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BACHELOR THESIS

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Hamiltonovské kružnice v hyperkrychlích s odstraněnými vrcholy

Katedra teoretické informatiky a matematické logiky

Supervisor of the bachelor thesis:	Mgr. Petr Gregor, Ph.D.
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Dedication. I would like to thank my supervisor Petr Gregor for his valuable comments and for his guidance throughout the world of hypercubes. I would also like to thank my family, friends and my girlfriend for their support and patience.

A special thanks deserve MetaCentrum Virtual Organization for allowing me to use their supercomputers to compute necessary results and the authors of the vector editor Ipe in which I drew the pictures.

I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Hamiltonovské kružnice v hyperkrychlích s odstraněnými vrcholy

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Abstrakt: V roce 2001 Stephen Locke [10] vyslovil hypotézu, že pro každou vyváženou množinu F obsahující $2k$ vadných vrcholů n-rozměrné hyperkrychle Q_n, kde $n \geq k+2$ a $k \geq 1$, je graf $Q_n - F$ hamiltonovský. Hypotéza je stále otevřená, byť jsou již známá částečná řešení, někdy i s různými podmínkami na F. V této práci prozkoumáme hamiltonovskost grafu $Q_n - F$, pokud množina vadných vrcholů F tvoří určitý izometrický podgraf v Q_n. Pro lichou (resp. sudou) izometrickou cestu P v Q_n je graf $Q_n - V(P)$ Hamiltonovsky laceabilní pro každé $n \geq 4$ (resp. $n \geq 5$). Přestože je znám silnější výsledek [15], metoda důkazu nám umožnila získat následující výsledky. Nechť C je izometrický cyklus v Q_n délky dělitelné čtyřmi pro $n \geq 6$. Pak je graf $Q_n - V(C)$ Hamiltonovsky laceabilní. Buď T izometrický strom v Q_n s lichým počtem hran a S izometrický strom v Q_m se sudým počtem hran. Pak pro každé $n \geq 4$, $m \geq 5$ jsou grafy $Q_n - T$ a $Q_m - S$ Hamiltonovsky laceabilní. Část důkazu je ověřena počítačem.

Klíčová slova: hyperkrychle, vadný vrchol, Hamiltonovská laceabilita, izometrický cyklus, izometrický strom

Title: Hamiltonian cycles in hypercubes with removed vertices

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Abstract: In 2001 Stephen Locke [10] conjectured that for every balanced set F of $2k$ faulty vertices in the n-dimensional hypercube Q_n where $n \geq k+2$ and $k \geq 1$ the graph $Q_n - F$ is hamiltonian. So far the conjecture remains open although partial results are known; some of them with additional conditions on the set F. We explore hamiltonicity of $Q_n - F$ if the set of faulty vertices F forms certain isometric subgraph in Q_n. For an odd (even) isometric path P in Q_n the graph $Q_n - V(P)$ is Hamiltonian laceable for every $n \geq 4$ (resp. $n \geq 5$). Although a stronger result is known [15], the method we use in proving the theorem allows us to obtain following results. Let C be an isometric cycle in Q_n of length divisible by four for $n \geq 6$. Then the graph $Q_n - V(C)$ is Hamiltonian laceable. Let T be an isometric tree in Q_n with odd number of edges and let S be an isometric tree in Q_m with even number of edges. For every $n \geq 4$, $m \geq 5$ the graphs $Q_n - T$ and $Q_m - S$ are Hamiltonian laceable. A part of the proof is verified by a computer.

Keywords: hypercube, fault tolerance, Hamiltonian laceability, isometric cycle, isometric tree

Contents

Introduction

Hypercubes have long been studied for their importance at parallel architectures [1] [2]. The vertices of a hypercube represent processors and edges represent links between them. Hypercubes can even simulate other networks, like trees or arrays. Their small diameter (compared to the number of vertices), edge and vertex-symmetry, a recursive structure and other properties have made them a popular model for interconnecting networks.

After several theoretical proposals, the first hypercube based computer called the Cosmic Cube was built in 1983 in Caltech. It had 64 processors that were connected to a 6-dimensional hypercube topology. Few years later, Intel started offering hypercube computers commercially. Some other attempts (let us mention the nCUBE architecture) were made to create successful hypercube computers. But due to their bad scalability (the degree of a hypercube grows logarithmically) hypercube computers were abandoned. More on this topic can be find in various books regarding parallel architecture and development of parallel algorithms [2]. Nowadays hypercube networks are still widely used and researched. Let us mention a P2P network HyperCuP [3] and a bluetooth network BlueCube [4]. In 2011 a hypercube topology for dynamic distributed databases called HyperD [5] was introduced.

A (cyclic) *binary Gray code* of dimension n is a sequence of binary strings of length n such that two consecutive strings differ in precisely one coordinate. It is easy to see that Gray codes correspond to Hamiltonian cycles in hypercubes. Gray codes were patented in 1953 by Frank Gray [6], a researcher at Bell Labs. Binary Gray codes and their variations also appear in surprising places such as solutions to puzzles like Tower of Hanoi or Chinese ring puzzle, signal encoding, data compression, graphic and image processing, hashing and many more [7].

As processors need to communicate with each other, a routing problem arises. That is, delivering a packet between two processors (one-to-one model) from one processor to more than one destinations (one-to-many model) and from many processors to a common destination (many-to-one model). Since a processor may fail, it is a subject of study on how many faulty vertices can hypercube tolerate and what conditions on the set of faulty vertices can increase the number of tolerable faults.

In this thesis we explore an existence of a Hamiltonian path in $Q_n - F$ for a set of faulty vertices $F \subseteq V(Q_n)$ that forms a certain isometric subgraph in Q_n. In the first chapter such isometric subgraph is a path. We show that if P is an isometric path in Q_n of odd (even) length and $n \geq 4$ ($n \geq 5$), then the graph $Q_n - P$ is Hamiltonian laceable. Although stronger result by Sun and Jou [15] is known, the proof of this theorem demonstrates method used in following two chapters.

In the second chapter we prove that if C is an isometric cycle in Q_n of length divisible by four and $n \geq 6$, then $Q_n - C$ is Hamiltonian laceable. This allows us to remove up to $2n$ faulty vertices. In the third chapter we consider isometric trees.

Let T be an isometric tree in Q_n where $n \geq 4$ that has the same number of vertices in each bipartite class. Let S be an isometric tree in Q_m where $m \geq$

5 with one vertex more in one bipartite class than the other. We show that the graphs $Q_n - T$ and $Q_m - S$ are Hamiltonian laceable. This was proved by induction. The part of the base of the induction, namely Lemma 21 stating that $Q_5 - T$ is Hamiltonian laceable where T is an isometric tree on Figure 5.2, was verified by a computer. The program that verified the lemma, and its output are discussed in Attachments.

Finally, we conjecture that if C is an isometric cycle in Q_n of length not divisible by four and $n \geq 6$, then $Q_n - C$ is Hamiltonian laceable.

1. Preliminaries

I assume that the reader has a basic knowledge of graph theory. For terminology, I recommend reading the chapter on graphs in the excellent book by Matoušek and Nešetřil [8]. Nevertheless, I would like to remind some standard notation that I will use.

1.1 Basic graph theory notation

All graphs that appear in this thesis are undirected. For a graph G let $V(G)$ and $E(G)$ stand for a vertex and an edge set of G, respectively. We require that graphs have at least one vertex. For $u, v \in V(G)$ let $d(u, v)$ denote the length of a shortest path between u and v in G.

Let $U \subseteq V(G)$, $F \subseteq E(G)$ and H be a subgraph of G. We say that H is an *induced subgraph* of G by a set of vertices U if $H = (U, E(G) \cap \binom{U}{2})$. Let $G - U$ denote the subgraph of G induced by $V(G) \setminus U$, let $G - F$ denote the graph $(V(G), E(G) \setminus F)$ and let $G - H$ denote the graph $(G - V(H)) - E(H)$. The *Cartesian product* $R \,\square\, S$ of graphs R and S is the graph with the vertex set $V(R \,\square\, S) = V(R) \times V(S)$ and the edge set

$$E(R \,\square\, S) = \{\, (u, v)(u', v) \mid uu' \in E(R)\} \cup \{\, (u, v)(u, v') \mid vv' \in E(S)\} \,.$$

Let $[n] = \{1, 2, \ldots, n\}$ and $[n]_0 = [n] \cup \{0\}$ where n is a positive integer. We now introduce some notation to describe paths and cycles in graphs.

A sequence $(v_0, v_1, \ldots, v_k)$ is a *path* in a graph G (sometimes denoted by P_{k+1}) if $v_i \in V(G)$ for all $i \in [k]_0$ and $v_{j-1}v_j \in E(G)$ for all $j \in [k]$. Let $P = (v_0, v_1, \ldots, v_k)$ be a path in a graph G. We call v_0 and v_k *endvertices* of P. For $j \in [k]_0$ and $i \in [j]_0$ the path $(v_i, v_{i+1}, \ldots, v_{j-1}, v_j)$ is a *subpath* of P. We can connect two subpaths R and S of a path P if there exists an edge $uv \in E(P)$ such that u is an endvertex of R and v is an endvertex of S. A sequence $(v_0, v_1, \ldots, v_{k-1})$ is a *cycle* in a graph G (sometimes denoted by C_k) if $v_i \in V(G)$ for all $i \in [k-1]_0$ and $v_{j-1}v_j \in E(G)$ for all $j \in [k-1]$ and $v_{k-1}v_0 \in E(G)$.

Sometimes we consider paths and cycles to be graphs instead of sequences of vertices. This slight abuse of notation should not cause any confusion. For example, it allows us to easily define subpaths of any graph.

1.2 Hypercubes

Let $\mathbb{Z}_2^n$ denote the group of n-dimensional vectors of zeroes and ones. The neutral element is the vector $(0, 0, \ldots, 0)$ of length n. Addition of two vectors of $\mathbb{Z}_2^n$ is addition of their coordinates modulo 2 and we denote it by $\oplus$. For $i \in [n]$ let e_i denote the vector in $\mathbb{Z}_2^n$ with one in the ith coordinate and zeroes in the rest of the coordinates.

Definition 1 *For $n \geq 0$ the n-dimensional hypercube Q_n is an undirected graph with $V(Q_n) = \{0, 1\}^n$ and*

$$E(Q_n) = \{\, uv \mid u \oplus v = e_i \text{ for some } i \in [n]\} \,.$$

Since hypercubes are the main topic of this thesis, we will explore them more. Sometimes the n-dimensional hypercube Q_n is called n-cube, a Boolean cube or a discrete cube. Nevertheless, we will always refer to it as the (n-dimensional) hypercube. From now on let n denote the dimension of the hypercube Q_n. Unless it is specified otherwise in the text, we assume $n \geq 1$. We will not consider the hypercube Q_0 in this thesis, since it is a single vertex, which is not interesting.

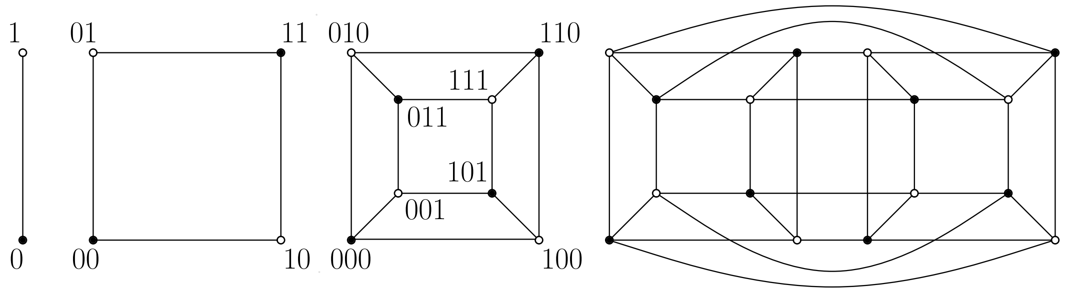

Figure 1.1: Examples of hypercubes Q_n for $n = 1, 2, 3, 4$.

To measure the distance between two vertices of the hypercube, we define the *Hamming distance* of $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n) \in \mathbb{Z}_2^n$ by

$$d_H(u, v) = |\{i \in [n] \mid u_i \neq v_i\}| .$$

Notice that $uv \in E(Q_n)$ if and only if the Hamming distance of vertices u and v is one. That is,

$$E(Q_n) = \{uv \mid d_H(u, v) = 1\} .$$

This means that two vertices of the hypercube that are connected with an edge differ in exactly one coordinate which can be attributed to this edge. If $u \oplus v = e_i$, the edge $uv \in E(Q_n)$ is said to have the *direction* $i \in [n]$. Each vertex $u \in V(Q_n)$ has degree n and each edge in the set $\{ux \in E(Q_n) \mid x \in V(Q_n)\}$ has a different direction that ranges from 1 to n, see Figure 1.2.

