Charles University in Prague

Faculty of Mathematics and Physics

MASTER THESIS

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Kreditní přirážka k tržnímu ocenění: přístupy k výpočtu a modelování

Credit Valuation Adjustment: Approaches to Modeling and Computations

Department of Probability and Mathematical Statistics

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Kreditní přirážka k tržnímu ocenění:přístupy k výpočtu a modelování

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Abstrakt:

V této práci se zabýváme rizikově neutrální oceňovací formulí pro kreditní přirážku k tržnímu oceňování v případech, kdy jsou jedna nebo obě strany kontraktu vystaveny riziku úpadku. V případě, kdy čelí úpadku jen jedna strana, je toto riziko kvantifikováno pomocí tzv. Credit Valuation Adjustment (CVA). Pokud této možnosti čelí obě strany, je riziko reprezentováno pomocí tzv. Bilateral Risk Adjustment (BVA). Tyto kreditní přirážky (CVA, BVA) jsou zapracovány ve vzorcích pro oceňování bezkupónových dluhopisů, kupónových dluhopisů a úrokových swapú. Rizikově neutrální pravděpodobnosti bankrotu, potřebné k zahrnutí těchto přirážek do cen daných kontraktů, jsou odvozeny z tržních kotací tzv. Credit default swaps. Pro jejich odvození použijeme tzv. bootstrap metodu. Pro modelování pravděpodobností úpadku používáme tzv. reduced form approach. V praktické části práce jsme se zaměřili na odvození rizikové neutrálních pravdépodobností úpadku pro Řecko a Českou Republiku v období let 2008-2010 včetně bouřlivého období v květnu 2010 na řeckém trhu. Následně je kvantifikováno CVA pro 18 státních dluhopisů kotovaných na trhu a ceny upraveny o CVA jsou porovnávány s reálnými tržními cenami. Také studujeme vliv výběru bezrizikové úrokové míry na tento výpočet. V poslední sekci konstruujeme úrokový swap mezi Českem a Řeckem, pro který určujeme CVA a BVA.

Klíčová slova: kreditní přirážku k tržnímu oceňování, rizikově neutrální pravděpodobnosti úpadku protistrany, riziko úpadku

Title: Credit Valuation Adjustment: approaches to computation and modeling

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Abstract: In this work we are introducing a risk neutral valuation formula for counterparty default risk adjustments in an unilateral and in a bilateral case. In the unilateral case the adjustment is represented by a Credit Valuation Adjustment(CVA) and in the bilateral case the adjustment is quantified by a Bilateral Risk Adjustment(BVA). We are incorporating these adjustments into the values of zero coupon bonds, coupon bearing bonds and interest rate swaps. For such an incorporation, risk neutral default probabilities extracted from the market quotes of Credit Default Swaps are needed. A Bootstrap method is used to derive them and a reduced form approach is used to model the default times. In the practical part, we are calculating Greek and Czech risk neutral default probabilities during the years 2008-2010. We are calculating CVA for 18 quoted Greek government bonds and we are comparing the adjusted prices with the market quoted prices of these bonds. We study the impact of a risk free interest rate curve choice on such a valuation. In the last sections, we construct an interest rate swap between the Czech and the Greece. We compute and study CVA and BVA for this interest rate swap.

Keywords:credit valuation adjustment, debit valuation adjustment, counterparty default risk

Contents

1. Introduction

Recently, we are witnesses to big private corporations bankruptcies and even whole countries are balancing on the edge of state bankrupt. These situations with connection of rapidly increasing so called over-the-counter(OTC) deals in financial markets are yielding following question. What is the impact of such a risk on a financial contract and its value? We are trying to answer this question through an adjustment of default free prices of financial instruments. Measures for quantification of default risk are expressed by CVA(Credit Valuation Adjustment) and BVA(Bilateral Valuation Adjustment) respectively. The key relations in this work that are representing these adjustments are

$$\mathrm{CVA} = \mathrm{V}_{\text{default free}} - \mathrm{V}_{\text{exposed to default risk}}$$

$$\mathrm{BVA} = \mathrm{V}^{Bi}_{\text{exposed to default risk}} - \mathrm{V}_{\text{ default free}}$$

where $V_{\text{default free}}$ is the value for a contract in default free environment and $V_{\text{exposed to default risk}}$ and $V^{Bi}_{\text{exposed to default risk}}$ stand for the values of the instruments in which is already incorporated unilateral and bilateral default risk exposure, respectively. CVA adjustment represents the case, where we are assuming that just one counterparty in the contract can default, thus CVA can be just nonnegative. In more realistic case, where both counterparties are exposed to default risk, BVA serves as a risk measure and it may be either positive or negative one. This thesis is divided into two main parts, theoretical and practical. In the theoretical part, we are summarizing the risk neutral approach to the valuation of bonds, swaps etc. . In Chapter 4 we are introducing key theorems that later in Chapter 5 help us to incorporate default risk into the price of financial contracts. We are dealing with two types of risks here: unilateral default risk and bilateral default risk. For quantification of default risk are crucial risk neutral default probabilities. We are using implied risk neutral default probabilities from the market, extracted from CDS quotes, this procedure is described in Section 8.1. Some of these theoretical parts, mainly about unilateral default risk and risk neutral valuation, are based on my prior diploma thesis that was submitted to Faculty of Sciences, Vrije University in Amsterdam in 2010.

In the practical part, we apply the whole theory on real market data. We use distressed situation around Greece in May 2010 to study CVA for Greek government bonds. We are extracting risk neutral default probabilities from CDS quotes on these bonds and then we use them to compute CVA for 18 coupon bonds quoted on the market. Market implied risk neutral default probabilities are extracted also for Czech Republic and are used in the last chapter. By the means of CVA, we are trying to approach real market prices of bonds through our theory, from default free price(cash-flows discounted by risk free interest rate) to real market prices of these bonds. Here in this thesis we also study the impact of default free interest rate curve choice. We are working with two most common curves; Treasury curve and interest rate swap curve. In the last Chapter 11 we apply the theory for unilateral and bilateral default risk for interest rate swaps. We work with artificially constructed interest rate swap between Greece and Czech Republic and computing CVA and DVA for it. From the price already adjusted for credit risk we subsequently compute fixed rates, that already include default risk exposure of counterparties, payed in the interest rate swap contract.

2. Valuation of Default Free Bonds

2.1 Zero-coupon Bonds

Zero coupon bond (or discount bond) is a financial instrument that at the maturity time, let denote it T, pays a certain amount of money (principal or face value) to its holder; we will consider the face value that equals to 1 without loss of generality. There are no payments between the issue of the bond and its maturity. We are considering a continuous trading economy with trading within interval $[0, \tau]$ for $\tau > 0$. We denote $P(t, T)$ the value of zero coupon bond at time $t, t \in [0, T]$, for $t \leq T \leq \tau$ and hence we require $P(T, T) = 1$ and $P(t, T) > 0$ for all $T \in [0, \tau]$ and $t \in [0, T]$. We are excluding the existence of non-trivial arbitrage opportunity in the market.

2.2 Interest Rate Term Structure

We are considering a zero-coupon bond with its maturity at time $T \leq \tau$ and price $P(t, T)$ for $t \in [0, T]$, trading in the market where bond price is strictly positive and adapted process on probability space $(\Omega, \mathbb{F}, \mathbb{P})$. Where Ω is state space, $\mathbb{F}$ is a σ-algebra representing measurable events and $\mathbb{P}$ is a probability measure. The term structure of interest rates is also known as a *yield curve* and it relates yields to maturity times. Yield-to-maturity on zero coupon bond is defined as

$$Y(t, T) = -\frac{1}{T - t} \ln P(t, T) \text{ for all } t \in [0, T) \tag{2.1}$$

To a given yield curve uniquely corresponds the bond price process according to the following formula

$$P(t, T) = \exp\left(-Y(t, T) \cdot (T - t)\right) \text{ for all } t \in [0, T] \tag{2.2}$$

There are two possibilities how the initial interest rate term structure is represented. Either it is represented by the current bond prices $P(0, T)$ or by the initial yield curve given by the following formula

$$P(0, T) = \exp(-Y(0, T)T) \text{ for all } T \in [0, \tau] \tag{2.3}$$

2.2.1 Forward Interest Rates

The general view on bond price methodology is through the spot rates to the forward rates. The interest rate in infinitesimal time interval $[T, T + dT]$ observed from time t is called *instantaneous continuously compounded forward rate* or simply *instantaneous forward rate* and is denoted by $f(t, T)$. This rate is not observable in the market. Starting from this concept the zero-copon bond price is

$$P(t, T) = \exp\left(-\int_t^T f(t, u)du\right) \text{ for all } t \in [0, T] \tag{2.4}$$

Alternatively, assume that prices $P(t,T)$ are continuously differentiable in T. Then we define a forward rate by the way of bond price definition as follows

$$f(t,T) = -\frac{\partial \text{ ln P(t,T)}}{\partial T} \tag{2.5}$$

We can look at the forward rate as a limit case of forward rate $f(t,T,U)$ that is observed at time t for borrowing over the future time interval $[T,U]$ for $t \leq T \leq U$. If we use the zero coupon bond price it follows that

$$\frac{P(t,U)}{P(t,T)} = \exp(-f(t,T,U)(U-T)) \text{ for all } t \leq T \leq U \tag{2.6}$$

or equivalently

$$f(t,T,U) = \frac{P(t,T) - P(t,U)}{U-T} \tag{2.7}$$

Granting continuity, we have $Y(t,T) = f(t,t,T)$, that follows from introduced formulas, but also from argument that lending money from time t to time T, $t \leq T$ is equivalent to investing money at time t to bonds with maturity T.

2.2.2 Short-term interest rates

We consider stochastic interest rate models that, are then used, in pricing of bonds based on *short-term interest rate*. Short-term interest rate is interpreted as an interest rate for borrowing or lending money during time interval $[t, t+dt]$ observed at time t. From the previous part we write $r_t = f(t,t)$.
In stochastic set up r is defined on probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and generally it is assumed that stochastic process r is modeled as an adapted process with almost all sample paths integrable on $[0,\tau]$ with respect to the Lebegue measure.
We introduce the price process of continuously compounded risk free security with interest rate r(we can refer to it as a saving account)

$$B_t = \exp\left(\int_0^t r_u du\right) \text{ for } t \in [0,\tau] \tag{2.8}$$

It is also possible to express B_t as a solution of stochastic differential equation. For almost all $\omega \in \Omega$, the function $B_t = B_t(\omega)$ solves the equation $dB_t = r_t B_t dt$ with initial condition $B_0 = 1$.
The B_t is amount that corresponds to the cash that is accumulated from time 0 to time t, starting with one unit of cash and rolling over the bonds with infinitesimal time to maturity.

2.3 Coupon Bearing Bonds

Coupon bearing bonds are financial instruments that pay to a holder $c_1, c_2, \ldots, c_m$ amounts of money at times $T_1, T_2, \ldots, T_m = T$ respectively. Typically, bonds pay a fixed coupon c and repay the principal M; $c_j = c$ for j=$1, 2, \ldots m-1$ and $c_m = M + c$. If we are considering different cash flows during the dates $T_1, T_2, \ldots, T_m$, then the expression for a bond price $P_c(t,T)$ is given by

$$P_c(t,T) = \sum_{j=1}^{m} c_j P(t,T_j) \tag{2.9}$$

For comparative purposes we extend the notation of yield to maturity from zero-coupon bonds to coupon bearing bonds. We are considering bonds that pay the same amount of money c at times $T_1, T_2, \ldots, T_m$ and the principal M is paid at time T_m. If the coupon is expressed as a preassigned interest rate times face value $c := r \cdot M$, we called it *coupon rate* and then we have

$$P_c(0,T) = \sum_{j=1}^{m} ce^{-Y_c(0)j} + Me^{-mY_c(0)} \tag{2.10}$$

where $P_c(0,T)$ is the current market price of the coupon bearing bond at time 0.

3. Default Free Valuation

This chapter is dealing with well known topic of risk neutral valuation of financial instruments. We assume default free environment here. We introduce here shortly, the standard theory about a risk neutral valuation of bonds, coupon bearing bonds and interest rate swaps. This serves as preliminaries to the valuation of such instruments in the markets, where counterparties are exposed to a default risk.

3.1 Risk-neutral Valuation Formula

In this section, we very shortly summarize the valuation process of contingent claims, that is used further in this work for valuation of various financial instruments. For full derivation of Risk-neutral Valuation Formula, we refer to Bingham&Kiesel [4] Chapter 6 and also we are following this publication here. We are considering a market, let's denote it $\mathcal{M}$, where an investor can trade continuously in time interval $[0, T]$. The uncertainty in the market is modeled by probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and filtration $\mathbb{F} = (\mathcal{F})_{0 \leq t \leq T}$, which satisfies the usual conditions of completeness and right continuity. In our market, we have $n+1$ primary traded assets that have price processes modeled by stochastic processes $S_0, S_1, \ldots, S_n$. We assume that $S = (S_0, \ldots, S_n)$ follows an adapted, right continuous, with left limits and strictly positive semimartingale on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$. As it is usual we are considering the numeraire

$$B_t = \exp\left(\int_0^t r(s)ds\right)$$

where $r(s)$ is well defined integrable process.

Definition 3.1.1. A stochastic process $S = S(t), t = 0, 1, \ldots T$ is called a martingale under the probability measure $\mathbb{Q}$, sometimes called $\mathbb{Q}$-martingale, and with respect to filtration $\mathbb{F}$, if the conditional expectation

$$E_{\mathbb{Q}}\left[S(t)|\mathcal{F}_{t-1}\right] = S(t-1), \text{ for all t,t} = 1 \ldots, T$$

Definition 3.1.2. Probability measure $\mathbb{Q}$ defined on $(\Omega, \mathcal{F})$ is called the equivalent martingale measure if $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and if the discounted process $\tilde{S}$ ($\tilde{S}(t) := S(t)/B_t$) is $\mathbb{Q}$-martingale.

We assume that in our market model exists equivalent martingale measure and denote our reference measure $\tilde{\mathbb{P}}$ and we restrict our attention to contingent claim C such that $C/B_T \in L^1(\mathcal{F}, \tilde{\mathbb{P}})$.
The next important thing is definition of attainable contingent claim. Freely speaking attainable contingent claim C is such a claim, that its value is possible to replicate in the market $\mathcal{M}$. It means that there exists trading strategy that by the combination of assets $S_0, S_1, \ldots, S_n$ replicates the value of our desired claim. It is possible to prove that the existence of a martingale measure implicates no

arbitrage. Next we introduce the fundamental theorem of risk neutral pricing. Since we have not built up mathematically rigorous risk neutral pricing theory here, we are presenting it without a proof. For the proof see Bingham&Kiesel [4].

Proposition 3.1.3. Risk-Neutral Valuation Formula *For any attainable claim C is given the unique arbitrage price process by the the risk neutral valuation formula*

$$\Pi_C(t) = B_t E_{\tilde{\mathbb{P}}}\left[\frac{C}{B_T}\Big|\mathcal{F}_t\right] \text{ for any } t \in [0,T] \tag{3.1}$$

This proposition tells us that the expectation under the equivalent martingale measure gives us the unique arbitrage free price of contingent claim.

3.2 Bonds

Basic building element in mathematical finance is zero coupon bond. From our risk neutral valuation framework just by putting in Theorem 3.1.3 $C = 1$ we get following proposition.

Proposition 3.2.1. *In our risk neutral framework and existence of.. equivalent martingale measure $\tilde{\mathbb{P}}$ we have following:*

i) the price of zero coupon bond with maturity T, at time $t \leq T$ is given by

$$P(t,T) = E_{\tilde{\mathbb{P}}}\left[\frac{B_t}{B_T}\Big|\mathcal{F}_t\right] = E_{\tilde{\mathbb{P}}}\left[\exp\left(-\int_t^T r(u)du\right)\Big|\mathcal{F}_t\right] \tag{3.2}$$

ii) the price of coupon bond with maturity T, at time $t \leq T$ defined as in Section 2.3 is given by

$$P_c(t) = E_{\tilde{\mathbb{P}}}\left[\sum_{j=1}^{m}\frac{B_t}{B_{T_j}}c_j\Big|\mathcal{F}_t\right] = E_{\tilde{\mathbb{P}}}\left[\sum_{j=1}^{m}c_j\exp\left(-\int_t^{T_j} r(u)du\right)\Big|\mathcal{F}_t\right] \tag{3.3}$$

Proof. In case i) we simply use Theorem 3.1.3 for $C = 1$
In case ii) we use that for $C = \sum_{j=1}^{m} c_j P(t, T_j)$ □

3.3 Swaps

A *swap* is a contract or agreement between two parties to exchange(swap) cash flows at some future prescribed dates. Under the usual conditions, the value of the swap at inception date and at the end of the swap's life is zero. We will consider the most usual *fixed-for-floating forward swap* called *plain vanilla* interest rate swaps. In this contract, there are two positions. The first position, *long position*, is when investor(payer) makes predetermined payments by a fixed interest rate from nominal value M and receives the cash that is based on some pre-specified floating rate. The second position is short and cash-flows are vice versa. The usual floating rate in interest rate swaps agreements is LIBOR[1].

[1]The **L**ondon **I**nter**B**ank **O**ffered **R**ate, or LIBOR, is the average interest rate at which banks in the London interbank market can borrow unsecured funds from other banks

Swaps can be settled in *arrears* and in *advance* .
•*Settlement in advance:* floating rate is determined at the beginning of the period and payments are also settled at the beginning.
• *Settlement in arrears:* payments are settled in the end of the period and the floating rate is determined at the beginning of the period.

Convention *Now we work with the settlement in arrears.*

Let's denote future dates T_j, $j = 0, \ldots, m$, called reset dates and dates when payments are settled(settlement dates) T_j, $j = 1, \ldots, m$, where $\delta_j = T_j - T_{j-1} > 0$ for every $j = 1, \ldots, m$. The floating rate for time period $[T_j, T_{j+1}]$ is determined at time T_j. At time T_{j+1} received cash-flow is, let's denote it $L(T_j)$, by reference to the price of zero coupon bond prevailing in this period, given as follows

$$L(T_j) = \left(\frac{1}{P(T_j, T_{j+1})} - 1 \right) \frac{1}{\delta_{j+1}} \text{ for } j = 0, \ldots, m-1 \tag{3.4}$$

This is simply the return of one unit investment per period $[T_j, T_{j+1}]$. The cash flows at any settlement date T_j, $j = 1, \ldots, m$ are from payer(long position) point of view, $L(T_{j-1})\delta_{j+1}M$ and $-\kappa\delta_{j+1}M$, where M is a notional principal and the κ is predetermined fixed rate.
Convention We assume for simplicity and without loss of generality that $M = 1$. According to our formula from Section 3.1 we have following,

Proposition 3.3.1. *Let us consider the payer interest rate swap with payments defined as above. Then the value of such a swap at time t, $t \leq T_0$ in arbitrage free framework is given by*

$$V_\kappa(t) = B_t E_{\tilde{\mathbb{P}}} \left\{ \sum_{j=1}^{m} B_{T_j}^{-1} \left((L(T)_{j-1})\delta_j - \kappa\delta_j) \right) \Big| \mathcal{F}_t \right\} \tag{3.5}$$

Proof. Proof follows from Proposition 3.1.3 where C is replaced by interest rate swap cash-flows. □

Furthermore we can write

$$
\begin{aligned}
V_\kappa(t) &= E_{\tilde{\mathbb{P}}}\left\{\sum_{j=1}^{m}\frac{B_t}{B_{T_j}}(L(T_{j-1})\delta_j - \kappa\delta_j\Big|\mathcal{F}_t\right\} \\
&\quad \text{using the term (3.4)} \\
&= \sum_{j=1}^{m} E_{\tilde{\mathbb{P}}}\left\{\frac{B_t}{B_{T_j}}\left(P(T_{j-1},T_j)^{-1} - (1+\kappa\delta_j)\right)\Big|\mathcal{F}_t\right\} \\
&\quad \text{using the formula for zero-coupon bond pricing} \\
&= \sum_{j=1}^{m} E_{\tilde{\mathbb{P}}}\left\{P(T_{j-1},T_j)^{-1}\frac{B_t}{B_{T_j-1}}E_{\tilde{\mathbb{P}}}\left(\frac{B_{T_{j-1}}}{B_{T_j}}|\mathcal{F}_{T_{j-1}}\right)\Big|\mathcal{F}_t\right\} \\
&- \sum_{j=1}^{m}(1+\kappa\delta_j)E_{\tilde{\mathbb{P}}}\left\{\frac{B_t}{B_{T_j}}\Big|\mathcal{F}_t\right\} \\
&\quad \text{using the information from filtration and equation (3.2)} \\
&= \sum_{j=1}^{m} E_{\tilde{\mathbb{P}}}\left\{e^{-r(T_j-T_{j-1})}\frac{e^{rt}}{e^{rT_{j-1}}}\Big|\mathcal{F}_t\right\} - \sum_{j=1}^{m}(1+\kappa\delta_j)P(t,T_j)
\end{aligned}
$$

From the previous lines we have proved the following result,

Proposition 3.3.2. *The price of interest rate swap settled in arrears with the same properties as were defined in the beginning of this chapter at time t,$t \leq T_0$ is*

$$V_\kappa(t) = \sum_{j=1}^{m}\left(P(t,T_{j-1}) - (1+\kappa\delta_j)P(t,T_j)\right) \tag{3.6}$$

Small rearrangement of the proposition 3.3.2 leads to the following.

Proposition 3.3.3. *For the price of interest rate swap settled in arrears holds following,*

$$V_\kappa(t) = P(t,T_0) - \sum_{j=1}^{m} c_j P(t,T_j),\ \textit{for } t \in [0,T_0] \tag{3.7}$$

where $c_j = \kappa\delta_j$ for $j = 1,2,\dots m-1$ and $c_m = 1+\kappa\delta_m$.

Remark 3.3.4. *As we see, the relationship 3.7 is just a combination of delivering the specific coupon bearing bond and to receive zero-coupon bond at the same time. It could easily be derived from a comparison of these two bonds that*

$$V_\kappa(t) = \underbrace{P(t,T_0)}_{\textit{zero-coupon bond}} - \underbrace{\sum_{j=1}^{m} c_j P(t,T_j)}_{\textit{coupon bearing bond}}$$

As it is mentioned above, the value of a swap at its time of initiation is 0. This important swap property leads to the following definition.

Definition 3.3.5. The forward swap rate $\kappa(t,T_0,m)$ at time t for the date T_0 is the value of the fixed rate κ that makes the value of the m-period forward swap zero,

$$V_\kappa(t) = 0$$

Using the Equation (3.7) we obtain the explicit formula for m-period forward swap rate, that is formulated in next proposition.

Proposition 3.3.6. *The m-period interest rate swap rate settled in arrears at time t,$t \leq T_0$ is given by*

$$\kappa(t, T_0, m) = \frac{(P(t, T_0) - P(t, T_n))}{\sum_{j=1}^{m} \delta_j P(t, T_j)} \tag{3.8}$$

3.4 Swaptions

In the next step, we consider an option on interest rate swap(IRS), *swaption*. We assume same setup for IRS as in previous chapter. The owner of a payer(receiver) swaption with strike κ maturing at time $T = T_0$, has the right to enter at time T to the underlying payer(receiver) swap settled in arrears. The payer swaption is an option that gives the buyer the right to enter into a swap from which he/she receives fixed payments. On the other hand the seller of a payer swaption gives to the investor right to enter into the payer swap in which the investor will receive payments based on floating rate and will pay fixed payments. From our standard risk neutral theory follows.

Proposition 3.4.1. *Let's denote $Swap(\kappa, T)$ the value of the payer interest rate swap, at time T with fixed swap rate κ and with the same payments and their pattern as defined in the beginning of Section 3.3. Then the price of the payer swaption with maturity T, at time t, $t \leq T$ equals to*

$$PS(t) = E_{\tilde{\mathbb{P}}}\left(\frac{B_t}{B_T}(Swap(\kappa, T))_+ \middle| \mathcal{F}_t\right) \tag{3.9}$$

Analogously for the receiver swaption is

$$RS(t) = E_{\tilde{\mathbb{P}}}\left(\frac{B_t}{B_T}(-Swap(\kappa, T))_+ \middle| \mathcal{F}_t\right) \tag{3.10}$$

Convention: for simplicity and without loss of generality here we assume $M = 1$ as well. If we plug in for the swap expression from Proposition 3.3.2 then we obtain following,

$$PS(t) = E_{\tilde{\mathbb{P}}}\left(\frac{B_t}{B_T}\left(E_{\tilde{\mathbb{P}}}\left\{\sum_{j=1}^{m}\frac{B_T}{B_{T_j}}(L(T_{j-1})\delta_j - \kappa\delta_j)\middle|\mathcal{F}_T\right\}\right)_+ \middle| \mathcal{F}_t\right) \tag{3.11}$$

and for the receiver swaption

$$RS(t) = E_{\tilde{\mathbb{P}}}\left(\frac{B_t}{B_T}\left(E_{\tilde{\mathbb{P}}}\left\{\sum_{i=1}^{m}\frac{B_T}{B_{T_j}}(\kappa\delta_{j+1} - L(T_{j-1})\delta_{j+1})\middle|\mathcal{F}_T\right\}\right)_+ \middle| \mathcal{F}_t\right) \tag{3.12}$$

If we use property, that any function can be spread as follows $f = f^+ - f^- = f^+ - (-f)^+$ [2] and combine two formulas from above it yields following swap parity:

[2] $f : \mathbb{R} \to \mathbb{R}$ and $f^+ := f^+(x) = \max(f(x), 0)$ and $f^- := f^-(x) = -\min(f(x), 0)$

Proposition 3.4.2. *For the value of receiver swaption and the value of payer swaption, both are expiring at the same date, holds following "parity" to a value of interest rate swap*

Payer Swaption-Receiver Swaption= Interest Rate Forward Swap

Remark 3.4.3. Structure of Swaption Trading strategy
If we look at the payer swaption, we can describe the strategy of the swaption owner, who wants to enter into a swap as follows,
Strategy
1st case *If κ is less then a swap rate of a market swap at time T, then the owner exercises the swaption and in the future his predetermined payments will be less than the market fix payments at the time of swap initiation.*
2nd case *If a swap rate is less than κ of the swaption at time T, the owner does not exercise the swaption but he has the possibility to enter into the swap anyway at the market swap rate.*
From these two cases, we can conclude that the fixed rate paid by the owner of the swaption who wanted to enter into swap is always smaller or equal to the κ.

Formulas from above give us a good picture what is the price of swaption, but for our purposes in practical part of this work we need an analytical formula that is easily computable. In the beginning of this section we have started that swaption is an option on IRS. From this fact one can expect that the easily computable pricing tool comes from Black Scholes formula. Black Scholes formula is proved in Appendix 12A in Hull(2002) [13] and in the same publication is derived formula for european swap option. This formula is called Black's Model(sometimes known as the Black-76 model).

