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Immersions and edge-disjoint linkages

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I would like to thank to my advisor for many interesting ideas and to Tomáš Gavenčiak for all the help and support.

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Abstrakt: Grafové imerze jsou přirozená analogie k intenzivně zkoumanému konceptu grafových minorů a topologických grafových minorů, ale teorie v této oblasti je mnohem méně rozvinutá. V práci se zabýváme hledáním postačujících podmínek pro existenci imerzí a vlastnostmi grafů, které neobsahují imerzi daného grafu.

Dokazujeme, že velká stromová šířka hranově čtyřsouvislého grafu implikuje existenci imerze libovolného čtyřregulárního grafu na malém počtu vrcholů, a že velký maximální stupeň hranově třisouvislého grafu implikuje existenci imerze libovolného třiregulárního grafu na malém počtu vrcholů.

Klíčová slova: teorie grafů, imerze, stromová šířka

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Abstract: Graph immersions are a natural counterpart to the widely studied concepts of graph minors and topological graph minors, and yet their theory is much less developed. In the present work we search for sufficient conditions for the existence of the immersions and the properties of the graphs avoiding an immersion of a fixed graph.

We prove that large tree-with of 4-edge-connected graph implies the existence of immersion of any 4-regular graph on small number of vertices and that large maximum degree of 3-edge-connected graph implies existence of immersion of any 3-regular graph on small number of vertices.

Keywords: graph theory, immersion, tree-width

Contents

Chapter 1

Introduction

The concept of graph immersions is a generalisation of subgraphs, similar to the concept of graph minors. We say that a graph H contains an immersion of a graph G if G can be obtained from H by series of lifting pairs of adjacent edges, vertex deletions and edge deletions, where lifting of the pair of edges uv, vw means deleting both edges uv and vw and replacing them by the edge uw.

We begin with a short informal survey of the most important results for immersions and comparison to similar results for minors. The definitions and results relevant for the rest of the thesis will be formally stated later in this chapter.

The notion of an immersion of a graph was introduced by Nash-Williams in 1960's [6]. He also proposed the famous conjecture, analogical to the Wagner's conjecture for graph minors, that graphs are well-quasi-ordered with respect to the immersion relation. Both conjectures have been proven by Robertson and Seymour [12], [13]. Another problem, studied but unsolved for both for minors and immersions is, whether every k-chromatic graph contains K_k as a minor or an immersion of K_k. For minors, the problem is called Hadwiger's conjecture and is known to be true for $k \leq 6$ [14]. For immersions, the conjecture was proven for $k \leq 7$ [2].

In the rest of this chapter we introduce the notation, terminology and some results.

1.1 Basic definitions

In this section, we survey basic definitions, notation and some theorems from graph theory in particular which are used throughout the thesis. Most of these topics are described in more detail, e.g., in Diestel's book [3].

Definition 1. A *simple graph* (or just a *graph*) G is a pair $(V(G), E(G))$, where $V(G)$ is a set of vertices and $E(G) \subseteq \binom{V(G)}{2}$ is a set of edges. An edge $e = \{u, v\}$ is usually denoted as uv and the vertices u end v are called *ends* of the edge e. A graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ is a *subgraph* of G. Two vertices $u, v \in V(G)$ are *adjacent* if there exists an edge $e = uv$ in $E(G)$. Vertices adjacent to v are also called *neighbors* of v and the set of all vertices adjacent to v is the *neighborhood* of v. Edges are *adjacent* if they have an end in common. Pairwise nonadjacent vertices or edges are called *independent*. Independent set of edges is also called a *matching*. A *vertex cover* of a graph G is a set of vertices C such that every edge in $E(G)$ has an end in C.

The following theorem shows a relation between a matching and a vertex cover.

Theorem 1. *(König 1931) The maximum size of a matching in G is equal to the minimum size of a vertex cover.*

Definition 2. A *multigraph* M is a pair $(V(M), E(M)$ of disjoint sets, where $V(M)$ is a set of vertices and $E(M)$ is a set of edges, and a map $\mathcal{E} : E(M) \to V(M) \cup \binom{V(M)}{2}$ that assigns to every edge one or two vertices, its *ends*. An edge with only one end is called a *loop*. We write $e = xy$ to express that the edge e has ends x and y, but it does not uniquely determine the edge. A *multiedge* of a multigraph is a nonempty set of edges $\mathcal{E}^{-1}(m)$ for some $m \in V(M) \cup \binom{V(M)}{2}$. If $x, y \in V(M)$, the multiedge $\mathcal{E}^{-1}(\{x, y\})$, is denoted $\widehat{xy}$, the multiedge $\mathcal{E}^{-1}(\{x\})$ is denoted $\widehat{xx}$. The *multiplicity* of a multiedge is a number of its elements.

Definition 3. An *directed graph* D is a pair $(V(D), E(D))$ of disjoint sets, where $V(D)$ is a set of vertices and $E(D)$ is a set of edges, and maps $\mathrm{init} : E(D) \to V(D)$ and $\mathrm{term} : E(D) \to V(D)$. For an edge $e \in E(D)$, a vertex $\mathrm{init}(e)$ is called the *initial* vertex of e, $\mathrm{term}(e)$ is called the *terminal* vertes of e and we say that e is *directed from* $\mathrm{init}\, e$ *to* $\mathrm{term}\, e$.

Definition 4. If G is a simple graph, a directed graph or a multigraph, we say that a vertex v has *degree* k if v is contained in k edges in G. We write it $\deg v = k$. If all the vertices of G have the same degree k, G is *k-regular*. The number $\delta(G) = min\{\deg v | v \in V(G)\}$ is the *minimum degree* of G, the number $\Delta(G) = max\{\deg v | v \in V(G)\}$ is the *maximum degree* of G and the number

$$\mathrm{d}(G) = \frac{\sum_{v \in V(G)} \deg v}{|V(G)|}$$

is the *average degree* of G.

If G is an directed graph, a vertex v has *in-degree* k if v is the terminal vertex of k edges of G, we write $\deg^- v = k$. A vertex v has *out-degree* k if v is the initial vertex of k edges of G, we write $\deg^+ v = k$.

Definition 5. Graphs G and H are *isomorphic* if there exists a bijection $\varphi : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(H)$.

In the thesis, we consider isomorphic graphs to be equal and graph classes to be closed under isomorphism.

Definition 6. A *path* P of length n is a graph with $n+1$ vertices $v_0, \dots, v_n$ and edges $e_i = v_i v_{i+1}$ for $i = 0, \dots, n-1$. The vertices v_0 and v_n are *ends* of the path P and the vertices $v_1, \dots, v_{n-1}$ are *internal vertices* of the path. A path in a graph G between vertices u and v or from u to v is a subgraph of G which is a path with ends u and v. *Distance* between vertices u, v in a graph G is length of a shortest path between u and v or infinity if there is no path between u and v in G.

A *cycle* C of length n, where $n \geq 3$, is comprised of a path of length $n-1$ with ends u and v and an edge uv.

An *n-star* S is a graph with $n+1$ vertices $v_0, \dots, v_n$ and edges $v_0 v_i$ for $i = 1, \dots, n$. We say that there is an *n-star between vertices* $v_1, \dots, v_n$ in a graph G, if there exists a vertex v_0 in G such that there is an edge $v_0 v_i$ in G for every $i = 1, \dots, n$.

A graph with an edge between every two vertices, i.e. $(V, \binom{V}{2})$, is called a *complete graph* or a *clique* and is denoted K_n, where $n = |V|$.

Definition 7. Let $G = (V, E)$ be a graph, $e = uv$ an edge and w a vertex in G. Then the graph obtained from G by *deleting the edge* e is the graph $G \setminus e = (V, E \setminus \{e\})$. The graph obtained from G by *deleting the vertex* w is the graph $G \setminus w = (V \setminus \{w\}, E \cap \binom{V \setminus \{w\}}{2})$.

Definition 8. A graph G is *connected* if there exists a path between any two vertices of G. Maximal connected subgraphs of a graph G are *connected components*.

A *vertex cut* in a connected graph G is a set W of vertices of G, such that the graph $G \setminus W$ is not connected. A vertex cut W is called a *k-cut* if its size is k, i.e., $|W| = k$. The only vertex in a 1-cut is called an *articulation*. G is called *k-connected* if its minimum vertex cut has size at least k. *The connectivity* of a graph G is the size of its minimum vertex cut if G is not complete and it is n if G is a complete graph on n vertices.

An *edge cut* in a connected graph G is a set W of edges of G, such that the graph $G \setminus W$ is not connected. The only edge in a 1-edge-cut is called an *cut edge*. G is called *k-edge-connected* if its minimum edge cut has size at least k. The *edge-connectivity* of a graph G is the size of its minimum edge cut.

