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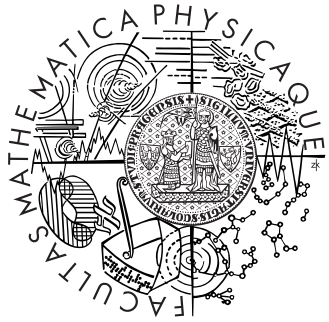

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Study of exact spacetimes

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In Prague, July 7, 2012 Robert Švarc

Název práce: Studium přesných prostoročasů

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Abstrakt: V této práci studujeme různé aspekty chování volných testovacích částic v Einsteinově obecné teorii relativity a analyzujeme fyzikální vlastnosti prostoročasů, v nichž se tyto částice pohybují. V první části zkoumáme geodetické pohyby ve čtyřrozměrných prostoročasech konstantní křivosti, tj. Minkowského a (anti-)de Sitterově vesmíru, s obecnou expandující impulsní gravitační vlnou. Jsou odvozeny jednoduché refrakční vztahy pro částice přecházející impuls a popsán vliv nenulové kosmologické konstanty. V druhé části této práce prezentujeme obecnou metodu použitelnou ke geometrické a fyzikální interpretaci prostoročasů v libovolné dimenzi. Tato metoda je založena na systematickém analyzování relativních pohybů volných testovacích částic. Rovnice geodetické deviace je vyjádřena vzhledem k přirozené ortonormální bázi. Diskutujeme jednotlivé příspěvky odpovídající různé algebraické struktuře tenzoru křivosti a obsažené hmotě. Tento formalizmus je následně užit ke zkoumání velké třídy netwistujících prostoročasů. Především pak analyzujeme pohyby částic v neexpandující Kundtově a expandující Robinsonově–Trautmanově rodině řešení.

Klíčová slova: Geodetiky, gravitační vlny, algebraická klasifikace prostoročasů, Kundtův prostoročas, Robinsonův–Trautmanův prostoročas.

Title: Study of exact spacetimes

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Abstract: In this work we study various aspects of the behaviour of free test particles in Einstein's general relativity and analyze specific physical properties of the background spacetimes. In the first part we investigate geodesic motions in the four-dimensional constant curvature spacetimes, i.e., Minkowski and (anti-)de Sitter universe, with an expanding impulsive gravitational wave. We derive the simple refraction formulae for particles crossing the impulse and describe the effect of nonvanishig cosmological constant. In the second part of this work we present a general method useful for geometrical and physical interpretation of arbitrary spacetimes in any dimension. It is based on the systematic analysis of the relative motion of free test particles. The equation of geodesic deviation is rewritten with respect to the natural orthonormal frame. We discuss the contributions given by a specific algebraic structure of the curvature tensor and the matter content of the universe. This formalism is subsequently used for investigation of the large class of nontwisting spacetimes. In particular, we analyse the motions in the nonexpanding Kundt and expanding Robinson–Trautman family of solutions.

Keywords: Geodesics, gravitational waves, algebraic classification of spacetimes, Kundt spacetime, Robinson–Trautman spacetime.

CONTENTS

CHAPTER

1

INTRODUCTION

A hundred years ago, in July 1912, Albert Einstein left Prague after his sixteen months appointment as a regular professor at the German Charles–Ferdinand University. During this stay he found the basic ideas of his future gravitational theory. Yet, it took him another three years until general relativity was completely finished (see the original paper [1]) and brought a fully new viewpoint into the understanding of such fundamental concepts as space, time and gravity.

In this theory, the three dimensional space is inseparably connected with the time dimension, and together they form the unified continuum called spacetime. In Einstein's description, the coordinates lose their directly measurable meaning, and the extraction of physically important information becomes more complicated. The effects of gravity elegantly correspond to the curvature of the spacetime which is naturally induced by the presence of matter and energy. On the other hand, the spacetime curvature necessarily induces motion of the matter distribution, and the theory thus becomes nonlinear. The gravitational field represented by such curved spacetime then, in the mathematical terms, corresponds to the manifold with Lorentzian metric. Its ten independent components have to satisfy a set of nonlinear partial differential equations of the second order describing the Einstein gravitational law. Due to the inherent nonlinearity, it is obviously difficult to analytically solve these field equations except in very special cases which are usually characterized by a high degree of symmetry. Therefore, perturbative approaches, or recently also sophisticated numerical simulations, are often used to study the more realistic astrophysical situations. For pedagogical descriptions of the Einstein gravity theory and the related topics see, e.g., the textbooks [2, 3].

In spite of the mathematical difficulties and the necessary idealizations, the role of exact analytical solutions of Einstein's field equations is really essential. They bring more profound insight into the structure of the theory, and help to elucidate the completely new aspects which follow from the understanding of gravity as an inherent property of the surrounding space and time. Such "exotic" phenomena as the black holes, bending of the light rays, evolution of the universe, and gravitational waves were first introduced by investigations of specific exact solutions. The detailed summary of the known four-dimensional exact spacetimes can be found in the classical book [4], and physical interpretation of the most important of them in the work [5].

Let us briefly mention some of the most interesting effects predicted by the Einstein general relativity. The examples listed below of the completely new physically relevant effects have been tested precisely and confirmed during the last century, and they established the general relativity as the best fitting theory of gravitation that we know. However, unification of the general relativity with the quantum theory still remains an open problem.

Einstein's work was initially motivated by the anomalous precession of Mercury which was immediately explained using his new theory [6]. In the case of Mercury the relativistic correction is only approximately 43 arc seconds per century. However, such effects are much stronger in the recently discovered compact and massive systems of binary neutron stars, e.g., the periaspis shift of the binary pulsar PSR 1913+16 is about 4.2 arc degree per year, and for the double pulsar PSR J0737+3039 is about 16.9 arc degree per year.

The influence of a gravitational field on the propagation of the light rays was already studied by Einstein during his Prague stay [7]. The first direct test of general relativity thus naturally became the measurement of the light bending which Eddington realized during the total solar eclipse in 1919. The specific deflection of light rays caused by the strong gravitational field of the compact masses such as galaxies is nowadays used for exploring the deep universe (gravitational lensing). In the case where the lens is created by a star, the effect of light bending can be used for the detection of exoplanets possibly orbiting the lensing star (gravitational microlensing).

Another important relativistic prediction is the existence of objects so massive and dense that everything, even the light, is prevented from escaping out of the surrounding area called event horizon. The exact spacetime describing the simplest (static and spherically symmetric) such object was (surprisingly) the first nontrivial solution of the Einstein equations found by Schwarzschild in 1916, see [8]. However, this geometry was better understood much later during the golden age of general relativity in the sixties. More realistic rotating generalization was presented by Kerr [9] in 1963. In our universe, these "black holes" can be formed as a final state of the evolution of sufficiently massive stars when the internal pressure is not able to resist the star's own gravity. The supermassive black hole (about four millions solar masses) is observed indirectly via the motion of orbiting stars in the center of the Milky Way Galaxy, and reside in the centers of almost all galaxies.

The Einstein gravitational law can be also applied for simplified global modeling of the whole universe. Employing the observed large scale isotropy and homogeneity of the known surrounding space, the general relativity immediately predicted the non-stationary evolution since the possibly static Einstein solution found in 1917 is not stable under perturbations. The universe is thus necessarily either expanding or collapsing (for the review of the first cosmological attempts see the work [10]). In 1929, the expansion of the universe was experimentally discovered by Hubble's measurements of the red-shift of the light emitted by distant galaxies, which is proportional to the distance between the source and the observer.

One of the most fascinating features of Einstein's relativity is the propagation of the gravitational interaction with the finite speed of light in the form of weak ripples in the curvature of the spacetime. Their existence was predicted by Einstein using the weak field limit in 1918, see [11]. The particular example of large exact family of gravitational waves was subsequently found by Brinkmann [12], however, the physical meaning of these solutions was understood much later. The gravitational waves have quite similar properties as the electromagnetic waves. They are transverse and (in four dimensions) have two independent polarization modes. The main difference is that the electromagnetic waves are of dipole character while the gravitational radiation is quadrupole. Typical astrophysical source of the gravitational waves is thus an asymmetric collapse of a star (ideally the explosion of supernova). Gravitational waves are also produced by the system of two massive objects orbiting each other. General relativity then predicts that during this motion the orbital energy is converted into the radiation and emitted away in the form of ripples in the curvature. This necessarily leads to the decrease of the orbital period, and finally to the merging of the orbiting objects. The first discovered system of this type was the binary pulsar PSR 1913+16 detected by Hulse and Taylor using the Arecibo antenna in 1974. It consists of two neutron stars with the orbital period 7.75 hours. The measured decrease of this period is about 76.5 microseconds per year. By the precise observations of this binary system during the following

years it was shown that the cumulative decrease of the orbital period exactly corresponds to the predictions given by the Einstein general relativity. Such measurements thus give us an indirect but very strong evidence of the existence of gravitational waves, and the discovery made by Hulse and Taylor was awarded the Nobel Prize in 1993.

The first attempts to directly detect the gravitational waves were performed by Joseph Weber in the late sixties using his own resonance detectors which were able to measure the relative deformations caused by the wave of the order 10^{-16}. Although the sensitivity was pretty high, the gravitational waves were not detected. The current models for the possible wave amplitudes from the common astrophysical sources predict the relative deformations about 10^{-22} and lower. This sensitivity corresponds to the measurements of the distance between the Sun and the Earth with the precision a single atom.

In the last decade, there has been a considerable effort dedicated to the direct detection of gravitational waves. The international network of detectors containing LIGO observatories in the United States, VIRGO in Italy and GEO 600 in Germany, has been built. However, even using these huge ground interferometric detectors with sensitivity about 10^{-22}, the gravitational waves have not yet been directly observed. Nowadays the LIGO and VIRGO detectors are upgraded for the higher sensitivity up to 10^{-23} (Advanced LIGO) which should lead to the significant increase of the possibly detectable sources. Also, the future space project of the European Space Agency called the New Gravitational wave Observatory (NGO) should be prepared for launch in 2020. In fact, it is based on the revised previous project LISA of ESA and NASA. The direct detection of gravitational waves should thus be expected in the following years.

Regardless of these observable astrophysical and cosmological consequences of general relativity, our work presented in this thesis has purely theoretical character. We are mainly interested in the physical interpretation of exact solutions of Einstein's field equations in any dimension. A special attention is paid to the idealized models of gravitational waves and to the interactions of such waves with geodesic observers. We hope this may help us to understand the theory of gravitational radiation in four and possibly any higher number of spacetime dimensions.

In the first part (Chapter 2) we analyze the behaviour of freely falling test particles in the four-dimensional spacetimes of constant curvature with an expanding impulsive gravitational wave. The class of solutions can be obtained as a null limit of two point masses accelerated in opposite directions, see [5]. We provide a detailed description of the observer's transition across the expanding wave surface with the δ-distribution profile. The effects given by the presence of a nonvanishing cosmological constant are also discussed.

In the second part we introduce the general method describing the relative motions of free test particles in an arbitrary spacetime of any dimension (Chapter 3). This method can be useful for a deeper understanding of the specific properties induced by the particular algebraic structure of the given spacetime. We demonstrate that the Weyl components cause the specific effects corresponding to their boost weight. In Chapter 4 we apply this method to investigation of the general nontwisting D-dimensional class of spacetimes. In Chapter 5 we investigate the relative behaviour of geodesics in the shearfree and nonexpanding Kundt class. The properties of important subclasses, namely pp-waves, VSI spacetimes and simple gyratons, are discussed in detail. In Chapter 6 we analyze the relative motion in the nontwisting, shearfree, and expanding Robinson–Trautman family of solutions. The differences between four and any higher dimension are described from the viewpoint of the geodesic observers.

CHAPTER

2

GEODESICS IN IMPULSIVE GRAVITATIONAL WAVES

In this chapter we analyze geodesic motion in spacetimes of constant curvature with an impulsive spherical gravitational wave. Our results are presented in the form of the paper *Refraction of geodesics by impulsive spherical gravitational waves in constant-curvature spacetimes with a cosmological constant* which was published in *Physical Review D* in *2010*, for a full citation see [13].

In the classical work [14] Penrose described an elegant geometric construction of impulsive gravitational waves. This approach is the so-called "cut" and "paste" method which is based on the identification of two pieces of Minkowski spacetime with a suitable wrap, obtaining thus an impulsive wave surface. Later Nutku and Penrose in [15] and Hogan in [16, 17] found explicit continuous coordinate systems covering the whole spacetime, i.e., both Minkowski halfspaces and the bordering impulsive surface. These results were extended for the case of a nonvanishing cosmological constant in the works of Hogan [18] and Podolský and Griffiths [19, 20]. More details and references can be found in [5].

The geodesic motion in such impulsive spherical gravitational waves propagating on a flat Minkowski background spacetime was studied by Podolský and Steinbauer in [21]. Their main attention was paid to an important family of explicit continuous $Z = const.$ geodesics.

We subsequently investigated a more general class of $\mathcal{C}^1$ geodesics, i.e., continuous curves with continuous first derivatives. Our results naturally cover and extend the previous case of $Z = const.$ geodesics on a flat background described in [21], and also generalize the discussion for motion affected by a presence of an arbitrary cosmological constant, i.e., for de Sitter or anti-de Sitter backgrounds.

Using only the assumption that these geodesics belong to the $\mathcal{C}^1$ class, and employing an explicit form of coordinate transformations between the continuous coordinate system and natural background coordinates on the subspaces "in front of" and "behind" the impulse, we found a simple refraction formulae fully describing the effects of the wave. These relations identify position and velocity of an observer before its interaction with the impulse, i.e., on the border of the halfspace "in front of" the wave, with those resulting from the influence of the impulsive gravitational wave, i.e., on the border of halfspace "behind" the impulse. Because the geodesics in the background spaces of constant curvature are well known, these refraction formulae thus completely describe the behaviour of continuous $\mathcal{C}^1$ observers in Minkowski, de Sitter and anti-de Sitter universe with

arbitrary impulsive spherical gravitational waves.

We also apply the derived refraction relations to the axially symmetric case of an impulse generated by a snapped cosmic string. The influence of the wave on free test particles than results in focusing of particles in the directions of moving ends of the snapped string. These effects depend only on the initial positions and velocities of the particles, and on the deficit angle characterizing the cosmic string. Combination of a homogeneous expansion given by the presence of a positive cosmological constant in de Sitter background spacetime and the refraction effects of the wave was also described.

These results provide us with a physical and geometrical intuition of possible behaviour of free test particles in spacetimes with impulsive spherical gravitational waves and can serve as a heuristic ansatz for a rigorous mathematical discussion based on the algebras of generalized functions.

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Refraction of geodesics by impulsive spherical gravitational waves in constant-curvature spacetimes with a cosmological constant

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We investigate motion of test particles in exact spacetimes with an expanding impulsive gravitational wave which propagates in a Minkowski, a de Sitter, or an anti-de Sitter universe. Using the continuous form of these metrics we derive explicit junction conditions and simple refraction formulas for null, timelike, and spacelike geodesics crossing a general impulse of this type. In particular, we present a detailed geometrical description of the motion of test particles in a special class of axially symmetric spacetimes in which the impulse is generated by a snapped cosmic string.

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I. INTRODUCTION

In the fundamental work [1] Roger Penrose introduced an elegant geometric ''cut and paste'' method for construction of impulsive spherical gravitational waves in a flat background. This is based on cutting Minkowski space along a null cone and then reattaching the two pieces with a suitable warp. An explicit class of such spacetimes, using coordinates in which the metric functions are continuous across the impulse, was subsequently given by Nutku and Penrose [2], Hogan [3,4] and, to include a nonvanishing cosmological constant, in [5–7]. An additional acceleration parameter can also be introduced [8].

This gives the complete family of expanding spherical waves of a very short duration which propagate in a Minkowski, a de Sitter, or an anti-de Sitter universe, that is in spacetimes with a constant curvature (zero, positive, or negative, respectively). Such solutions can naturally be understood as impulsive limits of Robinson-Trautman type-N vacuum solutions [9,10], namely, a suitable family of spherical sandwich waves of this type [6,11].

A stereographic interpretation of complex spatial coordinate involved in the Penrose junction condition across the impulse can be used for an explicit construction of specific solutions of this type, in particular, those which describe impulsive spherical waves generated by colliding and snapping cosmic strings [7]. A first such solution given already in [2] represents the snapping of a cosmic string, identified by a deficit angle in the region outside the spherical impulsive gravitational wave. The collision and breaking of a pair of cosmic strings can also be described in this way.

The particular solution for a spherical gravitational impulse generated by a snapping cosmic string in Minkowski space was alternatively described by Bičák and Schmidt [12]. This was obtained as a limiting case of the Bonnor-Swaminarayan solution for an infinite acceleration of a pair of Curzon-Chazy particles (see Chapter 15 of [10]). It was observed in [13] that such a situation is equivalent to the splitting of an infinite cosmic string as described in [14] or, rather, of two semi-infinite cosmic strings approaching at the speed of light and separating again at the instant at which they ''collide.''

The same explicit solution was also obtained in the limit of an infinite acceleration in the more general class which represents a pair of uniformly accelerating particles with an arbitrary multipole structure [15], or as an analogous limit of the C metric which describes accelerating black holes [16]. In the latter case, a nonvanishing cosmological constant can also be considered. This leads to a specific expanding spherical impulse generated by a snapping cosmic string in the (anti)de Sitter universe [17].

More details concerning these impulsive metrics and other references can be found in the review works [18,19] and in Chapter 20 of [10]. Note also that particle creation and other quantum effects in such spacetimes were investigated, e.g., by Hortaçsu and his collaborators [20–23].

The main objective of the present work is to study specific properties of these spacetimes, namely, the motion of test particles influenced by the spherical impulsive waves. In fact, Podolský and Steinbauer in [24] already investigated and described the behavior of exact geodesics in the case when the impulse expands in Minkowski flat space. Here we will generalize this study to any value of the cosmological constant, i.e., we will analyze the effects on geodesics when the spherical impulse expands in a de Sitter or an anti-de Sitter universe. Moreover, we will present the results in a form which is more convenient for physical and geometric interpretation.

Our paper is organized as follows. In Sec. II we review the class of spacetimes under consideration and describe the geometry of the expanding impulses. By employing a continuous form of the metric, in Sec. III we investigate a large class of C^1 geodesics crossing the spherical impulse.

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We explicitly derive the junction conditions and the refraction formulas, we study a subfamily of privileged global geodesics, and rewrite the junction conditions in a convenient five-dimensional formalism when $\Lambda \neq 0$. In Sec. IV we focus on impulsive waves generated by a snapped cosmic string. We discuss in detail the physical and geometric interpretation of the motion of test particles influenced by an impulse of such type.

II. EXPANDING IMPULSIVE WAVES IN CONSTANT-CURVATURE BACKGROUNDS

As a natural background for constructing the family of spherical expanding impulsive waves, we consider the conformally flat metric

$$ds_0^2 = \frac{2d\eta d\bar{\eta} - 2d\mathcal{U}d\mathcal{V}}{[1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V})]^2}. \tag{2.1}$$

This is a unified form for all spaces of constant curvature, namely, Minkowski space when $\Lambda = 0$, de Sitter space when $\Lambda > 0$, and anti-de Sitter space when $\Lambda < 0$.

Indeed, with the standard representation of the double null coordinates

$$\mathcal{U} = \frac{1}{\sqrt{2}}(t - z), \qquad \mathcal{V} = \frac{1}{\sqrt{2}}(t + z), \qquad \eta = \frac{1}{\sqrt{2}}(x + iy), \tag{2.2}$$

the metric (2.1) reads

$$ds_0^2 = \frac{-dt^2 + dx^2 + dy^2 + dz^2}{[1 + \frac{1}{12}\Lambda(-t^2 + x^2 + y^2 + z^2)]^2}, \tag{2.3}$$

which for $\Lambda = 0$ is the familiar form of the flat space. In the case $\Lambda \neq 0$, it is well known that the corresponding de Sitter and anti-de Sitter spaces can be represented as a four-dimensional hyperboloid,

$$-\mathrm{Z}_0^2 + \mathrm{Z}_1^2 + \mathrm{Z}_2^2 + \mathrm{Z}_3^2 + \varepsilon\mathrm{Z}_4^2 = \varepsilon a^2, \tag{2.4}$$

embedded in a flat five-dimensional spacetime,

$$ds_0^2 = -d\mathrm{Z}_0^2 + d\mathrm{Z}_1^2 + d\mathrm{Z}_2^2 + d\mathrm{Z}_3^2 + \varepsilon d\mathrm{Z}_4^2, \tag{2.5}$$

where $\varepsilon = 1$ for the de Sitter space ($\Lambda > 0$), $\varepsilon = -1$ for the anti-de Sitter space ($\Lambda < 0$), and $a = \sqrt{3/|\Lambda|}$. The specific parametrization of (2.4) given as

$$\begin{aligned}
\mathrm{Z}_0 &= \frac{1}{\sqrt{2}}(\mathcal{V} + \mathcal{U})\left[1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V})\right]^{-1},\\
\mathrm{Z}_1 &= \frac{1}{\sqrt{2}}(\mathcal{V} - \mathcal{U})\left[1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V})\right]^{-1},\\
\mathrm{Z}_2 &= \frac{1}{\sqrt{2}}(\eta + \bar{\eta})\left[1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V})\right]^{-1},\\
\mathrm{Z}_3 &= \frac{-i}{\sqrt{2}}(\eta - \bar{\eta})\left[1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V})\right]^{-1},\\
\mathrm{Z}_4 &= a\left[1 - \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V})\right]\left[1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V})\right]^{-1},
\end{aligned} \tag{2.6}$$

or inversely

$$\begin{aligned}
\mathcal{U} &= \sqrt{2}a\frac{\mathrm{Z}_0 - \mathrm{Z}_1}{\mathrm{Z}_4 + a},\\
\mathcal{V} &= \sqrt{2}a\frac{\mathrm{Z}_0 + \mathrm{Z}_1}{\mathrm{Z}_4 + a},\\
\eta &= \sqrt{2}a\frac{\mathrm{Z}_2 + i\mathrm{Z}_3}{\mathrm{Z}_4 + a},
\end{aligned} \tag{2.7}$$

takes (2.5) to the metric form (2.1). Consequently, for $\mathcal{U}$, $\mathcal{V} \in (-\infty, +\infty)$, and η an arbitrary complex number, these coordinates cover the entire (anti)de Sitter manifold (except the coordinate singularities at $\mathcal{U}$, $\mathcal{V} = \infty$). For more details about these coordinates and other properties of maximally symmetric spacetimes, see Chapters 3–5 of [10].

The Penrose "cut and paste" method [1] for constructing impulsive spherical waves in such backgrounds of constant curvature can now be performed explicitly as follows (see [5,7]).

In the region $U \geq 0$, let us consider the transformation

$$\begin{aligned}
\mathcal{V} &= \mathcal{V}^+ = AV - DU,\\
\mathcal{U} &= \mathcal{U}^+ = BV - EU,\\
\eta &= \eta^+ = CV - FU,
\end{aligned} \tag{2.8}$$

to coordinates $(U, V, Z, \bar{Z})$, where

$$\begin{aligned}
A &= \frac{1}{p|h'|}, \qquad B = \frac{|h|^2}{p|h'|}, \qquad C = \frac{h}{p|h'|},\\
D &= \frac{1}{|h'|}\left\{\frac{p}{4}\left|\frac{h''}{h'}\right|^2 + \epsilon\left[1 + \frac{Z}{2}\frac{h''}{h'} + \frac{\bar{Z}}{2}\frac{\bar{h}''}{\bar{h}'}\right]\right\},\\
E &= \frac{|h|^2}{|h'|}\left\{\frac{p}{4}\left|\frac{h''}{h'} - 2\frac{h'}{h}\right|^2\right.\\
&\quad \left. + \epsilon\left[1 + \frac{Z}{2}\left(\frac{h''}{h'} - 2\frac{h'}{h}\right) + \frac{\bar{Z}}{2}\left(\frac{\bar{h}''}{\bar{h}'} - 2\frac{\bar{h}'}{\bar{h}}\right)\right]\right\},\\
F &= \frac{h}{|h'|}\left\{\frac{p}{4}\left(\frac{h''}{h'} - 2\frac{h'}{h}\right)\frac{\bar{h}''}{\bar{h}'}\right.\\
&\quad \left. + \epsilon\left[1 + \frac{Z}{2}\left(\frac{h''}{h'} - 2\frac{h'}{h}\right) + \frac{\bar{Z}}{2}\frac{\bar{h}''}{\bar{h}'}\right]\right\},
\end{aligned} \tag{2.9}$$

with

$$p = 1 + \epsilon Z\bar{Z}, \qquad \epsilon = -1, 0, +1 \tag{2.10}$$

(the parameter ϵ is the Gaussian curvature of the spatial 2-surfaces in the closely related Robinson-Trautman foliation of the spacetimes, cf. Sec. 19.2 of [10]). Here

$$h \equiv h(Z) \tag{2.11}$$

is an *arbitrary* complex function, and the derivative with respect to its argument Z is denoted by a prime. The Minkowski and (anti)de Sitter metric (2.1) then becomes

$$ds_0^2 = \frac{2|(V/p)dZ + Up\bar{H}d\bar{Z}|^2 + 2dUdV - 2\epsilon dU^2}{[1 + \frac{1}{6}\Lambda U(V - \epsilon U)]^2}, \tag{2.12}$$

where H is the Schwarzian derivative of h given as

$$H(Z) = \frac{1}{2}\left[\frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2\right]. \tag{2.13}$$

In the complementary region $U \leq 0$, we apply a highly simplified form of the transformation (2.8) which arises for the special choice of the function $h(Z) = Z$. In view of (2.9), this implies relations

$$\begin{aligned} \mathcal{V} &= \mathcal{V}^- = \frac{V}{p} - \epsilon U, \\ \mathcal{U} &= \mathcal{U}^- = \frac{|Z|^2}{p} V - U, \\ \eta &= \eta^- = \frac{Z}{p} V. \end{aligned} \tag{2.14}$$

Since $H = 0$ in this case, by applying the transformation (2.14) the metric (2.1) takes the form

$$ds_0^2 = \frac{2(V/p)^2 dZd\bar{Z} + 2dUdV - 2\epsilon dU^2}{[1 + \frac{1}{6}\Lambda U(V - \epsilon U)]^2}. \tag{2.15}$$

Both in the coordinates of (2.12) and in the ones used in (2.15), the boundary hypersurface $U = 0$ is a *null cone* given by $\eta\bar{\eta} - \mathcal{U}\mathcal{V} = 0$. Using (2.2), it is obviously an expanding sphere $x^2 + y^2 + z^2 = t^2$ in flat Minkowski space. In view of (2.7), it is also an expanding sphere $Z_1^2 + Z_2^2 + Z_3^2 = Z_0^2$ in the (anti)de Sitter universe. Considering the relation (2.4), it follows that such null hypersurface $U = 0$ is the vertical cut $Z_4 = a$ through the de Sitter and anti-de Sitter hyperboloid in a flat five-dimensional spacetime, as shown in Fig. 1. This represents a spherical impulse which originates at time $Z_0 = 0$ and subsequently for $Z_0 > 0$ expands with the speed of light in these backgrounds (alternatively, for $Z_0 < 0$ the impulse is contracting).

An explicit *global* metric which is continuous across the impulse at $U = 0$ is now easily obtained by attaching the line element (2.15) for $U < 0$ to (2.12) for $U > 0$. The resulting metric takes the form

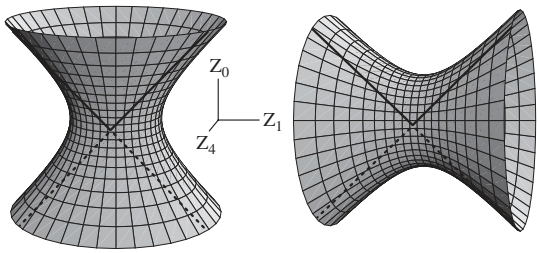

FIG. 1. An expanding spherical impulse can be visualized as a section $Z_4 = a$ of the four-dimensional hyperboloids representing de Sitter (left) and anti-de Sitter (right) spaces. The bold lines are trajectories of opposite poles of an expanding spherical wave surface given by $Z_2 = 0 = Z_3$. The time-reversed situation in the region $Z_0 < 0$, indicated by dashed lines, corresponds to contracting impulsive waves.

$$ds^2 = \frac{2|(V/p)dZ + U\Theta(U)p\bar{H}d\bar{Z}|^2 + 2dUdV - 2\epsilon dU^2}{[1 + \frac{1}{6}\Lambda U(V - \epsilon U)]^2}, \tag{2.16}$$

where $\Theta(U)$ is the Heaviside step function. Such a combined metric is continuous, but the discontinuity in the derivatives of the metric functions across $U = 0$ yields an impulsive gravitational wave term in the curvature proportional to the Dirac δ distribution. More precisely, in a suitable null tetrad, the only nonvanishing component of the Weyl tensor is $\Psi_4 = (p^2H/V)\delta(U)$ (for more details see [7]). The spacetime is thus conformally flat everywhere except on the impulsive-wave surface $U = 0$. Also, the only nonvanishing tetrad component of the Ricci tensor is $\Phi_{22} = (p^4H\bar{H}/V^2)U\delta(U)$. This demonstrates that the spacetime is vacuum everywhere, except on the impulse at $V = 0$ and at possible singularities of the function p^2H.

The expanding spherical impulse located at $U = 0$ obviously splits the spacetime into two separate conformally flat vacuum regions (Minkowski, de Sitter, or anti-de Sitter, according to Λ). For brevity, in the following we shall denote the constant-curvature half-space $U > 0$ as being "in front of the wave", and the other constant-curvature half-space $U < 0$ as being "behind the wave."

III. GEODESIC MOTION IN SPACETIMES WITH EXPANDING IMPULSIVE WAVES

The purpose of this paper is to investigate the effect of expanding impulsive waves on motion of freely moving test particles. We start by recalling geodesics in Minkowski and (anti)de Sitter spaces, then we will derive junction conditions for complete geodesics in the impulsive spacetimes summarized in the previous section and we will present the refraction formulas.

A. Geodesics in the backgrounds

Geodesic motion in spaces of constant curvature (2.3), the background spaces in which an impulse propagates, is well known.

When $\Lambda = 0$, this is just flat Minkowski space. General geodesics are, of course, given by

$$\begin{aligned} t &= \gamma\tau, \qquad x = x_i + \dot{x}_i(\tau - \tau_i), \\ y &= y_i + \dot{y}_i(\tau - \tau_i), \qquad z = z_i + \dot{z}_i(\tau - \tau_i), \end{aligned} \tag{3.1}$$

with $\gamma = \sqrt{\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 - e}$, i.e., τ is a normalized affine parameter of timelike ($e = -1$) or spacelike ($e = +1$) geodesics. For null geodesics ($e = 0$) it is always possible to scale the factor γ to unity. The constants x_i, y_i, z_i and $\dot{x}_i$, $\dot{y}_i$, $\dot{z}_i$ characterize the position and velocity, respectively, of each test particle at the instant

$$\tau_i = \frac{1}{\gamma}\sqrt{x_i^2 + y_i^2 + z_i^2}, \tag{3.2}$$

when the geodesic intersects the null cone $U = 0$. At τ_i each particle is hit by the impulse and its trajectory is refracted, see Sec. III C.

In the case of a nonvanishing cosmological constant Λ, to express all geodesics in the corresponding de Sitter and anti-de Sitter spaces it is very useful to employ the five-dimensional formalism. It can be shown [25] that, using the coordinates of (2.5), the explicit geodesic equations have a very simple and unified form, namely $\ddot{\mathrm{Z}}_\mathrm{p} + \frac{1}{3}\Lambda e \mathrm{Z}_\mathrm{p} = 0$, where $\mathrm{p} = 0, 1, 2, 3, 4$. Thus, explicit geodesics on the hyperboloid (2.4) are

$$\mathrm{Z}_\mathrm{p} = \mathrm{Z}_{\mathrm{p}i} + \dot{\mathrm{Z}}_{\mathrm{p}i}(\tau - \tau_i) \quad \text{when } \varepsilon e = 0, \tag{3.3}$$

$$\mathrm{Z}_\mathrm{p} = \mathrm{Z}_{\mathrm{p}i}\cosh\left(\frac{\tau - \tau_i}{a}\right) + a\dot{\mathrm{Z}}_{\mathrm{p}i}\sinh\left(\frac{\tau - \tau_i}{a}\right) \quad \text{when } \varepsilon e < 0, \tag{3.4}$$

$$\mathrm{Z}_\mathrm{p} = \mathrm{Z}_{\mathrm{p}i}\cos\left(\frac{\tau - \tau_i}{a}\right) + a\dot{\mathrm{Z}}_{\mathrm{p}i}\sin\left(\frac{\tau - \tau_i}{a}\right) \quad \text{when } \varepsilon e > 0, \tag{3.5}$$

where $a = \sqrt{3/|\Lambda|}$. The relation (3.3) describes null geodesics, expression (3.4) represents timelike geodesics in de Sitter space ($\varepsilon = 1$) or spacelike geodesics in anti-de Sitter space ($\varepsilon = -1$), whereas (3.5) corresponds to spacelike/timelike geodesics in de Sitter/anti-de Sitter space, respectively. Here τ is an affine parameter and $\mathrm{Z}_{\mathrm{p}i}, \dot{\mathrm{Z}}_{\mathrm{p}i}$ are constants of integration, namely, the positions and velocities at the instant of interaction with the impulse $\tau = \tau_i$. These ten constants are constrained by the following three conditions:

$$-(\dot{\mathrm{Z}}_{0i})^2 + (\dot{\mathrm{Z}}_{1i})^2 + (\dot{\mathrm{Z}}_{2i})^2 + (\dot{\mathrm{Z}}_{3i})^2 + \varepsilon(\dot{\mathrm{Z}}_{4i})^2 = e, \tag{3.6}$$

$$-(\mathrm{Z}_{0i})^2 + (\mathrm{Z}_{1i})^2 + (\mathrm{Z}_{2i})^2 + (\mathrm{Z}_{3i})^2 + \varepsilon(\mathrm{Z}_{4i})^2 = \varepsilon a^2, \tag{3.7}$$

$$-\mathrm{Z}_{0i}\dot{\mathrm{Z}}_{0i} + \mathrm{Z}_{1i}\dot{\mathrm{Z}}_{1i} + \mathrm{Z}_{2i}\dot{\mathrm{Z}}_{2i} + \mathrm{Z}_{3i}\dot{\mathrm{Z}}_{3i} + \varepsilon\mathrm{Z}_{4i}\dot{\mathrm{Z}}_{4i} = 0. \tag{3.8}$$

Equation (3.6) is the normalization of the affine parameter, Eq. (3.7) follows from the constraint (2.4), and Eq. (3.8) from its derivative.

By combining relations (2.7), (3.3), (3.4), and (3.5) it is now straightforward to express explicitly all geodesics in the four-dimensional metric representation of the (anti) de Sitter universe (2.1). Considering (2.2), which implies

$$\begin{aligned} t &= \frac{2a\mathrm{Z}_0}{\mathrm{Z}_4 + a}, \qquad z = \frac{2a\mathrm{Z}_1}{\mathrm{Z}_4 + a}, \\ x &= \frac{2a\mathrm{Z}_2}{\mathrm{Z}_4 + a}, \qquad y = \frac{2a\mathrm{Z}_3}{\mathrm{Z}_4 + a}, \end{aligned} \tag{3.9}$$

we also obtain geodesics in the metric (2.3), and by using other parametrizations of the hyperboloid (2.4), as summarized in [10], we may easily derive geodesics in any standard metric form of these constant-curvature spacetimes. Some of them will be given below.

Notice finally that close to the impulse (where $\tau - \tau_i$ is small) and also in the limit $\Lambda \to 0$ (so that $1/a$ is small) expressions (3.3), (3.4), and (3.5) take the same linear form $\mathrm{Z}_\mathrm{p} \approx \mathrm{Z}_{\mathrm{p}i} + \dot{\mathrm{Z}}_{\mathrm{p}i}(\tau - \tau_i)$. In view of (3.9) this is fully consistent with Eq. (3.1).

B. Explicit continuation of geodesics across the impulse

Now we will investigate geodesics in complete spacetimes (2.16) with the wave localized on $U = 0$. Geodesics which pass through the impulse have the same form (3.1) or (3.3), (3.4), and (3.5) both in front of the impulse and behind it. However, the constants of integration $\mathrm{Z}_{\mathrm{p}i}, \dot{\mathrm{Z}}_{\mathrm{p}i}$ may have *different values* on both sides.

We thus have to find explicit relations between these constants. To apply the appropriate junction conditions, we *assume that the geodesics are C^1 across the impulse* in the continuous coordinate system of (2.16). It means that the corresponding functions $Z(\tau)$, $V(\tau)$, $U(\tau)$ and also their first derivatives with respect to the affine parameter τ, evaluated at the interaction time $\tau = \tau_i$ [such that $U(\tau_i) = 0$], are continuous across the impulse. With this assumption, the constants

$$\begin{aligned} Z_i &\equiv Z(\tau_i), \qquad V_i \equiv V(\tau_i), \qquad U_i \equiv U(\tau_i) = 0, \\ \dot{Z}_i &\equiv \dot{Z}(\tau_i), \qquad \dot{V}_i \equiv \dot{V}(\tau_i), \qquad \dot{U}_i \equiv \dot{U}(\tau_i), \end{aligned} \tag{3.10}$$

describing positions and velocities at τ_i have the *same* values when evaluated in the limits $U \to 0$ both from the region in front ($U > 0$) and behind the impulse ($U < 0$).

To express the corresponding values in the conformally flat coordinates of (2.1), it is now straightforward to substitute (3.10) into the transformations (2.8) and (2.14),

$$\mathcal{V}_i^+ = AV_i, \qquad \mathcal{V}_i^- = \frac{V_i}{p},$$
$$\mathcal{U}_i^+ = BV_i, \qquad \mathcal{U}_i^- = \frac{|Z_i|^2}{p}V_i, \tag{3.11}$$
$$\eta_i^+ = CV_i, \qquad \eta_i^- = \frac{Z_i}{p}V_i,$$

and their derivatives,

$$\dot{\mathcal{V}}_i^+ = V_i(A_{,Z}\dot{Z}_i + A_{,\bar{Z}}\dot{\bar{Z}}_i) + A\dot{V}_i - D\dot{U}_i,$$
$$\dot{\mathcal{U}}_i^+ = V_i(B_{,Z}\dot{Z}_i + B_{,\bar{Z}}\dot{\bar{Z}}_i) + B\dot{V}_i - E\dot{U}_i, \tag{3.12}$$
$$\dot{\eta}_i^+ = V_i(C_{,Z}\dot{Z}_i + C_{,\bar{Z}}\dot{\bar{Z}}_i) + C\dot{V}_i - F\dot{U}_i,$$
$$\dot{\mathcal{V}}_i^- = -\frac{\epsilon V_i}{p^2}(Z_i\dot{\bar{Z}}_i + \bar{Z}_i\dot{Z}_i) + \frac{\dot{V}_i}{p} - \epsilon\dot{U}_i,$$
$$\dot{\mathcal{U}}_i^- = \frac{V_i}{p^2}(Z_i\dot{\bar{Z}}_i + \bar{Z}_i\dot{Z}_i) + \frac{|Z_i|^2}{p}\dot{V}_i - \dot{U}_i, \tag{3.13}$$
$$\dot{\eta}_i^- = \frac{V_i}{p^2}(\dot{Z}_i - \epsilon Z_i Z_i\dot{\bar{Z}}_i) + \frac{Z_i}{p}\dot{V}_i,$$

respectively [here A, B, C, D, E, F, p and their derivatives are constants, namely, the coefficients (2.9) and (2.10) evaluated at $Z = Z_i$].

Since we wish to express the "$-$" parameters behind the impulse in terms of the "$+$" parameters in front of the impulse, we invert expressions (3.11) and (3.12) in the half space in front of the wave, which yields

$$h(Z_i) = \frac{\eta_i^+}{\mathcal{V}_i^+}, \qquad V_i = \frac{\mathcal{U}_i^+}{B} = \frac{\mathcal{V}_i^+}{A} = \frac{\eta_i^+}{C},$$
$$\dot{Z}_i = \frac{p^2}{V_i}(\bar{C}_{,\bar{Z}}\dot{\eta}_i^+ + C_{,\bar{Z}}\dot{\bar{\eta}}_i^+ - A_{,\bar{Z}}\dot{\mathcal{U}}_i^+ - B_{,\bar{Z}}\dot{\mathcal{V}}_i^+),$$
$$\dot{V}_i = D\dot{\mathcal{U}}_i^+ + E\dot{\mathcal{V}}_i^+ - \bar{F}\dot{\eta}_i^+ - F\dot{\bar{\eta}}_i^+ + 2\epsilon\dot{U}_i,$$
$$\dot{U}_i = \frac{1}{V_i}(\bar{\eta}_i^+\dot{\eta}_i^+ + \eta_i^+\dot{\bar{\eta}}_i^+ - \mathcal{V}_i^+\dot{\mathcal{U}}_i^+ - \mathcal{U}_i^+\dot{\mathcal{V}}_i^+). \tag{3.14}$$

In order to obtain these relations, we employed the identities valid for the coefficients (2.9),

$$AB - C\bar{C} = 0, \qquad DE - F\bar{F} = \epsilon,$$
$$AE + BD - C\bar{F} - \bar{C}F = 1, \tag{3.15}$$

and also for their derivatives

$$DE_{,Z} + D_{,Z}E - F\bar{F}_{,Z} - F_{,Z}\bar{F} = 0, \qquad A_{,Z}E + B_{,Z}D - C_{,Z}\bar{F} - \bar{C}_{,Z}F = 0, \qquad AE_{,Z} + BD_{,Z} - C\bar{F}_{,Z} - \bar{C}F_{,Z} = 0,$$
$$A_{,Z}B + AB_{,Z} - C_{,Z}\bar{C} - C\bar{C}_{,Z} = 0, \qquad A_{,Z}B_{,Z} - C_{,Z}\bar{C}_{,Z} = 0, \qquad D_{,Z}E_{,Z} - F_{,Z}\bar{F}_{,Z} = 0,$$
$$A_{,Z}E_{,Z} + B_{,Z}D_{,Z} - C_{,Z}\bar{F}_{,Z} - \bar{C}_{,Z}F_{,Z} = H, \qquad C_{,Z}\bar{C}_{,\bar{Z}} + C_{,\bar{Z}}\bar{C}_{,Z} - A_{,Z}B_{,\bar{Z}} - A_{,\bar{Z}}B_{,Z} = \frac{1}{p^2},$$
$$F_{,Z}\bar{F}_{,\bar{Z}} + F_{,\bar{Z}}\bar{F}_{,Z} - D_{,Z}E_{,\bar{Z}} - D_{,\bar{Z}}E_{,Z} = p^2|H|^2,$$
$$A_{,Z}E_{,\bar{Z}} + A_{,\bar{Z}}E_{,Z} + B_{,Z}D_{,\bar{Z}} + B_{,\bar{Z}}D_{,Z} - C_{,Z}\bar{F}_{,\bar{Z}} - C_{,\bar{Z}}\bar{F}_{,Z} - \bar{C}_{,Z}F_{,\bar{Z}} - \bar{C}_{,\bar{Z}}F_{,Z} = 0, \tag{3.16}$$

plus their complex conjugates.

Now it only remains to substitute (3.14) into the expressions for positions (3.11) and velocities (3.13) behind the impulse. For the positions we thus obtain

$$\mathcal{V}_i^- = |h'|\mathcal{V}_i^+,$$
$$\mathcal{U}_i^- = |h'|\frac{|Z_i|^2}{|h|^2}\mathcal{U}_i^+, \tag{3.17}$$
$$\eta_i^- = |h'|\frac{Z_i}{h}\eta_i^+,$$

while for the velocities, after straightforward but somewhat lengthy calculation, we get

$$\dot{\mathcal{V}}_i^- = b_{\mathcal{V}}\dot{\mathcal{V}}_i^+ + a_{\mathcal{V}}\dot{\mathcal{U}}_i^+ + \bar{c}_{\mathcal{V}}\dot{\eta}_i^+ + c_{\mathcal{V}}\dot{\bar{\eta}}_i^+,$$
$$\dot{\mathcal{U}}_i^- = b_{\mathcal{U}}\dot{\mathcal{V}}_i^+ + a_{\mathcal{U}}\dot{\mathcal{U}}_i^+ + \bar{c}_{\mathcal{U}}\dot{\eta}_i^+ + c_{\mathcal{U}}\dot{\bar{\eta}}_i^+, \tag{3.18}$$
$$\dot{\eta}_i^- = b_\eta\dot{\mathcal{V}}_i^+ + a_\eta\dot{\mathcal{U}}_i^+ + \bar{c}_\eta\dot{\eta}_i^+ + c_\eta\dot{\bar{\eta}}_i^+,$$

where

$$b_{\mathcal{V}} = \frac{|h|^2}{4|h'|}\left|\frac{h''}{h'} - 2\frac{h'}{h}\right|^2,$$
$$a_{\mathcal{V}} = \frac{1}{4|h'|}\left|\frac{h''}{h'}\right|^2, \tag{3.19}$$
$$c_{\mathcal{V}} = -\frac{h}{4|h'|}\left(\frac{h''}{h'} - 2\frac{h'}{h}\right)\frac{\bar{h}''}{\bar{h}'},$$

$$b_{\mathcal{U}} = \frac{|h|^2}{|h'|}\left|1 + \frac{Z_i}{2}\left(\frac{h''}{h'} - 2\frac{h'}{h}\right)\right|^2,$$
$$a_{\mathcal{U}} = \frac{1}{|h'|}\left|1 + \frac{Z_i}{2}\frac{h''}{h'}\right|^2, \tag{3.20}$$
$$c_{\mathcal{U}} = -\frac{h}{|h'|}\left[1 + \frac{Z_i}{2}\left(\frac{h''}{h'} - 2\frac{h'}{h}\right)\right]\left[1 + \frac{\bar{Z}_i}{2}\frac{\bar{h}''}{\bar{h}'}\right],$$

$$
\begin{aligned}
b_\eta &= \frac{|h|^2}{2|h'|}\Big[1+\frac{Z_i}{2}\Big(\frac{h''}{h'}-2\frac{h'}{h}\Big)\Big]\Big(\frac{\bar h''}{\bar h'}-2\frac{\bar h'}{\bar h}\Big),\\
a_\eta &= \frac{1}{2|h'|}\Big(1+\frac{Z_i}{2}\frac{h''}{h'}\Big)\frac{\bar h''}{\bar h'},\\
\bar c_\eta &= -\frac{\bar h}{2|h'|}\Big(1+\frac{Z_i}{2}\frac{h''}{h'}\Big)\Big(\frac{\bar h''}{\bar h'}-2\frac{\bar h'}{\bar h}\Big),\\
c_\eta &= -\frac{h}{2|h'|}\Big[1+\frac{Z_i}{2}\Big(\frac{h''}{h'}-2\frac{h'}{h}\Big)\Big]\frac{\bar h''}{\bar h'},
\end{aligned}
\tag{3.21}
$$

and, naturally, $\bar c_\mathcal{V} = \overline{c_\mathcal{V}}$, $\bar c_\mathcal{U} = \overline{c_\mathcal{U}}$. Again, all these coefficients are constants which are obtained by evaluating the function h and its derivatives at $Z = Z_i$. Interestingly, they do not depend on the parameter ϵ. For $h(Z) = Z$ there is no refraction effect, which is consistent with the fact that $H = 0$, i.e., the impulse is absent.

Finally, using the transformation (2.2) we may rewrite the expressions for junction conditions (3.17) and (3.18) in the natural conformally flat background coordinates, namely,

$$
\begin{aligned}
x_i^- &= |h'|\frac{Z_i+\bar Z_i}{h+\bar h}x_i^+, \qquad y_i^- = |h'|\frac{Z_i-\bar Z_i}{h-\bar h}y_i^+,\\
z_i^- &= |h'|\frac{|Z_i|^2-1}{|h|^2-1}z_i^+, \qquad t_i^- = |h'|\frac{|Z_i|^2+1}{|h|^2+1}t_i^+,
\end{aligned}
\tag{3.22}
$$

for positions and

$$
\begin{aligned}
\dot x_i^- &= a_x\dot x_i^+ + b_x\dot y_i^+ + c_x\dot z_i^+ + d_x\dot t_i^+,\\
\dot y_i^- &= a_y\dot x_i^+ + b_y\dot y_i^+ + c_y\dot z_i^+ + d_y\dot t_i^+,\\
\dot z_i^- &= a_z\dot x_i^+ + b_z\dot y_i^+ + c_z\dot z_i^+ + d_z\dot t_i^+,\\
\dot t_i^- &= a_t\dot x_i^+ + b_t\dot y_i^+ + c_t\dot z_i^+ + d_t\dot t_i^+,
\end{aligned}
\tag{3.23}
$$

for velocities. The coefficients in (3.22) and (3.23) are somewhat complicated functions of Z_i, $h \equiv h(Z_i)$ and its derivatives $h' \equiv h'(Z_i)$, $h'' \equiv h''(Z_i)$:

$$
\begin{aligned}
&a_x = \frac{1}{2}(c_\eta + c_{\bar\eta} + \bar c_\eta + \bar c_{\bar\eta}), \quad b_x = \frac{i}{2}(-c_\eta - c_{\bar\eta} + \bar c_\eta + \bar c_{\bar\eta}), \quad c_x = \frac{1}{2}(-a_\eta - a_{\bar\eta} + b_\eta + b_{\bar\eta}),\\
&d_x = \frac{1}{2}(a_\eta + a_{\bar\eta} + b_\eta + b_{\bar\eta}), \quad a_y = \frac{1}{2i}(c_\eta - c_{\bar\eta} + \bar c_\eta - \bar c_{\bar\eta}), \quad b_y = \frac{1}{2}(-c_\eta + c_{\bar\eta} + \bar c_\eta - \bar c_{\bar\eta}),\\
&c_y = \frac{1}{2i}(-a_\eta + a_{\bar\eta} + b_\eta - b_{\bar\eta}), \quad d_y = \frac{1}{2i}(a_\eta - a_{\bar\eta} + b_\eta - b_{\bar\eta}), \quad a_z = \frac{1}{2}(-c_\mathcal{U} - \bar c_\mathcal{U} + c_\mathcal{V} + \bar c_\mathcal{V}),\\
&b_z = \frac{i}{2}(c_\mathcal{U} - \bar c_\mathcal{U} - c_\mathcal{V} + \bar c_\mathcal{V}), \quad c_z = \frac{1}{2}(a_\mathcal{U} - a_\mathcal{V} - b_\mathcal{U} + b_\mathcal{V}), \quad d_z = \frac{1}{2}(-a_\mathcal{U} + a_\mathcal{V} - b_\mathcal{U} + b_\mathcal{V}),\\
&a_t = \frac{1}{2}(c_\mathcal{U} + \bar c_\mathcal{U} + c_\mathcal{V} + \bar c_\mathcal{V}), \quad b_t = \frac{i}{2}(-c_\mathcal{U} + \bar c_\mathcal{U} - c_\mathcal{V} + \bar c_\mathcal{V}), \quad c_t = \frac{1}{2}(-a_\mathcal{U} - a_\mathcal{V} + b_\mathcal{U} + b_\mathcal{V}),\\
&d_t = \frac{1}{2}(a_\mathcal{U} + a_\mathcal{V} + b_\mathcal{U} + b_\mathcal{V}),
\end{aligned}
\tag{3.24}
$$

where the constants on the right-hand sides are given by expressions (3.19), (3.20), and (3.21).

To complete the derivation, it only remains to express the complex number Z_i explicitly in terms of the initial position of the test particle in front of the impulse. From Eqs. (3.14) and (2.2) it follows immediately that $h(Z_i) = \eta_i^+/\mathcal{V}_i^+ = (x_i^+ + iy_i^+)/(t_i^+ + z_i^+)$, i.e.,

$$
Z_i = h^{-1}\Big(\frac{x_i^+ + iy_i^+}{t_i^+ + z_i^+}\Big),
\tag{3.25}
$$

where h^{-1} denotes the complex inverse function to h.

