Abstract of doctoral thesis

Study of Arithmetical Structures and Theories with Regard to Representative and Descriptive Analysis

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We are motivated by a problem of understanding relations between local and global properties of an operation o in a structure of the form $\langle \mathcal{B}, o \rangle$, with regard to an application for the study of models $\langle \mathcal{B}, \cdot \rangle$ of Peano arithmetic, where $\mathcal{B}$ is a model of Presburger arithmetic. We are particularly interested in a dependency problem, which we formulate as the problem of describing the dependency closure

$$\mathrm{icl}^O(E) = \{\overline{d} \in B^n; (\forall o, o' \in O)(o \restriction E = o' \restriction E \Rightarrow o(\overline{d}) = o'(\overline{d}))\},$$

where $\mathcal{B}$ is a structure, O a set of n-ary operations on B, and $E \subseteq B^n$. We show, that this problem can be reduced to a definability question in certain expansion of $\mathcal{B}$. In particular, if $\mathcal{B}$ is a saturated model of Presburger arithmetic, and O is the set of all (saturated) Peano products on $\mathcal{B}$, we prove that, for $a \in B$, $\mathrm{icl}^O(\{a\} \times B)$ is the smallest possible, i.e. it contains just those pairs $(d_0, d_1) \in B^2$ for which at least one of d_i equals $p(a)$, for some polynomial $p \in \mathbb{Q}[x]$.

We show that the presented problematics is closely connected to the descriptive analysis of linear theories. That are theories, models of which are – up to a change of the language – certain discretely ordered modules over specific discretely ordered integral domains. We prove a quantifier elimination result in linear theories, and we find the prime models of their simple complete extensions. We perform a detailed analysis of definable sets in a model $\mathcal{A}$ of a linear theory, and show that definable sets are unions of linear images of polyhedra in A^n, with $n \in \mathbb{N}$.

A particularly important example of linear theories is the linear arithmetic LA (more precisely, its "$\mathbb{Z}$-like" variant ZLA). That is an arithmetical theory with the full induction, which extends Presburger arithmetic by multiplication by a single nonstandard element. As a corollary of the results above, we show that LA is model-complete (elimination set consists of primitive positive formulas) and decidable, we find its simple complete extensions and construct their prime models. We also prove that models of LA are, up to elementary equivalence, exactly all non-principal ultraproducts of the structures $\langle \mathbb{N}, 0, 1, +, \leq, n \cdot \_ \rangle$, with $n \in \mathbb{N}$.

As an algebraic application of the presented results, we show that the prime models of the simple complete extensions of LA determine 2^ω different integral domains R, with $\mathbb{Z}[x] \subseteq R \subseteq \mathbb{Q}[x]$, which are ω-stage Euclidean, but not k-stage Euclidean, for any $0 < k \in \mathbb{N}$. This solves the problem posed by G. E. Cooke in [Coo76].

Reference

[Coo76] G. E. Cooke, *A weakening of the Euclidean property for integral domains and applications to algebraic number theory. I*, Journal für die reine und angewandte Mathematik **282** (1976), 133–156.