A graph G is *k-linked* if G has at least $2k$ vertices and for every $2k$ distinct vertices $s_1, \dots s_k$, $t_1, \dots t_k$, there exist k pairwise vertex disjoint paths $P_1, \dots P_k$ such that s_i and t_i are the ends of the path P_i for each i.

For a subgraph G of H and $X \subseteq V(G)$, a *G-bridge over X* is a maximal connected subgraph of $H \setminus E(G)$ such that it does not contain any vertex cut consisting of vertices of X (i.e. vertices of $H \setminus X$ are in the same G-bridge over X if there exists a path between them in $H \setminus E(G) \setminus X$).

Theorem 2 (Mader)**.** *For every $k \in \mathbb{N}$, every graph with average degree $d(G) \geq 4k$ has a $(k+1)$-connected subgraph H such that $d(H) > d(G) - k/2$.*

Theorem 3. *(Thomas, Wollan [17]) Let G be a graph and $k \in \mathbb{N}$. If $d(G) \geq 4k$ and G is $2k$-connected, then G is k-linked.*

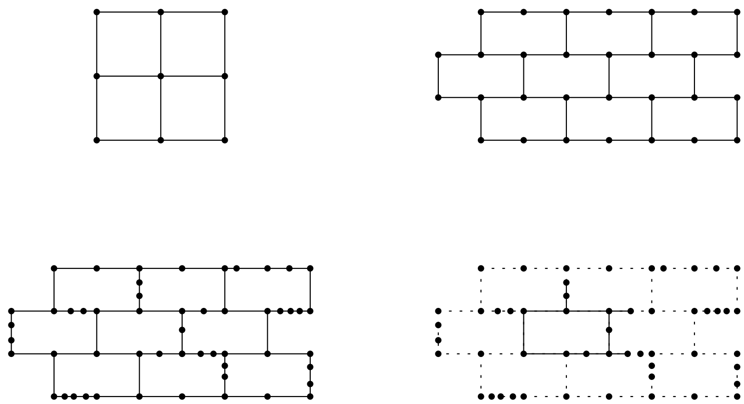


Figure 1.1: A grid, an elementary wall, a wall and its subwall

Definition 9. A *forest* is a graph that does not contain any cycle as a subgraph. A connected forest is called a *tree*. A *spanning tree* of a connected graph G is a tree T such that $V(T) = V(G)$ and $E(T) \subseteq E(G)$. A *spanning forest* of a graph G is a forest consisting of spanning trees of all the connected components of G. Vertices of degree 1 in a tree are *leaves*.

A *rooted tree* is a tree with one distinguished vertex, a *root*. In a rooted tree, the root is not a leaf, even if it has degree 1.

Definition 10. A *grid* of size $n \times m$ is a graph with vertices corresponding to a vertices in a plane with integer coordinates $[x, y]$ with $0 \leq x \leq n$, $0 \leq y \leq m$ and edges between vertices in (Euclidean) distance 1. A *subgrid* of a grid is subgraph of a grid that is a grid.

An *elementary wall of height h* is obtained from agrid of size $2h+1 \times h$ by deleting edges between vertices of the grid that can be written as $[2i, 2j]$ and $[2i, 2j+1]$, or $[2i+1, 2j+1]$ and $[2i+1, 2j+2]$ for $i, j \in \mathbb{N}$ and deleting vertices of degree one. I.e., an elementary wall of height h has h rows and h faces in every row as in the Figure 10.

A *wall of height h* is any subdivision of an elementary wall of height h.

A *subwall* of height h' of a wall W is a connected subgraph of W that consists of a wall W' of height h' smaller than height of W and a (possibly empty) set of paths such that each path has one end in W' and the rest of the path is disjoint from W'. *Boundary vertex* of a subwall S of a wall W is a vertex s of S such that there exists a vertex v in $V(W) \setminus V(S)$ such that sv is an edge of W. The set of all boundary vertices of S is denoted ∂_S.

Two vertices u and v of a wall W are in *wall distance k*, if on every path from u to v in W is at least $k+1$ branching vertices of W. I.e., distance between two vertices of elementary wall is the same as their wall distance.

Definition 11. A graph G is *planar* if it can be drawn into a plane ($\mathbb{R}^2$) so that vertices correspond to points of a plane and edges correspond to arcs that are pairwise disjoint except for the endpoints. By an arc we mean an image of an injective continuous function from interval $[0, 1]$ to $\mathbb{R}^2$.

The notion of embeddability into the surface naturally generalizes the notion of planarity.

Definition 12. By a *surface* we mean a 2-dimensional topological manifold. An *embedding of a graph* into a surface Σ is representation of a graph on Σ such that vertices correspond to points of the surface and the edges correspond to arcs that are pairwise disjoint except for the endpoints. By arcs we mean an image of an injective continuous function from $[0, 1]$ to Σ.

A *face* is a maximal connected subset of the space obtained from the surface by removing the points corresponding to the vertices and edges of the graph.

1.2 Immersions, graph minors and tree-width

In this section, we survey definitions and results related to immersions, graph minors and tree-width, focusing on the relation between immersions and graph minors and their connection to tree-width.

Definition 13. A graph obtained from G by *subdividing* an edge $uv \in V(G)$ is the graph obtained by adding a vertex w into $V(G)$ and replacing the edge uv by edges uw and vw. A graph H is a *subdivision* of a graph G, if H can be obtained from G by zero or more edge subdividings. A set B of vertices of H is the set of *branching vertices* for G, if H can be obtained from G by subdividing edges and $B = V(G)$.

Definition 14. A graph obtained from a graph G by *contracting an edge* $e = uv$ is the graph obtained by deleting the edge e and identifying the vertices u and v. The resulting vertex $\overline{uv}$ is adjacent to all the vertices which are adjacent to u or to v in $G \setminus e$.

Definition 15. A graph G is a *minor* of a graph H, if G can be obtained from H by a sequence of vertex deletions, edge deletions and edge contractions. If this sequence is nonempty, G is a *proper minor* of H.

Definition 16. A graph resp. multigraph obtained from a graph resp. multigraph G by *lifting a pair of adjacent edges* $e = uv$ and $e' = vw$ is the graph obtained by deleting the edges e and e' and adding the edge $f = uw$. A graph resp. multigraph obtained from a graph resp. multigraph G by *lifting a path* P between u and v is the graph obtained by deleting all the edges in P and adding the edge $e = uv$.

Note that the result of lifting in a simple graph can be a multigraph. Lifting edges $e = uv$ and $e' = vw$ where v is a vertex of degree 2 and deleting v afterwards is an inverse process to subdividing an edge $f = uw$.

Definition 17. A graph G is *immersed* in a graph H, if G can be obtained from H by a sequence of vertex deletions, edge deletions and lifts of pairs of adjacent edges. Alternatively, we say that H contains an *immersion* of G.

Throughout our considerations we often use the word immersion in the context when some of the vertices of graphs G and H are distinguished; we then require that the distinguished vertices of G correspond to the appropriate distinguished vertices of H.

Note that both relations "contains as a minor" and "contains an immersion" are transitive.

It is easy to see that if H contains a subdivision of G as a subgraph, H also contains G as a minor and an immersion. However, there is no relation between minor and immersion containment in general. For example, a planar graph G_1 cannot contain a nonplanar graph as a minor because all minor operations preserve embeddability into plane but G_1 can contain an immersion of nonplanar graph as shown in Fig.1.2. Complete graph on 5 vertices that is nonplanar can be obtained from G_1 by lifting paths $v_1u_1u_2v_3$, $v_2u_2u_3v_4$, $v_3u_3u_4v_5$, $v_4u_4u_5v_1$ and $v_5u_5u_1v_2$ and deleting vertices $u_1, \ldots u_5$, i.e., K_5 is immersed in $G1$.

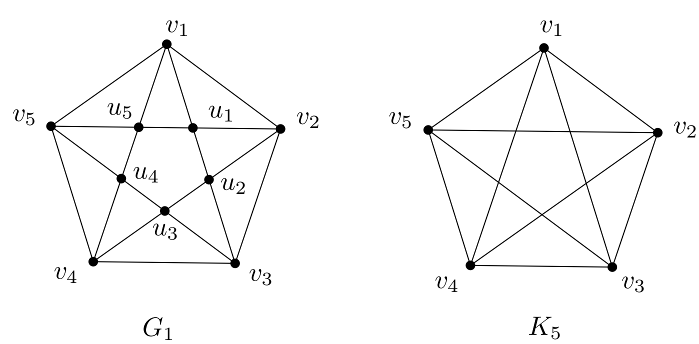


Figure 1.2: Planar graph containing immersion of K_5

On the other hand, a graph G_2 with maximum degree Δ cannot contain an immersion of a graph with degree greater than Δ because none of immersion operations can increase degree of a vertex, while G_2 can contain a graph with degree greater than Δ as shown in Fig. 1.3. G_2 of maximum degree 3 contains as 4-star with a vertex of degree 4. A 4-star can be obtained from G_1 by contracting the edge uv.

