Univerzita Karlova v Praze, Filozofická fakulta
Katedra logiky

Alexandra Kuncová

Bakalářská práce

Srovnání logického a psychologického pojetí čísla

Comparison of logical and psychological perspectives on the concept of number

Vedoucí práce: Mgr. Vít Punčochář

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Poděkování

V prvom rade by som chcela poďakovať Mgr. Vítovi Punčochářovi, vedúcemu práce, za to, že sa ujal mojej neurčitej predstavy a dodal jej potrebnú štruktúru. Aj keď je kognitívna veda pomerne mladá, rozrastá sa rýchlo, a tak mi každé jeho nasmerovanie padlo veľmi vhod. Rovnako chcem poďakovať aj profesorovi na belgickej univerzite KU Leuven, Walterovi Schaekenovi, ktorého prednáška ma priviedla k samotnej kognitívnej vede a následne teda aj k téme tejto práce. V neposlednom rade patrí vďaka aj mojim priateľom a blízkym, ktorí mi boli pri písaní viac ako len morálnou podporou.

Prohlašuji, že jsem bakalářskou práci vypracovala samostatně a že jsem uvedla všechny použité prameny a literaturu.

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Alexandra Kuncová

Abstrakt

Táto práca je rozdelená na tri hlavné časti. V prvej časti predstavujeme logický prístup k pojmu čísla na základe Fregeho diela Základy Aritmetiky. Popri snahe definovať a klasifikovať číslo ako také, rozoberáme aj Husserlove ťažkosti s psychologizmom, Fregeho logicizmus, či konštrukciu postupnosti prirodzených čísel. V druhej časti sa pozeráme na psychologický prístup k pojmu čísla, a to cez teórie a experimenty kognitívnej vedy. Zameriavame sa na detské chápanie čísel a množstiev, ich schopnosť počítať a neskôr získané všeobecné znalosti. V poslednej časti sumarizujeme rozdiely, ako aj podobnosti týchto dvoch prístupov.

Kľúčové slová: číslo, Frege, Husserl, identita, kognitívna veda, pojem, logicizmus, Piaget, psychologizmus, vzájomne jednoznačná korešpondencia.

"Kým zákony matematiky popisujú realitu, tak nie sú presné; a keď sú presné, tak nepopisujú realitu."
(Albert Einstein, Geometry and Experience, 1921)

Abstract

This paper is divided into three main parts. In the first part, we propose a logical approach to the concept of number based on Frege's Foundations of Arithmetic. Besides the main attempt to define and classify number per se, we also discuss Husserl's struggle with psychologism, Frege's logicism, and the construction of the series of natural numbers. In the second part, we look at a psychological approach to the concept of number through theories and experiments of cognitive science. We focus on infants' understanding of numbers and amounts, their counting abilities and later conventional skills. In the third part, we summarise differences as well as similarities of these two approaches.

Keywords: cognitive science, concept, Frege, Husserl, identity, logicism, number, one-to-one correspondence, Piaget, psychologism.

"As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."
(Albert Einstein, Geometry and Experience, 1921)

Contents

Introduction

We all have a certain idea of what number is. We have been taught how to use it as an essential part of a counting process. We are surrounded by it on a daily basis and in comparison to language differences many of us do share 'the mother tongue of numbers'. But what if we were asked to explain what number is to someone who had never heard of it? Can we even imagine a person without a concept of number? Probably not, as many of us cannot get their heads around the surreal ground it requires. The main problem would be to explain what it is about two cats and two cookies that makes them represent the same number. Though, it seems very unnatural to contemplate numbers like this. In some way or other we consider ourselves inherently connected to their concept. However, as with many other aspects inseparable from our life, knowing how to use it does not necessarily mean knowing what it actually is and where it comes from. . . .
In this paper we look at the concept of number from two quite different points of view. We plan to shed light upon the issue with the logical perspective in one hand and the psychological in the other. There might appear certain tension between the two approaches, but they are by no means each other's opposites. For the most part we would expect logic to provide us with the abstract side and psychology with the concrete side of the scrutiny. However, we want to show that in the case of something so natural yet unreachable as numbers, the ice between abstract and concrete might be considerably thin. As abstraction needs its limitations but can never get too fettered, concrete may head towards objective but will never reach it. Where do these two perspectives meet and where do they differ? Before we consider their potential cooperation, let us first introduce logical and psychological approaches separately. They were each challenged with a compelling question:

What is number? and Where does it come from?

Chapter 1

Logical perspective on the concept of number

1.1 Introduction

To begin with the comparison of logical and psychological approach to the concept of number there is one name that stands out from the rest and it is Gottlob Frege. As we will further see, he is very clear about his own anti-psychologistic perspective, the same way he tries to point out misconceptions of those dealing with the same issue. According to Frege the main problem of defining the concept of number came with one of the biggest enemies of his time – with the ease that many blended elements of psychology in the logical approach. To call the shadow by its real name we shall first shed light upon the tense spots of the relationship between logic and psychology.

1.2 Psychologism and its opponents

In this section we offer a brief introduction to psychologism. What does the term *psychologism* mean? Where did it come from? Why was it so popular? And last but not least, why did Frege stand against it in the time of its prime?
The influence of psychologism was the most significant in German-speaking philosophy at the end of the 19^{th} century. It is not a coincidence that it is the same period when empirical psychology (with tendency to subordinate philosophy) was recognised as a science. The term *psychologism* was usually used in the context of the German phrase Psychologismus-Streit, translated as the 'psychologism dispute'. These were debates over the idea that logic and epistemology actually belong under psychology. Thus, the logical truths (and those of mathematics) were considered dependent on people grasping them. The basic argument asserts that whatever deduction steps we do, we always get to a certain dead end from where there is no other way but to turn to the mental aspects of a person. It makes sense that many were tempted to establish such foundations at the point where letting the fundamental truths be self-verifying was out of the question.
There are two well-known figures closely associated with the critique of psychol-

ogism in the turn of the 20th century. We have already mentioned Frege and the second one is Edmund Husserl. Although eventually they were both on the same side Husserl addressed the critique only later on and his first work Philosophy of Arithmetic can be considered psychologistic. And as such Frege criticises it in the corresponding review. We return to this subject in the following section.[1]
Frege is a very persistent anti-psychologist. He argues that there are such thoughts which have their truth-value simply given for good[2] and "... regardless of whether or not anyone believes [them] and even whether or not anyone has grasped [them] at all.... They are neither created by our uses of language or acts of thinking, nor destroyed by their cessation."[Kle] He considers these thoughts to be abstract objects which are neither subjective nor objective (in a sense of physical entity such as a chair). On a ground of relativism they are rejected to exist in each one of us individually, for there would be no normativity (i.e., a set of valid norms), and it would be impossible to err since a truth (one of many) would depend purely on one's beliefs. Obviously, Frege could not accept such a view for 'normativity-embracing' logic. In comparison to that, the argument against certain kind of objectivity might seem a bit weaker. Frege merely denies that these objects exist in the physical realm. Consequently, he places them beyond the both known – the inner and the outer world – to a "third realm"[F-Ge, p. 302]. Whereas this Fregean third realm might look like a clever way to avoid the problem of the objects (seemingly) falling between the private mental realm and the tangible physical one we all share, psychologism would, understandably, place Fregean thoughts into the former realm since we cannot see or touch them (for they do not physically exist). We discuss this issue in detail later in this chapter.[3]
Naturally, this 'logico-psychologistic fight' was to be expected to emerge also outside of the German-speaking countries, spread out and consequently reach its glory. It was John Stuart Mill, a British philosopher, who played a big role, paradoxically, on both of the fields of the battle over 'the custody of logic'. Even though his ideas were recognised as a crucial contribution, he was often criticised for their mutual inconsistency, for his way of balancing with one foot in psychologistic philosophy and the other in anti-psychologistic.
To sum it up, psychologism is a quite powerful theory offering precious support and connections from the psychological sphere. Whether we actually need them calls for a deeper investigation which we aim to do in this paper.

[1] See section 1.3.
[2] Frege also calls such thoughts "timeless"[F-Ge, p. 309].
[3] See mainly section 1.5.

1.3 Frege on Husserl's Philosophy of Arithmetic

In this section we deal with Frege's Review of Dr. E. Husserl's Philosophy of Arithmetic. To prevent misunderstanding right at the beginning we shall recognise Husserl's main philosophical approach as anti-psychologistic. However, it was not always that clear. His former ambiguity is quite obvious in his Philosophy of Arithmetic on a basis of which Frege aptly considered him a logico-psychologist. Husserl seemed to take the right way, but needed some time to overcome its obstacles.

In the review Frege expresses his disappointment from discovering the evident traces of psychology interwoven into logic throughout Husserl's reflections; to use Frege's own words:

> "In reading this book, I have been able to see how very difficult it is for the sun of truth to penetrate the fog which arises out of the confusion of psychology and logic."[F-Re, p. 336]

Step by step, with emphasis on the first part of the book, Frege is dealing with drawbacks and serious misconceptions of Husserl's work. We do not intend to go very deeply into the subject as the ground of Frege's objections is described in the following sections. Thus, we shall focus here mainly on the psychological aspects at issue.