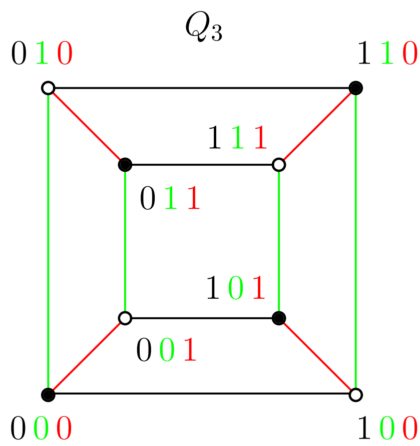

Figure 1.2: Edges of Q_3 with directions 1 (black), 2 (green) and 3 (red).

We define the *size* $|u|$ of a vertex $u \in V(Q_n)$ by number of one's in u. That is, $|u| = |\{i \in [n] \mid u_i = 1\}|$. With this property in mind, we can now distinguish two disjoint sets of vertices of the hypercube. The first set is the set of vertices of Q_n with even size. We call those vertices *black*, denote this set B_n and we will draw them as black dots (i.e. $\bullet$) in pictures. Analogously, we call the vertices of Q_n with odd size *white*, we denote their set by W_n and draw them as white circles (i.e. $\circ$) in pictures. Notice that $B_n \cap W_n = \emptyset$ and $B_n \cup W_n = V(Q_n)$. Since every edge of the hypercube connects a white vertex with a black one, it follows that Q_n is a bipartite graph with bipartition B_n, W_n. We say that $U \subseteq V(Q_n)$ is *balanced* if U has the same number of black and white vertices.

From the definition of the vertex set of the hypercube it is easy to see that $|V(Q_n)| = 2^n$. On the other hand, the number of edges of Q_n is $|E(Q_n)| = n2^{n-1}$. The edges of each direction form a perfect matching. Each direction splits Q_n into two copies of Q_{n-1}. For a detailed explanation on partitioning the hypercube into two smaller copies, see Section 1.2.1 below. Because Q_n has precisely n directions and Q_{n-1} has 2^{n-1} vertices, the number of edges of Q_n is $n2^{n-1}$.

1.2.1 Subcubes of hypercubes

Let us now focus on subcubes of the hypercube Q_n. A k-dimensional subcube is a subgraph isomorphic to Q_k. We show how to represent k-dimensional subcubes of Q_n by strings $\{0, 1, *\}^n$. Let $w \in \{0, 1, *\}^n$ and let us denote the number of $*$ in w by k. Then w uniquely represents a k-dimensional subcube (denoted by $Q_n[w]$) induced by the vertex set

$$\{(v_1, \dots, v_n) \in \mathbb{Z}_2^n \mid v_i = w_i \text{ if } w_i \neq *\}.$$

That is, $n-k$ coordinates of the vertices of Q_n are fixed and the rest k coordinates are every combination of zeroes and ones of length k, see Figure 1.3 for some examples of subcubes of Q_3.

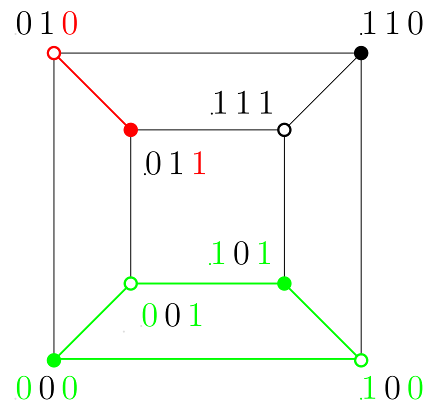

Figure 1.3: A 1-dimensional subcube $Q_3[01*]$ (red) and a 2-dimensional subcube $Q_3[*0*]$ (green).

Every hypercube is a Cartesian product of two hypercubes of smaller dimension. Precisely, $Q_{n+m} \simeq Q_n \square Q_m$ for every $n, m \geq 1$. One way to look at this formula is to imagine that we replace each vertex $u \in V(Q_n)$ with an m-dimensional hypercube denoted by Q_m^u. We then replace each edge $uv \in E(Q_n)$ with 2^m edges to connect two cubes Q_m^u and Q_m^v, see Figure 1.4 for an example.

Often we will use a special case of this property $Q_n \simeq Q_{n-1} \square Q_1$ to partition Q_n into two copies $Q^{0,i}_{n-1}$ and $Q^{1,i}_{n-1}$ where $i \in [n]$ is a direction of edges removed in order to cut Q_n into two parts. Formally, for $i \in [n]$ we define

$$Q^{0,i}_{n-1} = Q_n[w] \text{ where } w = (w_1, \dots, w_n) \in \{0, *\}^n \text{ with } w_i = 0 \text{ and } w_j = * \text{ for } j \neq i,$$
$$Q^{1,i}_{n-1} = Q_n[w] \text{ where } w = (w_1, \dots, w_n) \in \{1, *\}^n \text{ with } w_i = 1 \text{ and } w_j = * \text{ for } j \neq i.$$

If i is fixed we usually omit it and refer to these two subcubes simply as Q^0_{n-1} and Q^1_{n-1}.

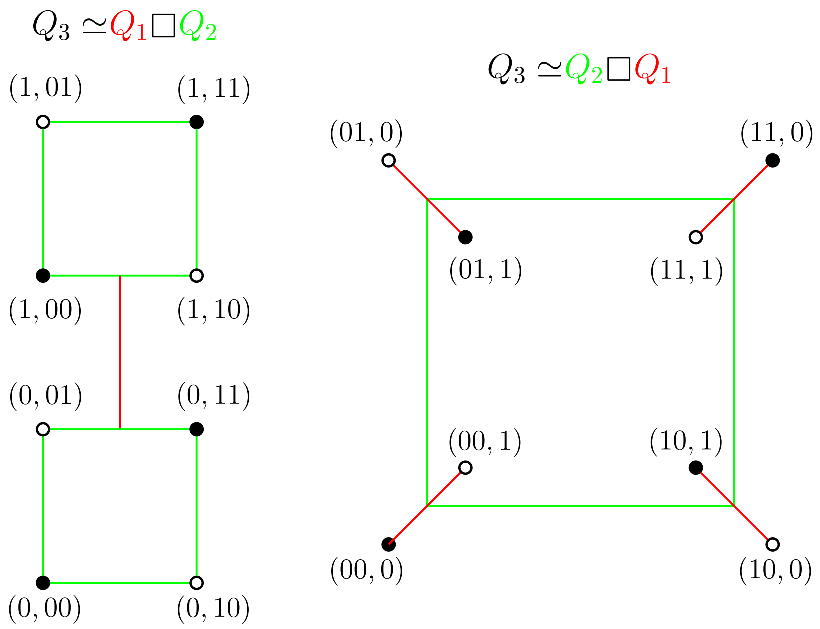

Figure 1.4: A scheme of Cartesian products $Q_1 \square Q_2$ (left) and $Q_2 \square Q_1$ (right). Red edges correspond to the edges of Q_1 and green edges correspond to the edges of Q_2.

1.2.2 Isometric subgraphs of hypercubes

We have defined everything necessary to draw our attention to the key topic of this thesis, isometric subgraphs of Q_n.

Definition 2 *A subgraph H of G is isometric if $d_H(u, v) = d_G(u, v)$ for every $u, v \in V(H)$.*

That is, the subgraph H preserves all distances from G. It is easy to see that every isometric subgraph is an induced subgraph, but the converse implication is not true. A counterexample to the converse is a path P_4 in the cycle C_5, see Figure 1.5. Three isometric subgraphs of Q_n will be important for us, an isometric path, an isometric cycle and an isometric tree. We will introduce and explore each of them in Chapter 3, Chapter 4 and Chapter 5, respectively.

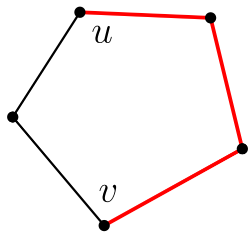

Figure 1.5: A path P_4 as an induced subgraph of C_5, that is not isometric.

1.3 Hamiltonicity

Let G be a graph and let u, v be its two distinct vertices. A *Hamiltonian cycle* in G is a cycle in G that contains every vertex of G. A *Hamiltonian path* between u and v in G is a path in G with its endvertices u and v that contains every vertex of G exactly once. We say that G is *hamiltonian* if it has a Hamiltonian cycle.

Let H be a bipartite graph with bipartitions U and V. We say that H is *balanced* if $|U| = |V|$ and we say it is *nearly balanced* if sizes of U and V differ by one.

Definition 3 *A bipartite graph H with its bipartition U, V is* Hamiltonian laceable *if*
(a) H is balanced and there exists a Hamiltonian path between every $u \in U$ and every $v \in V$, or
(b) H is nearly balanced with U its larger bipartite set and there exists a Hamiltonian path between every two distinct $u, u' \in U$.

Note that Hamiltonian laceability of balanced bipartite graphs implies hamiltonicity.

2. Previous results

It is well known that Q_n contains a Hamiltonian cycle for every $n \geq 2$ [9]. If we removed some vertices of Q_n (such vertices are often called *faulty vertices*), would it still contain a Hamiltonian cycle or a Hamiltonian path?

2.1 Locke's conjecture

Let $F \subseteq V(Q_n)$ be a balanced set of $2k$ vertices for $k \geq 1$. In 2001 Stephen Locke [10] conjectured an existence of a Hamiltonian cycle in $Q_n - F$ if $n \geq k+2$. Two years later, Richard Stong [10] solved it for $k = 1$ and published it in the American Monthly. The journal claims that Stong also sent the proof for $n \geq 2k+3\log_2 k+4$, but they have not published it. Next step in proving the conjecture was done by Harborth and Kemnitz [11], proving it for $k = 2, 3$ and by Dvořák and Gregor [12], proving it for $k \leq \frac{n-5}{6}$. So far, Locke's conjecture remains unproven. Yet in 2009, Gotchev and Castañeda claimed to have proved this conjecture and submitted the first part of the proof, which has not been published, but can be found on Prof. Gotchev's website [13]. There is also mentioned that they are (together with F. Latour) working on the second part of the proof.

2.2 Hypercubes with faulty vertices

Although above conjecture remains unproven, various results can be achieved with additional conditions on the set of faulty vertices $F \subseteq V(Q_n)$. One example of such condition is the minimum Hamming distance between every two vertices of F, which we denote by $d(F)$. Similarly, if $M \subseteq E(Q_n)$ is a set of edges, then $d(M)$ denotes the minimum distance between every two edges of M. The following theorem and two conjectures are work of Gregor and Škrekovski.

Theorem 4 (Gregor and Škrekovski [14]) *Let M be a set of edges of Q_{n+2} ($n \geq 1$) with $d(M) \geq 3$. Then $Q_{n+2} - V(M)$ contains a Hamiltonian cycle.*

Conjecture 5 (Gregor and Škrekovski [14]) *Let F be a balanced set of vertices of Q_n with $d(F) \geq 3$. Then $Q_n - F$ contains a Hamiltonian cycle.*

Conjecture 6 (Gregor and Škrekovski [14]) *Let A and B be equal-sized sets of black and white (respectively) vertices of Q_n with $d(A) \geq 4$ and $d(B) \geq 4$. Then $Q_n - (A \cup B)$ contains a Hamiltonian cycle.*

Another example of a condition on the set of faulty vertices F is that they form some special graph in Q_n. Sun and Jou [15] proved that if faulty vertices F form a path or a cycle in Q_n for $n \geq 4$ and $|F| \leq 2n-4$, then $Q_n - F$ is Hamiltonian laceable. Furthermore, they showed that the bound $|F| \leq 2n-4$ is tight.

Here we present three propositions that we will often use.

Proposition 7 (Havel [9]) *For every $n \geq 2$ there exists a Hamiltonian path between every $b \in B_n$ and every $w \in W_n$ in Q_n.*

Proposition 8 (Lewinter-Widulski [16]) *For every $n \geq 2$ and for every $b \in B_n$ there exist a Hamiltonian path between every two distinct vertices $w, w' \in W_n$ in $Q_n - \{b\}$.*

Proposition 9 (Dvořák-Gregor [17]) *Let P be a set of at most $2n-4$ prescribed edges in Q_n and $n \geq 5$. There exists a Hamiltonian path between every $b \in B_n$ and every $w \in W_n$ in Q_n which passes through all edges of P if and only if the subgraph induced by P consists of pairwise vertex-disjoint paths, none of them having b or w as internal vertices, or both of them as endvertices.*

3. Isometric paths in hypercubes

In this chapter we explore properties of isometric paths in hypercubes regarding directions of edges. Then we prove for large enough n and an isometric path P in Q_n that every $Q_n - P$ is Hamiltonian laceable. Finally, we present examples to show that this result does not hold for certain small dimensions of Q_n.