Black's Model

$$\begin{aligned} V_{call} &= A[F_0 N(d_1) - \kappa N(d_2)] \\ &\text{where} \\ d_1 &= \frac{\ln(F_0/\kappa) + \sigma^2 T/2}{\sigma\sqrt{T}} \\ d_2 &= \frac{\ln(F_0/\kappa) - \sigma^2 T/2}{\sigma\sqrt{T}} \end{aligned} \tag{3.13}$$

- T:Time to maturity of the option
- F_0: forward swap rate
- κ: Strike price of the option
- V_T: Value of V at time T
- σ : Volatility of F_0
- A=$\sum_{i=1}^{m} \frac{P(0,T_i)}{n}$ where n is number of payments during the year

This formula is used to compute payer swaption price. It gives the owner of the swaption right to enter into a swap in which he/she is paying fixed rate κ. The equivalent situation is for receiver swaption, where the swaption value is given by

$$V_{put} = A[\kappa N(-d_2) - F_0 N(-d_1)] \tag{3.14}$$

All notations remains the same as for the payer swaption. We will see, that this formula is essential for a computation of defaultable IRS price.

4. Counterparty Credit Risk Framework

In this section, we introduce the framework for treating with the counterparty credit risk, also referred to as a default risk or a counterparty default risk[1]. By the term credit quality we mean an exposure of given entity to this risk. It means a possibility that a counterparty in a financial contract will not fulfill its contractual commitment to meet his or her obligation stated in the contract. For later purposes we need to distinguish two types of counterparty risk, because in each contract exists at least two sides: an investor(who buy defaultable financial instrument) and a counterparty (who sell/issue instrument) and hence there exist two default events as well. So we need to distinguish

- Unilateral counterparty risk
- Bilateral counterparty risk

Unilateral risk means, that just one counterparty faces default risk and on the other hand, the bilateral counterparty risk exposes both sides to default risk. We need to consider unilateral counterparty risk in instruments, that are either assets or liabilities for an investor, during the whole life of the contract. To this category of instruments belongs e.g. bonds; it does not matter if the investor will default, because only he/she can lose money. For a bond issuer bond is booked on his/her liability site as a credit, so issuer can not lose any money. So there is no need to assume bilateral credit risk for this type of instrument, because even if we assume that also bond investor can default, it does not affect the price of the bond in any way.
On the other hand there, exist contracts that can be during their lifetime a liability and also an asset for both participants, depending on market conditions. For this type of contracts we need to consider bilateral credit risk because both sides can lose money, in the case of others default. To this group belong e.g. a swap; for a buyer of the swap it can be either assets or liabilities depends on the difference between floating rate and fixed rate. So in this case, both the buyer and the seller are exposed to counter party risk and can lose money and therefore the price of swap is affected. The reason why we are introducing this kind of risk besides the others, is that there is high increase in *over the counter*(OTC) trades during last decades. This kind of trades are directly negotiated between two private parties, there is no supervised intermediary between them or any clearing house. So there is no middle step between contractual sides that can alleviate the loss. The possible default exposure of counterparty is crucial in evaluating the contract. Good evidence of steep increasing OTC trades is the half year period from June 2008 to December 2008, when was recorded one of the steepest increase, gross market value increased from \$ 20 trillion to \$34 trillion (70 %)[2].
The concept of the default risk can be easily illustrated on the following consideration: an investor, who is trading on OTC market with entities that can

[1] we are using these terms equivalently with any further notice in the text
[2] data from Bloomberg publication **Counterparty Valuation adjustments** http://ssm.com/abstract=1463042

default, wants to be rewarded for this risk. This investor requires an additional *risk premium.* Clearly, it is possible to see on the bond markets, even if the bonds are not usual OTC instruments, that market participants who are investing into bonds, issued by lower credit quality entities, require to have higher yields. The difference between yields from risky instruments and non risky , in the meaning of default risk, is called credit spread.

To include the impact of the default risk to the value of financial instrument we need to model it somehow firstly. There exists two main categories of credit risk models
(i) Reduced form models also known as *intensity-based models*
(ii) Structural models also known as *models based on the value of the firm* .

Reduced form models describes the default by the means of exogenous jump process. This type of models are also called *intensity models* or *hazard rate models.* The main tool in this approach is the exogenous specification of conditional probability of default, given that default has not yet happened. The family of reduced form models, is suited to model credit spreads and it is in relatively simple way calibrate-able to the Credit Default Swap (CDS), see Section 6.1. Default events are modeled directly by probabilities of occurrence of such an event. We are using also this type of model to calculate risk neutral default probabilities.

Structural models represent different approach how to model defaults. These type of models deal with the firm's economics fundamentals. The market value of the company is the main stream of uncertainty that drives credit risk. They are mostly based on the work of Merton(1974) [19]. We can motivate to use such a model by the following consideration. Let us assume, that the company has issued a bond to finance its business activities. If at the maturity date T the firm can not repay all its commitments to the bond holders, we can conclude, that there has been a default. We can also consider more sophisticated and more reality capturing models, where the default can happen not only at the maturity date but also before this date. They are called *first-passage-time approach model* see Black ans Cox(1976)[5]. The default happens when the value of asset hit from above the certain barrier, this barrier may be either deterministic or a random process itself calling *barrier process.* First-passage-time approach allows to adapt the model to the real world conditions such as those mentioned; default before maturity or in many other possible ways, precisely specified the recovery payoff associated with default event or bankruptcy costs or taxes. Structural models assume that the company value follows the stochastic process, that is similar to the random process that describes the behavior of stocks. Because of this feature, the value of the company can be observed at any time and the default process can be completely monitored on default free market. For more properties of these model see Chapter 3 of *Bielicky and Rutkowsky(2002)* [3]. The disadvantage of these models is that they can not be easily calibrated on publicly accessible data, for example as CDS quotes.

4.1 General Pricing Formula

In this section, we introduce two types of general pricing formula for unilateral credit risk and also for bilateral credit risk. In the process of deriving the pricing formula of general payoff with no specified default, we follow the paper Brigo and Masseti(2004)[11] for unilateral case and for bilateral case we follow the article Brigo and Capponi(2008) [9]. In the beginning, we set up the framework for the probabilistic part and then we derive the pricing formula for general payoff.

4.1.1 Probabilistic set up

We construct the probability space $(\Omega, \mathcal{G}, \mathcal{G}_t, \tilde{\mathbb{P}})$. The measure $\tilde{\mathbb{P}}$ is risk neutral measure and $\mathcal{G}_t$ represents the flow of all information. The sub filtrations $\mathcal{F}_t$, that is right continuous and complete, represents the information about market but without default ($\mathcal{F}_t \subseteq \mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$), where $\mathcal{H}_t = \sigma(\{\delta_1 \leq u\}, \{\delta_2 \leq u\}); u \leq t$ is the right filtration generated by the default events. We use two default events since, we are going to use both of them for bilateral credit risk pricing formula.

In previous sections there was no need to monitor default events because we did not use this information. The information included in filtration $\mathcal{F}_t$ was sufficient. Now we want to incorporate the information about default events into the price of financial instrument. We need to model it by $\sigma-$ algebra $\mathcal{G}_t$. The difference in the probability space for non-default case and default case of the market is in information included in filtration. To the further notice, we are assuming all expectation with respect to risk neutral probability measure $\tilde{\mathbb{P}}$.

Assumption: *We are assuming that default times are independent from the interest rate term structure .*

4.1.2 Valuation Formula:unilateral counterparty risk

Here we are considering just the possibility of one counterparty default, so we are working with filtration $\mathcal{H}_t = \sigma(\{\delta \leq u\}; u \leq t)$, generated by default event of the counterparty. The next step is to determine a pricing formula of a general payoff with maturity τ. We do not need, at least now, to specify a model for the default time. Now we introduce required notation

$C(t,\tau)$	Net cashflow from the claim during the time interval $[t,\tau]$ discounted to time t
$NPV(\delta) := E_{\tilde{\mathbb{P}}}[C(\delta,\tau)\|\mathcal{G}_\delta]$	Net Present Value, expectation with respect to a risk neutral measure
R	The fraction of the payoff from a face value paid to an investor in case of default
$LGD := 1 - R$	Loss Given Default;loss in a value if default occures
$D(t,\delta) := \frac{B_t}{B_\delta}$	Discounting factor at time t for maturity δ

In the process of payoff determination we need to distinguish two cases of possible default time. If $\delta > \tau$ or $\delta \leq \tau$ respectively. If the default happened and the counterparty could not fulfill its obligations, or there was no default and all obligations from the contract were fullfiled.
• $\delta > \tau$ there was no default up to time τ, so the payoff is

$$\mathbf{1}_{\{\delta>\tau\}}C(t,\tau) \tag{4.1}$$

where $\mathbf{1}_{\{\delta>\tau\}}$ is indicator function of the set of all events in Ω for which $\delta > \tau$.

• $\delta \leq \tau$ The counterparty defaulted during the life of the contract so, could not fully repay the investment. We need to compute the net present value(NPV) of the residual payoff from time of default δ until maturity τ. If the NPV is positive for the defaulted counter party it is completely received by the counterparty itself.
If it is negative for defaulted counterparty, only the recovered part is payed.

$$\mathbf{1}_{\{\delta\leq\tau\}}(R(NPV(\delta))_+) \tag{4.2}$$

where $(x)_+ := max\{0, x\}$ for any real x.
If we combine two terms from above, we get the following pre-result

Proposition 4.1.1. *The payoff of the defaultable claim, discounted to time t, let's denote it $\Pi(t)^D$, is given by*

$$\Pi^D(t) = \mathbf{1}_{\{\delta>\tau\}}C(t,\tau) + \mathbf{1}_{\{\delta\leq\tau\}}\left[C(t,\delta) + D(t,\delta)(R(NPV(\delta)_+) - (-NPV(\delta))_+)\right] \tag{4.3}$$

Remark 4.1.2. *We can see that, if a default does not occur, the formula is reducing to standard expression for non-defaultable payoff, because the last term vanishes.*

From the previous lines we formulate the *General Counterparty Risk Pricing Formula* as an expectation of 4.3.

Theorem 4.1.3. General Unilateral Counterparty Risk Pricing Formula. *Let assume that a counterparty does not default before valuation date t and LGD is deterministic, then the price of payoff under the counterparty risk exposure is*

$$E_{\tilde{\mathbb{P}}}\{\Pi^D(t)|\mathcal{G}_t\} = E_{\tilde{\mathbb{P}}}\{\Pi(t)|\mathcal{G}_t\} - LGD \times E_{\tilde{\mathbb{P}}}\{\mathbf{1}_{\{t<\delta\leq\tau\}}D(t,\delta)(NPV(\delta))_+|\mathcal{G}_t\} \quad (4.4)$$

where $E_{\tilde{\mathbb{P}}}\{\Pi(t)|\mathcal{G}_t\}$ is expected value of the net cash-flows, discounted to time t of the claim without, any default risk consideration, nor investor's nor counterparty's(further just a default free cash-flows).

Proof. We can expand the terms inside the right hand expectation of 4.4 as follows

$$\mathbf{1}_{\{\delta>\tau\}}C(t,\tau)+\mathbf{1}_{\{\delta\leq\tau\}}C(t,\tau)+R\mathbf{1}_{\{\delta\leq\tau\}}D(t,\delta)(NPV(\delta)_+)-\mathbf{1}_{\{\delta\leq\tau\}}D(t,\delta)(NPV(\delta))_+ \quad (4.5)$$

since we know that

$$\Pi(t) = C(t,\tau) = \mathbf{1}_{\{\delta>\tau\}}C(t,\tau) + \mathbf{1}_{\{\delta\leq\tau\}}C(t,\tau)$$

Now we take the expected value of the second and the fourth term of 4.5 conditioned at time δ and use these two facts.

$$f = f^+ - f^- = f^+ - (-f)^+$$

and

$$\mathbf{1}_{\{\delta\leq\tau\}}C(t,T) = \mathbf{1}_{\delta\leq\tau}(C(t,\delta) + D(t,\delta)C(\delta,\tau))$$

it follows

$$\begin{aligned}
& E_{\tilde{\mathbb{P}}}[\mathbf{1}_{\{\delta\leq\tau\}}C(t,\tau) - \mathbf{1}_{\{\delta\leq\tau\}}D(t,\delta)(NPV(\delta))_+|\mathcal{G}_\delta] \\
= \; & E_{\tilde{\mathbb{P}}}[\mathbf{1}\{\delta\leq\tau\}[C(t,\delta) + D(t,\delta)C(\delta,\tau) - D(t,\delta)E_{\tilde{\mathbb{P}}}([C(\delta,\tau)]|\mathcal{G}_\delta)_+]|\mathcal{G}_\delta] \\
= \; & \mathbf{1}_{\{\delta\leq\tau\}}[C(t,\tau) - D(t,\delta)(E_{\tilde{\mathbb{P}}}[C(\delta,\tau)]|\mathcal{G}_\delta)_-] \\
= \; & \mathbf{1}_{\{\delta\leq\tau\}}[C(t,\tau) - D(t,\delta)(E_{\tilde{\mathbb{P}}}[-C(\delta,\tau)]|\mathcal{G}_\delta)_+]
\end{aligned}$$

When we combine result from the above computation with 4.5 we get

$$\begin{aligned}
& \mathbf{1}_{\{\delta>\tau\}}C(t,\tau) + \\
& \mathbf{1}_{\{t<\delta\leq\tau\}}[C(t,\tau) + D(t,\delta)(R(NPV(\delta))_+ - (-NPV(\delta))_+)]
\end{aligned}$$

which is the 4.3. □

Remark 4.1.4. *From Theorem 4.4, the price of defaultable instrument is default free price minus a discounted option term, more precisely a call option with zero strike price on the residual NPV, in case of $\delta \leq \tau = T$. From this follows that even if the original non default payoff is interest rate term structure independent, the optionality in the formula requires for some other valuations interest rate term structure model. We will see it in the following chapter, where we are dealing with defaultable interest rate swaps.*

4.1.3 Valuation Formula:bilateral counterparty risk

A logical extension of the unilateral counterparty risk pricing formula is incorporation of the second possible default into the the payoff. Let's denote investor's default time δ_1 and counterparty's default time δ_2. Now we are working with

$$\mathcal{H}_t = \sigma(\{\delta_1 \leq u\}, \{\delta_2 \leq u\}); u \leq t$$

as it was previously defined. The whole notation is the same as before, just it needs to be distinguished between two recovery rates R_1 and R_2 and corresponding LGD_1 and LGD_2. It is because the participants of the contract can come from different environments with different bankrupt laws and policies. We make an analysis of possible outcomes that come from different combinations of default times δ_1, δ_2 and maturity time τ. All statements in the following bullet points are from investor's perspective. From the perspective, of the counterparty perspective each cashflow is with different sign(e.g. investor pays so counterparty receives, for investor positive NPV, it is negative for counterparty etc).

- $\delta_1 > \tau$ and $\delta_1 > \tau$ The investor nor the counterparty defaults, the payoff is original default free payoff.
- $\delta_1 \leq \delta_2 \leq \tau$ If the NPV is positive for a defaulted investor, it is completely received by the investor. If the NPV is negative for defaulted investor, only recovery fraction is paid(R_1).
- $\delta_2 \leq \delta_1 \leq \tau$ If the NPV is positive for an investor, the investor receives just a recovery fraction from the residual payoff(R_2). If it is negative for the investor it is completely paid by the investor.

We denote the discounted net cashflows from a claim exposed to bilateral default risk as follows $C_B^D(t, \tau_i)$, for $i = 1, 2$ and the discounted net cashflow that does not face any credit risk remains the same as $C(t, \tau)$. From the previous statements follows next pre-result.

Proposition 4.1.5. *The payoff of a defaultable claim facing to the bilateral credit risk is given as follows*

$$\begin{aligned} C_B^D(t) &= \mathbf{1}_{\{\delta_1 > \tau, \delta_2 > \tau\}} C(t, \tau) \\ &+ \mathbf{1}_{\{\delta_2 \leq \delta_1 \leq \tau \cup \delta_2 \leq \tau \leq \delta_1\}} \left[C(t, \delta_2) + D(t, \delta_2)(R_2(NPV(\delta_2)_+) - (-NPV(\delta_2))_+)\right] \\ &+ \mathbf{1}_{\{\delta_1 \leq \delta_2 \leq \tau \cup \delta_1 \leq \tau \leq \delta_2\}} \left[C(t, \delta_1) + D(t, \delta_1)((NPV(\delta_1)_+) - R_1(-NPV(\delta_1))_+)\right] \end{aligned} \tag{4.6}$$

Finally we are approaching the pricing formula of the claim that is exposed to bilateral the default risk.

Theorem 4.1.6. General Bilateral Counterparty Risk Pricing Formula *The payoff at time t, of the claim with maturity time τ,$\tau \geq t$ under the bilateral default risk is given by*

$$\begin{aligned} E_{\tilde{\mathbb{P}}}\{C^D(t, \tau)|\mathcal{G}_t\} &= E_{\tilde{\mathbb{P}}}\{C(t, \tau)|\mathcal{G}_t\} \\ &+ LGD_1 \times E_{\tilde{\mathbb{P}}}\{\mathbf{1}_{\{\delta_1 \leq \delta_2 \leq \tau \cup \delta_1 \leq \tau \leq \delta_2\}} D(t, \delta)(-NPV(\delta_1))_+ |\mathcal{G}_t\} \\ &- LGD_2 \times E_{\tilde{\mathbb{P}}}\{\mathbf{1}_{\{\delta_2 \leq \delta_1 \leq \tau \cup \delta_2 \leq \tau \leq \delta_1\}} D(t, \delta)(NPV(\delta_2))_+ |\mathcal{G}_t\} \end{aligned} \tag{4.7}$$

where LGD_1 and LGD_2 are corresponding Loss Given Defaults of the investor and the counterparty respectively. Defaultable claim value consists of the a value of a default free claim plus a long position in a put option, with a zero strike price on the residual NPV, when the investor defaults before counterparty and before the maturity of the contract. The last term that contributes to the value of the defaultable claim is a short position in a call option with zero strike price on the residual NPV, conditioned by the earliest default of the counterparty and before the contract maturity.

Proof. Since the proof is just technical manipulation with cash-flows and expectations, same as for the proof of the formula 4.4, we just sketch here the main steps of the proof. For the full proof see Appendix A of Brigo and Capponi(2008) [9].
The expectation is linear, so we can rewrite the right hand side term in 4.7 as follows

$$\begin{aligned} & E_{\tilde{\mathbb{P}}}\{C(t,\tau) + LGD_1\{\mathbf{1}_{\{\delta_1 \le \delta_2 \le \tau \cup \delta_1 \le \tau \le \delta_2\}} D(t,\delta)(-NPV(\delta_1))_+ \\ - & LGD_2 \times \{\mathbf{1}_{\{\delta_2 \le \delta_1 \le \tau \cup \delta_2 \le \tau \le \delta_1\}} D(t,\delta)(NPV(\delta_2))_+ | \mathcal{G}_t\} \end{aligned}$$

We decompose cash-flows as follows

$$\begin{aligned} C(t,\tau) &= \mathbf{1}_{\{\delta_2 \le \delta_1 \le \tau \cup \delta_2 \le \tau \le \delta_1\}} C(t,\tau) \\ &+ \mathbf{1}_{\{\delta_1 \le \delta_2 \le \tau \cup \delta_1 \le \tau \le \delta_2\}} C(t,\tau) \\ &+ \mathbf{1}_{\{\delta_2 > \tau, \delta_1 > \tau\}} C(t,\tau) \end{aligned}$$

By using the relation from above, we can rewrite the formula 4.8 as follows

$$\begin{aligned} & E_{\tilde{\mathbb{P}}}\{\mathbf{1}_{\{\delta_1 \le \delta_2 \le \tau \cup \delta_1 \le \tau \le \delta_2\}} C(t,\tau) + (1-R_1)\{\mathbf{1}_{\{\delta_1 \le \delta_2 \le \tau \cup \delta_1 \le \tau \le \delta_2\}} D(t,\delta_1)(-NPV(\delta_1))_+ \\ & + \mathbf{1}_{\{\delta_2 \le \delta_1 \le \tau \cup \delta_2 \le \tau \le \delta_1\}} C(t,\tau) + (R_2 - 1)\{\mathbf{1}_{\{\delta_2 \le \delta_1 \le T \cup \delta_2 \le T \le \delta_1\}} D(t,\delta_2)(NPV(\delta_2))_+ \\ & + \mathbf{1}_{\{\delta_2 > T, \delta_1 > \tau\}} C(t,\tau) | \mathcal{G}_t\} \\ = & E_{\tilde{\mathbb{P}}}\{\mathbf{1}_{\{\delta_2 > \tau, \delta_1 > \tau\}} C(t,\tau) | \mathcal{G}_t\} \\ + & E_{\tilde{\mathbb{P}}}\{\mathbf{1}_{\{\delta_2 \le \delta_1 \le \tau \cup \delta_2 \le T \le \delta_1\}} C(t,\tau) \\ + & (R_2 - 1)\{\mathbf{1}_{\{\delta_2 \le \delta_1 \le \tau \cup \delta_2 \le \tau \le \delta_1\}} D(t,\delta_2)(NPV(\delta_2))_+ | \mathcal{G}_t\}\} \\ + & E_{\tilde{\mathbb{P}}}\{\mathbf{1}_{\{\delta_1 \le \delta_2 \le \tau \cup \delta_1 \le \tau \le \delta_2\}} C(t,\tau) \\ + & (1-R_1)\{\mathbf{1}_{\{\delta_1 \le \delta_2 \le \tau \cup \delta_1 \le \tau \le \delta_2\}} D(t,\delta_1)(-NPV(\delta_1))_+ | \mathcal{G}_t\}\} \end{aligned} \tag{4.8}$$

With some reformulation of the above expected values and cash-flows it is possible to show that the above expectations corresponds exactly with expected values of cash flows from Equation 4.6, in the same order. So this equality proves the theorem.

□

5. Incorporation of Counterparty Risk

In this section, we incorporate a possibility of a default before maturity of the financial instruments to our previous valuation model. For the first two bond cases we are using just the unilateral version of pricing formula for a defaultable claim. Since only an investor is exposed to a default risk of bond issuer. This chapter basically combines results from Chapter 3 and Chapter 4.

In the case of interest rate swap, we derive a formula for the unilateral default risk and for the bilateral default risk as well. We use these formulas in the second part of this work where we study particular interest rate swaps and their prices. We assume the same probabilistic framework and notation as we have developed earlier in Chapter 4.

Assumption: For the simplicity, in the whole chapter we are assuming independence between interest rates and risk neutral default probabilities.

5.1 Defaultable Zero Coupon Bonds

Now we are implementing to our valuation framework of zero coupon bonds the possibility of default by adapting formulas for a zero coupon bond price derived in Section 3.2. We are aware that there is no need to incorporate the default risk into the bond prices because the mechanism of the market already incorporating such risk into the bond prices, but as we see, the concept of CVA serves us a good tool to quantify the contribution of a credit risk into the bond prices. It is clear that the price of a bond is affected just by credit quality of a bond issuer. So as it is already mentioned, we need to assume just unilateral default risk in this case. The zero coupon bonds $P(t,T)$ are one of the possible fundamental quantities describing the interest rate curve. We consider the defaultable zero bond $P_D(t,T)$ as one of the possible instrument describing the defaultable market yields.

In non-default case a bond at maturity time T delivers one unit of money to a bond holder and the value of such a bond is given by 3.2. In this case it is

$$P_D(t,T) = B_t E_{\tilde{\mathbb{P}}}[B_T^{-1}\mathbf{1}_{\{\delta>T\}}|\mathcal{G}_t] \tag{5.1}$$

where $\mathcal{G}$ represents the flow of information about the possible occurrence of a default before time t. In this particular case we have ignored any recovery, so at time T the contract pays the desired payoff. If the date of default is after T, then on the other hand contract does not deliver any payoff, if $\delta \leq T$(defaulted before maturity date T). So such a payoff of zero coupon bonds satisfies the description of defaultable zero coupon bonds.

In the view of previous section we assume that bond holder not receiving anything after the bond's default is not reflecting the reality. We need to include recovery rate R, so the payoff and the price of such a bond will be given by the following result

Proposition 5.1.1. *In the case of a risk neutral framework and an existence of* *equivalent martingale measure $\tilde{\mathbb{P}}$ for a defaultable zero coupon bond with maturity T and with payoff discounted to time $t, t \leq T$ given as*

$$\frac{B_t}{B_T}\mathbf{1}_{\{\delta>T\}} + R\frac{B_t}{B_\delta}\mathbf{1}_{\{\delta\leq T\}}$$

where R is recovery rate, the value of such bond is given by

$$P_D(t,T) = E_{\tilde{\mathbb{P}}}\left[\frac{B_t}{B_T}\mathbf{1}_{\{\delta>T\}} + R\frac{B_t}{B_\delta}\mathbf{1}_{\{\delta\leq T\}}\Big|\mathcal{G}_t\right] \tag{5.2}$$

.

Proof. The proof is straight forward using Risk-Neutral Valuation Formula for $C = \mathbf{1}_{\{\delta>T\}} + R\mathbf{1}_{\{\delta\leq T\}}$ □

We can also use our General Counterparty risk pricing formula to prove the following proposition

Proposition 5.1.2. *The price of the defaultable zero coupon bond at time t with maturity T is*

$$P_D(t,T) = P(t,T) - LGD \times \int_t^T \tilde{\mathbb{P}}[\delta \in [u, u+du)|\mathcal{G}_t]\frac{B_t}{B_u}P(u,T)du \tag{5.3}$$

where $P(t,T)$ is the value of the default free zero coupon bond at time $t, t \leq T$ with maturity T.
Probabilities are with respect to the filtration $\mathcal{G}_t$ that means, that they are observed at time t with information up to this time.

Proof. We simply use Theorem 4.1.3, from that follows

$$P_D(t,T) = P(t,T) - LGD \times E_{\tilde{\mathbb{P}}}\left[\mathbf{1}_{\{\delta\leq T\}}\frac{B_t}{B_\delta}\left(E_{\tilde{\mathbb{P}}}\left[\frac{B_\delta}{B_T}\Big|\mathcal{G}_\delta\right]\right)_+ |\mathcal{G}_t\right]$$

and the last term can be rewritten as follows

$$\begin{aligned}
& E_{\tilde{\mathbb{P}}}\left[\mathbf{1}_{\{\delta\leq T\}}\frac{B_t}{B_\delta}\left(E_{\tilde{\mathbb{P}}}\left[\frac{B_\delta}{B_T}\Big|\mathcal{G}_\delta\right]\right)_+ \Big|\mathcal{G}_t\right] \\
& \text{default can happen in time interval from t to T} \\
= \; & E_{\tilde{\mathbb{P}}}\left[\int_t^T \mathbf{1}_{\{\delta\in[u,u+du)\}}\frac{B_t}{B_u}\left(E_{\tilde{\mathbb{P}}}\left[\frac{B_u}{B_T}\Big|\mathcal{G}_u\right]\right)_+ \Big|\mathcal{G}_t\right] du \\
& \text{thanks to the independence between} \\
& \text{the interest rate and the default time we have} \\
= \; & \int_t^T E_{\tilde{\mathbb{P}}}[\mathbf{1}_{\{\delta\in[u,u+du)\}}|\mathcal{G}_t]\frac{B_t}{B_u}P(u,T)du
\end{aligned}$$

□

As we see we get to the value of defaultable bond in two different ways and with a small reformulation of 5.2, these two results coincides.