Firstly, turning to the word *multiplicity* for help in defining the concept of number was definitely not the best idea of Husserl's. It has immediately become Frege's target. To answer the question how do we get to work with something like multiplicities in the first place, Frege summarises Husserl's explanation as follows: "Totalities are wholes whose parts are collectively connected.... The collective connection consists neither in the contents' being simultaneously in the awareness, nor in their arising in the awareness one after another.... The connection consists ... in the unifying act itself."[F-Re, p. 322] Husserl subsequently presents mental act which is supposed to connect contents simply by being directed towards them. However, the exact nature of this process stays unclear. Once again Frege shakes his head and leaves this erroneous path with the following words:

> "The most naive opinion is that according to which a number is something like a heap.... Next comes the conception of a number as a property of a heap, aggregate, or whatever else one might call it.

> Thereby one feels the need for cleansing the objects of their particularities. The present attempt belongs to those which undertake this cleansing in the psychological wash-tub."[F-Re, p. 323]

Secondly, even if this was a promising way to go, Husserl very soon stumbles across the contradictory requirement of abstraction so needed in his method. To get the overall number, on the one side, we have to abstract the contents from all of their particularities, but on the other side, we also need to leave something that would keep the information of them being connected. However, conveniently enough, on the ground of psychology "[n]umber-abstraction simply has the wonderful and very fruitful property of making things absolutely the same as one another without altering them."[F-Re, p. 332] Tempting, is it not?
Thirdly, Frege reproaches Husserl for his ambiguous use of the word *presentation*. It is very easy to get lost in the flow of Husserl's thoughts, because "... the difference between presentation and concept, between presenting and thinking, is blurred. Everything is shunted off into the subjective. But it is precisely because the boundary between the subjective and the objective is blurred, that conversely the subjective also acquires the appearance of the objective."[F-Re, p. 324] It is one's power over a presentation that we simply cannot control. It seems as if each one of us could modify the presentation in quite fundamental ways. It lies in contrast with Fregean conception of thought which is fully independent and stays 'untouched' whether anybody grasps it or not. Throughout Frege's investigation "[i]t becomes overwhelmingly evident that our presentations matter very little here, but that instead it is the very thing which we seek to make presentations to ourselves that is the subject of our concern, and that our assertions are about it."[F-Re, p. 336] Our presentation of a number itself is not the number we are looking for, because one number may have more than one presentation. Since logic deals directly with the nature of numbers, Husserl's presentations of numbers will not suffice and even might land on the ground of psychology.
In the end, Frege tests Husserl's resistance to "[t]hree reefs [which] spell danger for naive, and particularly for psychological, views of the nature of numbers. The first lies in the question, how the sameness of the units is to be reconciled with their distinguishability. The second consists in the numbers zero and one; and the third, in the large numbers."[F-Re, p. 330] Since at the end of this chapter we discuss further only the first two reefs, we briefly sketch the third one here.[4]
The problem consists in the impossibility to imagine (sufficiently) large numbers; neither all at once because of the spacial issues (i.e., the memory limitations),

[4] For the first and the second reef see subsections 1.5.2 and 1.5.3.

nor one by one because of our life expectancy. Limitations of Husserl's presentation and its formation would consequently deny the assertion that there are infinitely many numbers. Since we know that the assertion is true, there has to be a different way from Husserl's to account for large numbers, as a matter of fact, for all numbers altogether.
To sum it up, Frege did not have to be the (only) influence who once for all diverted Husserl from the psychological path, but he did offer him a very constructive criticism through this review. Although we have not seen Frege's arguments about the concept of number yet, this has given us some perspective on how easy it is to mix logical and psychological approach even with the strong commitment to the former.

1.4 Frege's logicism

Logicism is a theory claiming complete reducibility of arithmetic to logic – not only of arithmetic truths to logical truths, but also of arithmetic concepts to those of logic. Frege, as one of the representatives of logicism, intends to find out how to reduce the concept of number only to logical concepts. The following philosophical idea is that all we need is logical laws and logical truths and we can build the whole mathematics upon them. This is the point where Frege could not agree with Kant's classification of arithmetic sentences as synthetic a priori.
Firstly, let us introduce Kant's dual distinction between synthetic and analytic, a priori and a posteriori. Frege offers us this brief and clear explanation for each one of them: "If, in carrying out [a] process [of finding the proof of the proposition and of following it up right back to the primitive truths], we come only on general logical laws and on definitions, then the truth is an analytic one, bearing in mind that we must take account also of all propositions upon which the admissibility of any of the definitions depends. If, however, it is impossible to give the proof without making use of truths which are not of a general logical nature. . . , then the proposition is a synthetic one. For a truth to be a posteriori, it must be impossible to construct a proof of it without including an appeal to facts, i.e., to truths which cannot be proved and are not general, since they contain assertions about particular objects. But if, on the contrary, its proof can be derived exclusively from general laws, which themselves neither need nor admit of proof, then the truth is a priori."[F-Gr, § 3]
Frege is determined to show his classification of arithmetic sentences into the proposed system, in contrast with Kant, as analytic a priori. His plan is to build

it on the ground of purely logical definition of the concept of number. To explain the overall idea we need to focus on the justification for making a judgement and thus we have to watch carefully each step of inference in a deduction process. Because, as Frege puts it, "[w]hen a proposition is called a posteriori or analytic. . . , this is not a judgement about the conditions, psychological, physiological and physical, which have made it possible to form the content of the proposition in our consciousness; . . . rather, it is a judgement about the ultimate ground upon which rests the justification for holding it to be true."[F-Gr, § 3]

How exactly shall we imagine a method sufficient for the mentioned purposes? Frege suggests that ". . . the fundamental propositions of arithmetic should be proved, if in any way possible, with the utmost rigour; for only if every gap in the chain of deductions is eliminated with the greatest care can we say with certainty upon what primitive truths the proof depends; and only when these are known shall we be able to answer our original questions."[F-Gr, § 4] However, the cut in logic is not always as clear as in mathematics, because it is surrounded by philosophy from one side and psychology from the other, where both spheres have the ability to get involved more than is asked from them.

How does such a gap elimination work? When do we know that a certain step of deduction is already elementary and we do not have to continue disassemble it into even smaller steps? And then, how should we justify such an elementary step? The idea is to produce ". . . a chain of deductions with no link missing, such that no step in it is taken which does not conform to some one of a small number of principles of inference recognized as purely logical. . . . [However] the mathematician rests content if every transition to a fresh judgement is self-evidently correct, without enquiring into the nature of this self-evidence, whether it is logical or intuitive. A single such step is often really a whole compendium, equivalent to several simple inferences, and into it there can still creep along with these some element from intuition."[F-Gr, § 90] Self-evidence is well-known for carrying many difficulties and ambiguities, for there are no clear rules how to decide whether its nature is of a kind 'approved or forbidden'. Moreover, Frege admits that we often miss logical basis for a certain judgement and let our intuition to erroneously consider the proposition synthetic. Last but not least, ". . . the excessive variety of logical forms that has gone into the shaping of our language makes it difficult to isolate a set of modes of inference which is both sufficient to cope with all cases and easy to take in at a glance"[F-Gr, § 91]. It is a very good-looking manual, but it is too idealistic to be also a feasible one. Deductions could clearly become much longer. However, the main question we should be asking ourselves: Using only logical concepts and principles are we able to express (or deduce) everything that would normally be a subject of arithmetic? After all, it is the central task of logicism to show that it is possible.

Both psychologists and anti-psychologists follow the same goal – to get to the foundation of the truth. They carefully disassemble a deduction block by block, inference by inference. However, sooner or later, they all find themselves at the stage, where the former group thinks they ran out of the supporting pillars and therefore need to borrow some from psychology. They simply turn to psychology if a justification is needed. The latter group, on the other side, claims that once they truly understand the whole construction of the deduction, the way it was built, then they can decide which pillars are not necessarily lost. Using the latter's approach Frege 'sneaks' behind the need for justifications. As far as deduction steps are concerned, he suggests considering only conditions for facts instead of facts themselves. "By substituting in this way conditions for facts throughout the whole of a train of reasoning, we shall finally reduce it to a form in which a certain result is made dependant on a certain series of conditions. This truth would be established by thought alone. . . ."[F-Gr, § 17] It is something like a thought experiment, so to speak. Obviously, we do not need actual pillars to design the final construction, but we certainly do need thoughts, ability to reason and probably even some imagination. Does it not send us to the arms of psychology then? Not at all, because according to Frege thoughts and objects, we are dealing here with, are objective and the accompanying psychological processes (e.g., reasoning, judging, imaging, . . .) do not jeopardise this fact.

Even though he manages to stay safely on a side of anti-psychologism, Frege does not exactly prove the hypothesis of logicism. However, at the end of a quite deep investigation in his Foundations of Arithmetic he claims ". . . [he] made it probable that the laws of arithmetic are analytic judgements and consequently a priori. Arithmetic thus becomes simply a development of logic, and every proposition of arithmetic a law of logic, albeit a derivative one."[F-Gr, § 87]

1.5 Concept of number in Frege's Foundations of Arithmetic

First, we try to narrow down the area where to search for numbers. The attempt is to exclude what is not a number as there are logical reasons simply preventing certain cases. To actually define and construct numbers might not be as obvious as it seems. We focus on Frege's journey towards this goal. Throughout the process we meet both – the need for psychological support and the reasons for its rejection. We also deal with objectivity and potential subjectivity of numbers, the concept of unit, and the number 0 as an indication of non-existence. In the end, we offer the core of Frege's construction of the series of natural numbers.