Observation 10 *Edges of an isometric path in Q_n have all directions distinct.*

Proof Case $n = 1$ is trivial. Let $n \geq 2$, we prove the observation by contradiction. Let R be an isometric path in Q_n such that two distinct edges of R have an identical direction. Let the path $S = (v_0,\ v_1, \dots, v_k,\ v_{k+1})$ of length $k + 1$ denote the shortest subpath of R such that two distinct edges of S have identical direction. It is easy to see that those edges are v_0v_1 and v_kv_{k+1}.

Let $C = (v_0,\ v_1, \dots, v_k,\ v_{k+1},\ v_{k+2}, \dots, v_{2k-1})$ be a cycle in Q_n such that opposite edges have identical directions. That is, edges v_jv_{j+1} and $v_{j+k}v_{j+k+1}$ where $j \in [k-1]_0$ have identical direction (we assume that $v_{2k} = v_0$). Note that S is a subpath of C. Then the subpath $(v_{k+1},\ v_{k+1}, \dots .v_{2k-1},\ v_0)$ of C between $v_{k+1}v_0$ of length $k-1$ is shorter than the path S between v_0 and v_{k+1} of length $k+1$ which is a contradiction. ■

Since Q_n has n directions, it follows that every isometric path in Q_n has at most n edges. Let P be an isometric path of length k in Q_n and let i be a direction of an edge $uv \in E(P)$. We partition Q_n into $Q^{0,i}_{n-1}$ and $Q^{1,i}_{n-1}$. Notice that P splits into two subpaths P^0 and P^1 connected by the edge uv. The subpath P^0 is in $Q^{0,i}_{n-1}$ and subpath P^1 is in $Q^{1,i}_{n-1}$.

3.1 Avoiding an isometric path by a Hamiltonian path

We start by proving a simple lemma stating that for every $n \geq 2$ the graph Q_n without two adjacent vertices is Hamiltonian laceable. Note that Richard Stong [10] already proved that for every $n \geq 3$ the graph Q_n without two vertices of opposite parity is hamiltonian.

Lemma 11 *Let $b_0w_1 \in E(Q_n)$ where $n \geq 2$, $b_0 \in B_n$ and $w_1 \in W_n$. For every $b \in B_n \backslash \{b_0\}, w \in W_n \backslash \{w_1\}$ there exists a Hamiltonian path in $Q_n - \{b_0, w_1\}$.*

Proof We prove the statement by induction on n. For $n = 2$ the edge bw forms a Hamiltonian path in $Q_n - \{b_0w_1\}$ so it trivially holds. Assuming the lemma is true for $n-1$ we prove it for n. We choose $d \in B_n$, such that it is a neighbor of w_1 different from b_0. We denote the direction of the edge w_1d by i. Then $\{b_0w_1\} \in E(Q^{i,0}_{n-1})$ and we fix the direction i. There are two cases depending on whether the vertices b and w are in the same $(n-1)$-dimensional subcube of Q_n.

Case 1: *The vertices b, w are either both in Q^0_{n-1} or both in Q^1_{n-1}; say $b, w \in$* $V(Q^0_{n-1})$. By the induction hypothesis, there exists a Hamiltonian path H between b and w in $Q^0_{n-1} - \{b_0, w_1\}$. Let uv be an arbitrary edge of H, $u \in B_n$

and $v \in W_n$. We denote the neighbors of u, v in Q^1_{n-1} by x, y, respectively. By Proposition 7, there exists a Hamiltonian path between x and y in Q^1_{n-1}. We join this path with the path H by removing the edge uv and adding the edges ux and vy. That is, we obtain a Hamiltonian path $(b, \ldots, u, x, \ldots, y, v, \ldots, w)$ in $Q_n - \{b_0, w_1\}$.

Case 2: *The vertices b, w are in different subcubes; say $b \in V(Q^0_{n-1})$ and $w \in V(Q^1_{n-1})$.* We choose an edge $uv \in E(Q_n)$ such that $u \in V(Q^0_{n-1}$ is white and $v \in V(Q^1_{n-1}$ is black. By the induction hypothesis, there exists a Hamiltonian path between b and u in Q^0_{n-1} and by Proposition 7, there exists a Hamiltonian path between v and w in Q^1_{n-1}. By joining these paths, we obtain a Hamiltonian path $(b, \ldots, u, v, \ldots, w)$ in $Q_n - \{b_0, w_1\}$. ■

Now we prove the main theorem of this chapter stating that if P is an isometric path in Q_n of odd (even) length, then for every $n \geq 4$ ($n \geq 5$) the graph $Q_n - P$ is Hamiltonian laceable. This result is divided into two theorems. Theorem 12 deals with isometric paths of odd lengths and Theorem 13 deals with isometric paths of even lengths. It is clear that Lemma 11, which we will use in the proof of the following theorem, is just a special case when $|P| = 1$.

Note that far stronger result is known. Sun and Jou [15] showed that for every $n \geq 4$ and an arbitrary path R in Q_n of length at most $2n-4$ the graph $Q_n - R$ is Hamiltonian laceable. They also showed that the bound $2n-4$ on the length of faulty path is tight. Still, our theorem has its place here. The method we use for proving it will be reused later on, achieving new results. I believe it is better to illustrate the use of this method on simpler graphs such as paths before we use it for more complicated graphs.

Theorem 12 *Let $P = (b_0, w_1, b_2, w_3, \ldots, b_{k-1}, w_k)$ be an isometric path of odd length k in Q_n where $1 \leq k \leq n$, $n \geq 4$, every b_i is in B_n and every w_i is in W_n. There exists a Hamiltonian path between every $b \in B_n \setminus V(P)$ and every $w \in W_n \setminus V(P)$ in $Q_n - P$.*

Proof If $k = 1$ we use Lemma 11. Assuming that $k \neq 1$, we prove the theorem by induction on n. The proof of the base case is very similar to the proof of the induction step. To avoid repeating some parts, we first prove the induction step and then we prove the base case for $n = 4$.

Assuming the theorem holds for $n - 1$ we prove it for n. Let

$$l = \begin{cases} \frac{k-1}{2} & \text{if } k \equiv 3 \bmod 4 \\ \frac{k-1}{2} - 1 & \text{if } k \equiv 1 \bmod 4 \end{cases}.$$

We denote the direction of the edge $w_l b_{l+1}$ by i. Then the subpaths $P^0 = (b_0, w_1, \ldots, w_l)$ and $P^1 = (b_{l+1}, w_{l+2}, \ldots, w_k)$ are in $Q^{i,0}_{n-1}$ and $Q^{i,1}_{n-1}$, respectively. Since l and $k - (l+1)$ are odd, both subpaths P^0 and P^1 have odd length. After fixing the direction i, there are two cases to consider.

Case 1: *The vertices b, w are in different subcubes Q^0_{n-1}, Q^1_{n-1}; say $b \in V(Q^0_{n-1})$ and $w \in V(Q^1_{n-1})$, see Figure 3.1.* We choose an edge uv of direction i such that $u \in W_n \setminus V(P^0)$ is in Q^0_{n-1} and $v \in B_n \setminus V(P^1)$ is in Q^1_{n-1}. Since the number of edges of direction i in $Q_n - P$ is for $n \geq 4$ at least $2^{n-1} - k \geq 2^{n-1} - n \geq 1$, such edge exists.

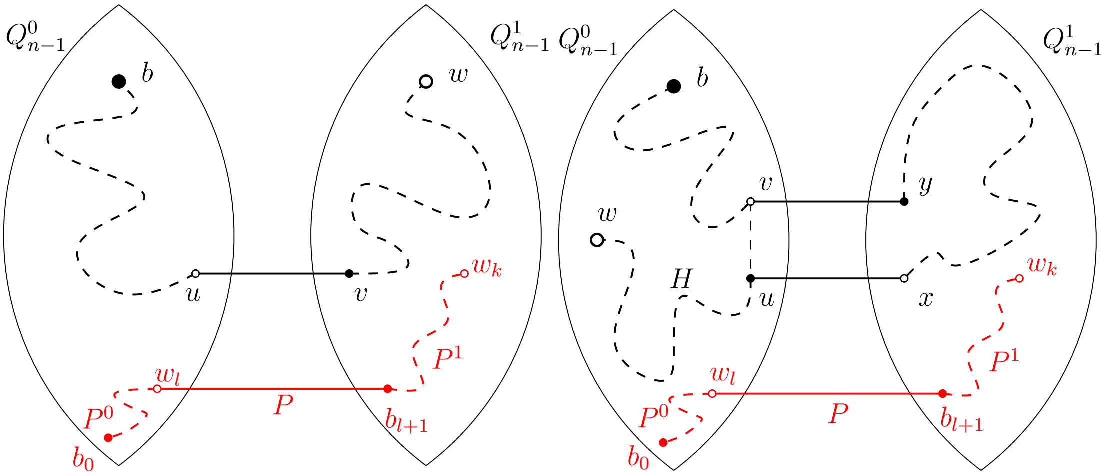

Figure 3.1: Cases 1 and 2 in the proof of Theorem 12.

By the induction hypothesis, there exists a Hamiltonian path between b and u in $Q^0_{n-1} - P^0$ and a Hamiltonian path between v and w in $Q^1_{n-1} - P^1$. By joining these paths we obtain a Hamiltonian path $(b, \ldots, u, v, \ldots, w)$ in $Q_n - P$.

Case 2: *The vertices b, w are in the same subcube; say in Q^0_{n-1}, see Figure 3.1.* By the induction hypothesis, there exists a Hamiltonian path H between b and w in $Q^0_{n-1} - P^0$. Let uv be an edge of H, $u \in B_n$ and $v \in W_n$ such that the neighbors of u, v in Q^1_{n-1} do not belong to P^1. We claim that such edge $uv \in E(H)$ exists.

We say that a vertex $p \in V(P^1)$ blocks an edge $rs \in E(H)$ if p is a neighbor of r or s in Q^1_{n-1}. Each vertex in the path P^1 blocks at most two edges of the path H. Remember that $|P^1| \leq \frac{n+1}{2}$. Since $|H| \geq 2^{n-1} - \frac{n-1}{2} - 1$, the number of edges which are not blocked by the path P^1 is for $n \geq 5$ at least

$$2^{n-1} - \frac{n-1}{2} - 1 - 2\left(\frac{n+1}{2} + 1\right) \geq \frac{2^n - 3n - 7}{2} \geq 1. \tag{3.1}$$

Therefore, uv exists and we denote neighbors of u and v in Q^1_{n-1} by x and y, respectively.

By the induction hypothesis, there exists a Hamiltonian path between x and y in $Q^1_{n-1} - P^1$. We join this path with the path H by removing the edge uv and adding the edges ux and vy. That is, we obtain a Hamiltonian path $(b, \ldots, v, y, \ldots, x, u, \ldots, w)$ in $Q_n - P$.

It remains to prove the base case for $n = 4$. Since $k \neq 1$ the only possible length of the path P is $k = 3$. The proof is analogous to the proof of the induction step for $n > 4$ above. Instead of using an induction hypothesis to find a Hamiltonian path in Q_3, we use Lemma 11 since $l = 1$. The only problem is that Equation (3.1) does not hold for $n = 4$. This is easily fixable if we precisely compute the number of edges that are not blocked by the path $P^1 = (b_2, w_3)$ instead of using a rough estimate. Since $|H| = 2^3 - 3$ and b_2 does not block any edge of H, we have that the number of edges in Q_4 that are not blocked by w_3 is $2^3 - 3 - 2 \cdot 1 = 3$. Thus the base of induction holds as well. ■

Theorem 13 *Let $P = (b_0, w_1, b_2, w_3, \ldots, w_{k-1}, b_k)$ be an isometric path of even length k in Q_n, where $2 \leq k \leq n$, $n \geq 5$, every b_i is in B_n and every w_i is in W_n.*

For every distinct $w, w' \in W_n \setminus V(P)$ there exists a Hamiltonian path between w and w' in $Q_n - P$.

