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Nerovnosti pro integrální operátory

Katedra matematické analýzy

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Název práce: Nerovnosti pro integrální operátory

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Abstrakt: Předložená práce obsahuje shrnutí dosud známých výsledků o operátorových nerovnostech typu „good λ", „better good λ" a „rearranged good λ" na prostorech funkcí nad Eukleidovským prostorem s Lebesgueovou mírou a jejich důsledky, v podobě složitějších operátorových nerovností a normových odhadů na Lebesguevých prostorech. Hlavním cílem práce ovšem je vybudovat podobnou teorii pro operátor Rieszova potenciálu na prostorech funkcí nad kvazi-metrickým prostorem s takzvanou „zdvojovací" mírou. Kombinací důsledků této teorie s již známými normovými odhady dostáváme omezenost operátoru Rieszova potenciálu na Lebesguesových a Lorentzových prostorech.

Klíčová slova: good λ-nerovnost, better good λ-nerovnost, Lebesgueův prostor, Lorentzův prostor, Rieszův potenciál, frakční maximální operátor.

Title: Inequalities for integral operators

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Abstract: The presented work contains a survey of the so far known results about the operator inequalities of the type "good λ", "better good λ" and "rearranged good λ" on the function spaces over the Euclidean space with the Lebesgue measure and their corollaries in the form of more complex operator inequalities and norm estimates. However, the main aim is to build similar theory for the Riesz potential operator on the function spaces over the quasi-metric space with the so-called "doubling" measure. Combining the corollaries of this theory with the known norm estimates we obtain the boundedness for the Riesz potential operator on the Lebesgue and Lorentz spaces.

Keywords: good λ-inequality, better good λ-inequality, Lebesgue space, Lorentz space, Riesz potential, fractional maximal operator.

Contents

Preface

All the results in this work are based on the paper of D.L. Burkholder and R.F. Gundy and papers of many other authors, who continued their work and proved various extensions. All of them worked with the basic idea of relating two operators, the first of which is “good” in a certain sense, while the other one is “bad”, by the suitable inequality, that allows to transfer some of the properties of the good operator to the bad one. This kind of inequalities was later called the good λ-inequalities.

The aim of this thesis is to compile and generalize scattered results about various λ-inequalities and norm estimates for an integral operator T and a corresponding maximal operator M on the function spaces over the quasi-metric spaces with the so called “doubling” measure and supplement it with some new results extended from function spaces over the metric spaces or the n-dimensional Euclidean space $\mathbb{R}^n$. Unlike $\mathbb{R}^n$, where T is considered to be quite general, we restrict ourselves to the case when T equals to a particular form of the Riesz potencial.

The first chapter contains definitions of some fundamental objects and an introduction to the theory of Banach function spaces with a particular focus on the Lebesgue and Lorentz spaces and after it as well their weighted variants. The rest of the chapter is devoted to the various covering lemmas, whose use is essential in the third chapter.

In the second chapter we compile the previously known results for the function spaces over $\mathbb{R}^n$, i.e. we formulate three important λ-inequalities for the Lebesgue measurable functions on $\mathbb{R}^n$ along with their corollaries which mostly concern various norm estimates for the Lebesgue spaces.

The third and main chapter defines the Riesz potential I and, in this case, a corresponding pair of fractional maximal operators on the function spaces over the quasi-metric space with the doubling measure. The main aim is the proof of the so-called “better good λ-inequality”, which allows to derive the other two λ-inequalities very easily and from them also a few corollaries in the form of norm estimates between I and one of the fractional maximal operators. The last section of this chapter extends the last mentioned norm estimates and survey the boundedness of the operator I on the Lebesgue and Lorentz spaces.

Regrettably, a generalization to the quasi-metric space brings some complications that does not enable us to run an analogical progress as in $\mathbb{R}^n$. Some of the cases are solved by adjusting the method of proof or by creating a different one but a few results known in $\mathbb{R}^n$ still remain open in general.

1. Notation and preliminaries

1.1 Banach function spaces

In this section we introduce the idea of a quasi-metric space with the so-called "doubling" measure which is a measure that satisfies two growth conditions. On this slightly generalized metric space we build a theory of Banach function spaces which provides a general setting for all function spaces we need.

