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DIPLOMOVÁ PRÁCE

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Využití nestandardních metod pro oceňování finančních derivátů

Katedra pravděpodobnosti a matematické statistiky

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I dedicate this work to Petr Špecián, my friend, and to Marcela Všechovská, our study councellor and the person who persuaded me to finish this thesis. Mark Twain once wrote that the best helping hand that you will ever receive is the one at the end of your own arm. That is true but sometimes you need somebody to remind you that you have an arm.

I would like to express my deepest gratitude to my supervisor, doc. RNDr. Jiří Witzany, Ph.D., for his help and patience during those 5 years I worked on this thesis.

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Využití nestandardních metod pro oceňování finančních derivátů

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Abstrakt: V této diplomové práci využíváme nestandardní metody k ocenění elektrických derivátů. Vývoj spotové ceny elektřiny modelujeme pomocí mean-reverting modelů na hyperjemném binomickém stromě a přechodem do rizikově neutrálního prostředí odvozujeme vzorce pro cenu forwardových kontraktů. Oba naše modely aplikujeme na německý trh s elektřinou a provádíme porovnání predikovaných forwadových cen s cenami forwardů obchodovaných na burze. Z naší analýzy usuzujeme, že jak Ornstein-Uhlenbeckův, tak i Schwartzův jednofaktorový model dobře predikují dlouhodobé forwardové kontrakty, zatímco výsledky predikce krátkodobých kontraktů jsou kvůli nízké likviditě nejednoznačné a alternativní přístupy by mohly být vhodnější.

Klíčová slova: forward na elektřinu, rizikově neutrální oceňování, nestandardní metody, Schwartzův jednofaktorový model, Orntein-Uhlenbeckův process

Title: Application of nonstandard methods for valuation of derivatives

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Abstract: In this thesis we use nonstandard methods for the valuation of derivatives on electricity. We model the dynamics of electricity spot price as mean reverting processes on the hyperfinite binomial tree and by switching to the risk-neutral world we derive analytical formulas for the price of forward contracts. Both of our models are fitted to the German electricity market and forward price predictions are compared with forward products traded on the exchange. We conclude that both the Ornstein-Uhlenbeck and the Schwartz one factor model fit long-term forward contracts well while our prediction results for short-term forward products are not conclusive due to low liquidity and alternative approaches might be suitable.

Keywords: electricity forward, risk-neutral valuation, nonstandard methods, Schwartz one factor model, Ornstein-Uhlenbeck process

Contents

Introduction

In the presented thesis we occupy ourselves with electricity price developement in Germany and valuation of derivative contracts on electricity. The research of power market and its characteristics started in the 1990's with the deregulation of this sector which introduced huge wholesale and retail trading activity. The scale and volume of traded electricity products are overwhelming nowadays and the market is still becoming more transparent and liquid.

Electricity differs substantially from other commodities due to its very limited storability and transportation possibilities. The electricity price is strongly influenced by current power supply and demand. It is therefore impossible to use standard arbitrage arguments which are the core of valuation for easily storable products. Although we cannot use the classical cost-and-carry formula, we are able to evaluate derivative securities by imposing stochastic processes of mean-reversion type on the spot price and calculating risk-adjusted expected values of future payoffs.

There is number of papers which use mean-reverting processes to model spot price dynamics and which apply risk-neutral valuation framework for derivative pricing. Schwartz [33] uses Ornstein-Uhlenbeck process, among others, to model copper, oil and gold prices. Closed form solutions for forward prices derived here served as the basis of other theoretical works. Clewlow and Strickland [10] make use of the same spot price model as Schwartz did to derive values of European options on electricity. Pilipovic [27] propose a two-factor mean-reverting model where electricity price reverts to a long-term equilibrium level which itself is treated as a random variable. Lucia and Schwartz [25] include a predictable seasonal component, which represents systematic behavior of electricity prices, in their model. Finally, Geman and Roncoroni [19] introduce discontinuous process atop of the standard mean-reversion to account for price jumps which can be sometimes observed in electricity market.

Majority of studies concerning electricity prices concentrate on most developed markets like the U.S., Australia or Northern Europe. Papers which analyse German electricity market are quite rare which is suprising given the size of German market and its role in price formation in other Central European countries. To those studying German market belong for example Huisman and Mahieu [21] who estimate mean-reversion coefficients from German spot prices. Weron [39] asseses the distribution of German electricity prices and focus on price spikes and extreme volatility. Seifert and Uhrig-Homburg [32] calibrate their jump-diffusion model to EEX (European Energy Exchange) market data. Although they identify jump patterns and use a mean-reverting model equipped with a seasonal deterministic component similar to our approach, only the pricing of European options is discussed here and no comparison with market prices of derivatives is presented. In this thesis we are going to confront theoretical forward prices, which our spot model implies, with real electricity products traded on the exchange.

Stochastic processes describing price developement are usually treated in the standard $\epsilon - \delta$ theory. We are going to study them via the nonstandard analysis which was introduced by Abraham Robinson [29] in 1961. In contrast to the standard theory which uses notion of the limit, nonstandard analysis utilizes the concept of infinitesimals - infinitely small numbers. We believe that working with infinitesimals gives more insight into the dymamics of our stochastic processes as infinitely small "jumps" are more imaginable than the standard framework. The notion of small jumps should be especially attractive to practitioners who work with discrete versions of stochastic processes on a daily basis and who are familiar with Monte Carlo methods.

The nonstandard theory already proved to be useful in the field of derivative valuation. As an example, Anderson [2] constructed the nonstandard equivalent of Brownian motion which was used by Cutland, Kopp and Willinger [12] to proof the famous Black-Scholes option pricing formula via the technique of nonstandard analysis. In this paper we pick up threads of these works and develop nonstandard versions of mean-reverting processes which lead to the valuation of electricity derivatives.

The outline of this thesis is as follows. In Section 1 we give a brief introduction to the nonstandard analysis and its usage in derivatives pricing. Section 2 introduces two mean-reverting processes for the electricity spot price which are described in the nonstandard framework. A seasonal deterministic component is incorporated in the model and pricing formulas for forward contracts are derived in Section 3. Section 4 contains basic statistics of German electricity market and model parameters are estimated. Because of the existence of inherent price jumps which are not in line with assumptions of mean-reverting models and which could jeopardize parameter estimations, regime-switching approach is employed and data are separated into two independent groups - prices following a continuous mean-reveting process and prices with jump behaviour. A method which includes jumps into the valuation formula is then deviced. In Section 5 we compare actual forward prices with prices forecasted by our models. Last section summarizes results and concludes the paper with a few comments and suggestions for future research.

1. Nonstandard analysis

The idea of infinitesimals stood at the beginning of modern calculus. When Newton was deriving his concept of integrals, he was working with infinitesimals, i.e. very small numbers near zero, on an intuitive basis. The original approach of infinitesimals was later abandoned in favour of more rigorous mathematical concepts based on limiting arguments and $\epsilon - \delta$ reasoning. Though it was possible to find many contradictions when infinitesimals were used intuitively without proper rigorous mathematical background, the infinitesimal calculus has survived in almost all fields of physics, mainly thanks to its directness and easy usage. Even though the physicits can use other mathematical methods, the infinitesimals are so intuitive and natural that other possible means of calculation seem as an unnecessary complication.

Considering the long history of infinitesimals, it can be suprising that rigorous justification of the laws of infinitesimals was first achieved in 1961 by Abraham Robinson [29]. Using formal logic Robinson built his theory of infinitesimals which was self-consistent and complied with all intuitive features of infinitesimals. Once the infinitesimals were correctly described, the flaws and padodoxes of some calculations were clarified and further developement of the theory could go on. From many authors who worked on this topic we should at least mention Robert M. Anderson and his nonstandard representation of Brownian motion [2], Peter. A Loeb and his Loeb measure [24] or Nigel Cutland, Ekkehard Kopp and Walter Willinger for their applications to financial topics [12].

1.1 Nonstandard real numbers

In this paragraph we would like to present one of many possibilities how to construct a system which can represent the real line equiped with additional special numbers satisfying properties an infinitesimal should have. After we construct this system and show the most important properties of it, we will abandon this concrete representation and we will define general nonstandard numbers axiomatically.

Consider the space of real sequences $\mathbb{R}^{\mathbb{N}}$. We can see that $\mathbb{R}^{\mathbb{N}}$ incorporates real numbers by mapping $\mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ which assigns the sequence of constants $\bar{r} = (r, r, r, ...)$ to every $r \in \mathbb{R}$. When we look for an infinitesimal, we need to find an object which is nonzero positive but less than any positive real number. If we take for example sequence $(1, 1/2, 1/3, ...)$, it seems that this object satisfies our notion of an infinitesimal. Every coordinate is positive and an infinite number of its coordinates is less than any given posite real number. This is surely a step in the right direction but we need to specify how to compare two numbers in order to have a totally ordered set.

Definition A *free ultrafilter* $\mathcal{U}$ on $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$ satisfying

1. if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$
2. if $A \in \mathcal{U}$ and $A \subset B \subset \mathbb{N}$, then $B \in \mathcal{U}$
3. if A is finite, then $A \notin \mathcal{U}$
4. if $A \in \mathbb{N}$, either $A \in \mathcal{U}$ or $\mathbb{N} \setminus A \in \mathcal{U}$

Free ultrafilter $\mathcal{U}$ is therefore a collection of subsets which is closed under intersection and supersets, contains no finite set and exactly one of sets A and $\mathbb{N} \setminus A$ is the member of $\mathcal{U}$. When we now define equivalence relation $=_{\mathcal{U}}$ on $\mathbb{R}^{\mathbb{N}}$, we can build equivalence classes which can be totally sorted.

Definition The equivalence relation $=_{\mathcal{U}}$ on $\mathbb{R}^{\mathbb{N}}$ is defined by

$$x =_{\mathcal{U}} y \iff \{n : x_n = y_n\} \in \mathcal{U}. \tag{1.1}$$

We denote the equivalence class of $x \in \mathbb{R}^{\mathbb{N}}$ with respect to the equivalence relation $=_{\mathcal{U}}$ by $[x]$.

Definition The relation $<_{\mathcal{U}}$ is defined by

$$[x] <_{\mathcal{U}} [y] \iff \{n : x_n < y_n\} \in \mathcal{U}. \tag{1.2}$$

The relation $>_{\mathcal{U}}$ can be defined similarly and the system of equivalence classes is now fully ordered because either $[x] <_{\mathcal{U}} [y]$, $[x] =_{\mathcal{U}} [y]$ or $[x] >_{\mathcal{U}} [y]$ holds for every $[x]$, $[y] \in \mathbb{R}^{\mathbb{N}}/\mathcal{U}$. Our initial suggestion for an infinitesimal was therefore correct, $[(1, 1/2, 1/3, ...)]$ is indeed less in the sense of $<_{\mathcal{U}}$ than any image of a positive real $[(a, a, a, ...)]$, $a \in \mathbb{R}^+$. There are many numbers which possess the infinitesimal feature and we denote them by $[x] \approx 0$. We say that two numbers are infinitely close, denoted by $[x] \approx [y]$, if their difference is an infinitesimal. It can be proved that any finite $x \in \mathbb{R}^{\mathbb{N}}/\mathcal{U}$ is infinitely close to some unique $r \in \mathbb{R}$, we call r the standard part of x and write $r = st(x) = {}^{\circ}x$.

We usually write ${}^*\mathbb{R}$ instead of $\mathbb{R}^{\mathbb{N}}/\mathcal{U}$ to stress that $\mathbb{R}^{\mathbb{N}}/\mathcal{U}$ is the extension of real numbers and we call this set *nonstandard real numbers*. Any mathematical object defined on $\mathbb{R}$ can be extended to ${}^*\mathbb{R}$. Given a real function $f : \mathbb{R} \to \mathbb{R}$, we can define function ${}^*f :^* \mathbb{R} \to^* \mathbb{R}$ by

$${}^*f([x]) = [(f(x_1), f(x_2), ...)]. \tag{1.3}$$

The function f is then extended to function $^*f$ pointwise and it is easy to see that many properties of f are inherited by $^*f$. This feature is described by the so called *Transfer Principle* which connects real number statements with their hyperreal extensions.

Theorem 1.1.1. (Tranfer Principle) *Let ϕ be any first order statement. Then*

$$\phi \textit{ holds in } \mathbb{R} \quad \Leftrightarrow \quad {}^*\phi \textit{ holds in } {}^*\mathbb{R} \tag{1.4}$$

The *first order* statement is a statement which uses usual logic connectiveness *and* ($\wedge$), *or* ($\vee$), *implies* ($\rightarrow$), *not* ($\neg$) and quantifies over elements of $\mathbb{R}$ ($^*\mathbb{R}$) but not over relations and functions, which must remain fixed. The transfer principle allows us to switch between real and hyperreal worlds very swiftly and without much effort and it is therefore invaluable help in hyperreal analysis. Its another asset is that it can play the role of an axiom in the axiomatic definition of nonstandard numbers.