Proof We denote the direction of the edge b_0w_1 by i. Then the vertex b_0 is in $Q_{n-1}^{i,0}$ and the subpath $P^1 = (w_1, b_2, \ldots, b_k)$ of the path P is in $Q_{n-1}^{i,1}$. After fixing the direction i, there are three cases to consider.

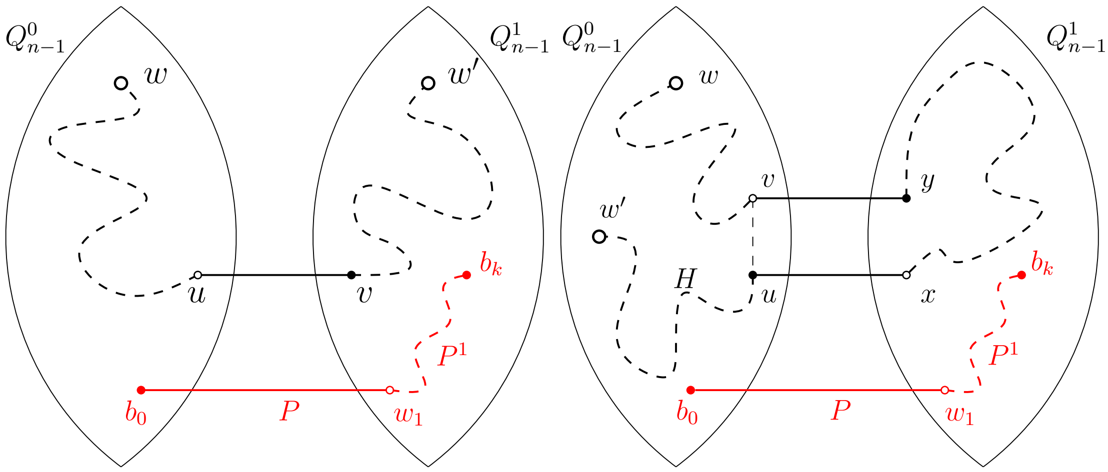

Figure 3.2: Cases 1 and 2 in the proof of Theorem 13.

Case 1: *The vertices w, w' are in different subcubes Q_{n-1}^0, Q_{n-1}^1; say $w \in V(Q_{n-1}^0)$ and $w' \in V(Q_{n-1}^1)$, see Figure 3.2.* We choose an edge uv of direction i distinct from b_0w_1 such that $u \in W_n$ is in Q_{n-1}^0 (and $v \in B_n$ in Q_{n-1}^1) and v does not belong to P^1. Since the number of edges of direction i in $Q_n - P$ is for $n \geq 5$ at least $2^{n-1} - k \geq 2^{n-1} - n \geq 1$, such edge exists. By Proposition 8, there exists a Hamiltonian path between w and u in $Q_{n-1}^0 - \{b_0\}$ and by Theorem 12, there exists a Hamiltonian path between v and w' in $Q_{n-1}^1 - P^1$. By joining these paths we obtain a Hamiltonian path $(w, \ldots, u, v, \ldots, w')$ in $Q_n - P$.

Case 2: *The vertices w, w' are in the subcube Q_{n-1}^0, see Figure 3.2.* By Proposition 8, there exists a Hamiltonian path H between w and w' in $Q_{n-1}^0 - \{b_0\}$. Let uv be an edge of H, $u \in B_n$ and $v \in W_n$ such that the neighbors of u, v in Q_{n-1}^1 do not belong to P^1. We claim that such edge $uv \in E(H)$ exists.

We say that a vertex $p \in V(P^1)$ blocks an edge $rs \in E(H)$ if p is a neighbor of r or s in Q_{n-1}^1. Each vertex in the path P^1 blocks at most two edges of the path H. Since $|H| = 2^{n-1} - 1$, the number of edges which are not blocked by the path P^1 is for $n \geq 5$ at least $2^{n-1} - 1 - 2(n-1) \geq 2^{n-1} - 2n + 1 \geq 1$. Therefore, uv exists and we denote neighbors of u and v in Q_{n-1}^1 by x and y, respectively.

By Theorem 12, there exists a Hamiltonian path between x and y in $Q_{n-1}^1 - P^1$. We join this path with the path H by removing the edge uv and adding the edges ux and vy. That is, we obtain a Hamiltonian path $(w, \ldots, v, y, \ldots, x, u, \ldots, w')$ in $Q_n - P$.

Case 3: *The vertices w, w' are in the subcube Q_{n-1}^1, see Figure 3.3.* We choose a vertex $u \in B_n$ in Q_{n-1}^1 such that $u \notin V(P^1)$. By Theorem 12, there exists a Hamiltonian path H between w and u in $Q_{n-1}^1 - P^1$. Let yw' denote the edge of H such that $y \in B_n$ is closer to w than to u on the path H. Let $x, v \in W_n$ be the neighbors of y, u in Q_{n-1}^0, respectively. By Proposition 8, there exists a Hamiltonian path between v and x in $Q_{n-1}^0 - \{b_0\}$. We join this path with

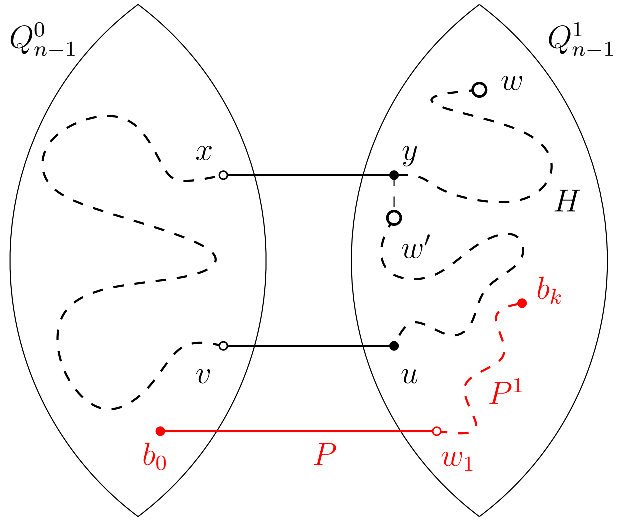

Figure 3.3: Case 3 in the proof of Theorem 13.

the path H by removing the edge yw' and adding the edges uv and xy. That is, we obtain a Hamiltonian path $(w, \dots, y, x, \dots, v, u, \dots, w')$ in $Q_n - P$. ■

3.2 Counterexamples in small hypercubes

The above theorems do not hold for certain small dimensions of hypercubes. We show several examples to support this.

It is obvious that both Theorem 12 and 13 do not hold for $n = 1$. Theorem 13 does not hold for $n = 2$. If P is an isometric path in Q_2 of length 2, then $Q_2 - P$ is a single vertex, which is not hamiltonian.

Remark Theorem 12 does not hold for $n = 3$, see Figure 3.4. Let $P = (b_0, w_1, b_2, w_3)$ where $b_0, b_2 \in B_n$ and $w_1, w_3 \in W_n$. Let us denote the common neighbor of w_1 and w_3 other than b_2 by b. Let w be the white neighbor of b that differs from w_1 and w_3. Thus the only path connecting b and w must be the edge bw, which is clearly not hamiltonian.

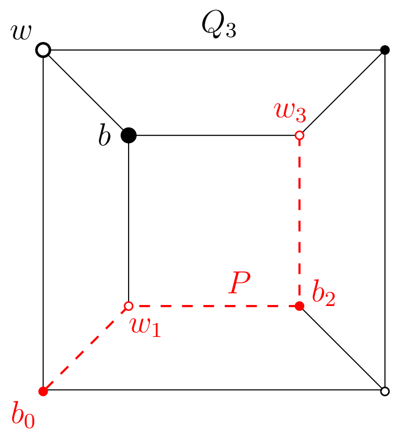

Figure 3.4: A nonexistence of a Hamiltonian path between b and w in $Q_3 - P$.

Remark Theorem 13 does not hold for $n = 3$, see Figure 3.5. Let $P = (b_0, w_1, b_2)$ where $b_0, b_2 \in B_n$ and $w_1 \in W_n$. Let x be the neighbor of w_1 that does not belong to the path P. Let us denote the white neighbors of x that do not belong to the path P by w and w'. Since the only neighbors of x that do not belong to the path P are w and w', the only path H between w and w' that visits x and avoids the vertices of P must be $H = (w, x, w')$, which is clearly not hamiltonian.

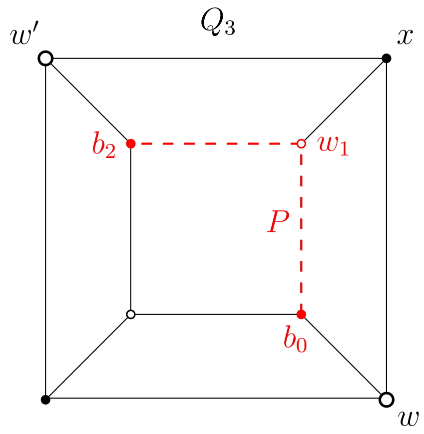

Figure 3.5: A nonexistence of a Hamiltonian path between w and w' in $Q_3 - P$.

Remark Theorem 13 does not hold for $n = 4$, see Figure 3.6. Let $P = (b_0, w_1, b_2, w_3, b_4)$ where $b_0, b_2, b_4 \in B_n$ and $w_1, w_3 \in W_n$. Let $x \in B_n$ be the neighbor of w_1 and w_3 that does not belong to the path P. Let us denote the (white) neighbors of x that do not belong to the path P by w, w'. Then there does not exist a Hamiltonian path between w and w' in $Q_4 - P$. We show this by a contradiction. Assume there is a Hamiltonian path H between w and w'. The path H contains the vertex x. Since only neighbors of x are the vertices w and w', it must be $H = (w, x, w')$ which is clearly not hamiltonian.

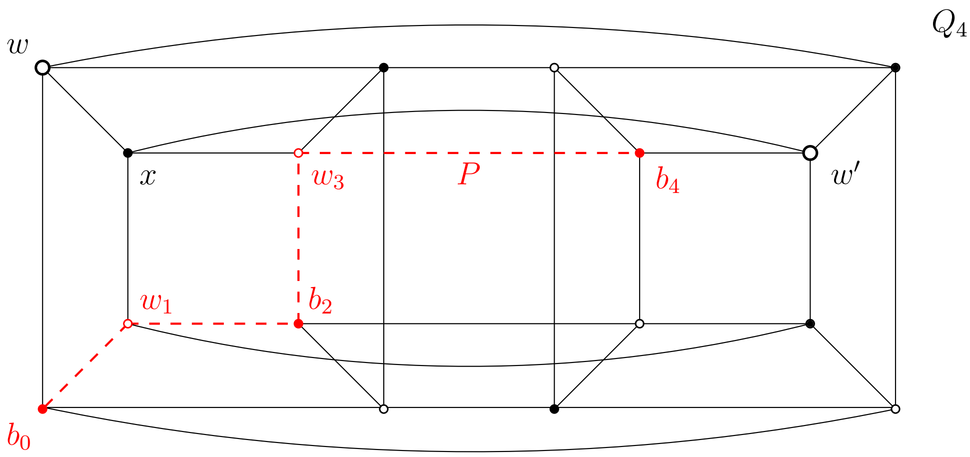

Figure 3.6: A nonexistence of a Hamiltonian path between w and w' in $Q_4 - P$.

4. Isometric cycles in hypercubes

In this chapter we explore properties of isometric cycles in hypercubes regarding directions of edges. Then we prove for an isometric cycle C in Q_n of length divisible by four and for every $n \geq 6$ that $Q_n - C$ is Hamiltonian laceable.

Observation 14 *Every isometric cycle in Q_n where $n \geq 2$ has exactly two distinct edges of identical direction.*

Proof For $n = 2$ it is obvious. We prove the statement for $n \geq 3$ by contradiction. The proof is similar to Observation 10. Notice that every cycle in Q_n has even number of edges of identical direction. For a contradiction assume that an isometric cycle D in Q_n has four distinct edges with identical directions. Let the path $S = (v_0,\, v_1, \ldots, v_k,\, v_{k+1})$ of length $k+1$ denote the shortest subpath of D such that exactly two distinct edges of S have identical direction. It is easy to see that those edges are v_0v_1 and v_kv_{k+1}.