Remark 5.1.3. *In practice, if we want to calculate the expression from 5.3,the integral needs to be approximated by the sums so assumptions about the time of possible defaults needs to be made.*
We divide the time interval from the beginning to the end of the bond life into m equidistant time intervals. If δ is in interval $(T_{i-1}, T_i]$, then we postpone it to the time T_i, for $i = 1, \ldots, m$. So we approximate the formula 5.3 by the following

$$P_D(t,T) \approx P(t,T) - LGD \times \sum_{i=1}^{m} \tilde{\mathbb{P}}[\delta \in (T_{i-1}, T_i]]_t \frac{B_t}{B_{T_i}} P(T_i, T) \tag{5.4}$$

The number of intervals depends on a default probability term structure that is available.

5.2 Defaultable Coupon Bearing Bonds

Analogously with section 2.3, we derive the price of a coupon bearing bond, paying coupons $c_1, c_2, \ldots c_m$ at times $T_1, T_2, \ldots T_m = T$ with a recovery rate R and we assume $M = 1$ as well.

Proposition 5.2.1. *In our risk neutral framework and under the existence of an equivalent martingale measure $\tilde{\mathbb{P}}$, we are considering a defaultable coupon bearing bond with maturity T, coupons as defined above and with payoff given by*

$$PayOff = \sum_{i=1}^{m} c_i \frac{B_t}{B_{T_i}} \mathbf{1}_{\{\delta > T_i\}} + \frac{B_t}{B_{T_m}} \mathbf{1}_{\{\delta > T_i\}} + R \sum_{i=1}^{m} \frac{B_t}{B_{T_i}} c_i \mathbf{1}_{\{\delta \in (T_{i-1}, T_i)\}} + R\mathbf{1}_{\{\delta \in (T_{m-1}, T_m)\}} \tag{5.5}$$

then the value at time t,$t \leq T_1$ of such bond is risk neutral expectation of its payoff

$$P_{C,D}(t,T) = E_{\tilde{\mathbb{P}}}[PayOff] \tag{5.6}$$

The first two terms correspond simply to-non defaultable coupon bearing bond and the last term is a recovery payment, if an early default happens. The notation correspond with the previous sections, so $P_{C,D}(t,T)$ means *the value of defaultable coupon bearing bond at time t with maturity T*.

By applying the general pricing theorem for defaultable securities we can prove easily following proposition.

Proposition 5.2.2. *Let's be $P_C(t,T)$ the default free value of the coupon bearing bond at time t, paying coupons $c_1, c_2, \ldots c_m$ at times $T_1, T_2, \ldots T_m$, then the value of the same coupon bearing bond with a defaultable counterparty(issuer of the bond) has price given by*

$$P_{C,D}(t,T) = P_C(t,T) - LGD \times \int_t^T \tilde{\mathbb{P}}(\delta \in [u, u+du) | \mathcal{G}_t) \frac{B_t}{B_u} P_c(u,T) du \tag{5.7}$$

where $P(t,T), t \leq T$ is the value of coupon bearing bond at time t with maturity T.

Proof. In this proof we are following exactly the same steps as in the proof of Theorem 5.1.2 . From Theorem 4.1.3 follows

$$\begin{aligned} P_{C,D}(t,T) &= P_C(t,T) \\ &- LGD \times E_{\tilde{\mathbb{P}}}\left[\mathbf{1}_{\{\delta \le T\}} \frac{B_t}{B_\delta} E_{\tilde{\mathbb{P}}}\left(\sum_{j=1}^{m} \mathbf{1}_{\{T_j \ge \delta\}} c_j \frac{B_\delta}{B_{T_j}} + \mathbf{1}_{\{T_m \ge \delta\}} \frac{B_\delta}{B_{T_m}} \Big| \mathcal{G}_\delta\right)_+ \Big| \mathcal{G}_t\right] \end{aligned}$$

The indicator functions in the second expectation secure that just the residual payments from the coupon bearing bonds are included. As the default can happen from the inception of the bond to its maturity we write

$$E_{\tilde{\mathbb{P}}}\left[\int_t^T \mathbf{1}_{\delta \in [u,u+du)} \frac{B_t}{B_u} \underbrace{E_{\tilde{\mathbb{P}}}\left(\sum_{j=1}^{m} \mathbf{1}_{\{T_j \ge \delta\}} c_j \frac{B_u}{B_{T_j}} + \mathbf{1}_{\{T_m \ge \delta\}} \frac{B_u}{B_{T_m}} \Big| \mathcal{G}_u\right)_+}_{\text{pay-off of coupon bearing bond}} \Big| \mathcal{G}_t\right] du$$

using the same computations and the same adjustments as in the proof of Theorem 5.1.2 we get

$$= \int_t^T \tilde{\mathbb{P}}(\delta \in [u, u+du) | \mathcal{G}_t) \frac{B_t}{B_u} P_C(u,T) du$$

that is exactly what we claim in theorem.

□

Remark 5.2.3. *In this case we also need to make the approximation about default times, if we want to use the formula in practice. If default occurs in time interval $(T_{i-1}, T_i]$ we put $\delta = T_i$ for all $i = 1, \dots m$, where time moments T_i's do not have to coincide with coupon payments. So we can approximate the integral in the expression 5.7 by a sum, the value of a defaultable coupon bearing bond in this case is given by*

$$P_{C,D}(t,T) \approx P_C(t,T) - LGD \times \sum_{i=1}^{m} \tilde{\mathbb{P}}(\delta \in (T_{i-1}, T_i] | \mathcal{G}_t) \frac{B_t}{B_{T_i}} P_C(T_i, T) \qquad (5.8)$$

In the case study in the end of this work we use a month as discretization step for Greek governments bonds. The same approach was made also in Brigo(2008) [7].

5.3 Defaultable Interest Rate Swap

We have already introduced the valuation of interest rates swap in Section 3.3. In this section we incorporate the default component to the pricing. Now it is important to distinguish what kind of counterparty risk we are going to consider, because as it has been mentioned earlier, swaps can be asset or liability for both participants of the contract. The situation for bonds has been straight forward since just bond investor can be rewarded for the default risk. For the interest rate swap it is different, since both participants of the contract can require price adjustment, in respect of the counterparty default risk. In the following two sections we compute prices of IRS with the consideration of both types of default risk; unilateral and bilateral.

5.3.1 Defaulable IRS: unilateral case

In this case we are considering an unilateral default risk, a default free counter party is exchanging fixed payments $\kappa(t, T_0, T_m)M$ for floating payments $L(T_{j-1})M$ for $0 \leq j \leq m$ from defaultable counterparty at time $T_1, T_2, \ldots T_m$. Where M is a notional principal and κ is predetermined fixed rate and $L(T_{j-1})$ is floating rate determined at time T_{j-1} for period $(T_{j-1}, T_j]$ for all j, $1 \leq j \leq m$. The cash settlements are at times $T_1, T_2, \ldots T_m = T$. As we have derived earlier in Section 3.3, the formula for discounted payoff for swaps in non default case looks as follows,

$$C(T_i, T_m) = \sum_{i=1}^{m} (L(T_{j-1})\delta_{j+1}M - \kappa\delta_{j+1}M)$$

and the swap rate that makes the value of the swap at time T_0 equal to zero is

$$\kappa = \kappa(t, T_0, T_m) = \frac{(P(t, T_0) - P(t, T_m))}{\sum_{j=1}^{m} \delta_j P(t, T_j)}, \text{ for } t \leq T_0$$

now we are considering the possibility of a counterparty default. The swap rate should be lower because the counterparty should be rewarded for bearing the risk of possible default.

Proposition 5.3.1. *If we consider the same IRS as defined in the beginning of this chapter where the fixed a leg payer is default free and a floating leg payer is exposed to default risk, the value of such IRS(let denote it $IRS_D(t)$) from the perspective of a default free counterparty, at time t, $t \leq T_0$ is given by*

$$IRS_D(t) = IRS(t) - LGD \times OptionPart \tag{5.9}$$

where $IRS(t)$ is the value of a default free IRS as it is defined in Chapter 3.3. The $OptionPart$ is the option term from Remark 4.1.4. After applying the approximation about possible default time; if default happened in $(T_{i-1}, T_i]$ then we put $\delta = T_i$ then the $OptionPart$ is approximately given by

$$OptionPart \approx \sum_{i=1}^{m} \tilde{\mathbb{P}}[\delta \in (T_{i-1}, T_i]] PayerSwaption_t(T_i, T_m, \kappa, \sigma_{T_i}, \kappa(t, T_i, T_m))\} \tag{5.10}$$

where $PayerSwaption_t(T_i, T_m, \kappa, \sigma_{T_i}, \kappa(t, T_i, T_m))$ is the value of a payer swaption at time t with maturity T_i and with underlying swap with maturity T_m, strike price κ, underlying swap rate $\kappa(t, T_i, T_m)$, volatility σ_{T_i}. Time moments $T_1, .., T_m$ are the same as the payment dates in original default free IRS. The value of defaultable IRS can be express as follows:

Default Free IRS +LGD×stream of default probability weighted Swaptions values at time t

Proof. From 4.4 follows that the payoff of defaultable IRS is in general as follows

$$IRS_{\mathrm{D}}^{approx}(t) = IRS(t) - LGD \times OptionPart$$

where OptionPart is expected value of the residual payoff after the default. In this case the residual payoff is just the residual payoff of the interest rate swap after time δ (by our approximation after first T_i that follows δ), so for case of the IRS we have

$$\begin{aligned}
OptionPart &= E_{\tilde{\mathbb{P}}}[\mathbf{1}_{\{\delta\le T_m\}}\frac{B_t}{B_\delta}(NPV(\delta))_+|\mathcal{G}_t] \\
&\quad \text{we divide the life of the contract into to the subintervals} \\
&\quad \text{ that correspond with the time period between payments} \\
&= E_{\tilde{\mathbb{P}}}\left[\sum_{i=1}^{m}\mathbf{1}_{\{\delta\in(T_{i-1},T_i]\}}\frac{B_t}{B_\delta}(E_{\tilde{\mathbb{P}}}[C(\delta,T_m)|\mathcal{G}_\delta])_+|\mathcal{G}_t\right] \\
&\quad \text{default times postponed to the following payment moments} \\
&\approx E_{\tilde{\mathbb{P}}}\left[\sum_{i=1}^{m}\mathbf{1}_{\{\delta\in(T_{i-1},T_i]\}}\frac{B_t}{B_{T_i}}(E_{\tilde{P}}[C(T_i,T_m)|\mathcal{G}_{T_i}])_+|\mathcal{G}_t\right] \\
&= E_{\tilde{\mathbb{P}}}\left[\sum_{i=1}^{m}\mathbf{1}_{\{\delta\in(T_{i-1},T_i]\}}\frac{B_t}{B_{T_i}}(E_{\tilde{\mathbb{P}}}[C(T_i,T_m)|\mathcal{G}_{T_i}])_+|\mathcal{G}_t\right] \\
&\quad \text{using independency of default time and the interest rate} \\
&= \sum_{i=1}^{m}E_{\tilde{\mathbb{P}}}\left[\mathbf{1}_{\{\delta\in(T_{i-1},T_i]\}}\right]\underbrace{E_{\tilde{\mathbb{P}}}\left[\frac{B_t}{B_{T_i}}(\text{Payer Swap})_+|\mathcal{G}_t\right]}_{\text{swaption's pay off}} \\
&= \sum_{i=1}^{m}\tilde{\mathbb{P}}[\delta\in(T_{i-1},T_i]]\text{PayerSwaption}_t(T_i,T_m,\kappa,\sigma_{T_i},\kappa(t,T_i,T_m))
\end{aligned}$$

□

If we have computed the value of IRS that already contains the credit risk of the counterparty, we are able to derive new fair fixed rate that also contains the adjustment for the default risk of the counterparty. We illustrate this method in the end of this work, where we are constructing imaginary IRS between Czech Republic and Greece.

5.3.2 Defaulable IRS: bilateral case

We have mentioned that in the case of IRS, both counterparties could be affected by a default of other entity from the contract. If the payer of the fixed leg is more exposed to a higher default risk(lower credit quality), the floating leg payer wants to be rewarded for such a risk and contrary. So it is fair, if the default risk of both is incorporated to the fixed rate. We use our standard formula for the incorporation of the default risk into the payoff of IRS. In the previous section, we have shown that the incorporation of default risk into the value of IRS is basically a subtraction of the swaption prices weighted by the risk neutral default probabilities from the value of default free IRS. Very similar it is for IRS in the environment of bilateral default risk. In the following proposition, we are assuming the same IRS as at the beginning of this chapter with same payments, but now both fixed leg payer and floating leg payer are exposed to default risk. For such a defaultable IRS, it holds following proposition.

Proposition 5.3.2. *For the IRS as defined in the beginning of this Chapter 5.3.1 and under the consideration of bilateral default risk the value of such IRS, let's denote it $IRS_D^B(t)$ at time $t, t \leq T$ is given by*

$$IRS_D^B(t) = IRS(t) + OptionPart_1 - OptionPart_2 \tag{5.11}$$

$$\begin{aligned}
& OptionPart_1 \approx \\
\approx \; & \sum_{i=1}^{m} LGD_1 \Big(\tilde{\mathbb{P}}[T_{i-1} < \delta_1 \leq T_i, T_i \leq \delta_2 \leq T_m] + \tilde{\mathbb{P}}[\, T_{i-1} < \delta_1 \leq T_i, \delta_2 \geq T_m] \Big) \\
\times \; & ReceiverSwaption_t(T_i, T_m, \kappa, \sigma_{T_i,T_b})\}
\end{aligned}$$

$$\begin{aligned}
& OptionPart_2 \approx \\
\approx \; & \sum_{i=1}^{m} LGD_2 \Big(\tilde{\mathbb{P}}[T_{i-1} < \delta_2 \leq T_i, T_i \leq \delta_1 \leq T_m] + \tilde{\mathbb{P}}[\, T_{i-1} < \delta_2 \leq T_i, \delta_1 \geq T_m] \Big) \\
\times \; & PayerSwaption_t(T_i, T_m, \kappa, \sigma_{T_i,T_b})\}
\end{aligned}$$

where Payer(Receiver)Swaption$_t(T_i, T_m, \kappa, \sigma_{T_i}, \kappa(t, T_i, T_m))$ is the value of payer(receiver) swaption at time t with maturity T_i and with underlying swap with maturity T_m, strike price κ, underlying swap rate $\kappa(t, T_i, T_m)$ and volatility σ_{T_i}. Time moments $T_1, .., T_m$ are the same as the payments dates in the original default free IRS.

Proof. The proof is straight forward by using the Theorem 4.1.6 and by using the same calculation as for optionpart in Proposition 5.3.1. From the equation 4.7 and the first right hand side expectation and replacing $C(t, T)$ by the IRS payoff, we get that it is equal to the value of default free IRS at time t.
The second right hand side expectation from equation 4.7 is giving us

$$\begin{aligned}
& LGD_1 \times E_{\tilde{\mathbb{P}}}\{\mathbf{1}_{\{\delta_1 \leq \delta_2 \leq T \cup \delta_1 \leq T \leq \delta_2\}} D(t, \delta_1)(-NPV(\delta_1))_+ | \mathcal{G}_t\} \\
& \text{we divide the life of the contract into to the subintervals} \\
& \text{ that correspond to the period between the payments} \\
= \; & LGD_1 \\
\times \; & E_{\tilde{\mathbb{P}}} \left[\sum_{i=1}^{m} \{\mathbf{1}_{\{\delta_1 \in (T_{i-1}, T_i], \delta_2 \in (T_i, T_m] \cup \delta_1 \in (T_{i-1}, T_i], \delta_2 \geq T_m\}} \frac{B_t}{B_\delta} (-E_{\tilde{\mathbb{P}}}[C(\delta_1, T_m) | \mathcal{G}_{\delta_1}])_+ | \mathcal{G}_t\} \right] \\
& \text{default times postponed to the following payment moments} \\
\approx \; & LGD_1 \\
\times \; & E_{\tilde{\mathbb{P}}} \left[\sum_{i=1}^{m} \{\mathbf{1}_{\{\delta_1 \in (T_{i-1}, T_i], \delta_2 \in (T_i, T_m] \cup \delta_1 \in (T_{i-1}, T_i], \delta_2 \geq T_m\}} \frac{B_t}{B_{T_i}} (-E_{\tilde{\mathbb{P}}}[C(T_i, T_m) | \mathcal{G}_\delta])_+ | \mathcal{G}_t\} \right] \\
& \text{we are considering } C(T_i, T_m) \text{ for IRS cash-flows from Chapter 3.3} \\
& \text{and assume once more independence between the default time and the interest rate} \\
= \; & LGD_1 \sum_{i=1}^{m} \Big(\tilde{\mathbb{P}}[T_{i-1} < \delta_2 \leq T_i, T_i \leq \delta_1 \leq T_m] + \tilde{\mathbb{P}}[\, T_{i-1} < \delta_2 \leq T_i, \delta_1 \geq T_m] \Big) \\
& \times \text{ReceiverSwaption}_t(T_i, T_m, \kappa, \sigma_{T_i,T_b})
\end{aligned}$$

The last term from the above is exactly the value of $optionpart_1$, The same we can do with the $optionterm_2$ and with the third right hand side expectation from the equation 4.7 and this proves our proposition. $\square$

For the unilateral case, we have prepared the formula for a defaultable IRS to such a form that it is almost straight forward to use it, if one has a marginal default probability term structure. For the bilateral case it is not so straight forward since we are dealing with joint default probabilities of the two counterparties. For a further practical use one need to choose appropriate form of a statistical structure to gain joint default probabilities from the marginal ones.

6. Credit Derivatives

Credit derivative is financial derivative, value of which is dependent on the credit-sensitive asset. This asset can be any financial instrument that is subjected to a default risk. Nowadays, this type of derivatives is hugely used to transfer or mitigate the credit risk. First agreements about credit risk transfer were signed in 1990s but the similar agreements about credit derivatives were use much more earlier. In these days, bonds issuers pay to the banks premium or annual fee in exchange for the bank's promise to make debt payments on behalf of the issuers. However these contracts have one big disadvantage and that is that they have not been sold separately from underlying assets. Current types of credit derivatives are traded on OTC market separately from the underlying credits.
Two main categories of the credit derivatives are *Single-name derivatives* and *Multi-name credit derivatives* .

• **Single-name credit derivatives:** are derivatives that protect against the risk of one particular counterparty for example, asset swap, credit linked notes or credit default swaps.
• **Multi-name credit derivatives:** are credit derivatives that protect against the default of one or more counterparties. The dependence on the pool of counterparties leads to a strong impact of default dependent structures of pool's members on derivatives price. Examples of this type are basket default swaps and collateralized debt obligation (CDO). For a different approaches to valuation of these instruments see Witzany(2010)[27] Chapter 5.
In the following part, we introduce basic product of a credit derivatives market and also one used at most for the purposes of extracting default probabilities from market data.

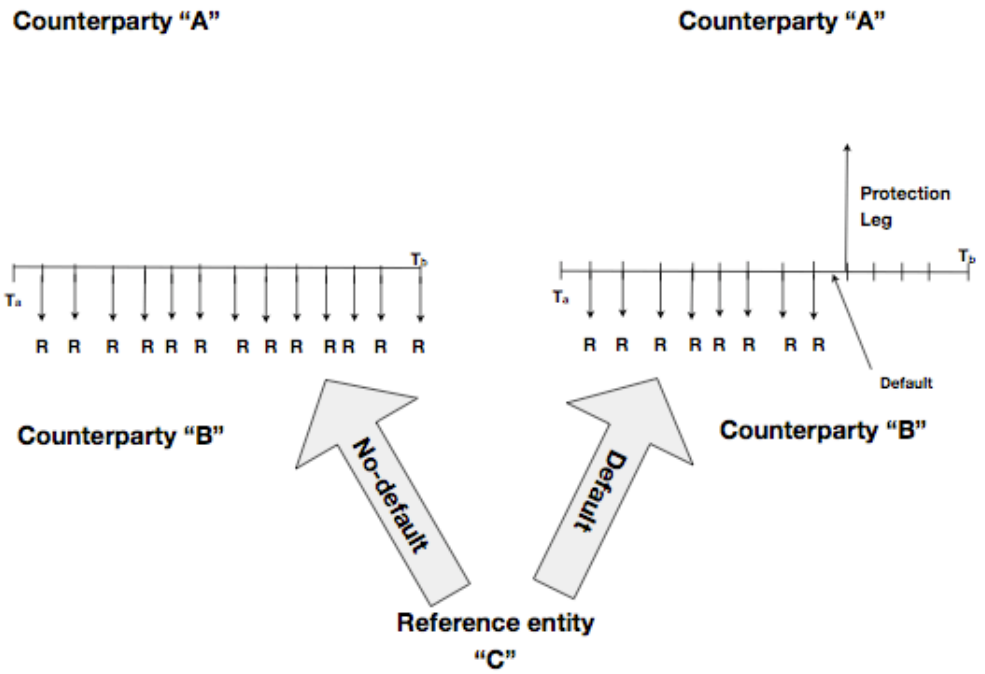
6.1 Credit Default Swaps

Credit default swap(CDS) is the basic protection contract that become quite liquid in recent years. It is used as an insurance against a default of the reference party. Let's describe the standard situation, when a company, let's denote it "A" buys protection from another company "B" against the possible default of a reference, third, company "C", in nominal value M. The contract secures, that if "C" will default, the company "B" will deliver to "A" the certain cash amount. This cash amount is denoted as LGD×M where LGD(Loss Given Default and has the same sense as was defined earlier). Typically company "C" is a bond issuer, so company "A" buys the protection against loss given by the bond[1](with nominal value M) issued by the company "C". In this case the LGD is equal to cash equivalents of the difference between a face value of a bond and post-default value of the bond or equal to the face value of the bond while taking the physical delivery of defaulted bond. Duration of the contract is up to the default time δ or up to the end of the contract T_b, so the life of the contract is represented either by interval (T_a, T_b) or by (T_a, δ). During the life of the contract "A" pays to "B" a

[1] It is not obligatory to own this bond. CDS buyer can buy these instrument from speculation purposes as well.

rate R[2](called CDS spread) from nominal value, at time $T_{a+1}, T_{a+2}, \dots T_b = \tau$ for protection. Usually these two cashflows from "A" to "B" respectively "B"to "A" are called *premium leg* and *protection leg* respectively. There exist two types of CDS's. When a protection payment is payed at time T_b, it is called "protection at maturity". If the protection is payed at the time δ(default time) it is called "protection at default".
The value R is sets at the beginning of the contract, in such a way that make the present value of the contract fair. It means that the difference between premium leg and protection leg equals to zero. For further purposes, we assume without loss of generality that $M = 1$.

Figure 6.1: How CDS works



Note: Here are illustrated cash-flows from CDS contract on not discounted value basis, R represents the whole premium leg here.

From the description of CDS cash-flows we can summarize

premium leg payments	$\sum_{i=a+1}^{b} \frac{(T_i - T_{i-1})}{\Delta} R \mathbf{1}_{\{\delta \geq T_i\}}$
accrued payment in case of default between last T_i before δ and δ	$\mathbf{1}_{\{T_a \leq \delta \leq T_b\}} R^{\frac{(\delta - T_{\{(\text{ of default the first i following } \delta) \ -1\}})}{\Delta}}$
protection payments at the time of default	$\mathbf{1}_{\{T_a \leq \delta \leq T_b\}} \times LGD$

[2]we need to emphasize that this is not recovery rate as was used in previous chapters

where Δ is a number of i during one year(e.g. if we consider four payments during the year then we have $\frac{(T_i - T_{i-1})}{\Delta} = 3/12$). The accrued term is the case when reference entity default between two premium payment dates and the investor should pay the accrued premium from the last payment to the contract seller; let's say a credit event occurs after 2 months from the previous payment, protection buyer still has to pay a proportion of the accumulated premium for 2 months.

6.2 Credit Default Swaps:valuation

In the following section we introduce pricing of CDS contract. We are assuming a default free case of CDS here. To avoid confusions, we need to explain what is the difference between *a default free case* and *a default case* of CDS pricing. We need to emphasis that the default free case has nothing to do with the reference entity. Here the default free case means that there is no default risk for either of protection buyer or protection seller, but the reference entity is still defaultable. We are discussing the possible outcomes of CDS pricing just in default free environment(from the point of view of protection seller and protection buyer) later. By putting together cash-flows from above we are obtaining following result

Proposition 6.2.1. *At time t, $t \leq T_a$ the discounted payoff of general CDS buyer is*

$$\Pi_{CDS}(t) = protectionleg_t + R \times premiumleg_t \tag{6.1}$$

where

$$\begin{aligned} protectionleg_t &= \frac{B_t}{B_\delta} \mathbf{1}_{\{T_a \leq \delta \leq T_b\}}\ LGD \\ premiumleg_t &= -\sum_{i=a+1}^{b} \frac{B_t}{B_{T_i}} \frac{(T_i - T_{i-1})}{\Delta} \mathbf{1}_{\{\delta > T_i\}} \\ &\quad - \frac{B_t}{B_\delta} \mathbf{1}_{\{T_a \leq \delta \leq T_b\}} \frac{(\delta - T_{\{(\text{ the first } i \text{ following } \delta)-1\}})}{\Delta} \end{aligned}$$

To avoid the last "accrued term " in the premium leg, there is a possibility to use two approaches how to deal with the accrued part as it is proposed in Brigo[7]
Approach (1) if the default occurs in interval $(T_{i-1}, T_i]$ we simply move it in to the time T_i and set the accrued part R equal to zero
Approach (2) we also move the default in to the time T_i, but set the accrual period to the whole period $(T_{i-1}, T_i]$ so the whole R is payed.
The third approach is, to approximate the last accrued term by the half of the whole premium for a given period that could be a good approximation if one can expect the uniform distribution of a default between payments.

By applying two approaches from above, it yields following preposition.