Real numbers are usually described by their two basic properties and not by their construction from rational numbers. It is sufficient enough to describe reals as an ordered field with the least upper bound property because other properties of real numbers can be deduced from these two axioms. The same approach can be applied to hyperreal numbers. It was already shown that we are able to extend real numbers by $\mathbb{R}^{\mathbb{N}}/\mathcal{U}$. There are many other ways how to extend real numbers and all of them fulfil the Transfer theorem. In fact, the hyperreal numbers can be axiomatically defined as an extension of real numbers which contains infinitesimals and obeys the Transfer Principle. Therefore we will usually talk about hyperreals without any reference to their concrete representation.

1.2 The nonstandard universe

Real numbers are the core of most mathematical objects and when we switch to hyperreals, one possibility how to construct structures known from real analysis would be to proceed step by step and develop the objects from scratch. Fortunately, we can avoid such tedious work and the whole standard universe can be mapped on nonstandard universe by another version of the Transfer principle.

Most of the classical mathematical universe can be sum up into the *superstructure over* $\mathbb{R}$. To build the superstructure we take atomic elements of some set, in this case real numbers, and produce repetitively power sets. Elements of this superstructure can be interpreted as well-known objects from real analysis.

Definition The *superstructure*, denoted by $\mathbb{V}$, is $V(\mathbb{R}) = \cup_{n\in\mathbb{N}} V_n(\mathbb{R})$ where

$$
\begin{aligned}
V_0(\mathbb{R}) &= \mathbb{R} &(1.5)\\
V_{n+1}(\mathbb{R}) &= V_n(\mathbb{R}) \cup \mathcal{P}(V_n(\mathbb{R})),\ n \in \mathbb{N}. &(1.6)
\end{aligned}
$$

The mechanism of how the superstructure can represent mathematical objects can be illustrated on functions. An ordered pair $(x, y) \in \mathbb{R}^2$ is defined in the set theory as $\{\{x\}, \{x, y\}\}$ and can be thought of as an element of $V_2(\mathbb{R})$. Every function $f : A \to B$ where $A, B \subset \mathbb{R}$ can be represented by its graph $\{(x, f(x)), x \in A\}$ which is a set of ordered pairs and therefore $f \in V_3(\mathbb{R})$. It is not suprising that sets of functions can be represented as $V_4(\mathbb{R})$.

The superstructure $\mathbb{V} = V(\mathbb{R})$ is constructed from real numbers and the superstructure $V({}^*\mathbb{R})$ is constructed from hyperreal numbers. Now we would like to connect same objects from these two superstructures together. This can be done through *superstructure embedding* ${}^* : V(\mathbb{R}) \to V({}^*\mathbb{R})$ that is injective, satisfies ${}^*x = x$ for every $x \in V_0(\mathbb{R})$ and for which the Transfer principle II (presented below) holds from definition. Before we state the second version of the Transfer principle we must differentiate between two kinds of sets in $V({}^*\mathbb{R})$. The superstructure embedding doesn't map onto $V({}^*\mathbb{R})$ and we can define so-called *internal* and *external* objects.

Definition Any object $A \in V({}^*\mathbb{R})$ is said to be *internal* if $A \in {}^*B$ for some $B \in V(\mathbb{R})$. The set of all internal objects is called *nonstandard universe* ${}^*\mathbb{V} = {}^*V(\mathbb{R})$. Sets other than internal are called *external*.

It is important to realize that ${}^*V(\mathbb{R})$ is not the same as $V({}^*\mathbb{R})$ and the inclusion ${}^*\mathbb{V} \subset V({}^*\mathbb{R})$ holds. The notion of internal sets is utilized in the Transfer priciple II that states which properties are inherited by ${}^*\mathbb{V}$.

Theorem 1.2.1. (Transfer Principle II) *Suppose that ϕ is a bounded quatifier statement. Then φ holds in $\mathbb{V}$ if and only if ${}^*\varphi$ holds in ${}^*\mathbb{V}$.*

The *bounded quantifier* statement is a statement that enables the quantifiers to range over prescribed sets only. Very important feature of the Transfer principle is its avoidance of external sets. Just internal sets inherit the properties of objects in $V(\mathbb{R})$ and therefore play a crucial role in the theory of nonstandard analysis.

1.3 Loeb measure

We would like to develop measure theory for the nonstandard space. Assume that we have an internal set Ω, an internal algebra $\mathcal{A}$ of subsets of Ω and a finite internal finitely additive measure $\mu : \mathcal{A} \to {}^*[0, \infty)$. We know from the property $\mu(\Omega) < \infty$ that the value of μ is always infinitely close to some real number and therefore we can define ${}^\circ\mu : \mathcal{A} \to [0, \infty)$ by ${}^\circ\mu(A) = {}^\circ(\mu(A))$, $A \in \mathcal{A}$. The triple $(\Omega, \mathcal{A}, {}^\circ\mu)$ is generally not a *measure space* but we are able to extend it to comply the requirements of σ-additivity for both $\mathcal{A}$ and μ.

It can be shown that when we have an increasing family of sets $(A_n)_{n\in\mathbb{N}}$, $A_n \in \mathcal{A}$, then a countable union of these sets $B = \cup_{n\in\mathbb{B}} A_n$ is again very close to some set A lying in $\mathcal{A}$. By "close" we mean that $A \setminus B$ is a so-called *Loeb null set*, a set for which, given $\epsilon > 0$, we can find a superset $C \in \mathcal{A}$ with property $\mu(C) < \epsilon$. With this on mind we make the following definition.

Definition We call $B \subseteq \Omega$ *Loeb measurable* if there is a set $A \in \mathcal{A}$ such that $A \Delta B$ is Loeb null. The set of all Loeb measurable sets we denote by $L(\mathcal{A})$.

We use in the definition the symmetric difference $A\Delta B = (A \setminus B) \cup (B \setminus A)$ because we want $L(\mathcal{A})$ to be closed under complementation. It can be proved that $L(\mathcal{A})$ is a σ-algebra and we call it *Loeb algebra*.

Another property of the sequence $(A_n)_{n\in\mathbb{N}}$ is that ${}^\circ\mu(A) = \lim_{n\to\infty} {}^\circ\mu(A_n)$ which motivates us to other definition.

Definition For $B \in L(\mathcal{A})$ we define its *Loeb measure* $\mu_L(B)$ by

$$\mu_L(B) = {}^\circ\mu(A) \tag{1.7}$$

where A is any member of $\mathcal{A}$ satisfying $A\Delta B$ null.

The existence of such $A \in \mathcal{A}$ with $A\Delta B$ null is ensured from the construction of $L(\mathcal{A})$. Since μ_L inherits finite additivity from μ and it is continuous from bellow, we can claim that μ_L is a σ-additive measure on $L(\mathcal{A})$. The following theorem summarizes our results.

Theorem 1.3.1. *The triplet* $\mathbf{\Omega} = (\Omega, L(\mathcal{A}), \mu_L)$ *is a measure space which is called the* Loeb space *given by* $(\Omega, \mathcal{A}, \mu)$.

One of the most important examples of the Loeb measure is the *Loeb counting measure* which is the nonstandard equivalent to the counting measure in real analysis. Let us have an infinite hyperfinite set $\Omega = \{1, 2, ..., N\}$ where $N \in {}^*\mathbb{N}\setminus\mathbb{N}$ and define the counting probability measure ν on Ω by

$$\nu(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{N} \tag{1.8}$$

for all $A \in {}^*\mathcal{P}(\Omega) = \mathcal{A}$. Both Ω and ${}^*\mathcal{P}(\Omega)$ are internal and the measure ${}^\circ\nu$ is finitely additive. Therefore from prior discussion we know that there exists the *Loeb counting measure* ν_L as the completion of ${}^\circ\nu$.

The Loeb counting measure can be used to alternatively construct the Lebesgue measure. We take an interval $[0,1]$ for example and mark equidistantly an infinite number of points in it. The measure of the interval will then be given by the number of points contained in the interval. The first step of this approach is the definition of the *hyperfinite time line* $\mathbf{T}$ corresponding to interval $[0,1]$.

Definition Choose some $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ and define the time step $\Delta t = N^{-1}$. The *hyperfinite time line* $\mathbf{T}$ is the set of points

$$\mathbf{T} = \{0, \Delta t, 2\Delta t, ..., N\Delta t = 1\}. \tag{1.9}$$

The set $\mathbf{T}$ is composed of countable number of points and therefore we can talk about the Loeb counting measure ν_L on this hyperfinite time line. Let us define the set of all subintervals of $[0,1]$ whose inverse image of the standard part mapping is Loeb measurable, i.e

$$\mathcal{M} = \{B \subseteq [0,1] : st_{\mathbf{T}}^{-1}(B) \text{ is Loeb measurable}\} \tag{1.10}$$

where $st_{\mathbf{T}}^{-1}(B) = \{t \in \mathbf{T} : {}^\circ t \in B\}$.

Theorem 1.3.2. *Define* $\lambda(B) = \nu_L(st^{-1}(B))$ *for* $B \in \mathcal{M}$. *Then* $([0,1], \mathcal{M}, \lambda)$ *is the Lebesgue measure space on* $[0,1]$.

1.4 Brownian motion

The well-known Wiener process (Brownian motion), a continuous stochastic process with independent and normally distributed increments, can be constructed as a limit of the standard random walk with appropriate scaling. It was shown by Anderson [2] that in the nonstandard setting the hyperfinite random walk is directly equivalent to the Brownian motion. This enables us to intuitively manipulate the Wiener process in terms of a more conceivable random walk but unlike standard analysis within rigorous mathematical background. In the following we will describe the nonstandard Wiener process along the lines of Keisler [23].

Assume a hyperfinite time line $\mathbf{T} = \{0, \delta t, 2\delta t, ..., N\delta t = 1\}$ as we already had above and a hyperfinite coin tossing $\Omega = \{-1, 1\}^N$. We define the *hyperfinite random walk* as an internal map $B : \mathbf{T} \times \Omega \to {}^*\mathbb{R}$ which counts random steps of length $\sqrt{\delta t}$

$$B(t, \omega) = \sum_{s=1}^{Nt} \omega(s)\sqrt{\delta t}, \quad \omega \in \Omega. \tag{1.11}$$

We can again construct the Loeb counting measure ν_L on Ω and implicitely assume that Ω has the measure structure given by $(\Omega, L(\mathcal{A}), \nu_L)$. The following theorem describes the relationship between the hyperfinite random walk and Brownian motion.

Theorem 1.4.1. *The process* $b : [0,1] \times \Omega \to \mathbb{R}$ *defined by*

$$b(^\circ t, \omega) = {}^\circ B(t, \omega) \tag{1.12}$$

where $(t, \omega) \in \boldsymbol{T} \times \Omega$ *is Brownian motion on the probability space* $(\Omega, L(\mathcal{A}), \nu_L)$.

The theorem states that when we observe the hyperfinite random walk B from the perspective of the standard world, that is when we transform the hyperfinite time line into a real line and take standard parts of the process values, we obtain Brownian motion b with its defining properties:

1. b is a stochastic process, i.e. $b(t, \cdot)$ is a ν_L-measurable function of ω for all $t \in [0,1]$, and $b(\omega, 0) = 0$ ν_L-a.s.

2. For $s < t \in [0,1]$ the increment $b(t, \cdot) - b(s, \cdot)$ is normally distributed with mean 0 and variance $t - s$.

3. The increments are independent: for any $s_1 < t_1 \leq s_2 < ... < t_n \in [0,1]$ the random variables $\{b(t_i, \cdot) - b(s_i, \cdot), i \leq n\}$ are independent.

It can be proved that b has almost all paths continuous as we would suspect. The hyperfinite construction also captures precisely the famous Einstein's formula that the root mean square displacement of a particle in any direction after time t is given by $\sqrt{2Dt}$ where D is the *diffusion coefficient*.

The scope of applicability of the Wiener process is vast. One area which utilizes the Wiener process and can therefore be researched by means of nonstandard analysis is finance. Models of stock or other asset prices usually contain the Wiener process to account for the underlying randomness.

1.5 Ito diffusion

It is possible to multiply the standard Wiener process by a volatility coefficient which allows for various standard deviation of the process or to add a drift function. Such a new process called an Ito diffusion is the cornerstone of modern financial mathematics and especially the properties of functions applied on an Ito diffusion are of high interest. Before we formulate the nonstandard version of the Ito-Doeblin formula which describes behaviour of a proces derived from an Ito diffusion, we need to do some preliminary work and define a few nonstadard terms. Definitions and theorems of this and next chapter are along the lines of the radically elementary approach of Herzberg [20].

Let us first specifically express the term Ito diffusion:

Definition *(Ito diffusion)* Let $\mathbf{T}$ be a hyperfinite time line and a, b: $\mathbb{R} \times [0,1] \to \mathbb{R}$. Stochastic process $\xi(t)$ is called an *Ito diffusion wih drift coefficient function* a and *diffusion coefficient function* b if and only if

$$\delta\xi(t) = a(\xi(t), t)\delta t + b(\xi(t), t)\delta Z(t) \tag{1.13}$$

for all $t \in \mathbf{T}$. If a, b don't depend explicitely on time t, then $\xi(t)$ is called a *time-homogeneous Ito diffusion.*

The nonstandard Ito intergral is defined with the help of an infinite sum:

Definition *(Stochastic integral)* For any two processes ξ, μ, the stochastic integral of μ with respect to ξ is the process $\int \mu\, d\xi$ defined by

$$\int_0^s \mu\, d\xi = \int_0^s \mu(t)\, d\xi(t) = \sum_{t<s} \mu(t)\, d\xi(t) \tag{1.14}$$

for all $s \in \mathbf{T}$.