Let $C = (v_0,\, v_1, \ldots, v_k,\, v_{k+1},\, v_{k+2}, \ldots, v_{2k-1})$ be a cycle in Q_n such that opposite edges have identical directions. That is, edges v_jv_{j+1} and $v_{j+k}v_{j+k+1}$ where $j \in [k-1]_0$ have identical direction (we assume that $v_{2k} = v_0$). Note that S is a subpath of C. Then the subpath $P = (v_{k+1},\, v_{k+1}, \ldots . v_{2k-1},\, v_0)$ of C between $v_{k+1}v_0$ of length $k-1$ is shorter than the path S between v_0 and v_{k+1} of length $k+1$, i.e. $|P| < |S|$.

Let $S^0 = (v_1,\, v_2, \ldots, v_k)$ denote the path S without its endvertices. To complete the contradiction, we have to show that the path $D - S^0$ between v_0 and v_{k+1} is longer than the path P. Recall that we have chosen S to be the shortest path in D that has precisely two edges of identical direction. Thus $|S| \leq |C - S_0|$ and using the inequality from the paragraph above we have $|P| < |C - S_0|$. ■

Observation 15 *Antipodal (opposite) edges of isometric cycles in Q_n where $n \geq 2$ have identical directions.*

Proof We show this by contradiction. Let C be an isometric cycle in Q_n and let uv and xy be its edges that have identical direction but are not opposite in C. We remove edges uv and xy from the cycle C splitting it into two paths, say $P^0 = (u, \ldots, x)$ and $P^1 = (v, \ldots, y)$. Since uv and xy are not opposite, one of the paths say P^0 is shorter. Then the path $(v, u \ldots, x)$ between v and x is shorter than the path $(v, \ldots, y, x)$ between v and x which is a contradiction. ■

Let C be an isometric cycle in Q_n of length $2k$. Since every Q_n has n directions, it follows that $k \leq n$ and the isometric cycle C has at most $2n$ edges. Let i be a direction of an edge $uv \in E(C)$. Let us partition Q_n into $Q^{0,i}_{n-1}$ and $Q^{1,i}_{n-1}$. Then C splits into two isometric paths P^0 and P^1. The path P^0 is in $Q^{0,i}_{n-1}$ and the path P^1 is in $Q^{1,i}_{n-1}$. Both of these paths have length k and they are connected together with edges uv and the other edge of C of direction i.

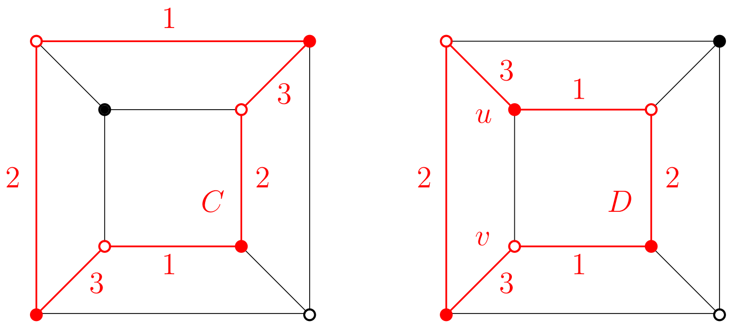

Figure 4.1: An isometric 6-cycle C in Q_3 (left) and a non-isometric 6-cycle D in Q_3 (right).

4.1 Avoiding an isometric cycle by a Hamiltonian path

Now we prove the main theorem of this chapter stating that if C is an isometric cycle in Q_n of length divisible by four, then for every $n \geq 6$ the graph $Q_n - C$ is Hamiltonian laceable. The proof is similar to the proof of Theorem 12.

Theorem 16 *Let $C = (b_0,\ w_1,\ b_2,\ w_3, \ldots, w_{k-1},\ b_k,\ w_{k+1},\ b_{k+2}, \ldots, w_{2k-1})$ be an isometric cycle of length $2k$ in Q_n where k is even, $2 \leq k \leq n$, $n \geq 6$, every b_i is in B_n, every w_i is in W_n. For every $b \in B_n \setminus V(C)$ and every $w \in W_n \setminus V(C)$ there exists a Hamiltonian path between b and w in $Q_n - C$.*

Proof We denote the direction of the edge b_0w_1 by i. Let $P^0 = (w_{k+1},\ b_{k+2}, \ldots, w_{2k-1},\ b_0)$ and $P^1 = (w_1,\ b_2,\ w_3, \ldots, b_k)$. Then P^0 is in $Q^{i,0}_{n-1}$ and P^1 is in $Q^{i,1}_{n-1}$. After fixing the direction i, there are two cases to consider.

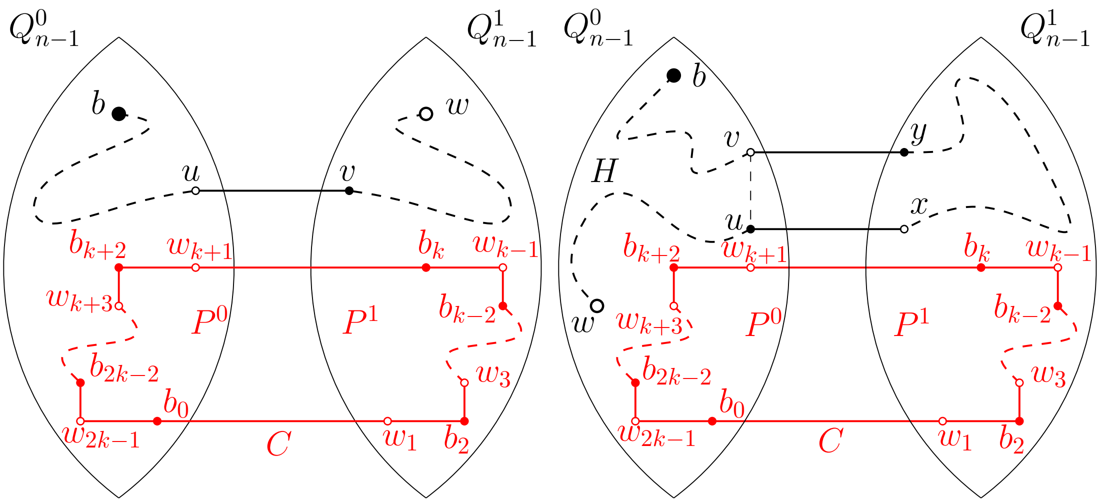

Figure 4.2: Cases 1 and 2 in the proof of Theorem 16.

Case 1: *The vertices b, w are in different subcubes Q^0_{n-1}, Q^1_{n-1}; say $b \in V(Q^0_{n-1})$ and $w \in V(Q^1_{n-1})$, see Figure 4.2.* We choose an edge uv of direction i such that $u \in W_n \setminus V(P^0)$ is in Q^0_{n-1} and $v \in B_n \setminus V(P^1)$ is in Q^1_{n-1}. Since the number of edges $xy \in E_n$ of direction i in $Q_n - C$ such that $x \in W_n$ is in Q^0_{n-1}

(and $y \in B_n$ is in Q^1_{n-1}) is for $n \geq 6$ at least $2^{n-2} - (k-1) \geq 2^{n-2} - n + 1 \geq 1$, such edge uv exists.

By Theorem 12, there exists a Hamiltonian path between b and u in $Q^0_{n-1} - P^0$ and a Hamiltonian path between v and w in $Q^1_{n-1} - P^1$. By joining these paths we obtain a Hamiltonian path $(b, \ldots, u, v, \ldots, w)$ in $Q_n - C$.

Case 2: *The vertices b, w are in the same subcube; say in Q^0_{n-1}, see Figure 4.2.* By Theorem 12, there exists a Hamiltonian path H between b and w in $Q^0_{n-1} - P^0$. Let uv be an edge of H, $u \in B_n$ and $v \in W_n$ such that the neighbors of u, v in Q^1_{n-1} do not belong to P^1. We claim that such edge uv exists.

We say that a vertex $p \in V(P^1)$ blocks an edge $rs \in E(H)$ if p is a neighbor of r or s in Q^1_{n-1}. Each vertex in the path P^1 different from w_1 and b_k blocks at most two edges of the path H. Since $|H| \geq 2^{n-1} - n - 1$, the number of edges which are not blocked by the path P^1 is for $n \geq 6$ at least $2^{n-1} - n - 1 - 2(n-2) \geq 2^{n-1} - 3n - 5 \geq 1$. Therefore, uv exists and we denote the neighbors of u and v in Q^1_{n-1} by x and y, respectively.

By Theorem 12, there exists a Hamiltonian path between x and y in $Q^1_{n-1} - P^1$. We join this path with the path H by removing the edge uv and adding the edges ux and vy. That is, we obtain a Hamiltonian path $(b, \ldots, v, y, \ldots, x, u, \ldots, w)$ in $Q_n - C$. ■

5. Isometric trees in hypercubes

In this chapter we explore properties of isometric trees in hypercubes regarding directions of edges and balance of bipartite set. Then we prove for an isometric tree T in Q_n such that it has odd (even) number of edges and for $n \geq 4$ ($n \geq 5$) that $Q_n - T$ is Hamiltonian laceable.

Observation 17 *All edges of an isometric tree in Q_n where $n \geq 1$ are distinct.*

Proof Let T be an isometric tree in Q_n and for contradiction assume that two distinct edges $uv, xy \in E(T)$ have the same direction. We choose a path P in T such that it contains both uv and xy. Since a tree does not contain a cycle, path P is an isometric path, which contradicts Observation 10. ■

Since every Q_n has n directions, it follows that every isometric tree in Q_n has at most n edges. Let T be an isometric tree with k edges in Q_n. Let i be the direction of an edge $uv \in E(T)$. We partition Q_n into $Q^{0,i}_{n-1}$ and $Q^{1,i}_{n-1}$. Then the tree T splits into trees T_0 and T_1. The tree T_0 is in $Q^{0,i}_{n-1}$ and T_1 is in $Q^{1,i}_{n-1}$ and these trees are connected by the edge uv. We will use a special case of this property when the vertex u is a leaf of T.

5.1 Black-white trees

Let us look at some special trees that we will use. Let T be a tree with U, V its bipartition. We say that the tree T is a *black-white tree*, if we call vertices of U black and vertices of V white. This naming will be useful when we consider trees in hypercubes, which have black and white vertices. A *balanced tree* is a black-white tree that has the same number of black and white vertices. A *black-balanced tree* is a black-white tree such that it has one more black vertex than it has white vertices. Analogously, a *white-balanced tree* is a black-white tree such that it has one more white vertex than it has black vertices.

We prove very useful property of balanced and black-balanced trees that both of them have a black leaf.

Lemma 18 *Every balanced tree has a black leaf.*

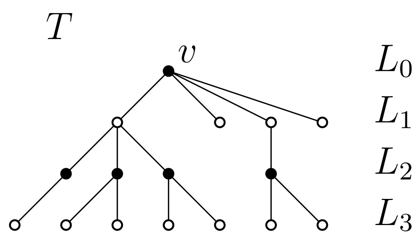

Figure 5.1: An example of the tree T with all leaves white for a contradiction.

Proof We denote the balanced tree by T. Assume for a contradiction that all leaves of T are white. We choose an arbitrary black vertex v of T and for illustration we imagine that v is the root of T, see Figure 5.1. Let k be the maximum of the set $\{d(v,u) \mid u \in V(T)\}$. We say that $L_i = \{u \in V(T) \mid d(v,u) = i\}$ is a layer for $i \in [k]$. Note that if $i \in [k]$ is even (odd) all vertices of L_i are black (white). Obviously, k is odd. Since there are no black leaves, for every nonzero even integer $i \in [k] : |L_i| \le |L_{i+1}|$ and since v is also not a leaf $|L_0| < |L_1|$. Hence T has more white than black vertices, which is a contradiction since T is balanced. ■

Corollary 19 *Every black-balanced tree has a black leaf.*

Proof We denote the black-balanced tree by S. Assume for a contradiction that all leaves of S are white. Remember that a black-balanced isometric tree has more black than white vertices. We add a white leaf to the tree S and denote this new tree by T. Note that T is balanced. Thus by Lemma 18, tree T has a black leaf and so does tree S since adding a white leaf to S does not change the number of black leaves. ■

Analogously, both balanced and white-balanced trees have a white leaf.

5.2 Avoiding small isometric trees in hypercubes

Before we prove the main theorem of this chapter, we need two lemmas which we use later on in the base of induction. Unfortunately, I have not been able to prove the second lemma, but I successfully verified it by a computer.

Figure 5.2: Trees T_1 and T_2 used in Lemma 20 and 21, respectively.