Proposition 6.2.2. *If we apply the Approach (1) the discounted payoff of CDS, let denote it Π_{CDS1} at time t is equal,*

$$\Pi_{CDS1}(t) = \sum_{i=a+1}^{b} \frac{B_t}{B_{T_i}} \mathbf{1}_{\{T_{i-1} \leq \delta \leq T_i\}}\ LGD - \sum_{i=a+1}^{b} \mathbf{1}_{\{\delta \geq T_i\}} \frac{B_t}{B_{T_i}} R \frac{(T_i - T_{i-1})}{\Delta} \tag{6.2}$$

and when we apply the Approach (2) the CDS payoff, let denote it Π_{CDS2} is given by

$$\Pi_{CDS2}(t) = \sum_{i=a+1}^{b} \frac{B_t}{B_{T_i}} \mathbf{1}_{\{T_{i-1} \le \delta \le T_i\}} \; LGD - \sum_{i=a+1}^{b} \mathbf{1}_{\{\delta > T_{i-1}\}} \frac{B_t}{B_{T_i}} R \frac{(T_i - T_{i-1})}{\Delta} \quad (6.3)$$

To decide which approximation is better we need to know if $\delta = T_i + \epsilon$ or if $\delta = T_i - \epsilon$. So the first approach or second approach will be more suitable.

Remark 6.2.3. *It is good to notice that $\mathbf{1}_{\{\delta > T_{i-1}\}} = \mathbf{1}_{\{\delta > T_i\}} + \mathbf{1}_{\{\delta \in (T_{i-1}, T_i]\}}$ for $i = \{a, a+1 \ldots, b\}$ so for $\Pi_{CDS1}(t)$ and $\Pi_{CDS2}(t)$ we have following*

$$\Pi_{CDS2}(t) - \Pi_{CDS1}(t) = \sum_{i=a+1}^{b} \frac{B_t}{B_{T_i}} R \mathbf{1}_{\{\delta \in (T_{i-1}, T_i]\}}$$

If we use the risk neutral valuation formula, the value of the CDS obtained at time t is the expected value of discounted payoff to time t. We summarize the pricing of CDS by three different approaches in the following proposition

Proposition 6.2.4. *The price of CDS at time t for three different approaches is*

$$CDS(t) = E_{\tilde{\mathbb{P}}} \left[\Pi_{CDS}(t) | \, \mathcal{G}_t \right] \quad (6.4)$$

$$CDS1(t) = E_{\tilde{\mathbb{P}}} \left[\Pi_{CDS1}(t) | \mathcal{G}_t \right] \quad (6.5)$$

$$CDS2(t) = E_{\tilde{\mathbb{P}}} \left[\Pi_{CDS2}(t) | \mathcal{G}_t \right] \quad (6.6)$$

7. Value of the Credit Risk

In this section we are discussing *value of the credit risk* that comes out from any failure to perform on agreements with the counterparty. We are introducing the *Credit Valuation Adjustment(CVA)* as a measure for the unilateral credit risk and BVA(Bilateral Valuation Adjustment) as a measure of the bilateral credit risk. This chapter is mostly summarizing our previous findings and giving previously derived formulas into the concept of these two adjustments.

7.1 CVA

Definition 7.1.1. Under the consideration of the unilateral default risk from investor's perspective(the investor is default free and the counterparty is exposed to the default risk), CVA is the difference between the value of a financial instrument on a default free basis and the value that includes in the price possibility of counterparty's default. So it is defined by following equation

$$\mathrm{P}_D = \mathrm{P}_{ND} - \mathrm{CVA} \tag{7.1}$$

where

- P_D is the value of financial instrument (or of a whole portfolio value), taking into account the unilateral counterparty risk
- P_{ND} is the value of financial instrument (or a whole portfolio value) without the counterparty risk

CVA may be represented as a cost of hedging, to insure the investor against the credit risk.
We have performed computations in Chapters 5 that yield analytical formulas for the instruments exposed to a credit risk. Unfortunately this procedure is only possible in some basic cases and some fundamental types of financial instruments. This obstacle leads to the fact that we can distinguish two different approaches for CVAs calculations: **i) Analytical approach** those performed here. The second one is using simulation technics, let's call it **ii) Simulation approach**. For more details about this approach see Pykhtin and Zhu(2007) [24]. We have already incorporated contribution of the counterparty default risk into the valuation of financial instruments. The main result that is shown below is that the value of financial instrument with the counterparty risk is the value of the default free instrument with subtracted option term on the residual value after the default.
With our definition of CVA and Theorem 4.1.3 we are getting to following preposition.

Proposition 7.1.2. *If we define CVA as in the Definition 7.1.1, then the value of CVA at time t is given by following formula*

$$CVA(t) = LGD \times E_{\tilde{\mathbb{P}}}\{\mathbf{1}_{\{t<\delta\leq\tau\}} D(t,\delta)(NPV(\delta))_+ | \mathcal{G}_t\} \tag{7.2}$$

Proof. Combine the result from Theorem 4.1.3 and Definition 7.1.1. □

CVA is an expected value of the positive residual payoff after the default. We have applied the General Counterparty Risk Pricing Formula 4.4 to the bonds and swaps where we have derived analytical, relatively easily computable formulas. For the summary of previous computations see Table 7.1.

Table 7.1: Credit Valuation Adjustment

Instrument	CVA
Zero-coupon Bond	$\text{LGD}\times \sum_{i=1}^{m} \tilde{\mathbb{P}}[\delta \in (T_{i-1}, T_i] \vert \mathcal{G}_t]_t \frac{B_t}{B_{T_i}} P(T_i, T)$
Coupon-bearing bond	$\text{LGD} \times \sum_{i=1}^{m} \tilde{\mathbb{P}}(\delta \in (T_{i-1}, T_i] \vert \mathcal{G}_t) \frac{B_t}{B_{T_i}} P_C(T_i, T)$
Interest rate Swap	$\text{LGD} \times \sum_{i=1}^{m} \tilde{\mathbb{P}}[\delta \in (T_{i-1}, T_i]]\text{PayerSwaption}_t(T_i, T_m, \kappa, \sigma_{T_i}, \kappa(t, T_i, T_m))\}$

Note: all notations are in coherence with previous findings and setups and instruments are with the same features as were defined earlier

As we see the value of CVA is always positive. The value of a financial instrument, from the investor's perspective in case of unilateral default risk incorporation into the prices, is always lower than non deaultable price. In the same contract, is adjustment from defaultable counterparty perspective equals to -CVA. Consequently, is the price of the instrument for defaultable counterparty after the incorporation of default risk(its own) higher than default free price.

7.2 BVA

As we have mentioned at the beginning of this chapter, in the case of bilateral default risk consideration, we are not speaking about the credit adjustment but about the bilateral adjustment. There is no debit or credit expression since, it is not predetermined if the adjustment is positive or negative for the value of a given instrument. From Theorem 4.1.6 we see that the value of adjustment depends on two non negative terms acting in opposite way. One is with a plus sign and second one is with a minus sign. We can define BVA in the similar way as CVA.

Definition 7.2.1. Under the consideration of the bilateral default risk, BVA is the difference between the value of financial instruments on the default free basis and the true value that includes in the price possibility of counterparties' default. So it is defined by the following equation

$$\text{P}_D^{Bi} = \text{P}_{ND} + \text{BVA} \tag{7.3}$$

where

- P_D^{Bi} is the value of a financial instrument (or of a whole portfolio value) taking into account the bilateral counterparty risk
- P_{ND} is the value of a financial instrument (or a whole portfolio value) without the counterparty risk

and further BVA can be split into

$$BVA{=}DVA_{bi} - CVA_{bi} \tag{7.4}$$

where CVA is an adjustment from one counterparty's perspective and DVA(Debit Valuation Adjustment) is an adjustment from the other counterparty's perspective.

Remark 7.2.2. *For an illustration how BVA works, let's assume following simplified example. Let's say that there are two counterparties in a contract. Counterparty "A" and counterparty "B". For a simplicity let's assume that the contract has the maturity one year and that A's credit quality is worse(probability of default within following one year) than credit quality of B. We assume that the value of CVA_{Bi} and DVA_{Bi} depends directly on the credit quality of the entity(if the credit quality is worse CVA_{Bi} or DVA_{Bi} is higher and other way around). For this simple example we have, that CVA_{Bi} represents a measure of the counterparty risk adjustment of A against the default risk of B and DVA_{Bi} represents the measure of counterparty risk adjustment of side B against default risk of A. In this particular case CVA_{Bi} is lower than DVA_{Bi} thus BVA is positive.*

Remark 7.2.3. Symmetry of risk *Sometimes the bilateral default risk is also called symmetric default risk. It is because, for BVA holds that, if it is computed from the perspective of the investor, the adjustment has value BVA(t) and if it is computed from the counterparty's perspective it has value -BVA(t). This holds because both counterparties aggree with this adjustment because there is no advantage for any of them.*

From Theorem 4.1.6 follows this general terms for BVA.

Proposition 7.2.4. *If we define BVA as in Definition 7.2.1, then the value of BVA at time t is given by following formula*

$$BVA(t){=}DVA(t)_{Bi} - CVA(t)_{Bi} \tag{7.5}$$

where

$$DVA(t)_{Bi}{=}LGD_1 \times E_{\tilde{\mathbb{P}}}\{\mathbf{1}_{\{\delta_1 \le \delta_2 \le T \cup \delta_1 \le T \le \delta_2\}} D(t,\delta)(-NPV(\delta_1))_+ | \mathcal{G}_t\} \tag{7.6}$$

$$CVA(t)_{Bi}{=}LGD_2 \times E_{\tilde{\mathbb{P}}}\{\mathbf{1}_{\{\delta_2 \le \delta_1 \le T \cup \delta_2 \le T \le \delta_1\}} D(t,\delta)(NPV(\delta_2))_+ | \mathcal{G}_t\} \tag{7.7}$$

With same notation as in Section 4.1.3. We have computed BVA for the interest rate swap and it is given as follows

$$BVA(t)_{IRS_D^B} = DVA(t)_{IRS_D^B} - CVA(t)_{IRS_D^B}$$

$$
\begin{aligned}
& DVA(t)_{IRS_D^B} \approx \\
\approx \; & \sum_{i=1}^{m} LGD_1 \left(\tilde{\mathbb{P}}[T_{i-1} < \delta_1 \leq T_i, T_i \leq \delta_2 \leq T_m] + \tilde{\mathbb{P}}[\; T_{i-1} < \delta_1 \leq T_i, \delta_2 \geq T_m] \right) \\
\times \; & \text{ReceiverSwaption}_t(T_i, T_m, \kappa, \sigma_{T_i,T_b})\}
\end{aligned}
$$

$$
\begin{aligned}
& CVA(t)_{IRS_D^B} \approx \\
\approx \; & \sum_{i=1}^{m} LGD_2 \left(\tilde{\mathbb{P}}[T_{i-1} < \delta_2 \leq T_i, T_i \leq \delta_1 \leq T_m] + \tilde{\mathbb{P}}[\; T_{i-1} < \delta_2 \leq T_i, \delta_1 \geq T_m] \right) \\
\times \; & \text{PayerSwaption}_t(T_i, T_m, \kappa, \sigma_{T_i,T_b})\}
\end{aligned}
$$

All terms from above are with the same notation as before. We are investigating the real development of DVA in the case of the IRS in the last chapter of this work.

8. Term Structure of Default Probabilities

We have already prepared the theory for valuation and quantification of the credit risk and now we need to introduce technics to gain default probabilities. As it is mentioned in Bluhm et al.[6] generally, there exist three possible approaches to obtain the default probabilities.

- From the historical default rates
- From Merton's option theoretic approach
- From the market(spread of defaultable bonds or swap, or other securities)- also known as implied default probabilities

Because we have already introduced the valuation of derivatives that are directly working with default probabilities, we are following in this work the approach mentioned as the last bullet point.
The main idea is to extract default probabilities from the values that are already quoted in the market. It is possible to do it from different contracts and securities, but here we are focusing on CDS's and their market quotes. As we have indicated at the beginning of Chapter 4, we are using here Reduced Form Model. This type of model is mostly based on the work of Jarrow and Turnbull (1995) [16], where is the credit event(default) considered to be the first event of Poisson counting process. This approach is commonly used and it is adapted by many other authors see e.g series of paper Brigo [9],[8], [10] or O'Kane and Turnbull(2003) [22]. As before, we denote the default time δ. In notation of Poisson process is the probability of default occurrence within a short time interval $[t, t + dt)$, conditioned that $\delta \geq t$, expressed as follows

$$\mathbb{P}(\delta < t + dt | \delta \geq t) = h(t)dt \tag{8.1}$$

where $h(t)$ is time dependent function known as a hazard rate. It can be shown that the survival probability to time T, conditioned on survival to time t, $t \leq \delta$ is given by

$$\mathbb{P}(\delta \geq T | \delta \geq t) = \exp\left\{-\int_t^T h(s)ds\right\} \tag{8.2}$$

This expression follows from the fact that the time between events in Poisson counting process has exponential distribution for a proof, see Prášková and Lachout (2003) [23] Chapter 3. Further we have

$$\mathbb{P}(\delta < T | \delta \geq t) = 1 - \exp\left\{-\int_t^T h(s)ds\right\} \tag{8.3}$$

If we are assuming that hazard rate is constant, $h(t) = \alpha$ for some t, it follows that

$$\mathbb{P}(\delta < T | \delta \geq t) = 1 - \exp\{\alpha(T - t)\} \tag{8.4}$$

For later purposes, let's assume that

$$h(u) = \alpha_k \text{ for } u \in [T_{k-1}, T_k]$$

So hazard rate is a piecewise constant function and the cumulative probability of default is given by

$$\tilde{\mathbb{P}}(\delta \leq T_j) = 1 - \exp\left\{ -\sum_{i=1}^{j} \alpha_i (T_i - T_{i-1}) \right\} \tag{8.5}$$

where $T_1, T_2, \ldots, T_j$ denotes the time moments in the time period of our interest. More precisely, these moments are maturities of available CDS's. For more details see following section, where we are connecting default probabilities and CDS valuation formulas.

8.1 Risk Neutral Default Probabilities Extracted from CDS Quotes

In Section 6.1 we have introduced general formulas for CDS payoffs, but these formulas are not really useful for our purposes. In the following two propositions, we make two assumptions that help us to prepare these formulas for using.

Proposition 8.1.1. *The value of CDS contract at time t, with the same cash-flows as are defined in 6.2.1 and under the assumption of independence between the default times and the interest rates, is given by*

$$CDS(t, \tilde{\mathbb{P}}(\delta \leq .|\mathcal{G}_t)) = ProtectionLeg_t + R \times PremiumLeg_t \tag{8.6}$$

where

$$ProtectionLeg_t = LGD \times \int_{T_a}^{T_b} \frac{B_t}{B_\delta} \tilde{\mathbb{P}}(u < \delta \leq u + \Delta u|\mathcal{G}_t) du \tag{8.7}$$

$$\begin{aligned} PremiumLeg_t \quad &= \quad -\sum_{i=a+1}^{b} \frac{B_t}{B_{T_i}} \frac{(T_i - T_{i-1})}{\Delta} (1 - \tilde{\mathbb{P}}(\delta \leq T_i|\mathcal{G}_t) \\ &\quad - \int_{T_a}^{T_b} \frac{B_t}{B_\delta} \frac{(\delta - T_{(\epsilon)-1})}{\Delta} \tilde{\mathbb{P}}(u \leq \delta < u + \Delta u|\mathcal{G}_t) du \end{aligned} \tag{8.8}$$

where $T_{(\epsilon)}$ is first T_i after default time δ.

Proof. From the risk neutral valuation framework follows that the value of financial derivative is an expected value from its payoff with respect to a risk neutral measure. Then from the Proposition 6.2.1, that gives us CDS payoff formula, follows that we need to compute following expected value

$$CDS(t, \tilde{\mathbb{P}}(\delta \leq .|\mathcal{G}_t)) = \text{Term1} + \text{R} \times \text{Term2} \tag{8.9}$$

where

$$Term1 = E\left[\frac{B_t}{B_\delta} \mathbf{1}_{\{T_a \leq \delta \leq T_b\}} \text{ LGD} \Big| \mathcal{G}_t \right]$$

and

$$Term2 =$$
$$- \quad E\left[\sum_{i=a+1}^{b}\frac{B_t}{B_{T_i}}\frac{(T_i-T_{i-1})}{\Delta}\mathbf{1}_{\{\delta>T_i\}}+\frac{B_t}{B_\delta}\mathbf{1}_{\{T_a\le\delta\le T_b\}}\frac{(\delta-T_{\{(\text{ the first i following }\delta)-1\}})}{\Delta}\Big|\mathcal{G}_t\right]$$

By assuming the independence of the default times and the interest rates, we can rewrite Term1 and Term2 as follows

$$\text{Term1} = LGD \times \int_{T_a}^{T_b}\frac{B_t}{B_\delta}\tilde{\mathbb{P}}(u<\delta\le u+\Delta u|\mathcal{G}_t)du \tag{8.10}$$

and

$$\begin{aligned}\text{Term2} &= R\times\sum_{i=a+1}^{b}\frac{B_t}{B_{T_i}}\frac{(T_i-T_{i-1})}{\Delta}(1-\tilde{\mathbb{P}}(\delta\le T_i|\mathcal{G}_t) \\ &- R\times\int_{T_a}^{T_b}\frac{B_t}{B_\delta}\frac{(\delta-T_{(\epsilon)-1})}{\Delta}\tilde{\mathbb{P}}(u\le\delta<u+\Delta u|\mathcal{G}_t)du\end{aligned} \tag{8.11}$$

□

Integrals from the above formulas represent that the default can happen at each time moment in the interval $[T_a, T_b]$, so we approximate this fact by following consideration. We divide the time interval $[T_a, T_b]$ into smaller time subintervals, during which a default may occur. If the default event happens in one of these subintervals, we shift it into the following right bound of this subinterval. It actually follows, what we have been assuming, that default can happen just in these discrete time moments. There need to be made a decision between sufficiently fine division of the interval and computational complexity. In Chapter 9.4 we are using monthly division of the year. As it is usual in practice we also assume that there is no accrual premium payments. Same assumptions were made in O'Kane and Turnbull(2003) [22].
Following result gives us very suitable and simple formula to use.

Proposition 8.1.2. *Let's assume that CDS contract has following properties:*

i) *n premium payments are at times*$\{T_{a'+1}, T_{a'+2}, ...T_{a+n}\}$

ii) $T_{a'+1}$ *is equal to the date of contract's inception and* $T_{a'+n}$ *is equal to the maturity date of the contract*

If we assume that reference entity can default only at discrete time moments $\{T_a, T_{a+2}, ...T_b\}$*, where* T_a *is the inception day of the contract and* $T_b = T_{a'+n}$ *is the maturity and at the same time* $\{T_{a'+1}, T_{a'+2}, ..., T_{a'+n}\}$ *are at the same time grids as* $\{T_a, T_{a+2}, .$ *then the price of such CDS at time* $t, t \le T_a$ *is approximated by*

$$CDS(t, \tilde{\mathbb{P}}(\delta \le .)) \approx Term_{protection} + R \times Term_{premium} \tag{8.12}$$

where

$$Term_{protection} = LGD \times \sum_{k=a+1}^{b} \frac{B_t}{B_{T_k}} (\tilde{\mathbb{P}}(\delta \le T_k) - \tilde{\mathbb{P}}(\delta \le T_{k-1}) | \mathcal{G}_t)) \qquad (8.13)$$

gives expression for approximated protection leg of the contract

$$Term_{premium} = - \sum_{k=a'+1}^{a'+n} \frac{B_t}{B_{T_k}} \frac{(T_k - T_{k-1})}{\Delta} (1 - \tilde{\mathbb{P}}(\delta \le T_k) | \mathcal{G}_t)$$

gives expression for approximated premium leg of the contract

Default probabilities are defined as in Section 8.

Proof. The premium leg remain the same, because premium was already payed at the discrete time moments. We know that

$$\tilde{\mathbb{P}}(u < \delta \le u + \Delta u) = (\tilde{\mathbb{P}}(\delta \le u + \Delta u) - \tilde{\mathbb{P}}(\delta \le u))$$

Since the default can happen only at the discrete time moments, we replace the integral in ProtectionLeg by the sum. So we have

$$\begin{aligned} & LGD \times \sum_{k=a+1}^{b} \frac{B_t}{B_{T_k}} \tilde{\mathbb{P}}(T_{k-1} < \delta \le T_k) \\ = \; & LGD \times \sum_{k=a+1}^{b} \frac{B_t}{B_{T_k}} (\tilde{\mathbb{P}}(\delta \le T_k) - \tilde{\mathbb{P}}(\delta \le T_{k-1})) \end{aligned}$$

□

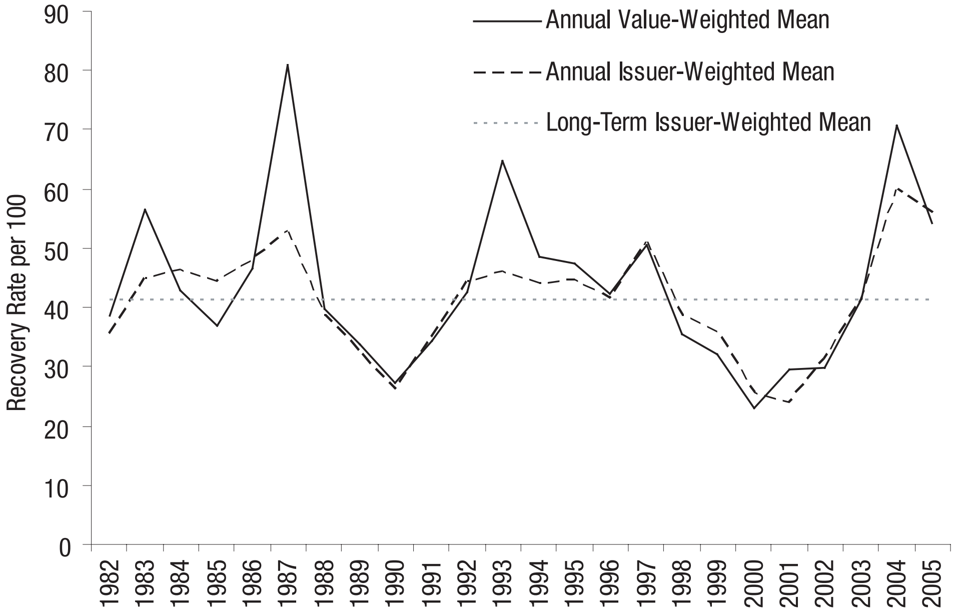
In this stage, if we know the interest rate term structure and CDS quotes(spreads), we are almost ready to extract default probabilities from the market. Just one last parameter in protection leg needs to be determined and it is LGD rate or a recovery rate, respectively.

Remark 8.1.3. Recovery rate assumption *we can see, to determine default probabilities from CDS prices, we need to determine a recovery rate as well. Those rates are mostly estimated on the basis of best "market knowledge" of a model developer or a trader and on the basis of historical values. If we go further, there is possibility to model these rates as stochastic processes.*

According to historical data and experience the market norm and the usual corporate recovery rate is 40%, *that is average historical rate. This number also support study that was made by rating agency Moody's Default and Recovery Rates of Corporate Bond Issuers, 1920-2005, see Figure 8.1.*

But for the purposes of this work, we need to estimate the proxy of recovery rates for sovereign bonds. This task is more complicated since the lack of data to estimate such rate historically. There was a relatively small number of countries that announced state bankrupt(see Table8.1). We use the estimation from Moody's(2007)[20], where they study 12 state defaults and estimate average recovery rate on the level of 54%. They use 30-day post-default price or pre-distressed exchange trading price as the measure of recovery rates.

Figure 8.1: Historical Recovery Rates



Note: source Default and Recovery Rates of Corporate Bond Issuers, 1920-2005,www.moodys.com.br/brasil/pdf/Default2006.pdf Exhibit 13

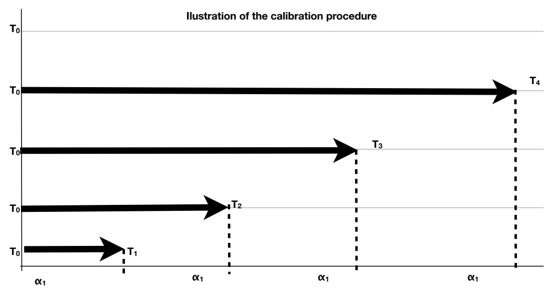
Table 8.1: Recovery Rates of Defaulted Countries

Country	Default Year	Recovery Rate
Argentina	2001	27%
Belize	2006	76%
Dominican Republic	2005	95%
Ecuador	1999	44%
Grenada	2004	65%
Ivory Coast	2000	18%
Moldova	2002	60%
Pakistan	1999	52%
Russia	1998	18%
Ukraine	2000	69%
Uruguay	2003	66%
	Average	**54%**

Note: source Moody's(2007)[20]

Now we know everything what we need to calibrate our model to market data.
Calibration process
First: Make assumption about recovery rate.
Progressively for each given CDS quote(subscript i stays for different CDS maturities) we are proceeding in following steps
Step 1: we take $CDS(T_i, \tilde{\mathbb{P}}(\delta \leq T_i)) = 0$ and compute α_i.
Step 2: compute probability of default $\tilde{\mathbb{P}}(\delta \leq T_i)$

Figure 8.2: Illustration of the bootstrap procedure



Bootstrapping of default probabilities step by step

Remark 8.1.4. *To compute particular hazard rates, one needs to compute a set of nonlinear equations. Because we can express α_i in terms of α_{i-1}, we can start from time interval $[T_0, T_1]$ to compute α_1 and so on. This approach is generally known as "bootstrap" method and it is illustrated in Figure 8.2*

From the bootstrap we get implied risk neutral default probabilities. We are talking about implied risk neutral default probabilities, because they are implied by the market and are coming from the risk neutral valuation formula for CDS's. For the rest of the work we are speaking about implied default probabilities, except where explicitly noted otherwise. See the discussion about another type of default probabilities in Remark 8.1.5.

Remark 8.1.5. Real World vs. Risk Neutral World
It need to be emphasized that there is a difference between the real world default probabilities estimated from historical data and probabilities implied by the market. These historical probabilities are mostly lower than probabilities gained from the market by the risk neutral valuation framework. Historical default probabilities can be given into the context of real expectations of market participants about defaults.

If we are talking about probabilities extracted from bonds[1] *the reason is that investors require higher yields for possible default of the issuer plus they also require some extra safety margin that is ultimately increasing extracted default probabilities. One of the first academic publication about this topic was Altman(1989) [1] where are described discrepancies between bonds prices and historical data. In Hull(2009) [14] are presented and compared historical and risk neutral default probabilities obtained from bonds of different rating classes. Results there show that the risk neutral default probabilities are more than ten times higher than historical default probabilities for studied bonds.*
The same situation is taking place for a protection seller in CDS contract and his view on default probability of the reference entity. Protection seller wants to be also protected by some risk margin against the possible wrong or not accurate estimate of the reference entity's default probability. This margin is enhancing default probabilities extracted from CDS's.
This argumentation is in line with the theory of risk neutral pricing as well. In this approach, probabilities of each possible price scenarios can be adjusted in such a way that expected value of a security's or a contract's discounted payoff, by risk free interest rate, is a martingale that implicates no arbitrage on the market, see Chapter 3.
Even though that risk neutral default probabilities does not give us the clear picture about the real market expectation about defaults, they are essential for the valuation of other contracts and securities in a risk neutral framework.