1.5.1 Psychological, physical or neither?

What exactly is number? Is it a property of external things? What does the concept of number stand for? Is there somewhere an object we refer to when calling for number? Do we see it? Can we touch it? Or, on the contrary, does it exist only in our mind? Do we all share it or is it specific for each one of us independently?
There are many questions that arise on one's mind when looking behind everything learned, deeper than any definition of arithmetic can reach, in order to get to the concept of that what answers the question, as Frege puts it, How many? For the ways we are used to think about and work with numbers, it should be a serious turn to the almost totally blank starting point without any general propositions to lead us. We are uncertain where to begin, what needs to be accounted for, what kind of reasoning is allowed or maybe even required. There is also a certain need for the purpose of the upcoming challenge – what could we possibly gain from this investigation? Is the desired concept not too obvious, too clear, to need a definition? After all, we usually do understand the contexts about numbers without any problem. The majority knows how to count and what to count. Still, we should give it a thought, because even though many of us would at first feel quite confident when being asked to explain what number is, we have to admit it is definitely not an easy task. We are almost predestined to end up in a trouble zone of the highly expected circular references. What is more, as Frege claims, the most frequent reason "... to hold that Number is indefinable ... [is simply] because attempts to define it have failed ... [not] because anything has been discovered in the nature of the case to show that it must be so."[F-Gr, § 20]
It is only understandable that many philosophers would turn for answers somewhere else. At the dark bottom, where it seems there is nothing more left to hold on to, psychological explanation is always at hand to help, to offer that much needed support. It often comes with a price of inappropriate use of notions such as *intuition*, *phenomenon*, *idea*, *presentation*, *sense*, and so on. Although Frege is fundamentally against any psychological influence in the territory 'purely designed' for logic, he recognises the temptation. We are getting to the unknown

environment, where there are only certain principles we can rely on and at the same time we strictly cannot get in conflict with them. It seems very natural to look for something physical or at least psychological, for something 'from the reachable reality', to justify our logical inferences. However, as far as these alternative approaches to numbers are concerned Frege sums it up as follows: "In the spatial sense [numbers] are, in any case, neither inside nor outside either the subject or any object.... Whereas each individual can feel only his own pain or desire or hunger, and can experience only his own sensations of sound and colour, numbers can be objects in common to many individuals, and they are in fact precisely the same for all, not merely more or less similar mental states in different minds."[F-Gr, § 93] We simply understand numbers as they are, 'in themselves and for themselves', so to speak.

Although we already know that numbers are exactly the same for everybody, does it mean they have to exist independently from us, therefore in the outside world, as let us say colours? Frege explains that the reason why we expect there to be this resemblance with a property such as colour is because they both take the same place in language, thus they seem to also play the same role. However, he goes further and illustrates the stumbling block of this frequent misconception via an example: "Is it not in totally different senses that we speak of a tree as having 1000 leaves and again as having green leaves? The green colour we ascribe to each single leaf, but not the number 1000. If we call all the leaves of a tree taken together its foliage, then the foliage too is green, but it is not 1000. To what then does the property 1000 really belong? It almost looks as though it belongs neither to any single one of the leaves nor to the totality of them all; is it possible that it does not really belong to things in the external world at all?"[F-Gr, § 22] Now, is there any way how to avoid subjectivity or once the property 1000 is excluded from the physical realm it simply has to fall into the mental one? The obvious problem with the mental realm is that all of us could have their own and mutually incompatible ways of defining the property 1000 (according to different objects that we assign it to).

What is more, Frege defends the objectivity of number emphasising once more that "[n]o description of ... the mental processes which precede the forming of a judgement of number ... can ever take the place of a genuine definition of the concept. It can never be adduced in proof of any proposition of arithmetic; it acquaints us with none of the properties of numbers."[F-Gr, § 26] It might not be the strongest argument, but subjectivity of numbers is completely unacceptable for Frege. He evidently admits the mental processes which handle the judgements of number, but also points out that they do not affect the nature of numbers whatsoever.

1.5.2 Number as a property of concepts

We shall take a better look at the concept of unit, for it is literally the basis of numbering per se. There would be no point in exploring what numbers are, if we did not know what to count in the first place. Therefore, we begin by investigating the way we determine individual objects before we can count them. What is it exactly that denotes what is one and what many (respectively none)?

As far as number is concerned there are different ways to regard one external phenomenon – e.g., we can see one pair of boots also as two separate boots, one deck of cards also as 4 suits of cards or as 32 separate cards. For the numbering to take place we first need to determine the initial unit. The result is obviously not the same when we count field of flowers in terms of individual pieces of flowers as if we counted it in terms of bouquets. Frege rejects Mill's opinion that there should be only one possible and therefore "... the characteristic manner in which the agglomeration is made up of, and may be separated into, parts" [F-Gr, § 23]. Not to mention, Frege stands strongly against using concepts such as agglomeration, multitude and plurality for their vagueness, difficulty to work with the numbers 0 and 1, and occasionally also for tempting but unfounded presupposition of their equivalence to numbers. Let us assume we want to count the number of units of a given indiscriminate set. We clearly need to put the separate things of the set on an equal footing. On this matter Frege chooses an idea of a mathematician Carl Johannes Thomae to illustrate the misconception of what we end up with if thinking that abstraction itself will serve the cause. Thomae claims we should "... disregard, in considering separate things, those characteristics which serve to distinguish them" [F-Gr, § 34], literally strip them off their differences and thus put them on the same level of pure units. Although it sounds very natural and we would easily expect a process like that to take place within us as a part of counting, Frege discovered an essential error there: "In that event we are not left ... with *the concept of the Number of the things considered*; what we get is rather a general concept under which the things in question fall." [F-Gr, § 34]

This actually comes hand-in-hand with Frege's crucial conclusion that number is a property of concepts. Therefore Thomae was not completely wrong, he just stopped one step before the finish line. Abstraction usually precedes the counting process and consequently results in a certain concept. However, we have to be careful what we assign numbers to. With the help of the number 0 Frege shows the difference as follows: "If I say 'Venus has 0 moons', there simply does not exist any moon or agglomeration of moons for anything to be asserted of; but what happens is that a property is assigned to the concept 'moon of Venus', namely that of including nothing under it."[F-Gr, § 46] Now, we can see why it is important to first define a concept by its characteristics and then determine all the objects that fall under it, rather than the other way around. Even though the need for abstraction persists, the latter method could not account for the number 0. Since there are no objects to start with, it would be impossible to construct any proposition denying existence of something (e.g., the proposition 'Venus has 0 moons'). However, the problem would automatically extend also to the inverse propositions, the ones conforming existence (i.e., with a concept of at least one object) as "[a]ffirmation of existence is in fact nothing but denial of the number nought"[F-Gr, § 53]. Thus, it clearly follows that also existence is a property of concepts.

It seems concepts offer a perfect area where to seek the judgements of number. As Frege points out, it is necessary to distinguish between the characteristics which make up a concept and are therefore asserted of the things falling under the concept and the ones which are asserted of the concept itself. The latter properties represent the classification of numbers proposed by Frege. However, this shift to the higher level (i.e., from the properties of things to the properties of concepts) also means that having only first-order logic at hand Frege would not manage.

We return to the term *unit* to finally give it a satisfactory definition. It could seem quite hopeless as choosing any approach led to the same problem – the problem already Husserl had to cope with: In an attempt to define the term, it seemed as if one was forced to abstract the particular objects or contents until all the units were identical but still remained distinguishable from each other. W. S. Jevons emphasises this point as follows: "It has often been said that units are units in respect of being perfectly similar to each other; but ... they must be different in at least one point, otherwise they would be incapable of plurality."[F-Gr, § 35] However, with a new background theory Frege formulates:

> "Only a concept which isolates what falls under it in a definite manner, and which does not permit any arbitrary division of it into parts, can be a unit relative to a finite Number."[F-Gr, § 54]

Two apparent enemies become a married couple when we realise that each of the two inconsistent requirements refer to different use of the word *unit*. On the one hand, when units are in the context such as Frege's formulation mentioned above, they are all naturally identical. On the other hand, speaking of their distinguishability, we actually refer to the distinguishability of the objects they stand for. Therefore, we can count the identical units and not worry that they would all merge into one, because each one stands for a different object.