However, for graphs with maximum degree at most 3, containing G as a minor implies existence of an immersion of G.

Proposition 4. *[3] Let G be a graph with maximum degree at most 3. If a graph H contains G as a minor, then H contains a subdivision of G as a subgraph.*

Corollary 5. *Let G be a graph with maximum degree at most 3. If a graph H contains G as a minor, then H contains a an immersion of G.*

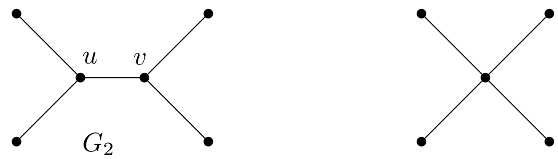


Figure 1.3: Graph of maximum degree 3 containing 4-star as a minor

The following three results compare sufficient conditions on average degree of a graph G to contain a subdivision K_r, resp. K_r as a minor, resp. an immersion of K_n.

Theorem 6. *(Kostochka, Thomasson) There exists a constant $\alpha \in \mathbb{R}$ such that for every $r \in \mathbb{N}$, every graph G of average degree $d(G) \geq \alpha r^2$ contains a subdivision of K_r as a subgraph. In particular, α can be* 10.

Theorem 7. *(Kostochka [4]) There exists a constant $\alpha \in \mathbb{R}$ such that for every $r \in \mathbb{N}$, every graph G of average degree $d(G) \geq \alpha r \log r$ contains K_r as a minor. Up to the value of α, this bound is best possible as a function of r.*

Quite recently, an analogical theorem was proved for containing an immersion of K_n.

Theorem 8. *[1] There exists a constant $\alpha \in \mathbb{R}$ such that for every $r \in \mathbb{N}$, every graph G of average degree $d(G) \geq \alpha r$ contains an immersion of K_r.*

In the rest of the section we focus on tree-width and its relation to graph minors. In all the results, the graphs are required to be simple.

Definition 18 (Robertson & Seymour [8])**.** A *tree decomposition* of a graph G is a pair $(T, \mathcal{V})$ where T is a tree (a *decomposition tree*) and $\mathcal{V} = \{V_t\}_{t \in V(T)}$ is a system of subsets $V_t \subseteq V(G)$ with the following properties:

- $\bigcup_{t \in V(T)} V_t = V(G)$
- for every edge $uv \in E(G)$ there exists $t \in V(T)$ such that $\{u, v\} \subseteq V_t$
- if $t, t', t'' \in V(T)$ and t'' is on the path between t and t', then $V_t \cap V_{t'} \subseteq V_{t''}$

The *width* of a tree decomposition $(T, \mathcal{V})$ is the size of the largest set V_t in the tree decomposition decreased by 1, i.e., $\max_{t \in V(T)}(|V_t| - 1)$.

The *tree-width* of a graph G is the minimum width of a tree decomposition of G.

The concept of tree decompositions and tree-width was introduced by Robertson and Seymour in 1980's. The relation of tree-width and graph minors was extensively studied in the series of their papers [7] and in many other works by other authors. Here, we restrict ourselves to a few results that are needed for proving the main result of the thesis.

Theorem 9. *[10] For every planar graph H, there is a number ω such that every graph with tree-width $\geq \omega$ has a minor isomorphic to H.*

Corollary 10. *For every number k, there is a number ω such that every graph with tree-width $\geq \omega$ contains a subdivision of a wall of height k.*

Lemma 11. *[9] For every n there exists d such that a grid with n prescribed vertices in distance at least d from each other contains a subdivision of any planar graph G with maximum degree 4 on n vertices with vertices of G mapped to the prescribed vertices.*

The following lemma is a well known consequence of tree-width duality theorem [15].

Lemma 12. *A grid $k \times k$ has tree-width k.*

1.3 Graph Structure Theorem

Definition 19. A graph G' is said to be obtained from a graph G by adding an *apex vertex* v, if $G' \setminus v = G$.

Definition 20. Let F be a face of an embedded graph G and let $v_0, v_1, \ldots, v_n = v_0$ be the vertices on the boundary of F, in the circular order. A *circular interval* for F is the set of vertices $\{v_i, v_{(i+1) \mod n}, \ldots, v_{(i+k) \mod n}\}$, where i and k are integers and $0 \leq i < n$. Let Λ be a finite list of circular intervals for F. We con struct a new graph as follows. For each circular interval $L \in \Lambda$, we add a new vertex v_L and edges between v_L and vertices of some (possibly empty) subset of L. If two intervals L and M of Λ have nonempty intersection, we may add an edge between v_L and v_M. If every vertex on the boundary of F appears in at most d intervals of Λ, we say that the resulting graph is obtained from G by adding a *vortex* of depth at most d to the face F .

Definition 21. let G and H be graphs and k a nonnegative integer. A *k-clique sum* of G and H is a graph obtained by identifying clique of size $m \leq k$ in G with a clique of size m in H and deleting some of the edges in this clique.

Theorem 13 (structural theorem)**.** *[11] For any graph G, there exists a positive integer k such that every graph H that does not contain G as a minor can be obtained as a k-clique sum of graphs $H_1, \ldots H_n$, where every H_i is a graph obtained from a graph H_i' that is embeddable on a surface on which H does not embed by adding at most k vortices, each of them of depth at most k, and at most k apex vertices.*

Chapter 2

Immersing graphs with maximum degree at most 4

In this chapter we prove the main result of the thesis that shows a relation between tree-width of a graph and containment of immersion of a graphs with small maximum degree.

Theorem 14. *For every integer n there exists m such that every 4-edge-connected graph H of tree-width at least m contains an immersion of any graph G on n vertices with maximum degree at most 4.*

The additional condition of 4-edge connectivity on H is necessary. A wall of height h contains a grid of size $h \times h$ as a minor. Therefore, by Lemma 12, if H contains a wall of height m^2, tree-width of H is at least m. Let $H_{counter}$ be a graph consisting of a wall W of height m^2 and $|V(W)|$ pairwise disjoint copies of K_5, such that every copy of K_5 contains exactly one vertex of W. Then $H_{counter}$ is a graph of tree-width at least m with minimal degree 4, but for arbitrarily large m, $H_{counter}$ does not contain an immersion of any 4-edge-connected graph on more than 5 vertices.

Before we start proving the theorem itself, we make a few simple observations. The following observation shows a correspondence of graphs with maximum degree at most 4 to planar graphs of maximum degree at most 4. In the proof of the theorem, we sometimes look for an immersion of a planar graph G^p that contains an immersion of G, instead of G.

Observation 15. *For every graph G on n vertices with maximum degree at most 4 there exists a planar graph G^p on at most $2n^2$ vertices with maximum degree at most 4 such that G is immersed in G^p.*

Proof. Since the maximum degree of G is at most 4, G has at most $2n$ edges. Consider a drawing of G into plane with minimal number of crossings such that no three edges cross in one point. There exist at most $\binom{|E|}{2} \leq \binom{2n}{2} = 2n^2 - n$ pairs of crossing edges. By replacing a pair of crossing edges u_1u_2 and v_1v_2 by a new vertex x and edges u_1x, u_2x, v_1x and v_2x we obtain a drawing of the graph G' that contains an immersion

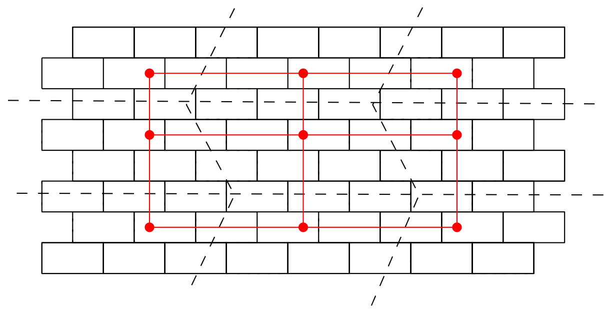


Figure 2.1: A wall of height 8 divided into 9 subwalls of height 2

of $G - G$ can be obtained from G' by lifting pairs of edges u_1x,xu_2 and v_1x, xv_2 and deleting the vertex x. Moreover, the drawing of G' has less pairs of crossing edges than the drawing of G. By repeatedly replacing crossing pairs of edges in this way, after at most $2n^2 - n$ replacements we obtain a plane graph G^p with at most $2n^2$ vertices that contains an immersion of G. $\square$

The following observation describes a construction of a grid from subwalls in a wall.