[1] see Chapter 26 of Hull(2002) [13] for this procedure

9. Case Study: Greek Debt Crisis

In the second half of this work we use the previous theory to study recent events and conditions on a government bond market. Circumstances on financial markets and financial health of some countries in Southern Europe gave us a good opportunity to study the default risk of a contract, where as an obligor is a whole country. Credit risk theory was mostly developed to determine and to manage the default risk of the big corporations. This fact causes that there are not many academic papers and publications dealing with sovereign default risk and so, there are not many academic papers as result benchmarks for our study. The debt crisis that almost caused the bankruptcy of Greece in May 2010 and events around Portugal and Spain during this year showed us an importance of being aware of a state default possibility as well.
We apply the framework from previous chapters to study default components, CVA in our notation, in case of real market quotes of sovereign Greek bonds. In the second part, we study CVA and BVA in the case of interest rate swap between Greece and Czech Republic.
This highly stressed period on the bond market provides us an opportunity to study interactions and relations between credit derivatives market, especially CDS market, and a sovereign bond market. The aim, is to use CVA concept to calculate modeled prices[1] of Greek bonds and compare them to real quotations on the market during the debt crisis in May 2010. The idea is to use default free interest rates for discounting of bond's cash-flows and subtract CVA, the default component. Modeled prices, computed by our formulas for incorporation of default risk, are mostly driven by default free interest rates, by which bond cash-flows are discounted and by risk neutral default probabilities, extracted from CDS premium quotes. Undoubtedly the choice of risk free curve has an impact on the results so we also investigate the impact of right choice of risk free interest rate curve.
The process of valuation includes an extraction of the risk neutral default probabilities from CDS quotes. This part of the process is also interesting because we can study reactions of CDS spreads and risk neutral default probabilities on events and political discussions about the solutions of Greek government debt.

The first task and data needed for it are good preliminary steps to calculate CVA and DVA for IRS between two countries. The concept of IRS between Greece and Czech Republic is not based on a real IRS quoted on the market. We use another data that we have to construct such a swap, see Section 11. On the following pages we would like to describe in detail the whole procedure, with emphasis on all obstacles and possible issues that accompany such procedures.

9.1 Data

Very often in such a study the main problem is not the computation process itself, but the problem is to gain enough data with appropriate quality. It is possible to get yield curves and some bond prices for free from different websites e.g European

[1]modeled by formulas derived in Chapter 5.2 by which we can calculate fair price of the bond. For this is just needed risk free interest rate term structure and default probabilities of the bond's issuer

Central Bank website. But it is very hard, if not impossible, to get quotation of CDS's for free, even for academic purposes. Because CDS contract is traded over the counter, data needs to be gained directly from traders and brokers. Of course there exist many different private companies that are collecting this sets of data, but they are gathering them for commercial purposes and they are not providing it, for academic purposes. In our case, all data used in this work are downloaded from Thomson Reuters[2]. For our calculation, we are using three main data sets: *CDS spreads quotes*,*Greek bonds prices/yields* and *default-free interest rate curves.*

•*CDS quotes:* to determined default risk of government bonds that are issued by Greece, we need to extract default probabilities from CDS's with Greece as a reference entity. We have downloaded CDS prices for 8 different maturities from 6 to 120 months. Data set contains quotes(spreads) from 5/11/2008 to 6/12/2010. Together 536 daily observations. CDS's are quoted in percentage basis points and a premium/spread is payed quarterly, which is the mostly used payment frequency. From CDS quotes, for different maturities and for each trading day we can extract the whole term structure of risk neutral default probabilities. By the term structure of default probabilities, we mean that for each day we have default probabilities for different default times(in our case default times are same as CDS maturities). For the second task of computing BVA for IRS we use CDS quotes with Czech Republic as the reference entity.

Table 9.1: Credit Default Swap:Greece as Reference Entity

Maturity(in months)	**Average premium**	**Premium Range**	
		Min	**Max**
6M	389.438	46.000	1267.680
12M	389.543	46.000	1268.700
24M	387.097	58.000	1191.586
36M	384.407	70.000	1122.575
48M	378.177	80.000	1049.817
60M	371.025	88.000	977.590
84M	360.940	91.200	945.600
120M	344.473	89.000	898.890

•*Bonds:* we apply our defeaultable coupon bearing bonds Formula 5.6 to all outstanding bonds issued by Greek government, that are possible to download from Thomson Reuters on 6th December 2010. We are working with daily yields of eighteen coupon bonds with maturities varying from June 2010 to June 2019. Bonds with maturities longer than 10 years are excluded. Coupon range is from 3.6% to 6.25%. Total number of trading days, during which we have at least one price/quotation of bond is 1225 (from 29/01/2008 to 6/12/2010). If we consider only the days containing yields of all 18 bonds, data shrinks to 143 observation points from 08/04/2010 to 6/12/2010. These 143 trading days contain almost 3000 yields observations and serve us as a good sample to our study(further just

[2]we get an access to Thomson Reuters Financial Lab terminal of University of Economics in Prague with kindly permission of the supervisor of this thesis Mr. doc. RNDr. Jiří Witzany, Ph.D.

main sample). Our sample exactly match with the distressed period on the Greek sovereign bond market. List of all bonds from main sample is in Table 9.2

Table 9.2: Greek Bonds Overview

Bond	**Maturity**	**Coupon Rate**	**Mean**	**Yields Range Min**	**Max**	**Mean**	**Price Range Min**	**Max**
GR010714	01/07/14	4.50%	10.61%	14.87%	6.58%	674,012	560,670	762,812
GR180511	18/05/11	5.35%	8.81%	18.27%	5.91%	995,581	935,529	1,040,864
GR180512	18/05/12	5.25%	9.96%	18.29%	6.14%	948,728	842,370	1,028,702
GR190719	19/07/19	6.00%	10.50%	13.05%	6.74%	409,867	322,426	546,523
GR200311	20/03/11	3.80%	17.41%	37.96%	11.74%	987,402	893,781	1,015,307
GR200312	20/03/12	4.30%	9.68%	17.96%	6.02%	939,439	801,358	982,715
GR200417	20/04/17	5.90%	10.52%	13.95%	6.69%	509,652	401,772	634,609
GR200513	20/05/13	4.60%	10.65%	17.37%	6.39%	878,929	757,344	989,677
GR200514	20/05/14	4.50%	10.81%	15.93%	6.48%	677,000	548,708	772,420
GR200715	20/07/15	3.70%	10.62%	14.43%	6.53%	605,551	493,458	715,217
GR200716	20/07/16	3.60%	10.17%	13.75%	6.57%	558,301	446,840	669,427
GR200717	20/07/17	4.30%	10.50%	13.65%	6.72%	497,558	395,438	621,965
GR200718	20/07/18	4.60%	10.24%	12.60%	6.72%	458,573	375,752	582,819
GR200811	20/08/11	3.90%	8.78%	18.28%	6.10%	974,239	873,617	1,008,185
GR200812	20/08/12	4.10%	10.16%	18.20%	6.19%	915,655	779,475	979,949
GR200813	20/08/13	4.00%	10.69%	16.42%	6.35%	854,413	726,492	953,883
GR200814	20/08/14	5.50%	10.89%	16.47%	6.62%	658,489	518,179	755,923
GR200815	20/08/15	6.10%	10.44%	14.57%	6.49%	604,210	485,584	714,097

To our analysis we also include 5 bonds, for which we have yields observations from 5/11/2008 to 6/12/2010. This second sample is used to investigate the default components during trading days with a smaller volatility of bond yields(further second sample). See the mean and standard deviation comparison of these 5 bonds in Table 9.3.

Table 9.3: Overview of Bonds from Second Sample

	Period (5/11/08-6/12/10)		**Period (5/11/08-31/03/10)**	
Bond ID	**Mean**	**Stdv**	**Mean**	**Std**
GR180512	5.79%	3.23%	3.77%	1.14%
GR200311	9.50%	5.96%	6.12%	2.54%
GR200513	6.34%	3.37%	4.20%	1.03%
GR200715	6.61%	3.08%	4.62%	0.83%
GR200716	6.61%	2.81%	4.81%	0.79%

•*Default Free Yield Curve:* many discussions about the yield curve that represents default free investments, lead us to use and to make computation for two different yield curves. One is a plain vanilla interest rate swap yield curve and other one is yield curve extracted from German Governments bonds, since these bonds can be considered as default free bonds. These bonds are also used for the curve that we have downloaded from Reuters and that is referred as EU Benchmark curve. As a basis for further work, it is used 38 quotes of swap rates with maturities varying from 1 to 120 months, for the period from 5/11/2008 to

2/12/2010. Actually, we are using these quotes for a shorter period. The second curve, treasury bond curve, is defined by 14 quotes for the same period as the swap rates with maturities also varying from 1 to 120 months. Later, we are discussing required interpolation procedure between this fixed maturities. Both these curves are downloaded from Thomson Reuters as well.

9.2 Default Free Interest Rate Term Structure

One of the most essential input parameter and also one of the most discussed input to any fixed income financial instrument pricing models is a default free interest rate term structure[3]. It is an essential for our valuation procedure and for the risk neutral default probabilities extractions, as well. Discussion about this issue is so complex and wide that it is not in the scope of this work to even fully list all academic and practitioners papers about it. Currently, there are two main groups of interest rate term structure users. Practitioners are using the plain vanilla interest rate swap curve(later just swap curve) on the other hand in academic sphere they are mostly using Treasury yield curves. In the academics and practitioners research articles and literature to secure a robustness of the results it is usual to use more definitions of default free interest rate curve. To make our analysis also more robust, we use two different yield curves, the swap curve and the Treasury curve. Swap curve, represents fixed payment rate against floating LIBOR rate. Swap curves are starting to be quite popular because they are considered to be less risky instruments than treasuries even though they are based on the LIBOR which is risky rate, since also big financial institutions are exposed to the default risk (e.g. Lehman Brother case). We can just mention that there is no face value to lose, one counterparty lose only if it is receiver position and there is very often a collateral required(security or a bond) which other party receive in case of the default.
The second yield curve is derived from high rated German Government bonds and it is also downloaded from Reuters [4]. Germany is the strongest economy in EU so it is reasonable to consider it as a non default-able bond issuer. It is commonly assumed that such bonds imply yields that are not containing any premium for default risk.

One big difference between these two instruments is the liquidity. Liquidity[5]risk influences the level of interest rates/yields[6] as well. The sensitiveness on this risk is not on the same level for swaps and for treasury bonds. Swaps are contracts not securities and this distinction is important, because there is just fixed amount of securities but swap as contracts can be in arbitrary large amount available on the market. So the swaps do not have a tendency to be influenced by any supply-demand premium. The difference in liquidity, one can see just from the number of quotation of different maturities in our data sets.

[3]further in the text we are equivalently using terms as benchmark curve, risk free curve

[4]Further in the text, we are using equivalently default free curve, risk free curve and sometimes also benchmark interest rate curve as the term for default free interest rate curve.

[5]In finance, liquidity risk is the risk that a given security or asset cannot be traded quickly enough in the market to prevent a loss (or make the required profit).

[6]It influence yields as well because yields are function of interest rates and time to maturity

9.3 Preparation of Default Free Interest Rate Curve for Modeling

For the exact time match of interest rates for all needed cashflows we need to interpolate values between downloaded fixed quotes of swap rates and treasury yields. As we have already mentioned in the data section, we have 38 swap yield quotes for each trading date, vary from 1 to 120 months maturity and for German treasuries we have 14 quotes varying from 1 to 120 months maturity. For the summary of basic statistical properties, see tables in Appendix A Tables A.2 and A.1.

For further use we need to interpolate both curves on a monthly basis. To interpolate monthly values from given set of observed rates we are using Nelson Siegel Model and following paper from Gilli et. al. 2010 [12]. They argue that the function used in Nelson Siegel Model, given as 9.1 can capture all types of yields curves(steep, flat, humped or inverted). This method is also used by European Central Bank to present daily benchmark curves[7]. We are following their suggestions and proposals and incorporate them in to the the interpolation procedure.

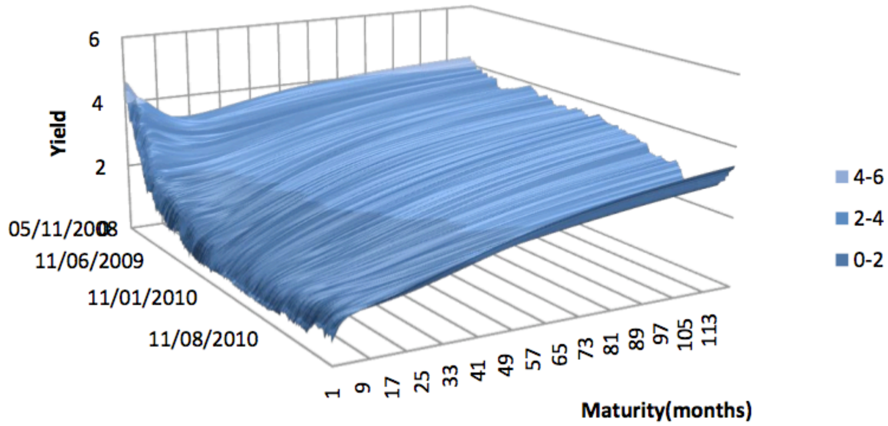
$$\begin{aligned} y_t(\tau) &= \beta_1 + \beta_2 \left[\frac{1-\exp(-\tau/\gamma_1)}{(\tau/\gamma_1)} \right] \qquad (9.1) \\ &+ \beta_3 \left[\frac{1-\exp(-\tau/\gamma_1)}{(\tau/\gamma_1)} - \exp(-\tau/\gamma_1) \right] + \beta_4 \left[\frac{1-\exp(-\tau/\gamma_2)}{(\tau/\gamma_2)} - \exp(\tau/\gamma_2) \right] \end{aligned}$$

In this equation $y_t(\tau)$ represents yields at time t with maturity τ. Parameters $\beta_1, \beta_2, \beta_3, \beta_4$ and γ_1, γ_2 are estimated for each day(for each t) in our samples. We are using this particular type of Nelson Siegel function but there are also used just 3 parameters variation of this function, without the last term after β_4 and γ_1 is considered to be a constant.
For the best estimates of six parameters, we are using build in function in Matlab program called $lsqcurvefit()$ for all six parameters for each day in the samples. This function finds the best estimate of the parameters in a least-squares sense. For parameters statistics overview see Appendix A. After the estimations of parameters we construct the complete interest rate term structure, see surfaces in Figure 9.1 and Figure 9.2.

CVA for coupon bearing bonds, in equation 10 is calculated as a sum of default free bond prices at each possible moment of a default, weighted by risk neutral default probabilities at these moments. In our case, it is one month interval in which defaults are possible, from the day of the bond inception to maturity of the bond. Because we have to price default free bonds not just as of today, but also in the future moments of their life time, we compute forward rates from our spot interest yield curves. This allow us to compute the price of the bond at future time moments. In both of our samples, bonds are with maturities less than 10 years, so we also need just forward rates up to 10 year maturity.

[7] for closer information see technical notes of ECB yields methodshttp://www.ecb.int/stats/money/yc/html

Figure 9.1: Swap rate term structure



9.4 From CDS Quotes to Default Probabilities

As we have already mentioned couple of times from CDS quotes for a given reference entity, we are able to extract default probabilities for this entity. To be more precise, we are able to extract a hazard rate and then compute risk neutral default probabilities. Let us recall the formula which is essential for the extraction of the risk neutral default probabilities and the whole process stands on it. From Proposition 8.1.2 we have

$$CDS(t, \tilde{\mathbb{P}}(\delta \leq .)) \approx Term_{protection} + Term_{premium} \tag{9.2}$$

where

$$Term_{protection} = LGD \times \sum_{k=a+1}^{b} \frac{B_t}{B_{T_k}} (\tilde{\mathbb{P}}(\delta \leq T_k) - \tilde{\mathbb{P}}(\delta \leq T_{k-1}) | \mathcal{G}_t)) \tag{9.3}$$

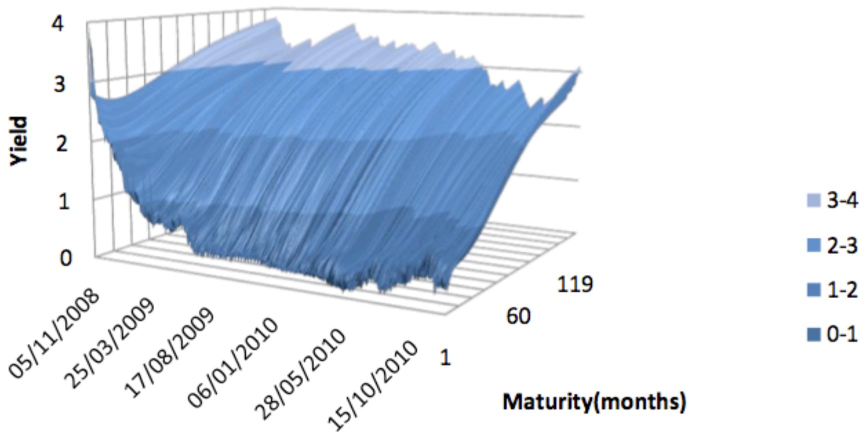
gives expression for approximated protection leg of a contract

$$Term_{premium} = -R \times \sum_{k=a'+1}^{a'+n} \frac{B_t}{B_{T_k}} \frac{(T_k - T_{k-1})}{\Delta} (1 - \tilde{\mathbb{P}}(\delta \leq T_k) | \mathcal{G}_t)$$

gives expression for approximated premium leg of a contract

where $\{T_{a'+1}, T_{a'+2}, ..., T_{a'+n}\}$ are the time moments when premium is payed and $\{T_a, T_{a+2}, ..., T_b\}$ are the possible default times. All variables have the same meaning as in the previous chapters and sections. Market value is quoted through the R, premium or spread. Bonds from the sample have the longest maturity of ten years, so we are focusing particularly on these quotes [6,12, 24, 36, 48, 69, 84, 120] in months. We have mentioned in Section 8 that it is appropriate to work with partially constant hazard rates. Also now we are assuming that the hazard rate is picewise flat function of maturity time. This implies that we are able to derive hazard rates valid on the following time intervals (0,6],(6,12], (12,24], (24,36], (36,48], (48,60], (60,84], (84,120] expressed in months. For each trading

Figure 9.2: Treasury rate term structure



day we get 8 hazard rates that are applicable on these intervals. Hazard rate on in each interval is computing from the spread for CDS with maturity given as the right bound of the interval e.g. hazard rate that is valid on time between today and next six months is computed from CDS quote with maturity of 6 months.
As reasonable approximation of possible default times, we are assuming time steps one month. Discretization of interval, index k, in $Term_{protection}$ is regarding to time step of one month given by the number of months from valuation day to maturity. The formula for the risk neutral default probability which is applicable with given 8 different quotes and monthly discretization of possible default times, is given as follows.

$$\tilde{\mathbb{P}}(\delta \leq T_j) = \begin{cases} 1 - \exp\left(-\frac{j}{12}\alpha_{0,6}\right) \text{ if } 0 < j \leq 6 \\ 1 - \exp\left(-\alpha_{0,6} - \frac{j}{12}\alpha_{6,12}\right) \text{ if } 6 < j \leq 12 \\ 1 - \exp\left(-\alpha_{0,6} - \alpha_{6,12} - \frac{j-12}{12}\alpha_{12,24}\right) \text{ if } 12 < j \leq 24 \\ ... \\ \cdots \\ \cdots \\ 1 - \exp\{-\alpha_{0,6} - \alpha_{6,12} - \alpha_{12,24} - \alpha_{24,36} - \alpha_{36,48} - \alpha_{48,60} \\ -\alpha_{60,84} - \frac{j-84}{36}\alpha_{84,120}\} \\ \text{ if } 84 < j \leq 120 \end{cases} \tag{9.4}$$

where $\alpha_{(i-1,i)}$, i $\in \{6, 12, 24, 36, 48, 69, 84, 120\}$ represents the piecewise constant hazard rate, in a period $(i-1, i]$.
As a computational platform, we use Microsoft Office Excel and its programming

extension VBA. We use this environment because we are working with lots of data and tabular software Excel is a very useful tool to handle such amount. In VBA it is easy to implement any needed structures for our computations.
Terms 9.4 together with CDS formulas are directly coded to VBA[8] to compute risk neutral default probabilities. By this way we can calculate default probability to any needed time horizon, from 0 to 120 months.
Premium R that is paid for a protection, is paid quarterly, so there is no need to adjust it, since it fits to our discretization steps. The last thing that needs to be answer before the calculation itself, is the recovery rate or LGD, respectively. We are assuming, that the recovery rate is 54 % as it is mentioned and reasoned in Remark 8.1.3. According to latter facts and assumptions, we build up the VBA program that compute hazard rates for Greece with both interest rate curves. As an input to this program are interpolated interest rate term structure on monthly basis(swap curve and treasury curve) and CDS quotes.

9.5 Greece Debt Crisis

To better understand recent circumstances around Greece, we should recapitulate events that almost caused the bankrupt of this country at the end of April and the beginning of May 2010.[9] This helps us to better interpret and take the results of our calculation into context of these days events. To shortly illustrate the atmosphere on financial markets, we use the average yield of bonds from the main sample[10] and their maximums and minimums. The following Chart 9.4 with comments is a sketch of the situation.

9.6 Greek Implied Default Probability Term Structure

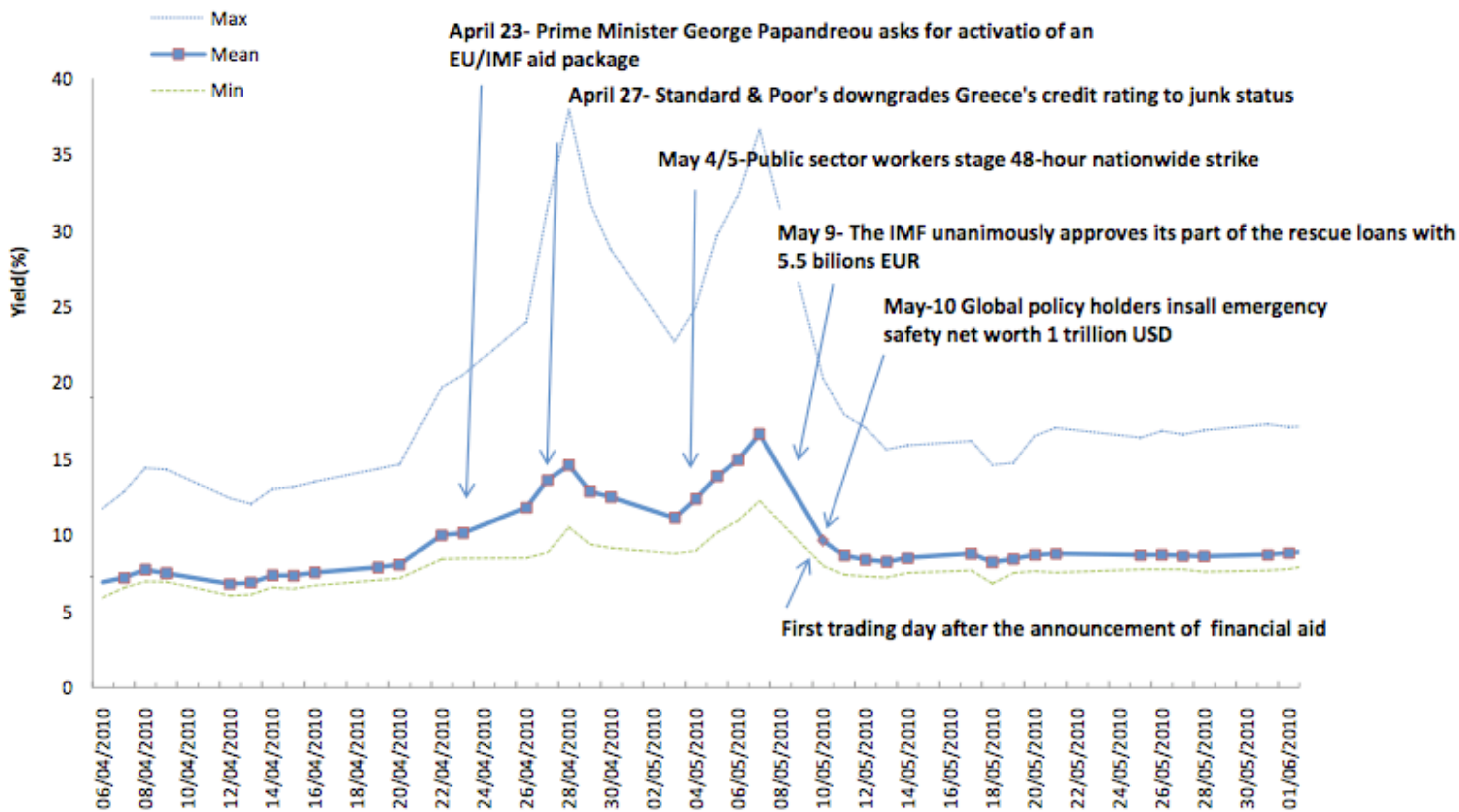
Implementation of the previously mentioned facts about default probabilities calculation, to VBA code, finds for each day in the CDS sample and for each maturity, corresponding hazard rate. As an example, see computed hazard rates as on 5/11/2008 and implied default probability curve, Figure 9.3.

In the example in Figure 9.3, we have computed hazard rates for the first day of our CDS sample, for 5/11/2008. These days was the sovereign bond market not so distressed as it is recently, so implied hazard rates and cumulative default probabilities are not so high. Ten years default probability of Greece at the level of 25% is pretty good comparing to 75% in May 2010. For more details to this curve see Table 9.5, where are some characteristic of this particular credit curve in detail. We see that hazard rate is piecewise constant between available CDS quotes.