Definition 1.1.1 (quasi-metric space with a doubling measure) Let X be a set endowed with a function $\varrho : X \times X \to \mathbb{R}$ satisfying

(i) $\varrho(x, y) \geq 0$,

(ii) $\varrho(x, y) = 0$ if and only if $x = y$,

(iii) $\varrho(x, y) = \varrho(y, x)$, (symmetry)

(iv) $\varrho(x, y) \leq d(\varrho(x, z) + \varrho(z, y))$ (d-relaxed triangle inequality)

for every $x, y, z \in X$ and some $d \geq 1$. The function ϱ is called a quasi-metric and the pair (X, ϱ) denotes a quasi-metric space. Then for the quasi-metric space (X, ϱ), $x \in X$ and a set $E \subset X$ we define the distance

$$\operatorname{dist}(x, E) = \inf_{y \in E} \varrho(x, y)$$

and the diameter

$$\operatorname{diam}(E) = \sup_{x,y \in E} \varrho(x, y).$$

Let $x \in X$ and $0 < r < 2 \operatorname{diam}(X)$. Then we define the ball $B(x, r)$ by

$$B(x, r) = \{y \in X : \varrho(x, y) < r\}.$$

Moreover we say that a set $E \subset X$ is bounded whenever there is a ball B such that $E \subset B$.

We further assume that there is a non-negative outer Borel-regular measure μ defined on (X, ϱ), i.e. such a measure μ on the quasi-metric space (X, ϱ) that all Borel sets are μ-measurable and for every set A there is a Borel set B such that $A \subset B$ and $\mu(A) = \mu(B)$, satisfying the following two properties:

(i) there is a doubling constant $D \geq 1$ such that for every $0 < r < \operatorname{diam}(X)$ and every $x \in X$ we have

$$\mu(B(x, 2r)) \leq D\mu(B(x, r)), \qquad \text{(doubling condition)}$$

(ii) there are two constants $C_\mu > 0$ and $n > 1$ such that for every $x \in X$ and every $0 < r < 2 \operatorname{diam}(X)$ we have

$$\mu(B(x, r)) \geq C_\mu r^n. \qquad \text{(lower bound condition)}$$

A non-negative outer Borel-regular measure satisfying the previous two conditions is called a doubling measure and (X, ϱ, μ) denotes a quasi-metric space X with a doubling measure μ, which will be further referred to as a space of homogeneous type.

Lemma 1.1.2 *Let (X, ϱ, μ) be a space of homogeneous type. Then for every $k > 1$ there is a constant D_k such that*

$$\mu(B(x, kr)) \leq D_k \mu(B(x, r))$$

for every $x \in X$ and $0 < r < \frac{2}{k} \operatorname{diam}(X)$.

Proof. For fixed k we find $m \in \mathbb{N}$ such that $2^{m-1} < k \leq 2^m$. Then, using the doubling condition, we obtain

$$\mu(B(x, kr)) \leq \mu(B(x, 2^m r)) \leq D^m \mu(B(x, r)).$$

Thus it suffices to set $D_k = D^m$. □

Remark: The letter D with the various subscripts will further denote the powers of the doubling constant D corresponding to the multiple of the radius in the same way as in the previous Lemma 1.1.2. Note that $D_2 = D$.

Remark: Let (X, ϱ, μ) be a space of homogeneous type, then

(i) every ball has a strictly positive measure,
(ii) if there is a ball $B \subset X$ with $\mu(B) = \infty$, then every ball has infinite measure.

Proof. The part (i) is a trivial consequence of the lower bound condition.
For the part (ii) let $B = B(x, r)$, $x \in X$, $r > 0$, be a fixed ball with $\mu(B) = \infty$ and let $B' = B(x', r')$, $x' \in X$, $0 < r' < 2 \operatorname{diam}(X)$, be an arbitrary ball. Then there is $k \geq 1$ such that $B \subset B(x', kr')$ and by Lemma 1.1.2 there is $D_k > 0$ such that $D_k \mu(B') \geq \mu(B(x', kr'))$. Therefore

$$\mu(B') \geq \frac{1}{D_k} \mu(B(x', kr')) \geq \frac{1}{D_k} \mu(B) = \infty.$$

□

In the view of (ii) in the last remark we omit the deformed spaces and further assume that every ball has finite measure.