1.5.3 Construction of the series of natural numbers

We have finished the picture of number classification and counting unit. It is time to finally take a look at the construction of numbers itself. We shall not go in great detail here and offer only a brief overview. Let us start with Frege's motivation for creating such a system. Despite discovering a gap in Leibniz's proof, Frege saw a real potential in his idea of a construction of all numbers reduced only to two basic steps – the number 1 and increase by one: We start with the number 1 and at a certain point by gradually increasing the current number by 1 we can get to any number from the infinite set of natural numbers.
At first, Frege gives us the adjective form of the definitions[5] of the numbers 0 and 1 (unlike Leibniz, Frege considers the number 0 as the first natural number) and also that of a successor (i.e., increase by one). He wants to point out that this might not be the way to successfully define numbers as he himself diverts from this path eventually. He proposes the following formulations:

(1) "[T]he number 0 belongs to a concept, if the proposition that a does not fall under that concept is true universally, whatever a may be." [F-Gr, § 55] (i.e., 0 belongs to 'an empty concept')

(2) "[T]he number 1 belongs to a concept F, if the proposition that a does not fall under F is not true universally, whatever a may be, and if from the propositions 'a falls under F' and 'b falls under F' it follows universally that a and b are the same." [F-Gr, § 55] (i.e., it is neither the number 0 nor anything greater than the number 1)

(3) "[T]he number $(n + 1)$ belongs to a concept F, if there is an object a falling under F and such that the number n belongs to the concept 'falling under F, but not a'." [F-Gr, § 55]

However, the question now is: What exactly does the expression 'the number n belongs to a concept F' mean? Frege further explains his doubts about this

[5] To learn more about Frege's adjective strategy see [Kol, pp. 192–193].

strategy: "[W]e cannot by the aid of our suggested definitions prove that, if the number a belongs to the concept F and the number b belongs to the same concept, then necessarily $a = b$. Thus we should be unable to justify the expression '*the* number which belongs to the concept F', and therefore should find it impossible in general to prove a numerical identity, since we should be quite unable to achieve a determinate number." [F-Gr, § 56]

Frege leaves the adjective strategy[6] and turns to "… the substantive strategy, i.e. he takes the number word for a proper name" [Kol, p. 193]. In order not to get ahead of ourselves we should first pay some attention to Fregean representation of numbers as "… self-subsistent objects that can be recognized as the same again" [F-Gr, § 56]. It directly leads him to the issue of the identity of numbers so each one of them can be assigned a number word as its proper name. There has to be something what makes three flowers and three leaves represent the same number. Could it be some kind of Fregean (i.e., subjective) idea of the number 3 that they have in common? Not really. A simple reflection confirms that it is impossible to form an idea of number as a self-subsistent object for it is not anything sensible. It is quite obvious if we try to imagine the number 0 as the number of apples. Any luck?… There are evidently some obstacles on our way to find a general criterion for the identity of numbers. Eventually, Frege decides to base his further formulations on Leibniz's definition of identity which says:

> "Things are the same as each other, of which one can be substituted for the other without loss of truth." [F-Gr, § 65]

Coming back to the adjective definitions from above and at the same time introducing extensions of concepts, Frege defines the problematic expression as follows:

> "[T]he Number which belongs to the concept F is the extension of the concept[7] 'equal to the concept F'." [F-Gr, § 68]

[6] As V. Kolman claims "Frege's arguments against the adjective strategy do not seem very convincing…" [Kol, p. 193]. After Russell's paradox severely stroke his system, Frege returned back to this crossroads (where he once chose the substantive strategy over the adjective one). However, even then he did not think it was possible to build arithmetic upon the adjective strategy.

[7] Frege notes that the expression 'extension of a concept' is simply assumed to be known here. Nevertheless, for those who are interested: Extension of a concept is a set of all the objects that the concept applies to.

Preventing the term *equality* from being founded on plain intuition Frege defines numerical equality (i.e., equinumerosity) as follows:

> "[T]he expression 'the concept F is equal to the concept G' is to mean the same as the expression 'there exists a relation ϕ which correlates one to one[8] the objects falling under the concept F with the objects falling under the concept G'."[F-Gr, § 72]

We should now have everything necessary to move on to Frege's substantive form of the definitions[9] of the natural numbers starting with the number 0. Firstly, he defines nought as follows:

> "0 is the Number which belongs to the concept 'not identical with itself'."[F-Gr, § 74]

He is aware that such a concept might seem quite nonsensical, but reminds us that the only thing logic requires is to be able to decide about every object whether it belongs to that concept or not. And we already know that there is no such object which would fall under the concept just mentioned, because everything is identical with itself. Frege chooses exclusively the concept of identity in order to make use of Leibniz's definition given above.

Secondly, Frege explains what it means to increase by one, namely, he defines the relation of successor "... in which every two adjacent members of the series of natural numbers stand to each other"[F-Gr, § 76]. He suggests the following definition:

> The natural number n succeeds directly after the natural number m if and only if "there exists a concept F, and an object falling under it x, such that the Number which belongs to the concept F is n and the Number which belongs to the concept 'falling under F but not identical with x' is m"[F-Gr, § 76].[10]

Last but not least, in order to complete the trio of Frege's adjective definitions, we should make sure that the number 1 fits in this construction as we would

[8] A relation between two sets is one-to-one correspondent, if each element from one set is paired (i.e., in the relation) with exactly one element from the other set and vice versa. No element from either set is left unpaired. (We discuss one-to-one correspondence in great detail in section 3.3.)

[9] In contrast to the previous adjective strategy, where the numbers were defined in a form of conditions, here they are directly assigned to certain object/-s (i.e., concept/-s), so to speak.

[10] Considering today's form of the inductive step from n to $(n + 1)$, this is obviously an essential definition for both – mathematicians and logicians.

expect it to. Using the two substantive definitions already introduced, naturally, we would like the number 1 to be the number succeeding directly after the number 0. Frege defines the number 1 as the number belonging to the concept 'identical with 0', because we know that exactly one object, namely the number 0, falls under it. We also know that there is no object falling under the concept 'identical with 0 but not identical with 0', so 0 is the number which belongs to this concept.
Now, let us look back at the definition of successor and place the concept 'identical with 0' as the concept F, the number 0 as the object x falling under it, and the number 1 as the Number n. 'Falling under F' then means being 'identical with 0' and as a result we can fill in the rest of the definition so that the Number which belongs to the concept 'identical with 0 but not identical with 0' is 0. From the left side of the definition we get that the natural number 1 succeeds directly after the natural number 0 and therefore "... the Number which belongs to the concept 'identical with 0' follows in the series of natural numbers directly after 0"[F-Gr, § 77]. So the number 1 fits the construction perfectly.
In addition, Frege also shows that there is no last member of the series of natural numbers, the series is therefore infinite. It directly follows from the proposition: "[T]he Number which belongs to the concept 'member of the series of natural numbers ending with n' follows in the series of natural numbers directly after n...."[F-Gr, § 83][11] Frege consequently introduces infinite numbers. First, he defines a finite number as "... a member of the series of natural numbers beginning with 0"[F-Gr, § 83]. And an infinite number is then defined as "[t]he Number which belongs to the concept 'finite Number'..."[F-Gr, § 84].

We have reached the end of Frege's enquiry into the concept of number. Throughout this last section we have given a brief insight into his idea of building arithmetic from scratch purely on logical ground. We have classified number as a property of concepts thus placed it outside of the physical and mental world, solved the mystery of contradictory qualities ascribed to units, and proposed three basic definitions in order to draw the construction of the series of natural numbers.

[11] We skip the proof of this proposition.

Chapter 2

Psychological perspective on the concept of number

2.1 Introduction

In this chapter we want to talk about the concept of number from a completely different point of view as we have until now. It is the view we all once shared – the view through eyes of a child. We would like to uncover some of the proposed theories about the quantitative development in infancy and its possible reflection on the conventional system learned a few years later. There are no clear proofs in the area of this issue. The question of the origins of cognition has to stay, it seems, yet unanswered. However, it does not have to stop us from looking for the methods of experiments that would give us more precise outcomes, for hypotheses that would explain existing results, and for the best-fit theories. It definitely has the scientific value until, if at all possible, the direct ways to the innate world of infants' reasoning are actually available.
There is a never-ending battle between the idea that children are born already with certain knowledge wired in and the idea of them being born completely blank as tabula rasa. The former assumes that infants are given the basic knowledge and their main task is to gradually discover its use in the world and build another knowledge on its ground. On the other hand, the latter denies any inborn knowledge, infants have to learn everything necessary 'from scratch'. As far as we are concerned we incline to one of the mild versions of the former idea which claims that children are born with certain mechanisms or modules to correctly process the incoming information rather than with pure knowledge. We hold the same opinion in the matter of infants' concept of number and their initial counting abilities. Let us now introduce the possible birth of number within us – humans – from the perspective of cognitive science.

2.2 Nature over nurture

In the previous chapter we have been analysing the concept of number from Frege's strictly logical point of view in an effort to show its anti-psychologistic aspects. However, one has to admit that certain room for second thoughts has been left there. Hence, let us move to the cognitive point of view which offers us a wide interdisciplinary perspective including partly that of psychology. The desired goal remains the same as for Frege – to get to the concept of number, to its nature, to its meaning. We shall not search for the answers within the given mathematical conventions or arithmetic itself as it is taught. We need to go deeper, to the absolute beginning. We have already peeked behind the definition of number, now we shall take a look behind the time when we were first told what number is. Although Frege probably did not have any intentions to go beyond the analysis of the concept of number we will set off this path and shed light upon the act of grasping the number itself. In order to get to the yet untouched nature of the relation between numbers and us, we have to avoid everything learned and literally turn for the answers to the infants.

A well-known figure, who initiated this kind of studies with children, was a developmental psychologist and philosopher – Jean Piaget. Without any doubts his work has been getting a lot of attention (also as a contribution to epistemology), but the majority consists of criticism for underestimating children's abilities. Let us briefly sketch Piaget's concept of number. He claims that a child constructs discrete numbers on the ground of ordering (i.e., organizing elements according to their increasing or decreasing size) and class inclusions (i.e., categorisation, grouping of same elements together). Thus, he introduces the psychological concept of number as the synthesis of ordering and classification.[1] His experiment described below proves how close the connection between numbers and spatial ordering might be in preschoolers. It means that unless children grasp the preservation of number sets independently of their spatial ordering they cannot be said to use the 'real' operational numbers.