Last three columns of Table 9.5 give us a good opportunity to shortly clarify outputs and relations of our model. From the previous pages we know, that

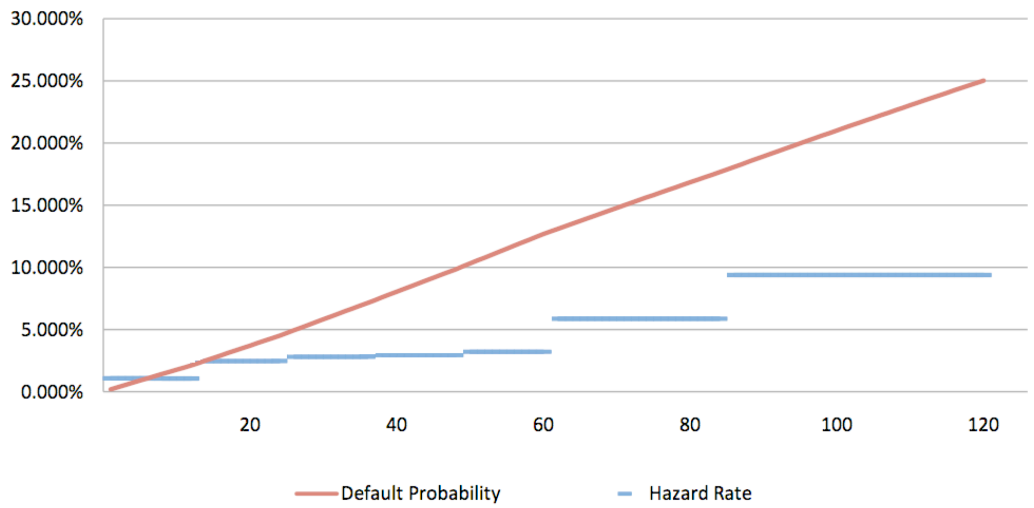
[8]All VBA codes used in this work are on accompanying CD to the master thesis.
[9]This is observed from December 2010, status in July 2011 is that Greece is still balancing on the edge of state bankrupt and receiving another billions of EUR from IMF and other countries
[10]Defined earlier as a the sample of 18 Greek governments bonds

Table 9.4: Greek Bond Crisis



Note: we are illustrating here the development of average yields, from our sample of Greek bonds, where mean value represents the mean of yields from our Main Sample. Source of comments wsj.com, NYTimes.com

Figure 9.3: Example of Hazard Rate and Default Probability Curve



Note: Hazard rate and default probabilities extracted from CDS quotes on 5/11/2008, using treasury curve as benchmark interest rate.

Table 9.5: An Example of Hazard Rates: premium leg and protection leg computed

Maturity	CDS Spread	Hazard Rates**	Premium Leg	Protection Leg	Modeled Spread
6 M	99.000	1.0701%	0.00484	48.855%	99.023
12M	98.500	1.0596%	0.00950	96.446%	98.511
24M	106.000	2.4627%	0.01991	187.791%	106.018
36M	113.500	2.8054%	0.03105	273.544%	113.513
48M	118.750	2.9506%	0.04198	353.465%	118.760
60M	124.000	3.2147%	0.05303	427.610%	124.008
84M	126.800	5.8619%	0.07104	560.215%	126.806
120M	131.000	9.4023%	0.09497	724.945%	131.005

*discounted by swap curve

Maturity	CDS Spread	Hazard Rates**	Premium Leg	Protection Leg	Modeled Spread
6 M	99.000	1.0728%	0.00487	49.097%	99.183
12M	98.500	1.0599%	0.00957	97.050%	98.590
24M	106.000	2.4598%	0.02007	189.383%	105.986
36M	113.500	2.8040%	0.03139	276.617%	113.491
48M	118.750	2.9473%	0.04257	358.497%	118.743
60M	124.000	3.2077%	0.05393	434.919%	123.995
84M	126.800	5.8592%	0.07259	572.445%	126.814
120M	131.000	9.3790%	0.09750	744.342%	130.991

*discounted by treasury curve

** hazard rates represent not annualized hazard rates they are valid between the last quote and current maturity, according to 9.4
Spreads are quoted in basis points (bpb).
Computed hazard rates for CDS quotes on Greek bonds as at 5/11/2008. In the column Protection Leg is the value of discounted protection leg for given maturity and in the column Premium Leg is discounted value of premium leg as it is described in the beginning of the Section 6.1. All values in these two tables are outputs of our VBA model.

we are computing hazard rate based on the assumption that the value of CDS at the inception of a contract is fair, the value is equal to zero. We use the formula for protection and for premium leg from Proposition 6.2.1. Value of $protectionleg - spread * premium$ should equal to zero. We see that for all maturities in Table 9.5 it holds. For example see 12 months maturity in the first part of the table(discounted by swap curve), $48.855\% \times 98.500bpb - 0.00484 = -0.000027$ that is close enough to zero. Difference is caused by a root finding[11] algorithm and its precision. The last column called Model Spread is the spread computed as the value of the protection leg divided by the premium leg. All modeled spreads in this table are equal to real market spreads. Such analysis was done for each quote to clarify results of our model.
From Tables B.2 and B.1, where basic statistical properties of both default probability term structures are presented, we clearly see that the choice of benchmark curve has not any significant effect on computed hazard rates and thus implied default probabilities. This is expected because discount rate has the similar impact on both legs of the contract.
Following Picture 9.4 presents Risk Neutral Implied Default Probability Term Structure with corresponding hazard rate term structure. For the same picture just for the swap curve see Appendix C Figure C.1.

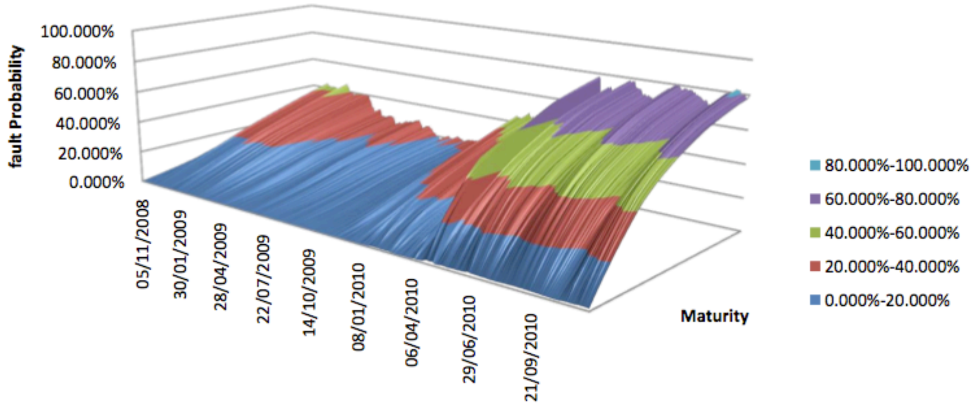
Pictures 9.4 give us a clear view on the reactions of default probabilities to sovereign debt crisis. Years 2008 and 2009 with quite flat default probability curves are replaced with highly distressed and steeply increasing credit curves in April 2010. Long term default probabilities dramatically increased from about 40% to almost 80% during April 2010. It means that CDS market implies that Greece will default during following ten years with a probability 80%. This number is quite high but could be expected according to circumstances around the Greek sovereign debt. Similar values for Greece can be found e.g. in a research European Liquidity Review(2008) [26], where 10 years cumulative default probabilities are around 70%[12] at the end of April 2010. In our case they are around 80% see Table C.2. But they are using recovery rate of 40% so this perhaps causes 10% difference. Also 5 years default probabilities are comparable with this study, one compare chart on the page 4 of this study and it is inline with Chart C.2(output from our model). So we see that recovery rate assumption could have perceptible impact on the risk neutral default probabilities computations.

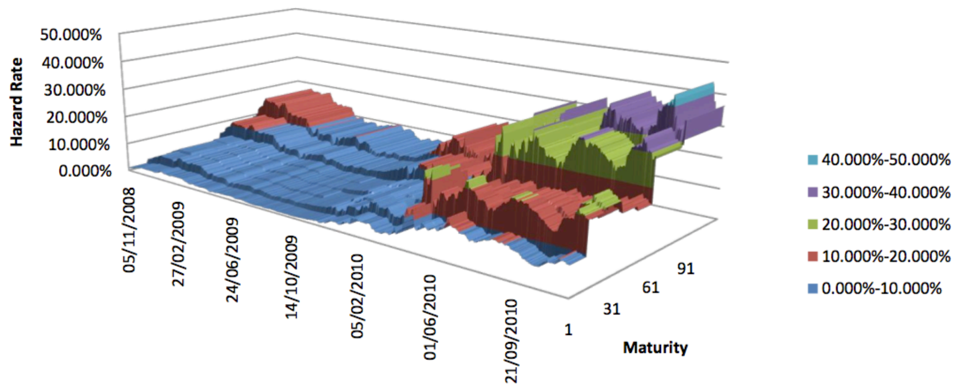
Confusing situation; hundreds of announcements of government heads of potential state rescuers of Athens, dramatical public riots and strikes of public sectors in the whole Greece contributed to rough movements of financial markets. One of the movements, more than 14% decrease in long term default probabil-

[11]we are using standard additional component to Microsoft Office Excel, Solver and this component is using Generalized Reduced Gradient algorithm to find solution to non linear equations

[12]another source can be website of CMA company http://www.cmavision.com/market-data/, leading source of independent OTC market data, where e.g. 8.7.2011 was Top 3 of Highest Default Probabilities leading by Greece, Ireland and Portugal with 5 year cumulative default probabilities from 54% to 84%. We are not able to present here the complete list of countries and their default probabilities since we have only access to demo account on this webpage.

Figure 9.4: Default Probability Term Structure with corresponding hazard rate term struture





Note:here the treasury curve is used as a benchmark curve. These two pictures show risk neutral default probability term structure and corresponding hazard rates as implied by data.

ities, was caused by the IMF approval of the rescue package during the second week in May 2010. This is illustrated in Figure 9.5

Figure 9.5: Comparison of Default Probabilities between 7th and 17th May

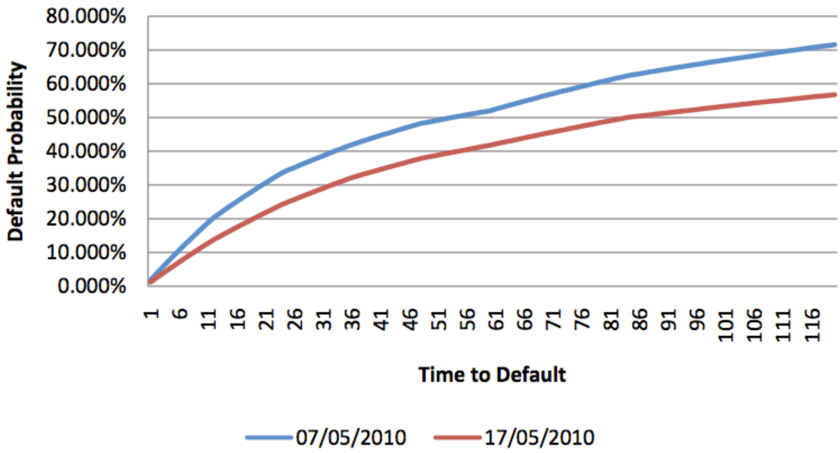


Illustration of how the approval of rescue package from IMF influences implied Greek default probabilities

To see the development of default probabilities for main 8 possible default time horizons see Figure C.2 in Appendix C.

10. Application Of Defaultable Coupon Bearing Bond Formula

Following section recapitulates and evaluates our approach to incorporate the default risk into the value of coupon bearing bonds. Here we are considering two types of risk free interest rate curves that serve us to get risk free discounted values of cash flows from given bonds. These values should be stripped of any risks that can influence the price of bonds. Let us refer to the risk free bond as to a bond value of which is not affected by any risk, simply cash flows corresponding with given bonds discounted by a risk free interest rate to the valuation date. The concept of CVA tells us that the the difference between the value of a risk free bond and risky bond should be caused by credit risk. Of course, bonds values already contain risks that are coming from the market in their prices, but our analytical formula for CVA lets us quantified just default component of bond prices. Let's recall the formula for a defaultable coupon bearing bond from Section 5.2.

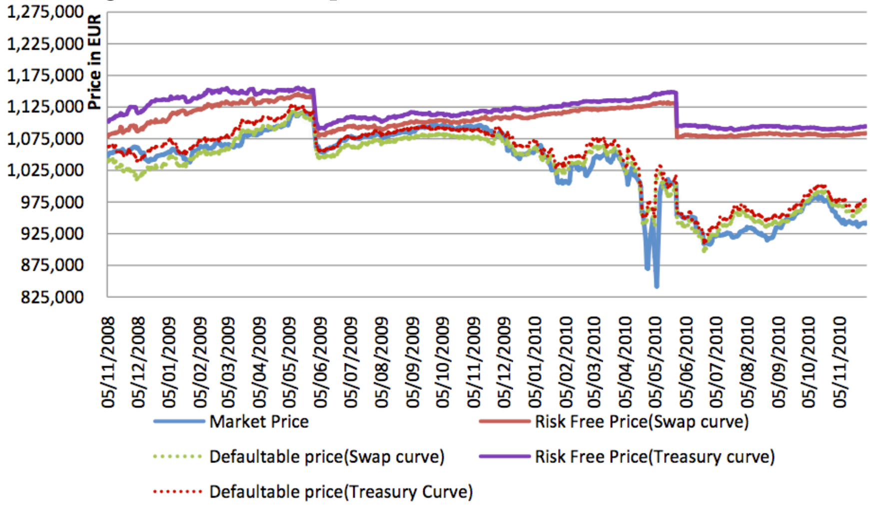
$$P_{C,D}(t,T) \approx P_C(t,T) - LGD \times \sum_{i=1}^{m} \tilde{\mathbb{P}}(\delta \in (T_{i-1}, T_i] | \mathcal{G}_t) \frac{B_t}{B_{T_i}} P_C(T_i, T) \qquad (10.0)$$

Where $P_{C,D}(t,T)$ is the price of defaultable coupon bearing bond at time t and $P_C(t,T)$ is a risk free bond price of coupon bearing bond at time t. As we have already mentioned and it is also possible to see from the previous formula CVA is given as a sum of a risk free price weighted by default probabilities at each possible time of a default multiplied by a Loss Given Default, in our case 46%. If we assume that the price of bond is sensitive only to credit risk, we expect that, if we calculate the risk free discounted value of cashflows from a given bond minus the corresponding CVA, it approaches the real price quoted on the market. Unfortunately it is not so straightforward. Since the price is determined by at more possible types of risks, like: inflation risk, interest rate risk, downgrade risk, liquidity risk and reinvestment risk. It implies that the spread between risk free bonds and risky bonds is not explainable just by credit risk. So we would expect that the price from our model will get closer to the real market price than risk free price of a bond, but the spread corresponding to other risks, remains. Our findings also support this expectation. This considerations is presented in the following example.

We clearly see in the Figure 10.1 how different prices behave. According to our expectations, the risk free prices are much more higher than the risky ones. Also the relation between price discounted by the swap curve and by the treasury curve is inline with our expectations. Since the treasury rates were slightly lower than the swap rates, price discounted by swap rates is, in average, under the price discounted by treasury rates.

As we see, modeled price that is using the treasury curve as riskless curve is always higher than market price. It implies that non default component contribute to the the market price as well. By nondefault component, we mean another type of risks than default risk. Modeled price computed with the swap

Figure 10.1: Example of Modeled Prices and Market Price



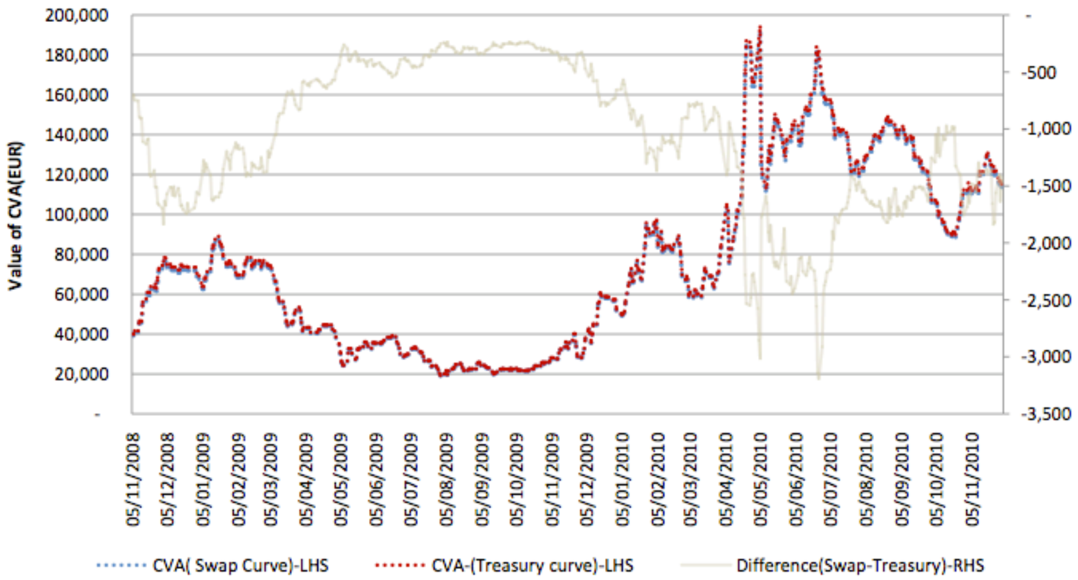
Here we are presenting different "types" of prices of Greek bond with maturity 18/05/2012 and coupon 5.25% of the nominal value of 1 Million EUR.This bond has code $GR18051210G$ in our main sample. In the legend it is always, in parenthesis, denoted with respect to what risk free curve is the price computed. Market price is the price observed on the market, defaultable price is price after applying of our formula 10 by using two different discount rates. Risk free price is a price computed as outstanding cash-flows discounted to the valuation date by the risk free interest rates.

curve is in some cases under the real market price. That implies overestimations of the default component in this case. This may be the consequence of having already included some part of default risk in the swap rates. That is coming from LIBOR curve. We can conclude that for this particular bond modeled price, at least, copied market prices by their shapes and trends. Following picture shows us development of CVA for the particular bond. Once more for two different benchmark curves, see Figure 10.2.

Development of CVA for both risk free curves for this particular bond, also corresponds with the development of the Greek Crisis. We see increasing trend in CVA value since the beginning of year 2010. Investors to Greek bonds already in January 2010, started to require a higher compensation for the possibility of thr default. The value of relation CVA computed by Swap curve minus CVA computed by Treasury curve as a benchmark curve is always negative. This is caused by above mentioned higher swap rates. However, it needs to be emphasized that this fact does not imply that modeled price computed with Swap curve as the benchmark is higher than one computed with Treasury curve as a benchmark curve. This holds, because the same interest rate is applied to the default free part of Equation 10 that also reduces the final modeled price of the bond.

Now we extend our analysis of a default component for the whole sample of bonds. We investigate how good is the spread between risk free bond and the market price of the bond covered by CVA. Basically, we investigate which part of the spread between these two types of bonds corresponds to the credit risk. The ratio in Table 10.1 and Table 10.2 is computed as: CVA divided by the spread between the risk free price and the market price. We are always comparing CVA

Figure 10.2: Example of CVA



Here we are presenting CVA for the Greek bond(right axis) with maturity 18/05/2012 and coupon 5.25% and nominal value of 1 Million EUR. In our sample it is bond with a code $GR18051210G$. In the legend it is, in parenthesis, denoted with respect to what risk free curve is the CVA computed. The difference(left axis) is computed as CVA computed by Swap curve as a benchmark curve minus CVA computed by Treasury curve as a benchmark curve.

to a default free price computed with respect to the same benchmark rates.

Ratio under 100% means that the spread between the risk free bond and the market price of such a bond is not explainable only by credit risk. On the other hand, the ratio bigger than 100% means that CVA computed by our model is overpricing the default risk for a given bond. From the previous tables, we see that such a situation, significantly occurring in the case of CVA computed on the basis of swap curve, as a benchmark curve. We have mentioned earlier that the swap rates include default risk, so it implies that they are already adjusted for this type of risk. This fact causes that for some bonds in our sample, is our modeled price under the market price and so the ratio is more than one hundred percent. More significantly is such a phenomena visible in Table 10.2 for our second sample. This sample also includes the dates before the debt crisis and the ratio is almost in all cases higher than 100%, so the CVA average value is much higher. We also need to take a closer look at the case, where the ratio is more than 100 percent, for bonds for which we have used Treasury curve as a benchmark. In this case, the ratio has slightly exceeded one hundred percent, for same bonds as for those discounted by the swap curve. This holds just for our main sample, which contains mostly highly distressed period, so such situation may causes this type of discrepancies. For a longer period, for the second sample Table 10.2, we do not observe such an exceeding Ratio above 100% for Treasury curve. On the other hand the Ratio for swap case increased significantly for each bond in the second sample. For such a small sample of bonds it is not possible to conclude anything and generalize it. We can just guess, why is the ratio for

Table 10.1: Credit Component Ratio of Bond Spreads

	SWAP		**TREASURY**	
	CVA(EUR)	**RATIO**	**CVA(EUR)**	**RATIO**
GR01071410YG	223,237	78.57%	227,964	73.58%
GR18051110G	67,576	113.99%	68,249	105.79%
GR18051210G	134,644	93.54%	136,461	86.93%
GR19071910G	336,658	66.56%	344,755	63.69%
GR2003113YG	54,153	57.26%	54,686	55.47%
GR2003123YG	123,311	97.16%	124,889	90.13%
GR2004177YG	301,356	71.11%	308,354	67.40%
GR20051310G	184,298	83.19%	187,598	77.64%
GR20051410G	219,997	77.41%	224,606	72.60%
GR20071510YG	242,274	73.69%	247,783	69.40%
GR20071610YG	261,120	74.36%	267,189	70.12%
GR20071710YG	285,748	70.42%	292,465	66.84%
GR20071810G	301,715	69.87%	308,899	66.49%
GR2008115YG	84,958	111.80%	85,853	103.47%
GR2008125YG	146,426	90.19%	148,575	83.90%
GR2008135YG	192,421	81.65%	196,055	76.28%
GR2008145YG	233,077	75.65%	238,057	71.09%
GR2008155YG	263,808	75.05%	269,747	70.59%
Average	**203,154**	**81.19%**	**207,344**	**76.19%**

Swap and Treasury denote which risk free curve was used. CVA values are averages for each bond, during the observed period in EUR.

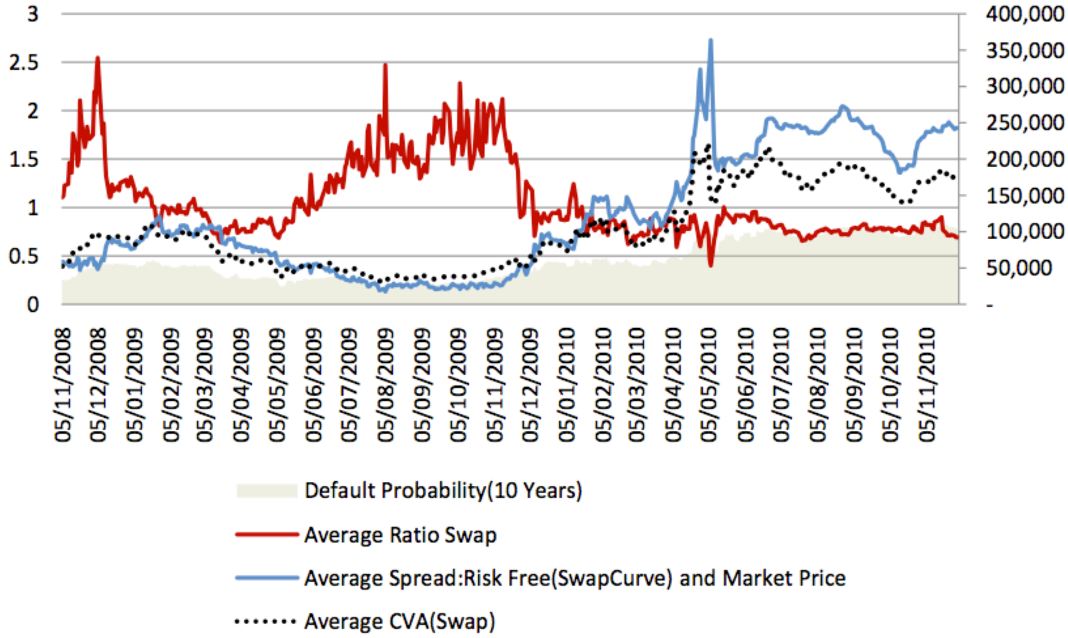
Table 10.2: Credit Component Ratio of Bond Spreads:Second Bond Sample

	SWAP		**TREASURY**	
	CVA(EUR)	**RATIO**	**CVA(EUR)**	**RATIO**
GR18051210G	75,002	128.99%	76,083.7	88.89%
GR2003113YG	34,808	47.81%	35,156.4	43.28%
GR20051310G	102,607	115.00%	104,447.9	84.80%
GR20071510YG	141,560	115.28%	144,669.3	87.34%
GR20071610YG	154,895	110.47%	158,405.4	86.26%
Average	**101,774**	**103.51%**	**103,753**	**78.11%**

Swap and Treasury denote which risk free curve was used. CVA values are averages for each bond, during the observed period.

our five bonds from the second sample so high. For better understanding of such increase of the ratio in this case, we plot four properties of bonds from the second sample, see Figure 10.3. This helps us to analyze such increase in the satio.

Figure 10.3: Detail of Ratio for Bonds from Second Sample



Here are presented values, computed as averages for bonds from second sample. Ratio and Default Probabilities are on the left axis and other values are on the right axis. Blue line represent the average difference between default free price of bond and defaultable price, discounted by swap curve.

We clearly see in the Picture 10.3 that in the second half of the year 2009 spread between the risk free price of Greek bonds (from second sample) and the market price of these bonds has been shrinking. It means that these bonds were considered to be less risky than before. However, risk neutral default probabilities implied from CDS market did not suggest any improvement in Greek credibility, they were not decreasing, thus CVA was not rising. From this consideration, we can conclude that such high ratio of spread coverage is implied by a small spread between the market price of bonds and the risk free price of bonds, computed with the swap rates as a benchmark. From the average ratios for each bond in our main and the second sample, we can conclude that our valuation formula for coupon bearing bond CVA covers more than 75 percent of spread between default risk exposed bonds and risk free bonds. Our results also support the fact that swap rates already include the adjustment of a default risk. In both tables we see the ratio computed with help of swap rates is higher than the ratio computed with help of treasury rates.
Similar results were achieved in Longstaff(2005) et. al. [18], where they computed such ratio on the sample of the corporate bonds for different ratings. They used three curves as benchmarks: treasury, swap curve and Refcorp[1] curve. They

[1]Resolution Funding Corporation:An agency established by the Financial Institutions Reform, Recovery and Enforcement Act of 1989 to fund the Resolution Trust Corporation, funded

argued that the spread between corporate bonds and benchmark curves are mostly caused by credit risk. They have computed that the coverage of the spread by credit risk varies from 51% to more than 81%. They noticed the same fact that the swap curve probably already include some credit risks and so the ratio is higher than one hundred percent sometimes. They are using the similar reduced form approach to compute default component of a bonds as we are using.