Lemma 1.1.3 *Let (X, ϱ, μ) be a space of homogeneous type, then*

(i) *the measure μ is σ-finite, in particular $X = \bigcup_{j=1}^{\infty} B_j$, where B_j are balls with $\mu(B_j) < \infty$,*
(ii) *X is a separable space.*

Proof. For the part (i) let $x \in X$.
If $\mu(X) < \infty$, then the space X has finite diameter, otherwise, using the lower bound condition, we can find a ball B in X with radius big enough such that $\mu(B) > \mu(X)$ and that is a contradiction. Thus the ball $B(x, r)$, where $r > \operatorname{diam}(X)$, contains whole X and has finite measure.
If $\mu(X) = \infty$, then $\operatorname{diam}(X)$ is obviously infinite and $\{B(x, r)\}_{r \in \mathbb{N}}$ is a countable collection of open concentric balls in X such that $X = \bigcup_{r \in \mathbb{N}} B(x, r)$ and $\mu(B(x, r)) < \infty$ for every $r \in \mathbb{N}$.

In the part (ii), using (i), we deduce that

$$X = \bigcup_{j=1}^{\infty} B(x_j, r_j),$$

where $B_j = B(x_j, r_j)$, $x_j \in X$ and $r_j > 0$. Then the countability of $\{B_j\}_{j\in\mathbb{N}}$ implies that it suffices to prove that for every j and $\varepsilon > 0$ there is a finite ε-net $A_j^\varepsilon \subset B_j$, i.e. for every $x \in B_j$ there is $y \in A_j^\varepsilon$ with $\varrho(x, y) < \varepsilon$. Let $\varepsilon > 0$.

If $\varepsilon \geq r_j$, then we set $A_j^\varepsilon = \{x_j\}$ and we are finished. If $\varepsilon < r_j$, then the collection $\{B(x, \frac{\varepsilon}{2d})\}_{x\in B_j}$ covers B_j and for every $x \in B_j$ and every $z \in B(x, \frac{\varepsilon}{2d})$ we have

$$\varrho(z, x_j) \leq d(\varrho(z, x) + \varrho(x, x_j)) < d(\frac{\varepsilon}{2d} + r_j) < d(\frac{r_j}{2d} + r_j) < 2dr_j,$$

which implies $B(x, \frac{\varepsilon}{2d}) \subset B(x_j, 2dr_j)$ for every $x \in B_j$. Now we inductively construct the set A_j^ε in the following way.

- For $k = 1$ we choose any $x_j^1 \in B_j$ and put it into A_j^ε.
- For a positive integer $k > 1$, either there exists $x_j^k \in B_j$ such that

  $$B(x_j^k, \frac{\varepsilon}{2d}) \cap B(x_j^i, \frac{\varepsilon}{2d}) = \emptyset$$

  for every $i = 1, \dots, k-1$ and we put it into A_j^ε, or the construction stops.

Since the ball $B(x_j, 2dr_j)$ has finite measure, $B(x, \frac{\varepsilon}{2d}) \subset B(x_j, 2dr_j)$ for every $x \in B_j$ and $\mu(B(x, \frac{\varepsilon}{2d})) \geq C_\mu \left(\frac{\varepsilon}{2d}\right)^n$ for every $x \in X$, the construction has to stop for a finite k. Thus the set A_j^ε is finite. Now $\{B(x, \varepsilon)\}_{x\in A_j^\varepsilon}$ covers B_j, otherwise there is $z \in B_j$ such that for every $x \in A_j^\varepsilon$ we have

$$\varepsilon \leq \varrho(z, x) \leq d(\operatorname{dist}(z, B(x, \frac{\varepsilon}{2d})) + \frac{\varepsilon}{2d}) \quad \Rightarrow \quad \operatorname{dist}(z, B(x, \frac{\varepsilon}{2d})) \geq \frac{\varepsilon}{2d}$$

and thus

$$B(z, \frac{\varepsilon}{2d}) \cap B(x, \frac{\varepsilon}{2d}) = \emptyset,$$

which is the contradiction with the construction of A_j^ε. Hence A_j^ε is a finite ε-net of B_j and taking $\varepsilon_m = \frac{1}{2^m}, m \in \mathbb{N}$, we have that $\bigcup_{m=1}^{\infty} \bigcup_{j=1}^{\infty} A_j^{\varepsilon_m}$ is a countable dense subset of X. □