C. Geometric interpretation and refraction formulas

In fact, relation (3.25) and its analogous counterpart in the region behind the impulse admits a nice geometric interpretation of the junction condition for positions across the impulse. Let us observe that from expressions (3.11), (3.14), and (2.2) it follows that

$$
Z_i = \frac{x_i^- + iy_i^-}{t_i^- + z_i^-},
\tag{3.26}
$$

$$
h(Z_i) = \frac{x_i^+ + iy_i^+}{t_i^+ + z_i^+}.
\tag{3.27}
$$

Therefore, the complex mapping $Z_i \leftrightarrow h(Z_i)$ can be understood as an identification of the corresponding positions of a test particle in the region behind the impulse ($U < 0$) and the region in front of the impulse ($U > 0$), which is uniquely determined by expressions (3.22). In other words, if the particle, moving along a geodesic, is located at (x_i^+, y_i^+, z_i^+) when it is hit by the impulsive wave ($U = 0$) at the time t_i^+, then it emerges from the impulse at the time t_i^- at the position (x_i^-, y_i^-, z_i^-).

Moreover, when the interaction time t_i^- is rescaled to be equal 1, expression (3.26) and its inverse

$$x_i^- = \frac{Z_i + \bar{Z}_i}{1 + |Z_i|^2}, \qquad y_i^- = i\frac{\bar{Z}_i - Z_i}{1 + |Z_i|^2}, \qquad z_i^- = \frac{1 - |Z_i|^2}{1 + |Z_i|^2}, \tag{3.28}$$

become the well-known relations for a stereographic one-to-one correspondence between a unit Riemann sphere and a complex Argand plane. As shown in Fig. 2, such mapping is obtained by projecting a straight line from the pole through P onto the equatorial plane. A point P on the sphere is thus uniquely characterized by a complex number Z in the complex plane (for more details see [7]).

Because of the stereographic relations (3.26) and (3.27), the complex mapping $Z_i \leftrightarrow h(Z_i)$ thus represents a geometric identification of the points $P^- \equiv (x_i^-, y_i^-, z_i^-)$ and $P^+ \equiv (x_i^+, y_i^+, z_i^+)$ on a unit sphere, which may be considered as a rescaled spherical impulsive surface $U = 0$. The mapping $Z \leftrightarrow h(Z)$ thus naturally encodes the junction conditions for position of a test particle on both sides of the impulse.

Interestingly, relations (3.26) and (3.27) do not involve a cosmological constant Λ. In other words, in the conformally flat coordinates (2.3), this geometric interpretation is valid for expanding spherical impulses in Minkowski, de Sitter, as well as in anti-de Sitter space.

For an illustrative geometrical description of the complete effect of the spherical impulsive wave on test particles moving along geodesics, it is useful to introduce suitable *angles* which characterize *position* of the particle and inclination of its *velocity vector* at the instant of interaction. Specifically, in the (x, z) section we define

$$\tan\alpha^\pm = \frac{x_i^\pm}{z_i^\pm}, \qquad \tan\beta^\pm = \frac{\dot{x}_i^\pm}{\dot{z}_i^\pm}, \tag{3.29}$$

while in the perpendicular (y, z) section we define

$$\tan\gamma^\pm = \frac{y_i^\pm}{z_i^\pm}, \qquad \tan\delta^\pm = \frac{\dot{y}_i^\pm}{\dot{z}_i^\pm}. \tag{3.30}$$

The superscript "+" applies to quantities in front of the expanding impulse (outside the sphere where $U > 0$), whereas the superscript "−" applies to the same quantities behind the impulse (inside the sphere where $U < 0$). Geometrical meaning of these angles is obvious from Fig. 3.

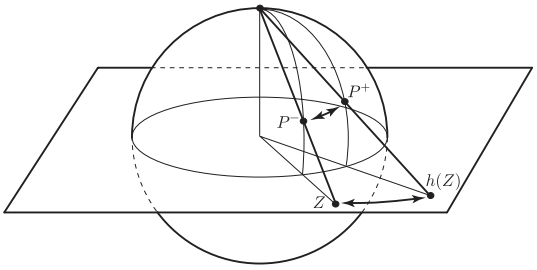

FIG. 2. Mapping in the complex plane $Z \leftrightarrow h(Z)$ is equivalent to identifying the points P^- inside the impulsive spherical surface with the corresponding points P^+ outside through the stereographic projection.

It is also useful to introduce *components of the velocity* of the test particle with respect to the frames outside and inside the impulse as

$$(v_x^\pm, v_y^\pm, v_z^\pm) \equiv \left(\frac{\dot{x}_i^\pm}{\dot{t}_i^\pm}, \frac{\dot{y}_i^\pm}{\dot{t}_i^\pm}, \frac{\dot{z}_i^\pm}{\dot{t}_i^\pm}\right). \tag{3.31}$$

If we now substitute the definitions (3.29), (3.30), and (3.31) into the equations (3.22) and (3.23), we obtain the following expressions which identify the positions:

$$\tan\alpha^- = \frac{(|h|^2 - 1)\,\mathrm{Re}Z_i}{(|Z_i|^2 - 1)\,\mathrm{Re}h}\tan\alpha^+, \qquad \tan\gamma^- = \frac{(|h|^2 - 1)\,\mathrm{Im}Z_i}{(|Z_i|^2 - 1)\,\mathrm{Im}h}\tan\gamma^+, \tag{3.32}$$

and inclinations of the velocity vector,

$$\tan\beta^- = \frac{v_z^+(a_x\tan\beta^+ + b_x\tan\delta^+ + c_x) + d_x}{v_z^+(a_z\tan\beta^+ + b_z\tan\delta^+ + c_z) + d_z}, \qquad \tan\delta^- = \frac{v_z^+(a_y\tan\beta^+ + b_y\tan\delta^+ + c_y) + d_y}{v_z^+(a_z\tan\beta^+ + b_z\tan\delta^+ + c_z) + d_z}, \tag{3.33}$$

on both sides of the impulse. These explicit relations are the general *refraction formulas* for motion of free test particles influenced by the expanding impulsive gravitational wave.

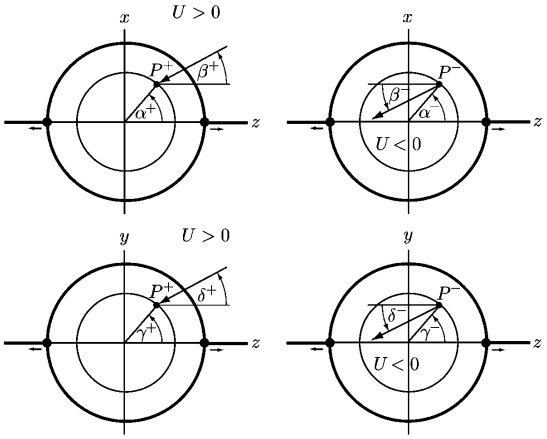

FIG. 3. Definition of the angles α, γ characterizing position of the particle and inclination β, δ of its velocity in the (x, z) section (top) and (y, z) section (bottom), respectively. Here the superscript "+" denotes quantities outside the spherical impulse (left), while "−" labels analogous quantities inside the impulse (right). The points of interaction $P^+ = (x_i^+, y_i^+, z_i^+)$ and $P^- = (x_i^-, y_i^-, z_i^-)$ correspond to those in Fig. 2. The impulsive gravitation wave is an expanding sphere indicated in each section by the bold outer circle.

D. Privileged exact geodesics $Z = \text{const}$

In this part of Sec. III we restrict our attention to a privileged class of exact global geodesics given by the condition

$$Z = Z^0 = \text{const}. \tag{3.34}$$

Indeed, using the continuous form of the impulsive-wave solution (2.16) it can easily be observed that the Christoffel symbols Γ^{μ}_{UU}, Γ^{μ}_{UV}, and Γ^{μ}_{VV} vanish identically when $\mu = Z, \bar{Z}$. Therefore, the geodesic equations always admit global solutions of the form (3.34), including across the impulse localized at $U = 0$ (i.e., without the necessity to *assume* that the geodesics are C^1).

In such a case, $\dot{Z}_i = 0$ and expressions (3.12) and (3.13) thus reduce to

$$\begin{aligned}
\dot{\mathcal{V}}_i^{+} &= A\dot{V}_i - D\dot{U}_i, \qquad & \dot{\mathcal{V}}_i^{-} &= \frac{\dot{V}_i}{p} - \epsilon\dot{U}_i, \\
\dot{\mathcal{U}}_i^{+} &= B\dot{V}_i - E\dot{U}_i, \qquad & \dot{\mathcal{U}}_i^{-} &= \frac{|Z_i|^2}{p}\dot{V}_i - \dot{U}_i, \\
\dot{\eta}_i^{+} &= C\dot{V}_i - F\dot{U}_i, \qquad & \dot{\eta}_i^{-} &= \frac{Z_i}{p}\dot{V}_i,
\end{aligned} \tag{3.35}$$

respectively. Using the relations (3.14) for velocities we obtain the (complex) constraint

$$\dot{\eta}_i^{+}\bar{C}_{,\bar{Z}} + \dot{\bar{\eta}}_i^{+}C_{,\bar{Z}} - \dot{\mathcal{U}}_i^{+}A_{,\bar{Z}} - \dot{\mathcal{V}}_i^{+}B_{,\bar{Z}} = 0, \tag{3.36}$$

and the following equations:

$$\begin{aligned}
\dot{\mathcal{V}}_i^{-} &= b^0_{\mathcal{V}}\dot{\mathcal{V}}_i^{+} + a^0_{\mathcal{V}}\dot{\mathcal{U}}_i^{+} + \bar{c}^0_{\mathcal{V}}\dot{\eta}_i^{+} + c^0_{\mathcal{V}}\dot{\bar{\eta}}_i^{+}, \\
\dot{\mathcal{U}}_i^{-} &= b^0_{\mathcal{U}}\dot{\mathcal{V}}_i^{+} + a^0_{\mathcal{U}}\dot{\mathcal{U}}_i^{+} + \bar{c}^0_{\mathcal{U}}\dot{\eta}_i^{+} + c^0_{\mathcal{U}}\dot{\bar{\eta}}_i^{+}, \\
\dot{\eta}_i^{-} &= b^0_{\eta}\dot{\mathcal{V}}_i^{+} + a^0_{\eta}\dot{\mathcal{U}}_i^{+} + \bar{c}^0_{\eta}\dot{\eta}_i^{+} + c^0_{\eta}\dot{\bar{\eta}}_i^{+},
\end{aligned} \tag{3.37}$$

where

$$\begin{aligned}
b^0_{\mathcal{V}} &= \frac{1}{p}(E - 2\epsilon B) + \epsilon B, \\
a^0_{\mathcal{V}} &= \frac{1}{p}(D - 2\epsilon A) + \epsilon A, \\
c^0_{\mathcal{V}} &= -\frac{1}{p}(F - 2\epsilon C) - \epsilon C,
\end{aligned} \tag{3.38}$$

$$\begin{aligned}
b^0_{\mathcal{U}} &= \frac{|Z_i|^2}{p}(E - 2\epsilon B) + B, \\
a^0_{\mathcal{U}} &= \frac{|Z_i|^2}{p}(D - 2\epsilon A) + A, \\
c^0_{\mathcal{U}} &= -\frac{|Z_i|^2}{p}(F - 2\epsilon C) - C,
\end{aligned} \tag{3.39}$$

$$\begin{aligned}
b^0_{\eta} &= \frac{Z_i}{p}(E - 2\epsilon B), \\
a^0_{\eta} &= \frac{Z_i}{p}(D - 2\epsilon A), \\
\bar{c}^0_{\eta} &= -\frac{Z_i}{p}(\bar{F} - 2\epsilon\bar{C}), \\
c^0_{\eta} &= -\frac{Z_i}{p}(F - 2\epsilon C).
\end{aligned} \tag{3.40}$$

The constants A, B, C, D, E, F, and p are given by the values of the functions (2.9) and (2.10) at $Z = Z_i = Z^0$.

In terms of the real conformally flat coordinates of metric (2.3), the velocities on both sides of the impulse are given by relations (3.23), where now

$$\begin{aligned}
a^0_x &= -2\frac{\mathrm{Re}Z_i}{p}\,\mathrm{Re}(F - 2\epsilon C), \qquad b^0_x = -2\frac{\mathrm{Re}Z_i}{p}\,\mathrm{Im}(F - 2\epsilon C), \qquad c^0_x = \frac{\mathrm{Re}Z_i}{p}[E - D + 2\epsilon(A - B)], \\
d^0_x &= \frac{\mathrm{Re}Z_i}{p}[E + D - 2\epsilon(A + B)], \qquad a^0_y = -2\frac{\mathrm{Im}Z_i}{p}\,\mathrm{Re}(F - 2\epsilon C), \qquad b^0_y = -2\frac{\mathrm{Im}Z_i}{p}\,\mathrm{Im}(F - 2\epsilon C), \\
c^0_y &= \frac{\mathrm{Im}Z_i}{p}[E - D + 2\epsilon(A - B)], \qquad d^0_y = \frac{\mathrm{Im}Z_i}{p}[E + D - 2\epsilon(A + B)], \\
a^0_z &= \frac{|Z_i|^2 - 1}{p}\,\mathrm{Re}(F - 2\epsilon C) + (1 - \epsilon)\,\mathrm{Re}C, \qquad b^0_z = \frac{|Z_i|^2 - 1}{p}\,\mathrm{Im}(F - 2\epsilon C) + (1 - \epsilon)\,\mathrm{Im}C, \\
c^0_z &= \frac{|Z_i|^2 - 1}{2p}[-E + D - 2\epsilon(A - B)] + \frac{1}{2}(1 - \epsilon)(A - B), \\
d^0_z &= \frac{|Z_i|^2 - 1}{2p}[-E - D + 2\epsilon(A + B)] - \frac{1}{2}(1 - \epsilon)(A + B), \qquad a^0_t = -\frac{|Z_i|^2 + 1}{p}\,\mathrm{Re}(F - 2\epsilon C) - (1 + \epsilon)\,\mathrm{Re}C, \\
b^0_t &= -\frac{|Z_i|^2 + 1}{p}\,\mathrm{Im}(F - 2\epsilon C) - (1 + \epsilon)\,\mathrm{Im}C, \qquad c^0_t = \frac{|Z_i|^2 + 1}{2p}[E - D + 2\epsilon(A - B)] - \frac{1}{2}(1 + \epsilon)(A - B), \\
d^0_t &= \frac{|Z_i|^2 + 1}{2p}[E + D - 2\epsilon(A + B)] + \frac{1}{2}(1 + \epsilon)(A + B).
\end{aligned} \tag{3.41}$$

Moreover, from the complex constraint (3.36) we may express two real components of the velocity in terms of the remaining two, namely,

$$\begin{aligned}\dot{y}_i^+ &= [\dot{x}_i^+[(B-A)\,\mathrm{Im}F-(E-D)\,\mathrm{Im}C]+2\dot{z}_i^+(\mathrm{Re}C\,\mathrm{Im}F-\mathrm{Im}C\,\mathrm{Re}F)]/[(B-A)\,\mathrm{Re}F-(E-D)\,\mathrm{Re}C],\\ \dot{t}_i^+ &= [\dot{x}_i^+(BD-AE)-\dot{z}_i^+[(B+A)\,\mathrm{Re}F-(E+D)\,\mathrm{Re}C]]/[(B-A)\,\mathrm{Re}F-(E-D)\,\mathrm{Re}C].\end{aligned} \tag{3.42}$$

Substituting these two relations and the coefficients (3.41) into (3.23), and using Eqs. (3.22) which relate the interaction positions, we finally obtain

$$\begin{aligned}\dot{x}_i^- &= x_i^-\frac{(E-D)\dot{x}_i^+ + 2\,\mathrm{Re}F\dot{z}_i^+}{(E-D)x_i^+ + 2\,\mathrm{Re}Fz_i^+},\\ \dot{y}_i^- &= y_i^-\frac{(E-D)\dot{x}_i^+ + 2\,\mathrm{Re}F\dot{z}_i^+}{(E-D)x_i^+ + 2\,\mathrm{Re}Fz_i^+},\\ \dot{z}_i^- &= [z_i^-[(E-D)\dot{x}_i^+ + 2\,\mathrm{Re}F\dot{z}_i^+]\\ &\quad -(1-\epsilon)(z_i^+\dot{x}_i^+ - \dot{z}_i^+x_i^+)]/[(E-D)x_i^+ + 2\,\mathrm{Re}Fz_i^+],\\ \dot{t}_i^- &= [t_i^-[(E-D)\dot{x}_i^+ + 2\,\mathrm{Re}F\dot{z}_i^+]\\ &\quad +(1+\epsilon)(z_i^+\dot{x}_i^+ - \dot{z}_i^+x_i^+)]/[(E-D)x_i^+ + 2\,\mathrm{Re}Fz_i^+].\end{aligned} \tag{3.43}$$

These relations are valid for any value of the cosmological constant Λ and for an arbitrary spherical impulse. They generalize Eqs. (4.5) obtained previously in [24] for a special impulse generated by a snapping cosmic string in the case when $\Lambda = 0$.

E. Junction conditions in the five-dimensional representation of (anti)de Sitter space

Finally, it will be illustrative to rewrite the explicit junction conditions for positions (3.22) and velocities (3.23) of test particles crossing the impulse in terms of the representation of de Sitter or anti-de Sitter space as the four-dimensional hyperboloid (2.4) in flat five-dimensional spacetime (2.5). Conformally flat coordinates of the metric (2.3) are obtained by the parametrization (2.6) with (2.2), i.e.,

$$\begin{aligned}\mathrm{Z}_0 &= \frac{t}{\Omega}, \qquad \mathrm{Z}_1 = \frac{z}{\Omega}, \qquad \mathrm{Z}_2 = \frac{x}{\Omega},\\ \mathrm{Z}_3 &= \frac{y}{\Omega}, \qquad \mathrm{Z}_4 = a\left(\frac{2}{\Omega}-1\right),\end{aligned} \tag{3.44}$$

where $\Omega = 1 + \frac{1}{12}\Lambda(-t^2+x^2+y^2+z^2)$, or inversely by

$$\begin{aligned}t &= \frac{2a\mathrm{Z}_0}{\mathrm{Z}_4+a}, \qquad z = \frac{2a\mathrm{Z}_1}{\mathrm{Z}_4+a},\\ x &= \frac{2a\mathrm{Z}_2}{\mathrm{Z}_4+a}, \qquad y = \frac{2a\mathrm{Z}_3}{\mathrm{Z}_4+a},\end{aligned} \tag{3.45}$$

with $\Omega = 2a/(\mathrm{Z}_4+a)$.

As explained in Sec. II, the expanding spherical impulse located at $U = 0$ corresponds to the cut $\mathrm{Z}_4 = a$ through the hyperboloid, see Fig. 1. Therefore, at the instant of interaction the particle is located at

$$\begin{aligned}t_i &= \mathrm{Z}_{0i}, \qquad z_i = \mathrm{Z}_{1i}, \qquad x_i = \mathrm{Z}_{2i},\\ y_i &= \mathrm{Z}_{3i}, \qquad \Omega_i = 1.\end{aligned} \tag{3.46}$$

The junction conditions (3.22) for positions thus imply

$$\begin{aligned}\mathrm{Z}_{0i}^- &= |h'|\frac{|Z_i|^2+1}{|h|^2+1}\mathrm{Z}_{0i}^+, \qquad \mathrm{Z}_{1i}^- = |h'|\frac{|Z_i|^2-1}{|h|^2-1}\mathrm{Z}_{1i}^+,\\ \mathrm{Z}_{2i}^- &= |h'|\frac{Z_i+\bar{Z}_i}{h+\bar{h}}\mathrm{Z}_{2i}^+, \qquad \mathrm{Z}_{3i}^- = |h'|\frac{Z_i-\bar{Z}_i}{h-\bar{h}}\mathrm{Z}_{3i}^+,\\ \mathrm{Z}_{4i}^- &= a = \mathrm{Z}_{4i}^+.\end{aligned} \tag{3.47}$$

By differentiating Eqs. (3.44) and evaluating them at the interaction time we obtain the relations

$$\begin{aligned}\dot{\mathrm{Z}}_{0i} &= \dot{t}_i - \dot{\Omega}_i t_i, \qquad \dot{\mathrm{Z}}_{1i} = \dot{z}_i - \dot{\Omega}_i z_i,\\ \dot{\mathrm{Z}}_{2i} &= \dot{x}_i - \dot{\Omega}_i x_i, \qquad \dot{\mathrm{Z}}_{3i} = \dot{y}_i - \dot{\Omega}_i y_i,\\ \dot{\mathrm{Z}}_{4i} &= -2a\dot{\Omega}_i,\end{aligned} \tag{3.48}$$

where $\dot{\Omega}_i = \frac{1}{6}\Lambda(-t_i\dot{t}_i + x_i\dot{x}_i + y_i\dot{y}_i + z_i\dot{z}_i)$, which are valid both in front and behind the impulse. From expressions (3.23) and (3.46) we thus obtain the following relations between velocities on both sides of the impulse:

$$\begin{aligned}\dot{\mathrm{Z}}_{\mathrm{p}i}^- &= a_\mathrm{p}\dot{\mathrm{Z}}_{2i}^+ + b_\mathrm{p}\dot{\mathrm{Z}}_{3i}^+ + c_\mathrm{p}\dot{\mathrm{Z}}_{1i}^+ + d_\mathrm{p}\dot{\mathrm{Z}}_{0i}^+ + n_\mathrm{p}\dot{\mathrm{Z}}_{4i}^+,\\ \dot{\mathrm{Z}}_{4i}^- &= -2a\dot{\Omega}_i = \dot{\mathrm{Z}}_{4i}^+,\end{aligned} \tag{3.49}$$

where we denoted $\mathrm{p} = 0, 1, 2, 3$. The constant coefficients $(a_0, a_1, a_2, a_3) \equiv (a_t, a_z, a_x, a_y)$, and similarly b_p, c_p, d_p, are given by (3.24). The coefficients n_p are defined as

$$n_\mathrm{p} = -\frac{1}{2a}(a_\mathrm{p}\mathrm{Z}_{2i}^+ + b_\mathrm{p}\mathrm{Z}_{3i}^+ + c_\mathrm{p}\mathrm{Z}_{1i}^+ + d_\mathrm{p}\mathrm{Z}_{0i}^+ - \mathrm{Z}_{\mathrm{p}i}^-), \tag{3.50}$$

where $\mathrm{Z}_{\mathrm{p}i}^-$ should be expressed using (3.47). Relations (3.49) can also be written in the matrix form:

$$\begin{pmatrix}\dot{\mathrm{Z}}_{2i}^-\\ \dot{\mathrm{Z}}_{3i}^-\\ \dot{\mathrm{Z}}_{1i}^-\\ \dot{\mathrm{Z}}_{0i}^-\\ \dot{\mathrm{Z}}_{4i}^-\end{pmatrix} = \begin{pmatrix}a_x & b_x & c_x & d_x & n_x\\ a_y & b_y & c_y & d_y & n_y\\ a_z & b_z & c_z & d_z & n_z\\ a_t & b_t & c_t & d_t & n_t\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\dot{\mathrm{Z}}_{2i}^+\\ \dot{\mathrm{Z}}_{3i}^+\\ \dot{\mathrm{Z}}_{1i}^+\\ \dot{\mathrm{Z}}_{0i}^+\\ \dot{\mathrm{Z}}_{4i}^+\end{pmatrix}. \tag{3.51}$$

Expressions (3.47) and (3.51) are explicit junction conditions which relate the positions and velocities of test particles when they cross an expanding spherical impulse. They are expressed in the natural five-dimensional coordinates of constant-curvature spaces with $\Lambda \neq 0$, namely,

the (anti)de Sitter half space in front of the impulse, and the analogous half space behind it. Obviously, the junction conditions depend on the complex function $h(Z)$ which defines the specific impulse of this type.

The advantage of expressing the junction conditions for geodesics in the ''geometrical'' five-dimensional formalism is that they may easily be applied to obtain the corresponding explicit conditions in terms of *any* standard coordinates of de Sitter or anti-de Sitter background space. We will demonstrate this procedure in the next section in which we concentrate of spherical impulses generated by a snapping cosmic string. Their influence on particles will most naturally be expressed in global coordinates in de Sitter space with a synchronous time coordinate, see Sec. IV C.

IV. GEODESICS CROSSING THE IMPULSE GENERATED BY A SNAPPED COSMIC STRING

The general results obtained above will now be applied to an important particular family of spacetimes in which the expanding spherical impulsive wave is generated by a snapped cosmic string (identified by a deficit angle in the region $U > 0$ in front of the impulse). Such exact vacuum solutions were introduced and discussed in a number of works, e.g. [2,7,12–17]. These can be written in the form of the metric (2.16) with

$$H(Z) = \frac{\frac{1}{2}\delta(1-\frac{1}{2}\delta)}{Z^2}, \tag{4.1}$$

which is obtained from the complex function

$$h(Z) = Z^{1-\delta}, \tag{4.2}$$

using the expression (2.13). Here $\delta \in [0, 1)$ is a real constant which characterizes the deficit angle $2\pi\delta$ of the snapped string that is located in the region outside the impulse along the z axis given by $\eta = 0$, as shown in Fig. 4 (see [2,7] for more details).

A. Explicit junction conditions

Expressions (3.22) which are the junction conditions for positions (in the natural conformally flat background coordinates on both sides of the impulse) are thus

$$\begin{aligned}
x_i^- &= (1-\delta)|Z_i|^{-\delta}\frac{Z_i+\bar{Z}_i}{Z_i^{1-\delta}+\bar{Z}_i^{1-\delta}}x_i^+,\\
y_i^- &= (1-\delta)|Z_i|^{-\delta}\frac{Z_i-\bar{Z}_i}{Z_i^{1-\delta}-\bar{Z}_i^{1-\delta}}y_i^+,\\
z_i^- &= (1-\delta)\frac{|Z_i|-|Z_i|^{-1}}{|Z_i|^{1-\delta}-|Z_i|^{\delta-1}}z_i^+,\\
t_i^- &= (1-\delta)\frac{|Z_i|+|Z_i|^{-1}}{|Z_i|^{1-\delta}+|Z_i|^{\delta-1}}t_i^+,
\end{aligned} \tag{4.3}$$

where, in view of relation (3.25),

$$Z_i^{1-\delta} = \frac{x_i^+ + iy_i^+}{t_i^+ + z_i^+}. \tag{4.4}$$

Let us also recall that $(x_i^\pm)^2 + (y_i^\pm)^2 + (z_i^\pm)^2 = (t_i^\pm)^2$ because the positions are evaluated on the impulse $U = 0$.

Similarly, it is straightforward to evaluate the specific form of the coefficients (3.24) which relate the velocities in Eqs. (3.23), namely,

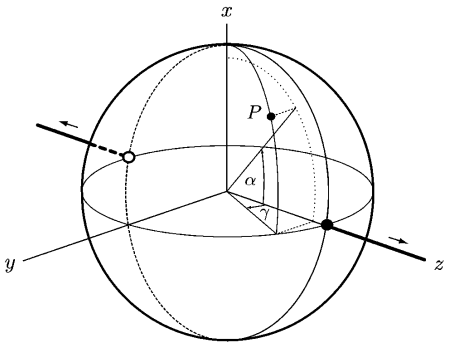

FIG. 4. Geometry of a spherical impulse expanding with the speed of light. It is generated by a snapped cosmic string, whose remnants are two semi-infinite strings located along the z axis outside the impulsive wave. Any point P on the impulse is described by two angles α and γ which characterize its projections to the (x, z) and (y, z) planes, respectively (cf. Fig. 3).

$$\begin{aligned}
a_x &= \frac{|Z_i|^\delta}{2(1-\delta)}\left[\left(1-\frac{1}{2}\delta\right)^2(Z_i^{-\delta}+\bar{Z}_i^{-\delta})+\frac{1}{4}\delta^2|Z_i|^{-2}(Z_i^{2-\delta}+\bar{Z}_i^{2-\delta})\right],\\
b_x &= \frac{|Z_i|^\delta}{2i(1-\delta)}\left[\left(1-\frac{1}{2}\delta\right)^2(Z_i^{-\delta}-\bar{Z}_i^{-\delta})+\frac{1}{4}\delta^2|Z_i|^{-2}(Z_i^{2-\delta}-\bar{Z}_i^{2-\delta})\right],\\
c_x &= \frac{\delta(1-\frac{1}{2}\delta)}{4(1-\delta)}|Z_i|^\delta[(Z_i^{-1}+\bar{Z}_i^{-1})-|Z_i|^{-2\delta}(Z_i+\bar{Z}_i)],\\
d_x &= -\frac{\delta(1-\frac{1}{2}\delta)}{4(1-\delta)}|Z_i|^\delta[(Z_i^{-1}+\bar{Z}_i^{-1})+|Z_i|^{-2\delta}(Z_i+\bar{Z}_i)],
\end{aligned} \tag{4.5}$$

$$
\begin{aligned}
a_y &= \frac{i|Z_i|^\delta}{2(1-\delta)}\left[\left(1-\frac{1}{2}\delta\right)^2(Z_i^{-\delta}-\bar{Z}_i^{-\delta})-\frac{1}{4}\delta^2|Z_i|^{-2}(Z_i^{2-\delta}-\bar{Z}_i^{2-\delta})\right],\\
b_y &= \frac{|Z_i|^\delta}{2(1-\delta)}\left[\left(1-\frac{1}{2}\delta\right)^2(Z_i^{-\delta}+\bar{Z}_i^{-\delta})-\frac{1}{4}\delta^2|Z_i|^{-2}(Z_i^{2-\delta}+\bar{Z}_i^{2-\delta})\right],\\
c_y &= \frac{i\delta(1-\frac{1}{2}\delta)}{4(1-\delta)}|Z_i|^\delta[(Z_i^{-1}-\bar{Z}_i^{-1})+|Z_i|^{-2\delta}(Z_i-\bar{Z}_i)],\\
d_y &= \frac{\delta(1-\frac{1}{2}\delta)}{4i(1-\delta)}|Z_i|^\delta[(Z_i^{-1}-\bar{Z}_i^{-1})-|Z_i|^{-2\delta}(Z_i-\bar{Z}_i)],
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
a_z &= \frac{\delta(1-\frac{1}{2}\delta)}{4(1-\delta)}|Z_i|^\delta[(Z_i^{1-\delta}+\bar{Z}_i^{1-\delta})-|Z_i|^{-2}(Z_i^{1-\delta}+\bar{Z}_i^{1-\delta})],\\
b_z &= \frac{\delta(1-\frac{1}{2}\delta)}{4i(1-\delta)}|Z_i|^\delta[(Z_i^{1-\delta}-\bar{Z}_i^{1-\delta})-|Z_i|^{-2}(Z_i^{1-\delta}-\bar{Z}_i^{1-\delta})],\\
c_z &= \frac{1}{2(1-\delta)}\left[\left(1-\frac{1}{2}\delta\right)^2(|Z_i|^\delta+|Z_i|^{-\delta})-\frac{1}{4}\delta^2(|Z_i|^{2-\delta}+|Z_i|^{\delta-2})\right],\\
d_z &= \frac{-1}{2(1-\delta)}\left[\left(1-\frac{1}{2}\delta\right)^2(|Z_i|^\delta-|Z_i|^{-\delta})+\frac{1}{4}\delta^2(|Z_i|^{2-\delta}-|Z_i|^{\delta-2})\right],
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
a_t &= -\frac{\delta(1-\frac{1}{2}\delta)}{4(1-\delta)}|Z_i|^\delta[(Z_i^{1-\delta}+\bar{Z}_i^{1-\delta})+|Z_i|^{-2}(Z_i^{1-\delta}+\bar{Z}_i^{1-\delta})],\\
b_t &= \frac{i\delta(1-\frac{1}{2}\delta)}{4(1-\delta)}|Z_i|^\delta[(Z_i^{1-\delta}-\bar{Z}_i^{1-\delta})+|Z_i|^{-2}(Z_i^{1-\delta}-\bar{Z}_i^{1-\delta})],\\
c_t &= \frac{-1}{2(1-\delta)}\left[\left(1-\frac{1}{2}\delta\right)^2(|Z_i|^\delta-|Z_i|^{-\delta})-\frac{1}{4}\delta^2(|Z_i|^{2-\delta}-|Z_i|^{\delta-2})\right],\\
d_t &= \frac{1}{2(1-\delta)}\left[\left(1-\frac{1}{2}\delta\right)^2(|Z_i|^\delta+|Z_i|^{-\delta})+\frac{1}{4}\delta^2(|Z_i|^{2-\delta}+|Z_i|^{\delta-2})\right].
\end{aligned}
\tag{4.8}
$$

Considering the structure of these relations, it is very convenient to reparametrize the complex number Z_i in the polar form as

$$
Z_i \equiv Re^{i\Phi}, \tag{4.9}
$$

where $R = |Z_i|$ and Φ are constants representing its modulus and phase, respectively. It immediately follows from the relation (4.4) that

$$
R = \left(\frac{(x_i^+)^2+(y_i^+)^2}{(t_i^++z_i^+)^2}\right)^{1/2(1-\delta)} = \left(\frac{t_i^+-z_i^+}{t_i^++z_i^+}\right)^{1/2(1-\delta)},
$$
$$
\tan((1-\delta)\Phi) = \frac{y_i^+}{x_i^+}. \tag{4.10}
$$

The junction conditions (4.3) for positions then take the form

$$
\begin{aligned}
x_i^- &= (1-\delta)\frac{\cos\Phi}{\cos((1-\delta)\Phi)}x_i^+,\\
y_i^- &= (1-\delta)\frac{\sin\Phi}{\sin((1-\delta)\Phi)}y_i^+,\\
z_i^- &= (1-\delta)\frac{R-R^{-1}}{R^{1-\delta}-R^{\delta-1}}z_i^+,\\
t_i^- &= (1-\delta)\frac{R+R^{-1}}{R^{1-\delta}+R^{\delta-1}}t_i^+,
\end{aligned}
\tag{4.11}
$$

and the coefficients (4.5), (4.6), (4.7), and (4.8) simplify to

$$
\begin{aligned}
a_x &= \frac{1}{1-\delta}\left[\left(1-\frac{1}{2}\delta\right)^2\cos(\delta\Phi)+\frac{1}{4}\delta^2\cos((2-\delta)\Phi)\right],\\
b_x &= \frac{-1}{1-\delta}\left[\left(1-\frac{1}{2}\delta\right)^2\sin(\delta\Phi)-\frac{1}{4}\delta^2\sin((2-\delta)\Phi)\right],\\
c_x &= \frac{\delta(1-\frac{1}{2}\delta)}{2(1-\delta)}(R^{\delta-1}-R^{1-\delta})\cos\Phi,\\
d_x &= -\frac{\delta(1-\frac{1}{2}\delta)}{2(1-\delta)}(R^{\delta-1}+R^{1-\delta})\cos\Phi,
\end{aligned}
\tag{4.12}
$$

$$
\begin{aligned}
a_y &= \frac{1}{1-\delta}\left[\left(1-\frac{1}{2}\delta\right)^2 \sin(\delta\Phi) + \frac{1}{4}\delta^2 \sin((2-\delta)\Phi)\right],\\
b_y &= \frac{1}{1-\delta}\left[\left(1-\frac{1}{2}\delta\right)^2 \cos(\delta\Phi) - \frac{1}{4}\delta^2 \cos((2-\delta)\Phi)\right],\\
c_y &= \frac{\delta(1-\frac{1}{2}\delta)}{2(1-\delta)}(R^{\delta-1} - R^{1-\delta})\sin\Phi,\\
d_y &= -\frac{\delta(1-\frac{1}{2}\delta)}{2(1-\delta)}(R^{\delta-1} + R^{1-\delta})\sin\Phi,
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
a_z &= \frac{\delta(1-\frac{1}{2}\delta)}{2(1-\delta)}(R-R^{-1})\cos((1-\delta)\Phi),\\
b_z &= \frac{\delta(1-\frac{1}{2}\delta)}{2(1-\delta)}(R-R^{-1})\sin((1-\delta)\Phi),\\
c_z &= \frac{1}{2(1-\delta)}\left[\left(1-\frac{1}{2}\delta\right)^2(R^{\delta}+R^{-\delta}) - \frac{1}{4}\delta^2(R^{2-\delta}+R^{\delta-2})\right],\\
d_z &= \frac{-1}{2(1-\delta)}\left[\left(1-\frac{1}{2}\delta\right)^2(R^{\delta}-R^{-\delta}) + \frac{1}{4}\delta^2(R^{2-\delta}-R^{\delta-2})\right],
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
a_t &= -\frac{\delta(1-\frac{1}{2}\delta)}{2(1-\delta)}(R+R^{-1})\cos((1-\delta)\Phi),\\
b_t &= -\frac{\delta(1-\frac{1}{2}\delta)}{2(1-\delta)}(R+R^{-1})\sin((1-\delta)\Phi),\\
c_t &= \frac{-1}{2(1-\delta)}\left[\left(1-\frac{1}{2}\delta\right)^2(R^{\delta}-R^{-\delta}) - \frac{1}{4}\delta^2(R^{2-\delta}-R^{\delta-2})\right],\\
d_t &= \frac{1}{2(1-\delta)}\left[\left(1-\frac{1}{2}\delta\right)^2(R^{\delta}+R^{-\delta}) + \frac{1}{4}\delta^2(R^{2-\delta}+R^{\delta-2})\right].
\end{aligned}
\tag{4.15}
$$

Notice finally that the terms involving R could also be conveniently expressed using the hyperbolic functions as

$$
\begin{aligned}
R - R^{-1} &= 2\sinh r,\\
R + R^{-1} &= 2\cosh r,\\
R^{\delta} - R^{-\delta} &= 2\sinh(\delta r),\\
R^{\delta} + R^{-\delta} &= 2\cosh(\delta r),\\
R^{1-\delta} - R^{\delta-1} &= 2\sinh((1-\delta)r),\\
R^{1-\delta} + R^{\delta-1} &= 2\cosh((1-\delta)r),\\
R^{2-\delta} - R^{\delta-2} &= 2\sinh((2-\delta)r),\\
R^{2-\delta} + R^{\delta-2} &= 2\cosh((2-\delta)r),
\end{aligned}
\tag{4.16}
$$

where

$$
r \equiv \log R = \frac{1}{2(1-\delta)}\log\left(\frac{t_i^+ - z_i^+}{t_i^+ + z_i^+}\right). \tag{4.17}
$$

Employing the relation $t_i^+ = \sqrt{(x_i^+)^2 + (y_i^+)^2 + (z_i^+)^2} = z_i^+\sqrt{1+\tan^2\alpha^+ + \tan^2\gamma^+}$, this can be written explicitly in terms of the initial position as

$$
r = \frac{1}{2(1-\delta)}\log\left(\frac{\sqrt{1+\tan^2\alpha^+ + \tan^2\gamma^+} - 1}{\sqrt{1+\tan^2\alpha^+ + \tan^2\gamma^+} + 1}\right). \tag{4.18}
$$

Moreover,

$$
\tan((1-\delta)\Phi) = \frac{\tan\gamma^+}{\tan\alpha^+}. \tag{4.19}
$$

The above formulas enable us to investigate behavior of arbitrary geodesics which cross the spherical impulse generated by a snapped cosmic string.

B. Analysis and description of the resulting motion

For simplicity, let us consider a family of test particles which are *at rest* in front of the impulse (i.e., in the constant-curvature region $U > 0$). Specifically, we will first assume that the velocities of the particles in the coordinates (2.3) of Minkowski, de Sitter or anti-de Sitter space vanish, $\dot{x}^+ = \dot{y}^+ = \dot{z}^+ = 0$.

Junction conditions (3.23) for the velocities across the impulse thus simplify considerably to

$$
\dot{x}_i^- = d_x\dot{t}_i^+, \qquad \dot{y}_i^- = d_y\dot{t}_i^+, \qquad \dot{z}_i^- = d_z\dot{t}_i^+, \qquad \dot{t}_i^- = d_t\dot{t}_i^+, \tag{4.20}
$$

where the constants d_x, d_y, d_z, d_t are given by (4.12), (4.13), (4.14), and (4.15), respectively. Using the definitions (3.29) and (3.30) and relations (4.11) for positions, it is straightforward to obtain the following refraction formulas:

$$
\begin{aligned}
\tan\alpha^- &= \frac{\sinh((1-\delta)r)}{\sinh r}\,\frac{\cos\Phi}{\cos((1-\delta)\Phi)}\,\tan\alpha^+,\\
\tan\beta^- &= \frac{\delta(1-\frac{1}{2}\delta)\cosh((1-\delta)r)\cos\Phi}{(1-\frac{1}{2}\delta)^2\sinh(\delta r) + \frac{1}{4}\delta^2\sinh((2-\delta)r)},
\end{aligned}
\tag{4.21}
$$

and

$$
\begin{aligned}
\tan\gamma^- &= \frac{\sinh((1-\delta)r)}{\sinh r}\,\frac{\sin\Phi}{\sin((1-\delta)\Phi)}\,\tan\gamma^+,\\
\tan\delta^- &= \frac{\delta(1-\frac{1}{2}\delta)\cosh((1-\delta)r)\sin\Phi}{(1-\frac{1}{2}\delta)^2\sinh(\delta r) + \frac{1}{4}\delta^2\sinh((2-\delta)r)}.
\end{aligned}
\tag{4.22}
$$

Because of the axial symmetry of the spacetime along the z axis (where the string is located in front of the impulse), it is natural to restrict attention to a *ring of test particles located in the* (x^+, z^+) plane, i.e., assuming $y^+ = 0$. From (3.30) it follows that $\gamma^+ = 0$ and, using (4.19), this implies $\Phi = 0$. Consequently, Eqs. (4.22) reduce to $\gamma^- = 0 = \delta^-$. It follows that $y_i^- = 0 = \dot{y}_i^-$, and motion of such particles will thus remain in the (x^-, z^-) plane behind the impulse. Relations (4.21), which describe the motion in the (x, z) plane, now reduce to

$$\tan\alpha^- = \frac{\sinh((1-\delta)r)}{\sinh r}\tan\alpha^+,$$
$$\tan\beta^- = \frac{\delta(1-\frac{1}{2}\delta)\cosh((1-\delta)r)}{(1-\frac{1}{2}\delta)^2\sinh(\delta r)+\frac{1}{4}\delta^2\sinh((2-\delta)r)}, \tag{4.23}$$

where the parameter r is given by Eq. (4.18). Because $\gamma^+ = 0$, this further simplifies to

$$r = \frac{1}{1-\delta}\log\left(\tan\frac{\alpha^+}{2}\right). \tag{4.24}$$

It is now possible to visualize the effect of the impulse generated by a snapped cosmic string on such a ring of test particles by plotting the corresponding graphs. In Figs. 5 and 6 we draw the functions $\alpha^-(\alpha^+)$ and $\beta^-(\alpha^+)$, respectively, which are given by (4.23) with (4.24), for several discrete values of the parameter δ. The geometrical meaning of these angles is described in Figs. 3 and 4. The angle α^+ parametrizes position of a particle of the ring in front of the impulse, while α^- and β^- determine, respectively, its position and velocity vector inclination behind the impulse.

Combining these two relations, we plot in Fig. 7 the motion of the (initially static) ring caused by the impulse. It can be seen that the particles are displaced towards the string, and directions of their velocities are oriented "along" the string. Particles located close to the string in front of the impulse are accelerated almost to the speed of light behind the impulse, and are "dragged" along the string (except those in the perpendicular plane $z = 0$ corresponding to $\alpha^+ = \frac{\pi}{2}$). In Fig. 8 we plot the magnitude $v^- = \sqrt{(v_x^-)^2 + (v_z^-)^2}$ of the resulting velocity vector as a function of α^+ for several values of the parameter δ.

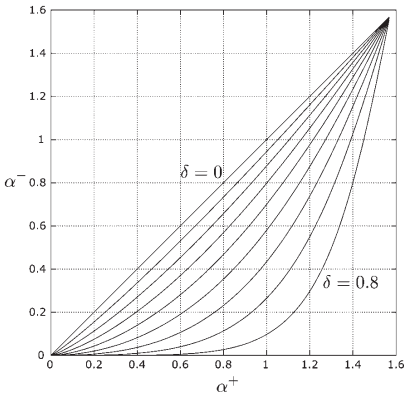

FIG. 5. The function $\alpha^-(\alpha^+)$ which determines the displacement of the position of a particle when it crosses the impulse generated by a snapped cosmic string. The curves correspond to different values of the deficit angle parameter $\delta = 0, 0.1, 0.2, \ldots, 0.8$.

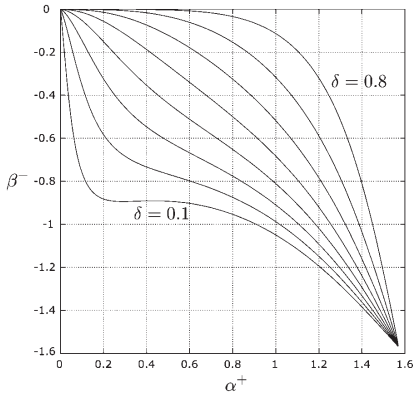

FIG. 6. The function $\beta^-(\alpha^+)$ which determines the dependence of the velocity vector inclination behind the impulse on the particle's position in front of the impulse. The curves plotted correspond to $\delta = 0.1, 0.2, \ldots, 0.8$.

Indeed, for small values of the angle α^+ the speed approaches that of light, $v^- \to 1$. The components $v_x^- = (\dot{x}_i^-/\dot{t}_i^-) = d_x/d_t$ and $v_z^- = (\dot{z}_i^-/\dot{t}_i^-) = d_z/d_t$ are separately drawn in Fig. 9. Since $v_x^-(0) = 0$, $v_z^-(0) = 1$ for any $\delta > 0$, the particles close to the string are accelerated "parallelly" along it. For $\delta \to 0$, $v_z^-(\alpha^+)$ becomes zero everywhere except at $\alpha^+ = 0$ where the string is located. Also, $v_z^-(\frac{\pi}{2}) = 0$ which means that the particles in the transverse plane $z = 0$ are accelerated "perpendicularly" and they thus stay in this plane, which is consistent with the symmetry of the system. The velocity vectors corresponding to such components are indicated in Fig. 7 by arrows.

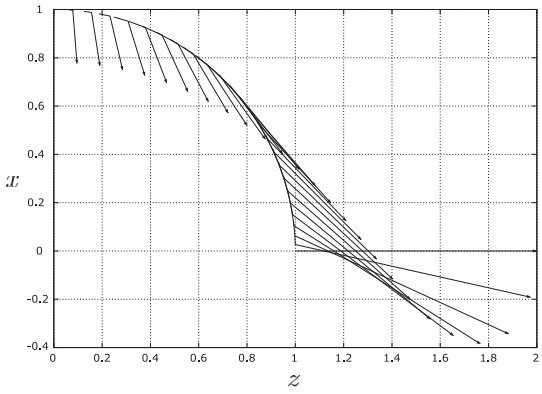

FIG. 7. The effect of the impulse with $\delta = 0.2$ on a ring of initially static test particles in the (x, z) plane. The particles are shifted and they start to move, as indicated by their velocity vectors with components (v_x^-, v_z^-) behind the impulse. The impulse is scaled here in such a way that it is given by a unit sphere on both sides of the impulse.

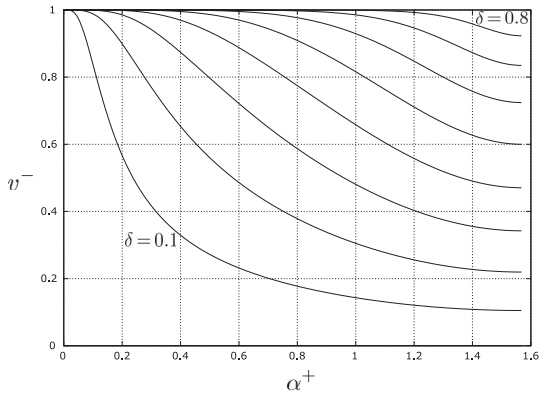

FIG. 8. The magnitude v^- of the velocity vector behind the impulse as a function of the particle's initial position α^+. The curves plotted correspond to different values of the parameter $\delta = 0.1, 0.2, \ldots, 0.8$.

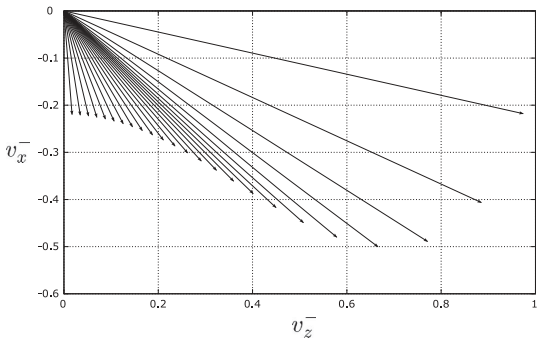

FIG. 10. The velocity vectors for $\delta = 0.2$ plotted as a function of the initial position α^+ of the particle in the ring.

To compare these velocity vectors for different values of the initial position α^+, we plot them in Fig. 10 from the common origin. The end points of these arrows for all $\alpha^+ \in [0, \frac{\pi}{2}]$ form a smooth curve, which is drawn in Fig. 11 for several discrete values of the conicity parameter δ. For small δ there is a single minimum in such curves, while for large values of δ the curves approach a unit circle since the particles are accelerated by the impulse almost to the speed of light in all directions.

Finally, in Figs. 12 and 13 we visualize the deformation of the ring of test particles, initially at rest, as it evolves with time. It can be concluded that the circle [which may be considered as a (x, z) section through a sphere] is deformed by the gravitational impulse into an axially symmetric pinched surface, elongated and expanding along the moving strings in the positive z direction. Also, the particles which initially started at $x > 0$ have $v_x^- < 0$, while those with $x < 0$ have $v_x^- > 0$. This explicitly demonstrates the "dragging" effect in such spacetimes caused by the moving strings and the corresponding impulse. With a growing value of the parameter δ, the deformation in the z direction is bigger.

In the complementary case, in which the *ring of static test particles is located in the* (x^+, y^+) plane perpendicular to the string (see Fig. 4), $z_i^+ = 0$ which corresponds to $\alpha^+ = \frac{\pi}{2}$. It thus follows from (4.17) that $r = 0$, i.e., $R = 1$. In such a case, the explicit junction conditions for positions simplify to

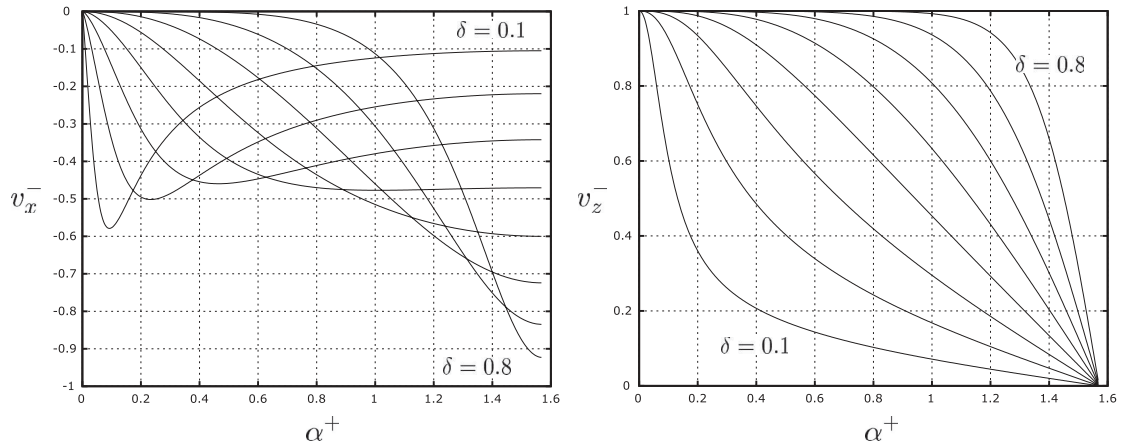

FIG. 9. The components v_x^- (left) and v_z^- (right) of the velocity vector behind the impulse as a function of initial position α^+. The curves correspond to $\delta = 0.1, 0.2, \ldots, 0.8$.