Observation 16. *For integers h' and k, every wall W of height $h = (h'+1)k$ contains k^2 disjoint subwalls of height h'. Let Gr be an auxiliary graph such that $V(Gr)$ is the set of k^2 subwalls and there is an edge between two subwalls W' and W'' in Gr if and only if there exists an edge between vertices of W and W' in W. Then the k^2 walls can be chosen in such a way that Gr contains a subgraph isomorphic to a grid $k \times k$.*

Every subwall W' of height h' contains $2(h'-1)(h'+1)$ branching vertices that are connected to other branching vertices of W' by three edge disjoint paths in W'. Let us call such vertices the inner branching vertices *of a subwall W'.*

In the rest of the chapter, we prove Theorem 14. We interrupt the proof several times, to formulate and prove auxiliary observations but get back to the main proof immediately afterwards. In the second part of the proof we discuss four separate cases **A.-D.**. The cases are not entirely disjoint but in each case we assume that none of the previous cases occurs.

Proof of Theorem 14. Let G be a graph on n vertices with maximum degree 4. Let d be a distance between prescribed vertices in the grid, required by Lemma 11 for embedding a planar graph on $2n^2$ vertices into a grid. Let $c = 4 \cdot 10^8 n^5$ and $k = 2c^2n^2d^2 = 2^5 \cdot 10^{16} n^{12} d^2$. By Observation 16, there exists h such that any wall of height h contains at least k disjoint subwalls that form a grid described in the observation and each subwall contains at least c inner branching vertices. By Corollary 10, there exists m such that every graph of tree-width at least m contains a wall of height h as a minor.

Let H be a 4-edge-connected graph with minimum degree at least 4 and of tree-width at least m. To prove the theorem we need to show that H contains an immersion of G. The structure of a graph H is too difficult to describe in general, therefore we first construct a graph H', consisting of a wall of height h and additional edges, that is immersed in H. Then we will look for an immersion of G in H'.

By Corollary 10 and the choice of m, H contains a wall of height h as a subgraph.

Observation 17. *If there exists a wall W' of height h in H, then there exists a wall W of height h such that for every branching vertex b of W, there exists a path in $H \setminus E(W)$ between b and a vertex $c \in V(W)$ that is in wall distance at least one from b.*

Proof. Take any wall W' of height h in H. Suppose that there exists a branching vertex b' of W' such that in $H \setminus E(W')$ there is no path from b' to any vertex of W' in wall distance at least one from b'. Color the components of $H \setminus E(W')$ containing vertices of W' in wall distance at least one from b' blue, the component containing b' red and all other components green.

Observe that if there is a path in $H \setminus E(W')$ between b different from b' and a vertex $c \in V(W)$ in wall distance at least one from b, this path is in a blue component.

We want to find two edge disjoint paths Q_1 and Q_2 such that one end of Q_1 and Q_2 is b' and the other end q of Q_1 is one of the branching vertices of W' in wall distance exactly one from b' and the other end of Q_2 is a vertex of W' in wall distance at least one from b' – to find such a path it is enough to find a path from b' to any blue vertex b, because then there exists a path in a blue component from b to a vertex in wall distance at least one from b'. By replacing a path P between b' and q in W' by Q_1, we obtain a wall W, in which there is a desired path between b' to a blue vertex in $H \setminus E(W)$.

Moreover, we find such a path Q_1, that for every branching vertex b, if it was connected to a vertex in wall distance at least one in $H \setminus E(W')$, it is connected in $H \setminus E(W)$, for W obtained from W' by replacing P by Q_1, too. Note that every path Q_1 such that $E(Q_1) \setminus E(P)$ contains only red and green edges has this property.

By repeating such a replacement we can decrease a number of branching vertices of a wall that are not connected to a vertex in wall distance one in $H \setminus E(W)$ to zero.

It remains to describe how to find a path Q_1.

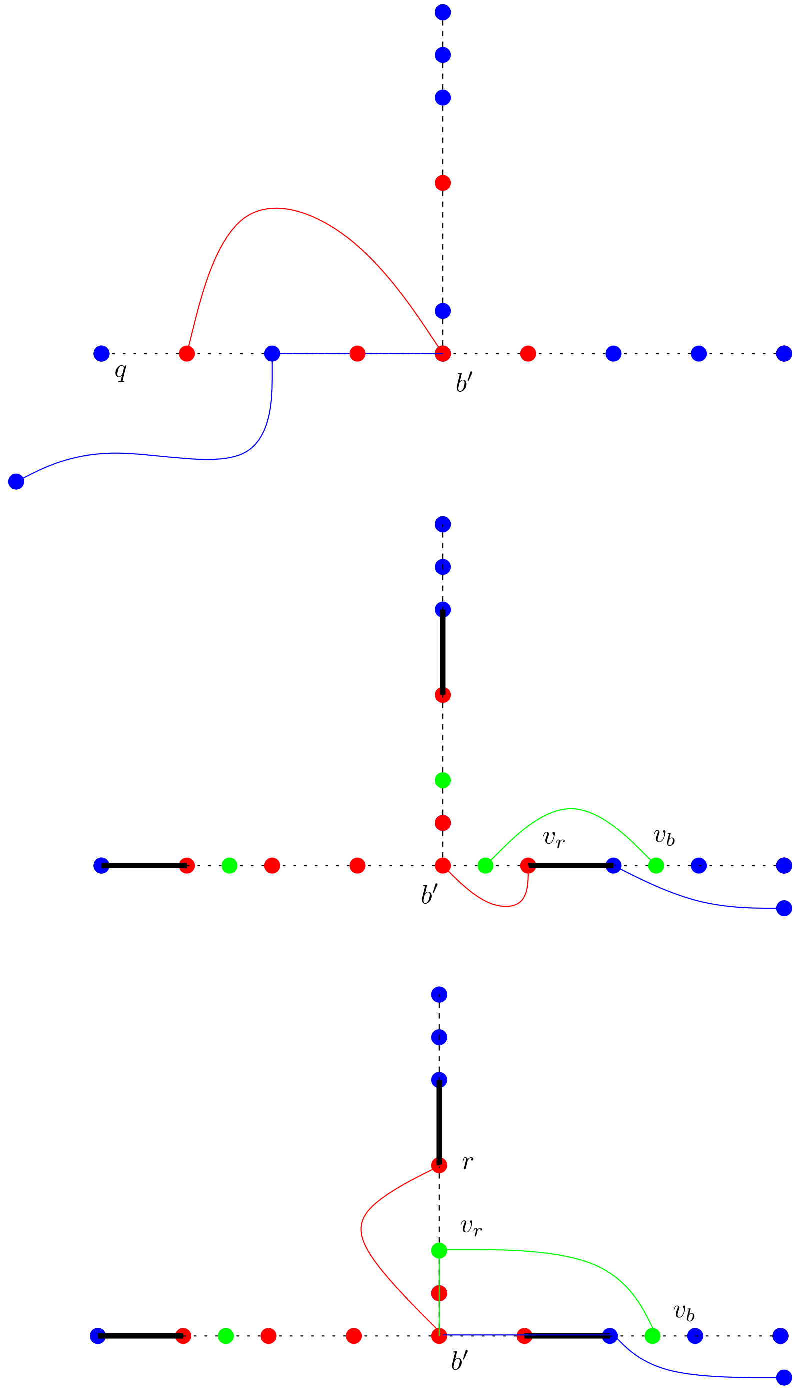
If there is a path P of W' between b' and its neighboring branching vertex containing a red vertex r between two blue vertices (by between we mean on the subpath between these two vertices not necessarily adjacent to any of them), we can obtain a path Q_1 from a path P by replacing a part of P between b' and r by a path in red component as shown in Figure A. We also get a path Q_2 that is edge disjoint with Q_1, from b' to a blue vertex.

If there is no such path between b' and its neighboring branching vertex, it means that on every path there are blue and red vertices separated as in Figure B. Since H is 4-edge-connected, there exists a green component containing at least one vertex

A

B

C



v_b between blue and one vertex v_r between red vertices. If v_r and v_b are on the same path P, we can obtain a path Q_1 from P by replacing a subpath of P between v_r and v_b by a path in the green component as shown in Figure B. If v_r and v_b are on different paths P_r and P_b of W' between b' and neighboring branching vertices of W', we first replace a path P_r containing v_r by a path Q_1' obtained from P_r by replacing the shortest subpath of P_r that contains all the red vertices in P_r by a path in the red component as shown in Figure C. By this replacement we obtain a wall W where the former green component is a part of component of $H \setminus E(W)$ containing b', so we can proceed as in the case where there is one red vertex between two blue vertices. □

Definition 22. Let us call a wall W in H such that for every branching vertex b of W, there exists a path in $H \setminus E(W)$ between b and a vertex $c \in V(W)$ that is in wall distance at least one from b *a good wall.*

(continued proof of Theorem 14)

Let W be a good wall of height h in H. Then W can be divided into k subwalls such that each such subwall $\mathbf{v}$ contains at least c inner branching vertices. Denote B a set of all the inner branching vertices in all the subwalls and $\mathbf{V}$ the set of the subwalls. Let us denote $C_1 \dots C_m$ all the W-bridges over B. For every vertex $v \in B$ we choose a shortest path P_v in $H' \setminus E(W)$ to a vertex of W in wall distance at least one.