Lemma 20 *Let* $T_1 = (\{b_0, w_1, b_2, w_3, b_4, w_5\}, \{b_0w_1, w_1b_2, b_2w_3, w_3b_4, b_2w_5\})$ *be a balanced isometric tree in* Q_5 *where* $b_0, b_2, b_4 \in B_5$ *and* $w_1, w_3, w_5 \in W_5$*, see Figure 5.2. For every* $b \in B_5 \setminus \{b_0, b_2, b_4\}$ *and every* $w \in W_5 \setminus \{w_1, w_3, w_5\}$ *there exists a Hamiltonian path between* b *and* w *in* $Q_5 - T_1$*.*

Proof We denote the direction of b_2w_3 by i and we fix it. Then the path $P_3 = (b_0, w_1, b_2, w_5)$ is in $Q_4^{0,i}$ and the path $P_1 = (w_3, b_4)$ is in $Q_4^{1,i}$. We fix the direction i. Then there are three cases to consider.

Case 1: *The vertices* b, w *are in different subcubes* Q_4^0, Q_4^1*; say* $b \in V(Q_4^0)$ *and* $w \in V(Q_4^1)$*, see Figure 5.3.* We choose an edge uv of direction i such that $u \in W_5$ is in Q_4^0 (and $v \in B_5$ is in Q_4^1) and u is distinct from w_1 and w_3 and $v \neq b_4$. Since the number of edges of direction i in $Q_5 - T_1$ is $2^{5-2} - 2 - 1 = 5$, such edge exists.

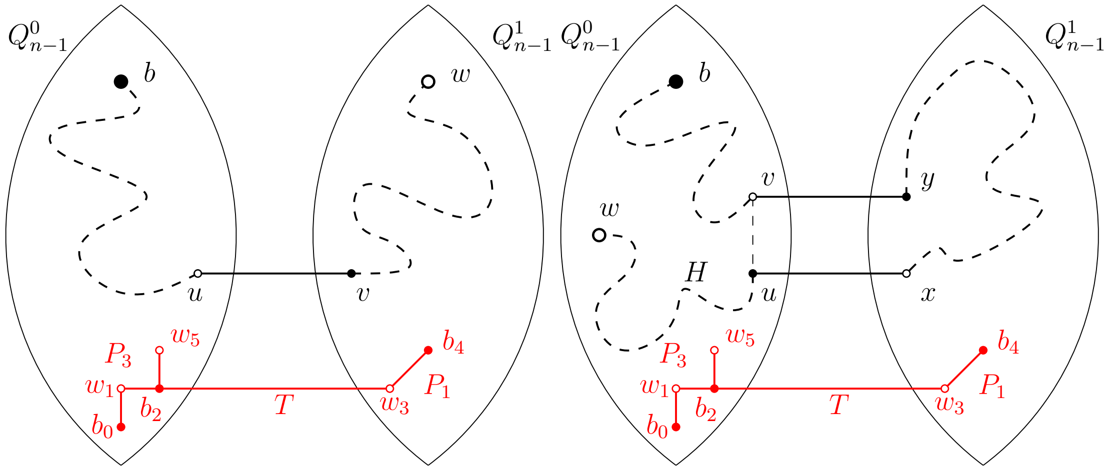

Figure 5.3: Cases 1 and 2 in the proof of Lemma 20.

By Theorem 12, there exists a Hamiltonian path between b and u in $Q_4^0 - P_3$ and a Hamiltonian path between v and w in $Q_4^1 - P_1$. By joining these paths we obtain a Hamiltonian path $(b, \dots, u, v, \dots, w)$ in $Q_5 - T_1$.

Case 2: *The vertices b, w are in Q_4^0, see Figure 5.3.* By Theorem 12, there exists a Hamiltonian path H between b and w in $Q_4^0 - P_3$. Let uv be an edge of H, $u \in B_5$ and $v \in W_5$ such that the neighbor of v in Q_4^1 differs from b_4. Since $|V(H)| = 2^4 - 4 = 11$ and b_4 can be neighbor of just one white vertex of H, such edge exists. Let x and y denote the neighbors of u and v in Q_4^1, respectively.

By Theorem 12, there exists a Hamiltonian path between x and y in $Q_4^1 - P_1$. We join this path with the path H by removing the edge uv and adding the edges ux and vy. That is, we obtain a Hamiltonian path $(b, \dots, v, y, \dots, x, u, \dots, w)$ in $Q_5 - T_1$.

Case 3: *The vertices b, w are in Q_4^1.* We prove this case in a same way as the Case 2 above. By Theorem 12, there exists a Hamiltonian path H between b and w in $Q_4^1 - P_1$. Let uv be an edge of H, $u \in B_5$ and $v \in W_5$ such that the neighbor of v in Q_4^0 differs from b_0 and the neighbor of u in Q_4^0 differs from w_1 and w_3. We show that such edge exists.

If b_0 is the neighbor of a white vertex $r \in V(H)$ it blocks at most two edges of H: $rs, qr \in E(H)$. Analogically with vertices w_1 and w_3. Thus b_0, w_1, w_3 can block at most six edges of H. Since $|E(H)| = 2^4 - 2 - 1 > 6$, such edge uv exists and we denote x and y the neighbors of u and v in Q_4^0, respectively.

By Theorem 12, there exists a Hamiltonian path between x and y in $Q_4^0 - P_3$. We join this path with the path H by removing the edge uv and adding the edges ux and vy. That is, we obtain a Hamiltonian path $(b, \dots, v, y, \dots, x, u, \dots, w)$ in $Q_5 - T_1$. ■

Lemma 21 *Let $T_2 = (\{b_0, w_1, b_2, w_3, b_4, w_5\}, \{b_0w_1, w_1b_2, b_2w_3, w_1b_4, b_2w_5\})$ be a balanced isometric tree in Q_5 where $b_0, b_2, b_4 \in B_5$ and $w_1, w_3, w_5 \in W_5$, see Figure 5.2. For every $b \in B_5 \setminus \{b_0, b_2, b_4\}$ and every $w \in W_5 \setminus \{w_1, w_3, w_5\}$ there exists a Hamiltonian path between b and w in $Q_5 - T_2$.*

I verified Lemma 21 by a supercomputer at the MetaCentrum VO [19] (virtual organization) by brute-force obtaining a list of all Hamiltonian paths between b

and w in $Q_5 - T_2$, see Attachment B. This is possible since there is only a finite number of configurations of T_2, b and w in Q_5. See Attachment A for further information about the program and where it can be found.

5.3 Avoiding an isometric tree by a Hamiltonian path

Now we prove the main theorem of this chapter stating that if T is an isometric tree in Q_n such that T has odd (even) number of edges for $n \geq 4$ ($n \geq 5$), then $Q_n - T$ is Hamiltonian laceable. This result is divided into two theorems considering balanced and black-balanced (also white-balanced) trees. We prove both theorems at once by a zig-zag induction.

Theorem 22 *Let T be a balanced isometric tree in Q_n with odd number k of edges where $1 \leq k \leq n$ and $n \geq 4$. For every $b \in B_n \setminus V(T)$ and every $w \in W_n \setminus V(T)$ there exists a Hamiltonian path between b and w in $Q_n - T$.*

Theorem 23 *Let S be a black-balanced isometric tree in Q_n with even number l of edges where $2 \leq l \leq n$ and $n \geq 5$. For every distinct $v, v' \in W_n \setminus V(S)$ there exists a Hamiltonian path between v and v' in $Q_n - S$.*

Obviously, Theorem 23 can be easily modified to hold for white-balanced trees as well. For simplicity, we will only use the version of Theorem 23 that uses black-balanced trees, unless we want to avoid confusion.

Proof We prove both Theorem 22 and Theorem 23 by two zig-zag inductions as indicated in Figure 5.4. Black and blue points represent bases of inductions and arrows represent induction steps. This proof is divided into four parts. In the first two parts we prove the bases of inductions and then in the latter two parts we prove both induction steps.

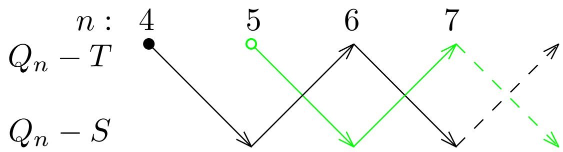

Figure 5.4: A scheme of the proof of Theorems 22 and 23 using two inductions.

Part I: The statement of Theorem 22 for $n = 4$.

Let T' be a balanced isometric tree in Q_4 with at most 3 edges. We prove an existence of a Hamiltonian path between every $b \in B_n \setminus V(T')$ and every $w \in W_n \setminus V(T')$ in $Q_5 - T'$. This is the base of induction which is represented by a black dot (i.e. ●) in Figure 5.4.

Let k' denote the number of edges of T'. Since k' is odd, there are two cases to consider. If $k' = 1$, then the tree T' is an edge and we use Theorem 12.

If $k' = 3$, then there exist only two trees with three edges; a path on four vertices P_3 and a star $K_{1,3}$ which is not balanced, see Figure 5.5. Thus we need to find a Hamiltonian path between every $b \in B_n \setminus V(P_3)$ and every $w \in W_n \setminus V(P_3)$ in $Q_n - P_3$, which holds by Theorem 12.

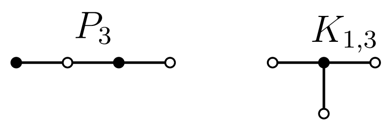

Figure 5.5: All trees with three edges [18]. Note that $K_{1,3}$ is not balanced.

Part II: The statement of Theorem 22 for $n = 5$.

Let T' be a balanced isometric tree in Q_5 with at most 5 edges. We prove an existence of a Hamiltonian path between every $b \in B_n \setminus V(T')$ and every $w \in W_n \setminus V(T')$ in $Q_5 - T'$. This is the base of induction which is represented by a green circle (i.e. ○) in Figure 5.4.

Let k' denote the number of edges of T'. Since k' is odd, there are three cases to consider. If $k' = 1$, then the tree T' is an edge and we use Theorem 12.

If $k' = 3$, then there exist two trees with three edges; a path on four vertices P_3 and a star $K_{1,5}$ which is not balanced, see Figure 5.5. Thus we need to find a Hamiltonian path between every $b \in B_n \setminus V(P_3)$ and every $w \in W_n \setminus V(P_3)$ in $Q_n - P_3$, which holds by Theorem 12.

If $k' = 5$, then there exist six trees with five edges, see Figure 5.6. Only three of those trees are balanced: P_5, T_1 and T_2. Since P_5 is an odd isometric path we use Theorem 12. To find a Hamiltonian path in $Q_5 - T_1$ and $Q_5 - T_2$, we use Lemma 20 and Lemma 21, respectively.

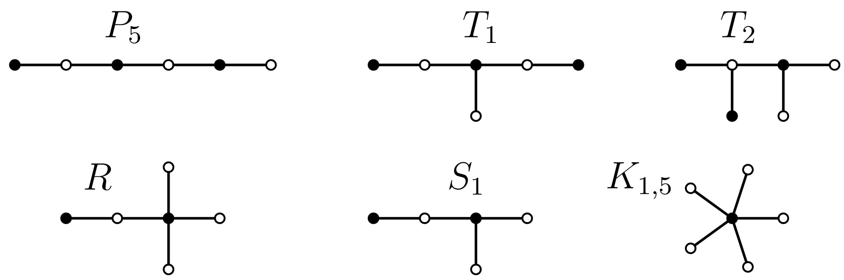

Figure 5.6: All trees with 5 edges [18]. Only P_5, T_1 and T_2 are balanced.

Part III: The statement of Theorem 22 for $n - 1$ implies the statement of Theorem 23 for n.

Recall that T is a balanced isometric tree in Q_{n-1} and that S is a black-balanced isometric tree in Q_n with even l denoting the number of its edges. We assume that $Q_{n-1} - T$ is Hamiltonian laceable and we prove that $Q_n - S$ is Hamiltonian laceable as well. This induction step is represented by an arrow pointing south-east (i.e. $\searrow$) in Figure 5.4.

By Corollary 19, there exists a black leaf b_0 of S and we denote its only neighbor by w_1. Let us denote the balanced isometric tree $S - \{b_0\}$ by T'. We denote the direction of b_0w_1 by i. Then b_0 is in $Q^{i,0}_{n-1}$ and T' is in $Q^{i,1}_{n-1}$. We fix the direction i. There are three cases to consider.