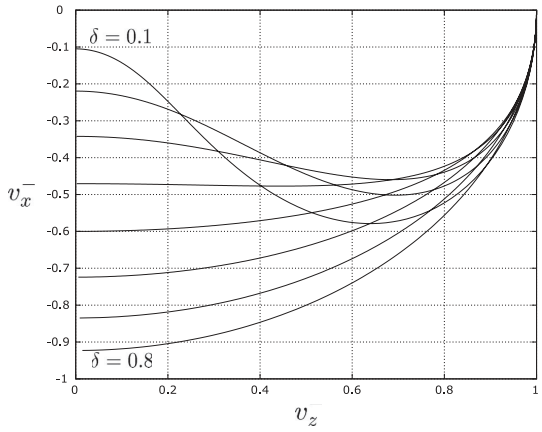

FIG. 11. Envelope of the velocity vectors for all $\alpha^+ \in [0, \frac{\pi}{2}]$, plotted for $\delta = 0.1, 0.2, \ldots, 0.8$.

$$
\begin{aligned}
x_i^- &= (1-\delta)\frac{\cos\Phi}{\cos((1-\delta)\Phi)}x_i^+, \\
y_i^- &= (1-\delta)\frac{\sin\Phi}{\sin((1-\delta)\Phi)}y_i^+, \\
z_i^- &= 0, \qquad t_i^- = (1-\delta)t_i^+,
\end{aligned}
\tag{4.25}
$$

and the coefficients in Eqs. (4.20) relating the velocities on both sides of the impulse become

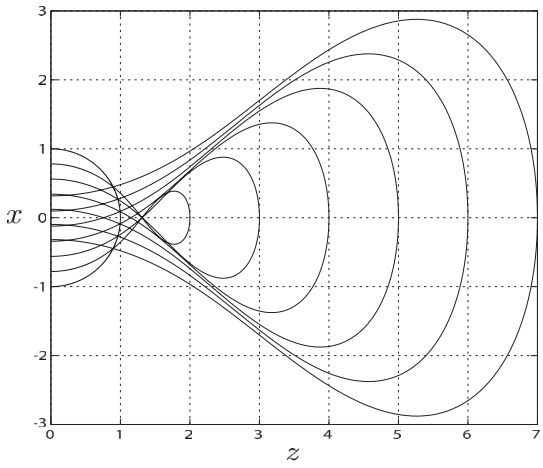

FIG. 12. Time sequence showing the deformation of the ring of test particles (indicated here by an initial semicircle of unit radius for $\alpha^+ \in [-\frac{\pi}{2}, \frac{\pi}{2}]$) caused by the spherical impulse generated by a snapping cosmic string with $\delta = 0.2$.

$$
d_x = -\delta\frac{1-\frac{1}{2}\delta}{1-\delta}\cos\Phi, \qquad d_y = -\delta\frac{1-\frac{1}{2}\delta}{1-\delta}\sin\Phi,
$$
$$
d_z = 0, \qquad d_t = \frac{1-\delta+\frac{1}{2}\delta^2}{1-\delta}. \tag{4.26}
$$

Since $\dot{z}_i^- = 0 = z_i^-$, motion of the particles will remain in the perpendicular (x^-, y^-) plane behind the impulse. In fact,

$$
\begin{aligned}
v^- &= \sqrt{(v_x^-)^2 + (v_y^-)^2} = \delta\frac{1-\frac{1}{2}\delta}{1-\delta+\frac{1}{2}\delta^2}, \\
\frac{v_y^-}{v_x^-} &= \tan\Phi.
\end{aligned}
\tag{4.27}
$$

This implies geometrically that all the particles will move radially inward with the same speed, and the circular ring will thus uniformly contract. This is in full agreement with the corresponding partial result obtained previously in Sec. IV B of [24].

For more general situations, in which the test particles in front of the impulse are not static in the x, y, z coordinates, the resulting motion can similarly be investigated using the relations (4.11) and (3.23). In particular, employing (4.12), (4.13), (4.14), (4.15), and (4.16) for the case when $\Phi = 0$, we obtain

$$
\begin{aligned}
x_i^- &= (1-\delta)x_i^+, \qquad y_i^- = 0 = y_i^+, \\
z_i^- &= (1-\delta)\frac{\sinh r}{\sinh((1-\delta)r)}z_i^+, \\
t_i^- &= (1-\delta)\frac{\cosh r}{\cosh((1-\delta)r)}t_i^+,
\end{aligned}
\tag{4.28}
$$

and

$$
\begin{aligned}
\dot{x}_i^- &= \frac{1-\delta+\frac{1}{2}\delta^2}{1-\delta}\dot{x}_i^+ - \delta\frac{1-\frac{1}{2}\delta}{1-\delta}[\sinh((1-\delta)r)\dot{z}_i^+ \\
&\quad + \cosh((1-\delta)r)\dot{t}_i^+], \\
\dot{y}_i^- &= \dot{y}_i^+, \\
\dot{z}_i^- &= \delta\frac{1-\frac{1}{2}\delta}{1-\delta}\sinh r\,\dot{x}_i^+ + \frac{(1-\frac{1}{2}\delta)^2}{1-\delta}[\cosh(\delta r)\dot{z}_i^+ \\
&\quad - \sinh(\delta r)\dot{t}_i^+] - \frac{\frac{1}{4}\delta^2}{1-\delta}[\cosh((2-\delta)r)\dot{z}_i^+ \\
&\quad + \sinh((2-\delta)r)\dot{t}_i^+], \\
\dot{t}_i^- &= -\delta\frac{1-\frac{1}{2}\delta}{1-\delta}\cosh r\,\dot{x}_i^+ + \frac{(1-\frac{1}{2}\delta)^2}{1-\delta}[-\sinh(\delta r)\dot{z}_i^+ \\
&\quad + \cosh(\delta r)\dot{t}_i^+] + \frac{\frac{1}{4}\delta^2}{1-\delta}[\sinh((2-\delta)r)\dot{z}_i^+ \\
&\quad + \cosh((2-\delta)r)\dot{t}_i^+],
\end{aligned}
\tag{4.29}
$$

where r is given by (4.24).

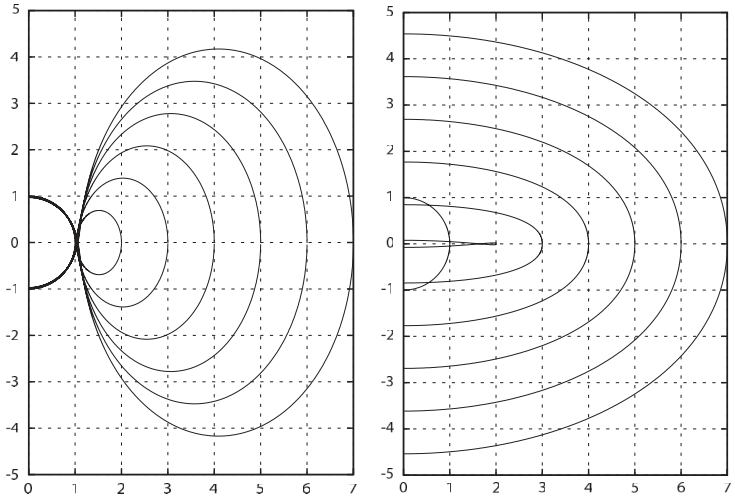

FIG. 13. Deformation of the ring of particles, as in Fig. 12, for $\delta = 0.005$ (left) and $\delta = 0.8$ (right).

C. Effect on particles comoving in de Sitter space

Finally, it will be illustrative to investigate the effect of the impulsive spherical wave generated by a snapped cosmic string on test particles which are comoving in the de Sitter (half)space in front of the impulse. Specifically, these particles are initially given by

$$\chi = \chi_i^+ = \chi_0, \qquad \theta = \theta_i^+ = \theta_0, \qquad \phi = \phi_i^+ = \phi_0, \tag{4.30}$$

where χ_0, θ_0, ϕ_0 are constants, in the coordinates which naturally cover the de Sitter universe in the standard form of the metric

$$ds_0^2 = -dt^2 + a^2\cosh^2\frac{t}{a} \times (d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)). \tag{4.31}$$

Such a parametrization of the de Sitter hyperboloid (2.4) is obtained by

$$\begin{aligned} \mathrm{Z}_0 &= a\sinh\frac{t}{a}, \\ \mathrm{Z}_1 &= a\cosh\frac{t}{a}\sin\chi\cos\theta, \\ \mathrm{Z}_2 &= a\cosh\frac{t}{a}\sin\chi\sin\theta\cos\phi, \\ \mathrm{Z}_3 &= a\cosh\frac{t}{a}\sin\chi\sin\theta\sin\phi, \\ \mathrm{Z}_4 &= a\cosh\frac{t}{a}\cos\chi, \end{aligned} \tag{4.32}$$

where $t \in (-\infty, +\infty)$, χ, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$. Inversely,

$$\begin{aligned} &\sinh\frac{t}{a} = \frac{\mathrm{Z}_0}{a}, \qquad \tan^2\chi = \frac{\mathrm{Z}_1^2 + \mathrm{Z}_2^2 + \mathrm{Z}_3^2}{\mathrm{Z}_4^2}, \\ &\tan^2\theta = \frac{\mathrm{Z}_2^2 + \mathrm{Z}_3^2}{\mathrm{Z}_1^2}, \qquad \tan\phi = \frac{\mathrm{Z}_3}{\mathrm{Z}_2}. \end{aligned} \tag{4.33}$$

The expanding impulse is located at $\mathrm{Z}_4 = a$ (see Fig. 1), i.e., it is given by $\cosh(t/a) = 1/\cos\chi$ which can be rewritten as

$$\tanh\frac{t}{a} = \sin\chi. \tag{4.34}$$

The snapped cosmic string is located at $\mathrm{Z}_2 = 0 = \mathrm{Z}_3$ in the de Sitter region in front of the impulse, which corresponds to $\theta^+ = 0, \pi$. The spacetime can thus be visualized as in Fig. 14.

Notice that the impulse is always located at the fixed value $\mathrm{Z}_4 = a$ but, as the spherical de Sitter universe expands, the impulse propagates from its north pole $\chi = 0$ at $t = 0$ to its equator $\chi = \frac{\pi}{2}$ as $t \to \infty$. The cosmic string was initially a closed loop around the whole meridian $\theta = 0, \pi$, but it snapped in the north pole at $t = 0$ (when the universe had the minimum radius a) generating the impulsive gravitational wave.

The convenient form of the junction conditions for geodesics is given in the five-dimensional representation by Eqs. (3.47) and (3.51). Using (4.4), (3.46), (4.32), and (4.34) we obtain a simple expression for the complex interaction parameter

$$Z_i^{1-\delta} = \tan\frac{\theta_0}{2}e^{i\phi_0} \tag{4.35}$$

(notice that this is consistent with the stereographic interpretation shown in Fig. 2). In view of (4.9) we thus obtain

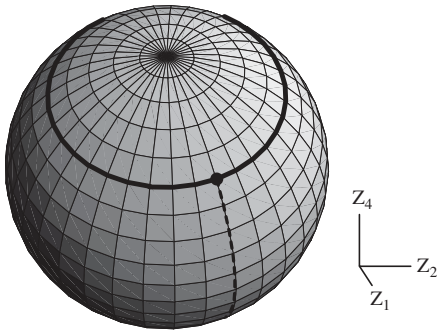

FIG. 14. The de Sitter universe with the snapped cosmic string (indicated by a dashed line at $Z_2 = 0 = Z_3$) and the related impulse ($Z_4 = a$) at a given time (the coordinate Z_3 is suppressed). As the universe expands, the impulse propagates from the north pole to the equator.

$$r = \frac{1}{1-\delta}\log\left(\tan\frac{\theta_0}{2}\right), \qquad \Phi = \frac{\phi_0}{1-\delta}. \tag{4.36}$$

In terms of these initial data we may rewrite (3.47) and (3.51), employing (4.16), explicitly as

$$\begin{aligned}
Z^-_{0i} &= (1-\delta)\frac{\cosh r}{\cosh((1-\delta)r)}Z^+_{0i},\\
Z^-_{1i} &= (1-\delta)\frac{\sinh r}{\sinh((1-\delta)r)}Z^+_{4i},\\
Z^-_{2i} &= (1-\delta)\frac{\cos\Phi}{\cos((1-\delta)\Phi)}Z^+_{2i},\\
Z^-_{3i} &= (1-\delta)\frac{\sin\Phi}{\sin((1-\delta)\Phi)}Z^+_{3i},\\
Z^-_{4i} &= a = Z^+_{4i}.
\end{aligned} \tag{4.37}$$

and

$$\begin{pmatrix} \dot{Z}^-_{2i}\\ \dot{Z}^-_{3i}\\ \dot{Z}^-_{1i}\\ \dot{Z}^-_{0i}\\ \dot{Z}^-_{4i}\end{pmatrix} = \begin{pmatrix} a_x & b_x & c_x & d_x & n_x\\ a_y & b_y & c_y & d_y & n_y\\ a_z & b_z & c_z & d_z & n_z\\ a_t & b_t & c_t & d_t & n_t\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix} \dot{Z}^+_{2i}\\ \dot{Z}^+_{3i}\\ \dot{Z}^+_{1i}\\ \dot{Z}^+_{0i}\\ \dot{Z}^+_{4i}\end{pmatrix}, \tag{4.38}$$

where the constant coefficients a_p, b_p, c_p, d_p are given by (4.12), (4.13), (4.14), (4.15), and (4.16) and n_p is determined by expression (3.50). It follows from (4.32) and (4.34) that

$$\begin{aligned}
Z^+_{0i} &= a\tan\chi_0,\\
Z^+_{1i} &= a\tan\chi_0\cos\theta_0,\\
Z^+_{2i} &= a\tan\chi_0\sin\theta_0\cos\phi_0,\\
Z^+_{3i} &= a\tan\chi_0\sin\theta_0\sin\phi_0.
\end{aligned} \tag{4.39}$$

Similarly, by differentiating (4.32) with respect to the proper time $\tau = t$ of a comoving particle we obtain

$$\begin{aligned}
\dot{Z}^+_{0i} &= \frac{1}{\cos\chi_0},\\
\dot{Z}^+_{1i} &= \frac{\sin^2\chi_0}{\cos\chi_0}\cos\theta_0,\\
\dot{Z}^+_{2i} &= \frac{\sin^2\chi_0}{\cos\chi_0}\sin\theta_0\cos\phi_0,\\
\dot{Z}^+_{3i} &= \frac{\sin^2\chi_0}{\cos\chi_0}\sin\theta_0\sin\phi_0,\\
\dot{Z}^+_{4i} &= \sin\chi_0.
\end{aligned} \tag{4.40}$$

These parameters explicitly satisfy the constraints (3.6), (3.7), and (3.8) for a timelike geodesic in de Sitter space ($e = -1$, $\varepsilon = 1$).

We can thus visualize the effect of the impulse on initially comoving particles in a de Sitter universe in the "five-dimensional" pictures shown in Figs. 15 and 16, where we plot the corresponding velocity vectors (with the spherical space, impulse and the snapped string as in Fig. 14).

In Fig. 15 the arrows indicate the velocities of different test particles, given by (4.38) and (4.40) with the same values of χ_0 (and ϕ_0 suppressed), behind the impulsive wave. The outer semicircle indicates the position of the same particles if the impulse would be absent—they would (comovingly) move because the de Sitter universe itself expands. Therefore, the *difference* gives the "net" effect of the impulse on these particles (by subtracting a natural comoving motion due to the global expansion of the uni-

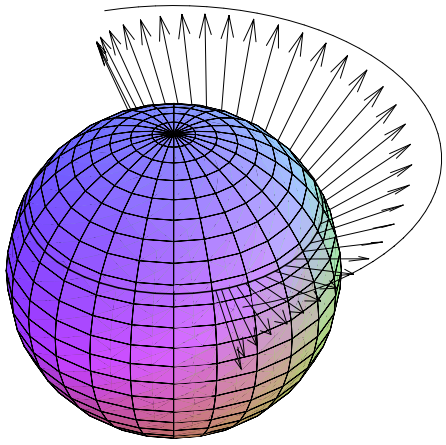

FIG. 15 (color online). The de Sitter universe with the snapped string and the impulsive wave, at a given time. The arrows indicate the velocities of different test particles behind the impulse. The outer semicircle locates the same comoving particles at a later time if the impulse would be absent, i.e., if they would move solely due to the expansion of the universe.

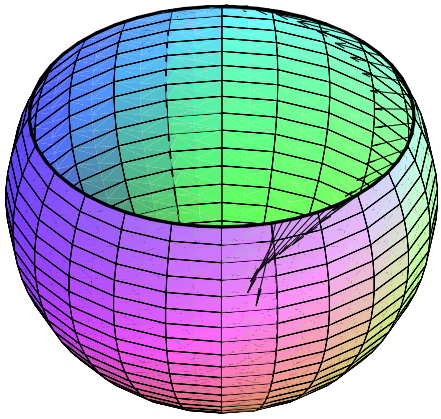

FIG. 16 (color online). The net effect of the impulse on the test particles, obtained by subtracting a comoving motion due to expansion of the universe. The particles are accelerated and dragged along the string.

verse). This is shown in Fig. 16. It can be seen that the particles close to the string are accelerated to higher speeds and are dragged along the string, while the particles in the transverse plane are accelerated perpendicularly to the string. In fact, Fig. 7 can be understood as a projection onto the horizontal section $Z_4 = a$ through Fig. 16.

V. CONCLUSIONS

We presented a complete and explicit solution of geodesic motion which describes the effect of expanding spherical impulsive gravitational waves propagating in constant-curvature backgrounds, provided the trajectories of test particles are of class C^1 in a continuous coordinate system. This generalizes results obtained previously for Minkowski background space [24] to any value of the cosmological constant, i.e., the de Sitter universe ($\Lambda > 0$) or anti-de Sitter universe ($\Lambda < 0$). Also, it is a counterpart of paper [25] in which motion of test particles in these background spaces with nonexpanding impulses was analyzed.

We derived a convenient form of the junction conditions (3.22), (3.23), and (3.24) and the corresponding refraction formulas (3.32) and (3.33), employing the natural coordinates in which the background metric (2.3) is conformally flat. Interestingly, the expressions are independent of the parameter $\epsilon = -1, 0, +1$ which occurs in the continuous metric (2.16) for the impulsive-wave spacetimes. We also considered the five-dimensional formalism which is suitable when $\Lambda \neq 0$, see Eqs. (3.47) and (3.51).

Subsequently, we discussed in detail the behavior of test particles in axially symmetric spacetimes in which the gravitational impulse is generated by a snapped cosmic string. In particular, we demonstrated that the particles are dominantly dragged by the impulse in the direction of the moving strings, and are accelerated to ultrarelativistic speeds in their vicinity, see Figs. 7–9. These results apply to any value of the cosmological constant. The strings and the associated impulse would thus effectively create opposite ''beams'' of particles, dominantly moving along the strings with the speed close to the speed of light, as visualized in Figs. 12 and 13.

ACKNOWLEDGMENTS

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CHAPTER

3

GEODESIC DEVIATION IN HIGHER DIMENSIONS

The main purpose of this chapter is to investigate the general properties of relative motions of free test particles in spacetimes of an arbitrary dimension. The obtained results are presented in the form of the paper *Interpreting spacetimes of any dimension using geodesic deviation* which was published in *Physical Review D* in *2012*. Full citation of this paper is [22].

Due to a freedom in the choice of particular coordinate system in general relativity (and also in other covariant theories) it is quite difficult to distinguish net effects of the gravitational field from those given by unsuitable choice of the coordinates. An important tool which can help us to analyze the physical properties of a given spacetime is a systematic study of relative accelerations between freely falling nearby test particles. Their relative motion is described by the equation of geodesic deviation (sometimes also called the Jacobi equation), see e.g. [3], which was generally analyzed in standard four-dimensional relativity by Szekeres [23] and many others. Rewriting the Szekeres results in a more convenient form using Newman–Penrose coefficients, see [24, 25] gives a direct connection between algebraic type of the spacetime and relative behaviour of free test particles.

Also the equation of geodesic deviation posses a tool how to directly measure the curvature of the spacetime since, e.g., the interferometric detectors of gravitational waves are based on precise measurements of relative changes of the test bodies positions.

Motivated by a growing interest in exact solutions of Einstein's equations in higher-dimensions in recent years, see [22] for references, we extend here the description of relative motion to an arbitrary spacetime in any dimension, and analyze the general behaviour of geodesic congruences with respect to the algebraic structure of a given spacetime. Such algebraic classification of higher-dimensional Lorentzian manifolds was developed by Coley, Milson, Pravda and Pravdová [26]. We use a fully equivalent notation introduced by Krtouš and Podolský in [27] because it is closer to those standardly used in four-dimensions.

To obtain invariant results we express the equation of geodesic deviation in an orthonormal frame connected to an observer. The components of Riemann curvature tensor are expressed by a traceless Weyl tensor representing a free gravitational field, specific combinations of energy–momentum tensor and its trace describing the matter content of a given spacetime. We rewrite the orthonormal frame components of the Weyl tensor in terms of the null fame aligned to the algebraic

structure of the spacetime. In correspondence with four-dimensional case we find that the overall effect of general gravitation field consists of purely transverse deformations, i.e., gravitational waves (in general propagating in opposite spatial directions), longitudinal effects, Newton-like deformations, isotropic influence of a cosmological constant, and effects connected with particular matter content of the universe. We also describe the dependence of our results on a particular observer, corresponding to a freedom in the choice of the frame given by Lorentz transformations.

The utility of the presented approach is demonstrated in our paper [22] by applying it on the case of pp-waves. More general and explicit examples will follow in this thesis, namely discussion of general nontwisting solutions of Einstein's equations in Chapter 4, nontwisting, nonexpanding and shearfree Kundt solutions in Chapter 5, and expanding Robinson–Trautman spacetimes in Chapter 6. For the transverse gravitational waves we found a more complex behaviour than in four spacetime dimensions.

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Interpreting spacetimes of any dimension using geodesic deviation

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We present a general method that can be used for geometrical and physical interpretation of an arbitrary spacetime in four or any higher number of dimensions. It is based on the systematic analysis of relative motion of free test particles. We demonstrate that the local effect of the gravitational field on particles, as described by the equation of geodesic deviation with respect to a natural orthonormal frame, can always be decomposed into a canonical set of transverse, longitudinal and Newton–Coulomb-type components, isotropic influence of a cosmological constant, and contributions arising from specific matter content of the Universe. In particular, exact gravitational waves in Einstein's theory always exhibit themselves via purely transverse effects with $D(D-3)/2$ independent polarization states. To illustrate the utility of this approach, we study the family of pp-wave spacetimes in higher dimensions and discuss specific measurable effects on a detector located in four spacetime dimensions. For example, the corresponding deformations caused by generic higher-dimensional gravitational waves observed in such physical subspace need not be trace-free.

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I. INTRODUCTION

In the last decade, there has been a growing interest in exact spacetimes within the context of higher-dimensional general relativity, primarily motivated by finding particular models for string theories, AdS/CFT correspondence, and brane-world cosmology. Such investigations thus concentrated mainly on various types of black holes and black rings, see [1–8] for reviews and further references. More general static or stationary axisymmetric [9–15], multi-black hole Majumdar–Papapetrou-type [16–23], and static solutions with cylindrical/toroidal symmetry [24–28] were also considered, including uniform and nonuniform black strings [29–36] with the aim to elucidate their instability [37–39]. Other important classes of higher-dimensional exact solutions of Einstein's equations have also been studied recently, for example, Robinson-Trautman and Kerr-Schild spacetimes [40–45], extensions of the Bertotti-Robinson, (anti-)Nariai, and Plebański-Hacyan universes [46], higher-dimensional Friedmann-type [47–51] and multidimensional cosmological models [20,52] (see also references therein), specific solitons [24,53,54], or various exact gravitational waves—in particular those that belong to nonexpanding Kundt family [55,56], namely, generalized pp-waves [57–63] (for a study of their collisions see [64]), vanishing scalar invariant (VSI) [62,63], and constant scalar invariant [65] spacetimes, or relativistic gyratons [66–71].

Fundamental general questions concerning the classification of higher-dimensional manifolds based on the algebraic structure of the curvature tensor have been clarified [72–75], including generalizations of the Newman-Penrose and the Geroch-Held-Penrose formalisms [76–80]. This paved the way for a systematic study of wide classes of algebraically special spacetimes in higher dimensions [81–84]. Investigation of asymptotic behavior of the corresponding fields and their global structure, in particular, properties of gravitational radiation, has also been initiated [85–99].

Nevertheless, in spite of the considerable effort devoted to this topic, there are still important aspects concerning the nature of gravitational fields in higher-dimensional gravity that remain open. Any sufficiently general method that could be used to probe geometrical and physical properties of a given spacetime would be useful. In the present work, we suggest and develop such an approach, which is based on investigation and classification of specific effects of gravity encoded in relative motion of nearby test particles.

In fact, in standard four-dimensional general relativity, this has long been used as an important tool for studies of spacetimes. Relative motion of close free particles helps us to clarify the structure of a gravitational field in which the test particles move. When they have no charge and spin, this is mathematically described by the equation of geodesic deviation (sometimes also called the Jacobi equation), which was first derived in the n-dimensional (pseudo-)Riemannian geometry by Levi-Civita and Synge [100–103], see [104] for the historical account. Shortly after its application to Einstein's gravity theory [105–114], it helped, for instance, to understand the behavior of test bodies influenced by gravitational waves or the physical fate of observers falling into black holes. Textbook descriptions of this equation, which is linear with respect to the separation vector connecting the

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test particles, are given, e.g., in [115–118]. Let us also mention that generalizations of the equation of geodesic deviation to admit arbitrary relative velocities of the particles were obtained in the works [119–127]. Further extensions, higher-order corrections to the geodesic deviation equation, their particular applications, and references can be found in the recent papers [127–134] and in the monograph [118].

In 1965, Szekeres [114] presented an elegant analysis of the behavior of nearby test particles in a *generic* four-dimensional spacetime. He demonstrated that the overall effect consists of specific transverse, longitudinal, and Newton–Coulomb-type components. This was achieved by decomposing the Riemann curvature tensor into the Weyl tensor and the terms involving the Ricci tensor (and Ricci scalar). While the former represents the "free gravitational field," the latter can be explicitly expressed, employing Einstein's field equations, in terms of the corresponding components of the energy-momentum tensor, which describes the matter content. In order to further analyze the Weyl tensor contribution, Szekeres used the formalism of self-dual bivectors [135,136] constructed from null frames. This enabled him to deduce the effects of gravitational fields on nearby test particles in spacetimes of various Petrov types. When these results are reexpressed in a more convenient Newman-Penrose formalism [137,138], explicit physical interpretation of the corresponding complex scalars Ψ_A is obtained. In particular, the Weyl scalar Ψ_4 (the only nontrivial component in type-N spacetimes) represents a purely transverse effect of exact gravitational waves, the scalar Ψ_3 (present, e.g., in type-III spacetimes) is responsible for longitudinal effects, and Ψ_2 (typical for spacetimes of type D) gives rise to Newton-like deformations of the family of test particles (see [139–141] for more details; inclusion of a nonvanishing cosmological constant was described in [128]).

It is the purpose of the present work to extend these results to arbitrary spacetimes in *any* dimension $D \geq 4$. The paper is organized as follows. In Sec. II, we recall the equation of geodesic deviation, including its invariant form with respect to the interpretation orthonormal frame adapted to an observer. In Sec. III, we perform the canonical decomposition of the curvature tensor using Einstein's equations and the real Weyl tensor components $\Psi_{A\ldots}$ with respect to an associated null frame. We thus derive an explicit and general form of the equation of geodesic deviation. Section IV analyses the character of all canonical components of a gravitational field. Section V is devoted to the discussion of uniqueness of the interpretation frame, and derivation of explicit relations that give the dependence of the field components on the observer's velocity. In Sec. VI, we describe the effect of pure radiation, perfect fluid and electromagnetic field on test particles. Final Sec. VII illustrates the method on the family of pp-waves in higher dimensions. There are also 3 Appendices: In Appendix A, we give relations to the standard complex formalism of $D = 4$ general relativity, and in Appendix B we summarize alternative notations commonly used in literature on $D \geq 4$ spacetimes. Finally, in Appendix C the Lorentz transformations of the $\Psi_{A\ldots}$ scalars are presented.

II. EQUATION OF GEODESIC DEVIATION

The main objective of the present work is to investigate and characterize the curvature of an arbitrary spacetime of dimension $D \geq 4$ by its local effects on freely falling test particles (observers). The gravitational field manifests itself, in Newtonian terminology, as specific "tidal forces" that cause the nearby particles to accelerate relative to each other. This leads to a deviation of corresponding geodesics whose separation thus changes with time: in various spatial directions the particles approach or recede from themselves, exhibiting thus the specific character of the spacetime in the vicinity of a given event.

In standard and also higher-dimensional general relativity, such a behavior of free test particles (without charge and spin) is described by the geodesic deviation equation [100–118]

$$\frac{\mathrm{D}^2 Z^\mu}{\mathrm{d}\tau^2} = R^\mu{}_{\alpha\beta\nu} u^\alpha u^\beta Z^\nu, \tag{1}$$

where $R^\mu{}_{\alpha\beta\nu}$ are components of the Riemann curvature tensor, u^α are components of the velocity vector $\boldsymbol{u} = u^\alpha \boldsymbol{\partial}_\alpha$ of the reference (fiducial) particle moving along a timelike geodesic $\gamma(\tau) \equiv \{x^0(\tau), \ldots, x^{D-1}(\tau)\}$, $u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}$, the parameter τ is its proper time (so that $\boldsymbol{u} \cdot \boldsymbol{u} \equiv g_{\alpha\beta} u^\alpha u^\beta = -1$), and Z^μ are components of the separation vector $\boldsymbol{Z} = Z^\mu \boldsymbol{\partial}_\mu$, which connects the reference particle with another nearby test particle moving along a timelike geodesic $\bar{\gamma}(\tau)$. The situation is visualized in Fig. 1.

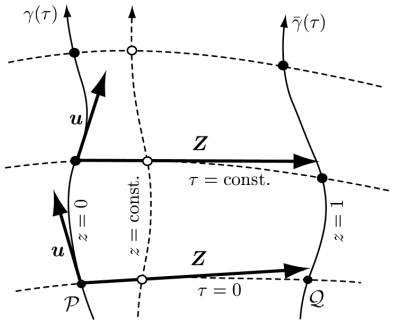

FIG. 1. In a curved D-dimensional spacetime, nearby test particles moving along geodesics accelerate toward or away from each other, as given by the equation of geodesic deviation (1). Here, $\boldsymbol{u}$ is the velocity vector of a reference particle, and $\boldsymbol{Z}$ is the separation vector that represents actual relative position of the second test particle at a given proper time τ.

Equation (1) explicitly expresses the relative acceleration of two nearby particles by the second absolute (covariant) derivative of the vector field $\boldsymbol{Z}$ along $\gamma(\tau)$,

$$\frac{\mathrm{D}^2 Z^\mu}{\mathrm{d}\tau^2} = (Z^\mu{}_{;\gamma} u^\gamma)_{;\delta} u^\delta = Z^\mu{}_{;\gamma\delta} u^\gamma u^\delta, \tag{2}$$

in terms of the local curvature tensor and the actual relative position of the particles, described by the separation vector $\boldsymbol{Z}(\tau)$ at the time τ.

To be geometrically more precise, the two geodesics should be understood as specific representatives of a congruence $\gamma(\tau, z)$, i.e., smooth one-parameter family of geodesics, such that $\gamma(\tau) \equiv \gamma(\tau, z = 0)$ and $\bar{\gamma}(\tau) \equiv \gamma(\tau, z = \text{const})$. The proper time τ and the parameter z, which labels the geodesics, can be chosen as coordinates on the submanifold spanned by the congruence. Thus, $\boldsymbol{u} = \boldsymbol{\partial}_\tau$ and $\boldsymbol{Z} = \boldsymbol{\partial}_z$, and the deviation vector field $\boldsymbol{Z}$ is Lie-transported along the geodesics generated by $\boldsymbol{u}$. Consider now the positions of two test particles at a given time, for example, $\mathcal{P}$ (located at $z = 0$) and $\mathcal{Q}$ (for which $z = 1$, say) at $\tau = 0$, as shown in Fig. 1. Their coordinates are related by the exponential map $x^\mu_{\mathcal{Q}} = \exp(z\boldsymbol{Z}) x^\mu_{\mathcal{P}}$ generated by $\boldsymbol{Z}$ at $\mathcal{P}$, where we set $z = 1$ to locate $\mathcal{Q}$. If the higher-order terms are negligible, this expression reduces to $x^\mu_{\mathcal{Q}} - x^\mu_{\mathcal{P}} \approx (\boldsymbol{Z} x^\mu)_{\mathcal{P}}$, demonstrating that the separation vector $\boldsymbol{Z}$ describes the relative position of the two test particles, and $\boldsymbol{Z}(\tau)$ gives its evolution that is obtained by solving the Eq. (1). Such linear approximation improves when the second test particle moves very close to the reference one, i.e., along the geodesic $z = \text{const} \ll 1$, in which case the separation is described by the vector field $z\boldsymbol{Z}(\tau)$.

It should also be recalled that the equation of geodesic deviation (1) is linear with respect to the components of the separation vector, neglecting higher-order terms in the Taylor expansion of exact expression for relative acceleration of free test particles. It can thus be used when the relative velocities of the particles are negligible, i.e., their geodesics are almost parallel. Generalizations of Eq. (1) to admit arbitrary relative velocities were obtained and applied in the works [119–127]. Further extensions, higher-order corrections to the geodesic deviation equation, and their specific applications can be found in [129–134] (for reviews and other references see [118,127,130,134]). Our aim, however, in this paper is to investigate local relative motion of nearby free test particles that are initially at rest with respect to each other. For such an analysis, the classical geodesic deviation Eq. (1) will be fully sufficient.

Now, in order to obtain invariant results independent of the choice of coordinates, it is natural to adopt the Pirani approach [105,106] based on the use of components of the above quantities with respect to a suitable orthonormal frame $\{\boldsymbol{e}_a\}$. At any point of the reference geodesic, this defines an observer's framework in which physical measurements are made and interpreted. In particular, the separation vector is expressed as $\boldsymbol{Z} = Z^a \boldsymbol{e}_a$. The timelike vector of the frame is identified with the velocity vector of the observer, $\boldsymbol{e}_{(0)} = \boldsymbol{u}$, and $\boldsymbol{e}_{(\mathrm{i})}$, where $\mathrm{i} = 1, 2, \ldots, D-1$, are perpendicular spacelike unit vectors that form its local Cartesian basis in the hypersurface orthogonal to $\boldsymbol{u}$ (see also Fig. 2),

$$\boldsymbol{e}_a \cdot \boldsymbol{e}_b \equiv g_{\alpha\beta} e^\alpha_a e^\beta_b = \eta_{ab} \equiv \mathrm{diag}(-1, 1, \ldots, 1). \tag{3}$$

Because of the fact that $\boldsymbol{u}$ is parallelly transported, for the zeroth frame component $Z^{(0)} \equiv \boldsymbol{e}^{(0)} \cdot \boldsymbol{Z} = -\boldsymbol{u} \cdot \boldsymbol{Z}$ we immediately obtain

$$\frac{\mathrm{d}^2 Z^{(0)}}{\mathrm{d}\tau^2} = -u_\mu \frac{\mathrm{D}^2 Z^\mu}{\mathrm{d}\tau^2} = -R_{\mu\alpha\beta\nu} u^\mu u^\alpha u^\beta Z^\nu = 0, \tag{4}$$

using the skew-symmetry of the Riemann tensor. Therefore, $Z^{(0)}(\tau)$ must be at most a linear function of the proper time. By a natural choice of initial conditions, consistent with the above construction of the geodesic congruence $\gamma(\tau, z)$, we set $Z^{(0)} = 0$. The temporal component of $\boldsymbol{Z}$ thus vanishes and the test particles always stay in the same spacelike hypersurfaces synchronized by τ.

Physical information about relative motion of the test particles is thus completely contained in the spatial frame components $Z^{(\mathrm{i})}(\tau) \equiv \boldsymbol{e}^{(\mathrm{i})} \cdot \boldsymbol{Z}$ of the separation vector $\boldsymbol{Z}$. These determine the actual relative spatial position of the two nearby particles. By projecting the geodesic deviation Eq. (1) onto $\boldsymbol{e}^{(\mathrm{i})} = \boldsymbol{e}_{(\mathrm{i})}$, we obtain

$$\ddot{Z}^{(\mathrm{i})} = R^{(\mathrm{i})}{}_{(0)(0)(\mathrm{j})} Z^{(\mathrm{j})}, \tag{5}$$

where $\mathrm{i}, \mathrm{j} = 1, 2, \ldots, \mathrm{D}-1$, and we denote the physical relative acceleration as

$$\ddot{Z}^{(\mathrm{i})} \equiv \boldsymbol{e}^{(\mathrm{i})} \cdot \frac{\mathrm{D}^2 \boldsymbol{Z}}{\mathrm{d}\tau^2} = e^{(\mathrm{i})}_\mu \frac{\mathrm{D}^2 Z^\mu}{\mathrm{d}\tau^2}. \tag{6}$$

The frame components of the Riemann tensor are $R_{(\mathrm{i})(0)(0)(\mathrm{j})} \equiv R_{\mu\alpha\beta\nu} e^\mu_{(\mathrm{i})} u^\alpha u^\beta e^\nu_{(\mathrm{j})}$. Let us note that Pirani [105,106] labeled, in $D = 4$, the frame components of the "tidal stress tensor" that occurs in Eq. (5) (with an opposite sign) as $K^a{}_b \equiv R^a{}_{0b0} = R^a{}_{cbd} u^c u^d$. They are equivalent to the electric part of the Riemann tensor $\mathcal{E}_{ab} \equiv R_{a0b0} = R_{acbd} u^c u^d$, see [118].

Following Pirani, it is also usually assumed that the orthonormal frame $\{\boldsymbol{e}_a\}$ is parallelly propagated along the reference geodesic. However, in our work we do not make such an assumption. In fact, as a key idea of the proposed interpretation method, we align the orthonormal frame with the algebraic structure of a given spacetime instead (see also Sec. V). This makes the investigation of its physical properties much easier.

III. CANONICAL DECOMPOSITION OF THE CURVATURE TENSOR

The next step is to express the frame components of the Riemann tensor $R_{(\mathrm{i})(0)(0)(\mathrm{j})}$. Using the standard

decomposition of the curvature tensor into the traceless Weyl tensor C_{abcd} and specific combinations of the Ricci tensor R_{ab} and Ricci scalar R,

$$R_{abcd} = C_{abcd} + \frac{2}{D-2}(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) - \frac{2}{(D-1)(D-2)}Rg_{a[c}g_{d]b}, \tag{7}$$

we immediately obtain

$$R_{(\mathrm{i})(0)(0)(\mathrm{j})} = C_{(\mathrm{i})(0)(0)(\mathrm{j})} + \frac{1}{D-2}(R_{(\mathrm{i})(\mathrm{j})} - \delta_{\mathrm{ij}}R_{(0)(0)}) - \frac{R\delta_{\mathrm{ij}}}{(D-1)(D-2)}. \tag{8}$$

Before substituting this into the geodesic deviation Eq. (5), we also employ the Einstein field equations, generalized to any dimension $D \geq 4$,

$$R_{ab} - \tfrac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi T_{ab}, \tag{9}$$

where Λ is a cosmological constant and T_{ab} is the energy-momentum tensor of the matter field. Using (9) and its trace $R = \frac{2}{2-D}(8\pi T - D\Lambda)$, we rewrite (8) as

$$R_{(\mathrm{i})(0)(0)(\mathrm{j})} = \frac{2\Lambda\delta_{\mathrm{ij}}}{(D-1)(D-2)} + C_{(\mathrm{i})(0)(0)(\mathrm{j})} + \frac{8\pi}{D-2}\left[T_{(\mathrm{i})(\mathrm{j})} - \delta_{\mathrm{ij}}\left(T_{(0)(0)} + \frac{2T}{D-1}\right)\right]. \tag{10}$$

The equation of geodesic deviation (5) thus takes the following invariant form:

$$\ddot{Z}^{(\mathrm{i})} = \frac{2\Lambda}{(D-1)(D-2)}Z^{(\mathrm{i})} + C_{(\mathrm{i})(0)(0)(\mathrm{j})}Z^{(\mathrm{j})} + \frac{8\pi}{D-2}\left[T_{(\mathrm{i})(\mathrm{j})}Z^{(\mathrm{j})} - \left(T_{(0)(0)} + \frac{2}{D-1}T\right)Z^{(\mathrm{i})}\right]. \tag{11}$$

The first term represents the isotropic influence of the cosmological constant Λ on free test particles, the second term describes the effect of a "free" gravitational field encoded in the Weyl tensor, while the second line in (11) gives a direct effect of specific matter present in a given spacetime.

The terms proportional to the coefficients $C_{(\mathrm{i})(0)(0)(\mathrm{j})}$ can further be conveniently expressed using the Newman–Penrose-type scalars, which are the components of the Weyl tensor with respect to an associated (real) null frame $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}_i\}$. This frame is introduced by the relations

$$\boldsymbol{k} = \frac{1}{\sqrt{2}}(\boldsymbol{u} + \boldsymbol{e}_{(1)}), \qquad \boldsymbol{l} = \frac{1}{\sqrt{2}}(\boldsymbol{u} - \boldsymbol{e}_{(1)}), \qquad \boldsymbol{m}_i = \boldsymbol{e}_{(i)} \quad \text{for } i = 2, \ldots, D-1, \tag{12}$$

where $\boldsymbol{u} \equiv \boldsymbol{e}_{(0)}$ is the velocity vector of the observer. Thus, $\boldsymbol{k}$ and $\boldsymbol{l}$ are future-oriented null vectors, and $\boldsymbol{m}_i$ are $D-2$ spatial Cartesian vectors orthogonal to them, satisfying

$$\boldsymbol{k}\cdot\boldsymbol{l} = -1, \qquad \boldsymbol{m}_i\cdot\boldsymbol{m}_j = \delta_{ij}, \qquad \boldsymbol{k}\cdot\boldsymbol{k} = 0 = \boldsymbol{l}\cdot\boldsymbol{l}, \qquad \boldsymbol{k}\cdot\boldsymbol{m}_i = 0 = \boldsymbol{l}\cdot\boldsymbol{m}_i. \tag{13}$$

Using the notation of [93], the components of the Weyl tensor in such a null frame are determined by the following scalars (grouped by their boost weight):

$$\begin{aligned}
&\Psi_{0^{ij}} = C_{abcd}k^a m_i^b k^c m_j^d, \\
&\Psi_{1^{ijk}} = C_{abcd}k^a m_i^b m_j^c m_k^d, && \Psi_{1T^i} = C_{abcd}k^a l^b k^c m_i^d, \\
&\Psi_{2^{ijkl}} = C_{abcd}m_i^a m_j^b m_k^c m_l^d, && \Psi_{2S} = C_{abcd}k^a l^b l^c k^d, \\
&\Psi_{2^{ij}} = C_{abcd}k^a l^b m_i^c m_j^d, && \Psi_{2T^{ij}} = C_{abcd}k^a m_i^b l^c m_j^d, \\
&\Psi_{3^{ijk}} = C_{abcd}l^a m_i^b m_j^c m_k^d, && \Psi_{3T^i} = C_{abcd}l^a k^b l^c m_i^d, \\
&\Psi_{4^{ij}} = C_{abcd}l^a m_i^b l^c m_j^d,
\end{aligned} \tag{14}$$

where $i, j, k, l = 2, \ldots, D-1$. All other frame components can be obtained using the symmetries of the Weyl tensor. The scalars in the left column are independent, up to the obvious constraints

$$\begin{aligned}
&\Psi_{0^{[ij]}} = 0, && \Psi_{0^k{}^k} = 0, \\
&\Psi_{1^{i(jk)}} = 0, && \Psi_{1^{[ijk]}} = 0, \\
&\Psi_{2^{ijkl}} = \Psi_{2^{klij}}, && \Psi_{2^{(ij)}} = 0, \\
&\Psi_{2^{(ij)kl}} = \Psi_{2^{ij(kl)}} = \Psi_{2^{[ijkl]}} = 0, \\
&\Psi_{3^{i(jk)}} = 0, && \Psi_{3^{[ijk]}} = 0, \\
&\Psi_{4^{[ij]}} = 0, && \Psi_{4^k{}^k} = 0,
\end{aligned} \tag{15}$$

while those in the right column of (14) are not independent because they can be expressed as the contractions (hence the symbol "T," which indicates "tracing")

$$\begin{aligned}
&\Psi_{1T^i} = \Psi_{1^k{}_i{}^k}, \\
&\Psi_{2S} = \Psi_{2T^k{}^k} = \tfrac{1}{2}\Psi_{2^{kl}{}^{kl}}, \\
&\Psi_{2T^{ij}} = \tfrac{1}{2}(\Psi_{2^{ikj}{}^k} + \Psi_{2^{ij}}) \\
&\text{where} \quad \Psi_{2T^{(ij)}} = \tfrac{1}{2}\Psi_{2^{ikj}{}^k}, \qquad \Psi_{2T^{[ij]}} = \tfrac{1}{2}\Psi_{2^{ij}}, \\
&\Psi_{3T^i} = \Psi_{3^k{}_i{}^k}.
\end{aligned} \tag{16}$$

In the case $D = 4$, these Weyl tensor components in the null tetrad reduce to the standard Newman-Penrose [137,138] complex scalars Ψ_A. Explicit expressions are given in Appendix A.

Using relations $\boldsymbol{e}_{(0)} = \frac{1}{\sqrt{2}}(\boldsymbol{k} + \boldsymbol{l})$, $\boldsymbol{e}_{(1)} = \frac{1}{\sqrt{2}}(\boldsymbol{k} - \boldsymbol{l})$, and the definition (14), a straightforward calculation then leads to the following expressions for the components $C_{(\mathrm{i})(0)(0)(\mathrm{j})}$ of the Weyl tensor, which appear in Eq. (11):

$$
\begin{aligned}
C_{(1)(0)(0)(1)} &= \Psi_{2S}, \\
C_{(1)(0)(0)(j)} &= \frac{1}{\sqrt{2}}(\Psi_{1T^j} - \Psi_{3T^j}), \\
C_{(i)(0)(0)(1)} &= \frac{1}{\sqrt{2}}(\Psi_{1T^i} - \Psi_{3T^i}), \\
C_{(i)(0)(0)(j)} &= -\tfrac{1}{2}(\Psi_{0^{ij}} + \Psi_{4^{ij}}) - \Psi_{2T^{(ij)}},
\end{aligned}
\tag{17}
$$

where $i, j = 2, \ldots, D-1$ label the spatial directions orthogonal to the privileged spatial direction of $\boldsymbol{e}_{(1)}$.

Putting this into (11), we obtain the final invariant and fully general form of the equation of geodesic deviation:

$$
\begin{aligned}
\ddot{Z}^{(1)} = &\frac{2\Lambda}{(D-1)(D-2)}Z^{(1)} + \Psi_{2S}Z^{(1)} + \frac{1}{\sqrt{2}}(\Psi_{1T^j} - \Psi_{3T^j})Z^{(j)} \\
&+ \frac{8\pi}{D-2}\Big[T_{(1)(1)}Z^{(1)} + T_{(1)(j)}Z^{(j)} \\
&- \Big(T_{(0)(0)} + \frac{2}{D-1}T\Big)Z^{(1)}\Big],
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
\ddot{Z}^{(i)} = &\frac{2\Lambda}{(D-1)(D-2)}Z^{(i)} - \Psi_{2T^{(ij)}}Z^{(j)} \\
&+ \frac{1}{\sqrt{2}}(\Psi_{1T^i} - \Psi_{3T^i})Z^{(1)} - \frac{1}{2}(\Psi_{0^{ij}} + \Psi_{4^{ij}})Z^{(j)} \\
&+ \frac{8\pi}{D-2}\Big[T_{(i)(1)}Z^{(1)} + T_{(i)(j)}Z^{(j)} \\
&- \Big(T_{(0)(0)} + \frac{2}{D-1}T\Big)Z^{(i)}\Big].
\end{aligned}
\tag{19}
$$

This completely describes relative motion of nearby free test particles in any spacetime of an arbitrary dimension D. In the next section, we will discuss the specific effects given by particular scalars that represent the contributions from various components of the gravitational and matter fields.

Finally, we remark that our notation, which uses $\Psi_{A\ldots}$ in any dimension, is simply related to the notations employed, e.g., in [72,73], in [76,81], and recently in [79]. The identifications for the components present in the invariant form of the equation of geodesic deviation are summarized in Table I. More details are given in Appendix B, in particular, see expressions (B8), (B11), and (B13).

TABLE I. Different equivalent notations used in the literature for the Weyl scalars that occur in the equations of geodesic deviation (18) and (19).

	Refs. [72,73]	Refs. [76,81]	Ref. [79]
Ψ_{2S}	$-C_{0101}$	$-\Phi$	$-\Phi$
$\Psi_{2T^{ij}}$	$-C_{0i1j}$	$-\Phi_{ij}$	$-\Phi_{ij}$
Ψ_{1T^j}	$-C_{010j}$		$-\Psi_j$
Ψ_{3T^j}	C_{101j}	Ψ_j	Ψ'_j
$\Psi_{0^{ij}}$	C_{0i0j}		Ω_{ij}
$\Psi_{4^{ij}}$	C_{1i1j}	$2\Psi_{ij}$	Ω'_{ij}

IV. EFFECT OF CANONICAL COMPONENTS OF A GRAVITATIONAL FIELD ON TEST PARTICLES

Let us consider a set of freely falling test particles, initially at rest relative to each other, which form, e.g., a small (hyper)sphere. In any curved spacetime, such a configuration undergoes tidal deformations that can be deduced from the accelerations measured by the fiducial observer attached to the reference test particle in the center. The resulting relative motion represents the effect of a given gravitational field, whose specific structure is explicitly characterized by the system (18) and (19).

First concentrating on the vacuum case, i.e., $T_{ab} = 0$, the system of equations describing purely gravitational interaction simplifies considerably to

$$
\begin{aligned}
\ddot{Z}^{(1)} = &\frac{2\Lambda}{(D-1)(D-2)}Z^{(1)} + \Psi_{2S}Z^{(1)} \\
&+ \frac{1}{\sqrt{2}}(\Psi_{1T^j} - \Psi_{3T^j})Z^{(j)},
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
\ddot{Z}^{(i)} = &\frac{2\Lambda}{(D-1)(D-2)}Z^{(i)} - \Psi_{2T^{(ij)}}Z^{(j)} \\
&+ \frac{1}{\sqrt{2}}(\Psi_{1T^i} - \Psi_{3T^i})Z^{(1)} - \frac{1}{2}(\Psi_{0^{ij}} + \Psi_{4^{ij}})Z^{(j)}.
\end{aligned}
\tag{21}
$$

The overall effect of the gravitational field on test particles is thus naturally decomposed into clearly identified components proportional to the cosmological constant Λ and the Weyl scalars $\Psi_{A\ldots}$. Of course, for algebraically special spacetimes some (or many) of these coefficients *vanish completely*, and even in algebraically general cases specific numerical values of the scalars $\Psi_{A\ldots}$ can distinguish the dominant terms from those that are negligible. Let us now briefly describe the character of each term separately, including its physical interpretation.

(i) Λ: *isotropic influence of the cosmological background*

The presence of the cosmological constant Λ is encoded in the term

$$
\begin{pmatrix} \ddot{Z}^{(1)} \\ \ddot{Z}^{(i)} \end{pmatrix} = \frac{2\Lambda}{(D-1)(D-2)} \begin{pmatrix} 1 & 0 \\ 0 & \delta_{ij} \end{pmatrix} \begin{pmatrix} Z^{(1)} \\ Z^{(j)} \end{pmatrix},
\tag{22}
$$

which can be written as $\ddot{Z}^{(\mathrm{i})} = \frac{2\Lambda}{(D-1)(D-2)}Z^{(\mathrm{i})}$ for *all* spatial components $\mathrm{i} = 1, 2, \ldots, D-1$. In parallelly propagated frames, this yields the following explicit solutions:

$$\Lambda = 0\text{: } Z^{(\mathrm{i})} = A_{\mathrm{i}}\tau + B_{\mathrm{i}},$$

$$\Lambda > 0\text{: } Z^{(\mathrm{i})} = A_{\mathrm{i}} \cosh\left[\sqrt{\frac{2\Lambda}{(D-1)(D-2)}}\tau\right] + B_{\mathrm{i}} \sinh\left[\sqrt{\frac{2\Lambda}{(D-1)(D-2)}}\tau\right],$$

$$\Lambda < 0\text{: } Z^{(\mathrm{i})} = A_{\mathrm{i}} \cos\left[\sqrt{\frac{2|\Lambda|}{(D-1)(D-2)}}\tau\right] + B_{\mathrm{i}} \sin\left[\sqrt{\frac{2|\Lambda|}{(D-1)(D-2)}}\tau\right],$$

where A_{i}, B_{i} are constants of integration. These are characteristic relative motions of test particles in spacetimes of constant curvature, namely, Minkowski space, de Sitter space and anti-de Sitter space, respectively, as derived by Synge [102,103].