Let B_i be a set of all the vertices $b \in B$ such the path P_b is contained in C_i, if the number of such vertices is even or one, otherwise let B_i be a set of all such vertices except one. Thus, for every i, B_i contains either one or an even number of vertices.

If there is only one vertex b in B_i, we lift P_b, and delete the rest of C_i. Let M_i be a matching consisting of a single edge obtained by lifting P_b.

Otherwise, for every C_i, we choose a minimal tree T_i containing B_i such that every vertex of B_i is a leaf of T_i (such a tree can be obtained from a spanning tree of $C_i \setminus B_i$, by adding vertices of B_i as leaves and deleting subtrees without vertices adjacent to B_i).

By the following observation, every T_i contains an immersion of a perfect matching M_i on B_i.

Observation 18. *Every tree with $2k$ leaves contains an immersion of a matching of size k on its leaves.*

Proof. We will proceed by induction. For a tree with only 2 leaves, the observation holds from the connectivity of a tree. Suppose it holds for every integer smaller than k. Let us have a tree T with $2k$ leaves. If T contains vertices of degree two, lift the pairs of edges adjacent to those vertices and delete the vertices. Now, every vertex either has degree at least 3 or is a leaf. There exists a vertex b of degree at least 3 adjacent to at least two leaves u, v. Lift the edges ub and bv and delete b if it becomes a leaf. The resulting graph Y is immersed in T and consists of an edge uv and a tree T' with $2(k-1)$ leaves. By induction hypothesis, T' contains an immersion of a matching of size $k-1$ on leaves. Therefore, Y contains an immersion of a matching of size k on leaves. $\square$

(continued proof of Theorem 14)

Let $M = \cup_{i=1}^{m} M_i$. Let H' be a multigraph $W \cup M$ with double edges where edges both in W and M exist. Then H' is immersed in H, $V(H') = V(W)$ and at least 2/3 of the vertices in B are connected by an edge in M with a vertex in wall distance at least one.

Define an auxiliary multigraph $\mathbf{H'} = (\mathbf{V}, \mathbf{E})$ such that $\mathbf{E} = \mathbf{E_W} \cup \mathbf{E_M}$, where $\mathbf{E_W} = \{\mathbf{e} = \mathbf{uv} | \exists u \in \mathbf{u}, v \in \mathbf{v}, uv \in E(W)\}$, i.e., $\mathbf{W} = (\mathbf{V}, \mathbf{E_W})$ is a grid with all edges simple, and $\mathbf{E_M} = E(M)$, such that $\mathcal{E}(uv) = \{\mathbf{u}, \mathbf{v}\}$ if $u \in \mathbf{u}$ and $v \in \mathbf{v}$.

For every natural number i, define a simple graph without loops $\mathbf{H'_i}$, such that $V(\mathbf{H'_i}) = \mathbf{V}$ and there is an edge $\mathbf{uv}$ in $\mathbf{H'_i}$, if and only if an edge $\widehat{\mathbf{uv}}$ has multiplicity at least i in $\mathbf{E_M}$.

The multigraph $\mathbf{H'}$ always has at least one of these four properties (as shown below):

A. At least $l = 2n^2d^2$ nodes of $\mathbf{V}$ are incident to a loop in $\mathbf{E_M}$

B. The average degree of $\mathbf{H'_1}$ is at least $\varphi = 10^5 n^3$

C. There is a matching of size l in $\mathbf{H'_\mu}$ where $\mu = 2000n^2$

D. There is a vertex cover of $\mathbf{H'_\mu}$ of size at most $l-1$

We prove that G is immersed in H' for each of these cases separately.

A. A loop $\mathbf{e}$ in $\mathbf{H'}$ corresponds to an edge in M between two vertices of the same subwall $\mathbf{v}$. Consequently $\mathbf{v}$ incident to a loop contains a vertex v of degree four, with all the neighbors in $\mathbf{v}$. We show that there is a subdivision of G^p in H', that uses these vertices as a vertices of G^p and edges of W as a paths between subwalls containing these vertices.

Let L be a set of nodes of $\mathbf{V}$ incident to a loop in $\mathbf{E_M}$. If L contains at least l nodes, we can find $2n^2$ disjoint subgrids in $\mathbf{W} = (\mathbf{V}, \mathbf{E_W})$ of size $d \times d$ with vertex of L in the middle. These vertices are in distance at least d from each other, therefore $\mathbf{W}$ contains a subdivision of G^p by lemma 11. We show that there is also a subdivision of G^p in H', using the following observations.

Observation 19. *For every subwall* $\mathbf{v}$ *and any three boundary vertices* v_1, v_2, v_3 *of* $\mathbf{v}$, $H'[V(\mathbf{v})]$ *contains an immersion of a 3-star between the vertices* v_1, v_2 *and* v_3. *The similar claim holds for 2-stars.*

Observation 20. *For every subwall* $\mathbf{v}$ *incident to a loop and for any four boundary vertices* v_1, v_2, v_3, v_4 *of* $\mathbf{v}$, $H'[V(\mathbf{v})]$ *contains an immersion of a 4-star between vertices* v_1, v_2, v_3, v_4.

Let $\bar{G^p}$ be a subdivision of G^p in $\mathbf{W}$. For every edge $\mathbf{uv}$ in $\bar{G^p}$, choose an edge $u_{\mathbf{v}}v_{\mathbf{u}} \in E(W)$ between boundary vertices $u_{\mathbf{v}} \in \partial_{\mathbf{u}}$ and $v_{\mathbf{u}} \in \partial_{\mathbf{v}}$. Let Δ be the set of these edges. By the previous observation, for every $\mathbf{v} \in L$, there exists a vertex $v \in \mathbf{v}$ that is a center of $\deg_{\bar{G^p}}(\mathbf{v})$-star $S_{\mathbf{v}}$ between all the vertices $v_{\mathbf{u}}$, such that $\mathbf{u}$ is a neighbor of $\mathbf{v}$. The union of the stars $S_{\mathbf{v}}$ for all $\mathbf{v} \in \bar{G^p}$ and Δ is a desired subdivision of G^p in H'.

Let O be a set of loops in $\mathbf{E_M}$. If there are less than l nodes incident to a loop, then the average degree of $\mathbf{H'} \setminus O$ is at least Φ, where

$$\Phi = \frac{\frac{2}{3}ck - cl}{k} = c\left(\frac{2}{3} - \frac{l}{k}\right) = c\left(\frac{2}{3} - \frac{l}{c^2 l}\right) = c\left(\frac{2}{3} - \frac{1}{c^2}\right) > \frac{c}{2}$$

The high average degree of a multigraph without loops implies either that the average number of neighbors of vertices is high or that there is a lot of multiedges with large multiplicity. The former case is discussed in **B.**, the latter in **C.** and **D.**.

B. Suppose that $\mathbf{H_1'}$ has average degree at least φ. First, we make a simple technical observation necessary for the proof of Lemma 22.

Observation 21. *Let G be an directed graph on n vertices, $c_1 > 1$ and $1 > c_2 > 0$. If $\Delta^+(G) \leq c_1 \cdot d^+(G)$, then G contains at least $\frac{1-c_2}{c_1-c_2}n$ vertices of out-degree at least $c_2 \cdot d^+(G)$.*

Proof. The observation is purely arithmetic and follows directly from estimating $d^+(G)$ from above. □

In Lemma 22 we show that if $\mathbf{H_1'}$ has average degree at least φ, then it contains a subdivision of K_n that can be used for finding an immersion of G in H'. The lemma is a modification of Theorem 6 and the proof of the lemma is analogical to the proof of Theorem 6 that can be found in [3].

Definition 23. Suppose that $\mathbf{H_1'}$ contains a subdivision $\bar{K_n}$ of K_n. We say that and edge uv in H' is a *presentable edge* for $\mathbf{uv}$, if $u \in \mathbf{u}$, $v \in \mathbf{v}$ and u is an inner branching vertex of $\mathbf{u}$ if $\mathbf{u}$ is a branching vertex of $\bar{K_n}$ and v is an inner branching vertex of $\mathbf{v}$ if $\mathbf{v}$ is a branching vertex of $\bar{K_n}$.

Lemma 22. *If $\mathbf{H_1'}$ has average degree at least φ, $\mathbf{H_1'}$ contains a subdivision $\bar{K_n}$ of K_n such that for every edge $\mathbf{uv}$ of $\bar{K_n}$, there exists a presentable edge uv in H'.*

Proof. For every edge $\mathbf{e}$ in $\mathbf{H'} \setminus O$, we choose one edge $e_p \in \mathbf{e}$. We find a subdivision $\bar{K_n}$ such that if $\mathbf{e}$ is $\bar{K_n}$, then e_p is a presentable edge for $\mathbf{e}$. We orient the edges in $\mathbf{H_1'}$ such that the edge $\mathbf{e} = \mathbf{uv}$ is directed from $\mathbf{u}$ to $\mathbf{v}$ if and only if e_p is adjacent to a branching vertex in $\mathbf{u}$. (I.e., some edges can be directed in both directions.) Denote $\overrightarrow{\mathbf{H_1'}}$ the directed graph obtained this way.