The nonstandard derivative is analogously the ratio of a function differential δy, $\delta y = f(a + \delta x) - f(a)$, to a nonzero infinitesimal variable change δx. If the standard part of $\delta y/\delta x$ exists and is the same for all δx, then f has a derivative at a and we write

$$f'(a) = st(\delta y/\delta x) \tag{1.15}$$

We say that a process ξ on $\mathbf{T}$ has limited trajectories if and only if there is some limited real C such that $\|\xi(t)\| \leq C$ for all $t \in \mathbf{T}$. We can now approach the Ito-Doeblin formula.

Lemma 1.5.1. (Ito-Doeblin formula) *Let ξ be an Ito diffusion with drift coefficient function a and diffusion coefficient function b built on $\boldsymbol{T}$, let $f : \mathbb{R} \times [0,1] \to \mathbb{R}$ be thrice differentiable and let stochastic processes $a(\xi, .)$, $b(\xi, .)$, $f''(\xi, .)$, $f'''(\xi, .)$ have limited trajectories. Then for all $s \in \boldsymbol{T}$*

$$\begin{aligned} f(\xi(s), s) - f(\xi(0), 0) \;\; \approx \;\; & \int_0^s \partial_1 f(\xi(t), t)\, d\xi(t) + \int_0^s \partial_2 f(\xi(t), t)\, dt + \\ & + \frac{1}{2} \int_0^s \partial_{1,1} f(\xi(t), t) b^2(\xi(t), t)\, dt \end{aligned} \tag{1.16}$$

When we perform partial derivatives in the Ito-Doeblin formula and calculate the change of f in one time step, we obtain a more compact, differential, form of the theorem:

$$\delta f \approx \left(\frac{\delta f}{\delta t} + a\frac{\delta f}{\delta \xi} + \frac{1}{2}b^2\frac{\delta^2 f}{\delta \xi^2} \right) \delta t + b\frac{\delta f}{\delta \xi}\delta Z \tag{1.17}$$

1.6 Change of measure

The evolution of each process is governed by some probability measure P. We can define a new process by switching to a different measure which in the context of the hyperfinitte tree means that we change probabilities of the up and down movements at each tree node. Since we will want to perform such measure transformation later in the text, we need to know effects of this operation. The nonstandard version of Girsanov's theorem provides us such information.

Theorem 1.6.1. (Girsanov's theorem) *Let μ be a limited stochastic process, let ξ be the process defined by $\xi(0) = 1$ and $\delta\xi(t) = \xi(t)\mu(t)\delta Z(t)$ for all $t \in \boldsymbol{T}$. When we introduce a new measure Q which has density ξ with respect to the uniform measure P*

$$Q(A) = \int_A \xi(t)\, dP \tag{1.18}$$

then a process Z^Q defined for all $t \in \boldsymbol{T}$ by

$$\delta Z^Q = \delta Z(t) - \mu(t)\delta t \tag{1.19}$$

is a Wiener martingale under Q.

Process $\xi(t)$ starts at one and evolves in time in accordance with its definition in Girsanov's theorem. As for the value of ξ at time $t + \delta t$ holds

$$\xi(t + \delta t) = \xi(t) \left(1 + \mu(t)\delta Z(t)\right) \tag{1.20}$$

we can express $\xi(t)$ in this compact form:

$$\xi(t) = \prod_{s \in \mathbf{T}, s \le t} \left(1 + \mu(s)\omega(s)\sqrt{\delta t}\right) \tag{1.21}$$

The probability of each node on our hyperfinite binomial tree is given by elementary branching probabilities which determine the probability of an upward and a downward jump from each node on the tree. From the definition of measure Q in (1.18) and the explicit form of ξ in (1.21) follows that changing the probability measure to Q is equivalent to adjusting branching probabilities by factor

$\frac{1}{2}\mu(t)\sqrt{\delta t}$

$$p_{up}(t) = \frac{1}{2}\left(1 + \mu(t)\sqrt{\delta t}\right) \tag{1.22}$$

$$p_{down}(t) = \frac{1}{2}\left(1 - \mu(t)\sqrt{\delta t}\right) \tag{1.23}$$

The relationship between branching probabilities and the drift of a process described by Girsanov's theorem will be used later when we switch from the physical to the risk-neutral world.

It can be shown that $\xi(t)$ is infinitely close to the exponential of a generalized Wiener process because

$$\prod_{s\in\mathbf{T},s\leq t}\left(1 + \mu(s)\omega(s)\sqrt{\delta t}\right) \approx exp\left(\int_0^t \mu(s)\, dZ(s) - \frac{1}{2}\int_0^t \mu^2(s)\, ds\right) \tag{1.24}$$

and therefore our nonstandard Girsanov theorem can be directly related to the standard version of this famous proposition in the form it is usually stated in literature.

2. Basic mean-reverting models

Electricity is a commodity with very limited possibility of storage. There exist couple of schemes how to store electricity but these techniques are expensive and therefore it is save to claim that most of the electricity produced is immediately consumed. For this reason the electricity spot price depends heavily on current electricity production and consumption and it abides the law of supply and demand. Even though the volatility of spot prices is high, total yearly consumption is stable and quite easy to predict. It is thus reasonable to assume that the price tends to its long-term equilibrium level even if it is currently above/under this level because of instantaneous lack/excess of supply. In this chapter we will investigate two models which force the spot price to return back to some constant long-term price level. Though constant long-term price level is an oversimplification of market reality, these models allow us to develop some intuition about market dynamics and presented findings will serve as a basis for further theory development.

2.1 Spot price processes

The mean-reverting tendency of spot prices can most easily be expressed either as the Ornstein-Uhlenbeck (OU) process or as the Schwartz one factor model (Schwartz). In the standard analysis the Ohrnstein-Uhlenbeck process can be written as

$$\mathrm{d}S = \alpha(\mu - S)\,\mathrm{d}t + \sigma\,\mathrm{d}z \tag{2.1}$$

where z represents the Wiener process, μ is a long-term level to which the price S tends and α is a strictly positive coefficient which influences speed of the mean reversion. We can see from (2.1) that when the price S is high, the drift term $\alpha(\mu - S)$ becomes negative and the price is pulled down. On the contrary when the price S is low, the drift becomes positive and the opposite holds - the price tends to rise. It should be noted that the tendency to revert back towards the long term level doesn't have to materialize as the random term $\sigma\,\mathrm{d}z$ may have the opposite sign and be greater in magnitude than the drift term.

The Schwartz one factor model introduced by Schwartz [33] assumes that not the price itself but its natural logarithm follows the Ornstein-Uhlenbeck process

$$\mathrm{d}(\ln S) = \alpha(\mu - \ln S)\,\mathrm{d}t + \sigma\,\mathrm{d}z \tag{2.2}$$

Applying Ito's lemma we obtain the master equation

$$\mathrm{d}S = \alpha\left(\mu + \frac{\sigma^2}{2\alpha} - \ln S\right) S\,\mathrm{d}t + \sigma S\,\mathrm{d}z \tag{2.3}$$

where α has the same interpretation as earlier but the spot price now reverts to a new long-term level $e^{\mu+\sigma^2/2\alpha}$.

2.2 Hyperfinite binomial tree

We will build our two models on a hyperfinite binomial tree similar to that used in [2]. Let us start at time $t = 0$ and let T be a given maturity of a derivative and the terminal time of our tree as well. Let us define $H \in {}^*N$ as an infinite number of steps so that the elementary time step $\delta t = T/H$ is an infinitesimal. The time line $\mathbf{T} = \{0, \delta t, 2.\delta t, ..., H.\delta t = T\}$ therefore consists of H time steps of length δt. By the hyperfinite binomial tree τ we will understand the union of power sets of time step sets, $\tau = \cup_{t\in\mathbf{T}}\{-1,+1\}^{\mathbf{T}\cap(0,t]} = \{-1,+1\}^{\leq\mathbf{T}}$. A member of the hyperfinite binomial tree $\omega \in \tau$ can therefore represent any individual node in the tree in contrast to the classical setting where only the nodes with path length T are considered. The path length of node ω will be denoted as $t(\omega) = \max(dom(\omega)) \in \mathbf{T}$. We could also truncate the path of node ω up to some earlier time $s < t(\omega)$. The node which arises from truncated ω will be denoted as $\omega|s$.

We build stochastic processes as internal functions on the tree $S : \tau \to {}^*R$ and proceed step by step from the initial value $S(0) = S_0$. For the Ornstein-Uhlenbeck process the change of S in one step $\delta S(\omega) = S(\omega^\frown\{j\}) - S(\omega)$, $j \in \{-1,+1\}$, is conditional on the value of S at node ω by the equation

$$\delta S(\omega) = \alpha(\mu - S(\omega))\delta t + \sigma j\sqrt{\delta t}, j \in \{-1,+1\}. \tag{2.4}$$

It is useful to denote the term $j\sqrt{\delta t}$ where j is either 1 or -1 as δZ.

The dynamics of the process is given by Loeb counting measure P on nodes with path length equal to T. This is equivalent to setting uniform branching probabilities for all nodes in the tree, i.e. to setting probability of an up movement p_{+1} to be 1/2 and the probability of a down movement p_{-1} to be complementary 1/2 for all nodes. As the elementary probabilities on each tree node uniquely define the process measure, we can move to another measure by simply adjusting branching probabilities which we will do soon.

Let us consider a general derivative V dependent on S. We will try to study its development via Ito-Doeblin lemma. Although it is not possible to literally apply the version of Ito-Doeblin lemma we stated earlier due to unknown upper bounds on $V''(S)$ and $V'''(S)$, the proof method using third-order Taylor expansion in [20] is applicable even for this case. The lemma states that the derivative V has the same source of uncertainty as its underlying S and V can be written in the form

$$\delta V(\omega) = \hat{\mu}(\omega)V(\omega)\delta t + \hat{\sigma}(\omega)V(\omega)\delta Z \tag{2.5}$$

where

$$\begin{aligned}
\hat{\mu}(\omega) &= \left(\frac{\delta V(\omega)}{\delta t} + \alpha(\mu - S(\omega))\frac{\delta V(\omega)}{\delta S(\omega)} + \frac{1}{2}\sigma^2\frac{\delta^2 V(\omega)}{\delta S(\omega)^2}\right)/V(\omega) && (2.6)\\
\hat{\sigma}(\omega) &= \sigma\frac{\delta V(\omega)}{\delta S(\omega)}/V(\omega) && (2.7)
\end{aligned}$$

and the equation holds up to an infinitesimal error almost surely (as will all following equations).

We assume that the market price of risk $\lambda = (\hat{\mu} - r)/\hat{\sigma}$ is constant for the electricity market. It was shown in [40] that when we change the splitting probabilitity p_{+1} from $\frac{1}{2}$ to $\frac{1}{2} - \lambda\frac{\sqrt{\delta t}}{2}$, we obtain a risk-neutral measure Q under which the process has zero price of risk. Furthermore, we have seen that the Girsanov theorem implies that changing the up probability this way is equivalent to adding a drift $-\lambda\sigma\delta t$ to the original equation with measure P. Therefore the risk-neutral process for S with uniform branching probabilities can be written in the form

$$\delta S(\omega) = \alpha(\mu_N - S(\omega))\delta t + \sigma\delta Z \tag{2.8}$$

$$\mu_N = \mu - \frac{\sigma\lambda}{\alpha}. \tag{2.9}$$

Let us shift the long-term level to zero by making a linear substitution $y = S - \mu_N$. Then for the new variable y holds

$$\delta y(\omega) = -\alpha y(\omega)\delta t + \sigma\delta Z \tag{2.10}$$

We will try to solve this equation now. We assume the solution in the form

$$y(\omega) = a(t(\omega))\left(y_0 + \sum_{s\in\mathbf{T}\cap[0,t(\omega)]} b(s)\omega(s)\sqrt{\delta t}\right) \tag{2.11}$$

where $t(\omega)$ indicates the length of node ω, $a(.)$, $b(.)$ are differentiable functions and y_0 is the starting value of our process. The change of y in one step δt should be therefore

$$\delta y(\omega) = \delta a(t(\omega)) \left(y_0 + \sum_{s \in \mathbf{T} \cap [0,t(\omega)]} b(s)\omega(s)\sqrt{\delta t} \right) + a(t(\omega))b(t(\omega))\delta Z \qquad (2.12)$$

From initial condition $y(0) = y_0$ we have $a(0) = 1$ and assuming $a(.) > 0$ we can rewrite the equation as

$$\delta y(\omega) = \frac{\delta a(t(\omega))}{a(t(\omega))\delta t} y(\omega)\delta t + a(t(\omega))b(t(\omega))\delta Z \qquad (2.13)$$

Comparing (2.13) with the original equation (2.10) we can see that we need

$$\frac{\delta a(t(\omega))}{\delta t . a(t(\omega))} = -\alpha \qquad (2.14)$$

$$a(t(\omega))b(t(\omega)) = \sigma \qquad (2.15)$$

and solving these two conditions we get the solution of the process in the form

$$y(\omega) = e^{-\alpha t(\omega)} \left(y_0 + \sigma \sum_{s \in \mathbf{T} \cap [0,t(\omega)]} e^{\alpha s}\omega(s)\sqrt{\delta t} \right) \qquad (2.16)$$

Function $y(\omega)$ is the sum of infinitesimal jumps of length $\sqrt{\delta t}$ which are multiplied by the factor $e^{-\alpha[t(\omega)-s)]}$ always less than 1. It is therefore possible to apply the nonstandard version of central limit theorem like in the case of pure Brownian motion and conclude that the process is Gaussian and as such fully determined by its expected value $E[y](t)$ and standard deviation $\mathrm{Var}[y](t)$.