(ii) $\Psi_{4^{ij}}$: *transverse gravitational wave propagating in the direction* $+\boldsymbol{e}_{(1)}$
This part of a gravitational field influences the test particles as

$$\begin{pmatrix} \ddot{Z}^{(1)} \\ \ddot{Z}^{(i)} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \Psi_{4^{ij}} \end{pmatrix}\begin{pmatrix} Z^{(1)} \\ Z^{(j)} \end{pmatrix}. \qquad (23)$$

Obviously, this is a purely transverse effect because there is no acceleration in the privileged spatial direction $\boldsymbol{e}_{(1)}$. The set of scalars $\Psi_{4^{ij}}$ forms a symmetric ($\Psi_{4^{ij}} = \Psi_{4^{ji}}$) and traceless ($\Psi_{4^{k}}{}_{k} = 0$) matrix of dimension $(D-2)\times(D-2)$, cf. the last line in (15), so that it has $\frac{1}{2}D(D-3)$ independent components corresponding to polarization modes (see also [72,78,79]). In direct analogy with a linearized Einstein gravity in four [115,116] and higher dimensions [86,93], $\Psi_{4^{ij}}$ represents the gravitational wave that propagates along the null direction $\boldsymbol{k}$, i.e., in the spatial direction $+\boldsymbol{e}_{(1)}$ [in view of relations (12) there is $k_{(1)} \equiv \boldsymbol{k}\cdot\boldsymbol{e}_{(1)} > 0$ while $k_{(i)} \equiv \boldsymbol{k}\cdot\boldsymbol{e}_{(i)} = 0$ for $i = 2,\ldots,D-1$]. Spacetimes of algebraic type N (for which only the components $\Psi_{4^{ij}} \equiv C_{1i1j}$ are nonvanishing [72,73]) can thus be interpreted as exact gravitational waves in any dimension $D \geq 4$.

(iii) $\Psi_{3T^{i}}$: *longitudinal component of a gravitational field with respect to* $+\boldsymbol{e}_{(1)}$
Such terms cause longitudinal deformations of a set of test particles given by

$$\begin{pmatrix} \ddot{Z}^{(1)} \\ \ddot{Z}^{(i)} \end{pmatrix} = -\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \Psi_{3T^{j}} \\ \Psi_{3T^{i}} & 0 \end{pmatrix}\begin{pmatrix} Z^{(1)} \\ Z^{(j)} \end{pmatrix}. \qquad (24)$$

These $(D-2)$ scalars $\Psi_{3T^{i}}$, which combine motion in the privileged spatial direction $\boldsymbol{e}_{(1)}$ with motion in the transverse directions $\boldsymbol{e}_{(i)}$, are also obtained using $\Psi_{3T^{i}} \equiv \Psi_{3^{k}}{}_{i}{}^{k}$, where $\Psi_{3^{ijk}} = -\Psi_{3^{ikj}}$ and $\Psi_{3^{ijk}} + \Psi_{3^{jki}} + \Psi_{3^{kij}} = 0$. Longitudinal effects of this type occur in spacetimes of type III and in algebraically more general cases.

(iv) Ψ_{2S}, $\Psi_{2T^{(ij)}}$: *Newton-Coulomb components of a gravitational field*
The terms

$$\begin{pmatrix} \ddot{Z}^{(1)} \\ \ddot{Z}^{(i)} \end{pmatrix} = \begin{pmatrix} \Psi_{2S} & 0 \\ 0 & -\Psi_{2T^{(ij)}} \end{pmatrix}\begin{pmatrix} Z^{(1)} \\ Z^{(j)} \end{pmatrix} \qquad (25)$$

give rise to deformations that generalize the classical Newton–Coulomb-type tidal effects in $D = 4$, namely, those in the vicinity of a spherically symmetric static source. Recall that $\Psi_{2S} = \Psi_{2T^{k}}{}^{k}$ [see (16) and (15) for further relations], so that the $(D-1)\times(D-1)$-dimensional matrix in (25) is symmetric and traceless. These terms are typically present in type-D spacetimes, for which the notation $\Psi_{2S} \equiv -\Phi$ and $-\Psi_{2T^{(ij)}} \equiv \Phi^{S}_{ij}$ is commonly used [76,78–82,99], see (B11). As shown in (A6), the only nonvanishing coefficients of this type in four dimensions are the diagonal elements $\frac{1}{2}\Psi_{2S} = \Psi_{2T^{(22)}} = \Psi_{2T^{(33)}} \equiv -\mathrm{Re}\,\Psi_{2}$.

(v) $\Psi_{1T^{i}}$: *longitudinal component of a gravitational field with respect to* $-\boldsymbol{e}_{(1)}$
The corresponding effect on test particles is

$$\begin{pmatrix} \ddot{Z}^{(1)} \\ \ddot{Z}^{(i)} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \Psi_{1T^{j}} \\ \Psi_{1T^{i}} & 0 \end{pmatrix}\begin{pmatrix} Z^{(1)} \\ Z^{(j)} \end{pmatrix}, \qquad (26)$$

which is very similar to the acceleration caused by the longitudinal component $\Psi_{3T^{i}}$, as described by (24). In fact, it is its counterpart: it follows from the definition (14) that the scalars $\Psi_{1T^{i}} \equiv \Psi_{1}{}_{k}{}^{k}{}_{i}$ (where $\Psi_{1^{ijk}} = -\Psi_{1^{ikj}}$ and $\Psi_{1^{ijk}} + \Psi_{1^{jki}} + \Psi_{1^{kij}} = 0$) are equivalent to $\Psi_{3T^{i}}$ under the interchange $\boldsymbol{k} \leftrightarrow \boldsymbol{l}$. Since $k_{(1)} \equiv \boldsymbol{k}\cdot\boldsymbol{e}_{(1)} > 0$ while $l_{(1)} \equiv \boldsymbol{l}\cdot\boldsymbol{e}_{(1)} < 0$, the scalars $\Psi_{1T^{i}}$ represent the longitudinal component of the field associated with the spatial direction $-\boldsymbol{e}_{(1)}$.

(vi) $\Psi_{0^{ij}}$: *transverse gravitational wave propagating in the direction* $-\boldsymbol{e}_{(1)}$
This component of a gravitational field is characterized by

$$\begin{pmatrix} \ddot{Z}^{(1)} \\ \ddot{Z}^{(i)} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \Psi_{0^{ij}} \end{pmatrix}\begin{pmatrix} Z^{(1)} \\ Z^{(j)} \end{pmatrix}, \qquad (27)$$

which is fully equivalent to (23) under $\boldsymbol{k} \leftrightarrow \boldsymbol{l}$. The scalars $\Psi_{0^{ij}}$ [which form a symmetric and traceless $(D-2)\times(D-2)$ matrix: $\Psi_{0^{ij}} = \Psi_{0^{ji}}$, $\Psi_{0^{k}}{}_{k} = 0$] thus describe the transverse gravitational wave

propagating along the null direction $\boldsymbol{l}$, i.e., in the spatial direction $-\boldsymbol{e}_{(1)}$. Superposition of gravitational waves that would propagate in *both* directions simultaneously (that is, an "outgoing" wave given by $\Psi_{4^{ij}}$ and an "ingoing" wave given by $\Psi_{0^{ij}}$) can only be present in spacetimes that are of algebraically general type.

V. UNIQUENESS OF THE INTERPRETATION FRAME AND DEPENDENCE OF THE FIELD COMPONENTS ON THE OBSERVER

The canonical components of a gravitational field described in the previous section are represented by the real coefficients $\Psi_{A\ldots}$. These are projections of the Weyl tensor onto particular combinations of the null frame $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}_i\}$, as defined in (14). They are spacetime scalars and in this sense the above physical interpretation is *invariant*. On the other hand, the values of $\Psi_{A\ldots}$ depend on the choice of the basis vectors of the frame. In this section, we will argue that such a dependence corresponds to simple local Lorentz transformations related to the choice of specific observer in a given event, and that the natural interpretation null frame is *essentially unique*.

Let us consider an observer attached to the reference (fiducial) test particle moving through some event in the spacetime, such as the point $\mathcal{P}$ in Fig. 1, whose velocity vector is $\boldsymbol{u}$. This timelike vector (normalized as $\boldsymbol{u}\cdot\boldsymbol{u} = -1$) defines an orthogonal spatial hypersurface of dimension $D-1$ spanned by the Cartesian vectors $\boldsymbol{e}_{(\mathrm{i})}$, where $\mathrm{i} = 1, 2, \ldots, D-1$. Assuming the spacetime is of an algebraic type I or more special, it is most natural to associate the corresponding Weyl-aligned null direction (WAND) with the null vector $\boldsymbol{k}$ of the interpretation reference frame, see Fig. 2.

The privileged unit vector $\boldsymbol{e}_{(1)}$, defining the longitudinal spatial direction, is then uniquely obtained by projecting $\boldsymbol{k}$ onto the spatial subspace orthogonal to $\boldsymbol{u}$. This also fixes the normalization of $\boldsymbol{k}$ [to satisfy the first relation in (12) we require $\boldsymbol{k}\cdot\boldsymbol{u} = -\frac{1}{\sqrt{2}}$]. The complementary null vector $\boldsymbol{l}$ of the frame is then also uniquely given via the relation $\boldsymbol{l} = \sqrt{2}\boldsymbol{u} - \boldsymbol{k}$. It only remains to choose the transverse spatial vectors $\boldsymbol{e}_{(2)}, \ldots, \boldsymbol{e}_{(D-1)}$, i.e., $\boldsymbol{m}_i = \boldsymbol{e}_{(i)}$. As shown in Fig. 2, these must lie in the $(D-2)$-dimensional subspace orthogonal both to $\boldsymbol{u}$ and $\boldsymbol{e}_{(1)}$, so that $\boldsymbol{k}\cdot\boldsymbol{m}_i = 0 = \boldsymbol{l}\cdot\boldsymbol{m}_i$ as required by (13). Neglecting possible inversions, the only remaining freedom is thus standard spatial rotations represented by the rotation group $SO(D-2)$, which acts on the space spanned by $\boldsymbol{m}_i$, see the explicit relation (C4) presented in Appendix C.

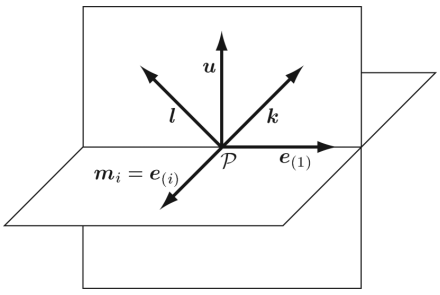

FIG. 2. Natural choice of the interpretation null frame and the related orthonormal frame (12) and (13). Up to spatial rotations of $\boldsymbol{m}_i = \boldsymbol{e}_{(i)}$, they are uniquely given by the velocity vector $\boldsymbol{u}$ of the observer and the WAND $\boldsymbol{k}$ at any event $\mathcal{P}$ of the spacetime.

For any spacetime of type N (in which the WAND has maximal alignment order) the null vector $\boldsymbol{k}$ is unique. In spacetimes of other algebraic types (namely III_i, II_i, I_i, and D), different WANDs exist. These can alternatively be used as the vector $\boldsymbol{k}$ of the interpretation null frame $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}_i\}$. Because the distinct WANDs can always be related using the null rotation with fixed $\boldsymbol{l}$, as given explicitly by Eq. (C2) in Appendix C, it is straightforward to evaluate the "new" values of the Weyl scalars $\Psi_{A\ldots}$ using the expressions (C6). Notice that the coefficients $\Psi_{4^{ij}}$, which are the amplitudes of transverse gravitational waves propagating along $\boldsymbol{k}$, are invariant under such a change.

Let us now consider another observer moving through the same event $\mathcal{P}$ with a different velocity $\tilde{\boldsymbol{u}}$. Locally, this transition is just the Lorentz transformation from the original reference frame $\{\boldsymbol{e}_a\}$ to $\{\tilde{\boldsymbol{e}}_a\}$ for which

$$\tilde{\boldsymbol{u}} = \frac{\boldsymbol{u} + \sum_{\mathrm{i}=1}^{D-1} v_{\mathrm{i}}\boldsymbol{e}_{(\mathrm{i})}}{\sqrt{1 - \sum_{\mathrm{i}=1}^{D-1} v_{\mathrm{i}}^2}}, \tag{28}$$

where $v_1, \ldots, v_{D-1}$ are components of the spatial velocity of the new observer with respect to the original Cartesian basis $\boldsymbol{e}_{(\mathrm{i})}$. This can be obtained as the combination of a boost in the $\boldsymbol{k}-\boldsymbol{l}$ plane followed by a null rotation with fixed $\boldsymbol{k}$, see Eqs. (C3) and (C1) in Appendix C, if we take the specific parameters

$$B = \frac{\sqrt{1 - \sum_{\mathrm{i}=1}^{D-1} v_{\mathrm{i}}^2}}{1 - v_1}, \qquad L_i = \frac{v_i}{\sqrt{1 - \sum_{\mathrm{i}=1}^{D-1} v_{\mathrm{i}}^2}}, \tag{29}$$

where $i = 2, \ldots, D-1$:

$$\begin{aligned}
\tilde{\boldsymbol{k}} &= \frac{\sqrt{1 - \sum_{\mathrm{i}=1}^{D-1} v_{\mathrm{i}}^2}}{1 - v_1}\boldsymbol{k}, \\
\tilde{\boldsymbol{l}} &= \frac{1}{\sqrt{1 - \sum_{\mathrm{i}=1}^{D-1} v_{\mathrm{i}}^2}}\left[(1 - v_1)\boldsymbol{l} + \sqrt{2}\sum_{i=2}^{D-1} v_i\boldsymbol{m}_i + \frac{\sum_{i=2}^{D-1} v_i^2}{1 - v_1}\boldsymbol{k}\right], \\
\tilde{\boldsymbol{m}}_i &= \boldsymbol{m}_i + \sqrt{2}\frac{v_i}{1 - v_1}\boldsymbol{k}.
\end{aligned} \tag{30}$$

Indeed, $\tilde{\boldsymbol{u}} \equiv \frac{1}{\sqrt{2}}(\tilde{\boldsymbol{k}} + \tilde{\boldsymbol{l}})$ gives exactly the relation (28). The corresponding change of the Weyl scalars $\Psi_{A\ldots}$ can thus be obtained by combining (C7) with (C5), which yields

$$
\begin{aligned}
&\frac{1}{B^2}\tilde{\Psi}_{0^{ij}} = \Psi_{0^{ij}}, \qquad \frac{1}{B}\tilde{\Psi}_{1^{ijk}} = \Psi_{1^{ijk}} - 2\sqrt{2}\Psi_{0^{[ij}}X_{k]}, \qquad \frac{1}{B}\tilde{\Psi}_{1T^i} = \Psi_{1T^i} + \sqrt{2}\Psi_{0^{ij}}X^j,\\
&\tilde{\Psi}_{2^{ijkl}} = \Psi_{2^{ijkl}} - 2\sqrt{2}(X_{[l}\Psi_{1^{k]ij}} - X_{[i}\Psi_{1^{j]kl}}) + 4(\Psi_{0^{i[k}}X_{l]}X_j + \Psi_{0^{j[l}}X_{k]}X_i), \qquad \tilde{\Psi}_{2S} = \Psi_{2S} - 2\sqrt{2}\Psi_{1T^i}X^i - 2\Psi_{0^{ij}}X^iX^j,\\
&\tilde{\Psi}_{2^{ij}} = \Psi_{2^{ij}} + \sqrt{2}\Psi_{1^{kij}}X^k - 2\sqrt{2}\Psi_{1T^{[i}}X_{j]} - 4\Psi_{0^{k[i}}X_{j]}X^k,\\
&\tilde{\Psi}_{2T^{ij}} = \Psi_{2T^{ij}} + \sqrt{2}\Psi_{1^{ikj}}X^k - \sqrt{2}\Psi_{1T^i}X_j - 2\Psi_{0^{ik}}X^kX_j + \Psi_{0^{ij}}|X|^2,\\
&B\tilde{\Psi}_{3^{ijk}} = \Psi_{3^{ijk}} + \sqrt{2}(\Psi_{2^{lijk}}X^l - \Psi_{2^{jk}}X_i + 2X_{[j}\Psi_{2T^{k]i}}) + 4\Psi_{1T^{[j}}X_{k]}X_i - 2(\Psi_{1^{jli}}X_k + \Psi_{1^{tjk}}X_i - \Psi_{1^{kli}}X_j)X^l\\
&\qquad + \Psi_{1^{ijk}}|X|^2 + 4\sqrt{2}\Psi_{0^{l[j}}X_{k]}X_iX^l - 2\sqrt{2}\Psi_{0^{i[j}}X_{k]}|X|^2,\\
&B\tilde{\Psi}_{3T^i} = \Psi_{3T^i} + \sqrt{2}\Psi_{2^{ij}}X^j - \sqrt{2}(\Psi_{2T^{ki}}X^k + \Psi_{2S}X_i) + 2(2\Psi_{1T^j}X_i - \Psi_{1^{kji}}X^k)X^j - \Psi_{1T^i}|X|^2\\
&\qquad + 2\sqrt{2}\Psi_{0^{jk}}X^jX^kX_i - \sqrt{2}\Psi_{0^{ij}}X^j|X|^2,\\
&B^2\tilde{\Psi}_{4^{ij}} = \Psi_{4^{ij}} + 2\sqrt{2}(\Psi_{3T^{(i}}X_{j)} - \Psi_{3^{(ij)k}}X^k) + 2\Psi_{2^{ikjl}}X^kX^l - 4\Psi_{2T^{k(i}}X_{j)}X^k + 2\Psi_{2T^{(ij)}}|X|^2 - 2\Psi_{2S}X_iX_j - 4\Psi_{2^{k(i}}X_{j)}X^k\\
&\qquad - 2\sqrt{2}(2\Psi_{1^{kl(i}}X_{j)}X^kX^l + \Psi_{1^{(ij)k}}X^k|X|^2 + \Psi_{1T^{(i}}X_{j)}|X|^2 - 2\Psi_{1T^k}X^kX_iX_j) + 4\Psi_{0^{kl}}X^kX^lX_iX_j\\
&\qquad - 4\Psi_{0^{k(i}}X_{j)}X^k|X|^2 + \Psi_{0^{ij}}|X|^4,
\end{aligned}
\tag{31}
$$

where we denoted

$$
X_i \equiv BL_i = \frac{v_i}{1 - v_1}. \tag{32}
$$

In particular, for spacetimes of algebraic type N, which admit a WAND of the maximal alignment order, the only nonvanishing component of the gravitational field is $\Psi_{4^{ij}}$ representing the transverse gravitational wave propagating in the spatial direction $\boldsymbol{e}_{(1)}$. It immediately follows from (31) and (29) that the transition to any other observer results just in a simple rescaling of the gravitational wave amplitudes

$$
\tilde{\Psi}_{4^{ij}} = \frac{(1 - v_1)^2}{1 - \sum_{\mathrm{i}=1}^{D-1} v_{\mathrm{i}}^2}\Psi_{4^{ij}}. \tag{33}
$$

If the new observer moves only in the spatial direction in which the wave propagates, $v_1 > 0$ and $v_i = 0$ for $i = 2, \ldots, D-1$, then $\tilde{\Psi}_{4^{ij}} = (1 - v_1)/(1 + v_1)\Psi_{4^{ij}}$, which is smaller than $\Psi_{4^{ij}}$. If the observer's velocity approaches the speed of light, $v_1 \to 1$, the amplitudes of the gravitational wave disappear, $\tilde{\Psi}_{4^{ij}} \to 0$. Contrarily, when the observer moves against the wave its amplitudes grow, and for $v_1 \to -1$ they diverge.

VI. THE EFFECT OF MATTER ON TEST PARTICLES

Let us now consider the direct effect of specific forms of matter on relative motion of test particles, as described by the invariant form of the equation of geodesic deviation (18) and (19). Setting the cosmological constant Λ and all components of the Weyl tensor to zero, it reduces to

$$
\begin{aligned}
\ddot{Z}^{(1)} &= \frac{8\pi}{D-2}\Big[T_{(1)(1)}Z^{(1)} + T_{(1)(j)}Z^{(j)} - \Big(T_{(0)(0)} + \frac{2}{D-1}T\Big)Z^{(1)}\Big],\\
\ddot{Z}^{(i)} &= \frac{8\pi}{D-2}\Big[T_{(i)(1)}Z^{(1)} + T_{(i)(j)}Z^{(j)} - \Big(T_{(0)(0)} + \frac{2}{D-1}T\Big)Z^{(i)}\Big].
\end{aligned}
\tag{34}
$$

It will be illustrative to investigate some important types of matter usually contained in the families of exact solutions of Einstein's equations, namely, pure radiation, perfect fluids, and electromagnetic fields.

(i) *pure radiation*

The energy-momentum tensor of a pure radiation field (or "null dust") aligned along the null direction $\boldsymbol{k}$ is

$$
T_{ab} = \rho k_a k_b, \tag{35}
$$

where ρ is a function representing the radiation density. Its trace vanishes, $T = 0$, and using (12) we derive that the only nonvanishing components of T_{ab} in the equation of geodesic deviation are $T_{(0)(0)} = T_{(1)(1)} = \frac{1}{2}\rho$. Equations (34) thus reduce considerably to

$$
\begin{pmatrix}\ddot{Z}^{(1)}\\ \ddot{Z}^{(i)}\end{pmatrix} = -\frac{4\pi\rho}{D-2}\begin{pmatrix}0 & 0\\ 0 & \delta_{ij}\end{pmatrix}\begin{pmatrix}Z^{(1)}\\ Z^{(j)}\end{pmatrix}. \tag{36}
$$

In an arbitrary dimension D, there is thus no acceleration in the longitudinal spatial direction $\boldsymbol{e}_{(1)}$. The effects in the transverse subspace are isotropic and (since $\rho > 0$) they cause the radial contraction that may eventually lead to an exact focusing.

(ii) *perfect fluid*
For a perfect fluid of energy density ρ and pressure p (which is assumed to be isotropic), the energy-momentum tensor is

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab}. \tag{37}$$

Provided the fluid is comoving, its velocity $\boldsymbol{u}$ coincides with the observer's velocity, which is the vector $\boldsymbol{e}_{(0)}$ of the orthonormal frame. The trace is $T = (D-1)p - \rho$, and the relevant nonvanishing frame components are $T_{(0)(0)} = \rho$, $T_{(1)(1)} = p$, and $T_{(i)(j)} = p\delta_{ij}$. The equation of geodesic deviation thus takes the form

$$\begin{pmatrix} \ddot{Z}^{(1)} \\ \ddot{Z}^{(i)} \end{pmatrix} = -8\pi \frac{(D-3)\rho + (D-1)p}{(D-1)(D-2)} \begin{pmatrix} 1 & 0 \\ 0 & \delta_{ij} \end{pmatrix} \begin{pmatrix} Z^{(1)} \\ Z^{(j)} \end{pmatrix}. \tag{38}$$

The resulting motion is isotropic, the same in the longitudinal and all transverse spatial directions. For positive ρ and p, the fluid matter causes a contraction, such as in the case of dust ($p = 0$), incoherent radiation ($p = \frac{D-3}{D-1}\rho$), or stiff fluid ($p = \rho$). However, for matter with a negative pressure, the set of test particles may expand. In particular, if the matter is described by the equation of state $p = -\rho = \text{const}$, it mimics the cosmological constant $\Lambda = 8\pi\rho$ since (38) is then completely equivalent to (22).

(iii) *electromagnetic field*
The energy-momentum tensor of an electromagnetic field is given by

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4} g_{ab} F_{cd}F^{cd}\right), \tag{39}$$

so that its trace is $T = \frac{1}{16\pi}(4-D)F_{ab}F^{ab}$. The frame components of T_{ab}, which occur in expressions (34), are

$$\begin{aligned} T_{(0)(0)} &= \frac{1}{4\pi}\left(F_{(0)c}F_{(0)}{}^c + \frac{1}{4}F_{ab}F^{ab}\right), \\ T_{(1)(1)} &= \frac{1}{4\pi}\left(F_{(1)c}F_{(1)}{}^c - \frac{1}{4}F_{ab}F^{ab}\right), \\ T_{(1)(i)} &= \frac{1}{4\pi}F_{(1)c}F_{(i)}{}^c, \\ T_{(i)(j)} &= \frac{1}{4\pi}\left(F_{(i)c}F_{(j)}{}^c - \frac{1}{4}\delta_{ij}F_{ab}F^{ab}\right). \end{aligned} \tag{40}$$

In this case, the equation of geodesic deviation takes the following more complicated form:

$$\begin{pmatrix} \ddot{Z}^{(1)} \\ \ddot{Z}^{(i)} \end{pmatrix} = \begin{pmatrix} \mathcal{T} & \mathcal{T}_j \\ \mathcal{T}_i & \mathcal{T}_{ij} \end{pmatrix} \begin{pmatrix} Z^{(1)} \\ Z^{(j)} \end{pmatrix}, \tag{41}$$

where

$$\begin{aligned} \mathcal{T} &= \frac{2}{D-2}(F_{(1)c}F_{(1)}{}^c - F_{(0)c}F_{(0)}{}^c) \\ &\quad - \frac{3}{(D-1)(D-2)}F_{ab}F^{ab}, \\ \mathcal{T}_i &= \frac{2}{D-2}F_{(1)c}F_{(i)}{}^c, \\ \mathcal{T}_{ij} &= \frac{2}{D-2}(F_{(i)c}F_{(j)}{}^c - \delta_{ij}F_{(0)c}F_{(0)}{}^c) \\ &\quad - \frac{3}{(D-1)(D-2)}\delta_{ij}F_{ab}F^{ab}. \end{aligned} \tag{42}$$

We observe that the clear distinction between the longitudinal and transverse spatial directions is not present, except at very special situations. Some important particular subcases can be easily identified and analyzed, for example, a null electromagnetic field for which the invariant vanishes, $F_{ab}F^{ab} = 0$, or purely electric aligned field in the vicinity of static black holes.

VII. AN EXPLICIT EXAMPLE: PP-WAVES IN HIGHER DIMENSIONS

We conclude this paper by demonstrating the usefulness of the above interpretation method on an important family of exact spacetimes, namely, the pp-waves. These are defined geometrically as admitting a covariantly constant null vector field $\boldsymbol{k}$. Such spacetimes thus form a special subclass of the Kundt spacetimes because the geodesic congruence generated by $\boldsymbol{k}$ is twist-free, shear-free, and nonexpanding.

In [55], we investigated general Kundt spacetimes in higher dimensions, admitting a cosmological constant Λ and a Maxwell field aligned with $\boldsymbol{k}$ (which is necessarily a multiple WAND). In natural coordinates, the metric of all such pp-waves can be written in the Brinkmann form [57]

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\mathrm{d}x^j + 2e_i\,\mathrm{d}x^i\mathrm{d}u - 2\mathrm{d}u\mathrm{d}r + c\,\mathrm{d}u^2, \tag{43}$$

where $\boldsymbol{k} \propto \boldsymbol{\partial}_r$ and g_{ij}, e_i, c are functions of the transverse spatial coordinates x^k and the null coordinate u. The explicit Einstein-Maxwell equations can be found in [55], namely, Eqs. (115)–(118).

For the metric (43), the interpretation null frame adapted to a general observer that has the velocity $\boldsymbol{u} = \dot{r}\boldsymbol{\partial}_r + \dot{u}\boldsymbol{\partial}_u + \dot{x}^2\boldsymbol{\partial}_{x^2} + \ldots + \dot{x}^{D-1}\boldsymbol{\partial}_{x^{D-1}}$ is

$$\begin{aligned} \boldsymbol{k} &= \frac{1}{\sqrt{2}\dot{u}}\boldsymbol{\partial}_r, \\ \boldsymbol{l} &= \left(\sqrt{2}\dot{r} - \frac{1}{\sqrt{2}\dot{u}}\right)\boldsymbol{\partial}_r + \sqrt{2}\dot{u}\boldsymbol{\partial}_u \\ &\quad + \sqrt{2}\dot{x}^2\boldsymbol{\partial}_{x^2} + \ldots + \sqrt{2}\dot{x}^{D-1}\boldsymbol{\partial}_{x^{D-1}}, \\ \boldsymbol{m}_i &= \frac{1}{\dot{u}}(e_k\dot{u} + g_{jk}\dot{x}^j)m_i^k\boldsymbol{\partial}_r + m_i^2\boldsymbol{\partial}_{x^2} + \ldots + m_i^{D-1}\boldsymbol{\partial}_{x^{D-1}}, \end{aligned} \tag{44}$$

where $g_{kl}m_i^k m_j^l = \delta_{ij}$, and nontrivial components of the Weyl tensor are

$$
\begin{aligned}
C_{ruru} &= -\frac{1}{(D-1)(D-2)}{}^{\mathrm{s}}R, \\
C_{riuj} &= \frac{1}{D-2}{}^{\mathrm{s}}R_{ij} - \frac{1}{(D-1)(D-2)}{}^{\mathrm{s}}R g_{ij}, \\
C_{ruui} &= \frac{1}{D-2}R_{ui} - \frac{1}{(D-1)(D-2)}{}^{\mathrm{s}}R e_i, \\
C_{ijkl} &= {}^{\mathrm{s}}R_{ijkl} - \frac{2}{D-2}(g_{i[k}{}^{\mathrm{s}}R_{l]j} - g_{j[k}{}^{\mathrm{s}}R_{l]i}) \\
&\quad + \frac{2}{(D-1)(D-2)}{}^{\mathrm{s}}R g_{i[k}g_{l]j}, \\
C_{uijk} &= R_{uijk} - \frac{2}{D-2}(e_{[j}{}^{\mathrm{s}}R_{k]i} - g_{i[j}R_{k]u}) \\
&\quad + \frac{2}{(D-1)(D-2)}{}^{\mathrm{s}}R e_{[j}g_{k]i}, \\
C_{iuju} &= R_{iuju} - \frac{1}{D-2}(c\,{}^{\mathrm{s}}R_{ij} - 2e_{(i}R_{j)u} + g_{ij}R_{uu}) \\
&\quad + \frac{1}{(D-1)(D-2)}{}^{\mathrm{s}}R(cg_{ij} - e_i e_j).
\end{aligned}
\tag{45}
$$

Using definition (14), we evaluate the Weyl tensor (45) in the interpretation null frame (44). Lengthy calculation (with some "miraculous" cancellations) gives the following nonvanishing Weyl scalars that enter the equations of geodesic deviation (18) and (19):

$$
\begin{aligned}
\Psi_{2S} &= \frac{1}{(D-1)(D-2)}{}^{\mathrm{s}}R, \\
\Psi_{2T^{ij}} &= \frac{1}{D-2}{}^{\mathrm{s}}R_{kl}m_i^k m_j^l - \frac{1}{(D-1)(D-2)}{}^{\mathrm{s}}R\delta_{ij}, \\
\Psi_{3T^i} &= -\frac{\sqrt{2}}{D-2}({}^{\mathrm{s}}R_{km}\dot{x}^m + R_{ku}\dot{u})m_i^k, \\
\Psi_{4^{ij}} &= 2\Bigg[\left({}^{\mathrm{s}}R_{kmln} - \frac{1}{D-2}g_{kl}{}^{\mathrm{s}}R_{mn}\right)\dot{x}^m\dot{x}^n \\
&\quad + 2\left(R_{kmlu} - \frac{1}{D-2}g_{kl}R_{mu}\right)\dot{x}^m\dot{u} \\
&\quad + \left(R_{kulu} - \frac{1}{D-2}g_{kl}R_{uu}\right)\dot{u}^2\Bigg]m_{(i}^k m_{j)}^l.
\end{aligned}
\tag{46}
$$

This is a general result valid for any pp-wave spacetime because no particular field equations have not yet been imposed.

Notice that $\Psi_{2T^{ij}} = \Psi_{2T^{(ij)}}$. Moreover, in accordance with the relations (16) and (15), $\Psi_{2S} = \Psi_{2T^k{}^k}$ and $\Psi_{4^k{}^k} = 0$ so that any pp-wave is traceless.

The relative tidal motion of nearby test particles in general pp-waves will thus be caused by the combination of the transverse gravitational wave (23) propagating along $\boldsymbol{k}$ with amplitude $\Psi_{4^{ij}}$, the longitudinal component (24) of the gravitational field with amplitude Ψ_{3T^i}, and the Newton-Coulomb contribution (25) determined by the scalars Ψ_{2S} and $\Psi_{2T^{ij}}$:

$$
\begin{aligned}
\ddot{Z}^{(1)} &= \Psi_{2S}Z^{(1)} - \frac{1}{\sqrt{2}}\Psi_{3T^j}Z^{(j)}, \\
\ddot{Z}^{(i)} &= -\Psi_{2T^{(ij)}}Z^{(j)} - \frac{1}{\sqrt{2}}\Psi_{3T^i}Z^{(1)} - \frac{1}{2}\Psi_{4^{ij}}Z^{(j)}.
\end{aligned}
\tag{47}
$$

There is also the isotropic background influence (22) if the cosmological constant Λ is present, or the interaction (41) with the electromagnetic field.

The scalars (46) that enter (47) combine kinematics (namely the velocity components $\dot{x}^m$, $\dot{u}$ of the observer) with the specific curvature of spacetime encoded in the only nonvanishing components of the Riemann and Ricci tensors, namely,

$$
\begin{aligned}
R_{ijkl} &= {}^{\mathrm{s}}R_{ijkl}, \\
R_{uijk} &= \tfrac{1}{2}(e_{k,ij} - e_{j,ik} + g_{ij,uk} - g_{ik,uj}) \\
&\quad + {}^{\mathrm{s}}\Gamma_{ij}^m(\tfrac{1}{2}g_{km,u} + e_{[m,k]}) - {}^{\mathrm{s}}\Gamma_{ik}^m(\tfrac{1}{2}g_{jm,u} + e_{[m,j]}), \\
R_{iuju} &= \tfrac{1}{2}(e_{i,uj} + e_{j,ui} - c_{,ij} - g_{ij,uu}) \\
&\quad + g^{kl}(\tfrac{1}{2}g_{ik,u} + e_{[k,i]})(\tfrac{1}{2}g_{jl,u} + e_{[l,j]}) \\
&\quad - {}^{\mathrm{s}}\Gamma_{ij}^k(e_{k,u} - \tfrac{1}{2}c_{,k}),
\end{aligned}
\tag{48}
$$

and

$$
\begin{aligned}
R_{ij} &= {}^{\mathrm{s}}R_{ij}, \\
R_{iu} &= [g^{jk}(\tfrac{1}{2}g_{ij,u} + e_{[j,i]})]_{,k} + [g^{jk}(\tfrac{1}{2}g_{ij,u} + e_{[j,i]})](\ln\sqrt{g})_{,k} \\
&\quad + g^{jk}g^{lm}(g_{im,k}e_{[l,j]} - \tfrac{1}{4}g_{km,i}g_{jl,u}) - (\ln\sqrt{g})_{,ui}, \\
R_{uu} &= -\tfrac{1}{2}(g^{ij}c_{,j})_{,i} - \tfrac{1}{2}(g^{ij}c_{,j})(\ln\sqrt{g})_{,i} + (g^{ij}e_{j,u})_{,i} \\
&\quad + (g^{ij}e_{j,u})(\ln\sqrt{g})_{,i} + g^{ij}g^{kl}e_{[i,k]}e_{j,l} \\
&\quad - \tfrac{1}{4}g^{ij}g^{kl}g_{ik,u}g_{jl,u} - (\ln\sqrt{g})_{,uu},
\end{aligned}
\tag{49}
$$

where ${}^{\mathrm{s}}R_{ijkl}$ and ${}^{\mathrm{s}}R_{ij}$ denote, respectively, the Riemann and Ricci tensors corresponding to the spatial metric g_{ij} only. The Ricci scalar ${}^{\mathrm{s}}R$ (equal to R) of this transverse $(D-2)$-dimensional Riemannian space enters, in fact, only the Newton-Coulomb scalars Ψ_{2S} and $\Psi_{2T^{ij}}$. Interestingly, these are also *independent of the velocity* of the observer.

There is a big simplification if we restrict ourselves to vacuum pp-waves. As shown in [55], the absence of an aligned electromagnetic field requires that the cosmological constant Λ also vanishes, so that the transverse Riemannian space must be Ricci flat, ${}^{\mathrm{s}}R_{ij} = 0$. In such a case, $\Psi_{2S} = 0 = \Psi_{2T^{ij}}$. Moreover, since $R_{iu} = 0 = R_{uu}$, the Weyl scalar Ψ_{3T^i} also vanishes and the gravitational wave amplitudes reduce to

$$
\Psi_{4^{ij}} = 2\left[{}^{\mathrm{s}}R_{kmln}\dot{x}^m\dot{x}^n + 2R_{kmlu}\dot{x}^m\dot{u} + R_{kulu}\dot{u}^2\right]m_{(i}^k m_{j)}^l.
\tag{50}
$$

Taking the simplest possibility of a flat transverse space,

$$g_{ij} = \delta_{ij}, \tag{51}$$

we obtain an important family of exact vacuum plane-fronted gravitational waves (possibly representing an external field of gyratons [66,67]), which propagate in Minkowski space. In fact, these metrics with

$$^{s}R_{ijkl} = 0, \qquad R_{uijk} = \tfrac{1}{2}(e_{k,ij} - e_{j,ik}),$$
$$R_{iuju} = \tfrac{1}{2}(e_{i,uj} + e_{j,ui} - c_{,ij}) + \delta^{kl}e_{[k,i]}e_{[l,j]}, \tag{52}$$

belong to the family of VSI spacetimes [63].

If the functions e_i can be globally removed by a suitable coordinate transformation (in the absence of gyratonic sources), the metric reduces to

$$\mathrm{d}s^2 = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j - 2\mathrm{d}u\mathrm{d}r + c(x^k, u)\mathrm{d}u^2. \tag{53}$$

In such a case, the spatial vectors of the null frame (44) are simply $\boldsymbol{m}_i = (\dot{x}^i/\dot{u})\boldsymbol{\partial}_r + \boldsymbol{\partial}_{x^i}$, and the frame is parallelly transported. This implies that the physical relative accelerations (6) are, in fact, ordinary time derivatives of the components of the separation vector, $\ddot{Z}^{(i)} = \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}Z^{(i)}$. Moreover, $\dot{u} = \text{const}$ along the geodesic since there is $\Gamma^u_{\alpha\beta} = 0$ for the metric (53).

The scalar components of the gravitational field (50) and (52) simplify to

$$\Psi_{4^{ij}} = -\dot{u}^2 c_{,ij}. \tag{54}$$

Using (49), the only remaining Einstein's vacuum equation $R_{uu} = 0$ reads $\Delta c \equiv \delta^{ij}c_{,ij} = 0$, which explicitly guarantees that the $(D-2)\times(D-2)$ symmetric matrix of the wave amplitudes $\Psi_{4^{ij}}$ is traceless. The equations of geodesic deviation thus reduce to

$$\frac{\mathrm{d}^2Z^{(1)}}{\mathrm{d}\tau^2} = 0, \qquad \frac{\mathrm{d}^2Z^{(i)}}{\mathrm{d}\tau^2} = \frac{1}{2}\dot{u}^2 c_{,ij}Z^{(j)}, \tag{55}$$

exhibiting the transverse character of the vacuum gravitational pp-waves propagating along $\boldsymbol{e}_{(1)}$. In general, there are $\frac{1}{2}D(D-3)$ independent polarization modes corresponding to the same number of free components of the matrix $\Psi_{4^{ij}}$.

In particular, if the metric function c is a quadratic form of the transverse spatial coordinates,

$$c = \sum_{i=2}^{D-1} \mathcal{A}_i (x^i)^2, \tag{56}$$

where the constant coefficients $\mathcal{A}_i$ must satisfy

$$\sum_{i=2}^{D-1} \mathcal{A}_i = 0, \tag{57}$$

$\Psi_{4^{ij}}$ is a traceless diagonal matrix with eigenvalues $\Psi_{4^{ij}} = -2\mathcal{A}_i\dot{u}^2$. The amplitudes are constant, i.e., the corresponding gravitational waves are homogeneous. If the test particles are initially at rest [$\dot{Z}^{(i)}(\tau = 0) = 0$, $Z^{(i)}(\tau = 0) = Z^{(i)}_0 = \text{const}$], equations of geodesic deviation (55) for (56) can be explicitly integrated to

$$Z^{(1)} = Z^{(1)}_0,$$
$$Z^{(i)} = \begin{cases} Z^{(i)}_0 \cosh(\sqrt{\mathcal{A}_i}|\dot{u}|\tau) & \text{for } \mathcal{A}_i > 0, \\ Z^{(i)}_0 \cos(\sqrt{-\mathcal{A}_i}|\dot{u}|\tau) & \text{for } \mathcal{A}_i < 0, \\ Z^{(i)}_0 & \text{for } \mathcal{A}_i = 0. \end{cases} \tag{58}$$

Therefore, in the transverse spatial directions $\boldsymbol{e}_{(i)}$ with $\mathcal{A}_i > 0$ the test particles *recede*, while in those directions with $\mathcal{A}_i < 0$ they *focus*. There is also a possibility that $\mathcal{A}_i = 0$, in which case there is no influence of the gravitational wave in the corresponding transverse spatial directions.

This results in completely new effects that are not allowed in classical $D = 4$ general relativity for which $i = 2, 3$ and the constraint (57) is simply $\mathcal{A}_2 = -\mathcal{A}_3$. Therefore, either a vacuum gravitational pp-wave in four-dimensional spacetime is absent ($\mathcal{A}_2 = -\mathcal{A}_3 = 0$), or it generates specific particle motions in *both* transverse directions $\boldsymbol{e}_{(2)}$ and $\boldsymbol{e}_{(3)}$ (focusing in one of them). In higher dimensions, however, the amplitudes are coupled via the D-dimensional constraint $\mathcal{A}_2 = -\mathcal{A}_3 - \sum_{i=4}^{D-1}\mathcal{A}_i$. From the point of view of a detector located on a $(1+3)$-dimensional brane with spatial directions $\boldsymbol{e}_{(1)}$, $\boldsymbol{e}_{(2)}$, $\boldsymbol{e}_{(3)}$, this would clearly exhibit itself as a violation of standard *TT*-property of gravitational waves (unless $\sum_{i=4}^{D-1}\mathcal{A}_i = 0$, which corresponds to a very special subcase). Such an anomalous behavior could possibly serve as a sign of the existence of higher dimensions (see also discussion of a similar effect within the context of linearized five-dimensional gravitational waves [91]).

It may also happen that $\mathcal{A}_k = 0$ for some k [in which case the metric function c given by (56) is independent of the corresponding spatial coordinate x^k] and thus there is no effect of the vacuum gravitational pp-wave on test particles in the transverse spatial direction $\boldsymbol{e}_{(k)}$. Even the special situations with $\mathcal{A}_2 = 0$ or $\mathcal{A}_3 = 0$ are allowed.

VIII. CONCLUSIONS

Let us conclude this work by quoting from the classic monograph [115], page 35: "In Einstein's geometric theory of gravity, the equation of geodesic deviation summarizes the entire effect of geometry on matter." This is true not only in standard $D = 4$ general relativity, but also in its extension to any higher number of dimensions. Indeed, we have explicitly demonstrated that the geodesic deviation equation, expressed in a suitable reference frame adapted

to the observer's geodesic and to the specific algebraic structure of a given spacetime, can be used as a useful tool for analyzing and understanding the specific effects of the gravitational field in an arbitrary dimension.

In particular, we derived the general canonical decomposition (18) and (19) of relative accelerations of nearby test particles freely falling in any spacetime. The gravitational contributions, identified and described in Sec. IV, consist of the isotropic background influence (22) of the cosmological constant Λ, transverse gravitational waves (23) and (27), complementary longitudinal effects (24) and (26), and the Newton-Coulomb component (25) of the gravitational field. The matter contributions were discussed in Sec. VI, namely, the influence of a pure radiation field (null dust) (36), perfect fluid (38), and generic electromagnetic field (41).

In the final Sec. VII, we also exemplified these results on an important family of exact pp-waves in higher dimensions (admitting a covariantly constant null vector field $\boldsymbol{k}$). Their nontrivial amplitudes are given by expressions (46). The vacuum VSI subclass of such Kundt spacetimes represents purely transverse gravitational waves propagating along the WAND $\boldsymbol{k}$ (in general associated with gyratonic sources). These exact gravitational waves have amplitudes $\Psi_{4^{ij}}$ determined by Eqs. (50) and (52), which form a $(D-2)\times(D-2)$ symmetric traceless matrix. Its $\frac{1}{2}D(D-3)$ components characterize the independent polarization modes. Explicit solution of the invariant equation of geodesic deviation for the metric function (56) is given in (58). Because of coupling between the eigenvalues of $\Psi_{4^{ij}}$, such higher-dimensional gravitational waves could possibly be identified observationally in $(1+3)$-dimensional brane as a violation of standard TT-property.

We hope that the presented general method of interpreting exact spacetimes, based of the study of geodesic deviation, will help to elucidate the physical and geometrical properties of various explicit solutions of Einstein's equations in an arbitrary dimension.

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APPENDIX A: RELATION TO COMPLEX NOTATION IN $D = 4$

In standard $D = 4$ general relativity, it is usual—instead of the real null frame $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}_2, \boldsymbol{m}_3\}$—to introduce a complex null tetrad $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ and to parametrize the Weyl tensor by the corresponding five *complex* components. These Newman-Penrose scalar quantities Ψ_A, first defined in [137], are closely related to the real quantities introduced in our text. Here, we present a dictionary relating these two notations. In $D = 4$, the transverse spatial index i runs only over two values 2, 3 and we can combine the real vectors $\boldsymbol{m}_i$ into the complex vectors

$$\boldsymbol{m} \equiv \frac{1}{\sqrt{2}}(\boldsymbol{m}_2 - \mathrm{i}\boldsymbol{m}_3), \qquad \bar{\boldsymbol{m}} \equiv \frac{1}{\sqrt{2}}(\boldsymbol{m}_2 + \mathrm{i}\boldsymbol{m}_3). \tag{A1}$$

Any real spatial vector $\boldsymbol{V}$ spanned on $\boldsymbol{m}_2$, $\boldsymbol{m}_3$ can be parametrized by a complex number V via the relation

$$\boldsymbol{V} = V^2\boldsymbol{m}_2 + V^3\boldsymbol{m}_3 = \frac{1}{\sqrt{2}}(\bar{V}\boldsymbol{m} + V\bar{\boldsymbol{m}}), \tag{A2}$$

so that $V = V^2 - \mathrm{i}V^3$ and $|V|^2 \equiv (V^2)^2 + (V^3)^2 = V\bar{V}$.

In four dimensions, there are only two real independent components of the Weyl tensor for each boost weight, namely,

$$\begin{aligned}
&\Psi_{0^{22}} = -\Psi_{0^{33}}, \qquad \Psi_{0^{23}} = \Psi_{0^{32}},\\
&\Psi_{1T^2} = \Psi_{1^{332}} = -\Psi_{1^{323}}, \qquad \Psi_{1T^3} = \Psi_{1^{223}} = -\Psi_{1^{232}},\\
&\Psi_{2^{2323}} = \Psi_{2^{3232}} = -\Psi_{2^{3223}} = -\Psi_{2^{2332}}\\
&\qquad = 2\Psi_{2T^{22}} = 2\Psi_{2T^{33}} = \Psi_{2S},\\
&\Psi_{2^{23}} = -\Psi_{2^{32}} = 2\Psi_{2T^{23}} = -2\Psi_{2T^{32}},\\
&\Psi_{3T^2} = \Psi_{3^{332}} = -\Psi_{3^{323}}, \qquad \Psi_{3T^3} = \Psi_{3^{223}} = -\Psi_{3^{232}},\\
&\Psi_{4^{22}} = -\Psi_{4^{33}}, \qquad \Psi_{4^{23}} = \Psi_{4^{32}}.
\end{aligned} \tag{A3}$$

These can be combined into five complex NP components [140,141] defined by

$$\begin{aligned}
&\Psi_0 = C_{abcd}k^a m^b k^c m^d, \qquad \Psi_1 = C_{abcd}k^a l^b k^c m^d,\\
&\Psi_2 = C_{abcd}k^a m^b \bar{m}^c l^d, \qquad \Psi_3 = C_{abcd}l^a k^b l^c \bar{m}^d,\\
&\qquad \Psi_4 = C_{abcd}l^a \bar{m}^b l^c \bar{m}^d,
\end{aligned} \tag{A4}$$

as

$$\begin{aligned}
&\Psi_0 = \Psi_{0^{22}} - \mathrm{i}\Psi_{0^{23}}, \qquad \Psi_1 = \frac{1}{\sqrt{2}}(\Psi_{1T^2} - \mathrm{i}\Psi_{1T^3}),\\
&\Psi_2 = -\frac{1}{2}(\Psi_{2^{2323}} + \mathrm{i}\Psi_{2^{23}}),\\
&\Psi_3 = \frac{1}{\sqrt{2}}(\Psi_{3T^2} + \mathrm{i}\Psi_{3T^3}), \qquad \Psi_4 = \Psi_{4^{22}} + \mathrm{i}\Psi_{4^{23}}.
\end{aligned} \tag{A5}$$

Notice the differences with respect to the notation used in [93]: here we have relabeled all transverse spatial indices as $i \to i+1$ to achieve that the privileged spatial direction is denoted as $\boldsymbol{m}_1 = \boldsymbol{e}_{(1)}$, and the scalars Ψ_{2S}, $\Psi_{2T^{ij}}$ are defined in (14) without the unnecessary factor 2. (Also, there is a missing factor $\frac{1}{2}$ in Eq. (A.7c) in [93].) Inversely, we obtain

$$\begin{aligned}
&\Psi_{0^{22}} = -\Psi_{0^{33}} = \mathrm{Re}\Psi_0, \qquad \Psi_{0^{23}} = \Psi_{0^{32}} = -\mathrm{Im}\Psi_0,\\
&\Psi_{1T^2} = \sqrt{2}\mathrm{Re}\Psi_1, \qquad \Psi_{1T^3} = -\sqrt{2}\mathrm{Im}\Psi_1,\\
&\Psi_{2S} = 2\Psi_{2T^{22}} = 2\Psi_{2T^{33}} = \Psi_{2^{2323}} = -2\,\mathrm{Re}\Psi_2,\\
&\Psi_{2^{23}} = -2\,\mathrm{Im}\Psi_2,\\
&\Psi_{3T^2} = \sqrt{2}\,\mathrm{Re}\Psi_3, \qquad \Psi_{3T^3} = \sqrt{2}\,\mathrm{Im}\Psi_3,\\
&\Psi_{4^{22}} = -\Psi_{4^{33}} = \mathrm{Re}\Psi_4, \qquad \Psi_{4^{23}} = \Psi_{4^{32}} = \mathrm{Im}\Psi_4.
\end{aligned} \tag{A6}$$

According to (17), the orthonormal components $C_{(\mathrm{i})(0)(0)(\mathrm{j})}$ of the Weyl tensor are

$$\begin{aligned}
&C_{(1)(0)(0)(1)} = -2\,\mathrm{Re}\Psi_2, \qquad C_{(1)(0)(0)(2)} = +\mathrm{Re}\Psi_1 - \mathrm{Re}\Psi_3, \qquad C_{(1)(0)(0)(3)} = -\mathrm{Im}\Psi_1 - \mathrm{Im}\Psi_3,\\
&C_{(2)(0)(0)(2)} = \mathrm{Re}\Psi_2 - \tfrac{1}{2}\mathrm{Re}\Psi_0 - \tfrac{1}{2}\mathrm{Re}\Psi_4, \qquad C_{(3)(0)(0)(3)} = \mathrm{Re}\Psi_2 + \tfrac{1}{2}\mathrm{Re}\Psi_0 + \tfrac{1}{2}\mathrm{Re}\Psi_4, \qquad C_{(2)(0)(0)(3)} = \tfrac{1}{2}\mathrm{Im}\Psi_0 - \tfrac{1}{2}\mathrm{Im}\Psi_4.
\end{aligned} \tag{A7}$$

Explicit equations of geodesic deviation (11) in $D = 4$ thus take the form

$$\begin{aligned}
\ddot{Z}^{(1)} &= \frac{\Lambda}{3}Z^{(1)} - 2\,\mathrm{Re}\Psi_2 Z^{(1)} + (\mathrm{Re}\Psi_1 - \mathrm{Re}\Psi_3)Z^{(2)} - (\mathrm{Im}\Psi_1 + \mathrm{Im}\Psi_3)Z^{(3)}\\
&\quad + 4\pi\Big[T_{(1)(1)}Z^{(1)} + T_{(1)(2)}Z^{(2)} + T_{(1)(3)}Z^{(3)} - \Big(T_{(0)(0)} + \frac{2}{3}T\Big)Z^{(1)}\Big],\\
\ddot{Z}^{(2)} &= \frac{\Lambda}{3}Z^{(2)} + \mathrm{Re}\Psi_2 Z^{(2)} + (\mathrm{Re}\Psi_1 - \mathrm{Re}\Psi_3)Z^{(1)} - \frac{1}{2}(\mathrm{Re}\Psi_0 + \mathrm{Re}\Psi_4)Z^{(2)} + \frac{1}{2}(\mathrm{Im}\Psi_0 - \mathrm{Im}\Psi_4)Z^{(3)}\\
&\quad + 4\pi\Big[T_{(2)(1)}Z^{(1)} + T_{(2)(2)}Z^{(2)} + T_{(2)(3)}Z^{(3)} - \Big(T_{(0)(0)} + \frac{2}{3}T\Big)Z^{(2)}\Big],\\
\ddot{Z}^{(3)} &= \frac{\Lambda}{3}Z^{(3)} + \mathrm{Re}\Psi_2 Z^{(3)} - (\mathrm{Im}\Psi_1 + \mathrm{Im}\Psi_3)Z^{(1)} + \frac{1}{2}(\mathrm{Im}\Psi_0 - \mathrm{Im}\Psi_4)Z^{(2)} + \frac{1}{2}(\mathrm{Re}\Psi_0 + \mathrm{Re}\Psi_4)Z^{(3)}\\
&\quad + 4\pi\Big[T_{(3)(1)}Z^{(1)} + T_{(3)(2)}Z^{(2)} + T_{(3)(3)}Z^{(3)} - \Big(T_{(0)(0)} + \frac{2}{3}T\Big)Z^{(3)}\Big].
\end{aligned} \tag{A8}$$

This fully agrees with the results presented in our previous work [128] [after permuting the indices as $1 \to 2 \to 3 \to 1$, and changing the signs of all imaginary parts due to a convention different from (A1)].