By Mader's theorem, if $d(\mathbf{H_1'}) \geq \varphi$, then there exists a subgraph $\mathbf{R}$ of $\mathbf{H_1'}$ that is $\frac{\varphi}{4}+1$-connected and its average degree is greater than $\frac{7}{8}\varphi$. Then $\mathbf{R}$ is $\frac{\varphi}{8}$-linked by Theorem 3.

Let $\overrightarrow{\mathbf{R}}$ be a graph $\mathbf{R}$ with edges directed as in $\overrightarrow{\mathbf{H_1'}}$. Note that $d^+(\overrightarrow{\mathbf{R}}) \geq d(\mathbf{R})/2 > 7\varphi/16$ and $\Delta^+(\overrightarrow{\mathbf{R}}) \leq c$ and $\mathbf{R}$ has at least $\varphi/4+1$ vertices. By Observation 21 for $c_1 = c/(7\varphi/16)$, $c_2 = 1/8$, there exist at least

$$D = \frac{7^2\varphi}{16\cdot 8c}\cdot\frac{\varphi}{4} = \frac{7^2\varphi^2}{2^9 c} = \frac{7^2 10^{10} n^6}{2^{11}10^8 n^5} > n$$

vertices of out-degree at least $1/8\varphi > n^2$ in $\overrightarrow{\mathbf{R}}$.

Then we can choose n vertices of out-degree at least $1/8\varphi$ in $\overrightarrow{\mathbf{R}}$ as branching vertices of $\bar{K}_n$ and for every such vertex $n-1$ of its neighbors in $\mathbf{R}$ as subdividing vertices of $\bar{K}_n$, such that all edges between the branching vertex and the neighboring subdividing vertices are directed from the branching vertex to its neighbors and all n^2 vertices are pairwise disjoint. Then we can find a linkage between subdividing vertices does not use branching vertices on linking paths (this can be done by finding linkage between subdividing vertices and between arbitrary pairs of branching vertices – to prevent them from being used by other linking paths between subdividing vertices) such that the paths of the linkage together with the set of vertices chosen as a branching vertices of $\bar{K}_n$ and edges between these vertices and subdividing vertices form a subdivision of K_n. .

□

Before continuing with the proof of Theorem 14, we observe the following:

Observation 23. *For any four inner branching vertices v_1, v_2, v_3, v_4 of a wall of hight at least 3, the wall contains an immersion of a 3-star between three of the vertices with the fourth vertex as a center. Similarly, for any 3 inner branching vertices, there is a 2-star between two of the vertices with the third as a center.*

Proof. Follows from the structure of a wall. □

(continued proof of Theorem 14)

Now, we use Lemma 22 to show that if $\mathbf{H_1'}$ has average degree at least φ, H' contains an immersion of G. We find a subdivision $\bar{K}_n$ in $\mathbf{H_1'}$ described in Lemma 22 and we choose $\bar{G}$ to be a subdivision of G in $\bar{K}_n$. Let P be a set of edges of H' such that for every edge $\mathbf{uv}$ in $\bar{G}$, P contains one presentable edge $uv \in H'$. We denote $v_1, \ldots v_{\deg_{\bar{G}}}$ vertices of $\mathbf{v}$ adjacent to the edges in P. Because maximum degree in $\bar{G}$ is 4, using observation 23, we find a $\deg(\mathbf{v}_{\bar{G}}-1)$-star $S_\mathbf{v}$ for every $\mathbf{v}$ in $\bar{G}$. The union of $S_\mathbf{v}$ for every $\mathbf{v}$ in $\bar{G}$ and the presentable edges for all the edges of $\bar{G}$ is a subdivision of G in H.

If $\mathbf{H_1'}$ does not have sufficiently large degree, edges of $\mathbf{H'}$ have average multiplicity greater than $\Phi/\varphi > \mu$. Note that they have multiplicity at most $2c$. Then we can either find a matching in $\mathbf{H}'_\mu$ of size at least l or there is a vertex cover of $\mathbf{H}'_\mu$ of size smaller than l, by1.

C. Suppose that there is a matching $\mathbf{N}$ of size l in $\mathbf{H}'_\mu$. Every edge $\mathbf{e} = \mathbf{uv} \in \mathbf{N}$ contains at least $\mu/2$ edges such that all their ends in $\mathbf{v}$ are branching vertices or at least $\mu/2$ edges such that all their ends in $\mathbf{u}$ are branching vertices, without loss of generality suppose the latter case. For every $\mathbf{e} \in \mathbf{N}$, choose one such subset of $\mathbf{e}$, denote it $\overrightarrow{\mathbf{e}}$ and assign it an orientation from $\mathbf{u}$ to $\mathbf{v}$. Let $\overrightarrow{\mathbf{N}}$ be an directed multigraph $(V(\mathbf{N}), \{\overrightarrow{\mathbf{e}} | \mathbf{e} \in E(\mathbf{N})\})$.

Denote N^- a set of vertices of $\overrightarrow{\mathbf{N}}$ that are sources and N^+ a set of vertices of $\overrightarrow{\mathbf{N}}$ that are sinks. From matching $\overrightarrow{\mathbf{N}}$, we can select a matching $\overrightarrow{\mathbf{N}'}$ of size $|V(G^p)|$ such that for every multiedge $\overrightarrow{\mathbf{uv}} \in \overrightarrow{\mathbf{N}'}$, vertex $\mathbf{u} = N^- \cap \vec{\mathbf{e}}$ is in distance at least d from all vertices of $\overrightarrow{\mathbf{N}'} \setminus \mathbf{v}$ in $(\mathbf{V}, \mathbf{E_W})$.

For every $\overrightarrow{\mathbf{uv}} \in N'$, where $\mathbf{v} \in N'^+$, we divide a wall $\mathbf{v}$ into nine disjoint subwalls of height at least $\lfloor h'/3 \rfloor$, where h' is height of h, $\widetilde{\mathbf{v}}_1 \dots \widetilde{\mathbf{v}}_9$ that form a grid 3×3 in a similar way as in Observation 16, such that for every $i = 1 \dots 9$, $\mathbf{v} \setminus \widetilde{\mathbf{v}}_i$ is a connected graph and for every set $B_{\mathbf{vw}}$ of boundary vertices between $\mathbf{v}$ and $\mathbf{w}$, $|B_{\mathbf{vw}} \cap \widetilde{\mathbf{v}}_i|$ is less than $|B_{\mathbf{vw}}|/2$. By pigeonhole principle, one of these nine subwalls, which we will denote $\widetilde{\mathbf{v}}$, contains endpoints of at least two edges e_1, e_2 in $\overrightarrow{\mathbf{e}}$. Choose a path between endpoints of e_1 and e_2 in $\widetilde{\mathbf{v}}$ and lift it together with e_1 and e_2 to obtain an edge between two branching vertices of $\mathbf{u}$. Denote a graph obtained by this process H'' and a corresponding auxiliary multigraph $\mathbf{H}''$.

By construction, there are at least $|V(G^p)|$ nodes in $\mathbf{H}''$ that contain an edge between two of its branching vertices, and these nodes are in distance at least d in $(\mathbf{V}, \mathbf{E_W})$.

To repeat the construction of immersion of G from **A.**, we observe that Observation 19 holds also for $\mathbf{v} \setminus \widetilde{\mathbf{v}}$ instead of $\mathbf{v}$:

Observation 24. *For every $\mathbf{v} \setminus \widetilde{\mathbf{v}}$ and any three vertices v_1, v_2, v_3 on the boundary of $\mathbf{v}$ that are not in $\widetilde{\mathbf{v}}$, $\mathbf{v} \setminus \widetilde{\mathbf{v}}$ contains an immersion of a 3-star between the vertices v_1, v_2 and v_3. Again, the similar claim holds for 2-stars.*

If an immersion of G^p in $(\mathbf{V}, \mathbf{E_W})$ contains an edge $\mathbf{uv}$ such that $\overrightarrow{\mathbf{uv}} \in \overrightarrow{\mathbf{N}'}$, there exists an edge $e \in E(W)$ between vertices $u_\mathbf{v} \in B_{\mathbf{uv}} \setminus \widetilde{\mathbf{u}}$ and $v_\mathbf{u} \in B_{\mathbf{vu}} \setminus \widetilde{\mathbf{v}}$ because $\widetilde{\mathbf{u}}$ contains less than $|B_{\mathbf{uv}}|/2$ vertices of $B_{\mathbf{uv}}$ and $\widetilde{\mathbf{v}}$ contains less than $|B_{\mathbf{vu}}|/2$ vertices of $B_{\mathbf{vu}}$.