It is relatively easy to calculate the expected value. As we use the uniform measure, the up and down jumps cancel each other and the expected value of x at time t is $E[y](t) = y_0 e^{-\alpha t}$. When we calculate variance of the process, we can use the independence of jumps on the tree so we have

$$E\left(\sum_{s \in \mathbf{T} \cap [0,t(\omega)]} e^{\alpha s(\omega)} j\sqrt{\delta t} \right)^2 = E\left(\sum_{s \in \mathbf{T} \cap [0,t(\omega)]} e^{2\alpha s}\delta t \right) = \frac{e^{2\alpha t} - 1}{2\alpha} \qquad (2.17)$$

and therefore

$$\mathrm{Var}\left[y\right](t) = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}) \qquad (2.18)$$

Going back to original price S we can see that the expected value of S at t is a weighted average of S_0 and μ_N and that variance is bounded by its limit value $\frac{\sigma^2}{2\alpha}$:

$$E[S](t) = S_0 e^{-\alpha t} + \mu_N(1 - e^{-\alpha t}) \tag{2.19}$$

$$\mathrm{Var}[S](t) = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}) \tag{2.20}$$

As we work in the risk-neutral world, the forward price of a contract with maturity T can be calculated as the expected value of the spot price at T (i.e. as the expected value under the equivalent martingale measure). Since S is normally distributed with expected value given by (2.19), the forward price for the Ornstein-Uhlenbeck process is

$$F(T) = \mu + (S_0 - \mu)e^{-\alpha T} - \frac{\lambda\sigma}{\alpha}(1 - e^{-\alpha T}). \tag{2.21}$$

In this final formula we have rearraged terms of the expected value in order to allow for another interpretation of the forward price. We can see that if we neglect the market price of risk, the forward price is long-term price level μ plus time attenuated deviation of current spot price from μ. Therefore the impact of current price deviation on the forward price gets weaker with increasing maturity.

The Schwartz one factor model can be built on the same hyperfinite tree as the Ornstein-Uhlenbeck process. The change of S in one step conditional on the value of S at node ω is given by the master equation

$$\delta S(\omega) = \alpha\left(\mu + \sigma^2/2\alpha - \ln S(\omega)\right) S(\omega)\delta t + \sigma S(\omega)\delta Z \tag{2.22}$$

The dynamics of the process is again given by the uniform measure P. When we define $x = \ln S$ and apply the nonstandard version of Ito's lemma, we obtain the Ornstein-Uhlenbeck process on the binomial tree

$$\delta x(\omega) = \alpha(\mu - x(\omega))\delta t + \sigma\delta Z \tag{2.23}$$

From prior calculations we know that x has these moments under the risk-neutral measure:

$$E[x](t) = x_0 e^{-\alpha t} + \mu_N(1 - e^{-\alpha t}) \tag{2.24}$$

$$\mathrm{Var}[x](t) = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}). \tag{2.25}$$

where $\mu_N = -\sigma\lambda/\alpha$ as before.

Since $x = \ln S$ is normally distributed, S has a lognormal distribution with mean $exp(E[x] + 0.5\mathrm{Var}[x])$ and therefore the forward price F(T) at time 0 for the Schwartz one factor model is given by

$$F(T) = \exp\left[\mu + (\ln S_0 - \mu)e^{-\alpha T} - \frac{\sigma\lambda}{\alpha}(1 - e^{-\alpha T}) + \frac{\sigma^2}{4\alpha}(1 - e^{-2\alpha T})\right] \quad (2.26)$$

It could be checked that the solution is in line with the result of Schwartz [33].

3. Models incorporating seasonal patterns

We have assumed so far that the spot price follows a process which reverts to some fixed long-term level. Though producing tractable and compact results, models assuming constant long-term level are oversimplification of economic reality. Electricity, like other commodities, shows strong seasonal patterns and we need to take this behaviour into account. The following path characteristics are most prominent:

- Daily shape - the spot price is heavily influenced by electricity consumption of the manufactoring industry. Spot prices are highest during hours when most of the companies produce their products which is in standard working hours. For this reason the market distinguishes between two types of forward contracts - base load and peak load. While the base load forward product is being delivered every hour in the week, the peak load forward product is being delivered in hours of industrial activity only.

- Weekly shape - weekend prices are significantly lower than working day prices for the same reason as for the daily shape difference. Production facilities are usually inactive at the weekend and consumption is lower than during working days.

- Yearly shape - the price level is also different between months. Energy consumption is higher during winter months when households use heating systems while lower during summer months when the heatings are off. As people start using air-conditioning in the summmer, this effect becomes less prominent.

In the following we will add deterministic seasonal components to our stochasic processes for the spot prices and derive formulas for forward prices similar to those we already discussed earlier. We will further define indicative functions for certain time periods which will be used for the seasonal component description.

3.1 Inclusion of deterministic component

We start with a simple Ohrstein-Uhlenbeck process equipped with a deterministic function reflecting seasonality in price movements. Following Lucia and Schwartz

[25] we represent the spot price $P(\omega)$ as a sum of its predictible seasonal component $f(t(\omega))$ and random OU deviation $S(\omega)$

$$P(\omega) = f(t(\omega)) + S(\omega) \tag{3.1}$$

Since the spot price P_t should be drawn to its typical level represented by $f(t)$, we model the mean-reverting process S_t with zero long-run mean

$$\delta S(\omega) = -\alpha S(\omega)\delta t + \sigma \delta Z \tag{3.2}$$

We can plug (3.1) into (3.2) and we receive

$$\delta \big(P(\omega) - f(\omega)\big) = \alpha \big(f(\omega) - P(\omega)\big)\, \delta t + \sigma j \sqrt{\delta t}, j \in \{-1, 1\} \tag{3.3}$$

After rearranging the terms the equation seems similar to the equation (2.4) we had for constant long-term price level:

$$\delta P(\omega) = \alpha \big(L(\omega) - P(\omega)\big)\, \delta t + \sigma j \sqrt{\delta t}, j \in \{-1, 1\} \tag{3.4}$$

where

$$L(\omega) = \frac{1}{\alpha}\frac{\delta f}{\delta t}(\omega) + f(\omega) \tag{3.5}$$

The long-term price level μ was replaced by a deterministic function $L(\omega)$ which depends on the seasonal component $f(\omega)$. Similar reasoning as in the case of μ yields a formula for the forward price in the form

$$F(T) = f(T) + \big(P_0 - f(0)\big)\, e^{-\alpha T} - \frac{\lambda\sigma}{\alpha}(1 - e^{\alpha T}) \tag{3.6}$$

We can see that the structure of the forward formula remains the same even with the inclusion of a deterministic trend. It has natural interpretation that the forward price is the usual seasonal price plus correction for the current spot price level and for the market price of risk.

The Schwartz one factor model can be equipped with a seasonal component as well. Assuming that the logarithmic price is the sum of a seasonal component $f(t(\omega))$ and the Ornstein-Uhlenbeck process S with zero μ

$$\ln P(\omega) = f(t(\omega)) + S(\omega) \tag{3.7}$$

we obtain

$$\delta P(\omega) = \alpha \big(L(\omega) - \ln P(\omega)\big)\, P(\omega)\delta t + \omega P(\omega) j \sqrt{\delta t} \tag{3.8}$$

where

$$L(\omega) = \frac{1}{\alpha}\left(\frac{\sigma^2}{2} + \frac{\delta f}{\delta t}(\omega)\right) + f(\omega) \tag{3.9}$$

Applying the same arguments as before the forward price for the Schwartz one factor model with a seasonal component is

$$F(T) = exp\left[f(T) + \left(\ln P_0 - f(0)\right)e^{-\alpha T} - \frac{\lambda\sigma}{\alpha}\left(1 - e^{-2\alpha T}\right) + \frac{\sigma^2}{4\alpha}\left(1 - e^{-2\alpha T}\right)\right] \tag{3.10}$$

3.2 Modeling deterministic component

There are several ways how to model the deterministic component of spot prices. While some authors impose a prescribed function on the shape of the deterministic component, we are going to use indicator functions for certain time periods. Although fitting data to an analytical function generally yields more robust and stable solution with fewer parameters, we do not want to a priori assume any shape or characteristics of seasonality. We would like to let the data speak for themselves and possibly explain our results with regards to what was said earlier about reasons for seasonality.

Because we are going to investigate real market data, the time step δt will not be an infinitesimal but some real time period. Therefore we need to move from the hyperfinite binomial tree to a discretization scheme which uses a calendar time line $(.., t-1, t, t+1, ..)$. In this sense we model the deterministic component $f(t)$ as a sum of indicator functions

$$f(t) = A + \sum_j B_j M_j(t) \tag{3.11}$$

where

$$M_j(t) = \begin{cases} 1 & \text{if date } t \text{ belongs to time period } j \\ 0 & \text{otherwise} \end{cases}$$

B_j represents a coefficent for the time period j and A is the initial price level. The time step and the type of the time period for which the indicator function M_j is 1 will be determined by the granularity of prices we would like to use. We will use either monthly granularity with monthly indicators or daily granularity with monthly and week-daily indicators.

In order to be able to estimate parameters of our model we rewritte equations which hold for the hyperfinite binomial tree into their discrete equivalents on the time line $(.., t-1, t, t+1, ..)$. The seasonal Ornstein-Uhlenbeck process in its discrete form could be written as

$$P(t) = A + \sum_j B_j M_j(t) + S(t) \quad (3.12)$$

$$S(t) = \phi S(t-1) + u(t) \quad (3.13)$$

and for the seasonal Schwartz one factor model must hold

$$\ln P(t) = A + \sum_j B_j M_j(t) + S(t) \quad (3.14)$$

$$S(t) = \phi S(t-1) + u(t) \quad (3.15)$$

where parameter $\phi \equiv 1 - \alpha$ enabled us to express mean-reverting processes as autoregressive models and $u(t)$ is the white noise with deviation σ, $u(t) \sim \mathcal{N}(0, \sigma^2)$. Therefore our task is to estimate parameters of multi-linear model with AR(1) error term which we will do in the next section.

4. Estimation of parameters

4.1 Basic statistics

We will use German settlement prices published on the EEX (European Energy Exchange) between years 2000 and 2010 as our base dataset for the estimation of model parameters. EEX prices are quoted in EUR/MWh format and have hourly granularity. As the electricity unit will always be 1 MWh, we will use only monetary quotes (EUR) for the electricity price. Since the smallest forward product we will encounter is a daily product, we first aggregate original spot data in daily average prices P_t. Table 4.1 displays basic descriptive statistics of our daily data sample.

	P_t	$P_t - P_{t-1}$	$\ln P_t$	$\ln P_t - \ln P_{t-1}$
Observations	4018	4017	4016	4013
Mean	56.75	0.01	3.91	-0.00
Median	50.35	-1.01	3.92	-0.02
Min	-53.46	-299.84	1.55	-1.95
Max	472.82	314.86	6.16	2.37
SD	31.15	19.63	0.50	0.30
Skewness	2.22	0.82	-0.01	0.75
Kurtosis	16.06	48.97	3.31	6.73

Table 4.1: Descriptive statistics of German spot market

It is typical for the electricity market that prices are not bounded from zero and negative prices are not only theoretically possible but they can be even often encountered. For example we can observe from Table 4.1 that it was possible to observe price -53.46 EUR in the time period of our interest.

Price distribution is skewed to the right which is not suprising as even though negative prices are possible, they are not so common as price spikes due to electricity generation shortfalls or extreme weather conditions. High kurtosis suggests that there were huge price jumps present which produce fat tails in price distribution. Figure 4.1 plots the daily settlement price development in 2000-2010 and exposes typical jump behaviour of electricity prices. It is common that from time to time the price spikes (usually) upwards and consequently drops down to its prior level. Until now we have dealt with continuous processes only but evidence suggests that the inclusion of a discontinuous process might be necessary.