In Piaget's famous non-conventional task children are first shown two identical rows of items of the same kind (e.g., sweets or coins) and asked whether the rows have the same number of items or one of them has more. The children have no problem answering correctly. However, then the items of one of the rows are spread out so the row becomes longer. The children are asked the same question as before. Only this time they erroneously answer that there are more items in the modified (i.e., the longer) row. In contrast with Piaget's conclusion that children lack basic quantitative skills, Gelman and Gallistel observed that children of this age, namely preschoolers before mastering in conventional number skills, already know the principles how to count.[2] Thus, there is a quite real possi-

[1] See [Pia, pp. 92–95].
[2] See [Mix, p. 4].

bility that the children in Piaget's task are simply reacting to the change itself. They want to show the experimenters that they have noticed the transformation. Unfortunately, this effort might result in the wrong answer. They also might be overwhelmed and confused when being asked the same question twice.

Through this example we can see the difficulties and pitfalls of the research methods. Not only there are many factors that may affect a child's fragile opinion, but sometimes also researchers' desire for a certain outcome may cross the boundaries of objectivity. At least these children were already able to speak, whereas working with infants is a completely different class of challenge. The obvious question striking here is: How can we get to know what infants think? One of the things that infants are very good at is getting bored. That is the reason why all the related research is based on their attention rate. The experiments using a sequence of pictures of the same kind (i.e., with the same number of items) showed that at the beginning infant's attention increases then it stabilises for a while until it slowly starts to decrease. This period is called habituation. When then the infant sees something unexpected, she gets surprised and her attention increases again, she so called dishabituates (e.g., when the number of items on a displayed picture suddenly changes).[3] Therefore, researchers use the looking (or reaction) time of infants to draw the conclusions in the experiments. The longer looking time is obviously caused by infant's detecting the novelty of the test stimulus, she might be said to disagree with the outcome (e.g., in case of 1+1=1). Thanks to this method certain sensitivity to numerosity has been uncovered even in a-few-hours-old newborns.[4]

It has been proven that people do have some kind of "... a sense of quantity – what Dehaene, following Tobias Dantzig, refers to as *the number sense*" [Lak, p. 29] from a very early age (or maybe even from birth), far before they are taught anything about numbers. "Infants react to changes in set size whether the sets are made up of heterogeneous items..., items that vary in size and type..., items that are in motion..., or temporal sequences of events...." [Mix, p. 20] The assumption that the concept of number is wired in might not be so far-fetched after all. However, there are still many intriguing questions that might be worth giving a thought. We will discuss the following ones: Is infants' quantitative reasoning based on discrete number, continuous amount, or both? Can infants count without conventional skills? If so, how do later-coming mathematical conventions map onto pre-existing quantitative representations?

[3] See [Pyl, pp. 29–30].

[4] See [Mix, p. 10].

In order to judge the extent of infants' competence researchers have focused on these three quantitative skills:[5]

(1) discriminating between set sizes,

(2) understanding the results of quantitative transformations,

(3) recognising equivalence and ordinality relations.

In the case of equivalence the idea is that children are supposed to recognise that two sets are in the same numerical class even if they differ in every other way (what in turn calls for the ability to categorise). It means that children might be asked to choose or construct an equivalent set to demonstrate this ability. However, it is not surprising that infants as young as one year were able to recognise equivalence only when the sets were of the same modality (i.e., either both visual or both audio) and very similar to each other. As far as ordinality is concerned children's task is to show understanding of the particular order in which numbers are organised using the comparisons 'less than' and 'greater than'. The one-year-olds evaluated the sets only in terms of 'equal' and 'unequal' (i.e., 'different than'), which might also indicate that ordinality judgements rise from those of equivalence. The ability to fully judge these relations naturally emerges between 2nd and 3rd year of age and gradually extends to increasingly complex comparisons.[6]

2.3 Discrete number vs. continuous amount

Initially, the research has been based on an attempt to find out whether infants can quantify small sets using discrete number. It does not come as a shock that numbers were privileged given that at first glance they seem more natural and considerably simpler than amounts. However, the major motivation was hidden behind the effort to find "*true* number concepts" [Mix, p. 21] in humans. Only later the idea of infants using actually continuous amounts instead has attracted researchers' attention. One can easily see how it would remain unnoticed, because usually "… when the number of items changes, the amount of stuff changes, too" [Mix, p. 11]. "Conversely, two sets may be the same in total amount, but differ in number. …" [Mix, p. 50] This is the discrepancy to focus on.
Let us take a look at the results of the following experiments. Feigenson and

[5] See [Mix, p. 8].
[6] On infants' equivalence and ordinality recognition skills see [Mix, chs. 3–4].

Spelke modified Wynn's experiment in order to control overall amount. Originally, in Wynn's famous task[7] 5-month-old infants were first shown one doll on a stage, then a screen was raised so they could not see the whole scene, but they could see as a hand put another doll behind the screen. In the end, the screen dropped and on the stage there was either one doll or two dolls. In this particular $1+1$ version, infants did look longer when there was only one doll instead of the (obviously) expected outcome of two dolls. Together with $2-1$ and $2+1$ versions Wynn concluded that infants are able to compute the numerical results of these arithmetical operations. However, Feigenson and Spelke's version of the task[8] was designed to decide whether children did not use continuous amount instead of arithmetic as such. For this purpose researchers used two different sizes of dolls, where the large one consisted of the same overall amount as two small ones together. They put two small dolls on a stage the same way as Wynn did before, but this time when the screen dropped there was either one large doll representing incorrect number but correct amount, or two large dolls representing correct number but incorrect amount. Surprisingly, infants were looking longer at the latter outcome. It was a clear sign that they did actually respond to continuous amount rather than to discrete number. Another evidence to back up this assumption is that with sets getting more complex it is becoming less obvious and therefore more difficult to estimate the overall amounts – this could help explain infants' limited counting abilities to very small sets.

There are conceptual and functional differences between continuous amount and discrete number. It all becomes more visible when we consider the unit measures. As far as discrete sets are concerned subdivision process is fixed and given 'for free' by the definition of discrete items as bounded and separate. On the other hand, there is no basic unit for continuous amount as all substances are infinitely divisible, meaning we can always take a smaller unit to be more precise. Thus, discrete sets can be counted "… exactly by mapping the items in a set onto an ordered set of tags, such as the count words"[Mix, p. 58], via one-to-one correspondence.[9] Whereas continuous amount will always be measured only on a certain level of approximation and the measuring unit needs to be chosen in advance. It might be a significant problem for children as they "… lack a fundamental notion of unit until well into elementary school"[Mix, p. 113].

Is there any chance that both types of quantity could be based on a single principle? Do they emerge simultaneously or one of them precedes the other?

[7] See [Mix, p. 18].
[8] See [Mix, p. 19].
[9] On one-to-one correspondence see section 3.3.

All kinds of research have been suggesting that children first begin by using continuous amount and a few years later, as preschoolers, they are already able to count on a ground of discrete number. Something interesting must be happening during that interval between the two achievements. However, this journey might not be without difficulties. As Gelman showed when preschoolers were asked to place a single amount (e.g., 1½ circles) on a beforehand numbered scale, they were confused and usually did not succeed.[10] It is not very obvious why as the majority of the same age category was able to order three similar items according to their weight or length. These children might be able to use both kinds of quantity but evidently not together in the same task. At the end of the day, the beauty of accuracy and the ease of measuring (i.e., counting) both seem like sufficient reasons to prefer discrete quantification. However, another Piaget's experiment shows what can happen when a child leaves continuous amount behind and rely only on discrete number. Many preschoolers know quite well how to fairly divide a decent discrete set by *one for me, one for you* strategy, but they tend to overlook amounts. In Piaget's task children were first given one cookie and an experimenter took two cookies for herself. Of course, the children immediately objected that the distribution was unfair. However, they were so keen on the number of items that they were more than satisfied when the experimenter 'solved the problem' simply by breaking their cookie into two pieces. As long as they had the same number of items as the experimenter, regardless of the uneven quantity per person, they happily considered the distribution fair.[11]
It seems quite unbelievable that children once so close to continuous amount could become unable to detect such gross violation of it only a couple of years later. Is it all at the expense of the shift to discrete number? Again, we are left with more questions than answers, but there could not be the right answer without first asking the right question, could it? To sum it up, the latest research, contrary to the former one, indicates that infants are born with sensitivity to overall amount of substance.[12] Exposed to our number-based world and discouraged by difficulty of measuring the continuous amount, preschoolers gradually transfer to quantifying over discrete number. Eventually, they are ready to start building conventional skills on these foundations.

[10] See [Mix, p. 53].
[11] For other examples see [Mix, pp. 59–60].
[12] See [Mix, p. 62].

2.4 'Unconventional' counting process

How can children determine the precise cardinality of a set without conventional skills? To be sensitive to numbers or amounts is one thing, but to be actually able to count before being taught the necessary arithmetical procedures arises many questions along with slight suspicion. The assumption that these calculation skills are wired in from birth seems too ambitious. Hence, the competence to count might be quite a hard nut to crack.

What exactly do we imagine under the notion of *counting process*? Gelman and Gallistel identified five principles that an arbitrary counting system has to follow to be useful.[13] The first three principles belong to the 'how to count' category:

(1) the one-to-one principle – one count word is assigned to one item and as a result every item is paired with exactly one count word;

(2) the stable order principle – the same order of count words of a counting system is used each time;

(3) the cardinality principle – the count word assigned to the last item represents the cardinality of the whole set that is being counted.