APPENDIX B: RELATION TO OTHER NOTATIONS USED IN $D \geq 4$

In the literature on higher-dimensional spacetimes, it is common to use alternative conventions for the null frame and the corresponding components. In particular, in the fundamental papers on algebraic classification of the Weyl tensor [72,73] the null frame $\{\ell, \mathbf{n}, \mathbf{m}_i\}$, where $i = 2, \ldots, D-1$,

$$\ell \equiv \mathbf{m}_0, \qquad \mathbf{n} \equiv \mathbf{m}_1, \qquad \mathbf{m}_2, \ldots, \mathbf{m}_{D-1}, \tag{B1}$$

is employed such that the metric is $g_{ab} = 2\ell_{(a}n_{b)} + \delta_{ij}m^i_a m^j_b$, i.e.,

$$\begin{aligned}
&\ell \cdot \mathbf{n} = 1, \qquad \mathbf{m}_i \cdot \mathbf{m}_j = \delta_{ij},\\
&\ell \cdot \ell = 0 = \mathbf{n} \cdot \mathbf{n}, \qquad \ell \cdot \mathbf{m}_i = 0 = \mathbf{n} \cdot \mathbf{m}_i.
\end{aligned} \tag{B2}$$

Following [72,73], the Weyl tensor can be decomposed into the frame components

$$\begin{aligned}
C_{abcd} &= 4C_{0i0j}n_{\{a}m^i_b n_c m^j_{d\}}\\
&\quad + 8C_{010i}n_{\{a}\ell_b n_c m^i_{d\}} + 4C_{0ijk}n_{\{a}m^i_b m^j_c m^k_{d\}}\\
&\quad + 4C_{0101}n_{\{a}\ell_b n_c \ell_{d\}} + 8C_{0i1j}n_{\{a}m^i_b \ell_c m^j_{d\}}\\
&\quad + 4C_{01ij}n_{\{a}\ell_b m^i_c m^j_{d\}} + C_{ijkl}m^i_{\{a}m^j_b m^k_c m^l_{d\}}\\
&\quad + 8C_{101i}\ell_{\{a}n_b \ell_c m^i_{d\}} + 4C_{1ijk}\ell_{\{a}m^i_b m^j_c m^k_{d\}}\\
&\quad + 4C_{1i1j}\ell_{\{a}m^i_b \ell_c m^j_{d\}},
\end{aligned} \tag{B3}$$

where $T_{\{abcd\}} \equiv \frac{1}{2}(T_{[ab][cd]} + T_{[cd][ab]})$ is a useful notation representing the standard symmetries of the curvature tensor. The terms in the separate lines of (B3) are sorted according to their boost weight corresponding to the scaling

$$\tilde{\ell} = \lambda\ell, \qquad \tilde{\mathbf{n}} = \lambda^{-1}\mathbf{n}, \qquad \tilde{\mathbf{m}}_i = \mathbf{m}_i. \tag{B4}$$

Using (B2), we immediately infer that the various scalar components in (B3) are explicitly given as

$$
\begin{aligned}
&C_{0i0j} = C_{abcd}\ell^a m_i^b \ell^c m_j^d, \qquad C_{0ijk} = C_{abcd}\ell^a m_i^b m_j^c m_k^d,\\
&C_{010i} = C_{abcd}\ell^a n^b \ell^c m_i^d, \qquad C_{ijkl} = C_{abcd}m_i^a m_j^b m_k^c m_l^d,\\
&C_{0101} = C_{abcd}\ell^a n^b \ell^d n^c, \qquad C_{01ij} = C_{abcd}\ell^a n^b m_i^c m_j^d,\\
&C_{0i1j} = C_{abcd}\ell^a m_i^b n^c m_j^d, \qquad C_{1ijk} = C_{abcd}n^a m_i^b m_j^c m_k^d,\\
&C_{101i} = C_{abcd}n^a \ell^b n^c m_i^d, \qquad C_{1i1j} = C_{abcd}n^a m_i^b n^c m_j^d.
\end{aligned}
\tag{B5}
$$

They are subject to a number of mutual relations that follow from the symmetries and from the trace-free property of the Weyl tensor, see [72]:

$$
\begin{aligned}
&C_{0i0}{}^i = 0, \qquad C_{010j} = C_{0ij}{}^i, \qquad C_{0[ijk]} = 0,\\
&C_{0101} = C_{0i1}{}^i, \qquad C_{i[jkl]} = 0,\\
&C_{0i1j} = -\tfrac{1}{2}C_{ikj}{}^k + \tfrac{1}{2}C_{01ij}, \qquad C_{011j} = -C_{1ij}{}^i,\\
&C_{1[ijk]} = 0, \qquad C_{1i1}{}^i = 0.
\end{aligned}
\tag{B6}
$$

Now, by comparing (B2) with our definition (13) it follows that the two null frames (B1) and (12) are related simply as

$$
\boldsymbol{k} \equiv \ell, \qquad \boldsymbol{l} \equiv -\mathbf{n}, \qquad \boldsymbol{m}_i \equiv \mathbf{m}_i. \tag{B7}
$$

Putting this identification into (B5), and comparing with (14), we observe that

$$
\begin{aligned}
&\Psi_{0^{ij}} \equiv C_{0i0j},\\
&\Psi_{1^{ijk}} \equiv C_{0ijk}, \qquad \Psi_{1T^i} \equiv -C_{010i},\\
&\Psi_{2^{ijkl}} \equiv C_{ijkl}, \qquad \Psi_{2^{ij}} \equiv -C_{01ij},\\
&\Psi_{2S} \equiv -C_{0101}, \qquad \Psi_{2T^{ij}} \equiv -C_{0i1j},\\
&\Psi_{3^{ijk}} \equiv -C_{1ijk}, \qquad \Psi_{3T^i} \equiv C_{101i},\\
&\Psi_{4^{ij}} \equiv C_{1i1j}.
\end{aligned}
\tag{B8}
$$

Moreover, the relations (B6) are equivalent to the constraints (15) and (16).

Also, in [62,63,76,83] the notation

$$
\Psi_{ij} \equiv \tfrac{1}{2}C_{1i1j}, \qquad \Psi_{ijk} \equiv \tfrac{1}{2}C_{1kij}, \qquad \Psi_i \equiv C_{101i} \tag{B9}
$$

was introduced and employed, which is useful for studies of type-N and type-III spacetimes, and

$$
\begin{aligned}
&\Phi_{ij} \equiv C_{0i1j}, \qquad \Phi^A_{ij} \equiv \tfrac{1}{2}C_{01ij}, \qquad \Phi^S_{ij} \equiv -\tfrac{1}{2}C_{ikj}{}^k,\\
&\Phi \equiv C_{0101} = -\tfrac{1}{2}C_{ij}{}^{ij},
\end{aligned}
\tag{B10}
$$

(where Φ^A_{ij}, Φ^S_{ij}, Φ denote antisymmetric, symmetric parts of Φ_{ij} and its trace, respectively) which is convenient for type-D spacetimes [78,81,82,99]. In view of (B8), we thus easily identify

$$
\begin{aligned}
&\Psi_{2^{ijkl}} \equiv C_{ijkl}, \qquad \Psi_{2^{ij}} \equiv -2\Phi^A_{ij}, \qquad \Psi_{2S} \equiv -\Phi,\\
&\Psi_{2T^{ij}} \equiv -\Phi_{ij}, \qquad \Psi_{3^{ijk}} \equiv -2\Psi_{jki}, \qquad \Psi_{3T^i} \equiv \Psi_i,\\
&\Psi_{4^{ij}} \equiv 2\Psi_{ij}.
\end{aligned}
\tag{B11}
$$

Very recently, in the generalization of the Geroch-Held-Penrose formalism to higher dimensions [79], another convention was suggested, namely,

$$
\begin{aligned}
&\Omega_{ij} \equiv C_{0i0j}, \qquad \Psi_{ijk} \equiv C_{0ijk}, \qquad \Psi_i \equiv C_{010i},\\
&\Omega'_{ij} \equiv C_{1i1j}, \qquad \Psi'_{ijk} \equiv C_{1ijk}, \qquad \Psi'_i \equiv C_{101i},\\
&\Phi_{ijkl} \equiv C_{ijkl}.
\end{aligned}
\tag{B12}
$$

These scalars are straightforwardly related to the quantities used in the present paper:

$$
\begin{aligned}
&\Psi_{0^{ij}} \equiv \Omega_{ij},\\
&\Psi_{1^{ijk}} \equiv \Psi_{ijk}, \qquad \Psi_{1T^i} \equiv -\Psi_i,\\
&\Psi_{2^{ijkl}} \equiv \Phi_{ijkl}, \qquad \Psi_{2^{ij}} \equiv -2\Phi^A_{ij},\\
&\Psi_{2S} \equiv -\Phi, \qquad \Psi_{2T^{ij}} \equiv -\Phi_{ij},\\
&\Psi_{3^{ijk}} \equiv -\Psi'_{ijk}, \qquad \Psi_{3T^i} \equiv \Psi'_i,\\
&\Psi_{4^{ij}} \equiv \Omega'_{ij}.
\end{aligned}
\tag{B13}
$$

APPENDIX C: LORENTZ TRANSFORMATIONS OF THE NULL FRAME AND THE CHANGES OF Ψ_A...

It is well known (see, e.g., [72,73,93]) that general transformations between different null frames can be composed from the following simple Lorentz transformations:

(i) *null rotation with* $\boldsymbol{k}$ *fixed* (parametrized by $D-2$ real parameters L^i):

$$
\tilde{\boldsymbol{k}} = \boldsymbol{k}, \quad \tilde{\boldsymbol{l}} = \boldsymbol{l} + \sqrt{2}L^i\boldsymbol{m}_i + |L|^2\boldsymbol{k}, \quad \tilde{\boldsymbol{m}}_i = \boldsymbol{m}_i + \sqrt{2}L_i\boldsymbol{k}, \tag{C1}
$$

(ii) *null rotation with* $\boldsymbol{l}$ *fixed* (parametrized by $D-2$ real parameters K^i):

$$
\tilde{\boldsymbol{k}} = \boldsymbol{k} + \sqrt{2}K^i\boldsymbol{m}_i + |K|^2\boldsymbol{l}, \quad \tilde{\boldsymbol{l}} = \boldsymbol{l}, \quad \tilde{\boldsymbol{m}}_i = \boldsymbol{m}_i + \sqrt{2}K_i\boldsymbol{l}, \tag{C2}
$$

(iii) *boost in the* $\boldsymbol{k}-\boldsymbol{l}$ *plane* (parametrized by a real number B):

$$
\tilde{\boldsymbol{k}} = B\boldsymbol{k}, \qquad \tilde{\boldsymbol{l}} = B^{-1}\boldsymbol{l}, \qquad \tilde{\boldsymbol{m}}_i = \boldsymbol{m}_i, \tag{C3}
$$

(iv) *spatial rotation in the space of* $\boldsymbol{m}_i$ (parametrized by an orthogonal matrix $\Phi_i{}^j$):

$$
\tilde{\boldsymbol{k}} = \boldsymbol{k}, \qquad \tilde{\boldsymbol{l}} = \boldsymbol{l}, \qquad \tilde{\boldsymbol{m}}_i = \Phi_i{}^j\boldsymbol{m}_j, \quad \text{with} \quad \Phi_i{}^j\Phi_k{}^l\delta_{jl} = \delta_{ik}. \tag{C4}
$$

Because of (13), $L^i = L_i$, $K^i = K_i$, and we employ a shorthand $|L|^2 \equiv L^i L_i$, $|K|^2 \equiv K^i K_i$. Under these Lorentz transformations of the frame, the Weyl scalars change as

(i) null rotation with $\boldsymbol{k}$ fixed:

$$
\begin{aligned}
\tilde{\Psi}_{0^{ij}} &= \Psi_{0^{ij}},\\
\tilde{\Psi}_{1^{ijk}} &= \Psi_{1^{ijk}} - 2\sqrt{2}\Psi_{0^{[ij}}L_{k]},\\
\tilde{\Psi}_{1T^i} &= \Psi_{1T^i} + \sqrt{2}\Psi_{0^{ij}}L^j,\\
\tilde{\Psi}_{2^{ijkl}} &= \Psi_{2^{ijkl}} - 2\sqrt{2}(L_{[l}\Psi_{1^{k]ij}} - L_{[i}\Psi_{1^{j]kl}}) + 4(\Psi_{0^{[ik}}L_{l]}L_j + \Psi_{0^{j[l}}L_{k]}L_i),\\
\tilde{\Psi}_{2S} &= \Psi_{2S} - 2\sqrt{2}\Psi_{1T^i}L^i - 2\Psi_{0^{ij}}L^iL^j,\\
\tilde{\Psi}_{2^{ij}} &= \Psi_{2^{ij}} + \sqrt{2}\Psi_{1^{kij}}L^k - 2\sqrt{2}\Psi_{1T^{[i}}L_{j]} - 4\Psi_{0^{k[i}}L_{j]}L^k,\\
\tilde{\Psi}_{2T^{ij}} &= \Psi_{2T^{ij}} + \sqrt{2}\Psi_{1^{ikj}}L^k - \sqrt{2}\Psi_{1T^i}L_j - 2\Psi_{0^{ik}}L^kL_j + \Psi_{0^{ij}}|L|^2,\\
\tilde{\Psi}_{3^{ijk}} &= \Psi_{3^{ijk}} + \sqrt{2}(\Psi_{2^{li[jk}}L^l - \Psi_{2^{j]k}}L_i + 2L_{[j}\Psi_{2T^{k]i}})\\
&\quad + 4\Psi_{1T^{[j}}L_{k]}L_i - 2(\Psi_{1^{jli}}L_k + \Psi_{1^{lik}}L_j - \Psi_{1^{kli}}L_j)L^l + \Psi_{1^{ijk}}|L|^2 + 4\sqrt{2}\Psi_{0^{l[j}}L_{k]}L_iL^l - 2\sqrt{2}\Psi_{0^{i[j}}L_{k]}|L|^2,\\
\tilde{\Psi}_{3T^i} &= \Psi_{3T^i} + \sqrt{2}\Psi_{2^{ij}}L^j - \sqrt{2}(\Psi_{2T^{ki}}L^k + \Psi_{2S}L_i)\\
&\quad + 2(2\Psi_{1T^j}L_i - \Psi_{1^{kji}}L^k)L^j - \Psi_{1T^i}|L|^2 + 2\sqrt{2}\Psi_{0^{jk}}L^jL^kL_i - \sqrt{2}\Psi_{0^{ij}}L^j|L|^2,\\
\tilde{\Psi}_{4^{ij}} &= \Psi_{4^{ij}} + 2\sqrt{2}(\Psi_{3T^{(i}}L_{j)} - \Psi_{3^{(ij)k}}L^k)\\
&\quad + 2\Psi_{2^{ikjl}}L^kL^l - 4\Psi_{2T^{k(i}}L_{j)}L^k + 2\Psi_{2T^{(ij)}}|L|^2 - 2\Psi_{2S}L_iL_j - 4\Psi_{2^{k(i}}L_{j)}L^k\\
&\quad - 2\sqrt{2}(2\Psi_{1^{kl(i}}L_{j)}L^kL^l + \Psi_{1^{(ij)k}}L^k|L|^2 + \Psi_{1T^{(i}}L_{j)}|L|^2 - 2\Psi_{1T^k}L^kL_iL_j)\\
&\quad + 4\Psi_{0^{kl}}L^kL^lL_iL_j - 4\Psi_{0^{k(i}}L_{j)}L^k|L|^2 + \Psi_{0^{ij}}|L|^4,
\end{aligned}
\tag{C5}
$$

(ii) null rotation with $\boldsymbol{l}$ fixed:

$$
\begin{aligned}
\tilde{\Psi}_{0^{ij}} &= \Psi_{0^{ij}} + 2\sqrt{2}(\Psi_{1T^{(i}}K_{j)} - \Psi_{1^{(ij)k}}K^k)\\
&\quad + 2\Psi_{2^{ikjl}}K^kK^l - 4K_{(i}\Psi_{2T^{j)k}}K^k + 2\Psi_{2T^{(ij)}}|K|^2 - 2\Psi_{2S}K_iK_j + 4\Psi_{2^{k(i}}K_{j)}K^k\\
&\quad - 2\sqrt{2}(2\Psi_{3^{kl(i}}K_{j)}K^kK^l + \Psi_{3^{(ij)k}}K^k|K|^2 + \Psi_{3T^{(i}}K_{j)}|K|^2 - 2\Psi_{3T^k}K^kK_iK_j)\\
&\quad + 4\Psi_{4^{kl}}K^kK^lK_iK_j - 4\Psi_{4^{k(i}}K_{j)}K^k|K|^2 + \Psi_{4^{ij}}|K|^4,\\
\tilde{\Psi}_{1^{ijk}} &= \Psi_{1^{ijk}} + \sqrt{2}(\Psi_{2^{li[jk}}K^l + \Psi_{2^{j]k}}K_i - 2\Psi_{2T^{i[j}}K_{k]})\\
&\quad + 4\Psi_{3T^{[j}}K_{k]}K_i - 2(\Psi_{3^{jli}}K_k + \Psi_{3^{lik}}K_i - \Psi_{3^{kli}}K_j)K^l + \Psi_{3^{ijk}}|K|^2\\
&\quad + 4\sqrt{2}\Psi_{4^{l[j}}K_{k]}K_iK^l - 2\sqrt{2}\Psi_{4^{i[j}}K_{k]}|K|^2,\\
\tilde{\Psi}_{1T^i} &= \Psi_{1T^i} + \sqrt{2}\Psi_{2^{ji}}K^j - \sqrt{2}(\Psi_{2T^{ij}}K^j + \Psi_{2S}K_i)\\
&\quad + 2(2\Psi_{3T^j}K_i - \Psi_{3^{kji}}K^k)K^j - \Psi_{3T^i}|K|^2 + 2\sqrt{2}\Psi_{4^{jk}}K^jK^kK_i - \sqrt{2}\Psi_{4^{ij}}K^j|K|^2,\\
\tilde{\Psi}_{2^{ijkl}} &= \Psi_{2^{ijkl}} - 2\sqrt{2}(K_{[l}\Psi_{3^{k]ij}} - K_{[i}\Psi_{3^{j]kl}}) + 4(\Psi_{4^{i[k}}K_{l]}K_j + \Psi_{4^{j[l}}K_{k]}K_i),\\
\tilde{\Psi}_{2S} &= \Psi_{2S} - 2\sqrt{2}\Psi_{3T^i}K^i - 2\Psi_{4^{ij}}K^iK^j,\\
\tilde{\Psi}_{2^{ij}} &= \Psi_{2^{ij}} - \sqrt{2}\Psi_{3^{kij}}K^k + 2\sqrt{2}\Psi_{3T^{[i}}K_{j]} + 4\Psi_{4^{k[i}}K_{j]}K^k,\\
\tilde{\Psi}_{2T^{ij}} &= \Psi_{2T^{ij}} + \sqrt{2}\Psi_{3^{jki}}K^k - \sqrt{2}\Psi_{3T^j}K_i - 2\Psi_{4^{jk}}K^kK_i + \Psi_{4^{ij}}|K|^2,\\
\tilde{\Psi}_{3^{ijk}} &= \Psi_{3^{ijk}} - 2\sqrt{2}\Psi_{4^{i[j}}K_{k]},\\
\tilde{\Psi}_{3T^i} &= \Psi_{3T^i} + \sqrt{2}\Psi_{4^{ij}}K^j,\\
\tilde{\Psi}_{4^{ij}} &= \Psi_{4^{ij}},
\end{aligned}
\tag{C6}
$$

(iii) boost in the $\boldsymbol{k}-\boldsymbol{l}$ plane:

$$\begin{aligned}
&\tilde{\Psi}_{0^{ij}} = B^2\Psi_{0^{ij}},\\
&\tilde{\Psi}_{1^{ijk}} = B\Psi_{1^{ijk}}, \qquad \tilde{\Psi}_{1T^i} = B\Psi_{1T^i},\\
&\tilde{\Psi}_{2^{ijkl}} = \Psi_{2^{ijkl}}, \qquad \tilde{\Psi}_{2^{ij}} = \Psi_{2^{ij}}, \qquad \tilde{\Psi}_{2S} = \Psi_{2S}, \qquad \tilde{\Psi}_{2T^{ij}} = \Psi_{2T^{ij}},\\
&\tilde{\Psi}_{3^{ijk}} = B^{-1}\Psi_{3^{ijk}}, \qquad \tilde{\Psi}_{3T^i} = B^{-1}\Psi_{3T^i},\\
&\tilde{\Psi}_{4^{ij}} = B^{-2}\Psi_{4^{ij}},
\end{aligned} \tag{C7}$$

(iv) spatial rotation in the space of $\boldsymbol{m}_i$:

$$\begin{aligned}
&\tilde{\Psi}_{0^{ij}} = \Phi_i{}^p\Phi_j{}^q\Psi_{0^{pq}},\\
&\tilde{\Psi}_{1^{ijk}} = \Phi_i{}^o\Phi_j{}^p\Phi_k{}^q\Psi_{1^{opq}}, \qquad \tilde{\Psi}_{1T^i} = \Phi_i{}^p\Psi_{1T^p},\\
&\tilde{\Psi}_{2^{ijkl}} = \Phi_i{}^n\Phi_j{}^o\Phi_k{}^p\Phi_l{}^q\Psi_{2^{nopq}}, \qquad \tilde{\Psi}_{2S} = \Psi_{2S},\\
&\tilde{\Psi}_{2^{ij}} = \Phi_i{}^p\Phi_j{}^q\Psi_{2^{pq}}, \qquad \tilde{\Psi}_{2T^{ij}} = \Phi_i{}^p\Phi_j{}^q\Psi_{2T^{pq}},\\
&\tilde{\Psi}_{3^{ijk}} = \Phi_i{}^o\Phi_j{}^p\Phi_k{}^q\Psi_{3^{opq}}, \qquad \tilde{\Psi}_{3T^i} = \Phi_i{}^p\Psi_{3T^p},\\
&\tilde{\Psi}_{4^{ij}} = \Phi_i{}^p\Phi_j{}^q\Psi_{4^{pq}}.
\end{aligned} \tag{C8}$$

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CHAPTER

4

PROPERTIES OF GENERAL NON-TWISTING GEOMETRIES

In this chapter we will present specific geometrical properties of a general family of spacetimes admitting a nontwisting null congruence of geodesics. Spacetimes of this type are well known in standard four-dimensional relativity, see [4] for review, and some of their subclasses were also investigated in the context of higher-dimensional theory, see e.g. [28, 31, 32, 34]. Employing the general formalism derived in Chapter 3, we will decompose the equation of geodesic deviation into specific components with respect to their boost weight which will be useful for subsequent analysis of relative motions of free test particles in more specific classes of nontwisting spacetimes, namely the Kundt (Chapter 5) and the Robinson–Trautman (Chapter 6) family.

4.1 Geometric description of the spacetimes

Following a standard approach, first we would like to heuristically construct a line element of spacetimes which can be foliated by null hypersurfaces with normal (and also tangent) affinely parameterized geodesic null vector field. In section 4.2, we will show that the congruence of integral curves generated by this vector field is necessary nontwisting and, on the other hand, that the requirement of a nontwisting geodesic field induces the foliation by orthogonal null hypersurfaces.

Initially, we consider a general Lorentzian manifold $\mathcal{M}$ of arbitrary dimension $D \geq 4$ which is locally covered by coordinates x^a where $a = 0, \ldots, D-1$. We introduce a family of null hypersurfaces, labeled by u, by a constraint $u(x^a) = const.$ and we choose the function u as a new coordinate on this manifold. The tangent and also normal of these hypersurfaces is simply given[1] by $\tilde{k}_a = -u_{,a} = -\delta^u_a$ which must satisfy $g^{ab}\tilde{k}_a\tilde{k}_b = g^{ab}u_{,a}u_{,b} = 0$ by the definition, and we thus obtain restriction on the metric function $g^{uu} = 0$.

In the next step we construct a null vector field $\tilde{k}^b = g^{ab}\tilde{k}_a = -g^{ub}$ whose integral curves are affinely parameterized geodesics, i.e., $\tilde{k}_{a;b}\tilde{k}^b = 0$.[2] The corresponding affine parameter r will be taken as the other coordinate on the manifold $\mathcal{M}$, i.e., the null vector field $\tilde{\mathbf{k}}$ is simply $\tilde{\mathbf{k}} = \partial_r$. In

[1] We could consider a more general case $\tilde{k}_a = -f(x^b)u_{,a}$, but the corresponding integral curves will not be affinely parameterized. We assume the affine parameterization for a simpler construction of optical scalars.

[2] Proof: the covariant differentiation of the scalar product $0 = \tilde{k}^b\tilde{k}_b$ gives $0 = (\tilde{k}^b\tilde{k}_b)_{;a} = 2\tilde{k}_{b;a}\tilde{k}^b$, and the explicit form of the covariant derivative $\tilde{k}_{a;b} = -u_{,ab} + \Gamma^c_{ab}u_{,c}$ imply the $a \leftrightarrow b$ symmetry.

coordinate components it means $\tilde{k}^b = \delta^b_r$ which is also given by $\tilde{k}^b = -g^{ub}$. We thus immediately obtain additional requirements on the metric functions, namely $g^{ur} = -1$ and $g^{ui} = 0$.

Our set of adapted coordinates (r, u, x^i) thus consists of the affine parameter r along null rays generated by the vector field $\tilde{\mathbf{k}}$, u which labels null hypersurfaces orthogonal to $\tilde{\mathbf{k}}$, and $D-2$ spatial coordinates x^i covering the transverse Riemannian space. The resulting line element of the spacetime takes the form

$$\mathrm{d}s^2 = g_{ij}(r,u,x)\,\mathrm{d}x^i\mathrm{d}x^j + 2g_{ui}(r,u,x)\,\mathrm{d}x^i\mathrm{d}u - 2\mathrm{d}u\mathrm{d}r + g_{uu}(r,u,x)\,\mathrm{d}u^2 \ , \tag{4.1}$$

where g_{ij}, i.e., the metric on the transverse space, and components $g_{ui} = g_{ij}g^{rj}$ and $g_{uu} = -g^{rr}+g_{ui}g^{ri}$ can be arbitrary functions of all coordinates, and the indices i, j range from 2 to $D-1$.

Finally notice that in the case of general metric function g_{ru}, which corresponds to a nonaffinely parameterized vector field, we can always make a transformation of the parameter $r \to \tilde{r}(r,u,x)$ and thus obtain line element with the gauge $g_{ru} = -1$, and with other metric functions rescaled.

4.2 Optical scalars

In [29, 30] properties of the null geodesic congruences and optical scalars in a general D-dimensional spacetime were discussed. It is natural to introduce a frame[3] ($\mathbf{k}$, $\mathbf{l}$, $\mathbf{m}_i$) consisting of two future oriented null vectors $\mathbf{k}$ and $\mathbf{l}$, and $D-2$ orthonormal spacelike vectors $\mathbf{m}_i$,

$$\mathbf{k}\cdot\mathbf{l} = -1 \ , \qquad \mathbf{m}_i\cdot\mathbf{m}_j = \delta_{ij} \ , \qquad \mathbf{k}\cdot\mathbf{k} = \mathbf{l}\cdot\mathbf{l} = 0 \ , \qquad \mathbf{k}\cdot\mathbf{m}_i = \mathbf{l}\cdot\mathbf{m}_i = 0 \ . \tag{4.2}$$

The metric is thus given by $g_{ab} = -2k_{(a}l_{b)} + \delta_{ij}m^i_a m^j_b$, and the covariant derivative of the null vector $\mathbf{k}$ can be rewritten in terms of this frame as[4]

$$k_{a;b} = K_{11}k_ak_b + K_{10}k_al_b + K_{1i}k_am^i_b + K_{i1}m^i_ak_b + K_{i0}m^i_al_b + K_{ij}m^i_am^j_b \ . \tag{4.3}$$

The components K_{ab} can be expressed[5] inversely as

$$\begin{aligned} &K_{11} = k_{a;b}l^al^b \ , && K_{10} = k_{a;b}l^ak^b \ , && K_{1i} = -k_{a;b}l^am^b_i \ , \\ &K_{i1} = -k_{a;b}m^a_il^b \ , && K_{i0} = -k_{a;b}m^a_ik^b \ , && K_{ij} = k_{a;b}m^a_im^b_j \ . \end{aligned} \tag{4.4}$$

Using the decomposition (4.3), the condition of geodesicity can be rewritten as $k_{a;b}k^b = -K_{10}k_a - K_{i0}m^i_a$, and we find that general vector field $\mathbf{k}$ could be geodesic, i.e. $k_{a;b}k^b \sim k_a$, and even affinely parameterized, i.e. $k_{a;b}k^b = 0$, if and only if $K_{i0} = 0$ and $K_{10} = 0$.

Geometrical properties of the congruence of integral curves generated by the null vector field $\mathbf{k}$ are characterized by the matrix K_{ij} which remains invariant under null rotations with $\mathbf{k}$ fixed, see (4.44) below, and is only simply rescaled under boosts in the $\mathbf{k}-\mathbf{l}$ plane, given in (4.45); detailed description of the Lorentz transformations can be found in Chapter 3. For further analysis of the congruence we now decompose the matrix K_{ij} into its trace, traceless symmetric part, and antisymmetric part,

$$K_{ij} = \Theta\delta_{ij} + \sigma_{ij} + A_{ij} \ , \tag{4.5}$$

which are called expansion, shear matrix and twist matrix, respectively, and can be explicitly written as

$$\Theta = \frac{\mathrm{Tr}K_{ij}}{D-2} \ , \qquad \sigma_{ij} = K_{(ij)} - \frac{\mathrm{Tr}K_{ij}}{D-2}\delta_{ij} \ , \qquad A_{ij} = K_{[ij]} \ , \tag{4.6}$$

[3] Particular choice of such frame will be given in Section 4.5.

[4] The components K_{01}, K_{00} and K_{0i} are not present because of $(g^{ab}k_ak_b)_{;c} = 0$ and thus $k_{a;c}k^a = 0$.

[5] The coefficients K_{ab} correspond to L_{ab} used in [30] except for sign changes given by a different normalisation of the null vectors $\mathbf{k}\cdot\mathbf{l} = \pm 1$.

These quantities are also preserved under the null rotations with fixed $\mathbf{k}$. For affinely parameterized geodesics we can construct scalars corresponding to the standard expansion, shear and twist, expressed only in terms of covariant derivatives of the null vector $\mathbf{k}$:

$$\Theta = \frac{1}{D-2} k^{a}_{\ ;a} \ , \qquad \sigma^2 = k_{(a;b)}k^{a;b} - \frac{1}{D-2}(k^{a}_{\ ;a})^2 \ , \qquad A^2 = -k_{[a;b]}k^{a;b} \ . \tag{4.7}$$

Vanishing σ^2 and A^2 scalars are equivalent to $\sigma_{ij} = 0$ and $A_{ij} = 0$ for all values i, j.

We may calculate the optical scalars (4.7) for the geodesic and affinely parameterized vector field $\tilde{\mathbf{k}}$ introduced in the previous section and used for our initial construction of the line element (4.1). Its covariant derivative is $\tilde{k}_{a;b} = \frac{1}{2} g_{ab,r}$ and the optical scalars thus are

$$\begin{aligned} \Theta &= \frac{1}{2(D-2)} g^{ab} g_{ab,r} = \frac{1}{2(D-2)} g^{ij} g_{ij,r} = \frac{1}{2(D-2)} (\ln \det g_{ij})_{,r} \ , \\ \sigma^2 &= \frac{1}{4}\left(g^{ac}g^{bd} - \frac{1}{D-2} g^{ab}g^{cd} \right) g_{ab,r} g_{cd,r} = \frac{1}{4} g^{ik}g^{jl} g_{ij,r} g_{kl,r} - (D-2)\Theta^2 \ , \\ A^2 &= 0 \ , \end{aligned} \tag{4.8}$$

where $a\,, b\,, c\,, d = 0, \ldots, D-2$ and $i\,, j\,, k\,, l = 2, \ldots, D-2$. We conclude that the spacetime (4.1) admit nontwisting null congruence of geodesics generated by the vector field $\tilde{\mathbf{k}}$.

In a more general way, a congruence of null curves with tangent k^a is (locally) orthonormal to a family of null hypersurfaces u if and only if

$$k_{[a;b}k_{c]} = 0. \tag{4.9}$$

This is obviously satisfied by our tangent $\tilde{k}_a = -u_{,a}$ which generates the existence of a nontwisting congruence. On the other hand, the existence of hypersurfaces (locally) orthogonal to a general vector field satisfying this condition is guaranteed by the Frobenius theorem, see [4] for more details. Crucial observation is that the condition (4.9) for geodesic null congruence can be rewritten using (4.3) only in terms of the twist matrix A_{ij} as

$$k_{[a;b}k_{c]} = \frac{1}{3} A_{ij} m^{i}_{[a} m^{j}_{b} k_{c]} \ . \tag{4.10}$$

For an arbitrary null vector l^c we get $k_{[a;b}k_{c]}l^c = -\frac{1}{3}A_{ij}m^i_a m^j_b$, from which it follows that nontwisting requirement $A_{ij} = 0$ is thus equivalent to the condition (4.9).

Finally, for further discussion of special cases such as shearfree and nonexpanding Kundt and expanding Robinson–Trautman spacetimes it is useful to assume that spatial metric g_{ij} can be decomposed[6] as $\gamma_{ij} = p^2 g_{ij}$, see e.g. [34], where γ_{ij} is an unimodular matrix, i.e., $\det \gamma_{ij} = 1$. For the determinant of the complete metric we immediately get

$$\det g_{ab} = -\det g_{ij} = p^{-2(D-2)} \ , \qquad \text{where} \qquad g_{ij} = p^{-2}\gamma_{ij} \ . \tag{4.11}$$

The nonvanishing shear and expansion scalars (4.7) now become

$$\Theta = -(\ln p)_{,r} \ , \qquad \sigma^2 = \frac{1}{4}\gamma^{ik}\gamma^{jl}\gamma_{ij,r}\gamma_{kl,r} \ , \tag{4.12}$$

which will be useful for determination of r-dependence of the spatial part of the line element with respect to the optical properties of the geodetic congruence, see also [31, 34].

[6] This assumption is taken without loss of generality. Although this decomposition is not invariant with respect to the choice of the spatial coordinates x^i, it gives unique conditions for the r-dependence of the optical scalars.

4.3 Riemann and Weyl tensors

Considering the general form (4.1) of the line element of a nontwisting spacetime we can now calculate all quantities characterizing its curvature. Using the common expression for Christoffel symbols,

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(g_{db,c} + g_{dc,b} - g_{bc,d}) \ , \tag{4.13}$$

where nontrivial components of an inverse metric g^{ab} are

$$g^{ij} \ , \qquad g^{ri} = g^{ij} g_{uj} \ , \qquad g_{ru} = -1 \ , \qquad g^{rr} = -g_{uu} + g^{ij} g_{ui} g_{uj} \ , \tag{4.14}$$

we immediately obtain

$$\begin{aligned}
&\Gamma^r_{rr} = 0 \ , \qquad \Gamma^r_{ru} = -\frac{1}{2} g_{uu,r} + \frac{1}{2} g^{ri} g_{ui,r} \ , \qquad \Gamma^r_{rj} = -\frac{1}{2} g_{uj,r} + \frac{1}{2} g^{ri} g_{ij,r} \ , \\
&\Gamma^r_{uu} = \frac{1}{2}\left[-g^{rr} g_{uu,r} - g_{uu,u} + g^{ri}(2g_{ui,u} - g_{uu,i})\right] \ , \\
&\Gamma^r_{uj} = \frac{1}{2}\left[-g^{rr} g_{uj,r} - g_{uu,j} + g^{ri}(2g_{i(u,j)} - g_{uj,i})\right] \ , \\
&\Gamma^r_{jk} = \frac{1}{2}\left[-g^{rr} g_{jk,r} - 2g_{u(j,k)} + g_{jk,u} + g^{ri}(2g_{i(j,k)} - g_{jk,i})\right] \ ,
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
&\Gamma^u_{rr} = 0 \ , \qquad \Gamma^u_{ru} = 0 \ , \qquad \Gamma^u_{ri} = 0 \ , \\
&\Gamma^u_{uu} = \frac{1}{2} g_{uu,r} \ , \qquad \Gamma^u_{ui} = \frac{1}{2} g_{ui,r} \ , \qquad \Gamma^u_{ij} = \frac{1}{2} g_{ij,r} \ ,
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
&\Gamma^i_{rr} = 0 \ , \qquad \Gamma^i_{ru} = \frac{1}{2} g^{ij} g_{uj,r} \ , \qquad \Gamma^i_{rk} = \frac{1}{2} g^{ij} g_{jk,r} \ , \\
&\Gamma^i_{uu} = \frac{1}{2}\left[-g^{ri} g_{uu,r} + g^{ij}(2g_{uj,u} - g_{uu,j})\right] \ , \\
&\Gamma^i_{uk} = \frac{1}{2}\left[-g^{ri} g_{uk,r} + g^{ij}(2g_{j(u,k)} - g_{uk,j})\right] \ , \\
&\Gamma^i_{kl} = \frac{1}{2}\left[-g^{ri} g_{kl,r} + g^{ij}(2g_{j(k,l)} - g_{kl,j})\right] \ ,
\end{aligned} \tag{4.17}$$

where $i\,, j\,, k = 2, \ldots, D-1$. The standard definition of the Riemann curvature tensor,

$$R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed} \ , \tag{4.18}$$

the indices lowering by $R_{abcd} = g_{ae} R^e_{bcd}$, and long calculations lead to

$$R_{rprq} = -\frac{1}{2} g_{pq,rr} + \frac{1}{4} g^{ij} g_{ip,r} g_{jq,r} \ , \tag{4.19}$$

$$R_{rpru} = -\frac{1}{2} g_{up,rr} + \frac{1}{4} g^{ij} g_{ip,r} g_{uj,r} \ , \tag{4.20}$$

$$R_{ruru} = -\frac{1}{2} g_{uu,rr} + \frac{1}{4} g^{ij} g_{ui,r} g_{uj,r} \ , \tag{4.21}$$

$$R_{rpkq} = g_{p[k,q],r} + \frac{1}{4}\left(g_{pk,r}g_{uq,r} - g_{pq,r}g_{uk,r}\right) - \frac{1}{4}g^{ri}\left(g_{pk,r}g_{iq,r} - g_{pq,r}g_{ik,r}\right)$$
$$+\frac{1}{4}g^{ij}\left[g_{iq,r}\left(2g_{j(p,k)} - g_{pk,j}\right) - g_{ik,r}\left(2g_{j(p,q)} - g_{pq,j}\right)\right] \ , \tag{4.22}$$

$$R_{rpuq} = g_{p[u,q],r} + \frac{1}{4}\left(g_{up,r}g_{uq,r} - g_{pq,r}g_{uu,r}\right) - \frac{1}{4}g^{ri}\left(g_{up,r}g_{iq,r} - g_{pq,r}g_{ui,r}\right)$$
$$+\frac{1}{4}g^{ij}\left[g_{iq,r}\left(2g_{j(u,p)} - g_{up,j}\right) - g_{ui,r}\left(2g_{j(p,q)} - g_{pq,j}\right)\right] \ , \tag{4.23}$$

$$R_{rupq} = g_{u[p,q],r} - \frac{1}{4}g^{ri}\left(g_{up,r}g_{iq,r} - g_{uq,r}g_{ip,r}\right)$$
$$+\frac{1}{4}g^{ij}\left[g_{iq,r}\left(2g_{j(u,p)} - g_{up,j}\right) - g_{ip,r}\left(2g_{j(u,q)} - g_{uq,j}\right)\right] \ , \tag{4.24}$$

$$R_{ruup} = g_{u[u,p],r} - \frac{1}{4}g^{ri}\left(g_{uu,r}g_{ip,r} - g_{up,r}g_{ui,r}\right)$$
$$+\frac{1}{4}g^{ij}\left[g_{ip,r}\left(2g_{uj,u} - g_{uu,j}\right) - g_{ui,r}\left(2g_{j(u,p)} - g_{up,j}\right)\right] \ , \tag{4.25}$$

$$R_{kplq} = {}^{S}R_{kplq} + \frac{1}{4}g^{rr}\left(g_{kq,r}g_{pl,r} - g_{kl,r}g_{pq,r}\right)$$
$$+\frac{1}{4}g_{kl,r}\left[g_{pq,u} - 2g_{u(p,q)} + g^{ri}(2g_{i(p,q)} - g_{pq,i})\right]$$
$$+\frac{1}{4}g_{pq,r}\left[g_{kl,u} - 2g_{u(k,l)} + g^{ri}(2g_{i(k,l)} - g_{kl,i})\right]$$
$$-\frac{1}{4}g_{kq,r}\left[g_{pl,u} - 2g_{u(p,l)} + g^{ri}(2g_{i(p,l)} - g_{pl,i})\right]$$
$$-\frac{1}{4}g_{pl,r}\left[g_{kq,u} - 2g_{u(k,q)} + g^{ri}(2g_{i(k,q)} - g_{kq,i})\right] \ , \tag{4.26}$$

$$R_{upkq} = g_{p[k,q],u} - g_{u[k,q],p} + \frac{1}{4}g^{rr}\left(g_{uq,r}g_{pk,r} - g_{uk,r}g_{pq,r}\right) + \frac{1}{4}\left(g_{uu,q}g_{pk,r} - g_{uu,k}g_{pq,r}\right)$$
$$+\frac{1}{4}\left[g_{uk,r}\left(g_{pq,u} - 2g_{u(p,q)}\right) - g_{uq,r}\left(g_{pk,u} - 2g_{u(p,k)}\right)\right]$$
$$+\frac{1}{4}g^{ri}\left[g_{uk,r}\left(2g_{i(p,q)} - g_{pq,i}\right) - g_{pk,r}(2g_{i(u,q)} - g_{uq,i})\right]$$
$$-\frac{1}{4}g^{ri}\left[g_{uq,r}\left(2g_{i(p,k)} - g_{pk,i}\right) - g_{pq,r}(2g_{i(u,k)} - g_{uk,i})\right]$$
$$+\frac{1}{4}g^{ij}\left(2g_{j(u,q)} - g_{uq,j}\right)\left(2g_{i(p,k)} - g_{pk,i}\right)$$
$$-\frac{1}{4}g^{ij}\left(2g_{j(u,k)} - g_{uk,j}\right)\left(2g_{i(p,q)} - g_{pq,i}\right) \ , \tag{4.27}$$

$$
\begin{aligned}
R_{upuq} = {} & g_{u(p,q),u} - \frac{1}{2}\left(g_{pq,uu} + g_{uu,pq}\right) + \frac{1}{4}g^{rr}\left(g_{up,r}g_{uq,r} - g_{uu,r}g_{pq,r}\right) \\
& -\frac{1}{4}g_{uu,r}\left[2g_{u(p,q)} - g_{pq,u} - g^{ri}\left(2g_{i(p,q)} - g_{pq,i}\right)\right] \\
& -\frac{1}{4}g_{pq,r}\left[g_{uu,u} - g^{ri}\left(2g_{ui,u} - g_{uu,i}\right)\right] \\
& +\frac{1}{4}g_{up,r}\left[g_{uu,q} - g^{ri}\left(2g_{i(u,q)} - g_{uq,i}\right)\right] \\
& +\frac{1}{4}g_{uq,r}\left[g_{uu,p} - g^{ri}\left(2g_{i(u,p)} - g_{up,i}\right)\right] \\
& +\frac{1}{4}g^{ij}\left(2g_{j(u,p)} - g_{up,j}\right)\left(2g_{i(u,q)} - g_{uq,i}\right) \\
& -\frac{1}{4}g^{ij}\left(2g_{uj,u} - g_{uu,j}\right)\left(2g_{i(p,q)} - g_{pq,i}\right) \ , \qquad (4.28)
\end{aligned}
$$

where the superscript ^{S} denotes tensors calculated from the spatial metric g_{ij} only, with respect to the spatial coordinates x^i. The Ricci tensor is given by contraction $R_{ab} = g^{cd}R_{cadb} = R^c_{\ acb}$, namely,

$$
\begin{aligned}
R_{rr} = {} & -\frac{1}{2}g^{pq}g_{pq,rr} + \frac{1}{4}g^{pq}g^{ij}g_{ip,r}g_{jq,r} \ , \qquad (4.29) \\
R_{ru} = {} & -\frac{1}{2}g_{uu,rr} + \frac{1}{2}g^{ri}g_{ui,rr} + g^{pq}g_{p[u,q],r} \\
& +\frac{1}{2}g^{pq}g_{up,r}\left(g_{uq,r} - g^{ri}g_{iq,r}\right) - \frac{1}{4}g^{pq}g_{pq,r}\left(g_{uu,r} - g^{ri}g_{ui,r}\right) \\
& +\frac{1}{4}g^{pq}g^{ij}\left[g_{iq,r}(2g_{j(u,p)} - g_{up,j}) - g_{ui,r}(2g_{jp,q} - g_{pq,j})\right] \ , \qquad (4.30) \\
R_{rk} = {} & -\frac{1}{2}g_{uk,rr} + \frac{1}{2}g^{ri}g_{ik,rr} + g^{pq}g_{p[k,q],r} \\
& +\frac{1}{2}g^{pq}g_{pk,r}\left(g_{uq,r} - g^{ri}g_{iq,r}\right) - \frac{1}{4}g^{pq}g_{pq,r}\left(g_{uk,r} - g^{ri}g_{ik,r}\right) \\
& +\frac{1}{4}g^{pq}g^{ij}\left[g_{iq,r}(2g_{j(p,k)} - g_{pk,j}) - g_{ik,r}(2g_{jp,q} - g_{pq,j})\right] \ , \qquad (4.31)
\end{aligned}
$$

$$
\begin{aligned}
R_{uu} = {} & -\frac{1}{2}g^{rr}g_{uu,rr} - 2g^{ri}g_{u[u,i],r} + \frac{1}{2}g^{pq}\left(2g_{up,uq} - g_{pq,uu} - g_{uu,pq}\right) \\
& +\frac{1}{2}g^{rp}g^{rq}\left(g_{uu,r}g_{pq,r} - g_{up,r}g_{uq,r}\right) + \frac{1}{4}g^{rr}g^{pq}\left(2g_{up,r}g_{uq,r} - g_{uu,r}g_{pq,r}\right) \\
& -\frac{1}{2}g^{pq}g^{ri}\left[g_{ip,r}(2g_{uq,u} - g_{uu,q}) - g_{up,r}(2g_{q(u,i)} - g_{ui,q})\right] \\
& -\frac{1}{4}g^{pq}g_{uu,r}\left[2g_{up,q} - g_{pq,u} - g^{ri}\left(2g_{ip,q} - g_{pq,i}\right)\right] \\
& +\frac{1}{2}g^{pq}g_{up,r}\left[g_{uu,q} - g^{ri}\left(2g_{i(u,q)} - g_{uq,i}\right)\right] \\
& -\frac{1}{4}g^{pq}g_{pq,r}\left[g_{uu,u} - g^{ri}\left(2g_{ui,u} - g_{uu,i}\right)\right] \\
& +\frac{1}{4}g^{pq}g^{ij}\left(2g_{j(u,p)} - g_{up,j}\right)\left(2g_{i(u,q)} - g_{uq,i}\right) \\
& -\frac{1}{4}g^{pq}g^{ij}\left(2g_{uj,u} - g_{uu,j}\right)\left(2g_{ip,q} - g_{pq,i}\right) \ , \qquad (4.32)
\end{aligned}
$$

$$
\begin{aligned}
R_{uk} = &-\frac{1}{2}g^{rr}g_{uk,rr} - g_{u[u,k],r} + g^{ri}(g_{u[i,k],r} - g_{k[u,i],r}) + g^{pq}(g_{p[k,q],u} - g_{u[k,q],p}) \\
&+\frac{1}{4}g^{rr}g^{pq}\left(2g_{pk,r}g_{uq,r} - g_{pq,r}g_{uk,r}\right) + \frac{1}{2}g^{ri}(g_{uu,r}g_{ik,r} - g_{uk,r}g_{ui,r}) \\
&+\frac{1}{2}g^{rp}g^{rq}(g_{uk,r}g_{pq,r} - g_{kp,r}g_{uq,r}) \\
&+\frac{1}{2}g^{pq}g^{ri}\left[2g_{uq,r}g_{k[p,i]} + 2g_{pk,r}g_{u[q,i]} - g_{ip,r}(2g_{q(u,k)} - g_{uk,q})\right] \\
&+\frac{1}{4}g^{pq}g^{ri}\left[g_{uk,r}(2g_{i(p,q)} - g_{pq,i}) + g_{pq,r}(2g_{i(u,k)} - g_{uk,i})\right] \\
&+\frac{1}{4}g^{pq}\left[4g_{pk,r}g_{u[u,q]} + 2g_{up,r}g_{uq,k} - g_{uk,r}(2g_{up,q} - g_{pq,u}) - g_{pq,r}g_{uu,k}\right] \\
&+\frac{1}{4}g^{pq}g^{ij}\left(2g_{j(u,q)} - g_{uq,j}\right)\left(2g_{i(p,k)} - g_{pk,i}\right) \\
&-\frac{1}{4}g^{pq}g^{ij}\left(2g_{j(u,k)} - g_{uk,j}\right)(2g_{ip,q} - g_{pq,i}) \ , \qquad (4.33)
\end{aligned}
$$

$$
\begin{aligned}
R_{pq} = &\ {}^{S}R_{pq} - \frac{1}{2}g^{rr}g_{pq,rr} + g_{pq,ur} - g_{u(p,q),r} + g^{ri}(g_{i(p,q),r} - g_{pq,ir}) \\
&+\frac{1}{4}g^{rr}g^{kl}(2g_{kp,r}g_{lq,r} - g_{kl,r}g_{pq,r}) \\
&+\frac{1}{2}\left(g_{pq,r}g_{uu,r} - g_{up,r}g_{uq,r}\right) + \frac{1}{2}g^{ri}\left(g_{up,r}g_{iq,r} + g_{uq,r}g_{ip,r} - 2g_{pq,r}g_{ui,r}\right) \\
&+\frac{1}{2}g^{rk}g^{rl}\left(g_{pq,r}g_{kl,r} - g_{pk,r}g_{ql,r}\right) + \frac{1}{2}g^{kl}\left(g_{uk,r} - g^{ri}g_{ik,r}\right)\left(2g_{l(p,q)} - g_{pq,l}\right) \\
&-\frac{1}{4}g^{kl}g_{kl,r}\left[2g_{u(p,q)} - g_{pq,u} - g^{ri}(2g_{i(p,q)} - g_{pq,i})\right] \\
&-\frac{1}{4}g^{kl}g_{pq,r}\left[2g_{uk,l} - g_{kl,u} - g^{ri}(2g_{ik,l} - g_{kl,i})\right] \\
&-g^{kl}g_{kq,r}\left(g_{p[l,u]} + g^{ri}g_{p[i,l]}\right) - g^{kl}g_{kp,r}\left(g_{q[l,u]} + g^{ri}g_{q[i,l]}\right) \ . \qquad (4.34)
\end{aligned}
$$

Finally, the Ricci scalar defined as $R = g^{ab}R_{ab}$ is

$$
\begin{aligned}
R = &\ {}^{S}R + g_{uu,rr} + (g^{rp}g^{rq} - g^{pq}g^{rr})g_{pq,rr} - 2g^{ri}g_{ui,rr} + 4g^{pq}(g_{p[q,u],r} + g^{ri}g_{p[i,q],r}) \\
&-\frac{1}{2}g^{pq}\left(3g_{up,r}g_{uq,r} - 2g_{uu,r}g_{pq,r}\right) + g^{pq}g^{ri}\left(3g_{up,r}g_{iq,r} - 2g_{ui,r}g_{pq,r}\right) \\
&+\frac{1}{4}g^{rr}g^{pq}g^{kl}\left(3g_{kp,r}g_{lq,r} - g_{kl,r}g_{pq,r}\right) - \frac{1}{2}g^{pq}g^{rk}g^{rl}\left(3g_{pl,r}g_{kq,r} - 2g_{pq,r}g_{kl,r}\right) \\
&-\frac{1}{2}g^{pq}g^{kl}\left[g_{kl,r}(2g_{up,q} - g_{pq,u}) + g_{kq,r}(3g_{pl,u} - 2g_{up,l}) - 2g_{uk,r}(2g_{lp,q} - g_{pq,l})\right] \\
&+\frac{1}{2}g^{pq}g^{kl}g^{ri}\left[g_{kl,r}(2g_{ip,q} - g_{pq,i}) + g_{kq,r}(3g_{pl,i} - 2g_{ip,l}) - 2g_{ik,r}\left(2g_{lp,q} - g_{pq,l}\right)\right] . \qquad (4.35)
\end{aligned}
$$

Our main aim now is to combine the Riemann tensor, Ricci tensor and Ricci scalar into the traceless Weyl tensor which, in arbitrary dimension D, is given by formula

$$
C_{abcd} = R_{abcd} - \frac{1}{D-2}\left(g_{ac}R_{bd} - g_{ad}R_{bc} + g_{bd}R_{ac} - g_{bc}R_{ad}\right) + \frac{R\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)}{(D-1)(D-2)} \ , \qquad (4.36)
$$

the components of the Weyl tensor for the line element (4.1) are thus

$$
\begin{aligned}
C_{rprq} &= R_{rprq} - \frac{1}{D-2} g_{pq} R_{rr} \ , \\
C_{rpru} &= R_{rpru} - \frac{1}{D-2} \left(R_{rp} + g_{up} R_{rr} \right) \ , \\
C_{ruru} &= R_{ruru} - \frac{1}{D-2} \left(2R_{ru} + g_{uu} R_{rr} \right) - \frac{R}{(D-1)(D-2)} \ , \\
C_{rpkq} &= R_{rpkq} - \frac{1}{D-2} \left(g_{pq} R_{rk} - g_{pk} R_{rq} \right) \ , \\
C_{rpuq} &= R_{rpuq} - \frac{1}{D-2} \left(-R_{pq} + g_{pq} R_{ru} - g_{up} R_{rq} \right) - \frac{R g_{pq}}{(D-1)(D-2)} \ , \\
C_{rupq} &= R_{rupq} - \frac{1}{D-2} \left(g_{uq} R_{rp} - g_{up} R_{rq} \right) \ , \\
C_{ruup} &= R_{ruup} - \frac{1}{D-2} \left(-R_{up} + g_{up} R_{ru} - g_{uu} R_{rp} \right) - \frac{R g_{up}}{(D-1)(D-2)} \ , \\
C_{kplq} &= R_{kplq} - \frac{1}{D-2} \left(g_{kl} R_{pq} - g_{kq} R_{pl} + g_{pq} R_{kl} - g_{pl} R_{kq} \right) + \frac{R \left(g_{kl} g_{pq} - g_{kq} g_{pl} \right)}{(D-1)(D-2)} \ , \\
C_{upkq} &= R_{upkq} - \frac{1}{D-2} \left(g_{uk} R_{pq} - g_{uq} R_{pk} + g_{pq} R_{uk} - g_{pk} R_{uq} \right) + \frac{R \left(g_{uk} g_{pq} - g_{uq} g_{pk} \right)}{(D-1)(D-2)} \ , \\
C_{upuq} &= R_{upuq} - \frac{1}{D-2} \left(g_{uu} R_{pq} - g_{uq} R_{up} + g_{pq} R_{uu} - g_{up} R_{uq} \right) + \frac{R \left(g_{uu} g_{pq} - g_{up} g_{uq} \right)}{(D-1)(D-2)} \ . \qquad (4.37)
\end{aligned}
$$

For the following discussion it will also be useful to present the simplest of these expressions explicitly, namely

$$
C_{rprq} = -\frac{1}{2} g_{pq,rr} + \frac{1}{4} g^{ij} g_{ip,r} g_{jq,r} - \frac{g_{pq} g^{kl}}{D-2} \left(-\frac{1}{2} g_{kl,rr} + \frac{1}{4} g^{ij} g_{ik,r} g_{jl,r} \right) \ . \qquad (4.38)
$$

4.4 Einstein field equations

Einstein's equations for a metric tensor g_{ab} can be written in the standard form as

$$
R_{ab} - \frac{1}{2} R \, g_{ab} + \Lambda \, g_{ab} = 8\pi T_{ab} \ , \qquad (4.39)
$$

or, equivalently, substituting its trace $R = \frac{2}{D-2}(\Lambda D - 8\pi T)$ as

$$
R_{ab} = \frac{2}{D-2} \Lambda \, g_{ab} + 8\pi \left(T_{ab} - \frac{1}{D-2} T \, g_{ab} \right) \ . \qquad (4.40)
$$

At the moment we admit the presence of an arbitrary matter field given by its energy momentum tensor T_{ab}, and a nonvanishing cosmological constant Λ. The set of equations corresponding to

the line element (4.1) thus becomes

$$\begin{aligned}
R_{rr} &= 8\pi T_{rr} \ , \\
R_{ru} &= -\frac{2}{D-2}\Lambda + 8\pi\left(T_{ru} + \frac{1}{D-2}T\right) \ , \\
R_{rp} &= 8\pi T_{rp} \ , \\
R_{uu} &= \frac{2}{D-2}\Lambda\, g_{uu} + 8\pi\left(T_{uu} - \frac{1}{D-2}T\, g_{uu}\right) \ , \\
R_{up} &= \frac{2}{D-2}\Lambda\, g_{up} + 8\pi\left(T_{up} - \frac{1}{D-2}T\, g_{up}\right) \ , \\
R_{pq} &= \frac{2}{D-2}\Lambda\, g_{pq} + 8\pi\left(T_{pq} - \frac{1}{D-2}T\, g_{pq}\right) \ ,
\end{aligned} \tag{4.41}$$

where R_{ab} are explicitly given by (4.29)–(4.34).