Let us further investigate jump behavior of spot prices. Figure 4.2 compares empirical distribution of daily spot price changes with the normal distribution of the same mean and variance. We can see that there are price changes far away from the center where we wouldn't expect them under the normal distribution

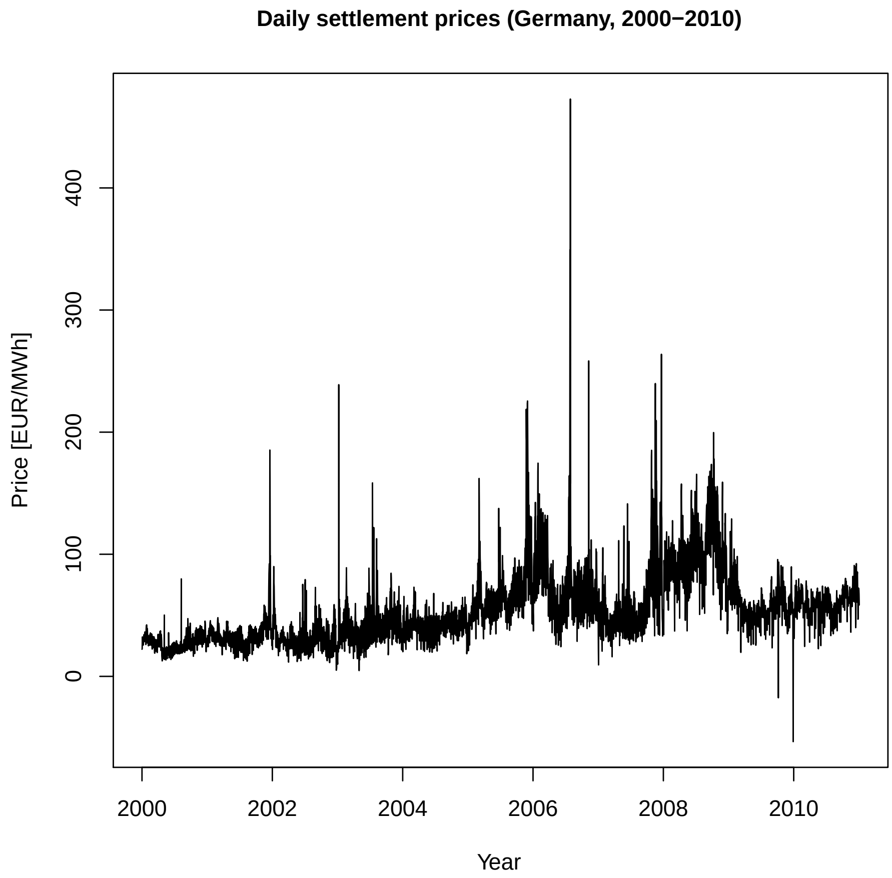

Figure 4.1: German daily settlement prices 2000-2010. Price spikes are charactrized by sudden upward movement of extreme magnitude followed by fast attenuation and fall to prior price level.

assumption. As large outliers are not in line with our assumption of normal error terms and can even jeopardize correct estimation of model parameters, we will develop framework which incorporates jump behaviour into our mean-reverting model. Fortunately, models with spike behaviour have been studied intensively in the energy literature so that we have a base to build on.

4.2 Overview of jump models

The presence of sudden price spikes is inherent in the electricity market and can be considered as a stylized fact. A typical jump is a sudden upward price movement of large magnitude which is followed by fast correction that brings the spot price to its prior level. Though price spikes can be found in other markets, rapid up and down discontinuities are fundamental feature of power prices due to the non-storable nature of electricity. The cause for a jump can be on both demand and supply side of market equilibrium. Severe weather conditions as well

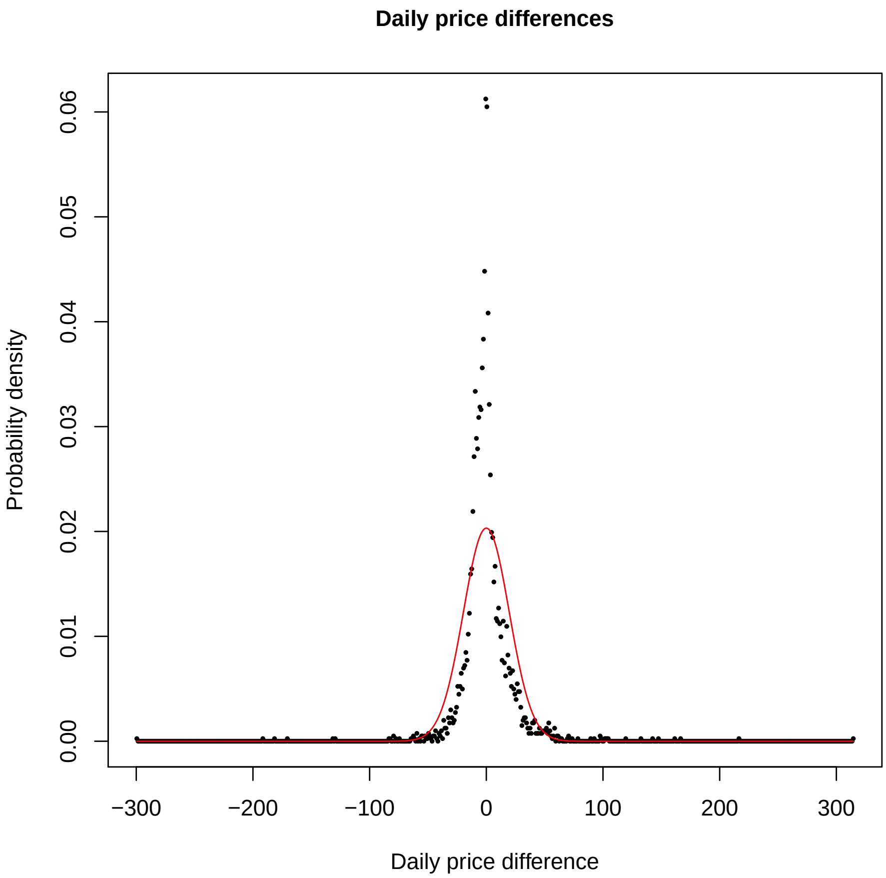

Figure 4.2: Histogram of empirical daily price changes compared to the normal distribution. Fat tails contained in the data sample can be filtered by application of the recursive jump filter.

as dropout of important power plants can result in price skyrocketing as the most expensive power sources need to be engaged. With the introduction of solar and wind power plants, which are guaranteed access to the distribution system at any time by the government, we can expect the opposite effect of sudden price drops to take place in the future. When it happens that weather conditions are most favourable for renewable energy production, uncontrollable excess of power in the system will drive prices deep below zero and downward drops will become more common than until now.

There are several ways how to model spikes. Prominent approaches which received most attention in the literature are Markov models, regime-switching models and multifactor models.

Markov models The representative of models with Markov property is the work of Geman and Roncoroni [19] who model the spot price behaviour by the stochastic differential equation

$$dE(t) = D\mu(t)\,dt + \theta\left[\mu(t) - E(t^{-})\right]\,dt + \sigma\,dZ(t) + h(t^{-})\,dJ(t) \tag{4.1}$$

which includes logarithmic price E, seasonal trend μ, parameter of mean-reversion θ and Brownian motion Z. Standard first order derivative is denoted as D and f(t-) is an abbreviation for the left limit of f at time t. The last term $h(t^{-})\,dJ(t)$ produces jump behavior when Poisson process triggers a series of random jumps whose number is determinded by a counting process and whose direction is detemined by a defined treshold τ - if the spot price is below τ, then there is an upwards movement and vice versa if the price is above the treshold. Similar models along these lines were devised by Roncori [30] and Deng [15].

Regime-switching models The drawback of Markov models (jump-diffusion processes) is that the jump declines to the normal price level immediately or very rapidly while it can be observed on the market that stressed prices can remain high for a nonnegligible time period which can account for several days. Regime-switching models offer possibility to keep unusually high prices for some time by introducing a Markov chain which determines if prices are in the standard base regime displaying mean reversion or in the spike regime with jump behaviour. Processes that are linked to each regime state are assumed to be independent of each other. Bierbrauer et al. [6] use a two state Markov chain with transition matrix $\mathbf{P}$ containing probabilities p_{ij} of switching from state i to state j between times t and $t+1$:

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{pmatrix} \tag{4.2}$$

While the first base regime is governed by a mean-reverting model, the second spike regime is based either on Gaussian, lognormal or Pareto distribution in order to cope with the heavy-tailed nature of severe jumps. Weron [37] also applies a two state regime-switching model but in his setting the spike regime follows a mean-reverting process with greater volatility and faster reversion than in the case of the base regime

$$dP(t) = \theta_1\left(\mu(t) - P(t)\right) + \sigma_1\,dZ(t) \qquad \text{(base regime)} \tag{4.3}$$

$$dP(t) = \theta_2\left(\mu(t) - P(t)\right) + \sigma_2\,dZ(t) \qquad \text{(spike regime)} \tag{4.4}$$

$$\theta_2 > \theta_1, \quad \sigma_2 > \sigma_1$$

Huisman and Mahieu [21] identify tree possible regimes: a normal mean-reverting regime, an initial jump regime whose values are drawn from lognormal distribution and a third regime which ensures that prices are pulled back to their prior level. The Markov transition matrix for the jump, reversal and normal mean-reverting regime is now specified as

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & p_{00} & 1 - p_{00} \\ 1 & 0 & 0 \end{pmatrix} \tag{4.5}$$

where p_{00} is the probability of staying in the base regime. As the initial jump is immediately followed by the reversal regime which brings the price back to its standard level, this model doesn't allow for consequtive extreme prices.

Multifactor models Multifactor models assume the price to be a sum of several factors which usually bear a jump-diffusion specification like in [4], [16] and [36]. In Benth [4], the spot price stripped off its seasonal component is a sum of independent Levy-driven Ornstein-Uhlenbeck processes

$$P(t) = \sum_{i=1}^{n} X_i(t) \tag{4.6}$$

$$dX_i(t) = -\theta X_i(t)\, dt + dL_i(t) \tag{4.7}$$

where $L_i(t)$ are independent Levy processes with finite second moments.

In our work we will use the regime-switching approach as a way how to deal with jumps. We assume that the spot price follows two independent processes governed by a 2×2 Markov transition matrix. If the system is in its base regime, then the spot price or the logarithm of the spot price follows the mean reverting process from equations (3.12)-(3.15). In the spike regime it would be possible to use any of heavy-tailed distributions for the dicontinuous process. The jump price would then be a series of i.i.d. random variables drawn, for example, from the Pareto distribution. As it will become clear from later discussion, we won't need any particular distributional representation of the spike regime for our purpose of forward pricing. The merit of the regime-switching model is that after we identify jumps and extract them, base regime data can be treated as a continuous time series without gaps due to the independence assumption.

4.3 Spike regime

Our ultimate task is to assess prices of forward contracts which are traded on the market. The approach we have chosen is that we calculate the expected value of future spot prices under the equivalent martingale measure. At this point we will make an extra assumption that the market price of jumps is zero and therefore for the spike regime holds $\lambda = 0$. The assumption of zero market price of risk for some short-term risk factors, which the spike regime belongs to, is common in the energy literature ([15], [8], [13]) and relates to the difficulties of risk premia estimation from historical commodity price data [34] as well as non-systematic nature of jumps which makes them diversifiable [9].

If we wanted to fully model spot price dynamics, it would be necessary to estimate coeficients of the Markov matrix by investigating the probablities of transition from one state to another and we would also need to estimate parameters of some heavy-tailed distribution which we decided to model jumps with. As our primary aim is not to mimic the spot price evolution but price forward contracts, it is sufficient to estimate spike intensity and average spike amplitude.

Following Cartea, Figueora [9] and Meyer-Brandis, Tankov [26], we define spike intensity ω as

$$\omega = \frac{\text{number of spikes detected}}{\text{number of data points}} \tag{4.8}$$

Since we don't observe any seasonality in the spike occurrence in our data, we can assume that the spike intensity is constant over time. Using ω we can define a spike correction to the forward price derived from the base mean-reverting process as

$$\text{spike correction} = \omega \times E(\text{jump magnitude}) \tag{4.9}$$

The interpretation of the spike correction (4.9) is that when we filter jumps out of the sample and work with the base regime data only (which we will soon do), we underestimate the future average spot price and thus we need to correct this bias. Since we assume zero market price of risk for jumps, the application of spike correction is straightforward.

In order to identify jumps in our data we will apply the so-called recursive filter [11]. If the daily price changes were normally distributed, then the probability of finding values further from the mean than 3 standard deviations is 0.3. Because our sample consists of more observations behind this threshold than we would expect, we consider all these outstreched observations as discontinuous jump behavior and exclude them. When we recalculate the sample standard deviation of filtered data, we can look for further price changes behind the chosen limit. This process can be repeated until we obtain data which follow the 3 standard deviation rule and which we consider to be the manifestation of our continuous base process with normal increments.

By applying the recursive filter we have identified 168 days out of total 4018 which contained a price jump according to our definition. This suggests that the spike intensity ω is equal to 4.6%. The average value of the spike magnitude was calculated as 110.34 EUR which means that the average price excess of a jump above the long-term price level (54.95 EUR) is 55.39 EUR. When we multiply the jump price excess by the spike intensity, we obtain correction 2.55 EUR to the base regime which accounts for the existence of jumps.