If we assume that the prices of bonds are sensitive to more risks than that represented by CVA, it seems that for our bond samples it is more appropriate to use Treasury curve as default free curve. It is better, since it gives better interpretation of possible spreads between the risky and risk free bonds. The main part of the spread is explainable by credit risk and the rest of the spread is explainable by another market risks.

It needs to be emphasize that even into the price of CDS it is possible to incorporate or separate the default component of the issuer or investor. In that case protection buyer wants to be rewarded for possible future default of protection seller, he/she wants to pay less premium for the protection. In that case lower premium implies lower default probabilities and thus lower CVA.

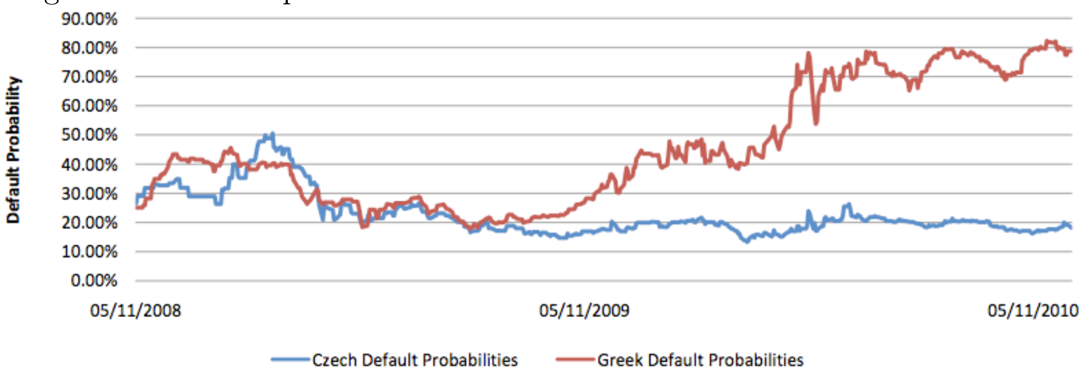
its own activities by issuing zero-coupon bonds through the US Treasury,U.S. Treasury Refcorp Strips. These bonds have the same credit risk as Treasury bonds, but they do not enjoy the same liquidity as Treasury bonds. (source: http://financial-dictionary.thefreedictionary.com/)

11. Interest Rate Swap Spread: Adjusted to Counterparty Risk

Up to now, we have been dealing with more or less only testing of the theory on the market data. Now we are focusing on one practical question that can arise in front of all types of the professionals dealing with interest rate derivatives, particularly with the interest rate swaps. What is a fair fixed rate that makes an interest rate swap contract fair, under consideration of a counterparty default risk? We are creating an artificial[1] interest rate swap contract between the Czech National Bank and the Greek National Bank. These two banks are stating as two counterparties in this contract. From our data set, we have used before, we have almost everything that we need for the calculation, except default probabilities of Czech Republic. We have derived these probabilities from CDS quotes in the same manner as we did it in Section 9.4 for Greek default probabilities. The following Chart 11.1 is comparing ten years default probabilities of Greece and Czech Republic.

Risk neutral default probabilities around 20% in December 2010 for Czech Republic are higher as one would expect, even if we take into consideration the Remark 8.1.5. Unfortunately, we have not found any academic paper about implied risk neutral default probabilities of Czech Republic, so we can not directly compare our results. But from the study that we have already mentioned, European Liquidity Review(2008) [26] follows that e.g. 10 year default probability of Italy is 29.1%, of France 14.3% (recovery rate 40%). In the context of other countries from Europe Region, 20% for 10 year Czech default probability seems to be reasonable[2]. For more details about the Czech default probabilities see Appendix B and Tables B.4 and B.3 and Chart C.3.

Figure 11.1: Development of 10 Years Greek and Czech Default Probabilities



Note: used swap curve for discounting

[1] We are not aware that such swap actually exists so we considered it as something created just for computation purposes

[2] Computation of Czech default probabilities was done with the same VBA program as for Greece and as we have already mentioned Greece risk neutral default probabilities are completely in line with other sources. It implies that there should not be any systematic error in our computation.

We divide this section into two parts. In the first one, we are assuming just the unilateral default risk (Section 5.3.1), once for Czech Republic and once for Greece as an investor. In the second part we are dealing with more realistic situation, with the bilateral default risk from Section 5.3.2.

11.1 Unilateral Credit Risk

From Section 5.3 we know how to compute the price of interest rate swap with one defaultable counterparty. It is a default free price of IRS minus CVA, where for IRS it is defined as the sum of swaptions on IRS spread weighted(assumption of independence of default and interest rate) by risk neutral default probabilities. Here, we are computing CVA for one theoretical IRS and deriving knew implied "fair" interest rate spread. Assumption about the contract on which one defaultable counterparty participates and one default free entity is mostly used when there is a big difference in the credit quality of participated entities. This can be the case of our two participants mainly in year 2010, as follows from Chart 11.1. Our imaginary swap can be between the the National Bank of Greece(GNB) and the Czech National Bank(CNB). We compute CVA for two cases; firstly, we are assuming that CNB is default free, Case I on the Picture 11.2, and the second case is when GNB is paying fixed rate and we computes CVA from its prospective, so we assume that it is default free[3], Case II on the Picture 11.2. Even though that both these assumptions, especially the second one, look not realistic we are using it to demonstrate the effect of different default probabilities term structure on fair swap price. For both of these cases we are assuming that default free entity is paying a fixed interest rate.

Swap Specification

Default free entity is paying the fixed interest rate semiannually against six month LIBOR rate that is also payed semiannually and in the same time-grids, so there is no time lag between these two payments. Maturities of the swap are 3,5,7 and 10 years.

We compute the new fair fixed rate that contains a reward for the possible default of a counterparty. We use already shown formulas from Chapter 3.6 for swap rate that makes the contract value equal to zero at the date of inception. We are considering different maturities of such swap, to demonstrate sensitivity of CVA on contract duration. We recall 5.3.1 for defaultable swap price.

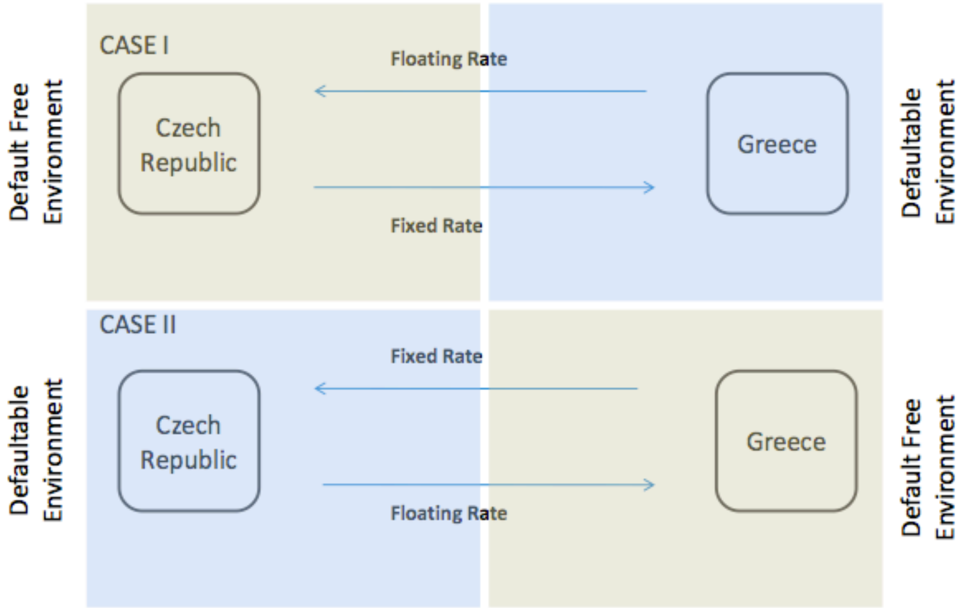
$$IRS_D(t) = IRS(t) - LGD \times OptionPart \tag{11.0}$$

where

$$OptionPart \approx \sum_{i=1}^{m} \tilde{\mathbb{P}}[\delta \in (T_{i-1}, T_i)] \text{PayerSwaption}_t(T_i, T_m, \kappa, \sigma_{T_i}, \kappa(t, T_i, T_m))\}$$

[3]this is so unrealistic assumption that reader can imagine in this case that a counterparty that is paying to Czech Republic fixed rate is e.g. Germany instead. We just did not want to confuse reader with another country.

Figure 11.2: Interest Rate Swap



We are using the same LGD value ($LGD = 0.46 = 1 - RecoveryValue$) estimated from historical data. For valuation of the swaption we use formula Black's3.13. All input parameters are known. Forward swap rates that are required as inputs are known since we have spot swap rates. The second required input is a volatility that we have estimated from time series of spot rates. We have estimated the volatility for different maturities of swaptions by standard method that is described e.g. in Hull(2002) [13][4]Chapter 17. The average of unbiased estimates through the all maturities is 37% and this constant is used in the valuation of swaption by Black's model.

Remark 11.1.1. *We are dealing here with interest rate swap paying both legs semiannually, but the standard swap on the European market is paying the floating leg semiannually and the fixed leg annually. So there is not the same number of payments during the contract validity. Frequency of payments would also be one of the factor affecting the CVA and BVA, respectively.*

We implement the formula 11.1 for CDS in to the VBA code. We are computing fair swap fixed rates for each trading day from 5/11/2008 to 6/12/2010. For each of these days, firstly we compute default free swap spread and then through CVA we derive adjusted fixed swap rates for default possibility of counterparty(further just adjusted swap rate). For each day we compute a corresponding CVA for a given swap, then we know that the price of such a swap is a default free swap price minus CVA. As we mentioned couple of times swap value should be at the day of inception equal to zero, thus by using the formula 3.8 we get implied, credit risk adjusted, swap rate. We are repeating this procedure for a swap with

[4]We have computed the unbiased estimate $\sigma_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(u_i - \mu_u)^2$ where $u_i = \frac{r_i - r_{i-1}}{r_{i-1}}$ and $\mu_u = \frac{1}{n}\sum_{i=1}^{n} u_i$, $r_i, i = 1, ...$ is spot rate time series. From daily volatility we have converted it to year volatility $\sigma_{year} = \sigma_{day}\sqrt{252}$,

Table 11.1: Unilateral Credit Risk Effect on Fixed Swap Rate

Default Free Swap Spread	Maturity	Period from 05/11/2008 Survival Probability	to 31/12/2009 Spread(in BPS)
2.331%	3Y	90.78%	-2.15
2.887%	5Y	83.40%	-5.18
3.230%	7Y	77.54%	-8.72
3.518%	10Y	69.96%	-13.86
		from 01/01/2010	**to 06/12/2010**
1.689%	3Y	65.68%	-8.30
2.207%	5Y	53.68%	-14.48
2.595%	7Y	44.29%	-20.60
2.974%	10Y	35.59%	-28.67

Note: Values presented in this table are computed as averages of given variables, for corresponding time periods. Default Free Swap Fixed Rate is computed without any exposure to default risk. Survival probabilities represent a probability of survival until the swap maturity. Spread is computed as a Risk Adjusted Rate - Default Free Rate and is given in basis percentage point unit. Negative spread implies that the adjusted rate is lower than the default free rate.

different maturities to see if and how the duration of the contract affects CVA and consequently the adjusted swap rate. Intuitively, we are are expecting that during periods with higher default probabilities and also with a longer duration of the contract, swap rate, that contains adjustment for a default risk, is decreasing. It is decreasing because the investor(fixed leg payer) wants to be rewarded for the higher risk of unfulfilled obligations from the contract.
Firstly, we are assuming that CNB is paying a fixed rate, so it is default free. We are making our analysis separately for the period before year 2010 and for the period from the beginning of 2010. This separation helps us to better understand the influence of different default probabilities term structure on the CVA for swaps and subsequently on fair fixed rates. Outputs from our model are in Table 11.1.

We see that the differences, caused by the credit risk of Greece are relatively small. They are basis points under the default free rates. It is more or less expectable, since there is no face value that is interchanges between the counterparties, so the possible loss is much smaller than e.g. the for bonds. To compare our results we found just one article Brigo and Masseti(2005)[10] where similar analysis was presented, but with closely unspecified swaps. Their results are presented in Table 11.2.

We do not have all information about parameters used for spreads calculation in Table 11.2, so we can not exactly say the reason of minor differences between our results and theirs. We see that our results are mostly inline, except of the 10 year swap with survival probability 53 % and spread 12.26BPS. From the last row of our Table 11.1 we perhaps can say that our model slightly overpricing the credit risk. Unfortunately, without more detailed information we can not conclude anything or determined which factor caused it.

Now we compute the fixed rate adjustment for case in which is the GNB paying fixed rate and want to be rewarded for default exposure of CNB. The situation is that GNB wants to step into the swap contract with CNB and wants

Table 11.2: Results from Brigo and Masseti(2005)[10]

Default Free Swap Fixed Rate	**Maturity**	**as at 10/03/2004 Survival Probability**	**Spread(in BPS)**
3.249%	5Y	96.380%	-0.51
4.074%	10Y	89.310%	-2.16
3.249%	5Y	88.570%	-1.80
4.074%	10Y	75.270%	-5.80
3.249%	5Y	74.780%	-4.25
4.074%	10Y	53.420%	-12.26

Note: Results are from Brigo and Masseti(2005)[10], computed at 10th March 2004 with flat volatility 15%. There are missing information about LGD used and about payment frequency

Table 11.3: Unilateral Credit Risk Effect on Fixed Swap Rate

		Period	
Default Free		**05/11/2008**	**31/12/2009**
Swap Spread	**Maturity**	**Survival Probability**	**Spread(in BPS)**
2.331%	3Y	92.184%	-1.74
2.887%	5Y	85.909%	-4.30
3.230%	7Y	81.142%	-7.20
3.518%	10Y	74.495%	-11.49
		01/01/2010	**06/12/2010**
1.689%	3Y	95.185%	-1.09
2.207%	5Y	90.830%	-2.78
2.595%	7Y	86.866%	-4.98
2.974%	10Y	80.871%	-8.67

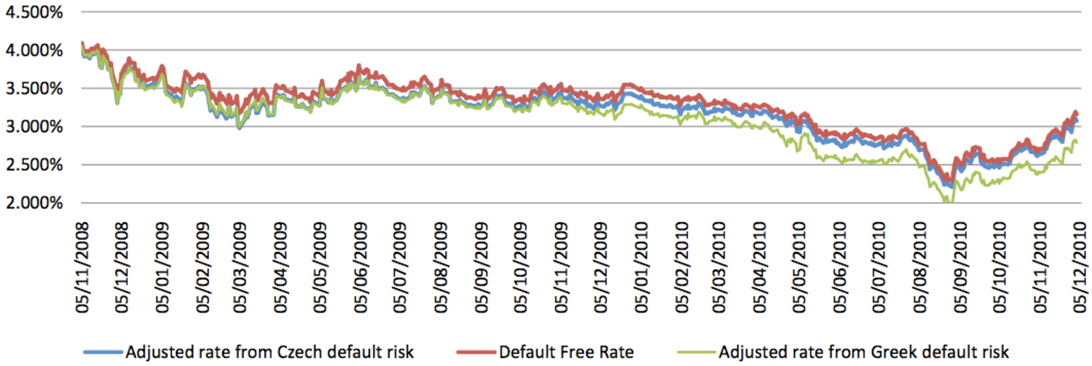
Note: Values presented in this table are computed as averages of given variables, for corresponding time periods. Default Free Swap Spread is computed without any exposure to default risk. Survival probabilities represent a probability of survival until the swap maturity. Spread is computed as Risk Adjusted Rate - Default Free Rate and is given in basis percentage point unit. Negative spread implies that adjusted rate is lower than default free rate.

to know the adjustment to fixed rate.

We clearly see that according to the expectation, if GNB wants to enter in to the swap contract with CNB the adjustment for fixed swap rate is much smaller than the adjustment for CNB. It is implied by the default probability term structures of both countries. Following chart compares adjusted rates for both cases.

It is obvious that such assumption about just one defaultable counterparty is not realistic. The biggest disadvantage of this approach is that it is certainly expensive for the defaultable counterparty. Because investor is exposed(mostly) to default risk as well and this fact can decrease the price of the instrument for the counterparty as well. We are dealing with such a situation in the following section.

Figure 11.3: 10 Year Fixed Swap Rates for CNB and GNB



Adjusted rate from Czech default risk is the fixed rate that is paid by GNB and Adjusted rate from Greek default risk is the fixed rate that is paid by CNB.

11.2 Bilateral Credit Risk

In this section we are assuming same conditions and setup as in previous section. One thing that is different is that both parties are facing the default risk. From Section 5.3.2 we know how to price this risk and let us recall the formula for the bilateral default risk for the interest rate swap.

$$IRS_D^B(t) = IRS(t) + DVA - CVA \tag{11.0}$$

$$\begin{aligned} DVA &\approx \sum_{i=1}^{m} LGD_1 \left(\tilde{\mathbb{P}}[T_{i-1} < \delta_1 \leq T_i, T_i < \delta_2 \leq T_m] + \tilde{\mathbb{P}}[\ T_{i-1} < \delta_1 \leq T_i, \delta_2 > T_m] \right) \\ &\times \ \text{ReceiverSwaption}_t(T_i, T_m, \kappa, \sigma_{T_i,T_b}) \} \end{aligned}$$

$$\begin{aligned} CVA &\approx \sum_{i=1}^{m} LGD_2 \left(\tilde{\mathbb{P}}[T_{i-1} < \delta_2 \leq T_i, T_i < \delta_1 \leq T_m] + \tilde{\mathbb{P}}[\ T_{i-1} < \delta_2 \leq T_i, \delta_1 > T_m] \right) \\ &\times \ \text{PayerSwaption}_t(T_i, T_m, \kappa, \sigma_{T_i,T_b}) \} \end{aligned}$$

Indexes 1 and 2 denote the investor and the counterparty, respectively. For the sake of simplicity, we are using same $LGD = 0.46$ for both participants. Adjustment that is made about the swap price can be positive or negative, as it has been already mentioned. From the investor's perspective, who pays a fixed rate, it is negative if the counterparty is more risky than the investor itself and it is positive if investor is more risky than the counterparty. As follows from IRS formula in a bilateral case we have to deal with a joint default distribution. So we need to incorporate into our valuation process the dependence between default probabilities of both banks/states. It would be not very realistic to assume that the probability of Greek default is totally independent from the Czech default probabilities and contrary. Today's mathematical finance especially with connection to CDO's has a great interest in the dependence structure between default events. We use one of the most elegant and straight forward method to use, Gaussian copula. This structure helps us to make one joint distribution from our two marginal distributions. Copula[5] approach is very useful when we need firstly to analyze and work with the marginal distributions separately and then when we need to incorporate their dependence structure in the relatively transparent manners. We are following here Li(2000) [17] one of the most famous and also most discussed working paper about using copulas in finance, more precisely in default correlation topic. For an introduction to copula theory see Nelsen(2006) [21]. The key theorem, Sklar's Theorem is telling us that for a given joint distribution and marginals distribution respectively exists a copula that binds these marginal distributions into one joint distribution. To see the original text we refer to original work Sklar(1959) [25]. The above statement is just freely speaking and does not claim to be complete or mathematically rigorous, We use it just to shortly present main idea of the copula theory. We have two entities to "bind" together so we need to deal with bivariate case of copula. More precisely, we are dealing with bivariate Gaussian copula. First we introduce general copula function and then follows Gaussian copula.

Definition 11.2.1. A function $\mathbf{C}$:$[0,1]^I \to [0,1]$ is a Copula function if there are uniform random variables $U_1, \ldots, U_I$ taking values in $[0,1]$ such that $\mathbf{C}$ is their joint distribution.

The copula $\mathbf{C}$ is a joint distribution of marginal random variables $U_1, \ldots, U_I$. In our case it is enough to have I=2. It can be easily shown that for marginal distribution function F_1, F_2, the function

$$F(x_1, x_2) = \mathbf{C}(F_1(x_1), F_2(x_2)) \ , \ x_1, x_2 \in \mathbb{R} \tag{11.-4}$$

defines a multivariate distribution function with marginals F_1, F_2

Definition 11.2.2. Bivariate Gaussian Copula: *let R be a positive symmetric matrix with main diagonal consists from 1 and Φ_R be the standardize multivariate normal distribution with correlation matrix R, then Bivariate Gaussian Copula is defined as follows*

$$\mathbf{C}(u_1, u_2, R) = \Phi_R(\Phi^{-1}(u_1), \Phi^{-1}(u_2))$$

where $\Phi^{-1}(u)$ denote the inverse of the normal cumulative distribution function.

In our case u_1, u_2 are representing default probabilities. Third input parameter to copula is correlation matrix as we see in copula definition. This matrix is reducing, in two dimensional case, to estimation of the one element of this matrix, since it contains four elements from which just one is unknown; two of them are ones(on main diagonal) and two of them are identical correlation coefficients, from symmetry. Estimation of the default correlation between two countries is another really wide and complicated topic. It is also hard to find the references to use already estimated results. New regulatory principles for bank institutions Basel III [2], propose to increase current[6] default correlation range $12\% - 24\%$ to

[5] from the Latin for 'join'.
[6] by current we mean Basel II

15% − 30% for big financial institutions that have asset in the value at least 100 billions US dollars. We use this number as a hint and since we are assuming even bigger entities, we consider the upper bound plus 10 %. Correlation on the level of 40% seems as reasonable value for our calculation. In article Chen et al.(2011) [28] they are dealing with correlation within South America countries. They estimated default correlation coefficients [7]between Argentina, Brazil, Mexico, Venezuela and they vary from 0.2 to almost 0.9. Of course these results tell us almost nothing since these countries are in totally different environment but we see that our hint value 40%, is more less somewhere in these bounds. From the previous definition easily follows that by using copula, we compute the probability as follows,

$$\tilde{\mathbb{P}}[T_{i-1} < \delta_1 \leq T_i, \delta_2 > T_m] = \tilde{\mathbb{P}}[\delta_1 \leq T_i] - \tilde{\mathbb{P}}[\delta_1 \leq T_{i-1}] - \mathbf{C}(p_{1,i}, p_{2,m}) + \mathbf{C}(\mathbf{p_{1,i-1}}, \mathbf{p_{2,m}})$$

where

$$p_{j,i} = \tilde{\mathbb{P}}[\delta_j \leq T_i], \text{ for } j = 1, 2$$

By this way we reach the point where each needed probability is known straight forward from our default probabilities term structures or is relatively easily computable by using copula. We are aware that the issue of dependence structure is much more complicated and certainly can be the key factor of further improvement of this approach to quantify the default component for financial derivatives. Here we are using such simplified approach just because of the limited scope of this work.

With increasing maturity the dependence structure is having higher and higher impact on the joint default probabilities. Longer time interval gives more opportunities to bankrupt together. To demonstrate the impact of dependence we draw joint default probabilities for independent case and for correlated case, see Chart 11.4.

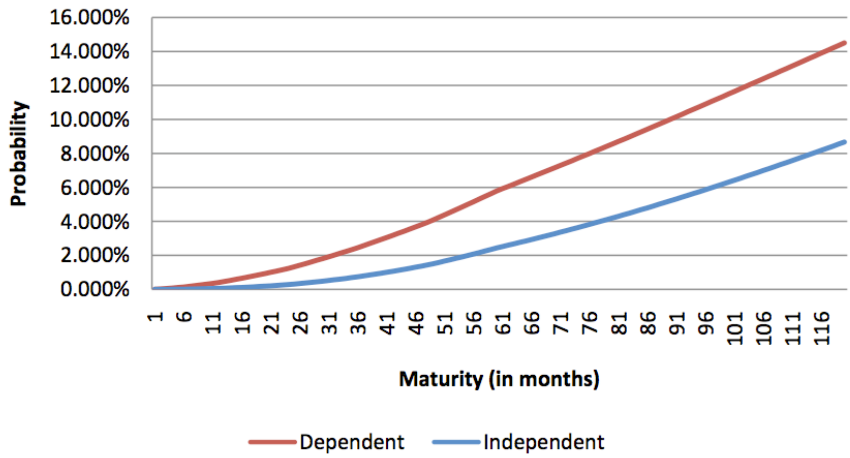
In Table 11.4 we are presenting comparison of CVA in the case of unilateral credit risk with the adjustment(DVA-CVA) computed in case of bilateral credit risk. Here, it is computed for the case when CNB is paying fixed rate to GNB. We are observing the high decrease of fixed rate because of BVA during the period before year 2010. Comparing the difference between default free fixed rate and "unilaterally" adjusted fixed to the difference between the default free fixed and "bilaterally" adjusted fixed rate we see the decrease about 50% e.g. from almost 13 bps in unilateral case to 6 bps in bilateral case(for ten year swap).

During the Greek Crisis when the difference between ten year default probability of Czech Republic and Greece was on average around 60% (see Figure 11.1) the incorporation of Czech credit exposure into the fixed rate has the minor impact, from 28,5 to 27,6 bps(for 10 year swap). This support our consideration from previous section that the assumption about just one defaultable counterparty is appropriate, when there is a big difference in credit quality of contract participants and can serves as a good upper bound estimation of swap rate.

From the development of 10 year default probabilities from Figure11.1 we would expect that at the end of 2008 and the beginning 2009 the fixed rate that CNB is paying should be higher than the default free rate(BVA is positive), or at least is closer to the rate(BVA is close to zero) that does not contain any default

[7]they used Pearson's correlation coefficient that is defined as the covariance of the two variables divided by the product of their standard deviations

Figure 11.4: Joint Default Probability of Greek and Czech Republic



Joint default probability of Czech Republic and Greece computed on 30/12/2009 by using Gaussian Copula with correlation 0.5(red line) and computed as two independent events(blue line). It represents the joint default probability that both Greece and Czech Republic default before time i where i is number of months on the horizontal axis.