4.5 Explicit null reference frames

In the previous discussion in section 4.2 we used some general properties of the real null frame consisting of two future oriented null vectors $\mathbf{k}$ and $\mathbf{l}$, and $D-2$ perpendicular spacelike vectors $\mathbf{m}_i$, namely,

$$\mathbf{k}\cdot\mathbf{l} = -1 \ , \qquad \mathbf{m}_i\cdot\mathbf{m}_j = \delta_{ij} \ , \qquad \mathbf{k}\cdot\mathbf{k} = \mathbf{l}\cdot\mathbf{l} = 0 \ , \qquad \mathbf{k}\cdot\mathbf{m}_i = \mathbf{l}\cdot\mathbf{m}_i = 0 \ . \tag{4.42}$$

Now we will show how to construct explicit such frames in general nontwisting spacetimes (4.1). In the adapted coordinates $(r,\, u,\, x^i)$ with $i = 2,\, 3, \ldots,\, D-1$, introduced at the beginning of this chapter, the metric can be rewritten in the matrix form as

$$g_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 & \dots \\ -1 & g_{uu} & g_{u2} & g_{u3} & \dots \\ 0 & g_{u2} & g_{22} & g_{23} & \dots \\ 0 & g_{u3} & g_{23} & g_{33} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \ . \tag{4.43}$$

There are several possibilities how to choose our reference frame which may be motivated either by mathematical simplicity or by specific relation to some physically preferred observer. The freedom in the choice is given only by the Lorentz transformations, and is described in Chapter 3. Now let us only recall the formulae for null rotations with $\mathbf{k}$ fixed, boosts in $\mathbf{k}-\mathbf{l}$ plane, and spatial rotations in the transverse space of $\mathbf{m}_i$ which will be important for further relations between the null frames (4.48), (4.49) and (4.50) introduced below:

- null rotations with $\mathbf{k}$ fixed are parameterized by $D-2$ real parameters $L_i = L^i$:

$$\tilde{\mathbf{k}} = \mathbf{k} \ , \qquad \tilde{\mathbf{l}} = \mathbf{l} + \sqrt{2}L^i\mathbf{m}_i + |L|^2\mathbf{k} \ , \qquad \tilde{\mathbf{m}}_i = \mathbf{m}_i + \sqrt{2}L_i\mathbf{k} \ , \tag{4.44}$$

- boost in $\mathbf{k}-\mathbf{l}$ plane is parameterized by a real number B:

$$\tilde{\mathbf{k}} = B\mathbf{k} \ , \qquad \tilde{\mathbf{l}} = B^{-1}\mathbf{l} \ , \qquad \tilde{\mathbf{m}}_i = \mathbf{m}_i \ , \tag{4.45}$$

- spatial rotation in the space of $\mathbf{m}_i$ is parameterized by an orthogonal matrix $\Phi_i{}^j$:

$$\tilde{\mathbf{k}} = \mathbf{k} \ , \qquad \tilde{\mathbf{l}} = \mathbf{l} \ , \qquad \tilde{\mathbf{m}}_i = \Phi_i{}^j\mathbf{m}_j \ , \qquad \text{where} \qquad \Phi_i{}^j\Phi_k{}^l\delta_{jl} = \delta_{ik} \ . \tag{4.46}$$

4.5.1 Natural null frames

In this subsection we identify the first vector $\mathbf{k}$ of the null frame with the nontwisting, affinely parameterized geodetic vector field $\tilde{\mathbf{k}} = \partial_r$ introduced in section 4.1. Employing the conditions (4.42) we find that (necessarily) $l^u = 1$ and $m_i^u = 0$, but there is still a freedom in the remaining components of $\mathbf{l}$ and $\mathbf{m}_i$, constrained by the conditions

$$0 = -2l^r + g_{uu} + 2g_{ui}l^i + g_{ij}l^i l^j \ , \qquad 0 = m_i^r - g_{uj}m_i^j - g_{jk}m_i^j l^k \ , \qquad \delta_{ij} = g_{kl}m_i^k m_j^l \ . \tag{4.47}$$

There are two natural choices

- assuming $l^i = 0$ we immediately get

$$\mathbf{k} = \partial_r \ , \qquad \mathbf{l} = \frac{1}{2}g_{uu}\partial_r + \partial_u \ , \qquad \mathbf{m}_i = g_{uq}m_i^q\partial_r + m_i^j\partial_j \ , \tag{4.48}$$

- if we wish the component m_i^r to be vanishing, the resulting frame must take the form

$$\mathbf{k} = \partial_r \ , \qquad \mathbf{l} = -\frac{1}{2}g^{rr}\partial_r + \partial_u - g^{ri}\partial_i \ , \qquad \mathbf{m}_i = m_i^j\partial_j \ . \tag{4.49}$$

These simplest possibilities are useful, e.g., for algebraic classification of spacetimes. If we assume identical spatial parts[7] of the vectors $\mathbf{m}_i$ in both these frames, the relation between (4.48) and (4.49) will be given only by the null rotation (4.44) with $\mathbf{k}$ fixed. The frame (4.48) can be obtained from (4.49) using $L_i = \frac{1}{\sqrt{2}}g_{uq}m_i^q$, and inversely by $L_i = -\frac{1}{\sqrt{2}}g_{uq}m_i^q$.

4.5.2 Interpretation frame

In the context of equation of geodesic deviation, we need to construct a frame connected with a particular observer moving along timelike geodesic $\gamma(\tau) = (r(\tau), u(\tau), x^i(\tau))$. The D-velocity of the observer is thus $\mathbf{u} = u^a\partial_a$ where $u^a = (\dot{r}, \dot{u}, \dot{x}^i)$, the dot denotes differentiation with respect to observer's proper time τ, i.e., $\frac{\mathrm{d}}{\mathrm{d}\tau}$. The observer's velocity $\mathbf{u}$ will be identified with the timelike vector constructed from the null vectors $\mathbf{k}$ and $\mathbf{l}$ as $\mathbf{e}_{(0)} \equiv \mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{k}+\mathbf{l})$, where the vector $\mathbf{k}$ of this null frame will be proportional to the nontwisting vector $\tilde{\mathbf{k}} = \partial_r$ introduced in section 4.1. It means that its only nonvanishing component will be k^r, i.e., $\mathbf{k} = k^r\partial_r$. Using the properties (4.42) we obtain

$$\begin{aligned}
\mathbf{k}\cdot\mathbf{u} = g_{ab}k^a u^b = \frac{1}{\sqrt{2}} \ , &\qquad \text{which implies} \quad k^r = \frac{1}{\sqrt{2}\dot{u}} \ , \\
\mathbf{e}_{(0)} = \mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{k}+\mathbf{l}) \ , &\qquad \text{and thus} \quad l^a = \sqrt{2}u^a - k^a \ , \\
\mathbf{k}\cdot\mathbf{m}_i = g_{ab}k^a m_i^b = g_{rb}k^r m_i^b \ , &\qquad \text{but only } g_{ru} \neq 0 \ , \text{ so that} \quad m_i^u = 0 \ , \\
\mathbf{l}\cdot\mathbf{m}_i = l_a m_i^a = 0 \ , &\qquad \text{but } m_i^u = 0 \ , \text{ so that} \quad m_i^r = -\frac{l_q m_i^q}{l_r} \ , \\
\mathbf{m}_i\cdot\mathbf{m}_j = g_{ab}m_i^a m_j^b = \delta_{ij} \ , &\qquad \text{but } g_{rr} = 0 \text{ and } m_i^u = 0 \ , \text{ so that } g_{pq}m_i^p m_j^q = \delta_{ij} \ ,
\end{aligned}$$

where the indices are a, $b = 0, \ldots, D-1$ and i, j, p, $q = 2, \ldots, D-1$. The resulting interpretation null frame is thus

$$\mathbf{k} = \frac{1}{\sqrt{2}\dot{u}}\partial_r \ , \qquad \mathbf{l} = \left(\sqrt{2}\dot{r} - \frac{1}{\sqrt{2}\dot{u}}\right)\partial_r + \sqrt{2}\dot{u}\,\partial_u + \sqrt{2}\dot{x}^i\partial_i \ , \qquad \mathbf{m}_i = -\frac{l_q m_i^q}{l_r}\partial_r + m_i^j\partial_j \ . \tag{4.50}$$

[7] If these parts are different we have to use spatial rotation (4.46).

In components we get

$$k^a = \left(\frac{1}{\sqrt{2}\dot{u}}, 0, 0, 0, \dots\right) ,$$
$$l^a = \left(\sqrt{2}\dot{r} - \frac{1}{\sqrt{2}\dot{u}}, \sqrt{2}\dot{u}, \sqrt{2}\dot{x}^2, \sqrt{2}\dot{x}^3, \dots\right) ,$$
$$m_i^a = \left(-\frac{l_q m_i^q}{l_r}, 0, m_i^2, m_i^3, \dots\right) , \tag{4.51}$$

where $l_r = -\sqrt{2}\dot{u}$, $l_q = \sqrt{2}(g_{uq}\dot{u} + g_{qk}\dot{x}^k)$, and m_i^r can thus be rewritten as $m_i^r = \frac{1}{\dot{u}}(g_{uq}\dot{u} + g_{qk}\dot{x}^k)m_i^q$. It is always possible to construct the spatial parts of the vectors $\mathbf{m}_i$ from an arbitrary basis by standard Gram-Schmidt orthogonalization procedure.

This interpretation frame (4.50) is also related to other frames, namely (4.49), by a simple Lorentz transformations. Assuming again the same spatial parts of $\mathbf{m}_i$ in both these frames, we obtain (4.50) from the natural null frame (4.49) by the boost in $\mathbf{k} - \mathbf{l}$ plane (4.45) and the null rotation with $\mathbf{k}$ fixed (4.44), where

$$B = \frac{1}{\sqrt{2}\dot{u}} , \qquad L_i = (g_{uq}\dot{u} + g_{qk}\dot{x}^k)m_i^q . \tag{4.52}$$

4.6 Geodesic deviation

We are going to follow the procedure introduced in Chaper 3 and investigate the relative motions of free test particles in nontwisting spacetimes of type (4.1). Let us briefly recall the main concepts and principal results of this approach, detailed description can be found in Chapter 3. The equation of geodesic deviation is

$$\frac{\mathrm{D}^2 Z^a}{\mathrm{d}\tau^2} = R^a{}_{bcd} u^b u^c Z^d , \tag{4.53}$$

where $R^a{}_{bcd}$ is the Riemann curvature tensor, u^b are components of the velocity vector of the reference particle (geodesic observer), the parameter τ is a proper time of the observer's timelike geodesic, and Z^a are components of the separation vector which connects the reference particle with another nearby test particle. We express this equation in coordinate-independent form. We thus consider an orthonormal frame $(\mathbf{e}_{(a)})$, i.e.,

$$\mathbf{e}_{(a)} \cdot \mathbf{e}_{(b)} \equiv g_{cd}\, e^c_{(a)} e^d_{(b)} = \eta_{ab} \equiv \mathrm{diag}(-1, 1, \dots, 1) , \tag{4.54}$$

consisting of one timelike vector $\mathbf{e}_{(0)}$, which we identify with observer's velocity $\mathbf{e}_{(0)} = \mathbf{u}$, and $D-1$ perpendicular spatial vectors $\mathbf{e}_{(\mathrm{i})}$. The only nontrivial components of the projected equation of geodesic deviation (4.53) onto the orthonormal frame $(\mathbf{e}_{(a)})$ than are

$$\ddot{Z}^{(\mathrm{i})} = R^{(\mathrm{i})}{}_{(0)(0)(\mathrm{j})} , \tag{4.55}$$

where i, j $= 1, \dots, D-1$ and $\ddot{Z}^{(\mathrm{i})} = e_a^{(\mathrm{i})} \frac{\mathrm{D}^2 Z^a}{\mathrm{d}\tau^2}$. For more precise discussion of the relative motion of free test particles it is natural to decompose Riemann tensor using (4.36) into the traceless Weyl tensor, Ricci tensor and scalar curvature, and to rewrite the orthonormal components of the Weyl tensor in terms of the null components, i.e., with respect to the real null frame (4.42). Also the Ricci tensor and Ricci scalar are expressed using the Einstein equations (4.40) and their trace. The null frame ($\mathbf{k}$, $\mathbf{l}$, $\mathbf{m}_i$) satisfying the conditions (4.42) is than related to the orthonormal frame $(\mathbf{e}_{(a)})$ by

$$\mathbf{k} = \frac{1}{\sqrt{2}}(\mathbf{e}_{(0)} + \mathbf{e}_{(1)}) , \qquad \mathbf{l} = \frac{1}{\sqrt{2}}(\mathbf{e}_{(0)} - \mathbf{e}_{(1)}) , \qquad \mathbf{m}_i = \mathbf{e}_{(i)} , \tag{4.56}$$

and orthonormal frame is inversely given by

$$\mathbf{e}_{(0)} = \frac{1}{\sqrt{2}}(\mathbf{k}+\mathbf{l}) \ , \qquad \mathbf{e}_{(1)} = \frac{1}{\sqrt{2}}(\mathbf{k}-\mathbf{l}) \ , \qquad \mathbf{e}_{(i)} = \mathbf{m}_i \ . \tag{4.57}$$

The components of the Weyl tensor $\Psi_{A\bullet}$ with respect to the null frame ($\mathbf{k}$, $\mathbf{l}$, $\mathbf{m}_i$) are defined[8] as

$$\begin{aligned}
\Psi_{0^{ij}} &= C_{abcd}k^a m_i^b k^c m_j^d \ , \\
\Psi_{1^{ijk}} &= C_{abcd}k^a m_i^b m_j^c m_k^d \ , \qquad & \Psi_{1T^i} &= C_{abcd}k^a l^b k^c m_i^d \ , \\
\Psi_{2^{ijkl}} &= C_{abcd}m_i^a m_j^b m_k^c m_l^d \ , \qquad & \Psi_{2S} &= C_{abcd}k^a l^b l^c k^d \ , \\
\Psi_{2^{ij}} &= C_{abcd}k^a l^b m_i^c m_j^d \ , \qquad & \Psi_{2T^{ij}} &= C_{abcd}k^a m_i^b l^c m_j^d \ , \\
\Psi_{3^{ijk}} &= C_{abcd}l^a m_i^b m_j^c m_k^d \ , \qquad & \Psi_{3T^i} &= C_{abcd}l^a k^b l^c m_i^d \ , \\
\Psi_{4^{ij}} &= C_{abcd}l^a m_i^b l^c m_j^d \ ,
\end{aligned} \tag{4.58}$$

where the scalars in the left column are independent while the right column represents their contractions:

$$\begin{aligned}
\Psi_{1T^i} &= \Psi_{1^{k}{}_{k}{}^{}{}_{i}} \ , \\
\Psi_{2T^{ij}} &= \frac{1}{2}(\Psi_{2^{ik}{}_{j}{}^{k}} + \Psi_{2^{ij}}) \ , \quad \Psi_{2S} = \Psi_{2T^{k}{}_{k}} = \frac{1}{2}\Psi_{2^{kl}{}_{kl}} \ , \\
\Psi_{3T^i} &= \Psi_{3^{k}{}_{k}{}^{}{}_{i}} \ .
\end{aligned} \tag{4.59}$$

The obvious symmetries of the Weyl tensor in the notation of (4.58) take the form

$$\begin{aligned}
\Psi_{0^{[ij]}} &= 0 \ , \qquad & \Psi_{0^{k}{}_{k}} &= 0 \ , \\
\Psi_{1^{i(kl)}} &= 0 \ , \qquad & \Psi_{1^{[ikl]}} &= 0 \ , \\
\Psi_{2^{ijkl}} &= \Psi_{2^{klij}} \ , \qquad & \Psi_{2^{(ij)kl}} &= \Psi_{2^{ij(kl)}} = \Psi_{2^{i[jkl]}} = 0 \ , \qquad \Psi_{2^{(ij)}} = 0 \ , \\
\Psi_{3^{i(kl)}} &= 0 \ , \qquad & \Psi_{3^{[ikl]}} &= 0 \ , \\
\Psi_{4^{[ij]}} &= 0 \ , \qquad & \Psi_{4^{k}{}_{k}} &= 0 \ .
\end{aligned} \tag{4.60}$$

Using (4.57) the Weyl tensor frame components with respect to the orthonormal frame ($\mathbf{e}_{(a)}$), which are important in the equation of geodesic deviation (4.55), can be expressed in terms of the null frame components (4.58) as

$$\begin{aligned}
C_{(1)(0)(0)(1)} &= C_{abcd}e^a_{(1)}e^b_{(0)}e^c_{(0)}e^d_{(1)} = \frac{1}{4}C_{abcd}(k-l)^a(k+l)^b(k+l)^c(k-l)^d \\
&= C_{abcd}k^a l^b l^c k^d \\
&= \Psi_{2S} \ , \\
C_{(1)(0)(0)(j)} &= C_{abcd}e^a_{(1)}e^b_{(0)}e^c_{(0)}e^d_{(j)} = \left(\frac{1}{\sqrt{2}}\right)^3 C_{abcd}(k-l)^a(k+l)^b(k+l)^c m_j^d \\
&= \frac{1}{\sqrt{2}}C_{abcd}(k^a l^b k^c m_j^d - l^a k^b l^c m_j^d) \\
&= \frac{1}{\sqrt{2}}(\Psi_{1T^j} - \Psi_{3T^j}) \ , \\
C_{(i)(0)(0)(j)} &= C_{abcd}e^a_{(i)}e^b_{(0)}e^c_{(0)}e^d_{(j)} = \frac{1}{2}C_{abcd}m_i^a(k+l)^b(k+l)^c m_j^d \\
&= \frac{1}{2}C_{abcd}\left(k^a m_i^b k^c m_j^d + l^a m_i^b l^c m_j^d + 2k^a m^b_{(i} l^c m^d_{j)}\right) \\
&= -\frac{1}{2}(\Psi_{0^{ij}} + \Psi_{4^{ij}}) - \Psi_{2T^{(ij)}} \ .
\end{aligned} \tag{4.61}$$

[8] Their relations to the other commonly used conventions are given in Chapter 3.

Therefore, the geodesic deviation equation (4.55) can be written in the invariant and decomposed form as

$$\begin{aligned}
\ddot{Z}^{(1)} &= \frac{2\Lambda}{(D-2)(D-1)}Z^{(1)} + \Psi_{2S}Z^{(1)} + \frac{1}{\sqrt{2}}(\Psi_{1T^j} - \Psi_{3T^j})Z^{(j)} \\
&\quad + \frac{8\pi}{D-2}\left[T_{(1)(1)}Z^{(1)} + T_{(1)(j)}Z^{(j)} - \left(T_{(0)(0)} + \frac{2T}{D-1}\right)Z^{(1)}\right] , \\
\ddot{Z}^{(i)} &= \frac{2\Lambda}{(D-2)(D-1)}Z^{(i)} - \Psi_{2T^{(ij)}}Z^{(j)} + \frac{1}{\sqrt{2}}(\Psi_{1T^i} - \Psi_{3T^i})Z^{(1)} - \frac{1}{2}(\Psi_{0^{ij}} + \Psi_{4^{ij}})Z^{(j)} \\
&\quad + \frac{8\pi}{D-2}\left[T_{(i)(1)}Z^{(1)} + T_{(i)(j)}Z^{(j)} - \left(T_{(0)(0)} + \frac{2T}{D-1}\right)Z^{(i)}\right] ,
\end{aligned} \tag{4.62}$$

where the indices i, j range form 2 to $D-1$. The overall effects of gravitational field on the relative behaviour of free test particles is thus given by influence of the matter content of universe represented by projections of its energy-momentum tensor T_{ab}, isotropic influence of the cosmological constant Λ, transverse gravitational waves described by the symmetric traceless matrices $\Psi_{0^{ij}}$ and $\Psi_{4^{ij}}$, longitudinal components Ψ_{1T^i} and Ψ_{3T^i}, and Newton components of the gravitational field given by Ψ_{2S} and $\Psi_{2T^{(ij)}}$. Discussion of specific properties of these components can be found in Chapter 3. Explicit form of the Weyl scalars $\Psi_{A^\bullet}$ will be given in the following subsection 4.6.1.

4.6.1 The null frame components of the Weyl tensor

We now consider a general frame of the form (4.51), i.e., the null vector $\mathbf{k}$ has a component only in the r direction, all components of the null vector $\mathbf{l}$ are nonvanishing, and spatial vectors $\mathbf{m}_i$ have only the u-component vanishing. Using these specific assumptions, which involve all three frames (4.48), (4.49) and (4.50), and employing the definition of the $\Psi_{A^\bullet}$ scalars (4.58), the frame components of the Weyl tensor relevant to the equations (4.62) will be

$$\begin{aligned}
\Psi_{0^{ij}} &= C_{rprq}k^r m_i^p k^r m_j^q , \\
\Psi_{1T^j} &= C_{rprq}k^r l^p k^r m_j^q + C_{rpru}k^r l^u k^r m_j^p , \\
\Psi_{2S} &= -C_{rprq}k^r l^p l^q k^r - 2C_{rpru}k^r l^u l^p k^r - C_{ruru}k^r l^u l^u k^r , \\
\Psi_{2T^{ij}} &= C_{rprq}k^r m_i^p (l^r m_j^q - l^q m_j^r) - C_{rpru}k^r m_i^p l^u m_j^r \\
&\quad + C_{rpuq}k^r m_i^p l^u m_j^q + C_{rpkq}k^r m_i^p l^k m_j^q , \\
\Psi_{3T^j} &= C_{rprq}l^p k^r (l^q m_j^r - l^r m_j^q) + C_{rpru}l^u k^r (2l^p m_j^r - l^r m_j^p) + C_{ruru}l^u k^r l^u m_j^r \\
&\quad - C_{rpuq}l^p k^r l^u m_j^q - C_{rpkq}l^p k^r l^k m_j^q - C_{ruup}l^u k^r l^u m_j^p - C_{rupq}l^u k^r l^p m_j^q , \\
\Psi_{4^{ij}} &= C_{rprq}(l^r m_i^p l^r m_j^q - l^r m_i^p l^q m_j^r + l^p m_i^r l^q m_j^r - l^p m_i^r l^r m_j^q) \\
&\quad + C_{rpru}l^u (m_i^r l^p m_j^r - m_i^r l^r m_j^p + m_i^r l^p m_j^r - m_i^p l^r m_j^r) \\
&\quad + C_{rpuq}l^u (m_i^q l^r m_j^p - m_i^q l^p m_j^r + m_i^p l^r m_j^q - m_i^r l^p m_j^q) \\
&\quad + C_{rpkq}l^k (m_i^q l^r m_j^p - m_i^q l^p m_j^r + m_i^p l^r m_j^q - m_i^r l^p m_j^q) \\
&\quad + C_{ruru}l^u m_i^r l^u m_j^r - C_{ruup}l^u (m_i^r l^u m_j^p + m_i^p l^u m_j^r) - C_{rupq}l^u (m_i^r l^p m_j^q + m_i^q l^p m_j^r) \\
&\quad + C_{upuq}l^u m_i^p l^u m_j^q + C_{upkq}l^u (m_i^p l^k m_j^q + m_i^q l^k m_j^p) + C_{kplq}l^k m_i^p l^l m_j^q .
\end{aligned} \tag{4.63}$$

For the sake of completeness we also present all the remaining Weyl components,

$$\begin{aligned}
\Psi_{1^{ijk}} &= C_{rprq}k^r m_i^p(m_j^r m_k^q - m_j^q m_k^r) + C_{rpmq}k^r m_i^p m_j^m m_k^q \ , \\
\Psi_{2^{ijkl}} &= C_{rprq}(m_i^r m_j^p m_k^r m_l^q - m_i^r m_j^p m_k^q m_l^r + m_i^p m_j^r m_k^q m_l^r - m_i^p m_j^r m_k^r m_l^q) \\
&\quad + C_{rpmq}(m_i^r m_j^p m_k^m m_l^q - m_i^p m_j^r m_k^m m_l^q + m_i^m m_j^q m_k^r m_l^p - m_i^m m_j^q m_k^p m_l^r) \\
&\quad + C_{mpnq}m_i^m m_j^p m_k^n m_l^q \ , \\
\Psi_{2^{ij}} &= C_{rprq}k^r l^p(m_i^r m_j^q - m_i^q m_j^r) + C_{rpmq}k^r l^p m_i^m m_j^q \\
&\quad + C_{rurp}k^r l^u(m_i^r m_j^p - m_i^p m_j^r) + C_{rupq}k^r l^u m_i^p m_j^q \ , \\
\Psi_{3^{ijk}} &= C_{rprq}(l^r m_i^p m_j^r m_k^q - l^r m_i^p m_j^q m_k^r + l^p m_i^r m_j^q m_k^r - l^p m_i^r m_j^r m_k^q) \\
&\quad + C_{pmq}(l^r m_i^p m_j^m m_k^q - l^p m_i^r m_j^m m_k^q + l^m m_i^q m_j^r m_k^p - l^m m_i^q m_j^p m_k^r) \\
&\quad + C_{rurp}l^u m_i^r(m_j^p m_k^r - m_j^r m_k^p) + C_{rpuq}l^u m_i^q(m_j^r m_k^p - m_j^p m_k^r) \\
&\quad - C_{rupq}l^u m_i^r m_j^p m_k^q + C_{upmq}l^u m_i^p m_j^m m_k^q + C_{mpnq}l^m m_i^p m_j^n m_k^q \ ,
\end{aligned} \tag{4.64}$$

and a general form of the frame components of an arbitrary energy–momentum tensor,

$$\begin{aligned}
T_{(0)(0)} &= T_{ab}e^a_{(0)}e^b_{(0)} = \frac{1}{2}T_{rr}(k^r k^r + l^r l^r + 2k^r l^r) + \frac{1}{2}T_{uu}l^u l^u + \frac{1}{2}T_{ij}l^i l^j \\
&\quad + T_{ru}(l^r l^u + k^r l^u) + T_{ri}(l^r l^i + k^r l^i) + T_{ui}l^u l^i \ , \\
T_{(1)(1)} &= T_{ab}e^a_{(1)}e^b_{(1)} = \frac{1}{2}T_{rr}(k^r k^r + l^r l^r - 2k^r l^r) + \frac{1}{2}T_{uu}l^u l^u + \frac{1}{2}T_{ij}l^i l^j \\
&\quad + T_{ru}(l^r l^u - k^r l^u) + T_{ri}(l^r l^i - k^r l^i) + T_{ui}l^u l^i \ , \\
T_{(1)(i)} &= T_{ab}e^a_{(1)}e^b_{(i)} = \frac{1}{\sqrt{2}}\left[T_{rr}(k^r m_i^r - l^r m_i^r) + T_{rj}(k^r m_i^j - 2l^{(r}m_i^{j)})\right] \\
&\quad - \frac{1}{\sqrt{2}}\left(T_{ru}l^u m_i^r + T_{uj}l^u m_i^j + T_{jk}l^j m_i^k\right) \ , \\
T_{(i)(j)} &= T_{ab}e^a_{(i)}e^b_{(j)} = T_{rr}m_i^r m_j^r + 2T_{rk}m^r_{(i}m^k_{j)} + T_{kl}m_i^k m_j^l \ .
\end{aligned} \tag{4.65}$$

Substituting now the explicit form of the interpretation frame components (4.51), which are connected to the geodesic observer with velocity vector $u^a = (\dot{r}, \dot{u}, \dot{x}^i)$, into equations (4.63) we get the following expressions which include the kinematic effects given by the specific motion of our observer.

The component of the gravitational field representing the effect of the transverse gravitational wave propagating along the null direction $\mathbf{l}$, i.e., in the spatial direction $-\mathbf{e}_{(1)}$, is

$$\Psi_{0^{ij}} = \frac{1}{2}m_i^p m_j^q \frac{1}{\dot{u}^2}C_{rprq} \ . \tag{4.66}$$

The terms which cause the longitudinal deformation of the congruence of free test particles in the spatial direction $-\mathbf{e}_{(1)}$ are

$$\Psi_{1T^j} = \frac{1}{\sqrt{2}}m_j^p\left(\frac{1}{\dot{u}^2}\dot{x}^k C_{rprk} + \frac{1}{\dot{u}}C_{rpru}\right) \ . \tag{4.67}$$

The Newton-like components of the gravitational field are given by

$$\Psi_{2T^{ij}} = m_i^p m_j^q \Bigg[\frac{1}{\dot{u}^2}\dot{x}^k\dot{x}^l \left(\frac{1}{2}C_{rprq}g_{kl} - C_{rprl}g_{kq}\right) + \frac{1}{\dot{u}}\dot{x}^k \left(C_{rprq}g_{uk} - C_{rprk}g_{uq} - C_{rpru}g_{kq} + C_{rpkq}\right) + \frac{1}{2}C_{rprq}g_{uu} - C_{rpru}g_{uq} + C_{rpuq}\Bigg] , \tag{4.68}$$

$$\Psi_{2S} = -\frac{1}{\dot{u}^2}\dot{x}^k\dot{x}^l C_{rkrl} - \frac{1}{\dot{u}}\dot{x}^k 2C_{rkru} - C_{ruru} . \tag{4.69}$$

The longitudinal deformation in the direction $+\mathbf{e}_{(1)}$ is represented by the set of scalars

$$\begin{aligned}\Psi_{3T^j} = \sqrt{2}m_j^p \Bigg[& \frac{1}{\dot{u}^2}\dot{x}^k\dot{x}^l\dot{x}^m \left(C_{rkrl}g_{mp} - \frac{1}{2}C_{rprk}g_{lm}\right) \\ & + \frac{1}{\dot{u}}\dot{x}^k\dot{x}^l \left(C_{rkrl}g_{up} - C_{rprk}g_{ul} + 2C_{rkru}g_{lp} - \frac{1}{2}C_{rpru}g_{kl} - C_{rklp}\right) \\ & + \dot{x}^k \left(-\frac{1}{2}C_{rprk}g_{uu} + 2C_{rkru}g_{up} - C_{rpru}g_{uk} + C_{ruru}g_{kp} - C_{rkup} - C_{rukp}\right) \\ & + \dot{u}\left(-\frac{1}{2}C_{rpru}g_{uu} + C_{ruru}g_{up} - C_{ruup}\right)\Bigg] .\end{aligned} \tag{4.70}$$

Finally, the transverse gravitational wave propagating along null direction $\mathbf{k}$, i.e., in the spatial direction $+\mathbf{e}_{(1)}$, corresponds to the deformation caused by matrix

$$\begin{aligned}\Psi_{4^{ij}} = 2m_{(i}^p m_{j)}^q \Bigg\{ & \frac{1}{\dot{u}^2}\dot{x}^k\dot{x}^l\dot{x}^m\dot{x}^n \left[\frac{1}{4}C_{rprq}g_{kl}g_{mn} + C_{rkrl}g_{pm}g_{qn} - C_{rlrp}g_{km}g_{qn}\right] \\ & + \frac{1}{\dot{u}}\dot{x}^k\dot{x}^l\dot{x}^m \Big[C_{rprq}g_{um}g_{kl} + 2C_{rkrl}g_{up}g_{qm} - C_{rlrp}\left(g_{uq}g_{km} + 2g_{uk}g_{qm}\right) \\ & \quad + 2C_{rlru}g_{pm}g_{qk} - C_{rpru}g_{qm}g_{kl} + C_{rplq}g_{mk} - 2C_{rklq}g_{mp}\Big] \\ & + \dot{x}^k\dot{x}^l \Big[\frac{1}{2}C_{rprq}\left(2g_{uk}g_{ul} + g_{uu}g_{kl}\right) + C_{rkrl}g_{up}g_{uq} - C_{rlrp}\left(2g_{uk}g_{uq} + g_{uu}g_{qk}\right) \\ & \quad + 4C_{rlru}g_{up}g_{qk} - C_{rpru}\left(2g_{uq}g_{kl} + g_{ul}g_{qk}\right) + C_{rpuq}g_{kl} - 2C_{rluq}g_{kp} \\ & \quad + 2C_{rplq}g_{uk} - 2C_{rklq}g_{up} + C_{ruru}g_{pk}g_{lq} - 2C_{rulq}g_{kp} + C_{kplq}\Big] \\ & + \dot{u}\dot{x}^k \Big[C_{rprq}g_{uu}g_{uk} - C_{rkrp}g_{uu}g_{uq} + 2C_{rkru}g_{up}g_{uq} - C_{rpru}\left(g_{uu}g_{qk} + 2g_{uq}g_{uk}\right) \\ & \quad + 2C_{rpuq}g_{uk} - 2C_{rkuq}g_{up} + C_{rpkq}g_{uu} \\ & \quad + 2\left(C_{ruru}g_{up}g_{qk} - C_{ruup}g_{qk} - C_{rukq}g_{up} + C_{upkq}\right)\Big] \\ & + \dot{u}^2 \Big[\frac{1}{4}C_{rprq}g_{uu} - C_{rpru}g_{uu}g_{uq} + C_{rpuq}g_{uu} \\ & \quad + C_{ruru}g_{up}g_{uq} - 2C_{ruup}g_{uq} + C_{upuq}\Big]\Bigg\} .\end{aligned} \tag{4.71}$$

General properties of these scalars $\Psi_{A\bullet}$ representing pure effects of the gravitational field are discussed in more details in Chapter 3.

4.6.2 Geodesics and parallel transport of the frame

Because of the connection of the interpretation frame (4.50) with a particular observer moving along the timelike geodesic $\gamma(\tau)$ let us present the explicit form of equations for general geodesic motion in the spacetime (4.1),

$$\begin{aligned}
&\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} + 2\Gamma^r_{ru}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}u}{\mathrm{d}\tau} + 2\Gamma^r_{rj}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau} + \Gamma^r_{uu}\frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}u}{\mathrm{d}\tau} + 2\Gamma^r_{uj}\frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau} + \Gamma^r_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\frac{\mathrm{d}x^k}{\mathrm{d}\tau} = 0 \ , \\
&\frac{\mathrm{d}^2 u}{\mathrm{d}\tau^2} + \Gamma^u_{uu}\frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}u}{\mathrm{d}\tau} + 2\Gamma^u_{uj}\frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau} + \Gamma^u_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\frac{\mathrm{d}x^k}{\mathrm{d}\tau} = 0 \ , \\
&\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} + 2\Gamma^i_{ru}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}u}{\mathrm{d}\tau} + 2\Gamma^i_{rj}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau} + \Gamma^i_{uu}\frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}u}{\mathrm{d}\tau} + 2\Gamma^i_{uj}\frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\frac{\mathrm{d}x^k}{\mathrm{d}\tau} = 0 \ ,
\end{aligned} \tag{4.72}$$

where the Christoffel symbols are given in (4.15)–(4.17). Notice that

- the second equation can be rewritten explicitly as

$$\ddot{u} + \frac{1}{2}g_{uu,r}\dot{u}^2 + g_{uj,r}\dot{u}\dot{x}^j + \frac{1}{2}g_{jk,r}\dot{x}^j\dot{x}^k = 0 \ , \tag{4.73}$$

- the first equation for r can be replaced by an explicit expression which follows from the normalization of the velocity $\mathbf{u}\cdot\mathbf{u} = -1$ as

$$\dot{r} = \frac{1}{2\dot{u}}\left(1 + g_{uu}\dot{u}^2 + 2g_{uj}\dot{u}\dot{x}^j + g_{jk}\dot{x}^j\dot{x}^k\right) \ . \tag{4.74}$$

For further discussion of the equation of geodesics deviation, it is sometimes useful to employ parallelly transported frames $(\mathbf{e}_{(a)})$ along observer's geodesic $\gamma(\tau)$ with tangent vector $\mathbf{e}_{(0)} = \mathbf{u}$. We thus explicitly express the corresponding conditions for the components of an arbitrary frame vector, i.e., $e^b_{(a);c}u^c = 0$:

$$\frac{\mathrm{d}e^r_{(a)}}{\mathrm{d}\tau} + \Gamma^r_{bc}u^b e^c_{(a)} = 0 \ , \qquad \frac{\mathrm{d}e^u_{(a)}}{\mathrm{d}\tau} + \Gamma^u_{bc}u^b e^c_{(a)} = 0 \ , \qquad \frac{\mathrm{d}e^i_{(a)}}{\mathrm{d}\tau} + \Gamma^i_{bc}u^b e^c_{(a)} = 0 \ . \tag{4.75}$$

Using $\mathbf{u}\cdot\mathbf{e}_{(a)} = \eta_{0a}$ and $u^a_{\ ;b}u^b = 0$ we find that $u_r e^r_{(a);b}u^b + u_u e^u_{(a);b}u^b + u_i e^i_{(a);b}u^b = 0$, i.e., if the components $e^u_{(a)}$ and $e^i_{(a)}$ are parallelly transported than $e^r_{(a)}$ is necessarily parallelly transported too.

4.6.3 On general algebraic type

The relation (4.66) for the Weyl tensor component $\Psi_{0^{ij}}$ can be rewritten using (4.37) as

$$\Psi_{0^{ij}} = \frac{1}{2\dot{u}^2}\left(R_{rprq}m^p_i m^q_j - \frac{R_{rr}}{D-2}\delta_{ij}\right) \ . \tag{4.76}$$

If the spatial vectors $\mathbf{m}_i$ are eigenvectors of the matrix $R_{r(i)r(j)}$, this becomes diagonal, and $\Psi_{0^{ij}} = 0$ prescribes simple conditions for the vanishing transverse gravitational wave propagating along $\mathbf{l}$, i.e., all such eigenvalues have to be equal to $\frac{1}{D-2}R_{rr}$. It is not obvious that these conditions are automatically satisfied for an arbitrary nontwisting spacetime (4.1) without further restrictions, and thus all possible effects of gravitational field on relative motion of test particles are generally present. Geometrically, $\Psi_{0^{ij}} \neq 0$ is equivalent to the condition when the spacetime is algebraically general (for $\Psi_{0^{ij}} = 0$ it is at least of type I, with $\mathbf{k}$ being the Weyl-aligned null direction – WAND).

The results of this chapter will now be used for description of geodesic deviation and algebraic structure of spacetimes in more specific situations in Chapter 5 and 6.

CHAPTER

5

KUNDT SPACETIMES

In this chapter we will investigate the relative motion of free test particles in the important family of Kundt spacetimes which are geometrically defined as solutions of Einstein's field equations admitting a nontwisting, nonexpanding and shearfree null congruence of geodesic. They thus form a particular subclass of general nontwisting geometries (4.1) discussed in the previous Chapter 4. Here, we will first recall general properties of the Kundt class given by its geometric definition and employ the field equations to obtain more specific line element, see [31] for more details. Then we will apply our general results of Chapter 3 and Chapter 4 and we will analyze the deviation of geodesics in the particular subclasses of Kundt spacetime, such as pp-waves, VSI spacetimes and simple gyratons.

5.1 Geometry of the Kundt spacetimes

In the case of general nontwisting spacetimes we introduced the scalars Θ (expansion), σ^2 (shear) and A^2 (twist) characterizing optical properties of the null affinely parameterized geodesic congruence, see equation (4.7) in section 4.2. We showed that line element (4.1) represents a D-dimensional nontwisting spacetime, but its shear and expansion are still in general nonvanishing. Using the decomposed form of the optical scalars (4.12),

$$\Theta = -(\ln p)_{,r} \ , \qquad \sigma^2 = \frac{1}{4}\gamma^{ik}\gamma^{jl}\gamma_{ij,r}\gamma_{kl,r} \ , \tag{5.1}$$

where $g_{ij} = p^{-2}\gamma_{ij}$, we find that the geometric definition of the Kundt family, i.e., spacetimes admitting a nontwisting ($A = 0$), nonexpanding ($\Theta = 0$) and shearfree ($\sigma = 0$) null geodesic congruence, imply the independence of the spatial part of metric g_{ij} on the coordinate r, see also [31]. The line element of general Kundt spacetime, in suitable coordinates, thus necessarily takes the form

$$\mathrm{d}s^2 = g_{ij}(u,x)\mathrm{d}x^i\mathrm{d}x^j + 2g_{ui}(r,u,x)\mathrm{d}x^i\mathrm{d}u - 2\mathrm{d}u\mathrm{d}r + g_{uu}(r,u,x)\mathrm{d}u^2 \ , \tag{5.2}$$

where the metric on the transverse space g_{ij} is a function only of the coordinates u and x^i. The other metric components g_{ui} and g_{uu} are still arbitrary functions of all coordinates, i.e., (r, u, x^i), and have to be specified employing the Einstein field equations (4.41).