The rest two principles belong to the 'what to count' category:

(4) the abstraction principle – any combination of discrete entities can be counted, because "... as long as an entity can be individuated, it can be counted" [Mix, p. 102];

(5) the irrelevance principle – an order in which we assign count words to items does not matter, as long as there is no item left without a count word assigned to it.

Children usually use conventional count words almost immediately after they learn to speak. They need one more year to comprehend the system and its arithmetical use. They tend to order the count word sequence incorrectly (or they use a completely different idiosyncratic sequence), but they do so with the full respect to the stable order principle. Even though they consequently determine the wrong count word for the role, they also express understanding of the cardinality principle.[14]

[13] See [Mix, p. 101].
[14] See [Mix, p. 103].

Let us now summarise what we know about the period when counting abilities happen to emerge in a child:[15]

(1) the level of accuracy and precision that can be demonstrated empirically improves;

(2) the set size that can be handled increases;

(3) the range of abstraction increases.

The following task is to find the best-fit explanation for all these and other peripheral conclusions. Out of four proposed models of non-verbal quantitative representation (i.e., subitizing,[16] the accumulator,[17] object representations,[18] and mental models) we further describe only the mental models for it is the only group of explanations which are not based on discrete number.
Huttenlocher et al. did an experiment[19] where first they showed children a set of blocks, hid it and transformed it (i.e., added or removed a block) so that children would see the transformation but not the resulting set. Then they asked them to lay out a set of blocks which would correspond to the modified hidden set. Astonishingly, children as young as three years could already produce the correct solution. Huttenlocher et al. think that not having conventional skills to help them, these children must have created something like a mental version of the hidden set.
This idea has been further developed to suit the majority of the known results.

[15] See [Mix, p. 40].

[16] It is "... a rapid, accurate process of enumeration that applies exclusively to small sets" [Mix, p. 82]. It has been observed in adults that they are able to enumerate sets of four or less items considerably faster than the greater sets.

[17] "This mechanism works by using an endogenous pacemaker to emit pulses at a constant rate.... To begin timing or counting, a switch is closed that gates pulses into a container. If the accumulator is counting, the switch gates in pulses one at a time. If the accumulator is timing, the switch stays open until timing has stopped. The resulting fullness of the container represents the total quantity or duration, depending on which mode has been operated." [Mix, p. 87]

[18] "These explanations involve assigning an abstract token to each individual in a set. Such accounts are based on hypothesized processes of spatial individuation in adults that have also been used to explain item subitizing.... When people perceive a visual scene, they encode both what objects are there and where the objects are located. The *where* system is faster and operates preattentively. This system encodes spatial location but does not add details about object features or identity. These details come later as the attentional mechanisms of the *what* system take over. It is the object individuation process of the where system that has been coopted by number researchers." [Mix, p. 93]

[19] See [Mix, p. 96].

"The mental model explicitly represents features of the situation critical for calculation, such as the number of countable entities in the initial set and the nature of the transformation, but does not preserve task-irrelevant features, such as the colour of the individual items or their spatial arrangement." [Mix, p. 96] It is presupposed that children start using the mental models at the same age as other symbolic abilities develop (i.e., language, model use, and pretend play). Similarly to subitizing and object representations, also here different abstract tokens are assigned to different items in a set that is being processed. Hence, the total number of items is represented by the group of tokens.

The object representations are thought to first offer bare individuation of items (i.e., via their locations) and only later other of their featural information might be added. In contrast to them the mental models work exactly the other way around – first they include any available featural information and only after certain level of abstraction they are left with bare symbols. Given the extent findings which have shown that children are better at recognising numerical equivalence when the two sets are very similar, the mental models fit evidently better. For object representations at the initial stage (i.e., individuation) there is no featural information to determine a level of similarity, and the difference in the children's abilities would therefore not occur. Furthermore, the mental models also account for the contemporary understanding of the crucial differences between discrete number and continuous amount as "[b]y applying [them] to different situations, children would begin to divide the world into quantities that can be represented exactly and those that cannot" [Mix, p. 127]. Last but not least, while learning how to work with a mental model and still using overall amount alongside (around the 2^{nd} year of age), it might be expected exactly what the current data shows, namely the inaccurate but highly approximate responses (in the non-verbal calculation task).[20] Therefore, the use of mental models might play a role of "... an intermediate step between infants' initial quantitative awareness and the subsequent acquisition of conventional skills" [Mix, p. 97].

2.4.1 Abstraction and categorisation

Infants' concept of number seems to be quite abstract. Before they can count, they have to determine what they are going to count, therefore, they first have to categorise the initial set. However, that might not be the case for the majority of experiments with infants as their visible field has been restricted for them

[20] See [Mix, p. 97].

(it has been 'categorised for free', so to speak). As infants' numerical competencies grow, the role of categorisation is becoming even more evident in their ability to map on a basis of "... more abstract, specific, dimensions... "[Mix, p. 132]. After all, categorisation is closely tied to all the concepts.
Both, Russell and Piaget, were quite attracted by the intriguing relation between number and class. Piaget's class-inclusion research showed that young children have difficulties with category hierarchies in part-whole relations. Until the age of seven or eight they were unable to decide whether there were more tulips or flowers in the presented set.[21] On a similar vein, Russell claims that "... each number itself can be defined as a class or category that contains sets of the particular number of items... "[Mix, p. 132]. It might take children some time to understand that even though numbers usually apply to that which has already been categorised (i.e., items are first classified into groups according to a certain property or properties such as colour or shape and only then counted), they also can be constructed as categories (e.g., all the sets of two items, alternatively characterised as the sets with the property of 'twoness'). It is not that straightforward to see that three cookies on a plate and three flowers in a vase have something in common. For their ambiguous reference to both – groups and properties of groups – number words do not necessarily clear the air right away. Even in the former case it is quite misty as "... they refer to a property that emerges from a grouping rather than pertaining inherently to an individual. For example, a red apple is red whether or not it is grouped together with other apples. However, three apples are only three apples when they are grouped somehow (i.e., spatially, temporally, or both)."[Mix, p. 137] We have already come across this puzzle in Chapter 1 as Frege illustrated the difference between properties of objects (e.g., colour) and properties of concepts (e.g., number).[22]
We have mentioned above that thanks to equivalence judgement it is obvious how children's abstraction skills gradually improve. Starting with the ability to recognise equivalence only for two identical sets (i.e., for cases of most literal comparisons), through comparisons of highly similar sets, children eventually become able to match heterogeneous sets with only number in common.[23] Not to mention that throughout this period of improvement children also gain general experience with classes of entities in the world. Their symbolic thought develops as they engage in different symbolic activities such as pretend play or use of the physical models. The whole progress in the skills of abstraction and categorisa-

[21] See [Mix, p. 76–77].
[22] See subsection 1.5.1.
[23] To learn more about this developmental pattern see [Mix, ch. 4].

tion takes place before children even start a school. One has to admit that they seem to be more than competent.

2.5 Conventional skills and measuring

It is the first time we have something 'tangible' in our hands to work with. Conventional skills do not emerge within us, they are simply taught. We gradually gain mathematical conventions and do our best to harmonise them with the pre-existing non-conventional skills which are assumed to emerge on their own. As the current research has shown infants are equipped with quantitative representations. They have abilities and tendencies to count, so to speak. A few years later, as preschoolers, they can arrive at consistently exact solutions. The majority of investigators seems to think that children understand the counting principles (most of the 'how to count' category) before they learn to count. We cannot expect all the principles to be grasped at once. After all, some have proposed that it might be impossible to comprehend the cardinality principle without first being able to count conventionally.[24] On the other hand, this way it would be inconsistent with the findings that 3-year-olds already understand cardinality.

Nevertheless, it brings us to the pressing question "... whether pre-verbal quantification helps or hinders the acquisition of conventional skills"[Mix, p. 104]. Researchers have noticed certain lag between non-verbal and verbal calculation skills. Since it might give us some answers it has become the main task to find out what is going on during this period. It is understandable that such an essential step 'from the inner to the outer world' does not happen overnight. One of the ideas proposes that children use the time "... to interpret conventional symbols for numbers without concrete support"[Mix, p. 112] (such as possibility to touch or see the objects that are to be counted).

Mathematical conventional skills include also measuring procedures. Preschoolers already know that there are things which can be counted and those which cannot. However, the latter still can be measured. Learning how to measure substances brings "... a better understanding of the relations among countables, masses, and units. More specifically, counting and measurement together act as a gateway that allows a bi-directional translation of continuous and discrete quantities."[Mix, p. 130] Young children have shown their ability to work with both types of quantities on a certain level, but it seemed as if they either used one

[24] See [Mix, p. 106].

or the other (e.g., inability to correctly place a single amount, such as $1^1/_2$ circles, on a beforehand numbered scale). Using both at the same time appeared too overwhelming and often resulted in errors caused by respecting principles of one but not the other type of quantity. To figure out that something like tomatoes might be not only counted in number but also measured in weight is a huge and challenging step. What is the most important, however, is that it sheds fruitful light upon the issue of a unit.