4.4 Estimation method for base regime

When we filter jumps which were identified by the recursive filter out of our data sample, we obtain a time series which should follow the mean-reverting process. We will now try to estimate parameters in (3.12)-(3.13) and (3.14)-(3.14) by the maximum likelihood method. The spot time series can be expressed in its vector form as

$$\begin{aligned} \vec{P} &= \mathbf{M}\vec{B} + \vec{S}, \\ \vec{S} &\sim \mathcal{N}(\vec{0}, \sigma^2 \mathbf{\Lambda}) \end{aligned} \tag{4.10}$$

where $\vec{P}$ is $n \times 1$ vector of observed spot prices, $\mathbf{M}$ is the matrix of indicator functions, $\vec{B}$ is the vector of seasonality coefficients we would like to find and $\vec{S}$ represents error terms which follow the first-order autoregressive AR(1) process. We have factored variance of white noise σ^2 ouf of positive-definite $\mathbf{\Lambda}$ which represents nonspherical disturbance of the AR(1) process

$$\mathbf{\Lambda} = \frac{1}{1-\phi^2} \begin{pmatrix} 1 & \phi & \cdots & \phi^{n-1} \\ \phi & 1 & \cdots & \phi^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \phi^{n-1} & \phi^{n-2} & \cdots & 1 \end{pmatrix}. \tag{4.11}$$

where ϕ is the autoregressive parameter we have defined in chapter 3.2. We can notice that costant variance of $\vec{S}_t$ is

$$\mathrm{Var}(\vec{S}_t) = \frac{\sigma^2}{1-\phi^2} \tag{4.12}$$

and the autocorrelation coefficients are

$$\rho_k = \phi^k \qquad k = 0, 1, 2, ... \tag{4.13}$$

Since $\mathbf{\Lambda}$ is positive definite, it is possible to find an invertible square root $\mathbf{\Lambda}^{1/2}$ such that

$$\mathbf{\Lambda} = \left(\mathbf{\Lambda}^{1/2}\right)^T \mathbf{\Lambda}^{1/2} \qquad \text{and} \qquad \mathbf{\Lambda}^{-1} = \mathbf{\Lambda}^{-1/2} \left(\mathbf{\Lambda}^{-1/2}\right)^T \tag{4.14}$$

It could be verified by multiplication that the inverse matrix $\mathbf{\Lambda}^{-1}$ is

$$
\mathbf{\Lambda}^{-1} = \begin{pmatrix} 1 & -\phi & 0 & \cdots & 0 & 0 & 0 \\ -\phi & 1+\phi^2 & -\phi & \cdots & 0 & 0 & 0 \\ 0 & -\phi & 1+\phi^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\phi & 1+\phi^2 & -\phi \\ 0 & 0 & 0 & \cdots & 0 & -\phi & 1 \end{pmatrix} \tag{4.15}
$$

and matrix $\left(\mathbf{\Lambda}^{-1/2}\right)^T$ can be represented as

$$
\left(\mathbf{\Lambda}^{-1/2}\right)^T = \begin{pmatrix} \sqrt{1-\phi^2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\phi & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\phi & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\phi & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\phi & \sqrt{1-\phi^2} \end{pmatrix} \tag{4.16}
$$

If we knew parameter ϕ, we could multiply equation (4.10) by $\left(\mathbf{\Lambda}^{-1/2}\right)^T$ from the left and obtain

$$
\vec{P}^* = \mathbf{M}^* \vec{B} + \vec{S}^* \tag{4.17}
$$

where

$$
\begin{aligned} \vec{P}^* &= \left(\mathbf{\Lambda}^{-1/2}\right)^T \vec{P} \\ \mathbf{M}^* &= \left(\mathbf{\Lambda}^{-1/2}\right)^T \mathbf{M} \\ \vec{S}^* &= \left(\mathbf{\Lambda}^{-1/2}\right)^T \vec{S} \end{aligned} \tag{4.18}
$$

Then

$$
\begin{aligned} \mathrm{Var}(\vec{S}^*) &= \sigma^2 \left(\mathbf{\Lambda}^{-1/2}\right)^T \mathbf{\Lambda}\mathbf{\Lambda}^{-1/2} \\ &= \sigma^2 \left(\mathbf{\Lambda}^{-1/2}\right)^T \left(\mathbf{\Lambda}^{1/2}\right)^T \mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{-1/2} \\ &= \sigma^2 \mathbf{I} \end{aligned} \tag{4.19}
$$

which means that $\vec{P}^*$ is described by a standard linear model with i.i.d. error terms

$$
\vec{S}^* \sim \mathcal{N}(\vec{0}, \sigma^2 \boldsymbol{I}) \tag{4.20}
$$

and ergo the maximum likelihood estimators of $\vec{B}$ and σ^2 can be obtained by solving the least squares problem which yields

$$\begin{aligned}
\hat{\vec{B}}(\phi) &= \left((\mathbf{M}^*)^T \mathbf{M}^*\right)^{-1} (\mathbf{M}^*)^T \vec{P}^* && (4.21)\\
\widehat{\sigma^2}(\phi) &= \frac{\left\|\vec{P}^* - \mathbf{M}^* \hat{\vec{B}}(\phi)\right\|}{n} && (4.22)
\end{aligned}$$

where $\|.\|$ is the vector norm. It is important to note that estimators $\hat{\vec{B}}$ and $\widehat{\sigma^2}$ are functions of parameter ϕ because substitution (4.18) depends on it. Although we have obtained intereting results, the value of ϕ still remains unknown and therefore it is necessary to express the full likelihood function of our problem.

We know that the error term of the original equation (4.10) has multivariate normal density

$$f\left(\vec{S}\right) = (2\pi)^{-n/2} \left|\sigma^2 \mathbf{\Lambda}\right|^{-1/2} \exp\left[-\frac{1}{2}\vec{S}^T (\sigma^2 \mathbf{\Lambda})^{-1} \vec{S}\right] \qquad (4.23)$$

Because $\left|\sigma^2 \mathbf{\Lambda}\right| = \sigma^{2n} \left|\mathbf{\Lambda}\right|$, we may rewrite the density as

$$f\left(\vec{S}\right) = (2\pi)^{-n/2} \left(\sigma^2\right)^{-n/2} \left|\mathbf{\Lambda}\right|^{-1/2} \exp\left[-\frac{1}{2}\vec{S}^T (\sigma^2 \mathbf{\Lambda})^{-1} \vec{S}\right] \qquad (4.24)$$

and the full log-likelihood function $l(\phi, \vec{B}, \sigma^2|\vec{P})$ corresponding to the original model (4.10) can be expressed as

$$\begin{aligned}
l(\phi, \vec{B}, \sigma^2|\vec{P}) &= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma^2 - \frac{1}{2}\ln|\mathbf{\Lambda}| - \\
&\quad -\frac{1}{2\sigma^2}\left(\vec{P} - \mathbf{M}\vec{B}\right)^T \mathbf{\Lambda}^{-1} \left(\vec{P} - \mathbf{M}\vec{B}\right) && (4.25)
\end{aligned}$$

We know from prior calculation how maximum likelihood estimators of $\vec{B}$ and σ^2, conditional on ϕ, look like. It is therefore possible to replace $\vec{B}$, σ^2 in $l(\phi, \vec{B}, \sigma^2|\vec{P})$ by their estimates $\hat{\vec{B}}(\phi)$, $\widehat{\sigma^2}(\phi)$ and we obtain log-likelihood as a function of ϕ only

$$l(\phi|\vec{P}) = const. - n\ln\left\|\vec{P}^* - \mathbf{M}^*\hat{\vec{B}}(\phi)\right\| - \frac{1}{2}\ln|\mathbf{\Lambda}| \qquad (4.26)$$

Optimizing $l(\phi|\vec{P})$ yields the maximum likelihood estimate of ϕ which in turn can be used to evaluate estimates of $\vec{B}$ and σ^2 in (4.21). Since the described method makes use of least squares estimators, it is sometimes refered to as the *generalized least squares* (GLS) method [22]. Equipped with the estimation procedure we can now start investigating the data.

4.5 Monthly granularity model

We will first start with a model based on data with monthly granularity. We consider that a month is our basic time step and we use only monthly indicator functions applied on monthly price averages in equation (3.11). For the estimation of parameters in (3.12)-(3.13) and (3.14)-(3.15) we use the generalised least squares method which was described in former section. Numerical details of the application of GLS can be found in Pinheiro [28].

Table 4.2 shows estimated paremeters of the deterministic seasonal component with their statistics for both models. Standard errors (SE) are substantial relative to the parameter magnitudes due to high volatility of sample prices. We can also notice that the level of significance for different indicator coefficients differ and some of them are not statistically significant.

The coefficient ϕ in AR(1) error process for the Ornstein-Uhlenbeck model was calculated as 0.93 with 95% confidence interval $(0.81, 0.98)$. The correspoding mean reversion coefficient α $(= 1 - \phi)$ is therefore 0.07 and is statistically significant. The coefficient ϕ for the Schwartz one factor model is 0.95 with 95% confidence interval $(0.82, 0.99)$. The mean reversion coefficient for this model is therefore 0.05 and again statistically significant. Standard deviations of white

	Ornstein-Uhlenbeck				Schwartz			
Parameter	Value	SE	t-value	p-value	Value	SE	t-value	p-value
A	57.02	11.18	5.10	0.00	3.96	0.23	17.07	0.00
Feb	0.53	2.71	0.20	0.85	0.00	0.04	0.00	1.00
Mar	-4.78	3.64	-1.31	0.19	-0.09	0.05	-1.70	0.09
Apr	-6.38	4.22	-1.51	0.13	-0.12	0.06	-1.97	0.05
May	-10.95	4.60	-2.38	0.02	-0.22	0.07	-3.20	0.00
Jun	-5.37	4.81	-1.12	0.27	-0.12	0.07	-1.65	0.10
Jul	-3.30	4.88	-0.68	0.50	-0.09	0.07	-1.24	0.22
Aug	-7.21	4.82	-1.49	0.14	-0.14	0.07	-1.89	0.06
Sep	1.72	4.63	0.37	0.71	0.01	0.07	0.16	0.87
Oct	3.79	4.27	0.89	0.38	0.03	0.06	0.54	0.59
Nov	1.92	3.71	0.52	0.61	0.02	0.06	0.39	0.70
Dec	-0.78	2.82	-0.28	0.78	-0.01	0.04	-0.14	0.89
ϕ	0.93	-	-	-	0.95	-	-	-
σ	9.08	-	-	-	0.14	-	-	-

Table 4.2: Estimates of the seasonal profile for models using monthly granularity. The seasonal profile starts in January at price A and its shape is determined by corresponding monthly coefficients.

noise σ are 9.08 (OU) and 0.14 (Schwartz).

Figure 4.3 plots normalized (average price is set to 1) annual seasonality gained from the Ornstein-Uhlenbeck process. The particular shape we can observe here is remarkable and reveals specific structure of the German market which can be interpreted in terms of influnce that weather has on electricity prices. First, we can notice that prices in the summer season are generally lower than in winter when people use heating systems. Next, it is obvious that the standard shape is not sinusoidal as some studies assume but there is a pronounced irregularity in June and July probably due to the usage of air conditioners and nuclear power plants maintenance. It is also interesting that the December price is lower compared to other winter months due to holidays in this month.

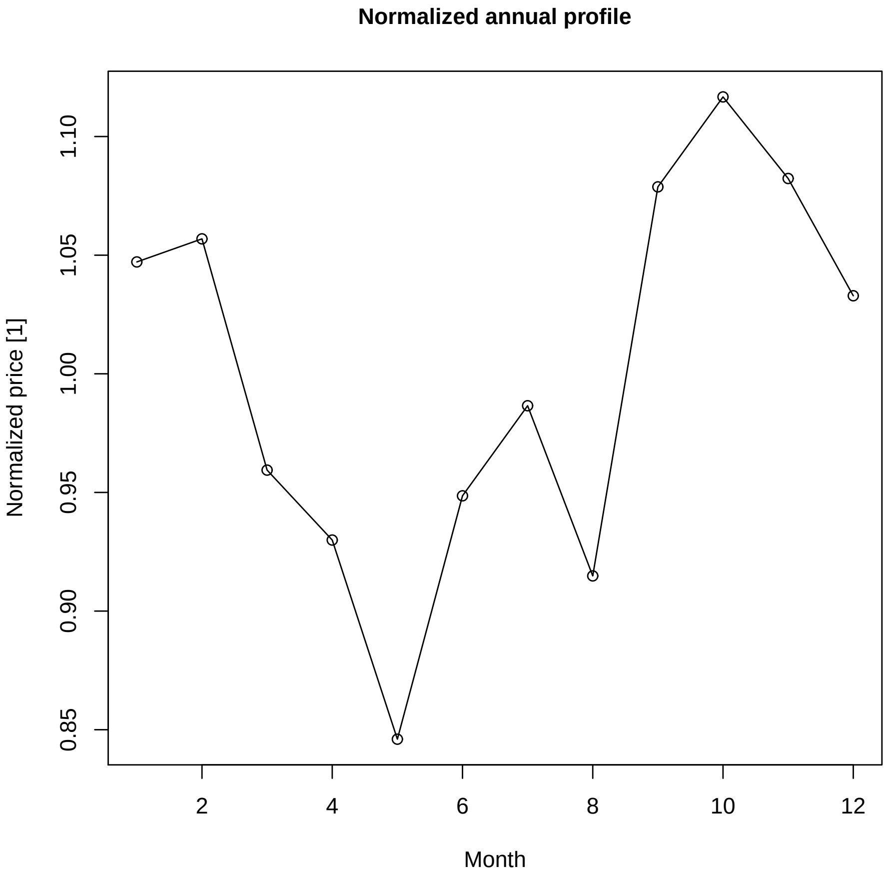

Figure 4.3: Normalized annual profile for the Ornstein-Uhlenbeck process using monthly granularity. The shape is fully implied by market data (virtue of indicator functions).