Considering the Kundt line element of the form (5.2), from expressions (4.15)–(4.17) for non-vanishing Christoffel symbols we immediately get

$$\begin{aligned}
\Gamma^r_{ru} &= -\frac{1}{2}g_{uu,r} + \frac{1}{2}g^{ri}g_{ui,r} \ , \\
\Gamma^r_{rj} &= -\frac{1}{2}g_{uj,r} \ , \\
\Gamma^r_{uu} &= \frac{1}{2}\left[-g^{rr}g_{uu,r} - g_{uu,u} + g^{ri}(2g_{ui,u} - g_{uu,i})\right] \ , \\
\Gamma^r_{uj} &= \frac{1}{2}\left[-g^{rr}g_{uj,r} - g_{uu,j} + g^{ri}(2g_{i(u,j)} - g_{uj,i})\right] \ , \\
\Gamma^r_{jk} &= \frac{1}{2}\left[-2g_{u(j,k)} + g_{jk,u} + g^{ri}(2g_{i(j,k)} - g_{jk,i})\right] \ ,
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
\Gamma^u_{uu} &= \frac{1}{2}g_{uu,r} \ , \\
\Gamma^u_{ui} &= \frac{1}{2}g_{ui,r} \ ,
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
\Gamma^i_{ru} &= \frac{1}{2}g^{ij}g_{uj,r} \ , \\
\Gamma^i_{uu} &= \frac{1}{2}\left[-g^{ri}g_{uu,r} + g^{ij}(2g_{uj,u} - g_{uu,j})\right] \ , \\
\Gamma^i_{uk} &= \frac{1}{2}\left[-g^{ri}g_{uk,r} + g^{ij}(2g_{j(u,k)} - g_{uk,j})\right] \ , \\
\Gamma^i_{kl} &= \frac{1}{2}g^{ij}(2g_{j(k,l)} - g_{kl,j}) \ ,
\end{aligned} \tag{5.5}$$

where the indices i, j, k, l range from 2 to $D-1$. Using the constraint $g_{ij,r} = 0$ in (4.19)–(4.28) for the Riemann tensor of a general nontwisting spacetime we immediately find that R_{rprq} and R_{rpkq} vanish identically,

$$R_{rprq} = 0 \ , \qquad \text{and} \qquad R_{rpkq} = 0 \ , \tag{5.6}$$

while the remaining nontrivial components of the Riemann tensor are

$$R_{rpru} = -\frac{1}{2}g_{up,rr} \ , \tag{5.7}$$

$$R_{ruru} = -\frac{1}{2}g_{uu,rr} + \frac{1}{4}g^{ij}g_{ui,r}g_{uj,r} \ , \tag{5.8}$$

$$R_{rpuq} = \frac{1}{2}g_{up,rq} + \frac{1}{4}g_{up,r}g_{uq,r} - \frac{1}{4}g^{ij}g_{ui,r}\left(2g_{j(p,q)} - g_{pq,j}\right) \ , \tag{5.9}$$

$$R_{rupq} = g_{u[p,q],r} \ , \tag{5.10}$$

$$R_{ruup} = g_{u[u,p],r} + \frac{1}{4}g^{ri}g_{up,r}g_{ui,r} - \frac{1}{4}g^{ij}g_{ui,r}\left(2g_{j(u,p)} - g_{up,j}\right) \ , \tag{5.11}$$

$$R_{kplq} = {}^{S}R_{kplq} \ , \tag{5.12}$$

$$
\begin{aligned}
R_{upkq} = {} & g_{p[k,q],u} - g_{u[k,q],p} \\
& + \frac{1}{4}\left[g_{uk,r}\left(g_{pq,u} - 2g_{u(p,q)}\right) - g_{uq,r}\left(g_{pk,u} - 2g_{u(p,k)}\right)\right] \\
& + \frac{1}{4}g^{ri}\left[g_{uk,r}\left(2g_{i(p,q)} - g_{pq,i}\right) - g_{uq,r}\left(2g_{i(p,k)} - g_{pk,i}\right)\right] \\
& + \frac{1}{4}g^{ij}\left(2g_{j(u,q)} - g_{uq,j}\right)\left(2g_{i(p,k)} - g_{pk,i}\right) \\
& - \frac{1}{4}g^{ij}\left(2g_{j(u,k)} - g_{uk,j}\right)\left(2g_{i(p,q)} - g_{pq,i}\right) \ , \qquad (5.13)
\end{aligned}
$$

$$
\begin{aligned}
R_{upuq} = {} & g_{u(p,q),u} - \frac{1}{2}\left(g_{pq,uu} + g_{uu,pq}\right) + \frac{1}{4}g^{rr}g_{up,r}g_{uq,r} \\
& - \frac{1}{4}g_{uu,r}\left[2g_{u(p,q)} - g_{pq,u} - g^{ri}\left(2g_{i(p,q)} - g_{pq,i}\right)\right] \\
& + \frac{1}{4}g_{up,r}\left[g_{uu,q} - g^{ri}\left(2g_{i(u,q)} - g_{uq,i}\right)\right] \\
& + \frac{1}{4}g_{uq,r}\left[g_{uu,p} - g^{ri}\left(2g_{i(u,p)} - g_{up,i}\right)\right] \\
& + \frac{1}{4}g^{ij}\left(2g_{j(u,p)} - g_{up,j}\right)\left(2g_{i(u,q)} - g_{uq,i}\right) \\
& - \frac{1}{4}g^{ij}\left(2g_{uj,u} - g_{uu,j}\right)\left(2g_{i(p,q)} - g_{pq,i}\right) \ , \qquad (5.14)
\end{aligned}
$$

where the superscript ^{S} denotes again tensor quantities calculated using only the spatial metric g_{ij}, and derivatives taken only with respect to the spatial coordinates x^i. Simplification of the expressions (4.29)–(4.34) for the Ricci tensor immediately gives $R_{rr} = 0$, and

$$
\begin{aligned}
R_{ru} = {} & -\frac{1}{2}g_{uu,rr} + \frac{1}{2}g^{ri}g_{ui,rr} + \frac{1}{2}g^{pq}g_{up,rq} \\
& + \frac{1}{2}g^{pq}g_{up,r}g_{uq,r} - \frac{1}{4}g^{pq}g^{ij}g_{ui,r}\left(2g_{jp,q} - g_{pq,j}\right) \ , \qquad (5.15)
\end{aligned}
$$

$$
R_{rk} = -\frac{1}{2}g_{uk,rr} \ , \qquad (5.16)
$$

$$
\begin{aligned}
R_{uu} = {} & -\frac{1}{2}g^{rr}g_{uu,rr} - 2g^{ri}g_{u[u,i],r} + \frac{1}{2}g^{pq}\left(2g_{up,uq} - g_{pq,uu} - g_{uu,pq}\right) \\
& - \frac{1}{2}g^{rp}g^{rq}g_{up,r}g_{uq,r} + \frac{1}{2}g^{rr}g^{pq}g_{up,r}g_{uq,r} \\
& + \frac{1}{2}g^{pq}g^{ri}g_{up,r}(2g_{q(u,i)} - g_{ui,q}) \\
& - \frac{1}{4}g^{pq}g_{uu,r}\left[2g_{up,q} - g_{pq,u} - g^{ri}\left(2g_{ip,q} - g_{pq,i}\right)\right] \\
& + \frac{1}{2}g^{pq}g_{up,r}\left[g_{uu,q} - g^{ri}\left(2g_{i(u,q)} - g_{uq,i}\right)\right] \\
& + \frac{1}{4}g^{pq}g^{ij}\left(2g_{j(u,p)} - g_{up,j}\right)\left(2g_{i(u,q)} - g_{uq,i}\right) \\
& - \frac{1}{4}g^{pq}g^{ij}\left(2g_{uj,u} - g_{uu,j}\right)\left(2g_{ip,q} - g_{pq,i}\right) \ , \qquad (5.17)
\end{aligned}
$$

$$
\begin{aligned}
R_{uk} = &-\frac{1}{2}g^{rr}g_{uk,rr} - g_{u[u,k],r} + g^{ri}(g_{u[i,k],r} - g_{k[u,i],r}) + g^{pq}(g_{p[k,q],u} - g_{u[k,q],p}) \\
&-\frac{1}{2}g^{ri}g_{uk,r}g_{ui,r} + \frac{1}{4}g^{pq}g^{ri}\left[4g_{uq,r}g_{k[p,i]} + g_{uk,r}(2g_{i(p,q)} - g_{pq,i})\right] \\
&+\frac{1}{4}g^{pq}\left[2g_{up,r}g_{uq,k} - g_{uk,r}(2g_{up,q} - g_{pq,u})\right] \\
&+\frac{1}{4}g^{pq}g^{ij}\left(2g_{j(u,q)} - g_{uq,j}\right)\left(2g_{i(p,k)} - g_{pk,i}\right) \\
&-\frac{1}{4}g^{pq}g^{ij}\left(2g_{j(u,k)} - g_{uk,j}\right)\left(2g_{ip,q} - g_{pq,i}\right) \ , \qquad (5.18)
\end{aligned}
$$

$$
R_{pq} = {}^{S}R_{pq} - g_{u(p,q),r} - \frac{1}{2}g_{up,r}g_{uq,r} + \frac{1}{2}g^{kl}g_{uk,r}\left(2g_{l(p,q)} - g_{pq,l}\right) \ , \qquad (5.19)
$$

Finally, using (4.35) the scalar curvature of any Kundt spacetime is given by

$$
R = {}^{S}R + g_{uu,rr} - 2g^{ri}g_{ui,rr} - 2g^{pq}g_{up,rq} - \frac{3}{2}g^{pq}g_{up,r}g_{uq,r} + g^{pq}g^{kl}g_{uk,r}(2g_{lp,q} - g_{pq,l}) \ . \qquad (5.20)
$$

5.2 Applying the field equations

We have not specified the matter content of the spacetime so far. However, we may recall the approach of [31] and simplify the r-dependence of the metric (5.2) using the Einstein field equations (4.41). Notice that

- the explicit value of the Ricci tensor component $R_{rr} = 0$ together with $g_{rr} = 0$ gives a restriction on the energy–momentum tensor, namely $T_{rr} = 0$,
- we can also directly integrate the Einstein's equation connected with R_{rk} assuming $T_{rk} = 0$,
- assuming $T_{rk} = 0$, the equation for R_{ru} implies that T_{ru} must be independent of r,
- using the trace of Einstein's equations we can determine the r-dependence of g_{uu}; if the trace of energy–momentum tensor does not depend on r, the metric function g_{uu} could be only quadratic in r.

Following [31], we restrict ourselves to the vacuum spacetimes, possibly with a cosmological constant Λ or spacetimes with the Maxwell field aligned with the null vector $\tilde{\mathbf{k}}$ ($F_{ab}\tilde{k}^b = Q\tilde{k}_a$, where Q is an arbitrary function). The energy–momentum tensor in such a case is

$$
4\pi T_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd} \ , \qquad (5.21)
$$

where F_{ab} in coordinates (r, u, x^i) adapted to the field $\tilde{\mathbf{k}} = \partial_r$ has the components $F_{ru} = Q$ and $F_{ri} = 0$, and thus $T_{rr} = T_{ri} = 0$. The components F_{ui} and F_{ij} are constrained by source-free Maxwell equations $F_{[ab;c]=0}$ and $F^{ab}{}_{;b} = 0$. Their precise discussion can be found in [31], recall only the main results:

- the components $F_{ru} = Q(u, x)$ and $F_{ij} = F_{ij}(u, x)$ are independent of r, and F_{ui} is given by $F_{ui} = -rQ_{,i} - \xi_i$ where $\xi_i(u, x)$ are arbitrary functions,
- the trace of energy–momentum tensor is $T = -\frac{D-4}{16\pi}F_{ab}F^{ab}$, where $F_{ab}F^{ab} = F_{ij}F^{ij} - 2Q^2 \equiv F^2 - 2Q^2$, and it does also not depend on r.

Concerning these restrictions and employing Einstein's field equations the line element of a general Kundt spacetime can be written in the form

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j + 2(e_i + f_i r)\mathrm{d}x^i\mathrm{d}u - 2\mathrm{d}u\mathrm{d}r + (ar^2 + br + c)\mathrm{d}u^2 \,, \tag{5.22}$$

with the r-dependence fully determined. All the remaining functions, g_{ij}, e_i, f_i, a, b and c, are constrained by the specific Einstein–Maxwell equations and only depend on the coordinates u and x^i. In particular, the function $a(u,x)$ reads

$$a(u,x) = \frac{1}{2}\left[f^i f_i + f^i{}_{,i} + f^i(\ln\sqrt{g})_{,i}\right] + \frac{2\Lambda}{D-2} - \frac{F^2 + 2(D-3)Q^2}{D-2} \,, \tag{5.23}$$

where $f^i = g^{ij}f_j$ and $g = \det g_{ij}$, see [31].

5.3 Geodesic deviation in general Kundt spacetime

The geometric restrictions leading to the line element (5.2) imply $C_{rprq} = 0$, see (4.38) with $g_{pq,r} = 0$, and (4.63) then gives $\Psi_{0^{ij}} = 0$. The Kundt spacetimes are thus necessarily of algebraic type I or more special, see subsection 4.6.3.

Moreover, the assumption $T_{rk} = 0$ and Einstein's equations guarantee that $g_{ui} = e_i + f_i r$, i.e., additionally $R_{rpru} = 0$, $R_{rp} = 0$ which imply $C_{rpru} = 0$ and $C_{rpkq} = 0$, see (4.37). Substituting these vanishing components of the Weyl tensor into (4.63) and (4.64) we immediately find[1] that $\Psi_{1T^j} = 0$ and $\Psi_{1^{ijk}} = 0$. Due to the vanishing Weyl tensor null components of boost weights 2 and 1, the line element (5.22) represents a Kundt spacetime which is (at least) of algebraic type II.

The equations (4.62) describing the geodesic deviation in general nontwisting spacetime, using $\Psi_{0^{ij}} = 0$ and $\Psi_{1T^j} = 0$ in the case of the Kundt class (5.22), reduce to

$$\begin{aligned}
\ddot{Z}^{(1)} &= \frac{2\Lambda}{(D-2)(D-1)}Z^{(1)} + \Psi_{2S}Z^{(1)} - \frac{1}{\sqrt{2}}\Psi_{3T^j}Z^{(j)} \\
&\quad + \frac{8\pi}{D-2}\left[T_{(1)(1)}Z^{(1)} + T_{(1)(j)}Z^{(j)} - \left(T_{(0)(0)} + \frac{2T}{D-1}\right)Z^{(1)}\right] \,, \\
\ddot{Z}^{(i)} &= \frac{2\Lambda}{(D-2)(D-1)}Z^{(i)} - \Psi_{2T^{(ij)}}Z^{(j)} - \frac{1}{\sqrt{2}}\Psi_{3T^i}Z^{(1)} - \frac{1}{2}\Psi_{4^{ij}}Z^{(j)} \\
&\quad + \frac{8\pi}{D-2}\left[T_{(i)(1)}Z^{(1)} + T_{(i)(j)}Z^{(j)} - \left(T_{(0)(0)} + \frac{2T}{D-1}\right)Z^{(i)}\right] \,,
\end{aligned} \tag{5.24}$$

where the trace of the energy–momentum electromagnetic tensor is $T = -\frac{D-4}{16\pi}F_{ab}F^{ab}$ with $F_{ab}F^{ab} = F^2 - 2Q^2$, and its frame components are

$$\begin{aligned}
T_{(0)(0)} &= \frac{1}{4\pi}\left(F_{(0)a}F_{(0)}{}^a + \frac{1}{4}F_{ab}F^{ab}\right) \,, & T_{(0)(i)} &= \frac{1}{4\pi}F_{(0)a}F_{(i)}{}^a \,, \\
T_{(1)(1)} &= \frac{1}{4\pi}\left(F_{(1)a}F_{(1)}{}^a - \frac{1}{4}F_{ab}F^{ab}\right) \,, & T_{(1)(i)} &= \frac{1}{4\pi}F_{(1)a}F_{(i)}{}^a \,, \\
T_{(0)(1)} &= \frac{1}{4\pi}F_{(0)a}F_{(1)}{}^a \,, & T_{(i)(j)} &= \frac{1}{4\pi}\left(F_{(i)a}F_{(j)}{}^a - \frac{1}{4}\delta_{ij}F_{ab}F^{ab}\right) .
\end{aligned} \tag{5.25}$$

[1] Notice that $\Psi_{1^{ijk}} = 0$ implies $\Psi_{1T^j} = 0$ by the definition.

The null interpretation frame (4.50) connected with a particular observer moving along timelike geodesic $\gamma(\tau)$ with velocity $\mathbf{u} = \dot{r}\partial_r + \dot{u}\partial_u + \dot{x}^i\partial_i$ is of the form

$$\begin{aligned}
\mathbf{k} &= \frac{1}{\sqrt{2}\dot{u}}\partial_r \ , \\
\mathbf{l} &= \left(\sqrt{2}\dot{r} - \frac{1}{\sqrt{2}\dot{u}}\right)\partial_r + \sqrt{2}\dot{u}\,\partial_u + \sqrt{2}\dot{x}^i\partial_i \ , \\
\mathbf{m}_i &= \frac{1}{\dot{u}}(g_{uq}\dot{u} + g_{qk}\dot{x}^k)\, m_i^q\,\partial_r + m_i^j\,\partial_j \ ,
\end{aligned} \tag{5.26}$$

where $g_{uq} = e_q + f_q r$. Using our previous results for the projections of the Weyl tensor on the null frame (4.68)–(4.71) and the exact values

$$C_{rprq} = 0 \ , \qquad C_{rpru} = 0 \ , \qquad C_{rpkq} = 0 \ , \tag{5.27}$$

we obtain the only nonvanishing null frame components of the Weyl tensor, which are relevant in equations (5.24), namely,

$$\begin{aligned}
\Psi_{2T^{ij}} &= m_i^p m_j^q\, C_{rpuq} \ , \\
\Psi_{2S} &= -C_{ruru} \ , \\
\Psi_{3T^j} &= \sqrt{2} m_j^p \left[\dot{x}^k\,(C_{ruru}\,g_{kp} - C_{rkup} - C_{rukp}) + \dot{u}\,(C_{ruru}\,g_{up} - C_{ruup})\right] \ , \\
\Psi_{4^{ij}} &= 2m_{(i}^p m_{j)}^q \Big\{ \dot{x}^k\dot{x}^l\,(C_{rpuq}\,g_{kl} - 2C_{rluq}\,g_{kp} + C_{ruru}\,g_{pk}g_{lq} - 2C_{rulq}\,g_{kp} + C_{kplq}) \\
&\quad + 2\dot{u}\dot{x}^k\,(C_{rpuq}\,g_{uk} - C_{rkuq}\,g_{up} + C_{ruru}\,g_{up}g_{qk} - C_{ruup}\,g_{qk} - C_{rukq}\,g_{up} + C_{upkq}) \\
&\quad + \dot{u}^2\,(C_{rpuq}\,g_{uu} + C_{ruru}\,g_{up}g_{uq} - 2C_{ruup}\,g_{uq} + C_{upuq})\Big\} \ .
\end{aligned} \tag{5.28}$$

We may also substitute the explicit expressions for the Weyl tensor (4.37):

$$\begin{aligned}
\Psi_{2T^{ij}} &= m_i^p m_j^q \left[R_{rpuq} - \frac{1}{D-2}\,(g_{pq}R_{ru} - R_{pq}) - \frac{R\,g_{pq}}{(D-1)(D-2)}\right] \ , \\
\Psi_{2S} &= -R_{ruru} + \frac{2}{D-2}R_{ru} + \frac{R}{(D-1)(D-2)} \ , \\
\Psi_{3T^j} &= \sqrt{2} m_j^p \Big\{ \dot{x}^k \left[R_{ruru}\,g_{kp} - R_{rkup} - R_{rukp} - \frac{1}{D-2}\,(g_{kp}R_{ru} + R_{kp})\right] \\
&\quad + \dot{u}\left[R_{ruru}\,g_{up} - R_{ruup} - \frac{1}{D-2}\,(g_{up}R_{ru} + R_{up})\right]\Big\} \ , \\
\Psi_{4^{ij}} &= 2m_{(i}^p m_{j)}^q \Big\{ \dot{x}^k\dot{x}^l \left[R_{rpuq}\,g_{kl} - g_{pk}(2R_{rluq} - g_{lq}R_{ruru} + 2R_{rulq}) + R_{kplq} - \frac{g_{pq}}{D-2}\,(g_{kl}R_{ru} + R_{kl})\right] \\
&\quad + 2\dot{u}\dot{x}^k \left[R_{rpuq}\,g_{uk} - R_{ruup}\,g_{qk} - g_{up}(R_{rkuq} - R_{ruru}\,g_{qk} + R_{rukq}) + R_{upkq} - \frac{g_{pq}}{D-2}(g_{uk}R_{ru} + R_{uk})\right] \\
&\quad + \dot{u}^2 \left[R_{rpuq}\,g_{uu} - g_{uq}\,(2R_{ruup} - g_{up}R_{ruru}) + R_{upuq} - \frac{g_{pq}}{D-2}\,(g_{uu}R_{ru} + R_{uu})\right]\Big\} \ .
\end{aligned} \tag{5.29}$$

where the components R_{abcd} are explicitly given by (5.8)–(5.14), R_{ab} by (5.15)–(5.19) and the scalar curvature R by (5.20).

In conclusion, concerning a general Kundt spacetime (5.22), the equations of geodesic deviation (5.24) imply that the behaviour of a set of free test particles will be determined by presence of

the spacetime matter content (for example by aligned Maxwell field (5.25)), by isotropic influence of cosmological constant Λ, and by pure gravitational effects consisting only of Newton-like tidal deformation represented by $\Psi_{2T^{(ij)}}$ and Ψ_{2S}, longitudinal accelerations with respect to $+\mathbf{e}_{(1)}$ given by Ψ_{3T^j}, and transverse gravitational wave propagating in $+\mathbf{e}_{(1)}$ encoded in traceless symmetric matrix $\Psi_{4^{ij}}$.

Notice finally that the Newton-type deformation does not directly depend on the velocity of observer's motion, see (5.29). On the other hand, longitudinal and radiative component are affected by observer's motion in the u direction by terms proportional to $\dot{u}$, but also in all spatial directions x^i, i.e., by terms proportional to the velocities $\dot{x}^i$. The kinematic effects given by motion in the transverse space and represented by terms proportional to $\dot{x}^i$ can be transformed away by transition to another observer moving with the same velocity in the opposite direction, for more details see Chapter 3 and particular example in section 5.6.

5.4 Properties of pp-waves

It is natural to start the investigation of subclasses of Kundt's spacetimes with an important exact gravitational wave model, namely "plane fronted waves with parallel rays" which are also called pp-waves. We only briefly recall the main results described in Chapter 3 and we will focus on the properties of vacuum pp-waves which represent exact higher-dimensional type N solutions. We show that these properties are in general different form those well known in classical four-dimensional general relativity, and similar for all higher-dimensional radiative spacetimes.

This class of solutions was discovered by Brinkmann in the context of conformal mapping of Einstein spaces, see [12]. Their geometric definition requests the existence of covariantly constant null vector field $\tilde{\mathbf{k}}$. The expressions (4.7) for optical scalars then implies that these solutions have to necessary belong to the Kundt class. The explicit form of the covariant derivative in the spacetime (5.22) with adapted coordinates (r, u, x^i) is $\tilde{k}_{a;b} = \frac{1}{2}g_{ab,r}$. If we require $\tilde{k}_{a;b} = 0$, the resulting line element must be independent of coordinate r. We thus get the line element

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j + 2e_i(u,x)\mathrm{d}x^i\mathrm{d}u - 2\mathrm{d}u\mathrm{d}r + c(u,x)\mathrm{d}u^2 \ , \tag{5.30}$$

where the functions $e_i(x,u) \equiv g_{ui}$ and $c(x,u) \equiv g_{uu}$ have to satisfy the Einstein-Maxwell equations (4.41). For construction of the $\Psi_{A\bullet}$ scalars we use (5.7)–(5.14) together with property $g_{ab,r} = 0$, and calculate the only nonvanishing components of Riemann tensor:

$$R_{kplq} = {}^{S}R_{kplq} \ , \tag{5.31}$$

$$\begin{aligned} R_{upkq} = {} & g_{p[k,q],u} - g_{u[k,q],p} \\ & + \frac{1}{4}g^{ij}\left(2g_{j(u,q)} - g_{uq,j}\right)\left(2g_{i(p,k)} - g_{pk,i}\right) \\ & - \frac{1}{4}g^{ij}\left(2g_{j(u,k)} - g_{uk,j}\right)\left(2g_{i(p,q)} - g_{pq,i}\right) \ , \end{aligned} \tag{5.32}$$

$$\begin{aligned} R_{upuq} = {} & g_{u(p,q),u} - \frac{1}{2}\left(g_{pq,uu} + g_{uu,pq}\right) \\ & + \frac{1}{4}g^{ij}\left(2g_{j(u,p)} - g_{up,j}\right)\left(2g_{i(u,q)} - g_{uq,i}\right) \\ & - \frac{1}{4}g^{ij}\left(2g_{uj,u} - g_{uu,j}\right)\left(2g_{i(p,q)} - g_{pq,i}\right) \ . \end{aligned} \tag{5.33}$$

For the Ricci tensor we immediately get from (5.15)–(5.19)

$$\begin{aligned} R_{uu} &= \frac{1}{2}g^{pq}\left(2g_{up,uq}-g_{pq,uu}-g_{uu,pq}\right) \\ &\quad +\frac{1}{4}g^{pq}g^{ij}\left(2g_{j(u,p)}-g_{up,j}\right)\left(2g_{i(u,q)}-g_{uq,i}\right) \\ &\quad -\frac{1}{4}g^{pq}g^{ij}\left(2g_{uj,u}-g_{uu,j}\right)\left(2g_{ip,q}-g_{pq,i}\right)\ , \qquad (5.34) \\ R_{uk} &= g^{pq}(g_{p[k,q],u}-g_{u[k,q],p}) \\ &\quad +\frac{1}{4}g^{pq}g^{ij}\left(2g_{j(u,q)}-g_{uq,j}\right)\left(2g_{i(p,k)}-g_{pk,i}\right) \\ &\quad -\frac{1}{4}g^{pq}g^{ij}\left(2g_{j(u,k)}-g_{uk,j}\right)\left(2g_{ip,q}-g_{pq,i}\right)\ , \qquad (5.35) \\ R_{pq} &= {}^{S}R_{pq}\ , \qquad (5.36) \end{aligned}$$

and the Ricci scalar is given only by its spatial part, $R = {}^{S}R$. The expressions (5.29) for the $\Psi_{A\bullet}$ scalars then take the simplified explicit form

$$\begin{aligned} \Psi_{2S} &= \frac{{}^{S}R}{(D-1)(D-2)}\ , \qquad \Psi_{2T^{(ij)}} = m^{p}_{(i}m^{q}_{j)}\left[\frac{1}{D-2}\,{}^{S}R_{pq}-\frac{{}^{S}R\,g_{pq}}{(D-1)(D-2)}\right]\ , \\ \Psi_{3T^{j}} &= -\frac{\sqrt{2}}{D-2}m^{p}_{j}\left(\dot{x}^{k}\,{}^{S}R_{kp}+\dot{u}\,R_{up}\right)\ , \\ \Psi_{4^{ij}} &= 2m^{p}_{(i}m^{q}_{j)}\Bigg[\dot{x}^{k}\dot{x}^{l}\left({}^{S}R_{kplq}-\frac{g_{pq}}{D-2}\,{}^{S}R_{kl}\right) \\ &\qquad +2\dot{u}\dot{x}^{k}\left(R_{upkq}-\frac{g_{pq}}{D-2}R_{uk}\right)+\dot{u}^{2}\left(R_{upuq}-\frac{g_{pq}}{D-2}R_{uu}\right)\Bigg]\ . \qquad (5.37) \end{aligned}$$

Notice that the Newton components of the gravitational field represented by Ψ_{2S} and $\Psi_{2T^{(ij)}}$ depend only on the curvature of transverse space, i.e., on the Ricci tensor ${}^{S}R_{pq}$ and the scalar ${}^{S}R$ calculated using the Riemannian metric g_{ij} on the transverse $(D-2)$-dimensional space with respect to coordinates x^{k}. In the case of the Ricci flat transverse space, these components vanish and such pp-waves will be necessarily of algebraic type III or more special.

5.4.1 Vacuum pp-waves

Now, we restrict our attention to the vacuum pp-waves. Because of $R_{ru}=0$, the Einstein equation $R_{ru} = -\frac{2\Lambda}{D-2}$ implies vanishing[2] cosmological constant Λ. Moreover, applying the remaining vacuum field equations we find that the only nonvanishing component of Weyl tensor is

$$\Psi_{4^{ij}} = 2m^{p}_{(i}m^{q}_{j)}\left(\dot{x}^{k}\dot{x}^{l}\,{}^{S}R_{kplq}+2\dot{u}\dot{x}^{k}R_{upkq}+\dot{u}^{2}R_{upuq}\right)\ . \qquad (5.38)$$

Vacuum pp-waves are thus necessarily of algebraic type N, and represent exact transverse gravitational waves propagating along the null direction $\tilde{\mathbf{k}}$ corresponding to the spatial direction $+\mathbf{e}_{(1)}$. The deformation of a set of free test particles is described only by

$$\ddot{Z}^{(1)} = 0\ , \qquad \ddot{Z}^{(i)} = -\frac{1}{2}\Psi_{4^{ij}}Z^{(j)}\ . \qquad (5.39)$$

Obviously, there is no acceleration in the privileged spatial direction $\mathbf{e}_{(1)}$. The set of scalars $\Psi_{4^{ij}}$ forms a symmetric and traceless matrix of dimension $(D-2)\times(D-2)$ which has in general

[2] In a case of an aligned Maxwell field, cosmological constant is necessarily positive, namely $2\Lambda = F^{2}+2(D-3)Q^{2}$.

$N \equiv \frac{1}{2}D(D-3)$ independent components corresponding to polarization modes of a gravitational wave. The freedom in a choice of the transverse part $\mathbf{m}_i$ of the interpretation frame is given just by spatial rotations (4.46), see Chapter 3,

$$\tilde{\mathbf{m}}_i = \Phi_i{}^j\, \mathbf{m}_j \ , \qquad \text{where} \qquad \Phi_i{}^j\, \Phi_k{}^l\, \delta_{jl} = \delta_{ik} \ , \tag{5.40}$$

which leave the null frame vectors unchanged, $\tilde{\mathbf{k}} = \mathbf{k}$ and $\tilde{\mathbf{l}} = \mathbf{l}$. These spatial rotations belong to SO$(D-2)$ group containing $N_{\rm rot} \equiv \frac{1}{2}(D-2)(D-3)$ independent parameters representing its generators. Therefore, the number of physical degrees of freedom is

$$N - N_{\rm rot} = D - 3 \ , \tag{5.41}$$

which corresponds to the number of independent eigenvalues of the symmetric and traceless matrix Ψ_{4ij}, and fully characterize the deformation of a test congruence. With respect to the signs of the eigenvalues we can distinguish

$$\binom{D-2}{2} = \frac{1}{2}(D-2)(D-3) \ , \tag{5.42}$$

physically different cases. Sum of all the eigenvalues must be vanishing (traceless property of gravitational waves), i.e., there is at least one positive and one negative eigenvalue in case of a nontrivial Ψ_{4ij}. The relation (5.42) gives the number of distinct possibilities how to divide the remaining eigenvalues into three groups with positive, null and negative sign, respectively. Such a discussion is primarily motivated by a particular example of pp-waves, but holds also for any other exact gravitational waves represented by traceless symmetric matrix.

Finally, we would like to illustrate these results in a particular case which can be completely integrated. We may choose the simplest possibility, discussed also in Chapter 3, which corresponds to plane fronted waves propagating in Minkowski spacetime, i.e., the spatial metric will be flat, $g_{ij} = \delta_{ij}$. We additionally assume that the nondiagonal metric functions $g_{ui} = e_i$ are vanishing or can be globally removed by a suitable coordinate transformation (gyratons are absent). The general line element (5.30) then becomes

$$\mathrm{d}s^2 = \delta_{ij}\, \mathrm{d}x^i \mathrm{d}x^j - 2\mathrm{d}u\mathrm{d}r + c(u,x)\mathrm{d}u^2 \ . \tag{5.43}$$

Our interpretation frame (5.26) will be of the form

$$\mathbf{k} = \frac{1}{\sqrt{2}\dot{u}}\partial_r \ , \qquad \mathbf{l} = \left(\sqrt{2}\dot{r} - \frac{1}{\sqrt{2}\dot{u}}\right)\partial_r + \sqrt{2}\dot{u}\partial_u + \sqrt{2}\dot{x}^i\partial_i \ , \qquad \mathbf{m}_i = \frac{\dot{x}^i}{\dot{u}}\partial_r + \partial_i \ . \tag{5.44}$$

The only nontrivial components of Riemann tensor in spacetime (5.43) are $R_{upuq} = -\frac{1}{2}c_{,ij}$ which imply that the scalars (5.38) simplify to

$$\Psi_{4ij} = -\dot{u}^2 c_{,ij} \ . \tag{5.45}$$

The remaining Einstein equation, namely $R_{uu} = \delta^{ij}c_{,ij} = 0$, guarantees that the amplitude matrix Ψ_{4ij} is traceless. From the equation of geodesics (4.73) it follows that $\ddot{u}$ vanishes identically and the velocity $\dot{u}$ is necessarily constant along the geodesic. Moreover, using the conditions for parallel transport (4.75) we find that the interpretation frame is parallelly transported and the relative accelerations in (5.39) can be thus replaced by ordinary time derivatives. Deformation of the test congruence of geodesics will be given by

$$\frac{\mathrm{d}^2 Z^{(1)}}{\mathrm{d}\tau^2} = 0 \ , \qquad \frac{\mathrm{d}^2 Z^{(i)}}{\mathrm{d}\tau^2} = \frac{1}{2}\dot{u}^2 c_{,ij} Z^{(j)} \ , \tag{5.46}$$

where we only have to specify the function $c(u,x)$ satisfying $\delta^{ij}c_{,ij}=0$. We will consider a simple quadratic form constructed only of spatial coordinates x^i, i.e.,

$$c=\sum_{i=2}^{D-1}\mathcal{A}_i{x^i}^2\ , \qquad \text{where} \qquad \sum_{i=2}^{D-1}\mathcal{A}_i=0\ , \tag{5.47}$$

$\mathcal{A}_i$ are constants. The amplitudes of corresponding gravitational waves are thus constants $\Psi_{4^{ij}}=-2\dot{u}^2\mathcal{A}_i\delta_{ij}$ (no summation), and the waves of this type are thus homogeneous. Explicit solution of equations of geodesic deviation (5.46) will take the form

$$Z^{(1)}(\tau)=A_1\tau+B_1\ , \qquad Z^{(i)}(\tau)=A_i\mathrm{Re}\left\{\mathrm{e}^{|\dot{u}|\sqrt{\mathcal{A}_i}\tau}\right\}+B_i\mathrm{Re}\left\{\mathrm{e}^{-|\dot{u}|\sqrt{\mathcal{A}_i}\tau}\right\}\ , \tag{5.48}$$

where A_i and B_i are constants of integration. The most illustrative case to investigate is the deformation of a sphere formed by static test particles. Such initially static configuration, $\dot{Z}^{(\mathrm{i})}(\tau=0)=0$, corresponds to the choice $A_1=0$ and $A_i=B_i=\frac{1}{2}Z^i(\tau=0)\equiv\frac{1}{2}Z^i_0$. The equation (5.48) can be rewritten as

$$Z^{(1)}(\tau)=Z^1_0\ , \qquad \text{and} \qquad \begin{array}{lll} Z^{(i)}(\tau)=Z^i_0\cosh\left(|\dot{u}|\sqrt{\mathcal{A}_i}\tau\right) & \text{for} & \mathcal{A}_i>0\ , \\ Z^{(i)}(\tau)=Z^i_0\cos\left(|\dot{u}|\sqrt{-\mathcal{A}_i}\tau\right) & \text{for} & \mathcal{A}_i<0\ , \\ Z^{(i)}(\tau)=Z^i_0 & \text{for} & \mathcal{A}_i=0\ , \end{array} \tag{5.49}$$

We immediately observe that the particles in spatial direction $\mathbf{e}_{(i)}$ with positive $\mathcal{A}_i$ recede, in directions with negative $\mathcal{A}_i$ are focused, and particles in directions where $\mathcal{A}_i$ vanish are not affected by the wave.

As an example, we will analyze a deformation of a three-dimensional test sphere in the transverse space induced by a *five-dimensional plane fronted* wave. There may occur *three nontrivial physically different situations*, see (5.42),

- two eigenvalues are positive and one is negative: $\Psi_{4^{22}}\geq\Psi_{4^{33}}>0>\Psi_{4^{44}}$, see Figure 5.1,
- one eigenvalue is positive and two are negative: $\Psi_{4^{22}}>0>\Psi_{4^{33}}\geq\Psi_{4^{44}}$, see Figure 5.2,
- one eigenvalue is positive, one is vanishing and one is negative: $\Psi_{4^{22}}>\Psi_{4^{33}}=0>\Psi_{4^{44}}$, see Figure 5.3.

Notice that there exists only one possibility in the classical four-dimensional general relativity, where necessarily $\Psi_{4^{22}}=-\Psi_{4^{33}}$. Its visualization is similar to the Figure 5.3, where $\mathbf{e}_{(3)}$ will become the direction of propagation of the wave.

If we restrict our measurements of gravitational waves in generally D-dimensional universe only into four-dimensional "real" subspace spanned by the vectors $(\mathbf{e}_{(0)},\mathbf{e}_{(1)},\mathbf{e}_{(2)},\mathbf{e}_{(3)})$, in this subspace we should observe the violation of traceless property of such waves[3], namely $\Psi_{4^{22}}=-\Psi_{4^{33}}-\sum_{i=4}^{D-1}\Psi_{4^{ii}}$. In the extreme cases when $\Psi_{4^{22}}>0$ and $\Psi_{4^{33}}\geq 0$ we will observe a behaviour which is not typical for gravitational waves in standard general relativity.

[3]Except very special cases when $\Psi_{4^{22}}=-\Psi_{4^{33}}$ and $\sum_{i=4}^{D-1}\Psi_{4^{ii}}=0$.

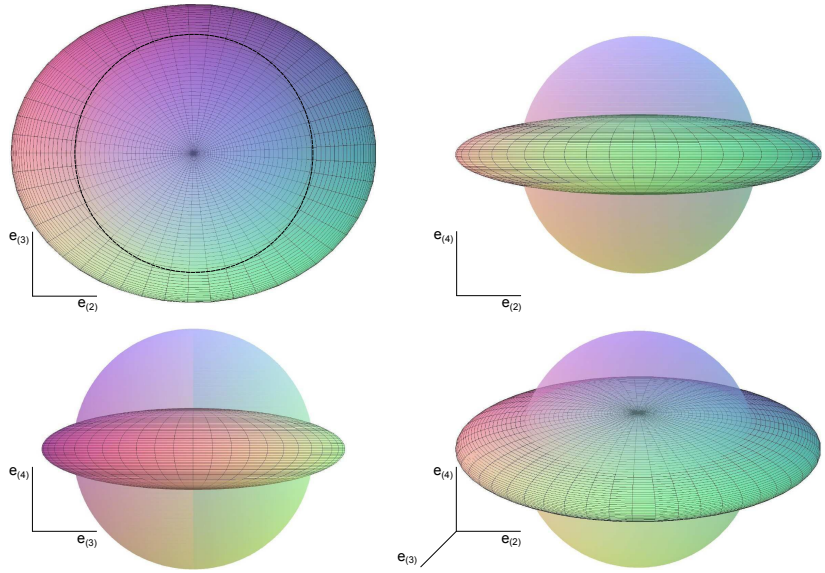

Figure 5.1: Deformation of a sphere of test particles in the case when $\Psi_{4^{22}} \geq \Psi_{4^{33}} > 0 > \Psi_{4^{44}}$. These pictures display planes $(\mathbf{e}_2, \mathbf{e}_3)$, $(\mathbf{e}_2, \mathbf{e}_4)$, $(\mathbf{e}_3, \mathbf{e}_4)$, and a global view.

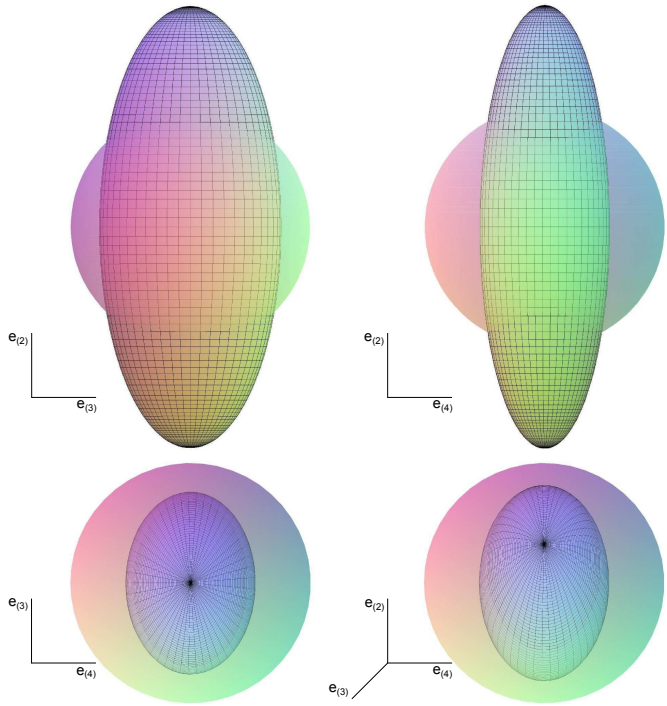

Figure 5.2: Deformation of a sphere of test particles in the case when $\Psi_{4^{22}} > 0 > \Psi_{4^{33}} \geq \Psi_{4^{44}}$. These pictures display planes $(\mathbf{e}_3, \mathbf{e}_2)$, $(\mathbf{e}_4, \mathbf{e}_2)$, $(\mathbf{e}_4, \mathbf{e}_3)$, and a global view.

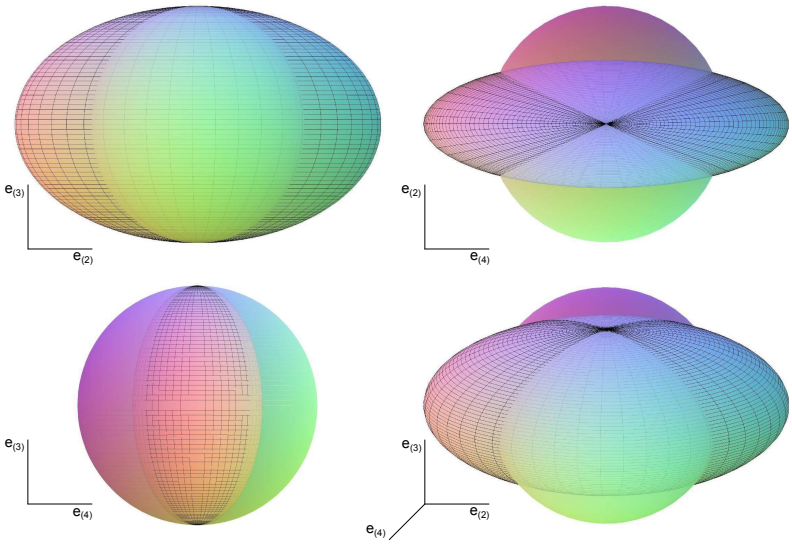

Figure 5.3: Deformation of a sphere of test particles in the case when $\Psi_{4^{22}} > \Psi_{4^{33}} = 0 > \Psi_{4^{44}}$. These pictures display planes $(\mathbf{e}_2, \mathbf{e}_3)$, $(\mathbf{e}_2, \mathbf{e}_4)$, $(\mathbf{e}_4, \mathbf{e}_3)$, and a global view.

5.5 VSI spacetimes

In the works [32, 33], spacetimes with vanishing scalar invariants of all orders were presented. These solutions of Einstein's equations belong to general Kundt class (5.22), in the case when the transverse metric g_{ij} is flat, i.e., $g_{ij} = \delta_{ij}$. The resulting line element is given by

$$\mathrm{d}s^2 = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + 2(e_i + f_i r)\mathrm{d}x^i\mathrm{d}u - 2\mathrm{d}u\mathrm{d}r + (ar^2 + br + c)\mathrm{d}u^2 \ . \tag{5.50}$$

Its obvious that pp-waves on a flat Minkowski background, discussed in the previous section, form a subclass of VSI spacetimes. The components of the Weyl tensor are almost of the same form as in the general case (5.29), only with R_{kplq} vanishing in the scalars $\Psi_{4^{ij}}$, namely,

$$\begin{aligned}
\Psi_{2T^{ij}} &= m_i^p m_j^q \left[R_{rpuq} - \frac{1}{D-2}\left(g_{pq}R_{ru} - R_{pq}\right) - \frac{R\, g_{pq}}{(D-1)(D-2)} \right] \ , \\
\Psi_{2S} &= -R_{ruru} + \frac{2}{D-2}R_{ru} + \frac{R}{(D-1)(D-2)} \ , \\
\Psi_{3T^j} &= \sqrt{2}m_j^p \Big\{ \dot{x}^k \left[R_{ruru}\, g_{kp} - R_{rkup} - R_{rukp} - \frac{1}{D-2}\left(g_{kp}R_{ru} + R_{kp}\right) \right] \\
&\qquad + \dot{u} \left[R_{ruru}\, g_{up} - R_{ruup} - \frac{1}{D-2}\left(g_{up}R_{ru} + R_{up}\right) \right] \Big\} \ , \\
\Psi_{4^{ij}} &= 2m_{(i}^p m_{j)}^q \Big\{ \dot{x}^k\dot{x}^l \left[R_{rpuq}\, g_{kl} - g_{pk}(2R_{rluq} - g_{lq}R_{ruru} + 2R_{rulq}) - \frac{g_{pq}}{D-2}\left(g_{kl}R_{ru} + R_{kl}\right) \right] \\
&\qquad + 2\dot{u}\dot{x}^k \left[R_{rpuq}\, g_{uk} - R_{ruup}\, g_{qk} - g_{up}(R_{rkuq} - g_{qk}R_{ruru} + R_{rukq}) + R_{upkq} - \frac{g_{pq}}{D-2}(g_{uk}R_{ru} + R_{uk}) \right] \\
&\qquad + \dot{u}^2 \left[R_{rpuq}\, g_{uu} - g_{uq}\left(2R_{ruup} - g_{up}R_{ruru}\right) + R_{upuq} - \frac{g_{pq}}{D-2}\left(g_{uu}R_{ru} + R_{uu}\right) \right] \Big\} \ ,
\end{aligned} \tag{5.51}$$

where, using the flatness condition $g_{ij} = \delta_{ij}$ and (5.8)–(5.14), the Riemann tensor becomes

$$R_{ruru} = -a + \frac{1}{4} f^i f_i \ , \tag{5.52}$$

$$R_{rpuq} = \frac{1}{2} f_{p,q} + \frac{1}{4} f_p f_q \ , \tag{5.53}$$

$$R_{rupq} = f_{[p,q]} \ , \tag{5.54}$$

$$R_{ruup} = \frac{1}{2}(2ra_{,p} + b_{,p} - f_{p,u}) + \frac{1}{4} g^{ri} f_i f_p - \frac{1}{2} f^i \left(e_{[i,p]} + r f_{[i,p]}\right) \ , \tag{5.55}$$

$$R_{upkq} = (e_{[q,k]} + r f_{[q,k]})_{,p} + \frac{1}{2}\left[f_q (e_{(p,k)} + r f_{(p,k)}) - f_k (e_{(p,q)} + r f_{(p,q)})\right] \ , \tag{5.56}$$

$$\begin{aligned} R_{upuq} = {} & (e_{(p,q)} + r f_{(p,q)})_{,u} - \frac{1}{2}(ar^2 + br + c)_{,pq} \\ & + \frac{1}{4} g^{rr} f_p f_q - \frac{1}{2}(2ar + b)(e_{(p,q)} + r f_{(p,q)}) \\ & + \frac{1}{4} f_p \left[(ar^2 + br + c)_{,q} - 2g^{ri}\left(e_{[i,q]} + r f_{[i,q]}\right)\right] \\ & + \frac{1}{4} f_q \left[(ar^2 + br + c)_{,p} - 2g^{ri}\left(e_{[i,p]} + r f_{[i,p]}\right)\right] \\ & + \delta^{ij}\left(e_{[j,p]} + r f_{[j,p]}\right)\left(e_{[i,q]} + r f_{[i,q]}\right) \ . \end{aligned} \tag{5.57}$$

Nontrivial components of the Ricci tensor given by (5.15)–(5.19) are

$$R_{ru} = -a + \frac{1}{2}\delta^{pq} f_{p,q} + \frac{1}{2} f^p f_p \ , \tag{5.58}$$

$$\begin{aligned} R_{uu} = {} & -g^{rr} a - g^{ri}(2ra_{,i} + b_{,i} - f_{i,u}) + \frac{1}{2}\delta^{pq}\left[2(e_p + r f_p)_{,uq} - (ar^2 + br + c)_{,pq}\right] \\ & - \frac{1}{2} g^{rp} g^{rq} f_p f_q + \frac{1}{2} g^{rr} f^p f_p \\ & + g^{ri} f^p (e_{[p,i]} + r f_{[p,i]}) - \frac{1}{2}\delta^{pq}(2ar + b)(e_{p,q} + r f_{p,q}) \\ & + \frac{1}{2} f^p \left[(ar^2 + br + c)_{,p} - 2g^{ri}(e_{[i,p]} + r f_{[i,p]})\right] \\ & + \delta^{pq}\delta^{ij}\left(e_{[j,p]} + r f_{[j,p]}\right)\left(e_{[i,q]} + r f_{[i,q]}\right) \ , \end{aligned} \tag{5.59}$$

$$\begin{aligned} R_{uk} = {} & -\frac{1}{2}(2ra_{,k} + b_{,k} - f_{k,u}) + \frac{1}{2} g^{ri}(f_{i,k} - 2f_{k,i}) - \delta^{pq}(e_{[k,q]} + r f_{[k,q]})_{,p} \\ & - \frac{1}{2} g^{ri} f_k f_i + \frac{1}{2}\delta^{pq}\left(f_p (e_{q,k} + r f_{q,k}) - f_k (e_{p,q} + r f_{p,q})\right) \ , \end{aligned} \tag{5.60}$$

$$R_{pq} = -f_{(p,q)} - \frac{1}{2} f_p f_q \ , \tag{5.61}$$

and from (5.20) for the Ricci scalar it follows that

$$R = 2a - 2\delta^{pq} f_{p,q} - \frac{3}{2} f^p f_p \ . \tag{5.62}$$

Notice that from (5.51) and the explicit relations for the components R_{ruru}, R_{rpuq}, R_{ru}, R_{pq} and the scalar R it immediately follows that $\Psi_{2T^{ij}}$ (and necessary also Ψ_{2S}) depend only on the metric functions a and f_i which are thus fully responsible for Newton-like deformations of a set of test particles in the VSI spacetimes.

In particular, in a vacuum case with $\Lambda = 0$, these Newton-like components are explicitly

$$\Psi_{2T^{ij}} = m_i^p m_j^q \frac{1}{2}\left(f_{p,q} + \frac{1}{2} f_p f_q\right) , \qquad \Psi_{2S} = a - \frac{1}{4} f^i f_i , \tag{5.63}$$

where, using (5.23), it immediately follows that $\Psi_{2S} = \Psi_{2T^k{}^k}$.

5.6 The simplest gyratons

Considering $a = b = f_i = 0$ in the line element (5.50) of the VSI spacetimes we obtain the simplest example of solution belonging to another important subclass of Kundt family which are physically interpreted as gyratons. These solutions represent a field of localized spinning null fluid and the simplest metric of this type is a particular pp-wave (5.30) with a flat transverse part,

$$\mathrm{d}s^2 = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + 2e_i\mathrm{d}x^i\mathrm{d}u - 2\mathrm{d}u\mathrm{d}r + c\mathrm{d}u^2 . \tag{5.64}$$

However, this solution is not in general vacuum, and presence of the gyratonic matter implies that the functions e_i can not be removed globally, see [31] for details. Using (5.37) with $g_{ij} = \delta_{ij}$, the only nonvanishing Weyl scalars become

$$\begin{aligned} \Psi_{3T^j} &= -\sqrt{2} m_j^p \, \dot{u} \, C_{ruup} , \\ \Psi_{4^{ij}} &= 2 m^p_{(i} m^q_{j)} \left[2\dot{u}\dot{x}^k \left(C_{upkq} - C_{ruup}\, g_{qk}\right) + \dot{u}^2 \left(C_{upuq} - 2 C_{ruup}\, g_{uq}\right)\right] , \end{aligned} \tag{5.65}$$

or equivalently by substituting for the Weyl tensor (4.37):

$$\begin{aligned} \Psi_{3T^j} &= -\frac{\sqrt{2}}{D-2} m_j^p \, \dot{u} \, R_{up} , \\ \Psi_{4^{ij}} &= 2 m^p_{(i} m^q_{j)} \left[2\dot{u}\dot{x}^k \left(R_{upkq} - \frac{g_{pq}}{D-2} R_{uk}\right) + \dot{u}^2 \left(R_{upuq} - \frac{g_{pq}}{D-2} R_{uu}\right)\right] , \end{aligned} \tag{5.66}$$

where the explicit expressions for all nontrivial components of the Riemann and Ricci tensors are

$$\begin{aligned} R_{upkq} &= \frac{1}{2}\left(e_{q,pk} - e_{k,pq}\right) , \\ R_{upuq} &= \frac{1}{2}\left(e_{p,uq} + e_{q,up} - c_{,pq}\right) + \delta^{ij} e_{[i,p]} e_{[j,q]} , \end{aligned} \tag{5.67}$$

and

$$\begin{aligned} R_{uu} &= -\frac{1}{2}\delta^{pq} c_{,pq} - \delta^{pq}\delta^{kl} e_{p,l} e_{[k,q]} + \delta^{pq} e_{p,uq} , \\ R_{up} &= \delta^{kq} e_{[q,p],k} . \end{aligned} \tag{5.68}$$

We immediately observe that the longitudinal term Ψ_{3T^j} depends only on the functions e_i corresponding to the gyratonic source, which thus necessary induce the longitudinal deformations of the test congruence. Due to the flatness of the transverse space, and $a = f_i = 0$, there are no components of boost weight zero, and this solution is thus in general of algebraic type III.

In a vacuum case, i.e., outside the gyratonic source, only the transverse components $\Psi_{4^{ij}}$ are nonvanishing, namely,

$$\Psi_{4^{ij}} = 2 m^p_{(i} m^q_{j)} \left[2\dot{u}\dot{x}^k e_{[q,k]p} + \dot{u}^2 \left(e_{(p,q)u} - \frac{1}{2} c_{,pq} + \delta^{kl} e_{[k,p]} e_{[l,q]}\right)\right] . \tag{5.69}$$

The pure effects of the gravitational field can be explicitly seen using a suitable Lorentz transformations. At the end of Section 5.3 we mentioned that those terms proportional to the spatial velocities x^i in the projections of the Weyl tensor represents only kinematic effects, and can be removed by a transition to another observer (as described in Chapter 3). We demonstrate such a transition in the simple case of the spacetime (5.64). In particular, the D-velocity of a new observer is

$$\tilde{\mathbf{u}} = \frac{\mathbf{u} + \sum_{\mathrm{i}=1}^{D-1} v_{\mathrm{i}}\, \mathbf{e}_{(\mathrm{i})}}{\sqrt{1 - \sum_{\mathrm{i}=1}^{D-1} v_{\mathrm{i}}^2}} \ , \tag{5.70}$$

where v_{i} are components of the spatial velocity of a new observer with respect to the old one. The appropriate changes of $(\mathbf{k}, \mathbf{l}, \mathbf{m}_i)$ frame correspond to the combination of a boost in the $\mathbf{k}-\mathbf{l}$ plane (4.45), and a null rotation with $\mathbf{k}$ fixed (4.44), where the parameters are

$$B = \frac{\sqrt{1 - \sum_{\mathrm{i}=1}^{D-1} v_{\mathrm{i}}^2}}{1 - v_1} \ , \qquad\qquad L_i = \frac{v_i}{\sqrt{1 - \sum_{\mathrm{i}=1}^{D-1} v_{\mathrm{i}}^2}} \ . \tag{5.71}$$

Using the frame $(\tilde{\mathbf{k}}, \tilde{\mathbf{l}}, \tilde{\mathbf{m}}_i)$ obtained by these Lorentz transformations, the relevant Weyl scalars (of boost weights -1 and -2) take the form

$$\begin{aligned} B\tilde{\Psi}_{3T^i} &= \Psi_{3T^i} \ , \\ B^2\tilde{\Psi}_{4^{ij}} &= \Psi_{4^{ij}} + 2\sqrt{2}\left(\Psi_{3T^{(i}}X_{j)} - \Psi_{3^{(ij)k}}X^k\right) \ , \end{aligned} \tag{5.72}$$

where $X_i = BL_i$. Due to $g_{ij} = \delta_{ij}$, we choose the spatial part of the vectors $\mathbf{m}_i$ in the interpretation frame (5.26) simply as $m_i^j = \delta_i^j$. Using (5.65) and (4.64) for $\Psi_{3^{ijk}}$ the coefficients in (5.72) thus become

$$\begin{aligned} \Psi_{3T^j} &= -\sqrt{2}\dot{u}\, C_{ruuj} \ , \qquad\qquad \Psi_{3^{ijk}} = \sqrt{2}\dot{u}\, C_{uijk} \ , \\ \Psi_{4^{ij}} &= 2\delta^p_{(i}\delta^q_{j)}\left[2\dot{u}\dot{x}^k\left(C_{upkq} - C_{ruup}\delta_{qk}\right) + \dot{u}^2\left(C_{upuq} - 2C_{ruup}\, g_{uq}\right)\right] \ . \end{aligned} \tag{5.73}$$

If we require the terms proportional to spatial velocities $\dot{x}^i$ in $\tilde{\Psi}_{4^{ij}}$ given by (5.72) to be vanishing, we have to take $X^i = -\dot{x}^i$. The Weyl components measured by such a new observer than become

$$\tilde{\Psi}_{3T^j} = -\frac{\sqrt{2}\dot{u}}{B} C_{ruuj} \ , \qquad\qquad \tilde{\Psi}_{4^{ij}} = \frac{2\delta^p_{(i}\delta^q_{j)}\dot{u}^2}{B^2}\left(C_{upuq} - 2C_{ruup}g_{uq}\right) \ , \tag{5.74}$$

where B is given by (5.71). If the new observer does not change its motion in the direction of propagation of the wave $\mathbf{e}_{(1)}$, i.e., $v_1 = 0$, the parameter B becomes $B = \sqrt{1 - \sum_{i=2}^{D-1} v_i^2}$, and from (5.74) we immediately see that the measured amplitude of the wave grows with observer's motion in the transverse space.