In the case of discrete quantity it is the object individuation that determines the unit for us. On a basis of location or movement, although not so much of colour or shape, infants automatically individuate items in a given set and thus the only thing left there is to count them. Substances are far less generous in this regard. We can see how essential it is to realise the role of individuation in counting act itself. How do we learn to individuate things in the first place? If an object does not move, how do we distinguish it from the rest of the scene? Do we sense its 'contours'? Pylyshyn has proposed an idea of FINST mechanism (FINgers of INSTantiation) as a part of the theory of visual indexing.[25] It works very similarly to mouse cursor on a screen – it assigns an index to an object in the visual field and follows the object as it moves until it disappears for a long time. As useful as FINST might be, it does not solve the mystery of individuation per se. Are we simply born with a sense of 'objecthood'? If so, would it not be too distracting for infants quantifying over continuous amount? Do they simply ignore the 'contours' when counting is being handled? For now we leave the questions hanging in the air as we shall return to the subject in the final chapter of this paper.[26]

By the time of their first day of school children already know how to count, they only miss mathematical conventions for it. Once children grasp the role of conventional symbols, their flexibility and universality, they leave non-verbal representations behind and head towards greater precision. "The mastery of number facts, along with notions of decomposition, therefore seems to involve a final move away from concrete referents toward an internalized representation of the counting sequence."[Mix, p. 111] Children are thus freed from the concrete visible and tangible world entering the one of thoughts only.

As far as mapping of the conventions onto pre-existing quantification is concerned, there is still a blind spot there. Obviously, children know too much not to use it as a reliable ground for further knowledge, but it is not clear how

[25] See [Pyl, pp. 41–44].

[26] See section 3.4.

smoothly the whole process goes. Many educational researchers engaged with this issue emphasise how much children might profit if they are encouraged to solve mathematical problems on a basis of their current knowledge instead of immediately being told a method how to do it.[27] Looking for their own path considerably strengthens the connection between the separate pieces of their knowledge. Hence, less information and more thinking might actually result in more overall knowledge.

[27] See, e.g., [Mar].

Chapter 3

Comparison of logical and psychological perspectives

3.1 Introduction

In Chapter 1 we have talked about Frege's logical perspective on the concept of number. The main intruder in his investigation was clearly psychologism. If there was no obvious justification for a certain inference or so called primitive truth, many philosophers would turn to psychology and basically place these on the ground of our mental environment rather than claim their self-verification or absolute independence of us. On the other hand, in Chapter 2 we have discussed cognitive perspective on the concept of number. It has given us some insight into behavioural patterns in a matter of infants' reaction to numbers as well as amounts. According to these patterns certain models of non-verbal quantitative representation were suggested. The reason why we have reached after this field was to take a better look at so called "the number sense"[Lak, p. 29] which we seem to be born with. It is not purely psychological approach and by no means does it have tendency to subordinate logic the way that psychologistic view demanded. To clear the air it is crucial to call attention to the fact that the logical and psychological perspectives have not been proposed in this paper as each other's opposites but rather as two different ways to crack the nutshell of the concept of number.

In this last chapter we first shed light upon mutual comparison of the two perspectives – we shall point out their possible similarities as well as their differences. After that we discuss the logic behind humans' reasoning and difference between competence and performance as our opinions and beliefs might never be totally out of the picture. To help us with this task we challenge children's skill of one-to-one correspondence. On the top of it, we briefly look at the individuation process, the process of recognising separate objects in space, in order to better understand how much is in fact happening at the backstage of the birth of a unit. Last but not least, we offer a very interesting perspective of modern cognitive science claiming that "... mathematics as we know it and teach it can only be humanly created and humanly conceptualized mathematics"[Lak, p. 2]. These last four topics shall outline the natural yet complex relation that logical and psychological aspects may appear in.

3.2 Overall comparison of the two perspectives

It is obvious that both perspectives – logical and psychological – were at a certain point deeply engaged in a debate over the concept of number. Even though their goals were the same, their journeys were fundamentally different. The logical perspective approaches the issue on the objective level of (Fregean) thought and its investigation is thus a priori. The only available experiments are the thought experiments. The majority of the 'material' for logic to work with could reside in Fregean "third realm"[F-Ge, p. 302] of the objective and non-sensible, in the realm placed 'between' the mental and the physical realms. By virtue of relativism logic simply cannot rely on the ambiguous mental world. "[P]ropositional logic, the syllogism, probability theory, and the like, describe some independent-of-people purpose, such as the preservation of truth when inferences are made (where inference itself is also to be understood as independent of people)"[Pel, p. 3]. However, there are opponents to psychologism who consider logic and mathematics in the physical world. Therefore, even though Fregean third realm seems to be the best fit for everything non-sensible and without a need for a bearer (i.e., objective), there are still those who believe we manage without it.

As the history draws the picture, logic is not exclusively fond of psychology trying to take over. It is crucial to understand that the use of the psychological terms, such as *mental act*, does not immediately yield psychologism. "Even Frege, the arch-anti-psychologist, could hold that our primary evidence for the truth of logic and mathematics was our psychological 'grasping' of statements about these fields. So, the *provenance of the evidence* could just be irrelevant to the issue of psychologism. What matters is the *source of the content* of the logical terms."[Pel, p. 6 – footnote 5]

The psychological perspective, on the other hand, goes to the outside world and looks for the answers through the actual experiments. Its investigation is clearly empirical and relies on human cognition. Although it is interested in the mental world, it does its best to stay as objective as possible. We have reached for the experiments with children, in particular with children from newborns up to preschoolers, because they have not been exposed to the mathematical conventions yet. The idea is to understand the number sense per se, basic arithmetic and the rest as it gradually emerges from them. One has to admit that uncovering the

nature of number through its innateness does sound appealing.
It is especially the interdisciplinary study of mind and intelligence, namely cognitive science, that might help tackling the balance between logic and psychology. This science deals with issues such as what knowledge or mechanisms we are born with, what might be taken for granted when referring to human logic, what competences we actually have, and the like. All these questions point out the same conclusion: "Arithmetic may seem easy once you've learned it, but there is an awful lot to it from the perspective of the embodied mind." [Lak, p. 50]
Despite all the differences between logical and psychological perspectives (e.g., a priori – empirical, physical/third realm – mental realm, independent of people – based on people, thought experiments – actual experiments, . . .), at the end of the day, they both have proposed a very colourful and quite objective look at the concept of number. There is no reason why, for example, the concept of number could not be inborn and at the same time the arithmetic built on it perfectly analytic. It is a different story that we might be still wondering where the line between logic and psychology is, but we do not dwell on this particular issue here.

3.3 One-to-one correspondence

Many topics associated with the issue of the concept of number have their logical and psychological aspect. It can be illustrated with the following example of one-to-one correspondence. One-to-one correspondence covers one of the three basic principles 'how to count'.[1] One item corresponds to one count word (i.e., one count word is assigned to one item). This relation might be explained more extensively through the set of its three properties as follows:

(1) Each item has exactly one count word assigned to it. It basically means that this relation is a function from a set of discrete items to a set of count words.

(2) Each count word from an adequate subset of all count words (i.e., the set of count words for all natural numbers in the range from 1 to n, where the number n denotes cardinality of the given set of items – e.g., for a set of 4 items it is the set of count words for 1, 2, 3, 4) is assigned to at least one item. Functions satisfying this property are said to be 'onto' and called surjective.

[1] The principles 'how to count' have been described in great detail in section 2.4.

(3) No count word is assigned to more than one item. Functions satisfying this property are said to be 'one-to-one'[2] and called injective.

This essential relation was put on a ground of mathematics under the name of a *bijection*. Even Frege could not define equinumerosity of two concepts without a requirement of them being one-to-one correspondent. Actually, what Frege suggests is simply a bijection between the extensions of the two concepts, which are in fact the sets of elements falling under those concepts. The use of this relation in everyday counting has been proved in cognitive science by practical experiments with preschoolers. However, as much as theory does meet practice, there is also something important in practice that theory cannot account for, namely, an opinion. Let us return to Piaget's theory and his experiment previously mentioned in Chapter 2.

"In accordance with set theory and logicians Frege, Whitehead and Russell, [Piaget claims that] we might further assume that the number simply originates by forming a one-to-one correspondence between two classes or sets." [Pia, p. 95] However, he points out that besides one-to-one correspondence as we know it from mathematics, therefore an arbitrary correspondence between two sets, there is another type of correspondence which is based on similarity of the corresponding elements (i.e., a copy corresponds to its original). Since it already contains a number unit, it is only the former type of correspondence which defines discrete numbers.