Definition 1.1.4 (Banach function space) Let (X, ϱ, μ) be a space of homogeneous type and let $\mathcal{M} = \mathcal{M}(X, \varrho, \mu)$ be the space of (equivalence classes of) μ-measurable functions on X.

Then a function norm on $\mathcal{M}$ is a function $\varphi : \mathcal{M} \to [0, \infty]$ satisfying the following properties:

(BF_1) $\varphi(f) = 0$ if and only if $f = 0$,
(BF_2) $\varphi(\alpha f) = |\alpha|\, \varphi(f)$ for every $\alpha \neq 0$,
(BF_3) $\varphi(f + g) \leq \varphi(f) + \varphi(g)$,
(BF_4) $|f| \leq |g|$ μ-a.e. implies $\varphi(f) \leq \varphi(g)$,
(BF_5) $0 \leq f_m \nearrow f$ μ-a.e. implies $\varphi(f_m) \nearrow \varphi(f)$,
(BF_6) $\varphi(\chi_E) < \infty$ for every μ-measurable set $E \subset X$, where $\mu(E) < \infty$,

(BF$_7$) for every $f \in \mathcal{M}$ and every μ-measurable set $E \subset X$ with $\mu(E) < \infty$, there is a constant C such that

$$\int_E |f| \, d\mu \leq C\varphi(f)$$

for all $f, g, f_m \in \mathcal{M}$, $m \in \mathbb{N}$, and $\alpha \in \mathbb{R}$. If φ is a function norm, then the space $Z = \{f \in \mathcal{M}, \varphi(f) < \infty\}$ is called a Banach function space. For $f \in Z$, we use $\|f\|_Z$ instead of $\varphi(f)$.

Remark: The above mentioned equivalence is meant as equality of functions almost everywhere.

Even though a Banach function space is a collection of functions $f \in \mathcal{M}$ such that the function norm of f, $\varphi(f)$, is finite, a function norm in general is defined for all $f \in \mathcal{M}$.

Remark: The letter C, without any subscript or superscript, will throughout this thesis denote an universal constant. It may change its value between the theorems and if it cannot cause any misunderstanding even from line to line.

We finish this section with two examples of classes of Banach function spaces.

Definition 1.1.5 (Lebesgue spaces) Let (X, ϱ, μ) be a space of homogeneous type, then for $1 \leq p \leq \infty$ and a μ-measurable function f on X we define the Lebesgue norm

$$\|f\|_p = \begin{cases} \left(\int_X |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \operatorname*{esssup}_{x \in X}\{|f(x)|\}, & p = \infty. \end{cases}$$

Then the Lebesgue space $L^p(X, \varrho, \mu)$ is a space consisting of all μ-measurable functions f on X for which the Lebesgue norm $\|f\|_p$ is finite. We further denote the space $L^p(X, \varrho, \mu)$ by L^p.

We also define the space of locally integrable functions $L^1_{\mathrm{loc}}(X, \varrho, \mu) = L^1_{\mathrm{loc}}$ consisting of μ-measurable functions f, where $f \in L^1(K)$ for every compact subset K of X.

Definition 1.1.6 (Lorentz spaces) Let (X, ϱ, μ) be a space of homogeneous type, then for $1 \leq p, q \leq \infty$ and a μ-measurable function f on X we define the Lorentz norm

$$\|f\|_{p,q} = \begin{cases} \left(\int_0^\infty q s^{q-1} \left(\mu(\{x \in X : |f(x)| > s\})\right)^{\frac{q}{p}} ds\right)^{\frac{1}{q}}, & 1 \leq p, q < \infty, \\ \sup_{s > 0}\{s \left(\mu(\{x \in X : |f(x)| > s\})\right)^{\frac{1}{p}}\}, & 1 \leq p \leq \infty, \ q = \infty. \end{cases}$$

Then the Lorentz space $L^{p,q}(X, \varrho, \mu)$ is a space consisting of all μ-measurable functions f on X for which the Lorentz norm $\|f\|_{p,q}$ is finite. We further denote the space $L^{p,q}(X, \varrho, \mu)$ by $L^{p,q}$.