4.6 Daily granularity model

In order to include even the standard weekly shape into the model and to be able to price short-term forward products, we move from monthly granularity to daily granularity. The seasonal function will now be composed of 11 monthly indicators and 6 week-daily indicators. All parameters are again assessed by the least squares method.

Table 4.3 summarizes seasonality estimation results for both models. It is worth mentioning that standard errors of daily coefficients are substantially lower than standard errors of monthly coefficients. This suggests that weekly pattern of daily prices is more pronounced and regular than annual patern of monthly prices.

Both coefficients of mean reversion α are statistically significant and their values are 0.10 and 0.09 for the Ornstein-Uhlenbeck and the Schwartz one factor model, respectively. Standard errors of white noise σ are 11.06 (OU) and 0.19 (Schwartz).

Figure 4.4 shows normalized weekly and annual price profiles for the Ornstein-

	Ornstein-Uhlenbeck				Schwartz			
Parameter	Value	SE	t-value	p-value	Value	SE	t-value	p-value
A	52.18	3.35	15.59	0.00	3.83	0.06	65.20	0.00
Feb	1.87	3.02	0.62	0.54	0.06	0.05	1.25	0.21
Mar	3.05	3.78	0.81	0.42	0.15	0.06	2.26	0.02
Apr	8.74	4.17	2.10	0.04	0.25	0.07	3.48	0.00
May	-2.15	4.37	-0.49	0.62	-0.10	0.08	-1.35	0.18
Jun	0.83	4.48	0.19	0.85	-0.01	0.08	-0.11	0.92
Jul	6.83	4.51	1.52	0.13	0.12	0.08	1.57	0.12
Aug	8.37	4.47	1.87	0.06	0.16	0.08	2.09	0.04
Sep	15.02	4.39	3.43	0.00	0.28	0.08	3.64	0.00
Oct	16.91	4.20	4.03	0.00	0.35	0.07	4.87	0.00
Nov	5.50	3.85	1.43	0.15	0.18	0.07	2.81	0.00
Dec	4.24	3.12	1.36	0.17	0.18	0.05	3.44	0.00
Mon	-2.46	0.64	-3.82	0.00	-0.03	0.01	-2.72	0.01
Sat	-12.82	0.45	-28.47	0.00	-0.23	0.01	-31.17	0.00
Sun	-23.81	0.57	-41.92	0.00	-0.48	0.01	-50.65	0.00
Thu	4.20	0.45	9.36	0.00	0.05	0.01	6.74	0.00
Tue	4.84	0.62	7.78	0.00	0.06	0.01	5.74	0.00
Wed	5.37	0.57	9.35	0.00	0.07	0.01	7.35	0.00
ϕ	0.90	-	-	-	0.91	-	-	-
σ	11.06	-	-	-	0.91	-	-	-

Table 4.3: Estimates of seasonal profile for models using daily granularity. The seasonal profile starts on January 1 at price A and its shape is determined by corresponding monthly and week-daily coefficients.

Uhlenbeck model. The weekly profile has a smooth concave shape and can be fully interpreted in terms of the amount of electricity consumed by factories. It is common that factories produce mostly during regular working days and less during weekends. Therefore the coefficients are above average, which is 1, from Monday to Friday and below average at the weekend. It can be also noticed that when the production starts and ends, i.e. on Monday and Friday, the coefficients are slightly lower than in the middle of the week. Finally, Sunday has the lowest coefficient of all week-days as one could expect from the day of the Lord.

The annual profile is interestingly different from the annual profile we calculated from monthly averages, especially first 3 monthly coefficients show opposite price development tendency. There might be a few issues when we mix week-daily and monthly indicator functions. For example, daily indicator functions

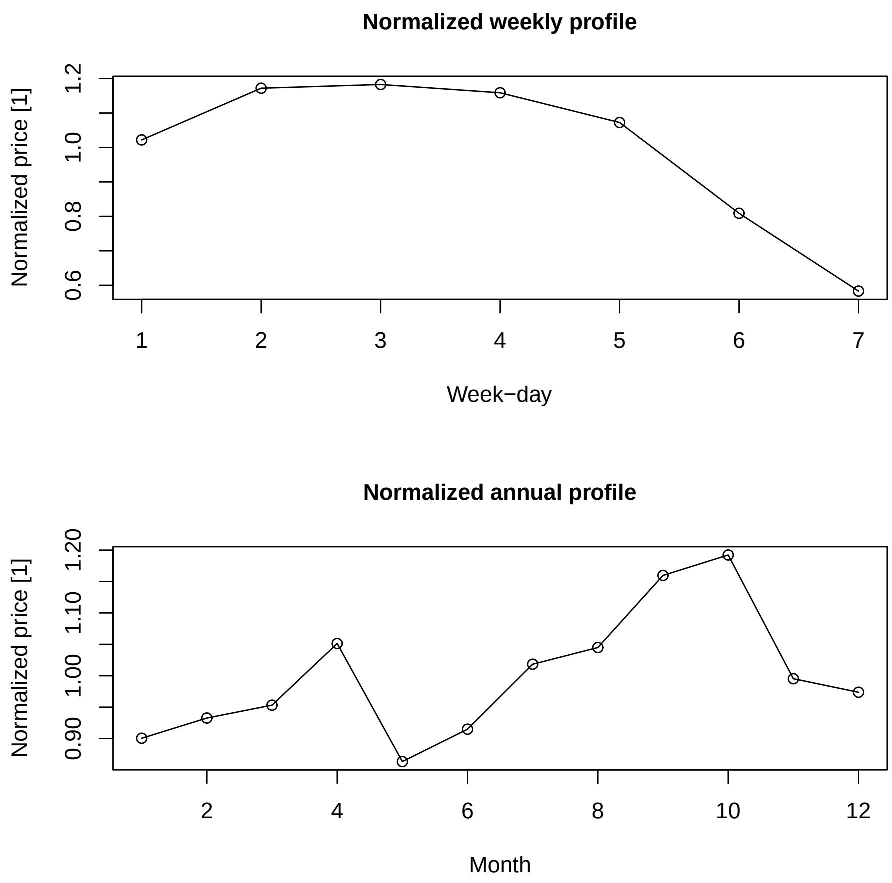

Figure 4.4: Normalized week-daily and annual profile for the Ornstein-Uhlenbeck process using daily granularity. The week-daily profile is fully in line with electricity consumption pattern of the manufactoring sector. A different frequency period between daily and monthly indicator functions could have resulted in an otherwise shaped annual profile compared to the profile that was calculated from monthly granularity.

have high frequency and change with every time step while monthly indicators change less often which could lead to weaker statistics. Another issue could be that daily shape is more pronounced than monthly shape and therefore monthly shape can be burdened with proportionately more noise than it is in the case of daily shape.

5. Forward valuation

Valuation formulas (3.6) and (3.10) which were derived in the theoretical part of the thesis calculate the forward price of a product which is assumed to be settled at a single future time point T. We can associate maturity T with some time step in our discretization scheme (3.12)-(3.15) but electricity unlike most commodities cannot be delivered instantaneously and it is being delivered during a certain time period. The delivery of many electricity forward products usually spans a longer time window than just a day or a month which are our basic time steps. Typically day, week, month, quarter and year products are traded on main exchanges. For these reasons we need to split forward products into appropriate granularity and calculate the theoretical price of each product as the average over its parts which we are able to value.

In the following we will compare prices of both short-term and long-term real forward contracts with their theoretical counterparts derived from our two stochastic processes.

5.1 Long-term forward contracts

In this section we will value long-term electricity forward products, specifically monthly, quarterly and yearly German base products. Base products are contracts which oblige the seller to deliver electricity uniformly and continuously over a defined time period. As it was already suggested, we cut quaterly and yearly products into collections of monthly products so that we have all data in monthly resolution. We use the annual seasonal shape and mean-reverting coefficients estimated in section 4.5 and plug them into formulas (3.6) and (3.10). We also make correction with respect to the existence of jumps from (4.9). The final theoretical product price is the average of its monthly pieces.

The sample of derivatives which will be used to compare our model with consists of complete term structure from 22 closing days between January 2011 and October 2012. The days were chosen as the last closing days in each month and quotes published by 3 major brokers active on the German market - Tradition Financial Services (TFS), Platts and GFI - were used as the term structure source. In case of multiple entries for the same product the average price is taken as the reference price.

We consider the current monthly spot price to be implied by the quote of the forward product for the next month and our valuation formula. In this way we align the theoretical and market term structure at their beginnings and avoid

	Ornstein-Uhlenbeck		Schwartz	
Date	RMSE	MAE	RMSE	MAE
2011-01-31	2.89	1.91	2.46	1.73
2011-02-28	2.64	2.07	3.14	2.62
2011-03-31	4.20	3.26	3.18	2.42
2011-04-29	4.53	3.83	5.02	4.37
2011-05-31	4.86	4.07	3.33	2.69
2011-06-30	5.16	4.44	4.09	3.52
2011-07-29	5.44	4.81	4.03	3.64
2011-08-31	7.68	6.87	7.41	6.55
2011-09-30	5.54	4.89	4.49	3.82
2011-10-31	3.13	2.47	2.14	1.92
2011-11-30	2.85	2.23	2.14	1.77
2011-12-30	3.75	3.11	3.68	3.15
2012-01-31	1.99	1.50	1.99	1.69
2012-02-29	1.98	1.40	1.80	1.48
2012-03-30	2.33	1.85	2.21	1.67
2012-04-30	4.02	3.51	3.70	3.35
2012-05-31	2.48	1.95	2.90	2.44
2012-06-29	2.77	2.36	2.80	2.31
2012-07-31	3.44	2.70	3.49	2.95
2012-08-31	3.65	3.14	5.18	4.51
2012-09-28	4.15	3.72	3.77	3.43
2012-10-31	2.66	2.09	2.93	2.38

Table 5.1: Valuation errors of long-term forwards for each closing date assuming zero market price of risk. The root mean square error (RMSE) and the mean absolute error (MAE) of the difference between predicted and real market forward prices are calculated for both models. Although both models produce forward prices which fit the market term structutre well, the Schwartz one factor model could be considered as slightly superior.

necessity of the estimation of the current monthly spot price which could prove to be tricky and could introduce another sources of error into our model.

We first start with the assumption that the market price of risk is zero which would reflect rational expectations of the power market. In order to assess goodness of our model predictions we calculate root mean squared error (RMSE) and mean absolute error (MAE) between the model prediction and observed market prices. The results of this comparison are shown in Table 5.1. We can see that the accurateness of predictions differs only slightly between the two models. While the root mean square error for the whole sample is 4.15 EUR in case of the Ornstein-Uhlenbeck model, it is 3.78 EUR in case of the Schwartz one factor model. The reason for this slight difference might lie in the difference between valuation formulas of our models. The forward price for the Schwartz one factor

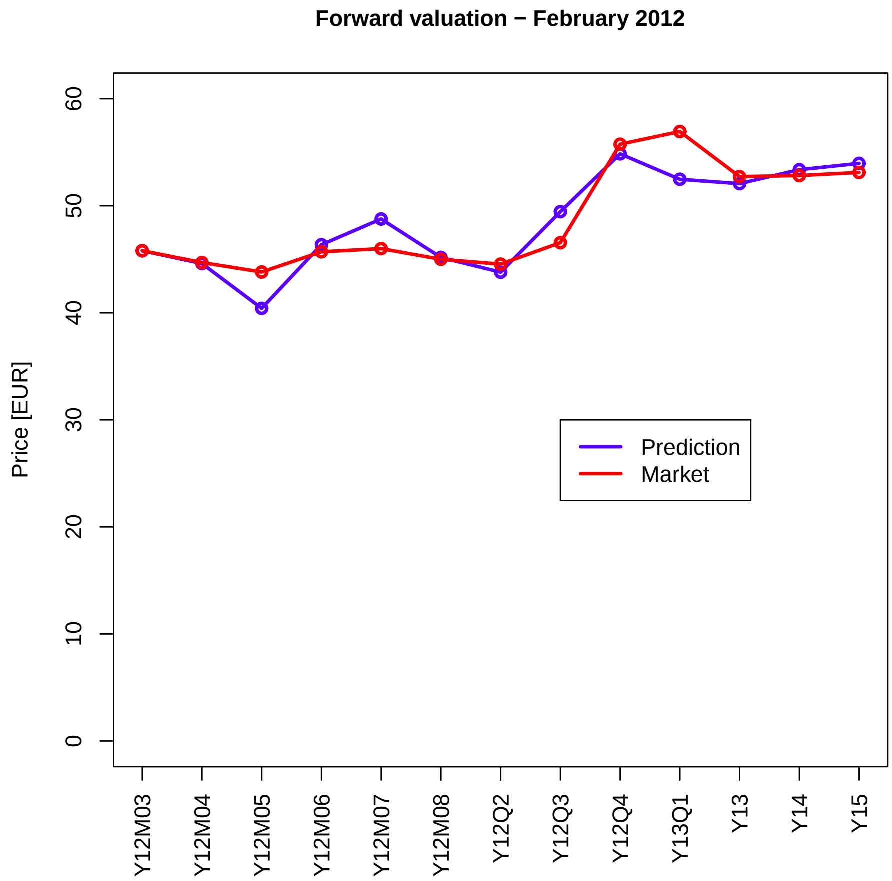

Figure 5.1: Comparison of real market prices with theoretical values provided by the Orstein-Uhlenbeck model for closing date 2/29/2012. It can be seen that theoretical values genuilly track the real term structure which is confirmed by the lowest RMSE value (1.98 EUR) of all observation days.

model depends on the variance of the error term among others while the forward price for the Ornstein-Uhlenbeck process is fully dependent on the standardized shape only. Although the Schwartz model is slightly superior, predictions obtained from the Ornstein-Uhlenbeck process seem reasonable as well. We can read that RMSE ranges from 1.98 (1.80) EUR on 2/29/2012 to 7.68 (7.41) EUR on 8/31/2011 for OU (Schwartz) process.