From Lemma 11 and the same construction as in **A.**, it follows that H'' contains an immersion of G.

D. Suppose that $\mathbf{H}'_\mu$ has a vertex cover C of size at most $l-1$. While the average degree of H' is Φ, i.e. the number of edges is $\Phi \cdot k/2$, the number of edges in multiedges of multiplicity less than μ is at most $(\mu - 1)\varphi k/2$. Thus, the number of edges in $\mathbf{H}'$ in multiedges of multiplicity at least μ and therefore incident to C, is at least $(\Phi - \varphi(\mu - 1))k/2$.

Recall that H' consists of the wall W and the matching M on the vertices of W. Let $H^\diamond$ be a graph obtained from H' in the following way:

(i) Delete all edges of M incident to C such that their endpoint in $H' \setminus C$ is not an inner branching vertex and all the edges of M between nodes of C. Observe that we deleted at most cl edges, because at least one of endpoints of every such edge is an inner branching vertex of $\mathbf{v} \in C$.

(ii) Delete all edges of M that are not incident to C. If there remain less than $\delta > 9\varepsilon + 1$ edges of M between some $\mathbf{v}$ in $\mathbf{H}' \setminus C$ and C, delete all of them.

(iii) Delete all edges of M that are not incident to C.

Note that $V(H^\diamond) = V(H')$ and $W \subseteq H^\diamond$. Let $\mathbf{H}^\diamond$ be a corresponding auxiliary multigraph and let $\mathbf{F}$ be a set of its nodes $\mathbf{v}$ such that $\mathbf{v}$ is not in C and there is an edge between $\mathbf{v}$ and C in $\mathbf{H}^\diamond \setminus \mathbf{E_W}$. Every node $\mathbf{v}$ of $\mathbf{F}$ has degree at most c in $\mathbf{H}^\diamond \setminus \mathbf{E_W}$ because every edge incident to $\mathbf{v}$ has its endpoint in one of the inner branching vertices of $\mathbf{v}$.

Observation 25. *The number of vertices in $\mathbf{F}$ is greater than $cl/2$.*

Proof. Consider a node $\mathbf{v}$ that is adjacent to C in $\mathbf{H}'_\mu \setminus \mathbf{E_W}$ and does not belong to $\mathbf{F}$. At least $\mu - \delta$ edges between $\mathbf{v}$ and C have been deleted during (i) and at most $\delta - 1$ edges between $\mathbf{v}$ and C have been deleted during (ii). Consequently, there exist at most $cl/(\mu - \delta)$ such vertices. Therefore, during (i) and (ii), at most $cl + cl(\delta - 1)/(\mu - \delta)$ edges have been deleted.

Since nodes in $\mathbf{F}$ have degree at most c in $\mathbf{H}^\diamond \setminus \mathbf{E_W}$ and there are at least

$$\frac{\Phi - (\mu - 1)\varphi}{2}k - c(l-1) - \frac{c(l-1)(\delta - 1)}{(\mu - \delta)}$$

edges incident to C, the number of nodes in $\mathbf{F}$ is at least

$$\frac{(\Phi - \varphi(\mu - 1))k/2 - cl(1 + \frac{\delta - 1}{\mu - \delta})}{c} > k/c - l(1 + \frac{\delta}{\mu - \delta}) > cl/2$$

because $\Phi - \varphi(\mu - 1) > 2$, $k = c^2 l$ and $1 + \frac{\delta}{\mu - \delta} < c/2$. □

(continued proof of Theorem 14)

Divide every $\mathbf{v} \in C$, into nine smaller subwalls and denote $\tilde{\mathbf{v}}$ one of them that is incident to the most of the edges of $H^\diamond \setminus W$. If there is an odd number of edges of $H^\diamond \setminus W$ incident to $\tilde{\mathbf{v}}$, delete one of them. Divide the remaining edges of $H^\diamond \setminus W$ incident to $\tilde{\mathbf{v}}$ into pairs, such that a path in $\tilde{\mathbf{v}}$ between their endpoints does not contain an endpoint of any other edge. For every such pair of edges e_1 and e_2, lift a path consisting of e_1, e_2 and the path between their endpoints in $\tilde{\mathbf{v}}$.

Denote the resulting graph $H^\times$ and the corresponding auxiliary multigraph $\mathbf{H}^\times$. The graph $H^\times$ is an immersion of $H^\diamond$ and a $V(H^\times) = V(H^\diamond)$. By construction, the average degree of the graph $\mathbf{H}^\times[\mathbf{F}]$ is at least $\frac{|\mathbf{F}|\delta/9 - l}{|\mathbf{F}|} \geq (\delta - 1)/9 = \varepsilon$. We can repeat the stages analogical to A., B. and C. of the proof for the graph $H^\times$, the situation D. cannot occur this time, because all the edges of $H^\times \setminus W$ are between inner branching vertices and therefore the maximum degree of a node of $\mathbf{H}^\times \setminus \mathbf{E_W}$ is at most c.

$\mathbf{A}^\times$. Let $\mathbf{L}^\times$ be a set of vertices of $\mathbf{F}$ incident to a loop in $\mathbf{H}^\times$. If there are at least l of vertices in $\mathbf{L}^\times$, we can find $|G^p|$ vertices of $\mathbf{L}^\times$ that are in distance at least d from each other in $(\mathbf{V}, \mathbf{E_W})$. By Lemma 11, there exists a subgraph of $\mathbf{H}^\times$ isomorphic to a subdivision of G^p with vertices of G^p corresponding to some vertices of $\mathbf{F}$. From Observation 24 and the same arguments as in C., it follows that there exists an immersion of G^p in $\mathbf{H}^\times$.

If there are less than l vertices in $\mathbf{F}$ incident to a loop, the average degree of vertices of $\mathbf{F}$ in $\mathbf{H}^\times \setminus \mathbf{L}^\times$ is at least

$$\Phi^\times \geq \varepsilon - \frac{cl}{2|\mathbf{F}|}$$

because every vertex incident to a loop is incident to at most $c/2$ loops.

$\mathbf{B}^\times$. If the average number of neighbors of vertices of $\mathbf{F}$ in $\mathbf{H}^\times[\mathbf{F}]$ is at least $\varphi^\times = 10n^2$, by Theorem 6, there exists a subdivision of K_n in $\mathbf{H}^\times[\mathbf{F}]$. By Observation 23, and the same arguments as in B., if follows that there exists an immersion of G in $H^\times$.

$\mathbf{C}^\times$. For $\mu^\times = 10$, $\mathbf{H}^\times[\mathbf{F}]_{\mu^\times}$ contains either a matching of size at least l, or it has a vertex cover $\mathbf{C}^\times$ of size at most l. In the latter case, if vertices in $\mathbf{H}^\times[\mathbf{F}]$ do not have the average number of neighbors at least $\varphi^\times$, then there exists at least $(\Phi^\times - \varphi^\times(\mu^\times))|\mathbf{F}|$ edges incident to $\mathbf{C}^\times$, i.e. the average degree of a vertex in C is

$$\frac{(\Phi^\times - \varphi^\times \mu^\times)|\mathbf{F}|}{l} = \frac{(\Phi^\times - 100n^2)|\mathbf{F}|}{l} > c.$$

This cannot happen, because maximum degree of $\mathbf{H}^\times$ is c. Therefore, there exists a matching of size at least l in $\mathbf{H}^\times[\mathbf{F}]$. By the same arguments as in C., we get that there exists an immersion of G^p in $H^\times$. $\square$

Chapter 3

Graphs without an immersion of a 3-regular graph

In this chapter, we prove that 3-edge-connected graphs that do not contain an immersion of a fixed 3-regular graphs have bounded maximum degree. We believe that this result can be generalized for k-edge-connected graphs and immersions of k-regular graphs.

For 3-regular graphs the result is particularly interesting, because by Corollary 5 every graph H that does not contain an immersion of a 3-regular graph G also does not contain G as a minor. Therefore, our result can be combined with graph structure theorem to obtain more precise characterisation of graphs without an immersion of a fixed 3-regular graph.

The generalisation of the result for 4-regular graph would improve the condition on 4-edge-connected graphs that do not contain an immersion of a fixed 4-regular graph, given by Theorem 14 in the previous chapter – such graphs would have both bounded tree-width and maximum degree.