Remark: Let (X, ϱ, μ) be a space of homogeneous type. Then

(i) the Lebesgue norms $\|.\|_p$, $1 \le p \le \infty$, are function norms,
(ii) the Lorentz norms $\|.\|_{p,q}$, $1 \le q \le p < \infty$, are function norms,
(iii) for every $1 < p < \infty$ and $1 \le q \le \infty$ there is another norm $\|.\|_X$ equivalent to the Lorentz norm $\|.\|_{p,q}$, i.e. there are two constants $C_l, C_u > 0$ satisfying $C_l \|.\|_{p,q} \le \|.\|_X \le C_u \|.\|_{p,q}$, such that the space $L^{p,q}$ with $\|.\|_X$ is a Banach function space,
(iv) the Lorentz space $L^{p,p} = L^p$ for every $1 \le p \le \infty$ and if we considered $L^{\infty,q}$ for every $1 \le q < \infty$, then $L^{\infty,q}$ would contain only zero function,
(v) for every $1 < q \le \infty$ the Lorentz norm $\|.\|_{1,q}$ is only a quasi-norm and there exists no norm $\|.\|_X$ equivalent to $\|.\|_{1,q}$ as in (iii).

Proof. For (i) – (iv) see e.g. Chapter 1 and Chapter 4 in Bennett and Sharpley [5] and for (v) see [12]. □

1.2 Weights and rearrangements

For both classes of function spaces defined in the previous section there is a weighted variant, but before we introduce the weighted Lebesgue and Lorentz spaces we need to recall a few facts about weights and also rearrangements. The Lorentz spaces are closely associated with the distributions and rearrangements and therefore they can be introduced by the means of the non-increasing rearrangement or the distribution function. We consider both approaches and prove their equivalence.

Definition 1.2.1 (weights and Muckenhoupt weight classes) Let (X, ϱ, μ) be a space of homogeneous type. Then a weight is a positive μ-measurable function $w \in L^1_{\text{loc}}$ and we set

$$w(E) = \int_E w(x) d\mu(x)$$

for any μ-measurable set $E \subset X$. Moreover we say that

(i) w belongs to the Muckenhoupt A_∞ class, $w \in A_\infty$, if for given $0 < \varepsilon < 1$, there exists $0 < \varepsilon' < 1$ such that for every ball B from X and every μ-measurable set $E \subset B$ we have

$$\mu(E) < \varepsilon' \mu(B) \Rightarrow w(E) < \varepsilon w(B),$$

(ii) w belongs to the Muckenhoupt A'_∞ class, $w \in A'_\infty$, if there are constants $C_{A'_\infty} \ge 1$ and $\delta \ge 1$ such that for every ball B from X and every μ-measurable set $E \subset B$ we have

$$\frac{\mu(E)}{\mu(B)} \le C_{A'_\infty} \left(\frac{w(E)}{w(B)} \right)^\delta,$$

(iii) w belongs to the Muckenhoupt A_p class, $w \in A_p$, $1 < p < \infty$, if there is a positive constant C_{A_p} such that for every ball B from X we have

$$\left(\int_B w(x) d\mu(x) \right) \left(\int_B w(x)^{-\frac{1}{p-1}} d\mu(x) \right)^{p-1} \le C_{A_p} \mu(B)^p.$$

(iv) w belongs to the Muckenhoupt A_1 class, $w \in A_1$, if there is a positive constant C_{A_1} such that for every ball B from X and $x \in B$ we have

$$\frac{1}{\mu(B)} \int_B w(x) d\mu(x) \leq C_{A_1} \operatorname*{essinf}_{x \in B} w(x).$$

Before we formulate and prove lemma that shows some relations between the A_p and A_∞ classes, we need to adopt one theorem from [18] (Theorem 13 on page 8). For the easier application and better understanding we reformulate it to the form more suitable for our purpose. This modification is only a slightly less general version of the original theorem and thus can be proved in the same way.