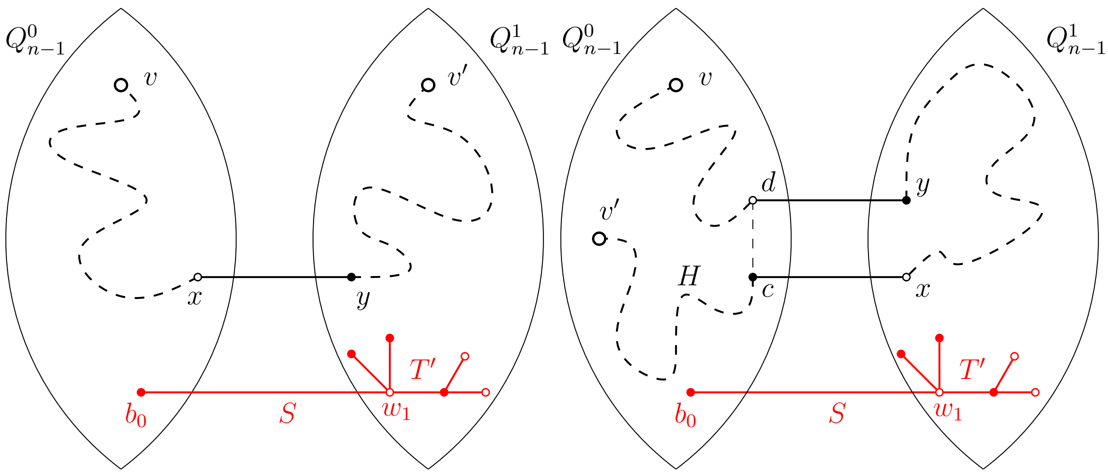

Figure 5.7: Cases 1 and 2 in the proof of Theorem 23 with S as an example of a black-balanced tree on 7 vertices.

Case 1: *The vertices v, v' are in different subcubes Q^0_{n-1}, Q^1_{n-1}; say $v \in V(Q^0_{n-1})$ and $v' \in V(Q^1_{n-1})$, see Figure 5.7.* We choose an edge xy of direction i such that $x \in W_n \setminus \{v\}$ is in Q^0_{n-1} (and $y \in B_n$ is in Q^1_{n-1}) and y does not belong to T'. The number of edges of direction i in Q_n that have its black vertex in Q^0_{n-1} is 2^{n-2}. These edges can be blocked by one of $\frac{l}{2}$ black vertices of T' or by a vertex v. Thus the number of choices for an edge xy is for $n \geq 5$ at least $2^{n-2} - \frac{l}{2} - 1 \geq 2^{n-2} - \lfloor\frac{n}{2}\rfloor - 1 \geq 1$.

By Proposition 8, there exists a Hamiltonian path between v and x in $Q^0_{n-1} - \{b_0\}$ and by the induction hypothesis, there exists a Hamiltonian path between y and v' in $Q^1_{n-1} - T'$. By joining these paths we obtain a Hamiltonian path $(v, \ldots, x, y, \ldots, v')$ in $Q_n - S$.

Case 2: *The vertices v, v' are in the subcube Q^0_{n-1}, see Figure 5.7.* By Proposition 8, there exists a Hamiltonian path H between v and v' in $Q^0_{n-1} - \{b_0\}$. Let cd be an edge of H, $c \in B_n$ and $d \in W_n$ such that the neighbors of c, d in Q^1_{n-1} do not belong to T'. We claim that such edge $cd \in E(H)$ exists.

We say that a vertex $p \in V(T')$ blocks an edge $rs \in E(H)$ if p is a neighbor of r or s in Q^1_{n-1}. Each of l vertices in the tree T' blocks at most two edges of the path H. Since $|H| = 2^{n-1} - 2$, the number of edges which are not blocked by the tree T' is for $n \geq 5$ at least $2^{n-1} - 2 - 2l \geq 2^{n-1} - 2n - 2 \geq 1$. Therefore, such edge cd exists and we denote the neighbors of c and d in Q^1_{n-1} by x and y, respectively.

By the induction hypothesis, there exists a Hamiltonian path between x and y in $Q^1_{n-1} - T'$. We join this path with the path H by removing the edge cd and adding the edges cx and dy. That is, we obtain a Hamiltonian path $(v, \ldots, d, y, \ldots, x, c, \ldots, v')$ in $Q_n - S$.

Case 3: *The vertices v, v' are in the subcube Q^1_{n-1}, see Figure 5.8.* We choose a vertex $c \in B_n$ in Q^1_{n-1} such that c does not belong to the tree T'. By the induction hypothesis, there exists a Hamiltonian path H between v and c in $Q^1_{n-1} - T'$. Let yv' denote the edge of H such that $y \in B_n$ is closer to v than to c on the path H. Let $x, d \in W_n$ be the neighbors of y, c in Q^0_{n-1}, respectively.

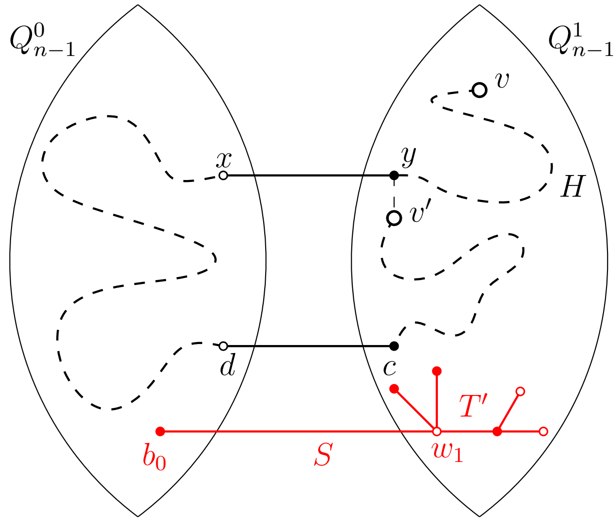

Figure 5.8: Case 3 in the proof of Theorem 23 with S as an example of a black-balanced tree on 7 vertices.

By Proposition 8, there exists a Hamiltonian path between x and d in $Q^0_{n-1} - \{b_0\}$. We join this path with the path H removing the edge yv' and adding the edges dc, xy. That is, we obtain a Hamiltonian path $(v, \ldots, y,\ x, \ldots, d,\ c, \ldots, v')$ in $Q_n - S$.

We have proved that for a black-balanced isometric tree S in Q_n is $Q_n - S$ Hamiltonian laceable. Please note, that by switching color black with white it is easy to see that all arguments above hold as well. That is, we have also proved Hamiltonian laceability of $Q_n - S_1$ where S_1 is a white-balanced isometric tree in Q_n.

Part IV: The statement of Theorem 23 for $n-1$ implies the statement of Theorem 22 for n.

Recall that S is a white-balanced isometric tree in Q_{n-1} and that T is a balanced isometric tree in Q_n with odd k denoting the number of its edges. We assume that $Q_{n-1} - S$ is Hamiltonian laceable. We prove a Hamiltonian laceability of $Q_n - T$. This induction step is represented by an arrow pointing north-east (i.e. $\nearrow$) in Figure 5.4.

By Lemma 18 there exists a black leaf b_0 of T and we denote its only neighbor in T by w_1. Let us denote the white-balanced isometric tree $T - \{b_0\}$ by S'. We denote the direction of b_0w_1 by i. Then b_0 is in $Q^{i,0}_{n-1}$ and S' is in $Q^{i,1}_{n-1}$. We fix the direction i. There are four cases to consider.

Case 1: *The vertex b is in Q^0_{n-1} and the vertex w is in Q^1_{n-1}, see Figure 5.9.* We need to find an isometric path $R = (b_0, x, z, y)$ in Q^0_{n-1} such that $z \neq b$ and the neighbors of x and y in Q^1_{n-1} do not belong to the tree S'. There are $n-1$ neighbors of b_0 that are in Q^0_{n-1}. We need to choose one of them such that its neighbor in Q^1_{n-1} is not among the $\frac{k-1}{2}$ black vertices of S'. The number of such neighbors of b_0 is for $n \geq 6$ at least $n - 1 - \frac{k-1}{2} \geq n - 1 - \lfloor \frac{n-1}{2} \rfloor \geq 1$. We choose one of them, denote it by x and we fix it. There are $n-3$ neighbors of x in Q^0_{n-1} different from b and b_0. We choose one of them, denote it by z and we fix it.

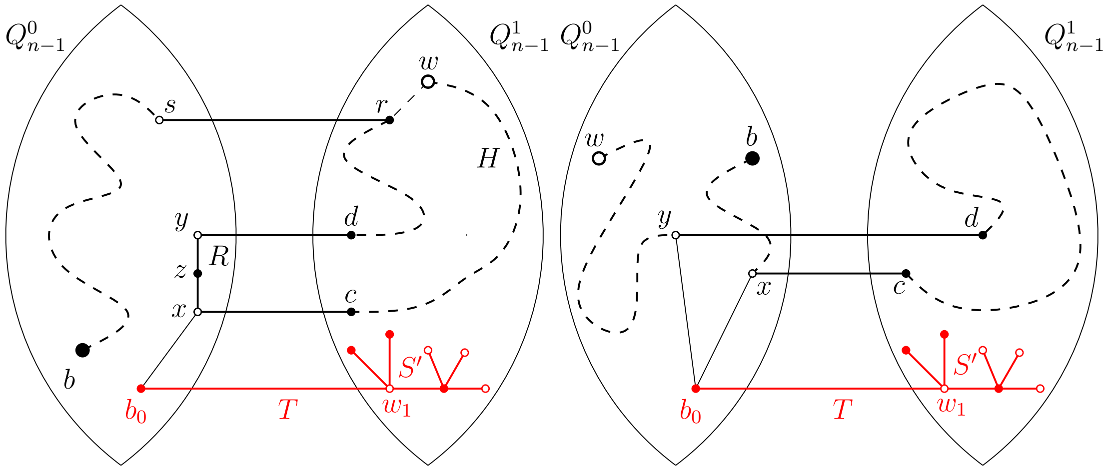

Figure 5.9: Cases 1 and 2 in the proof of Theorem 22 with T as an example of a balanced tree on 8 vertices.

There are at least $n-3$ neighbors of z in Q^0_{n-1} different from x and not having b_0 as a neighbor. The last condition says that the direction of the edge between z and any of these neighbors is different from the direction of b_0x and is needed for R to be isometric. Of those $n-3$ neighbors of z we need to choose a vertex such that its neighbor in Q^1_{n-1} is none of $\frac{k-1}{2}$ black vertices of S'. The number of such vertices is for $n \geq 6$ at least $n-3-\frac{k-1}{2} \geq n-3-\lfloor\frac{n-1}{2}\rfloor \geq 1$. We choose one of them and denote it by y. Thus we now have an isometric path $R = (b_0, x, z, y)$.

Let us denote the neighbors of x and y in Q^1_{n-1} by c and d, respectively. By the induction hypothesis, there exists a Hamiltonian path H in $Q^1_{n-1} - S'$. We denote a neighbor of w on the path H different from c and d by r. Without loss of generality r is closer to d than to c on the path H. We denote the neighbor of r in Q^0_{n-1} by s. By Theorem 12, there exists a Hamiltonian path between s and b in $Q^0_{n-1} - R$. By removing the edges b_0x, rw and adding the edges xc, yd, rs we obtain a Hamiltonian path $(b, \ldots, s, r, \ldots, d, y, z, x, c, \ldots, w)$ in $Q_n - T$.

Case 2: *The vertices b, w are in the subcube Q^0_{n-1}, see Figure 5.9.* We choose neighbors x and y of b_0 in Q^0_{n-1} such that their neighbors in Q^1_{n-1} do not belong to the tree S'. We show that such vertices x, y exist.

There are $n-1$ neighbors of b_0 in Q^0_{n-1}. The number of black vertices of S' is $\frac{k-1}{2}$. Thus the number of candidates for x and y is for $n \geq 6$ at least $n-1-\frac{k-1}{2} \geq n-1-\lfloor\frac{n-1}{2}\rfloor \geq 2$. By Proposition 9, there exists a Hamiltonian path between b and w in Q^0_{n-1} using the edges xb_0 and b_0y. We denote the neighbors of x and y in Q^1_{n-1} by c and d, respectively. By the induction hypothesis, there exists a Hamiltonian path between c and d in $Q^1_{n-1} - S'$. By removing the edges xb_0, b_0y and adding the edges xc and yd we obtain a Hamiltonian path $(b, \ldots, x, c, \ldots, d, y, \ldots, w)$ in $Q_n - T$.