Piaget's task discussed earlier targets exactly the preschoolers' understanding of one-to-one correspondence. There are two slightly different conditions and for each one of them children are asked whether two rows of coins are of the same number or one of them has more coins. In both conditions there is the same number of coins in the both rows. However, in the first condition the rows are also of the same length, whereas for the second condition one of the rows is spread to make it look longer. This manipulation is included in the experiment so that the children can see it. As a result, in the first condition, the children answer correctly and thus manifest their ability to use one-to-one correspondence. In the second condition, however, it seems the manipulation is too overwhelming and leads them astray. When they are asked to count the coins, they do discover that the rows are actually of the same number. They probably thought the change

[2] Even though they sound similar, it is important to distinguish between one-to-one correspondence, which is actually a synonym for bijection and therefore a function satisfying all three previously mentioned properties; and one-to-one function, which is a function that does not necessarily satisfy the second of the properties.

itself was too important to ignore it, so they decided to answer differently to point it out.
In various tasks preschoolers have confirmed their competence to count according to all three principles 'how to count' with the one-to-one principle among them. They do know how to use it, but for some reason, there are occasions when they do not use it. We can see how important the role of an opinion or belief might be in our otherwise quite logical reasoning. It points out the well-known distinction between competence and performance. Competence describes what we are able to do and performance, on the other hand, what we actually do. Our errors in the latter do not have to impair the former. Individual performances of the same person through various forms of the task might differ drastically as there are many possible factors to effect the person (e.g., "... short-term memory limitations or lack of sufficient time... "[Pel, p. 9]). It only shows that even through a whole series of experiments it still might not be easy to define the real extent of a certain competence.
In Piaget's task above we have clearly seen the drawback of humans' reasoning. We might have the right tools to think logically, but as long as there are occasions when we do not use them and let our opinions and beliefs jeopardise them, we might not be able to actually separate logical from psychological. Whereas logic is interested purely in our competence, psychology has more to do with our performance.[3] However, as we have already mentioned, sometimes it might be impossible to tell where the performance ends and the competence begins.

3.4 Individuation into units

Individuation into units is another topic concerned with the concept of number which has both – logical and psychological aspects. It is also closely related to the previous topic, since before we can even consider one-to-one correspondence (i.e., a correspondence between two sets), we need to understand a unit per se. Only then we can assign one unit to another. It definitely seems worth taking a glance at the issue of individuation process and units for discrete numbers.
We have been dealing with units in both previous chapters – with their mutual identity and distinguishability in Chapter 1 and with their different determination for continuous amounts and discrete numbers in Chapter 2. Units for

[3] Some might object that logic has nothing to do with humans' competence, with humans as such. There is no need to take the parable literally. It makes more sense, if we look at mathematics and logic from the perspective introduced further in section 3.5.

discrete numbers are naturally identical to each other as they are in fact items after being stripped of all their differences by abstraction. Their distinguishability consists simply in 'the special link' to those different items before being stripped. On this matter we should also recall that Frege reproached Husserl for this kind of idea. According to Husserl's theory in order to grasp the number, we have to use abstraction to turn different objects into mutually identical units and at the same time preserve their distinguishability.

How do these two seemingly contradictory qualities look like in real counting? When counting a certain grouping of objects (e.g., somehow limited or chosen in our visual field) on a level of 'falling into the grouping that is being counted at the moment' all the objects are considered equal to each other (i.e., identical as units) – if an object belongs to the grouping then it stands for one unit, otherwise it is completely irrelevant. However, counting yields also necessity to distinguish these objects so one can keep track which objects are already counted and which not. This requirement does not sound surprising as we normally distinguish objects primarily according to their different spatial locations. However, has it always been so evident for us? Let us look back at Chapter 2 and reflect on infants' capability to individuate and grasp units.

It is quite fascinating that already infants are able to individuate (i.e., to distinguish) separate objects in space (alternatively in time). It is suggested that they might be born with 'the object concept' – some kind of a sense of 'objecthood'.[4] However, there is one yet crucial twist connected to this idea. Different research has indicated that infants start with quantifying over continuous amounts and only later learn how to quantify over discrete numbers. Obviously, there is a certain clash between the sense of objecthood and quantification over continuous amounts. Ability to individuate objects, for example according to their location, seems to have something to do with the 'contours' of the objects. However, those would almost certainly jeopardise infants' primary sense of overall amount of 'the object-stuff', which is much likely to be needed for the proposed quantification over continuous amounts.

Nevertheless, infants' ability to individuate objects in space has been shown as well as their preference for continuous amounts over discrete numbers. The former yields for distinguishing the contours of objects and the latter, on the contrary, for overlooking the contours. To account for both ideas simultaneously infants' counting has to be represented by one process with two seemingly contradictory qualities. It might be the case that infants individuate separate

[4] To learn more about the infant's object concept see [Pyl].

objects in space with the help of their contours as we would imagine. However, once the grouping of objects to be counted is chosen, their definition of contours changes – they may be distinguishing between the elements of the particular grouping and the rest, and as the result they would draw the contours around the continuous amount consisting of all the elements together (i.e., 'the element-stuff'). Infants would still individuate, so to speak, but in the role of the unit it would actually be 'the totality of units'.
Our intention was to point out that counting discrete sets might not be as evident as it would seem. Individuation into units is essential for the counting process. Lucky for us that we are either born with the ability to individuate or come across it in a very young age. From then on subdivision of discrete sets into units happens automatically.

3.5 Metaphor not only for poets

In this last section we offer an alternative contemporary view. Modern cognitive science has come up with a very fresh and daring perspective on the nature of mathematics, which bravely brings psychology back into the picture. The trick comes with the realisation that as much as we want to leave logic 'psychology-free', we literally cannot even get it out of our mind. As these cognitive scientists would say:

> "Mathematics as we know it is human mathematics, a product of the human mind. Where does mathematics come from? It comes from us!" [Lak, p. 9]

Fregean objective and non-sensible thoughts, which also include arithmetic sentences, are definitely jeopardised by this kind of thinking. It is argued that human mind simply cannot yield fully-fledged objectivity in the world, only 'human objectivity', so to speak. How could we then place Fregean thoughts outside of the mental realm into the third realm and claim arithmetic sentences to be a priori? Only hardly. . . .
Mathematics has been obviously shifted to an empirical scientific level. However, it still reflects quite accurately on the objective external world and therefore there is a great demand for a certain 'bridge' between the physical and the abstract – the bridge called metaphor. Let us briefly sketch the real importance of metaphor in our everyday life. The thing about metaphor is that we usually think of it as a figure of speech for poets to 'decorate' their written words, but

we fail to realise that it is actually far more commonplace and widely used. Even the majority of our learning happens on its ground. Lakoff and Núñez put it as follows: "Metaphor is not a mere embellishment; it is the basic means by which abstract thought is made possible. One of the principal results in cognitive science is that abstract concepts are typically understood, via metaphor, in terms of more concrete concepts." [Lak, p. 39]
To be more precise, this mechanism "... grounded in the sensory-motor system ... is called *conceptual metaphor*" [Lak, p. 5]. We do not use such metaphor intentionally, nor consciously, it is "... part of our system of thought" [Lak, p. 42]. It is actually by its means that we "... associate the arithmetic of natural numbers with a huge range of experiences in the world: experiences of collections, structurings of objects, the manipulation of physical segments, and motion" [Lak, p. 96]. And it is exactly on the basis of this metaphorical 'bridge' that arithmetic seems so natural to us. Arithmetic is not discovered in the outside world, it is simply well built in by us, humans. The latest cognitive research has shown that we are born with the basic arithmetical abilities already wired in (e.g., very basic addition and subtraction) and it is suggested that the rest is built on them thanks to constant linking to our everyday non-mathematical thought via conceptual metaphor. As an illustrative example of "... spatial inference [being] mapped onto an abstract logical domain ... [t]here is a commonplace metaphor, Categories Are Containers, through which we understand a category as being a bounded region in space and members of the category as being objects inside that bounded region" [Lak, p. 43].
As we can see this kind of "... embodied mathematics will look very different from disembodied formal mathematics" [Lak, p. 45] that we usually come across. We can imagine that majority of logicians would strongly disagree with this perspective, since they consider psychological aspects far from welcome in the field of arithmetic. For example, Frege claims:

> "Neither logic nor mathematics has the task of investigating minds and the contents of consciousness whose bearer is a single person. Perhaps their task could be represented rather as the investigation of the mind, of the mind not of minds." [F-Ge, p. 308]

It is quite obvious that cognitive scientists have headed for the opposite direction on this matter as Frege did. The reason for it is that they avoid "... the all-too-common conception that mathematics is about calculation and about formal proofs from formal axioms and definitions... " [Lak, p. 49].

Cognitive science has been flourishing these last couple of decades which has resulted in many compelling theories. The most remarkable and evidently very contributive is its interdisciplinary focus. Throughout this paper it has helped us see the possible connections between logic and psychology and clarify their mutual attraction to the concept of number and arithmetic.

Conclusion

We have presented logical and psychological perspectives on the concept of number. Surely, we all have suspected from the beginning they would not end up walking hand-in-hand. What we had in mind, however, was the idea of their mutual profit if working together rather than each on its own or even against each other. Throughout our investigation we have met the potential as the psychological approach provides logic with the touch of reality it often lacks, and, in return, the logical approach brings the abstract framework into psychology. The line seems to be as thin between concrete and abstract here as we have seen it between subjective and objective, mental and physical, performance and competence. It is the well-known middle ground of Fregean third realm which requires some qualities of each component but cannot fully accept either of them – the ground that cognitive science has been balancing on.

Have we successfully answered what number is and where it comes from? Well, yes and no. We have discussed several intriguing logical ideas as well as those of cognitive science, but they would hardly help explain what number is to somebody who had never heard of it.... Whether we have the concept of number already wired in from birth or it only emerges later on, whether we define number as a property of concepts or as a product of the mental models, there is that *number sense* we all are equipped with. It is an inherent connection we cannot imagine not having, the ultimate web structuring the world into units to be eventually put back together into numbers. And as natural as it seems, it remains unreachable....

At the end of the day, we have to admit it is a quite complex cluster of thoughts behind such a simple question as "How many?"

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