Theorem 26 (Mader [5])**.** *Let $G = (V, E)$ be a graph that has at least $r(s,t)$ edge-disjoint paths between s and t for all $s, t \in V \setminus x$. If there is no cut edge incident to x and $d(x) \neq 3$, then some edge pair (xu, xv) can be lifted so that in the resulting graph there are still at least $r(s,t)$ edge-disjoint paths between s and t for all $s, t \in V \setminus x$.*

Observation 27. *If G contains an immersion of an n-star with edges of multiplicity k, then G contains an immersion of any k-regular graph on n vertices.*

Lemma 28. *Let G be a 3-edge-connected (multi)graph. If there is a vertex $c \in V(E)$ such that $\sum_{uc \in E(G)} \min(\mu(uc), 3) \geq 6n$, then G contains an immersion of an n-star with edges of multiplicity 3.*

Proof. We may assume that multiplicity of every multiedge incident to c in G is at most 3. Denote N_1 respectively N_2 sets of vertices of $N(c)$ that are connected to c by a multiedge of multiplicity 1 respectively 2 in G.

Let H be a graph obtained from a graph G by lifting pairs of edges by Mader's theorem 26 from all vertices of $V(G) \setminus \{c\}$ as long as possible without lifting a pair containing an edge incident to c and without decreasing the degree of vertices of N_1 and N_2 under 3, and deleting isolated vertices.

Observe that the graph H is still 3-edge-connected, because by lifting a pair of edges incident to a vertex v, we do not change the number of edge disjoint paths between vertices in $V(G) \setminus v$. Therefore, we create a 2-edge-cut only by decreasing degree of v to 2. But then we obtain 3-edge-connected graph again by lifting the remaining two edges incident to v, again without changing the number of edge disjoint paths between other vertices, and deleting v because it becomes isolated.

Now, all the vertices in $H \setminus \{c \cup N(c)\}$ have degree 3, but vertices in $N(c)$ can have arbitrarily large degree. By the following process we decrease the number of vertices in $N(c)$ that have degree greater than 3 in $H \setminus c$, preserving 3-edge-connectivity.

Consider a graph $H_v = H \setminus \{\widehat{cv}\}$ for some vertex $v \in N(c)$ that has degree greater than 3 in $H \setminus c$.

If H_v is 3-edge-connected, by Theorem 26 and previous reasoning we can lift pairs of edges adjacent to v in H_v until v has degree at most 3 in a way that the resulting graph is either 3-edge-connected or it has the only 2-edge-cut – the last two edges adjacent to v.

By lifting the same pairs of edges in H instead of H_v, we obtain a 3-edge-connected graph H'_v, such that v has degree at most 3 in $H'_v \setminus c$.

If H_v is not connected, we delete all vertices that are in the component K of H_v that contains v, except v. Again, by deleting these vertices in H, we obtain a 3-edge-connected graph H'_v.

If H_v is connected but not 3-edge-connected, let C to be an edge-cut of minimum size. Observe that in the component of $K = H_v \setminus C$ that contains v are at most two other vertices of $N(c)$.

There exist $|C|$ edge disjoint paths from v to $H_v \setminus (K \cup C)$ in $K \cup C$. We lift these paths in H and delete the rest of K. Observe that the resulting graph H'_v is 3-edge-connected and that v has degree at most 2 in $H'_v \setminus c$.

Note that if some of the edges of the cut C are incident to c, lifting decreases the number of the vertices adjacent to c but it does not decrease degree of c and the resulting multiedge between c and v has multiplicity at most 5.

By repeating this process as long as necessary for different vertices of $N(c)$ (at most once for each vertex), we obtain a 3-edge-connected graph H' that is immersed in H, such that every vertex of $N(c)$ have degree at most 3 in $H' \setminus c$. Note that if there is a multiedge of multiplicity 5 between c and a vertex v in H' that arose during the process, then v is not incident to any other edge. If there is a multiedge of multiplicity 4 between c and a vertex v that arose during the process, then v has degree at most 1 in $H' \setminus c$. By 3-edge-connectivity of H', the number of edges between c and any connected component of $H' \setminus c$ is at least 3.

Denote N'_1 respectively N'_2 sets of vertices of $N(c)$ that are connected to c by a multiedge of multiplicity 1 respectively 2 in H'. Denote N'_3 a set of vertices of $N(c)$ that are connected to c by a multiedge of multiplicity at least 3 in H'.

Now, we construct a star S with all multiedges of multiplicity at least 3 and with the vertex c as a center. Let us begin with an empty star consisting only from c. We add another vertices and edges as follows:

For every connected component K of $H' \setminus c$, if K contains a single vertex v, we add $\widehat{cv}$ into S (the multiedge $\widehat{cv}$ has multiplicity at least 3).

If K contains more than one vertex, we choose a minimal rooted tree T_K in K that contains all the vertices of $N(c) \cap K$ (i.e. every leaf of T_K is in $N(c)$), such that the root is a vertex of degree 1 in T_K. Note that every vertex in T_k has at most two children. We repeat the following algorithm modifying K, H', T_K, N'_1 N'_2 and N'_3 and adding vertices and multiedges to S, until T_K contains a single vertex.

Algorithm:
If T_K contains a leaf v that is not in any of N'_1, N'_2 and N'_3, delete v from T_K.
If T_K contains a leaf v in N'_3:

- if $\widehat{cv}$ has multiplicity greater than 3, lift edges cv and vu, where u is a predecessor of v
- delete v from T_k and add v and $\widehat{cv}$ to S

If all the leaves of T_K are in N'_1 or N'_2, choose a vertex u of T_K such that all its children are leaves. Then one of the following situations occurs:

Vertex u is not in any of N'_1, N'_2 and N'_3:

- u has two children v and w:
  - at least one of v, w, without loss of generality v, is in N'_2:
    lift edges uv and uw and cw to obtain a multiedge of multiplicity 3 between c and v and add it into S, delete vertices u, w from T_K
  - both v and w are in N'_1:
    lift edges cv, vu and cw, wu, delete vertices v and w from T_K and add vertex u into N'_2
- u has only one child v: if u has a predecessor p, lift edges uv, up and delete u from T_K, otherwise delete u from T_K

Vertex u is in N'_1:

- u has two children v and w:
  - both v and w are in N'_1:
    lift edges cv, vu and cw, wu to obtain a multiedge of multiplicity 3 between c and u, add it into S and delete v, w from T_K

  - at least one of v, w, without loss of generality v, is in N_2':
    lift edges uc, uv to obtain a multiedge of multiplicity 3 between c and v, add it into S and delete v from T_K

- u has only one child v:
  - v is in N_1': lift edges cv, uv to obtain a multiedge of multiplicity 2 between c and u, delete v (and move u from N_1' to N_2')
  - v is in N_2': lift edges cu, uv to obtain a multiedge of multiplicity 3 between c and v, add it into S and delete u from T_K

Vertex u is in N_2':

- u has two children v and w:
  - both of them are in N_2': lift edges uv, uc and uw uc to obtain multiedges of multiplicity 3 between c and v,w, add them into S and delete u, v and w from T_K.
  - exactly one of them, without loss of generality v, is in N_2': lift edges uv, uw cw to obtain a multiedge of multiplicity 3 between c and v, add it into S
  - both of them are in N_1': if u has a predecessor p, lift edges vu and up. Lift wc and uw to obtain a multiedge of multiplicity 3 between c and u, add it into S and delete u and w from T_K
- u has only one child v: lift edges cv, uv to obtain a multiedge of multiplicity 3 between c and u, add it into S and delete v from T_K

Vertex u is in N_3':

- u has two children v and w:
  - both of them are in N_2': lift edges uv, uc and uw uc to obtain multiedges of multiplicity 3 between c and v,w, add them into S and delete v and w from T_K. If u has a predecessor p, lift uc and up. Delete u from T_K.
  - exactly one of them, without loss of generality v, is in N_2': lift edges uv, uw cw to obtain a multiedge of multiplicity 3 between c and v, add it into S
  - both of them are in N_1': if u has a predecessor p, lift edges vu and up. Delete w from T_K. Add $\widehat{uc}$ into S and delete u from T_K.
- u has only one child v: If u has a predecessor p, lift edges uv and up. Add $\widehat{uc}$ into S and delete u from T_K.

After the end of the algorithm, T_K contains a single vertex v. If $\widehat{cv}$ has multiplicity at least 3, add it into S.

Observe that the algorithm processes each vertex in T_K at most twice. In each step of the algorithm, the degree of c in H' either does not change, or it decreases by 1. After every decreasing, we add at least one vertex into S. After the end of the algorithm, T_K contains a single vertex v such that multiplicity of $\widehat{cv}$ is at most 5. Since we know that the number of edges between c and any connected component of H' is at least 3, every such component provides at least one vertex to S.

Thus, for every 6 edges in multiedges of multiplicity at most 3 in incident to c in G, we added at least one vertex into S. This implies the statement of the lemma.

□

The following corollary for simple graphs follows directly from Observation 27 and Lemma 28.

Corollary 29. *Let H be a 3-edge-connected simple graph. If there $\Delta(H) > 6n$, then H contains an immersion of any 3-regular graph G on n vertices.*

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