To visualize how for example the Ornstein-Uhlenbeck predictions fit market data we plot the best and the worst predictions (in the sense of RMSE). Figure 5.1 shows the best fit from February 2012. A first casual look reveals that the prediction genuilly tracks the real market term structure. The worst prediction from August 2011 is displayed in Figure 5.2. In this case we can observe that the relative price changes are similar for both curves but the market price level is steadily higher than predicted.

We can compute the implicit market price of risk λ, which we assume to be

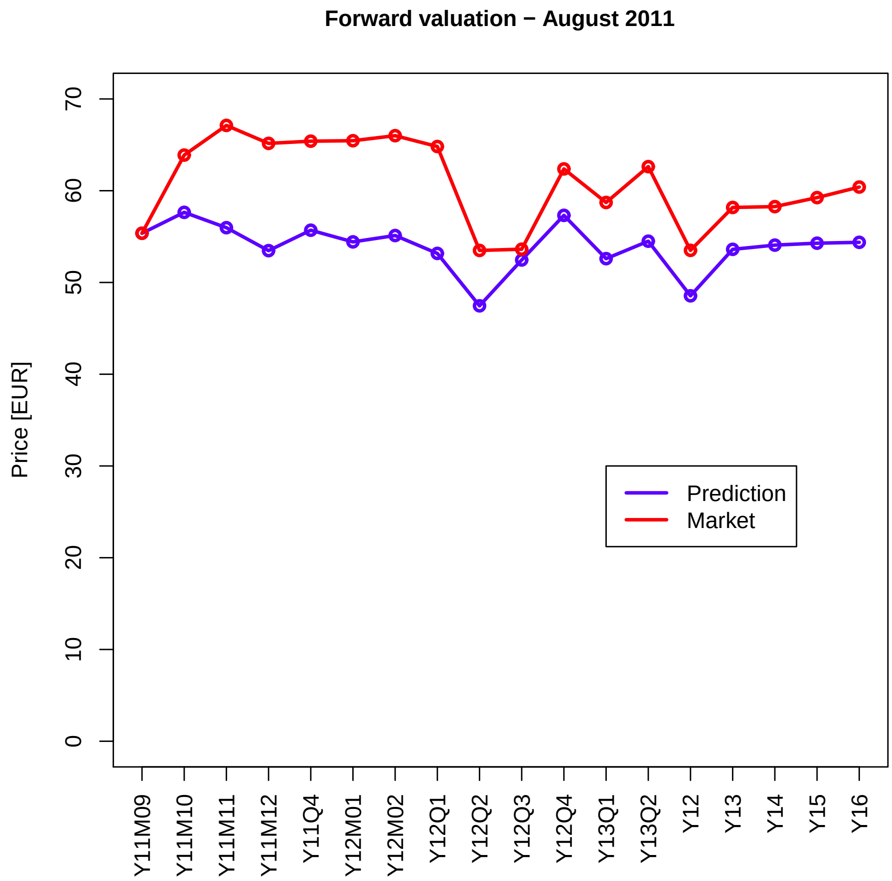

Figure 5.2: Comparison of real market prices with theoretical values provided by the Orstein-Uhlenbeck model for closing date 8/31/2011. Although both curves have similar shape, market prices are systematically above the prediction. The difference between real and theoretical price levels is captured in the highest RMSE (7.68 EUR) of all observation days.

constant, by manipulating with its value and trying to fit the predicted forward curve to the market forward curve while keeping all other parameters constant. We have chosen the root mean square error to be the measure of such closeness.

The implied market price of risk λ for the Ornstein-Uhlenbeck process obtained by minimizing sample RMSE is -0.024. The inclusion of this market price of risk in valuation formula (3.6) further decreases the sample root mean square error to 3.82. By assuming that the market price of risk for the Schwartz one factor model is -0.012 we can decrease its sample RMSE to its mimimal value 3.69. As we worked wih monthly spot prices whose value is implied by the next month forward price, the true market price of risk is partially included in the spot price.

The negative value of λ is in line with majority of power market literature and could be reasonably explained by higher incentive for hedging on the demand side relative to the supply side caused by the nonstorable nature of electricity. Similar to our findings Botterud [7] found negative market price of risk in the Nordpool market, Geman and Vasicek [18] demonstrated existence of negative risk premia in the U.S. market and Weron [38] concluded that even the market price of risk implied by Asian-style electricity options seems to be negative.

5.2 Short-term forward contracts

The same analysis as for the long-term forward contracts can be done for short-term forward contracts, i.e. for daily and weekly products. Forward quotes were taken from two data sources - Platts and Alpiq - and forward valuation is again performed by cutting products into daily granularity and subsequent averaging. As there are usually market data for only a few short-term products and it is easier to assess a daily spot price than a monthly spot price, we use directly the daily settlement price from closing dates as the reference price S in valuation formulas. One can argue that this value is not fully known at the time when exchanges close (round 5 p.m.) but we believe that due to regular daily consumption patterns the overall error is small.

Table 5.2 compares predicted values with market data under the zero market price of risk assumption. RMSE of the Ornstein-Uhlenbeck process for the whole sample is 7.93 EUR and the whole sample RMSE for the Schwartz one factor model is 6.89 EUR. This analysis suggests that the predictive power of our valuation formulas are poorer for short-term forward contracts. Explanation of higher deviation between predicted prices and the market can be twofold. First, the short-term end of the forward curve is very volatile with limited liquidity and only a few products are traded which worsenes the ability to perform comparison with our predictions. Second, the effects of current supply and demand might outweight mean-reverting nature of electricity in the short run. It might be also possible that market conditions temporarily shift the long-term price level and prices revert to this newly established equilibrium for some time.

The implied market price of risk for Ornstein-Uhlenbeck process is 0.10 and produces RMSE of 5.99 EUR, the market price of risk for Schwartz one factor model is 0.08 and produces RMSE of 5.95 EUR.

	Ornstein-Uhlenbeck		Schwartz	
Date	RMSE	MAE	RMSE	MAE
2011-01-31	9.05	8.11	8.96	7.76
2011-02-28	7.90	6.22	9.29	6.63
2011-03-31	3.18	3.15	3.66	3.38
2011-05-31	6.87	6.78	7.82	7.81
2011-06-30	3.22	2.69	5.34	4.92
2011-07-29	4.42	3.92	2.85	2.41
2011-08-31	4.24	4.16	3.14	2.75
2011-09-30	8.10	7.14	8.39	6.70
2011-10-31	7.61	7.61	6.97	6.97
2011-11-30	3.11	3.08	2.73	2.50
2011-12-30	4.99	4.78	4.69	4.38
2012-01-31	8.18	6.08	7.54	5.99
2012-02-29	3.29	3.19	4.74	3.66
2012-03-30	3.66	3.05	4.57	4.51
2012-04-30	13.11	12.08	10.33	10.31
2012-05-31	7.64	7.50	9.42	9.10
2012-06-29	5.23	4.51	6.16	5.92
2012-07-31	5.89	5.85	5.28	5.21
2012-08-31	5.73	4.58	4.52	3.97
2012-09-28	5.66	4.73	4.29	2.63

Table 5.2: Valuation errors of short-term forwards for each closing date assuming zero market price of risk. The root mean square error (RMSE) and the mean absolute error (MAE) of the difference between predicted and real market forward prices are calculated for both models. Valuation errors are higher than in the case of long-term forwards.

Results summary

In this paper we have studied the application of nonstandard methods on the valuation of electricity derivatives. Specifically, we have developed the nonstandard version of the Ornstein-Uhlenbeck and the Schwartz one factor models with deterministic seasonal component which should describe behaviour of electricity spot prices and which enabled us to value forwards on electricity in the risk-neutral world. We have included jump behaviour of electricity prices into the model via regime-switching framework and the assumption of zero risk premium for spikes. Both mean-reverting processes were calibrated to the German electricity market and predicted forward prices were compared with real market quotes.

During the course of forward price calculation we have assumed that the market price of risk is constant. There are some authors [38] who argue that this assumption is too restrictive and the market price of risk is rather an increasing function of time. It could be of course possible to model the market price of risk as a deterministic function of time $\lambda(t)$ with similar pricing formula results but we believe that the simplicity of model with constant λ outweighs possible improvement in pricing. We also need to keep in mind that such function $\lambda(t)$ would need to be specified and its coefficients would have to be estimated which could turn out to be a difficult task in the German electricity market.

Deterministic seasonal function $f(t)$ which mean-reverting processes are built around is modeled via time period indicators. The indicator approach allowed us to discover interesting patterns of electricity prices in Germany but the question is if the periodic seasonal function with no trend is appropriate representation of average future spot prices, especially in the long run. For example it could be possible to add a linear trend to the model and fit a new seasonal function with a trend to the data. We would then obtain either upward or downward sloping future spot price projection based on the historical price trend but we believe that it is problematic to forecast future price trends from historical data of the electricity market. Although electricity consumption is fairly constant, electricity generation is heavily influenced by two opposing factors. On the one side there is technology improvement either in generation efficiency or energy sources extraction which pushes price down, on the other side there are renewable energy penalties and certificate trading schemes imposed by governments which increase production costs and forces electricity prices to rise. We argue that it would be up to some fundamental analysis to assess these two factors in the past and in the future to project a long-term price level. As we perform only statistical analysis here, we believe that the assumption of no drift is appropriate.

Comparison of predicted forward prices with market data yielded interesting results. We have seen that both the Ornstein-Uhlenbeck and the Schwartz one factor model produced long maturity term structure forecast which fits market prices very well with the Schwartz model slightly outperforming OU. We have first investigated forward predictions under the assumption of zero market price of risk. After that we have included the market price of risk into the model by minimizing the root mean square error between the prediction and market prices. As an example, when we included the market price of risk in OU model, we obtained root mean square error of 3.81 EUR which we consider to be a very promising result considering that the average forward price in our sample is 53.32 EUR and prices range from 38.73 EUR to 68.05 EUR.

Both models perform poorer at forecasting short-term forward contracts relative to long-term forward contracts. This could be explained by the fact that the short-term forward market is not very liquid, it usually covers only a few products and therefore the comparison results should be treated with care. Alternative approaches to mean reversion get a lot of traction in recent years. Especially models based on ARIMA processes seem to be promising ([3], [1], [31]).

There are possibilities how to improve our stochastic models and consequently forward valuation formulas. We have treated those spot prices which we considered to be manifestation of jump behaviour relatively coarsely when we extracted only information about intensity and mean jump amplitude from the data. Many authors have focused on the modelling of electricity spikes recently ([35], [5] and [17] for example) so it would be feasible to follow one of proposed approaches and develop finer treatment of jumps. The valuation formula would then probably become more complex if even a closed form solution possible.

Another option would be to experiment with the structure of the seasonal function. Although we believe that the shape function presented in this work depicts the regular pattern of electricity spot prices best, there are certainly alternative ways how to model seasonality. We could for example use weekly indicators instead of monthly indicators to allow for finer annual shape or we could have included additional day types like certain holidays or special days with unusual characteristics into the weekly shape. Another possible strategy would be to fit the data to a prescribed functional dependence if we deviced some.

The reader should be informed about potential pitfalls connected with proposed model improvements and with the result of our model as well. Although all models which make forecast based on historical data need to consider the length of history which is relevant for the future, the issue of data relevance is especially troubling for the electricity market. Some electricity markets are relatively young (which the German electricity market is) and therefore still undergo a development phase. Also the impact of political decisions on the market functioning proved to be huge and politicians can and do influence electricity prices enourmously. For these reasons one should be very carefull about the model complexity and its data requirements because prices from not so distant past can be completely obsolete with respect to the current state of the market. Maybe here lies the curious thing why most authors study the Nord Pool market which is among the oldest and most developed electricity markets. In this thesis we have ventured to study the German electricity market which is not so developed as other markets but it is dominant in the Central Europe and particularly important for the Czech Republic.

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