Theorem 1.2.2 *Let (X, ϱ) be a quasi-metric space and suppose that μ and ν are two measures on (X, ϱ) satisfying the doubling condition and that there are $0 < \varepsilon_0, \varepsilon'_0 < 1$ such that for each ball $B \subset X$ and each μ-measurable set $E \subset B$ we have $\nu(E) < (1 - \varepsilon'_0)\nu(B)$ whenever $\mu(E) < \varepsilon_0 \mu(B)$. Then there are a constant $C > 0$ and an index $\delta \geq 1$ such that*

$$\frac{\nu(E)}{\nu(B)} \leq C \left(\frac{\mu(E)}{\mu(B)} \right)^{\frac{1}{\delta}}$$

for each ball $B \subset X$ and each μ-measurable set $E \subset B$.

Proof. See [18], pages 16 and 17. □

Lemma 1.2.3 *Let (X, ϱ, μ) be a space of homogeneous type. Then for a weight w and $1 < p < \infty$ we have the following set of implications:*

$$w \in A_p \Rightarrow w \in A'_\infty \Rightarrow w \in A_\infty.$$

Proof. Let $1 < p < \infty$ and let $w \in A_p$. Then, using the Hölder inequality, we obtain

$$\begin{aligned} \mu(E) &= \int_B \chi_E(y) w(y)^{\frac{1}{p}} w(y)^{-\frac{1}{p}} d\mu(y) \\ &\leq \left(\int_E w(y) d\mu(y) \right)^{\frac{1}{p}} \left(\int_B w(y)^{-\frac{1}{p-1}} d\mu(y) \right)^{\frac{p-1}{p}} \\ &\leq C_{A_p}^{\frac{1}{p}} w(E)^{\frac{1}{p}} \mu(B) w(B)^{-\frac{1}{p}} \end{aligned}$$

for any ball $B \subset X$, μ-measurable set $E \subset B$ and $1 < p < \infty$. Notice that the A_p condition was used in the last estimate and therefore indeed $w \in A_p$ implies $w \in A'_\infty$ with $\delta = \frac{1}{p}$ and $C_{A'_\infty} = C_{A_p}^{1/p}$.

For the proof of the second implication let $w \in A'_\infty$.Then, using the doubling condition of μ and $w \in A'_\infty$, we have

$$C_{A'_\infty} \left(\frac{w(B(x,r))}{w(B(x,2r))} \right)^\delta \geq \frac{\mu(B(x,r))}{\mu(B(x,2r))} \geq \frac{1}{D}$$

for any $x \in X$ and $r > 0$. Thus

$$(DC_{A'_\infty})^{1/\delta} w(B(x,r)) \geq w(B(x,2r))$$

and hence the measure ν defined by $d\nu(x) = w(x)d\mu(x)$ also satisfies the doubling condition with doubling constant $(DC_{A'_\infty})^{1/\delta}$.

From $w \in A'_\infty$ also follows that

$$\frac{\mu(B \setminus E)}{\mu(B)} \le C_{A'_\infty} \left(\frac{w(B \setminus E)}{w(B)} \right)^\delta \tag{1.1}$$

for each ball $B \subset X$ and μ-measurable set $E \subset B$. Hence, if we suppose that $\mu(E) < \varepsilon_0 \mu(B)$ for an arbitrary $0 < \varepsilon_0 < 1$, then certainly $(1 - \varepsilon_0)\mu(B) < \mu(B \setminus E)$. Now, using (1.1), we obtain

$$(1 - \varepsilon_0) < \frac{\mu(B \setminus E)}{\mu(B)} \le C_{A'_\infty} \left(\frac{w(B \setminus E)}{w(B)} \right)^\delta .$$