Case 3: *The vertices b, w are in the subcube Q^1_{n-1}, see Figure 5.10.* We choose a vertex $c \in B_n \setminus \{b\}$ in Q^1_{n-1} such that c does not belong to the tree S'. By the induction hypothesis, there exists a Hamiltonian path H between c and b in $Q^1_{n-1} - S'$. Let yw denote the edge of H such that $y \in B_n$ is closer to b than to c on the path H. Let $x, d \in W_n$ be the neighbors of y and c in Q^0_{n-1}, respectively. By Proposition 8, there exists a Hamiltonian path between d and x in $Q^0_{n-1} - \{b_0\}$. We join this path with the path H by removing the edge yw

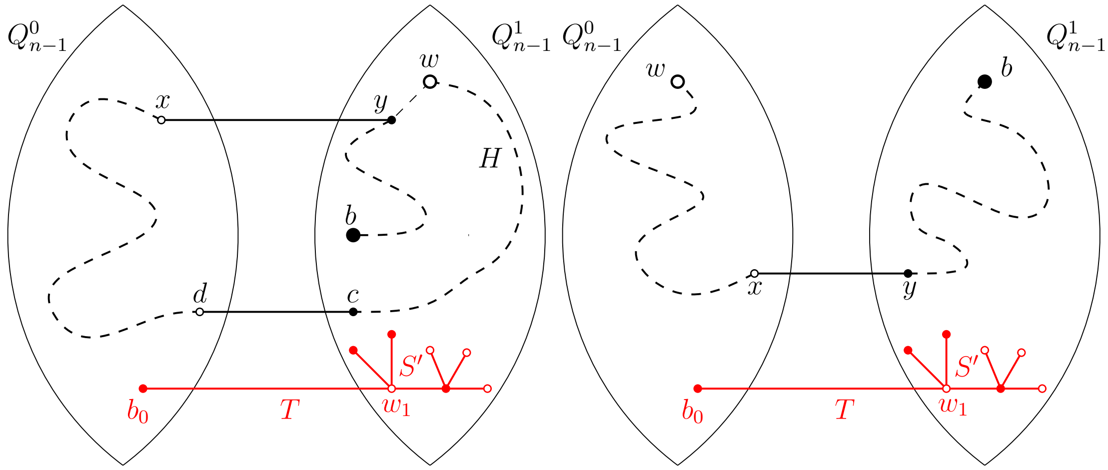

Figure 5.10: Cases 3 and 4 in the proof of Theorem 22 with T as an example of a balanced tree on 8 vertices.

and adding the edges dc, xy. That is, we obtain a Hamiltonian path $(w, \dots, c, d, \dots, x, y, \dots, b)$ in $Q_n - T$.

Case 4: *The vertex w is in Q^0_{n-1} and the vertex b is in Q^1_{n-1}, see Figure 5.10.* We choose an edge xy of direction i distinct from b_0w_1 such that $x \in W_n \setminus \{w\}$ is in Q^0_{n-1}, $y \in B_n \setminus \{b\}$ in Q^1_{n-1} and y does not belong to S'. The number of edges of direction i in Q_n that have its black vertex in Q^0_{n-1} is 2^{n-2}. These edges can be blocked by one of $\frac{k-1}{2}$ black vertices of S' or by b or w. Thus the number of choices for an edge xy is for $n \geq 6$ at least $2^{n-2} - \frac{k-1}{2} - 2 \geq 2^{n-2} - \lfloor \frac{n-1}{2} \rfloor - 2 \geq 1$. By Proposition 8, there exists a Hamiltonian path between w and x in $Q^0_{n-1} - \{b_0\}$ and by the induction hypothesis, there exists a Hamiltonian path between y and b in $Q^1_{n-1} - S'$. By joining these paths we obtain a Hamiltonian path $(w, \dots, x, y, \dots, b)$ in $Q_n - T$. ■

Theorems 22 and 23 do not hold for smaller dimensions than stated. In Chapter 3, we showed that hypercubes of small dimensions without isometric paths are not Hamiltonian laceable. Since a special case of a tree is a path, the examples in Section 3.2 can be used as counterexamples for small dimensions as well.

Conclusion

Let $F \subseteq Q_n$ be a set of faulty vertices. We have proved that for large enough n, the graph $Q_n - F$ is Hamiltonian laceable if F forms any of these isometric subgraphs of Q_n: a path, a cycle of length divisible by four and a tree.

We have presented two new results. Let C be an isometric cycle in Q_n such that its length is divisible by four and $n \geq 6$. We have proved that $Q_n - C$ is Hamiltonian laceable for $n \geq 6$. Since C can have up to $2n$ vertices, this extends the previous result by Sun and Jou[15]. They showed that Q_n without an arbitrary cycle on at most $2n-4$ vertices is Hamiltonian laceable for every $n \geq 4$. Next we have proved that if T is an isometric tree in Q_n with odd (even) number of edges and $n \geq 5$ (resp. $n \geq 6$), then the graph $Q_n - T$ is Hamiltonian laceable. This theorem was proved by an induction and a part of the base of the induction was verified by a computer. The program that verified the base and the program's output can be found on the attached CD, see Attachments for more information.

An observant reader may notice that among these proofs of the existence of a Hamiltonian path in Q_n without various isometric subgraphs of Q_n, an isometric cycle whose length is not divisible by four is missing. I have been unable to fully prove that if C is an isometric cycle of even length not divisible by four in Q_n, then $Q_n - C$ is Hamiltonian laceable for every large enough n. I state it here as a conjecture and I present its incomplete proof. I will be looking forward for anybody to prove the missing case or prove the whole conjecture by a different approach.

Conjecture 24 *Let $C = (b_0, w_1, b_2, w_3, \ldots, b_{k-1}, w_k, b_{k+1}, w_{k+2}, \ldots, w_{2k-1})$ be an isometric cycle of length $2k$ in Q_n where k is odd, $2 \leq k \leq n$, $n \geq 6$, every b_i is in B_n, every w_i is in W_n. For every $b \in B_n \setminus V(C)$ and every $w \in W_n \setminus V(C)$ there exists a Hamiltonian path between b and w in $Q_n - C$.*

Proof Attempt We denote the direction of the edge b_0w_1 by i. Then the subpath $P^0 = (w_1, b_2, w_3, \ldots, w_k)$ is in $Q^{i,0}_{n-1}$ and the subpath $P^1 = (b_{k+1}, w_{k+2}, \ldots, w_{2k-1})$ is in $Q^{i,1}_{n-1}$. After fixing the direction i, there are three cases to consider.

Case 1: *The vertices b, w are in different subcubes Q^0_{n-1}, Q^1_{n-1}; say $b \in V(Q^0_{n-1})$ and $w \in V(Q^1_{n-1})$, see Figure 5.11.* We choose an edge uv of direction i such that $u \in W_n$ is in Q^0_{n-1} and does not belong to P^0 and $v \in B_n$ is in Q^1_{n-1} and does not belong to P^1. Such edge exists since the number of edges of direction i in $Q_n - C$ is for $n \geq 6$ at least $2^{n-1} - 2k \geq 2^{n-1} - 2n \geq 1$. By Theorem 13, there exists a Hamiltonian path between b and u in $Q^0_{n-1} - P^0$ and a Hamiltonian path between v and w in $Q^1_{n-1} - P^1$. By joining these paths we obtain a Hamiltonian path $(b, \ldots, u, v, \ldots, w)$ in $Q_n - C$.

Case 2: *The vertices b, w are in the subcube Q^0_{n-1}, see Figure 5.11.* Let $u \in V(Q^0_{n-1})$ be a neighbor of b_{k+1} different from w and w_{k+2}. Let $x \in V(Q^0_{n-1})$ be a neighbor of b_0 different from w and w_{2k-1}. By Proposition 9, there exists a Hamiltonian path in Q^0_{n-1} between b and w passing through the edges ub_{k+1}, b_0x and all the edges in $E(P^0)$. Let v and y be the neighbors of u and x in Q^1_{n-1}, respectively. By Theorem 13, there exists a Hamiltonian path between v and y

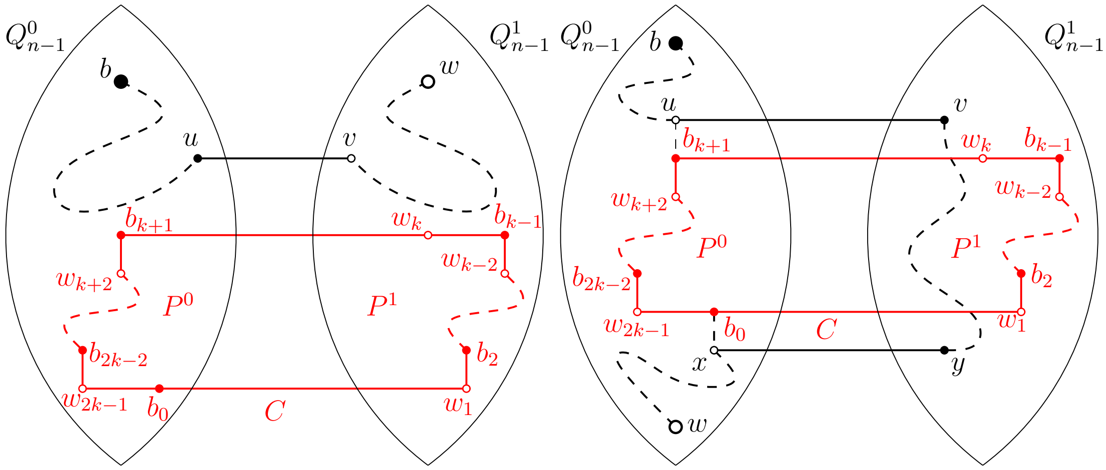

Figure 5.11: Cases 1 and 2 for Theorem 24.

in $Q^1_{n-1} - P^1$. We join these paths by adding the edges xy, uv and removing the edges ub_{k+1}, b_0x and removing the path P^0 as well. That is, we obtain a Hamiltonian path $(b, \ldots, x, y, \ldots, v, u, \ldots, w)$ in $Q_n - C$.

Case 3: *The vertices* b, w *are in the subcube* Q^1_{n-1}. I did not succeed in proving this case. ■

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Attachments

This is a list of attachments that can be found on the CD that comes with this thesis. Let $T = (\{b_0, w_1, b_2, w_3, b_4, w_5\}, \{b_0w_1, w_1b_2, b_2w_3, w_1b_4, b_2w_5\})$ be a black-white tree with vertices b_0, b_2, b_4 black and vertices w_1, w_3, w_5 white, see Figure 5.2 where T is denoted by T_2. Both attachments bellow concern Lemma 21 and this particular tree. Let us restate the lemma.

Lemma 25 *Let T be isometric in Q_5. For every $b \in B_5 \setminus V(T)$ and every $w \in W_5 \setminus V(T)$ there exists a Hamiltonian path between b and w in $Q_5 - T$.*

A The program verifying Lemma 21

The folder `A_program` on the attached CD contains: an executable program `hypercubes.exe`, its C++ source code `hypercubes.cpp`, `makefile` and a `README` file. The program verifies Lemma 21. That is, whether $Q_5 - T$ is laceable for T isometric in Q_5.

The program does not take any input. It generates an isometric tree T in Q_5, vertices $b \in B_5$ and $w \in W_5$ that do not belong to the tree T. We call T, b and w in Q_5 with such conditions a valid. Then it tries to find a Hamiltonian path from b to w in $Q_5 - T$ by depth first search. Next step depends on whether it succeeds.

Yes. The program finds a Hamiltonian path H between b and w in $Q_5 - T$. It adds b, w, T and H into a file `paths.out`. Then generates some new valid configuration of b, w and T (if such configuration exists) and repeats the process.

No. The program fails to find a Hamiltonian path between b and w in $Q_5 - T$. It outputs "A Hamiltonian path between b and w was not found for the configuration: b_0 w_1 b_2 w_3 b_4 w_5" and terminates.

If the program runs out of new valid configurations of b, w and T it outputs "The hypothesis was successfully verified." and terminates.

B List of Hamiltonian paths

The folder `B_paths` contains a `README` file and a text file `paths.out`. The file `paths.out` is a list of all Hamiltonian paths between b and w in $Q_5 - T$ for every isometric T in Q_5 and every vertices $b \in B_5 \setminus V(T)$, $w \in W_5 \setminus V(T)$. The structure of the file is a list of items. Each item consists of following three lines.

b w b_0 w_1 b_2 w_3 b_4 w_5
b v_1 d_2 v_3 d_4 ... v_{23} d_{24} w
empty line

Every such item means that there exists a Hamiltonian path $(b, v_1, d_2, v_3, d_4, \ldots, v_{23}, d_{24}, w)$ in $Q_5 - T$ where every d_i is in B_5 and every v_i is in W_5. This output was generated by the program `hypercubes.cpp` in Attachment A. It was computed by a supercomputer at MetaCentrum Virtual Organization [19]. I would like to thank them for allowing me to use their supercomputers.