CHAPTER

6

ROBINSON–TRAUTMAN SPACETIMES

In this chapter we will analyze the properties of nontwisting, shearfree, and expanding spacetimes which form the so-called Robinson–Trautman class. These solutions are well known in four dimensions where they represent an important family of exact radiative spacetimes (see e.g. the textbooks [4, 5] for their mathematical description, physical interpretations and list of references). Their higher-dimensional generalizations were investigated by Podolský, Ortaggio and Žofka in the works [34, 35], where it was surprisingly shown that the requirements imposed by Einstein's field equations are much more restrictive than in four dimensions. In particular, the higher-dimensional Robinson–Trautman family (vacuum or with an aligned electromagnetic field) is necessary of algebraic type D, i.e., it is not so rich as in four dimensions where it can be of type II or more special, see [4].

Following the discussion of relative motion of free test particles presented in the previous chapters, here we would like to emphasize the fundamental difference between the nontwisting, shearfree, and expanding Robinson–Trautman solutions in four and in any higher number of dimensions.

6.1 Geometry of the Robinson–Trautman spacetimes

In this subsection we briefly recall the main results of [34, 35]. We employ geometrical restrictions and field equations, and introduce the Robinson–Trautman line element in $D \geq 4$.

The general construction presented in Chapter 4 guarantees that the spacetimes described by metric (4.1) are nontwisting ($A = 0$), and the remaining optical scalars Θ (expansion) and σ (shear) can be written as (4.12), namely,

$$\Theta = -(\ln p)_{,r} \ , \qquad \sigma^2 = \frac{1}{4}\gamma^{ik}\gamma^{jl}\gamma_{ij,r}\gamma_{kl,r} \ . \tag{6.1}$$

The existence of a shearfree null congruence of geodesics leads to the condition $\gamma_{ij,r} = 0$, and the spatial part of the metric thus necessary becomes

$$g_{ij}(r,u,x) = p^{-2}(r,u,x)\,\gamma_{ij}(u,x) \ , \tag{6.2}$$

where γ_{ij} is unimodular. However, this decomposition does not significantly simplify the relations for the Riemann tensor, Ricci tensor and Ricci scalar (4.19)–(4.35). Therefore, we first employ the Einstein field equations (4.41) with specific energy–momentum tensor, in particular, with an

aligned Maxwell field $F_{ab}\tilde{k}^b = \mathcal{N}\tilde{k}_a$, where $\mathcal{N}$ is an arbitrary function[1] and $\tilde{\mathbf{k}} = \partial_r$. The energy–momentum tensor T_{ab} will thus be of the form (5.21), and in the adapted coordinates (r, u, x^i) its components T_{rr} and T_{rk} vanish. Using $F_{ru} = \mathcal{N}$ and $F_{rk} = 0$, then the trace of T_{ab} becomes $T = -\frac{D-4}{16\pi}\left(F_{ij}F^{ij} - 2\mathcal{N}^2\right)$. The Einstein–Maxwell field equations (4.41) then simply determine the r-dependence of the metric (4.1) in the Robinson–Trautman case:

- Due to the vanishing component T_{rr} and the factorization (6.2), the equation containing R_{rr}, given by (4.29), will be $R_{rr} = 0$, namely,

$$R_{rr} = (D-2)\left[\frac{p_{,rr}}{p} - 2\left(\frac{p_{,r}}{p^2}\right)^2\right] = -(D-2)\left(\Theta_{,r} + \Theta^2\right) = 0 \ , \tag{6.3}$$

  and the function p becomes $p = r^{-1}P(u,x)$. The spatial metric g_{ij} can thus be written as

$$g_{ij} = r^2 h_{ij}(u,x) \ , \qquad \text{where} \qquad h_{ij} = P^{-2}\gamma_{ij} \ . \tag{6.4}$$

- The equation $R_{rk} = 0$, with R_{rk} given by (4.31), written in the form $R_{rk} = -\frac{1}{2}h_{kl}r^{2-D}\left(r^D g^{rl}_{,r}\right)_{,r}$, implies that the components g^{rl} have to be powers of r, namely,

$$g^{rl} = e^l(u,x) + r^{1-D}f^l(u,x) \ . \tag{6.5}$$

- The source-free Maxwell equations $F_{[ab;c]=0}$ and $F^{ab}{}_{;b} = 0$ specify restrictions on the electromagnetic field, see [35] for details, and give the r-dependence of the Maxwell tensor F_{ab},

$$F_{ij} = F_{ij}(u,x) \ , \qquad F_{ru} = \mathcal{N} = r^{2-D}Q(u,x) \ , \qquad F_{ui} = r^{3-D}\frac{Q_{,i}}{D-3} - \xi_i(u,x) \ , \tag{6.6}$$

  where $\xi_i(u,x)$ are arbitrary functions and the invariant $F_{ab}F^{ab}$ thus becomes $F_{ab}F^{ab} = r^{-4}F^2 - r^{2(2-D)}2Q^2$ with F^2 defined as $F^2 = F_{ik}F_{jl}h^{ij}h^{kl}$.

- The Einstein equations containing R_{ij} determine the remaining r-dependence of the Robinson–Trautman metric. The detailed discussion can also be found in the works [34, 35]. Here, we only recall that the metric function g^{rr} is power of the coordinate r, functions f^i have to vanish identically, e^i can be (at least locally) set to zero, $e^i = 0$, and the spatial part of the metric h_{ij} may depend on the coordinate u only via conformal factor P^{-2},

$$h_{ij}(u,x) = \frac{\gamma_{ij}(x)}{P^2(u,x)} \ . \tag{6.7}$$

Applying these results on the metric (4.1), the line element of the Robinson–Trautman spacetimes with aligned Maxwell field will be of the simple form

$$\mathrm{d}s^2 = r^2 h_{ij}(u,x)\mathrm{d}x^i\mathrm{d}x^j - 2\mathrm{d}u\mathrm{d}r - g^{rr}(r,u,x)\mathrm{d}u^2 \ , \tag{6.8}$$

where $g_{rr} = -g^{uu}$ and $g_{uk} = g^{rk} = 0$. The component g^{rr} is, in general, a function of all coordinates, where its r-dependence is the combination of powers, and can be written as

$$\begin{aligned} g^{rr} = {} & \frac{\mathcal{R}}{(D-2)(D-3)} + \frac{2(\ln\sqrt{h})_{,u}}{D-2}r - \frac{2\Lambda}{(D-2)(D-1)}r^2 - \frac{\mu}{r^{D-3}} \\ & + \frac{2Q^2}{(D-2)(D-3)r^{2(D-3)}} - \frac{F^2}{(D-2)(D-5)r^2} \ , \end{aligned} \tag{6.9}$$

[1] In the context of Robinson–Trautman spacetimes, we follow the notation of [35] and we use $\mathcal{N}$ instead of Q which is have reserved for the r-independent part of $\mathcal{N}$.

where $\mathcal{R}$ represents a scalar curvature calculated with respect to spatial metric h_{ij}, parameter $h = \det h_{ij}$ so that $\sqrt{h} = P^{2-D}$, Λ is a cosmological constant, μ is an arbitrary function of u and x, and Q and F characterize the electromagnetic field. Other restrictions on these functions following from the Einstein–Maxwell equations depend on the number of dimensions D, namely,

- In any higher dimension $D > 4$ we get $\mathcal{R} = \mathcal{R}(u)$, $\mu = \mu(u)$, and $Q = Q(u)$.

- In even dimensions (except $D \neq 4$ and $D \neq 6$) the function F becomes $F = F(u)$ and the factor P can be written as $P(u, x) = P(x)U(u)$. Using the freedom in a choice of coordinates it can be (without loss of generality) set

$$\begin{aligned} &P = P(x)\ , && h_{ij} = h_{ij}(x)\ , && \mathcal{R} = \text{const.}\ , \\ &\mu = \text{const.}\ , && Q = \text{const.}\ , && F^2 = \text{const.}\ . \end{aligned} \tag{6.10}$$

- In odd dimensions the previous conditions (6.10) hold, and additionally $F^2 = 0$.

- In $D = 6$ it is necessary to distinguish the case $Q \neq 0$, in which $P(u, x)$ can be factorized and it is possible to set $h_{ij} = h_{ij}(x)$, $Q = const.$, and $F^2 = F^2(x)$, and the alternative case $Q = 0$, where P does not take a factorized form and, in general, $P = P(u, x)$, $h_{ij} = h_{ij}(u, x)$, and $F^2 = F^2(u, x)$.

- In $D = 4$ the spatial part of metric can always be written in a conformally flat form $h_{ij} = P^{-2}(u, x)\delta_{ij}$ and the remaining functions contained in g^{rr} depend on u and x^i coordinates.

6.2 Geodesic deviation in the Robinson–Trautman spacetimes of general dimension D

Following the general description of relative motion of free test particles discussed in Chapter 3, we now construct the frame components of the Weyl tensor (4.66)–(4.71) and then substitute them into the equation of geodesic deviation (4.62). We will consider a fully general form (6.8) of the (electro)vacuum Robinson–Trautamn metric. Specific conditions given by the particular number of dimensions will be employed later, see Section 6.3 and 6.4.

6.2.1 Riemann and Weyl tensors

For further discussion of possible observer's motion it is useful to explicitly express the Christoffel symbols. Using (4.15)–(4.17),

$$\begin{aligned} &\Gamma^r_{rr} = 0\ , && \Gamma^r_{rj} = 0\ , && \Gamma^i_{rr} = 0\ , && \Gamma^i_{ru} = 0\ , \\ &\Gamma^u_{rr} = 0\ , && \Gamma^u_{ru} = 0\ , && \Gamma^u_{ri} = 0\ , && \Gamma^u_{ui} = 0\ , \end{aligned} \tag{6.11}$$

and

$$\Gamma^r_{ru} = -\frac{1}{2}g_{uu,r} \ , \qquad \Gamma^r_{uu} = -\frac{1}{2}\left(g^{rr}g_{uu,r} + g_{uu,u}\right) \ ,$$
$$\Gamma^r_{uj} = -\frac{1}{2}g_{uu,j} \ , \qquad \Gamma^r_{jk} = \frac{1}{2}r\left(-2g^{rr}h_{jk} + rh_{jk,u}\right) \ ,$$
$$\Gamma^u_{uu} = \frac{1}{2}g_{uu,r} \ , \qquad \Gamma^u_{ij} = rh_{ij} \ ,$$
$$\Gamma^i_{rk} = r^{-1}\delta^i_k \ , \qquad \Gamma^i_{uu} = -\frac{1}{2}r^{-2}h^{ij}g_{uu,j} \ ,$$
$$\Gamma^i_{uk} = \frac{1}{2}h^{ij}h_{jk,u} \ , \qquad \Gamma^i_{kl} = \frac{1}{2}h^{ij}(2h_{j(k,l)} - h_{kl,j}) \ . \tag{6.12}$$

The general relations (4.19)–(4.28) for the Riemann tensor of nontwisting spacetimes in the case of (electro)vacuum Robinson–Trautman metric (6.8) give

$$R_{rprq} = 0 \ , \qquad R_{rpru} = 0 \ , \qquad R_{rpkq} = 0 \ , \qquad R_{rupq} = 0 \ , \tag{6.13}$$

and

$$\begin{aligned}
R_{ruru} &= -\frac{1}{2}g_{uu,rr} \ , \\
R_{rpuq} &= -\frac{1}{2}r\left(h_{pq}g_{uu,r} + h_{pq,u}\right) \ , \\
R_{ruup} &= \frac{1}{2}\left(g_{uu,rp} - r^{-1}g_{uu,p}\right) \ , \\
R_{kplq} &= r^2\mathcal{R}_{kplq} + g^{rr}r^2\left(h_{kq}h_{pl} - h_{kl}h_{pq}\right) \\
&\quad + \frac{1}{2}r^3\left(h_{kl}h_{pq,u} + h_{pq}h_{kl,u} - h_{kq}h_{pl,u} - h_{pl}h_{kq,u}\right) \ , \\
R_{upkq} &= r^2h_{p[k,q],u} + \frac{1}{2}r\left(g_{uu,q}h_{pk} - g_{uu,k}h_{pq}\right) \\
&\quad + \frac{1}{4}r^2h^{ij}\left[h_{jq,u}\left(2h_{i(p,k)} - h_{pk,i}\right) - h_{jk,u}\left(2h_{i(p,q)} - h_{pq,i}\right)\right] \ , \\
R_{upuq} &= -\frac{1}{2}\left(r^2h_{pq,uu} + g_{uu,pq}\right) - \frac{1}{2}rh_{pq}\left(g^{rr}g_{uu,r} + g_{uu,u}\right) \\
&\quad + \frac{1}{4}r^2g_{uu,r}h_{pq,u} + \frac{1}{4}h^{ij}\left[r^2h_{jp,u}h_{iq,u} + g_{uu,j}\left(2h_{i(p,q)} - h_{pq,i}\right)\right] \ ,
\end{aligned} \tag{6.14}$$

where $\mathcal{R}_{kplq}$ is the Riemann tensor calculated with respect to the metric h_{ij} and its coordinates x^i. From (4.29)–(4.34) for the Ricci tensor we immediately obtain $R_{rr} = 0$, $R_{rp} = 0$, and

$$\begin{aligned}
R_{ru} &= -\frac{1}{2}g_{uu,rr} - \frac{1}{2}r^{-1}\left[h^{pq}h_{pq,u} + (D-2)g_{uu,r}\right] \ , \\
R_{uu} &= -\frac{1}{2}g^{rr}g_{uu,rr} - \frac{1}{2}h^{pq}\left(h_{pq,uu} + r^{-2}g_{uu,pq}\right) - \frac{1}{2}(D-2)r^{-1}(g^{rr}g_{uu,r} + g_{uu,u}) \\
&\quad + \frac{1}{4}h^{pq}g_{uu,r}h_{pq,u} + \frac{1}{4}h^{pq}h^{ij}\left[h_{jp,u}h_{iq,u} + r^{-2}g_{uu,j}\left(2h_{ip,q} - h_{pq,i}\right)\right] \ , \\
R_{uk} &= -\frac{1}{2}g_{uu,rk} + h^{pq}h_{p[k,q],u} - \frac{1}{2}r^{-1}(D-4)g_{uu,k} \\
&\quad + \frac{1}{4}h^{pq}h^{ij}\left[h_{jq,u}\left(2h_{i(p,k)} - h_{pk,i}\right) - h_{jk,u}\left(2h_{ip,q} - h_{pq,i}\right)\right] \ , \\
R_{pq} &= \mathcal{R}_{pq} - (D-3)g^{rr}h_{pq} + rh_{pq}g_{uu,r} + \frac{1}{2}r\left[(D-2)h_{pq,u} + h^{kl}h_{pq}h_{kl,u}\right] \ ,
\end{aligned} \tag{6.15}$$

and the Ricci scalar given by (4.35) becomes

$$R = r^{-2}\mathcal{R} + g_{uu,rr} + 2(D-2)r^{-1}g_{uu,r} \\ -r^{-2}g^{rr}(D-2)(D-3) + r^{-1}\,(D-1)\,h^{pq}h_{pq,u}\ . \tag{6.16}$$

Notice that using the factorization (6.7) of spatial metric h_{ij} and the definition $g_{uu}(r,u,x) \equiv -2H(r,u,x)$, together with $g_{uu} = -g^{rr}$, we can rewrite these quantities into the standard form usually used in four dimensions, see e.g. [5], and also in the works [34, 35]. In particular, the Riemann tensor becomes

$$\begin{aligned}
R_{ruru} &= H_{,rr}\ , \\
R_{rpuq} &= r\,h_{pq}\,[H_{,r} + (\ln P)_{,u}]\ , \\
R_{ruup} &= -H_{,rp} + r^{-1}H_{,p}\ , \\
R_{kplq} &= r^2\mathcal{R}_{kplq} - 4r^2 h_{k[l}h_{q]p}\,[H + r(\ln P)_{,u}]\ , \\
R_{upkq} &= 2r\left[H_{[,k}h_{q]p} + r\,h_{p[q}(\ln P)_{,k]u}\right]\ , \\
R_{upuq} &= H_{,pq} - \frac{1}{2}H_{,k}h^{kl}(2h_{l(p,q)} - h_{pq,l}) \\
&\quad + r\,h_{pq}\left\{H_{,r}\,[2H + r(\ln P)_{,u}] + H_{,u} - r\,[(\ln P)_{,u}]^2 + r(\ln P)_{,uu}\right\}\ ,
\end{aligned} \tag{6.17}$$

the nonvanishing components of the Ricci tensor are given by

$$\begin{aligned}
R_{ru} &= H_{,rr} + r^{-1}(D-2)\,[H_{,r} + (\ln P)_{,u}]\ , \\
R_{uu} &= 2HH_{,rr} + r^{-2}h^{pq}R_{upuq}\ , \\
R_{uk} &= H_{,rk} - r^{-1}H_{,k} + r^{-2}h^{pq}R_{upkq}\ , \\
R_{pq} &= \mathcal{R}_{pq} - 2rh_{pq}\,[H_{,r} + (\ln P)_{,u}] - 2(D-3)h_{pq}\,[H + r(\ln P)_{,u}]\ ,
\end{aligned} \tag{6.18}$$

and the scalar curvature R is simply

$$R = r^{-2}\,\{\mathcal{R} - 2(D-2)(D-3)\,[H + r(\ln P)_{,u}]\} - 2H_{,rr} - 4r^{-1}(D-2)\,[H_{,r} + (\ln P)_{,u}]\ . \tag{6.19}$$

Now we employ the general expressions (4.37) for the components of the Weyl tensor and we immediately find that C_{rprq}, C_{rpkq}, C_{rpru}, C_{rupq} vanish identically while the remaining components are given by

$$\begin{aligned}
C_{ruru} &= R_{ruru} - \frac{2R_{ru}}{D-2} - \frac{R}{(D-1)(D-2)}\ , \\
C_{ruup} &= R_{ruup} + \frac{R_{up}}{D-2}\ , \\
C_{rpuq} &= R_{rpuq} - \frac{1}{D-2}\,(g_{pq}R_{ru} - R_{pq}) - \frac{R\,g_{pq}}{(D-1)(D-2)}\ , \\
C_{kplq} &= R_{kplq} - \frac{1}{D-2}\,(g_{kl}R_{pq} - g_{kq}R_{pl} - g_{pl}R_{kq} + g_{pq}R_{kl}) + \frac{R}{(D-1)(D-2)}\,(g_{kl}g_{pq} - g_{kq}g_{pl})\ , \\
C_{upjk} &= R_{upkq} - \frac{1}{D-2}\,(g_{pq}R_{uk} - g_{pk}R_{uq})\ , \\
C_{upuq} &= R_{upuq} - \frac{1}{D-2}\,(g_{uu}R_{pq} + g_{pq}R_{uu}) + \frac{R\,g_{uu}g_{pq}}{(D-1)(D-2)}\ .
\end{aligned} \tag{6.20}$$

Finally, combining these relations (6.20) for the Weyl tensor with explicit formulae for R_{abcd}, R_{ab} and R, see (6.14)–(6.16) or equivalently (6.17)–(6.19), we obtain the following useful identities

which express the Weyl tensor components C_{rpuq}, C_{kplq} and C_{upuq} as specific combinations of the component C_{ruru}, quantities characterizing the transverse-space metric h_{ij}, and the metric function H,

$$\begin{aligned}
C_{rpuq} &= -\frac{r^2 h_{pq}}{D-2} C_{ruru} \ , \\
C_{kplq} &= r^2 \mathcal{R}_{kplq} - \frac{2r^2 h_{k[l} h_{q]p}}{(D-2)(D-3)} \left(2r^2 C_{ruru} + \mathcal{R}\right) \ , \\
C_{upuq} &= 2H C_{rpuq} + H_{,pq} - \frac{1}{2} H_{,k} h^{kl} \left(2h_{l(p,q)} - h_{pq,l}\right) \\
&\quad - \frac{h_{pq}}{D-2} h^{ij} \left[H_{,ij} - \frac{1}{2} H_{,k} h^{kl} \left(2h_{l(i,j)} - h_{ij,l}\right)\right] \ .
\end{aligned} \tag{6.21}$$

The component C_{ruru} can be written as

$$C_{ruru} = \frac{D-3}{D-1} \left(-\frac{1}{2} g_{uu,rr} + r^{-1} g_{uu,r} - r^{-2} g_{uu}\right) - \frac{r^{-2}\mathcal{R}}{(D-1)(D-2)} \ , \tag{6.22}$$

and using the explicit relation (6.9) for the metric function g_{uu} it becomes

$$C_{ruru} = -(D-2)(D-3)\frac{\mu}{2r^{D-1}} + \frac{2(2D-5)Q^2}{(D-1)r^{2(D-2)}} - \frac{6(D-3)F^2}{(D-1)(D-2)(D-5)r^4} \ . \tag{6.23}$$

It is also useful to define the difference $w_{pq} \equiv C_{upuq} - 2HC_{rpuq}$, which employing the identities (6.21), takes the form

$$\begin{aligned}
w_{pq} &= H_{,pq} - \frac{1}{2} H_{,k} h^{kl} \left(2h_{l(p,q)} - h_{pq,l}\right) \\
&\quad - \frac{h_{pq}}{D-2} h^{ij} \left[H_{,ij} - \frac{1}{2} H_{,k} h^{kl} \left(2h_{l(i,j)} - h_{ij,l}\right)\right] \ ,
\end{aligned} \tag{6.24}$$

where we can substitute for the function H from (6.9) and use a shorthand $\Gamma^k_{pq} \equiv \frac{1}{2} h^{kl} \left(2h_{l(p,q)} - h_{pq,l}\right)$. For the first part we immediately get

$$\begin{aligned}
H_{,pq} - H_{,k}\Gamma^k_{pq} &= \frac{\mathcal{R}_{,pq} - \mathcal{R}_{,k}\Gamma^k_{pq}}{2(D-2)(D-3)} + \frac{(\ln\sqrt{h})_{,upq} - (\ln\sqrt{h})_{,uk}\Gamma^k_{pq}}{D-2}\, r - \frac{\mu_{,pq} - \mu_{,k}\Gamma^k_{pq}}{2r^{D-3}} \\
&\quad + \frac{(Q^2)_{,pq} - 2QQ_{,k}\Gamma^k_{pq}}{(D-2)(D-3)r^{2(D-3)}} - \frac{(F^2)_{,pq} - 2FF_{,k}\Gamma^k_{pq}}{2(D-2)(D-5)r^2} \ ,
\end{aligned} \tag{6.25}$$

while the second part subtracts its trace and thus guarantees that w_{pq} is traceless.

6.2.2 Equation of geodesic deviation

The interpretation frame (4.50) connected with a particular timelike observer moving in the non-twisting, shearfree and expanding Robinson–Trautman spacetime (6.8) becomes

$$\mathbf{k} = \frac{1}{\sqrt{2}\dot{u}}\partial_r \ , \qquad \mathbf{l} = \left(\sqrt{2}\dot{r} - \frac{1}{\sqrt{2}\dot{u}}\right)\partial_r + \sqrt{2}\dot{u}\,\partial_u + \sqrt{2}\dot{x}^i\partial_i \ , \qquad \mathbf{m}_i = \frac{g_{kp}\dot{x}^k m_i^p}{\dot{u}}\partial_r + m_i^j\partial_j \ , \tag{6.26}$$

where $g_{kp} = r^2 h_{kp}$. Employing the general form of the projections of the Weyl tensor (4.66)–(4.71) and the fact that the components C_{rprq}, C_{rpkq}, C_{rpru}, C_{rupq} are in this case vanishing, the terms

important for the equation of geodesic deviation (4.62) will be given by

$$\begin{aligned}
\Psi_{2S} &= -C_{ruru} \ , \\
\Psi_{2T^{ij}} &= m_i^p m_j^q C_{rpuq} \ , \\
\Psi_{3T^j} &= \sqrt{2} m_j^p \left[\dot{x}^k \left(g_{kp} C_{ruru} - C_{rkup} \right) - \dot{u}\, C_{ruup} \right] \ , \\
\Psi_{4^{ij}} &= 2 m_{(i}^p m_{j)}^q \Big\{ \dot{x}^k \dot{x}^l \left[g_{kl} C_{rpuq} - g_{pk} \left(2 C_{rluq} - g_{lq} C_{ruru} \right) + C_{kplq} \right] \\
&\qquad + 2 \dot{x}^k \dot{u} \left(C_{upkq} - g_{kq} C_{ruup} \right) + \dot{u}^2 \left(g_{uu} C_{rpuq} + C_{upuq} \right) \Big\} \ .
\end{aligned} \tag{6.27}$$

We can rewrite these scalars by substituting the explicit decomposition (6.20), namely,

$$\begin{aligned}
\Psi_{2S} &= -R_{ruru} + \frac{2}{D-2} R_{ru} + \frac{R}{(D-1)(D-2)} \ , \\
\Psi_{2T^{ij}} &= m_i^p m_j^q \left[R_{rpuq} - \frac{1}{D-2} \left(g_{pq} R_{ru} - R_{pq} \right) - \frac{R\, g_{pq}}{(D-1)(D-2)} \right] \ , \\
\Psi_{3T^j} &= \sqrt{2} m_j^p \left\{ \dot{x}^k \left[g_{kp} R_{ruru} - R_{rkup} - \frac{1}{D-2} \left(g_{kp} R_{ru} + R_{kp} \right) \right] - \dot{u} \left(R_{ruup} + \frac{R_{up}}{D-2} \right) \right\} \ , \\
\Psi_{4^{ij}} &= 2 m_{(i}^p m_{j)}^q \Big\{ \dot{x}^k \dot{x}^l \left[g_{kl} R_{rpuq} - g_{kp} \left(2 R_{rluq} - g_{lq} R_{ruru} \right) + R_{kplq} - \frac{g_{pq}}{D-2} \left(g_{kl} R_{ru} + R_{kl} \right) \right] \\
&\qquad + 2 \dot{x}^k \dot{u} \left(R_{upkq} - g_{kq} R_{ruup} - \frac{g_{pq}}{D-2} R_{uk} \right) \\
&\qquad + \dot{u}^2 \left[g_{uu} R_{rpuq} + R_{upuq} - \frac{g_{pq}}{D-2} \left(g_{uu} R_{ru} + R_{uu} \right) \right] \Big\} \ .
\end{aligned} \tag{6.28}$$

Using the relations (4.64) for the remaining Weyl components we find that $\Psi_{1^{ijk}}$ is vanishing and thus the Robinson–Trautman class of spacetimes described by the metric (6.8) is (at least) of algebraic type II in any dimension D.

The equation of geodesic deviation (4.62) will thus contain the same Weyl scalars $\Psi_{A\bullet}$ as in the Kundt spacetime (5.24), i.e.,

$$\begin{aligned}
\ddot{Z}^{(1)} &= \frac{2\Lambda}{(D-2)(D-1)} Z^{(1)} + \Psi_{2S} Z^{(1)} - \frac{1}{\sqrt{2}} \Psi_{3T^j} Z^{(j)} \\
&\qquad + \frac{8\pi}{D-2} \left[T_{(1)(1)} Z^{(1)} + T_{(1)(j)} Z^{(j)} - \left(T_{(0)(0)} + \frac{2T}{D-1} \right) Z^{(1)} \right] \ , \\
\ddot{Z}^{(i)} &= \frac{2\Lambda}{(D-2)(D-1)} Z^{(i)} - \Psi_{2T^{(ij)}} Z^{(j)} - \frac{1}{\sqrt{2}} \Psi_{3T^i} Z^{(1)} - \frac{1}{2} \Psi_{4^{ij}} Z^{(j)} \\
&\qquad + \frac{8\pi}{D-2} \left[T_{(i)(1)} Z^{(1)} + T_{(i)(j)} Z^{(j)} - \left(T_{(0)(0)} + \frac{2T}{D-1} \right) Z^{(i)} \right] \ ,
\end{aligned} \tag{6.29}$$

where the Weyl scalars are now given by (6.28). The components of the energy–momentum tensor describing aligned electromagnetic field are also given by (5.25), together with specific restrictions following from the geometry of Robinson–Trautman spacetime, see [35]. Relative motion in the geodesic congruence caused by a free gravitational field contained in the Weyl tensor thus again consists of the Newton-like tidal deformation encoded in the terms Ψ_{2S} and $\Psi_{2T^{ij}}$, longitudinal effects given by the set of scalars Ψ_{3T^j} and transverse gravitation waves described by the symmetric and traceless matrix $\Psi_{4^{ij}}$. However, these specific behaviour now significantly depends on the number of dimensions D, see sections 6.3 and 6.4.

Finally, notice that the terms containing the spatial velocities $\dot{x}^i$ in (6.27) and (6.28) can be removed by a suitable transition to another observer with its D-velocity (5.70) represented by specific Lorentz transformation, see Chapter 3 or an explicit example in Subsection 5.6. In particular, we substitute the general form of the Weyl scalars (6.27) and the remaining scalars $\Psi_{A\bullet}$ given by (4.64), namely,

$$\begin{aligned}
\Psi_{2^{ij}} &= 0 \ , \qquad\qquad \Psi_{2^{ijkl}} = C_{mpnq} m_i^m m_j^p m_k^n m_l^q \ , \\
\Psi_{3^{ijk}} &= \sqrt{2}\dot{x}^p \left[C_{rmun}\, g_{pq} m_i^n \left(m_j^q m_k^m - m_j^m m_k^q \right) + C_{pmqn} m_i^m m_j^q m_k^n \right] \\
&\quad + \sqrt{2}\dot{u}\, C_{uplq} m_i^p m_j^l m_k^q \ ,
\end{aligned} \tag{6.30}$$

into the relations describing the combination of the Lorentz boost in $\mathbf{k}-\mathbf{l}$ plane and null rotation with $\mathbf{k}$ fixed in the case of (electro)vacuum Robinson–Trautman spacetimes,

$$\begin{aligned}
\tilde{\Psi}_{2S} &= \Psi_{2S} \ , \qquad\qquad \tilde{\Psi}_{2T^{ij}} = \Psi_{2T^{ij}} \ , \\
B\tilde{\Psi}_{3T^i} &= \Psi_{3T^i} - \sqrt{2}\left(\Psi_{2T^{ki}} X^k + \Psi_{2S} X_i \right) \\
B^2\tilde{\Psi}_{4^{ij}} &= \Psi_{4^{ij}} + 2\sqrt{2}\left(\Psi_{3T^{(i}} X_{j)} - \Psi_{3^{(ij)k}} X^k \right) \\
&\quad + 2\Psi_{2^{ikjl}} X^k X^l - 4\Psi_{2T^{k(i}} X_{j)} X^k + 2\Psi_{2T^{(ij)}} |X|^2 - 2\Psi_{2S} X_i X_j \ ,
\end{aligned} \tag{6.31}$$

with

$$X_i = BL_i = -g_{pq}\dot{x}^p m_i^q \ , \tag{6.32}$$

where the parameters B and L_i are given by (5.71). The Weyl scalars measured by the new observer with D-velocity given by (5.70) in the equation of geodesic deviation (6.29) then simply become

$$\begin{aligned}
\tilde{\Psi}_{2S} &= -C_{ruru} \ , \\
\tilde{\Psi}_{2T^{ij}} &= m_i^p m_j^q C_{rpuq} \ , \\
B\tilde{\Psi}_{3T^j} &= -\sqrt{2} m_j^p \dot{u}\, C_{ruup} \ , \\
B^2\tilde{\Psi}_{4^{ij}} &= 2 m_{(i}^p m_{j)}^q \dot{u}^2 \left(g_{uu} C_{rpuq} + C_{upuq} \right) \ .
\end{aligned} \tag{6.33}$$

For investigation of pure gravitational effects, which are not affected by the transverse motion of the observer, we can also try to choose a particular class of 'radial' geodesic observers which are "static" in the spatial directions, i.e., their velocities $\dot{x}^i$ are vanishing.

6.3 Geodesic deviation in higher dimensions

In this section we will investigate the relative motion of test particles in the higher-dimensional (electro)vacuum Robinson–Trautman class. In particular, we restrict our attention only on the typical cases $D > 4$ and $D \neq 6$.[2] These solutions are characterized by the metric function g^{rr} independent on the spatial coordinates x^i and, on the other hand, the factor P becomes only the function of x^i. These restrictions immediately simplify the relations (6.17)–(6.19) for the Riemann and Ricci tensor and the Ricci scalar. For the nonvanishing components we get

$$\begin{aligned}
R_{ruru} &= H_{,rr} \ , \\
R_{rpuq} &= r\, h_{pq} H_{,r} \ , \\
R_{kplq} &= r^2 \mathcal{R}_{kplq} - 4r^2 h_{k[l} h_{q]p} H \ , \\
R_{upuq} &= r\, h_{pq} \left(2HH_{,r} + H_{,u} \right) \ ,
\end{aligned} \tag{6.34}$$

[2] Except some very special situations, the case $D = 6$ is also included in this discussion, see [35] for details.

$$
\begin{aligned}
R_{ru} &= H_{,rr} + r^{-1}(D-2)H_{,r} \ , \\
R_{uu} &= 2HH_{,rr} + r^{-2}h^{pq}R_{upuq} \ , \\
R_{pq} &= \mathcal{R}_{pq} - 2rh_{pq}H_{,r} - 2(D-3)h_{pq}H \ ,
\end{aligned} \tag{6.35}
$$

and

$$
R = r^{-2}\left[\mathcal{R} - 2(D-2)(D-3)H\right] - 2H_{,rr} - 4r^{-1}(D-2)H_{,r} \ . \tag{6.36}
$$

Employing these results, the identities (6.21) and the relations (6.20), for the Weyl tensor we find that the only only nonvanishing components are

$$
\begin{aligned}
C_{rpuq} &= -\frac{r^2 h_{pq}}{D-2}C_{ruru} \ , \\
C_{ruru} &= -(D-2)(D-3)\frac{\mu}{2r^{D-1}} + \frac{2(2D-5)Q^2}{(D-1)r^{2(D-2)}} - \frac{6(D-3)F^2}{(D-1)(D-2)(D-5)r^4} \ , \\
C_{kplq} &= r^2\mathcal{R}_{kplq} - \frac{2r^2 h_{k[l}h_{q]p}}{(D-2)(D-3)}\left(2r^2 C_{ruru} + \mathcal{R}\right) \ , \\
C_{upuq} &= 2HC_{rpuq} \ .
\end{aligned} \tag{6.37}
$$

The equations of geodesic deviation still retains of the form (6.29) where the Weyl scalars now become

$$
\begin{aligned}
\Psi_{2S} &= -C_{ruru} \ , \qquad\qquad \Psi_{2T^{ij}} = -\frac{\delta_{ij}}{D-2}C_{ruru} \ , \\
\Psi_{3T^j} &= \sqrt{2}m^k_j \dot{x}^l h_{kl} r^2 \frac{D-1}{D-2}C_{ruru} \ , \\
\Psi_{4^{ij}} &= 2r^2\left[\mathcal{R}_{kplq} - \frac{2\mathcal{R}h_{k[l}h_{q]p}}{(D-2)(D-3)}\right]\dot{x}^k\dot{x}^l m^p_i m^q_j \\
&\quad + 2r^2\frac{D-1}{D-3}C_{ruru}\left[r^2 h_{pk}h_{lq}\dot{x}^k\dot{x}^l m^p_i m^q_j - \frac{h_{kl}\dot{x}^k\dot{x}^l}{D-2}\delta_{ij}\right] \ .
\end{aligned} \tag{6.38}
$$

We immediately see that the general observer with D-velocity $\mathbf{u} = \dot{r}\partial_r + \dot{u}\partial_u + \dot{x}^i\partial_i$ will measure the Newton-like tidal deformations induced by Ψ_{2S} and $\Psi_{2T^{ij}}$, the longitudinal deformations given by Ψ_{3T^j}, and also the transverse behaviour described by the matrix $\Psi_{4^{ij}}$.

However, the effects represented by the terms Ψ_{3T^j} and $\Psi_{4^{ij}}$ are only kinematic, i.e., caused by the motion of the observer itself. This follows directly from the transformed quantities (6.33), where the scalars $\tilde{\Psi}_{3T^j}$ and $\tilde{\Psi}_{4^{ij}}$ vanish. We can also employ the explicit choice of observer's geodesic described by the equations (4.72), namely,

$$
\begin{aligned}
&\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} + 2\Gamma^r_{ru}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}u}{\mathrm{d}\tau} + \Gamma^r_{uu}\frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}u}{\mathrm{d}\tau} + 2\Gamma^r_{uj}\frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau} + \Gamma^r_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\frac{\mathrm{d}x^k}{\mathrm{d}\tau} = 0 \ , \\
&\frac{\mathrm{d}^2 u}{\mathrm{d}\tau^2} + \Gamma^u_{uu}\frac{\mathrm{d}u}{\mathrm{d}\tau}\frac{\mathrm{d}u}{\mathrm{d}\tau} + \Gamma^u_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\frac{\mathrm{d}x^k}{\mathrm{d}\tau} = 0 \ , \\
&\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} + 2\Gamma^i_{rj}\frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\frac{\mathrm{d}x^k}{\mathrm{d}\tau} = 0 \ .
\end{aligned} \tag{6.39}
$$

where the Christoffel symbols are given by (6.11)–(6.12) with $g_{uu,j} = 0$ and $h_{jk,u} = 0$. These equations admit the 'radially' falling observers characterized by fixed transverse positions $x^i(\tau) = x^i_0 = const.$, i.e., $\dot{x}^i = 0$. It follows from (6.38) that these will measure only the tidal deformations induced by the terms Ψ_{2S} and $\Psi_{2T^{ij}}$. This is in correspondence with the result of the works [34, 35], according to which the (electro)vacuum higher-dimensional Robinson–Trautman spacetimes are of algebraic type D.

6.4 Geodesic deviation in four dimensions

The most important difference between the four-dimensional and higher-dimensional cases is that the functions contained in g^{rr} component of the Robinson–Trautmann metric (6.8) depend, in general, on u and x^i coordinates. Also, it is always possible to express the transverse metric h_{ij} in a conformally flat form, see [34]. The complete Robinson–Trautman line element in $D = 4$ thus reads

$$\mathrm{d}s^2 = r^2 P^{-2}(u,x)\left[(\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2\right] - 2\mathrm{d}u\mathrm{d}r - 2H(r,u,x)\mathrm{d}u^2 \ . \tag{6.40}$$

with the function $H(r,u,x)$ given by (6.9),

$$2H = \frac{\mathcal{R}}{2} - 2r(\ln P)_{,u} - \frac{\Lambda}{3}r^2 - \frac{\mu}{r} + \frac{Q^2 + \frac{1}{2}F^2}{r^2} \ , \tag{6.41}$$

where $\mathcal{R}(u,x)$ is the scalar curvature of the transverse two-space with the metric h_{ij} related to the conformal factor $P(u,x)$ by $\mathcal{R} = 2\Delta \ln P = 2P^2[(\ln P)_{,22} + (\ln P)_{,33}]$, parameter Λ represents the cosmological constant, the functions $F^2 = F^2(u,x)$ and $Q^2 = Q^2(u,x)$ characterize the electromagnetic field, and $\mu = \mu(u,x)$ is an arbitrary function.

For simplicity, we restrict ourselves only to the vacuum solutions, i.e., $F = 0$ and $Q = 0$, with cosmological constant Λ and the constant function μ redefined as $\mu = 2m$. The components of the interpretation frame (6.26), using the four-dimensional metric (6.40) which implies the natural choice of the spatial parts of the vectors $\mathbf{m}_i$, namely $m_i^p = \frac{P}{r}\delta_i^p$, simply become

$$\begin{aligned}
k^\mu &= \left(\frac{1}{\sqrt{2}\dot{u}}, 0, 0, 0\right) \ , & l^\mu &= \left(\sqrt{2}\dot{r} - \frac{1}{\sqrt{2}\dot{u}}, \sqrt{2}\dot{u}, \sqrt{2}\dot{x}^2, \sqrt{2}\dot{x}^3\right) \ , \\
m_2^\mu &= \left(\frac{r}{P}\frac{\dot{x}^2}{\dot{u}}, 0, \frac{P}{r}, 0\right) \ , & m_3^\mu &= \left(\frac{r}{P}\frac{\dot{x}^3}{\dot{u}}, 0, 0, \frac{P}{r}\right) \ .
\end{aligned} \tag{6.42}$$

Combining the relations for the frame components of the Weyl tensor (6.28) with the expressions (6.17)–(6.19) for R_{abcd}, R_{ab} and R we obtain an explicit form of the Weyl scalars,

$$\begin{aligned}
\Psi_{2S} &= 2\frac{m}{r^3} \ , & \Psi_{2T^{ij}} &= \delta_{ij}\frac{m}{r^3} \ , \\
\Psi_{3T^j} &= -3\sqrt{2}\,\dot{x}^k\delta_{jk}\frac{m}{r^2 P} - \dot{u}\,\mathcal{R}_{,j}\frac{P}{2\sqrt{2}r^2} \ , \\
\Psi_{4^{ij}} &= 2\dot{x}^k\dot{x}^l\left[P^2\mathcal{R}_{kilj} - \frac{1}{P^2}\,\mathcal{R}\,\delta_{k[l}\delta_{j]i} - \frac{3m}{rP^2}\left(2\delta_{ki}\delta_{lj} - \delta_{ij}\delta_{kl}\right)\right] \\
&\quad + \frac{\dot{u}\dot{x}^k}{r}\left(\mathcal{R}_{,k}\delta_{ij} - 2\delta_{k(i}\mathcal{R}_{,j)}\right) + 2\dot{u}^2\frac{P^2}{r^2}w_{ij} \ ,
\end{aligned} \tag{6.43}$$

where the matrix w_{ij}, which is absent in the higher-dimensional case $D > 4$, describes gravitational radiation. It is explicitly given by (6.24), i.e.,

$$\begin{aligned}
w_{ij} = {} & \frac{\mathcal{R}_{,ij}}{4} - r(\ln P)_{,uij} + 2\delta^k_{(i}(\ln P)_{,j)}\left(\frac{\mathcal{R}_{,k}}{4} - r(\ln P)_{,uk}\right) \\
& - \frac{\delta_{ij}\delta^{pq}}{2}\left[\frac{\mathcal{R}_{,pq}}{4} - r(\ln P)_{,upq} + 2\delta^k_{(p}(\ln P)_{,q)}\left(\frac{\mathcal{R}_{,k}}{4} - r(\ln P)_{,uk}\right)\right] \ .
\end{aligned} \tag{6.44}$$

The explicit, invariant form of equations of geodesic deviation (6.29) in the four-dimensional

vacuum Robinson–Trautman class of spacetimes will thus simply read

$$\ddot{Z}^{(1)} = \frac{2\Lambda}{(D-2)(D-1)} Z^{(1)} + \Psi_{2S} Z^{(1)} - \frac{1}{\sqrt{2}} \Psi_{3T^j} Z^{(j)} \ ,$$
$$\ddot{Z}^{(i)} = \frac{2\Lambda}{(D-2)(D-1)} Z^{(i)} - \Psi_{2T^{(ij)}} Z^{(j)} - \frac{1}{\sqrt{2}} \Psi_{3T^i} Z^{(1)} - \frac{1}{2} \Psi_{4^{ij}} Z^{(j)} \ , \tag{6.45}$$

where $i, j = 2, 3$. The overall relative motion thus consists of the isotropic influence of the cosmological constant Λ, Newton-like deformation induced by the terms Ψ_{2S} and $\Psi_{2T^{(ij)}}$, the longitudinal effect given by Ψ_{3T^i} and the transverse deformation corresponding to $\Psi_{4^{ij}}$.

Pure influence of the gravitational field can be obtain again by the transition to a suitable observer, described by (6.33), which will measure the amplitudes

$$\tilde{\Psi}_{2S} = 2\frac{m}{r^3} \ , \qquad \tilde{\Psi}_{2T^{ij}} = \delta_{ij} \frac{m}{r^3} \ ,$$
$$B\tilde{\Psi}_{3T^j} = -\frac{\dot{u}}{\sqrt{2}r^2} P(\Delta \ln P)_{,j} \ , \qquad B^2 \tilde{\Psi}_{4^{ij}} = 2\dot{u}^2 \frac{P^2}{r^2} w_{ij} \ . \tag{6.46}$$

This is in accordance with the fact that four-dimensional vacuum Robinson–Trautman spacetime are of algebraic type II, or more special.

Finally, employing the relation $\mathcal{R} = 2\Delta \ln P$ and using notation $x^2 \equiv x$ and $x^3 \equiv y$, the independent components of the symmetric traceless matrix $\Psi_{4^{ij}}$ describing the transverse gravitational waves can be rewritten in a more convenient form,

$$\begin{aligned}
\Psi_{4^{xx}} = &-\frac{6m}{rP^2}(\dot{x}^2 - \dot{y}^2) - \frac{2\dot{u}}{r}\big[\dot{x}(\Delta \ln P)_{,x} - \dot{y}(\Delta \ln P)_{,y}\big] \\
&- \frac{P^2\dot{u}^2}{r}\big[(\ln P)_{,uxx} - (\ln P)_{,uyy} + 2(\ln P)_{,ux}(\ln P)_{,x} - 2(\ln P)_{,uy}(\ln P)_{,y}\big] \\
&+ \frac{P^2\dot{u}^2}{2r^2}\big[(\Delta \ln P)_{,xx} - (\Delta \ln P)_{,yy} + 2(\Delta \ln P)_{,x}(\ln P)_{,x} - 2(\Delta \ln P)_{,y}(\ln P)_{,y}\big] \ , \\
\Psi_{4^{xy}} = &-\frac{12m}{rP^2}\dot{x}\dot{y} - \frac{2\dot{u}}{r}\big[\dot{x}(\Delta \ln P)_{,y} + \dot{y}(\Delta \ln P)_{,x}\big] \\
&- \frac{2P^2}{r}\dot{u}^2\big[(\ln P)_{,uxy} + (\ln P)_{,ux}(\ln P)_{,y} + (\ln P)_{,uy}(\ln P)_{,x}\big] \\
&+ \frac{P^2}{r^2}\dot{u}^2\big[(\Delta \ln P)_{,xy} + (\Delta \ln P)_{,x}(\ln P)_{,y} + (\Delta \ln P)_{,y}(\ln P)_{,x}\big] \ ,
\end{aligned}$$

directly representing "+" (xx) and "×" (xy) amplitudes of the expanding Robinson–Trautman waves.

CHAPTER

7

CONCLUSIONS

In the first part of the thesis (Chapter 2) we analyzed the geodesic motion in four-dimensional Minkowski, de Sitter and anti-de Sitter universe in which expanding impulsive spherical gravitational waves propagate. Since the geodesics in the background spacetimes of constant curvature are well known we concentrated our attention on their correct connection across the null impulsive hypersurface. Employing the continuous form of the line element describing such waves, and using explicit transformations to the background coordinates, we derived and investigated the general refraction formulae fully characterizing the transition of general $\mathcal{C}^1$ observers (in the continuous coordinates) across any spherical impulse. The influence of the nonvanishing cosmological constant has been naturally included and emphasized by expressing the results in the suitable global five-dimensional parameterizations of the (anti-)de Sitter universe.

All the effects were explicitly illustrated in the case of impulsive waves generated by a snapped cosmic string. In full detail we elucidated the specific focusing properties of such waves and we described the dependence of the behaviour of free test particles on the deficit angle parameter characterizing the mass of the cosmic string. In the case of de Sitter spacetime, we analyzed the superposition of the impulsive spherical wave effects with the isotropic expansion of the background universe given by the presence of the positive cosmological constant.

These results were published in the journal *Physical Review D* (2010), see the full reference [13].

The second part of the thesis contains the analysis and discussion of properties of higher-dimensional equation of geodesic deviation and its applications in specific interesting situations.

In Chapter 3 we presented the general analysis of the relative motion in the congruence of free test particles in arbitrary spacetimes of any dimension. Employing the natural orthonormal frame connected with a particular timelike observer we obtained the invariant form of the equation of geodesic deviation. Its right-hand side, namely the Riemann curvature tensor, was expressed in terms of the Weyl tensor and specific combinations of an energy momentum tensor and its trace. The Weyl tensor was further decomposed into the null-frame scalar components naturally related to the algebraical structure of the spacetime. The overall behaviour of free particles in an arbitrary D-dimensional spacetime thus consists of the effects given by the presence of the specific matter content of the spacetime, the isotropic influence of the cosmological constant, and contributions from the free gravitational field represented by the null-frame components of the Weyl tensor, namely, the Newton-like tidal deformations, the longitudinal accelerations, and the effects due to the transverse gravitational waves.

These results were published in the journal *Physical Review D* (2012), see the full reference [22].

We applied this procedure and demonstrated its usefulness in the case of the general nontwisting spacetimes (Chapter 4), and their specific subclasses, namely the nonexpanding Kundt family (Chapter 5), and the expanding Robinson–Trautman family (Chapter 6). The work in progress is the application of these results also to the family of spacetimes constructed by the so-called warp product method, in which case it seems that the transverse effects caused by the additional warped dimension are trivial due to the fact that the relevant Weyl components vanish.

Specifically, in Chapter 4 we applied the method introduced in Chapter 3 to the wide class of higher-dimensional spacetimes admitting a nontwisting congruence of null geodesics. We have recalled the geometry of such spacetimes, and calculated the fully general explicit expressions for the corresponding Riemann and Ricci tensors and the Ricci scalar. We have then introduced the specific frame connected with a particular observer and expressed the equation of geodesic deviation in the invariant form. All the effects mentioned above are in this nontwisting case present, and such spacetimes are thus (without further restrictions) algebraically general.

The explicit geometric quantities derived in our work, characterizing the curvature of arbitrary nontwisting spacetimes, can also be used in the search for new solutions of this type in standard Einstein's relativity (or in the generalized theories), and for investigation of their physical properties.

In Chapter 5 we discussed the properties of nontwisting, nonexpanding and shearfree Kundt family of solutions. We employed the general discussion of relative motion of free test particles, and the results derived in the case of general nontwisting spacetimes (Chapter 4). The physical meaning of specific metric functions and their role in the equation of geodesic deviation was described, and the utility of this approach demonstrated in the various Kundt subclasses such as pp-waves, VSI spacetimes and simple gyratons. The richer structure of effects induced by the additional dimensions were observed in the behaviour of free test particles caused by the higher-dimensional gravitational waves. These results can be simply applied to explicit analyzes of the relative motion in any other spacetime of the nonexpanding Kundt type.

In Chapter 6 we investigated the family of nontwisting, shearfree and expanding Robinson–Trautman solutions. Employing the geodesic deviation equation we demonstrated the significant difference between the higher-dimensional solutions of this type and those well known from the standard four-dimensional theory. Suitable observer in the higher-dimensional Robinson–Trautman spacetime will measure only the Newton-like tidal components of the gravitational field (algebraic type D effects), but in the four-dimensional solutions the longitudinal effects and the transverse gravitational waves are also present (algebraic type II, or more special, effects). Applications of these results and their illustrations in the particular cases, e.g., the C-metric in four-dimensions, and the black hole spacetimes in higher-dimensions, are still the work in progress.

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