Thus $(1 - \varepsilon_0)^{\frac{1}{\delta}} C_{A'_\infty}^{-\frac{1}{\delta}} w(B) < w(B \setminus E)$ which implies

$$\nu(E) = w(E) < (1 - (1 - \varepsilon_0)^{\frac{1}{\delta}} C_{A'_\infty}^{-\frac{1}{\delta}}) w(B) = (1 - \varepsilon'_0)\nu(B),$$

where $\varepsilon'_0 = (1 - \varepsilon_0)^{\frac{1}{\delta}} C_{A'_\infty}^{-\frac{1}{\delta}}$. Evidently $0 < \varepsilon'_0 < 1$ and therefore we have satisfied all the assumptions of Theorem 1.2.2. Applying it we obtain

$$\frac{w(E)}{w(B)} = \frac{\nu(E)}{\nu(B)} \le C \left(\frac{\mu(E)}{\mu(B)} \right)^{\frac{1}{\delta}} \tag{1.2}$$

for each ball $B \subset X$ and each μ-measurable set $E \subset B$.

Now let $0 < \varepsilon < 1$, let $B \subset X$ be a ball and let $E \subset B$ be a μ-measurable set. Let $\mu(E) < \varepsilon' \mu(B)$, where $0 < \varepsilon' < 1$ with exact value to be specified later, then from (1.2) follows

$$\frac{w(E)}{w(B)} \le C \left(\frac{\mu(E)}{\mu(B)} \right)^{\frac{1}{\delta}} < C \varepsilon'^{\frac{1}{\delta}}.$$

Thus by setting ε' small enough such that $C\varepsilon'^{\frac{1}{\delta}} < \varepsilon$ we proved that $w \in A_\infty$.

□

Definition 1.2.4 (distribution and rearrangement) Let (X, ϱ, μ) be a space of homogeneous type, f be a μ-measurable function on X and let w be a weight. Then for $s > 0$ we define the distribution function of f with respect to w by

$$D_{f,w}(s) = w(\{x \in X : |f(x)| > s\}).$$

Furthermore, for $t > 0$ we define the non-increasing rearrangement of f with respect to w by

$$f^*_w(t) = \inf\{s > 0 : D_{f,w}(s) \le t\}$$

and the averaged rearrangement of f with respect to w by

$$f^{**}_w(t) = \frac{1}{t} \int_0^t f^*_w(s) ds.$$

Definition 1.2.5 (weighted Lebesgue spaces) Let (X, ϱ, μ) be a space of homogeneous type, then for a weight w, a μ-measurable function f on X and $1 \le p \le \infty$ we define the weighted Lebesgue norm

$$\|f\|_{p,w} = \begin{cases} \left(\int_X |f(x)|^p \, w(x) d\mu(x)\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \operatorname*{esssup}_{x \in X}\{|f(x)| \, w(x)\}, & p = \infty. \end{cases}$$

Then the weighted Lebesgue space $L^p_w(X, \varrho, \mu)$ consists of all μ-measurable functions f on X for which the weighted Lebesgue norm $\|f\|_{p,w}$ is finite. We further denote the space $L^p_w(X, \varrho, \mu)$ by L^p_w.

Definition 1.2.6 (weighted Lorentz spaces) Let (X, ϱ, μ) be a space of homogeneous type, then for a weight w, a μ-measurable function f on X and $1 \le p, q \le \infty$ we define the weighted Lorentz (quasi-)norm

$$\|f\|_{p,q,w} = \begin{cases} \left(\int_0^\infty q s^{q-1} \left(D_{f,w}(s)\right)^{\frac{q}{p}} ds\right)^{\frac{1}{q}}, & 1 \le p < \infty, \ 1 \le q < \infty, \\ \sup_{s>0}\{s \left(D_{f,w}(s)\right)^{\frac{1}{p}}\}, & 1 \le p \le \infty, \ q = \infty, \end{cases}$$

or, in the terms of the rearrangement,

$$\|f\|_{p,q,w} = \begin{cases} \left(\int_0^\infty \frac{q}{p} t^{\frac{q}{p}-1} \left(f^*_w(t)\right)^q dt\right)^{\frac{1}{q}}, & 1 \le p < \infty, \ 1 \le q < \infty, \\ \sup_{t>0}\{t^{\frac{1}{p}} f^*_w(t)\}, & 1 \le p \le \infty, \ q = \infty. \end{cases}$$

Then the weighted Lorentz space $L^{p,q}_w(X, \varrho, \mu)$ consists of all μ-measurable functions f on X for which the weighted Lorentz norm $\|f\|_{p,q,w}$ is finite. We further denote the space $L^{p,q}_w(X, \varrho, \mu)$ by $L^{p,q}_w$.