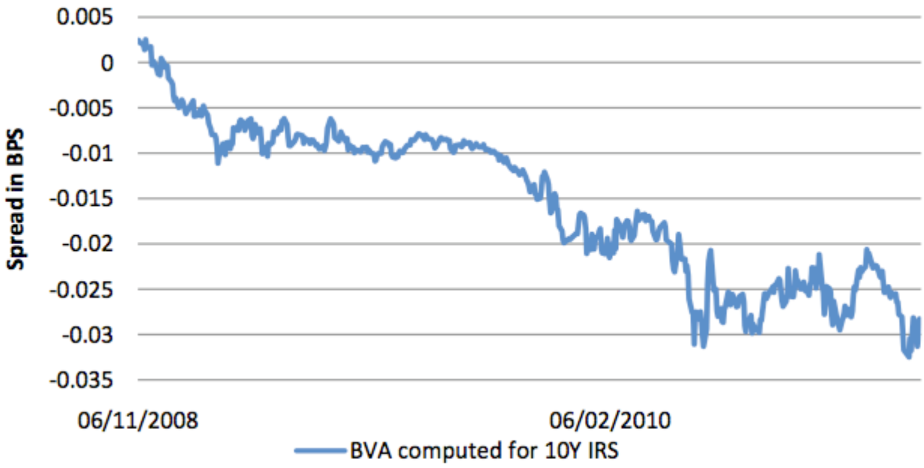
Table 11.4: Unilateral Credit Risk Versus Bilateral Credit Risk

Maturity	**Default Free Swap Rate**	**CVA in BSP**	**DVA-CVA in BSP**
	Period	**from 05/11/2008**	**to 30/12/2009**
3 Y	2.33%	-2.15	-1.23
5 Y	2.89%	-5.18	-3.49
7 Y	3.23%	-8.72	-6.26
10 Y	3.52%	-13.86	-10.16
	Period	**from 04/01/2010**	**to 06/12/2010**
3 Y	1.69%	-8.30	-6.82
5 Y	2.21%	-14.48	-13.18
7 Y	2.60%	-20.60	-19.24
10 Y	2.97%	-28.67	-27.18

Note: third and last column are presenting differences between default free fixed rates and rates adjusted by unilateral and bilateral credit credit risk. CVA is computed as in first case for Czech Republic paying fixed rate and in same for same case is BVA. Difference values are in BPS. Values are computes as averages for given period.

risk. It is because GNB wants to be rewarded for higher credit risk on the side of CNB. In Figure 11.5 we clearly see that bilateral DVA is higher than bilateral CVA at the beginning of year 2009, this implies that the difference adjusted rate - default free rate is positive or close to zero. It also needs to be emphasize that bilateral adjustments are also affected by forward swap rates that are inputs to Black's model. So there is no straight forward implication that if one company has higher default risk than another, adjustment has be positive or negative, respectively.

Figure 11.5: Bilateral Credit Risk Fixed Rate Adjustment



Here is presented the difference computed as Bilateral Risk Adjusted Spread-Default Free Spread. Positive numbers indicate that entity that is paying fixed rate is exposed to higher default risk than its counterparty.

We have computed also differences for three options for the correlation coefficient, that is a driving parameter of dependence in copula. As we see in the

Table 11.5: Comparison of different input correlation parameters $\rho = 0.1, 0.6, 0.8$

Maturity	**Default Free Swap Rate**	**CVA in BSP**	**BVA($\rho = 40\%$) in BSP**	**Correlation Variants** 0.1	0.3	0.8
	Period	**from 05/11/2008**	**to 30/12/2009**			
3 Y	2.33%	-2.15	-1.23	-1.31	-1.15	-3.21
5 Y	2.89%	-5.18	-3.49	-3.75	-3.27	-9.25
7 Y	3.23%	-8.72	-6.26	-6.74	-5.88	-16.40
10 Y	3.52%	-13.86	-10.16	-10.93	-9.59	-25.78
	Period	**from 04/01/2010**	**to 06/12/2010**			
3 Y	1.69%	-8.30	-6.82	-6.90	-6.80	-7.81
5 Y	2.21%	-14.48	-13.18	-13.27	-13.21	-13.26
7 Y	2.60%	-20.60	-19.24	-19.31	-19.37	-17.34
10 Y	2.97%	-28.67	-27.18	-27.12	-27.49	-21.04

Note: we compare here the effect of the change in the correlation coefficient from used 40% to another values.

Table 11.5 also the default correlation coefficient, as an input to copula function, has significant effect on fair fixed swap rate. This input seems to be one of the key factors that needs to be determined and studied properly.

12. Conclusion

In this work we went through a valuation process of defaultable financial instruments represented by a risk neutral valuation concept, from the very beginning, to the practical application of CVA and BVA framework Chapter 7 on a real market data in Chapter 9. In the theoretical part it turns out how many assumptions and approximations need to be done on the way to derive computable formulas for quantification of a default risk. The most discussed and used approximation is about the possible default times. This approximation was crucial during the bootstrap process from CDS's. For each of our formulas, from the summary table 7.1, we assume that a credit event can only occur on a finite number of discrete points per year, in our case 12 points are representing 12 months. For the bootstrap of risk neutral default probabilities and also for the formulas for defaultable cases, for each type of the introduced contract, one of the essential input parameter is LGD ratio. We have estimated the recovery rate and subsequently LGD ratio from historical data. We have used data about state bankruptcies of 12 countries, because in the practical part we are dealing with government bonds and interest rate swaps between two countries. We have estimated LGD ratio of 46% from the data used in the table 8.1.
In the practical part, firstly, we are estimating implied risk neutral default probabilities of Greece and also of Czech Republic. As it was expected almost 80% 10-year Greek risk neutral default probability follows from the CDS quotes in May 2010. Slightly unexpected was the 10 year implied risk neutral default probability of Czech Republic that is around 20 % in December 2010. Even thought the risk neutral default probabilities are not reflecting real expectations of market participants about the real default of Czech Republic, as it is explained in Remark 8.1.5, we have not expected such high values.
After the estimation of the whole probability term structure for both countries, we have started to study how a defaultable coupon bond formula is incorporating and covering the credit risk of Greek government bonds quoted on the market. We have tested it on 18 coupon bonds issued by Greece by using two risk free curves; a swap curve and a treasury curve to obtain some level of robustness. From our computations in Chapter 10 follows that approximately 75% of spread between the risk free bonds and the risky bonds(in this particular case Greek bonds) can be the consequence of credit risk. By using the swap curve as a benchmark curve, our results were even slightly higher than 75 percent, since the swap curve already contains some credit risk component coming from LIBOR. We were witnesses to the situation, when the credit risk implied by our formulas had overestimated the real market price of a default risk, table 10.1 and 10.2, represented by the difference between the risk free price and the market price of bonds. Perhaps this was another effect of having credit risk included in the swap curve.
In the very last section, we have constructed an artificial interest rate swap between Czech Republic and Greece. We have used the theory for IRS for default free case Section 3.3 and defaultable case Section 5.3 and computed a fixed leg payment rate that includes default exposure of the counterparty . We were considering both types of credit risk; unilateral and bilateral.

In the case of unilateral risk, difference between the default free fixed swap rate and the rate that already contains default risk(of Greece) was in basis points. It varies from one to almost thirty basis points on average. As we have expected, CVA while considering the Greek default exposure, see table 11.1, is higher than in the case of the Czech exposure to the default, see table 11.3. Later on, when Greece is computing unilateral CVA from its own perspective against Czech Republic, the adjustment varies from two to nine basis points on average.
For the bilateral risk consideration, we had to use a copula function to compute joint default probabilities of both countries. Bilateral case is more realistic, since not just Czech Republic wants to pay less in IRS because of the Greek bad credit quality. After the incorporation of Czech default exposure through BVA, the adjusted fixed rate differs from the default free fixed rate in the range from one to twenty seven basis points. The reduction in the value of BVA in comparison to CVA happened because of both entities default exposure. The difference between the unilateral CVA and BVA is greater during the periods, when both participants have a stable and similar level of risk neutral default probabilities. During the period when participants on the contract differ in credit quality significantly the incorporation of bilateral adjustment has not such high impact, in comparison to incorporation just unilateral adjustment.

As a possible improvement of our computations in this thesis can be, definitely, a deeper investigation of the recovery rates and an incorporation of another methods to represent them. The second thing that definitely can improve quantification of the bilateral default risk, is an incorporation of more sophisticated structures into the valuation of joint default probabilities. As we have already mentioned, dependence structure issue is vigorously discussed in practice and also in the academic sphere. It would be also interesting to investigate effects of incorporation of protection seller's and protection buyer's default risk exposure into the price of CDS and consequently the effect on implied risk neutral default probabilities.

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A. Descriptive Statistics of Used Data and Parameters Estimations

Table A.1: Summary of basic treasury yields' statistical properties from 5/11/2008 to 2/12/2010(in total 531 trading days)

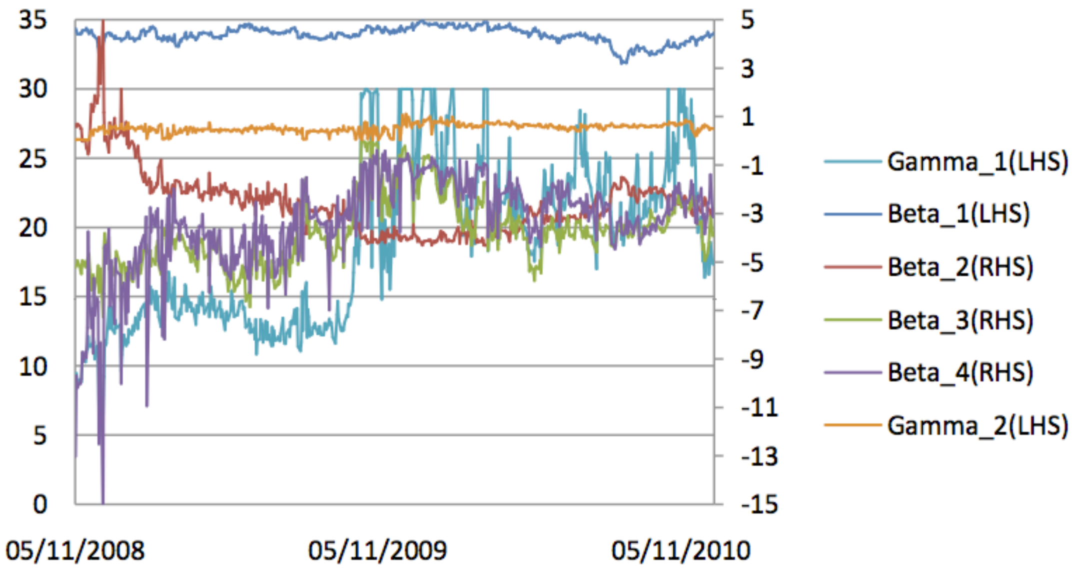
Months to maturity	**Mean**	**Standard Deviation**	**Yield Range**	
			Min	**Max**
1	0.672%	0.628%	0.133%	3.973%
3	0.628%	0.525%	0.130%	2.742%
6	0.703%	0.502%	0.160%	2.778%
9	0.766%	0.454%	0.322%	2.755%
12	0.856%	0.424%	0.354%	2.698%
24	1.167%	0.409%	0.407%	2.510%
36	1.490%	0.461%	0.586%	3.030%
48	1.872%	0.477%	0.917%	3.240%
60	2.136%	0.421%	1.198%	3.084%
72	2.414%	0.421%	1.492%	3.365%
84	2.639%	0.418%	1.675%	3.573%
96	2.826%	0.401%	1.860%	3.734%
108	2.940%	0.375%	2.005%	3.753%
120	3.041%	0.368%	2.109%	3.771%

Table A.2: Summary of basic statistical properties of swap rates(from 5/11/2008 to 2/12/2010(in total 577 trading days))

Months to			Yield Range	
maturity	Mean	Standard Deviation	Min	Max
1	0.869%	0.836%	0.296%	4.611%
2	0.991%	0.862%	0.371%	4.696%
3	1.154%	0.835%	0.428%	4.410%
6	1.387%	0.729%	0.767%	4.201%
9	1.420%	0.644%	0.783%	3.832%
12	1.420%	0.569%	0.869%	3.483%
15	1.468%	0.529%	0.902%	3.394%
18	1.526%	0.504%	0.929%	3.362%
21	1.627%	0.480%	1.039%	3.377%
24	1.761%	0.450%	1.187%	3.433%
27	1.868%	0.437%	1.263%	3.488%
30	1.951%	0.437%	1.297%	3.533%
33	2.023%	0.441%	1.363%	3.574%
36	2.092%	0.445%	1.393%	3.613%
39	2.163%	0.446%	1.391%	3.652%
42	2.234%	0.445%	1.427%	3.690%
45	2.305%	0.441%	1.484%	3.724%
48	2.373%	0.438%	1.532%	3.757%
51	2.438%	0.434%	1.578%	3.790%
54	2.500%	0.430%	1.623%	3.819%
57	2.560%	0.426%	1.667%	3.848%
60	2.618%	0.423%	1.711%	3.876%
63	2.674%	0.420%	1.755%	3.903%
66	2.728%	0.417%	1.800%	3.928%
69	2.781%	0.414%	1.842%	3.955%
72	2.832%	0.411%	1.884%	3.983%
75	2.881%	0.408%	1.922%	4.014%
78	2.928%	0.406%	1.959%	4.041%
81	2.973%	0.403%	1.996%	4.071%
84	3.016%	0.401%	2.032%	4.101%
87	3.058%	0.399%	2.066%	4.128%
90	3.097%	0.398%	2.099%	4.155%
93	3.135%	0.396%	2.132%	4.180%
96	3.172%	0.395%	2.164%	4.205%
102	3.240%	0.392%	2.223%	4.252%
108	3.304%	0.391%	2.277%	4.300%
114	3.364%	0.390%	2.329%	4.349%
120	3.422%	0.389%	2.377%	4.394%

Table A.3: Basic Properties of Nelson Siegel Interpolation Parameters for Swap Curve

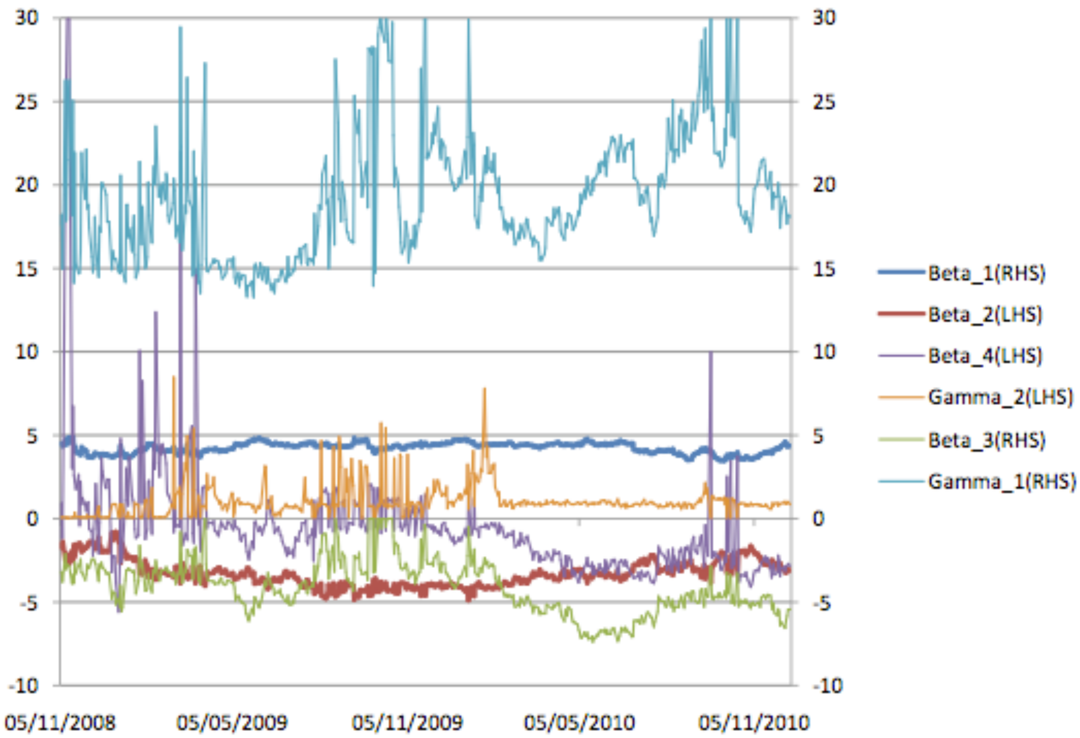
	Mean	**Standard deviation**	**Range**	
			Min	**Max**
Beta$_1$	4.365	0.327	3.199	4.985
Beta$_2$	-2.559	1.346	-4.312	5.009
Beta$_3$	-3.763	1.408	-7.258	0.000
Beta$_4$	-3.364	1.935	-15.648	-0.396
Gamma$_1$	18.785	5.902	6.380	30.000
Gamma$_2$	0.529	0.184	0.026	1.105



Note: The graph shows estimation of Nelson Siegel curve parameters interpolation model. RHS stands for Right Hand Side axis and LHS stands for Left Hand Side Axis

Table A.4: Basic Properties of Nelson Siegel Interpolation Parameters for Treasury Curve

	Mean	Standard Deviation	Range Min	Range Max
Beta$_1$	4.241	0.310	3.468	4.85791
Beta$_2$	-3.289	0.797	-4.878	-0.799
Beta$_3$	-4.101	1.587	-7.393	0.01912
Beta$_4$	-0.541	3.617	-5.595	30
Gamma$_1$	19.338	3.745	13.259	30
Gamma$_2$	1.040	0.998	0.028	8.48653



Note: The graph shows estimation of Nelson Siegel curve parameters interpolation model. RHS stands for Right Hand Side axis and LHS stands for Left Hand Side Axis

B. Computed Hazard Rates and Default Probabilities

Table B.1: Summary of Greek Hazard Rates and Default Probabilities: treasury curve used for discounting

		Hazard Rate		
Maturity	**Mean**	**Standard Deviation**	**Min**	**Max**
6	4.17%	3.65%	0.50%	13.32%
12	4.17%	3.67%	0.50%	13.34%
24	8.22%	6.66%	1.52%	23.32%
36	8.06%	6.20%	2.06%	23.03%
48	7.46%	4.72%	2.44%	19.17%
60	6.92%	3.86%	1.00%	17.83%
84	13.57%	8.69%	3.90%	33.96%
120	16.92%	9.60%	5.33%	43.47%

		Default Probability		
Maturity	**Mean**	**Standard Deviation**	**Min**	**Max**
6	4.02%	3.46%	0.50%	12.47%
12	7.76%	6.56%	0.99%	23.40%
24	14.47%	11.28%	2.49%	39.22%
36	20.34%	14.64%	4.48%	50.04%
48	25.37%	16.53%	6.78%	57.06%
60	29.76%	17.54%	9.26%	63.88%
84	37.18%	19.63%	13.13%	74.02%
120	45.26%	20.50%	17.64%	82.34%

Table B.2: Summary of Greek Hazard Rates and Greek Default Probabilities: swap curve used for discounting

		Hazard Rate		
Maturity	**Mean**	**Standard Deviation**	**Min**	**Max**
6	4.17%	3.65%	0.50%	13.32%
12	4.17%	3.67%	0.50%	13.33%
24	8.22%	6.66%	1.52%	23.29%
36	8.06%	6.20%	2.07%	23.05%
48	7.45%	4.71%	2.45%	19.15%
60	6.91%	3.84%	0.88%	17.79%
84	13.55%	8.66%	3.90%	33.89%
120	16.86%	9.55%	5.32%	43.30%

		Default Probabilitie		
Maturity	**Mean**	**Standard Deviation**	**Min**	**Max**
6	4.02%	3.46%	0.50%	12.47%
12	7.76%	6.56%	0.99%	23.39%
24	14.47%	11.28%	2.49%	39.19%
36	20.34%	14.64%	4.49%	49.99%
48	25.37%	16.52%	6.79%	56.97%
60	29.76%	17.52%	9.27%	63.85%
84	37.17%	19.60%	13.15%	73.97%
120	45.23%	20.48%	17.65%	82.28%

Table B.3: Summary of Czech Hazard Rates: swap curve used for discounting

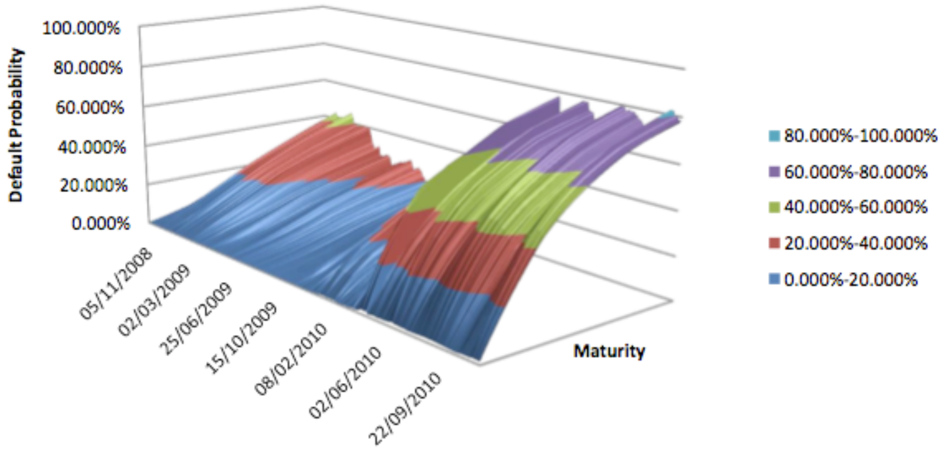
	Average	**Standard**	**Range**	
Maturity (months)		**Deviation**	**Min**	**Max**
6	0.755%	0.630%	0.163%	3.079%
12	0.927%	0.623%	0.316%	3.172%
24	2.290%	1.276%	0.870%	6.807%
36	2.785%	1.340%	1.423%	8.433%
48	2.988%	1.422%	1.430%	9.396%
60	3.100%	1.274%	9.099%	1.668%
84	5.257%	1.901%	3.060%	13.027%
120	8.104%	2.497%	5.135%	18.696%

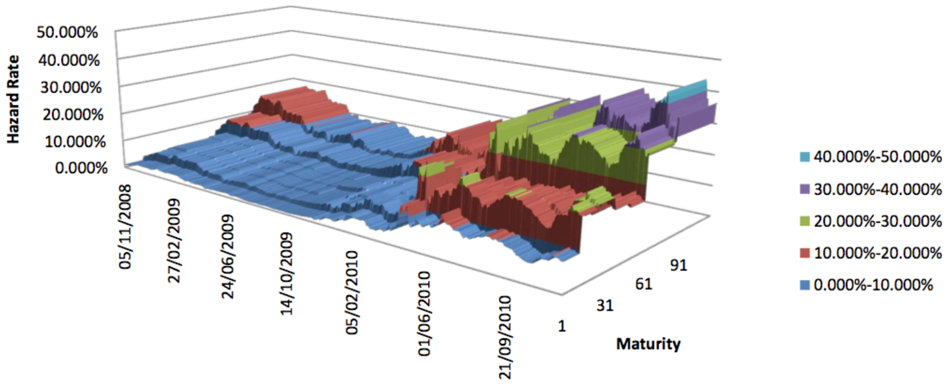
Table B.4: Summary of Czech Default Probabilities: swap curve used for discounting

	Average	**Standard**	**Range**	
Maturity (months)		**Deviation**	**Min**	**Max**
6	0.750%	0.622%	0.000%	3.032%
12	1.660%	1.217%	0.478%	6.060%
24	3.864%	2.364%	1.512%	12.078%
36	6.467%	3.459%	2.903%	18.300%
48	9.165%	4.514%	4.281%	25.627%
60	11.877%	5.359%	32.095%	5.864%
84	16.282%	6.447%	8.701%	40.389%
120	22.635%	7.529%	13.271%	50.554%

C. Charts

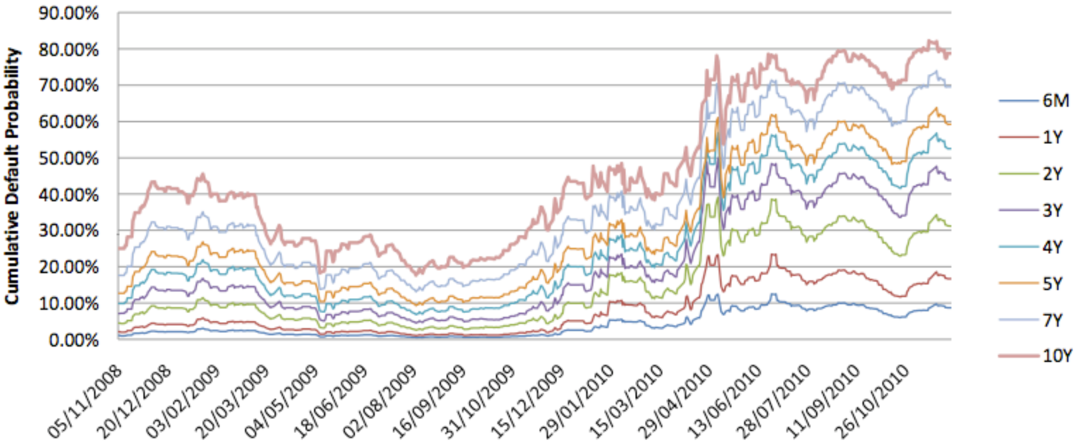
Figure C.1: Greek Default Probability Term Structure with Corresponding Hazard Rate Term Ttruture





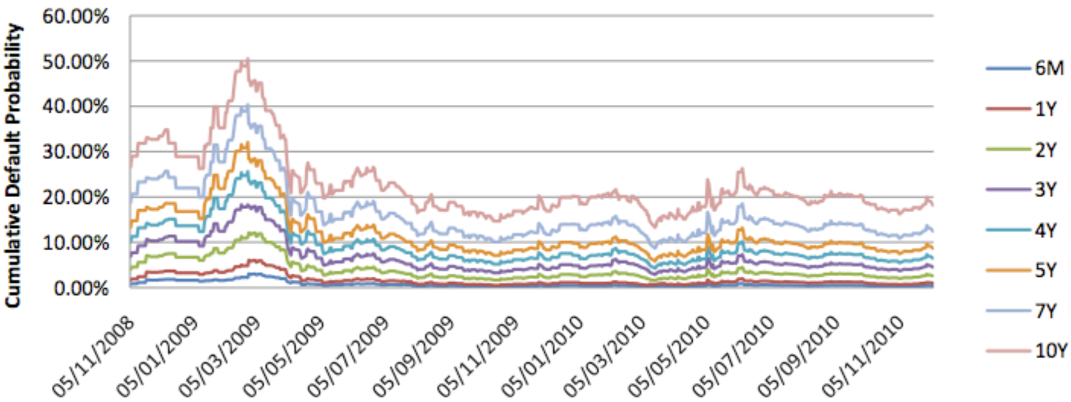
Note:here is used treasury swap curve as a benchmark curve. These two pictures are showing risk neutral default probability term structure and corresponding hazard rates as implied by the model

Figure C.2: Development of Greek Cumulative Default Probabilities



Implied cumulative risk neutral default probabilities for maturities from 6 months to 10 year. Recovery rate is 54% and swap curve is used as a benchmark default free curve, in this case

Figure C.3: Development of Czech Cumulative Default Probabilities



Implied cumulative risk neutral default probabilities for maturities from 6 months to 10 year. Recovery rate is 54% and swap curve is used as a benchmark default free curve, in this case

List of Tables

List of Abbreviations

BVA Bilateral valuation adjustment

CDO Collateralized debt obligations

CDS Credit default swap

CVA Credit valuation adjustment

DVA Debit valuation adjustment

IRS Interest rate swap

LIBOR The London Inter Bank Offered Rate

OTC Over the counter

Refcorp Resolution Funding Corporation