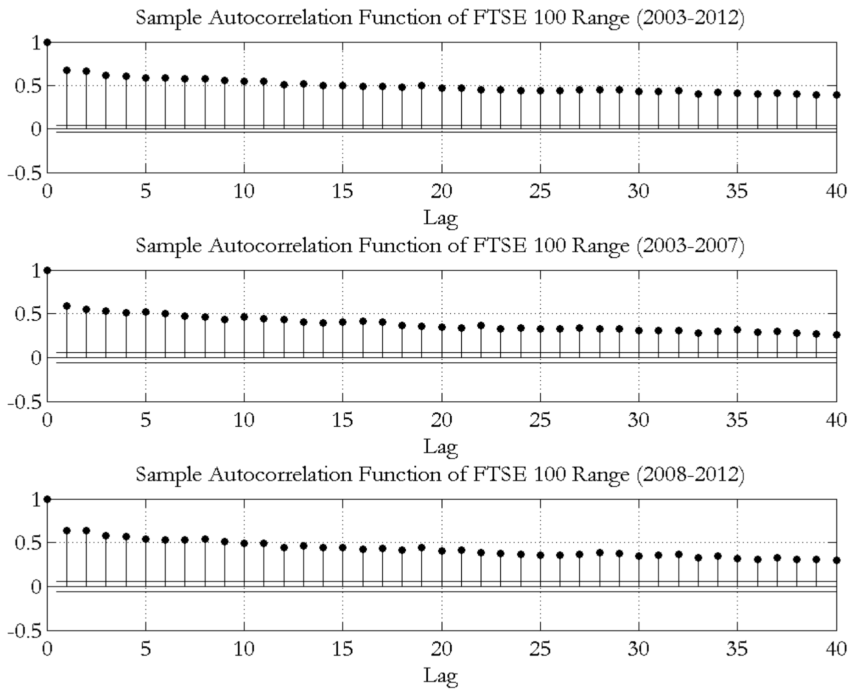


Figure 17: ACF of S&P 500 Range in All Periods

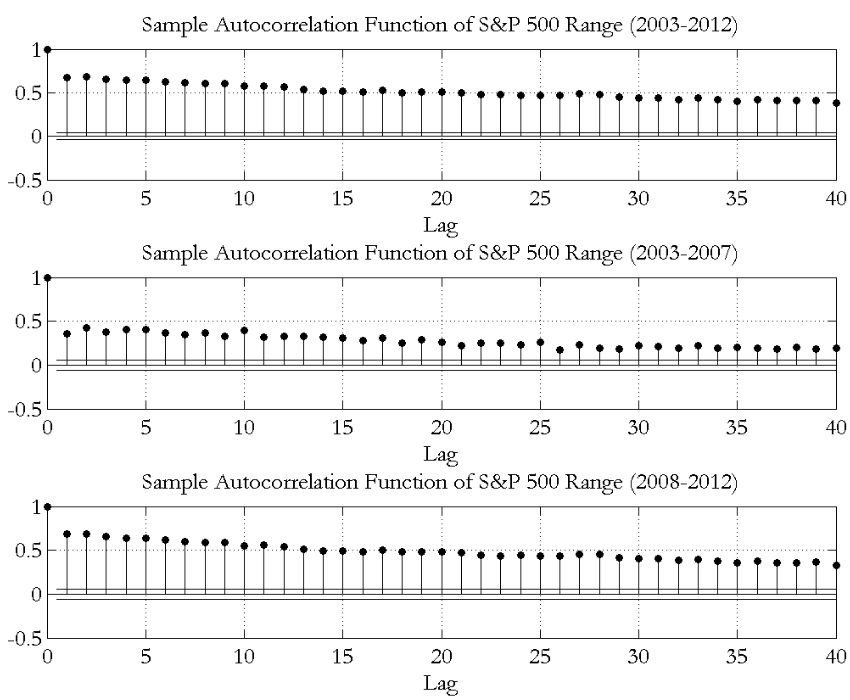


Figure 18: ACF of NIKKEI 225 Range in All Periods