Remark: The previous two definitions are equivalent.

Proof. For $q = \infty$ let $f_m = \sum_{j=1}^m c_j \chi_{E_j}$, $m \in \mathbb{N}$, be a simple function, where E_j are w-measurable sets, $\mu(E_j) > 0$ for every $j = 1, \ldots, m$, $E_j \cap E_k = \emptyset$ if $j \ne k$ and $c_1 > c_2 > \ldots > c_m > c_{m+1} = 0$. Let $a_j = w(E_1) + \ldots + w(E_j)$, $1 \le j \le m$, and define a_0 to be 0. Then the distribution function $D_{f_m,w}$ has the form

$$D_{f_m,w}(s) = \begin{cases} a_j, & c_{j+1} \le s < c_j, \ 1 \le j \le m, \\ 0, & c_1 \le s, \end{cases}$$

and the non-increasing rearrangement $(f_m)^*_w$ has the form

$$(f_m)^*_w(t) = \begin{cases} c_j, & a_{j-1} \le t < a_j, \ 1 \le j \le m, \\ 0, & a_m \le t. \end{cases}$$

Thus we see that for $p > 0$ we have

$$\sup_{s>0}\{s(D_{f_m,w}(s))^{\frac{1}{p}}\} = \sup_{1 \le j \le m}\{a_j^{\frac{1}{p}} c_j\} = \sup_{t>0}\{t^{\frac{1}{p}} (f_m)^*_w(t)\}. \tag{1.3}$$

It is easy to see that for every μ-measurable function f there is a sequence of simple function f_m, defined as above, such that $f_m(x) \nearrow |f(x)|$ for every $x \in X$. It is also clear that

$$E_s^m = \{x \in X : |f_m(x)| > s\} \subset E_s = \{x \in X : |f(x)| > s\}$$

and $\bigcup_{m=1}^{\infty} E_s^m = E_s$ for every $s > 0$. Thus also

$$D_{f_m,w}(s) = w(E_s^m) \leq w(E_s) = D_{f,w}(s) \quad \text{and} \quad \lim_{m\to\infty} D_{f_m,w}(s) = D_{f,w}(s) \quad (1.4)$$

for every $s > 0$.

From the definition of the non-increasing rearrangement it now follows that

$$(f_m)_w^*(t) \leq (f_{m+1})_w^*(t) \leq f_w^*(t) \quad (1.5)$$

for $m = 1, 2, 3, \ldots$ and every $t \geq 0$. For fixed $t \geq 0$ let $l = \lim_{m\to\infty}(f_m)_w^*(t)$. Since $(f_m)_w^*(t) \leq l$, we have $D_{f_m,w}(l) \leq D_{f_m,w}((f_m)_w^*(t)) \leq t$. Thus $D_{f,w}(l) = \lim_{m\to\infty} D_{f_m,w}(l) \leq t$ and since $f_w^*(t)$ is infimum over s, where $D_{f,w}(s) \leq t$, then $f_w^*(t) \leq l$. But from the inequality $(f_m)_w^*(t) \leq f_w^*(t)$ letting m tend to infinity we obtain $l \leq f_w^*(t)$. It therefore follows that

$$\lim_{m\to\infty} (f_m)_w^*(t) = l = f_w^*(t). \quad (1.6)$$

Now, using (1.3), (1.4), (1.5) and (1.6), we obtain

$$\begin{aligned}\sup_{s>0}\{s(D_{f,w}(s))^{\frac{1}{p}}\} &= \lim_{m\to\infty} \sup_{s>0}\{s(D_{f_m,w}(s))^{\frac{1}{p}}\} \\ &= \lim_{m\to\infty} \sup_{t>0}\{t^{\frac{1}{p}}(f_m)_w^*(t)\} = \sup_{t>0}\{t^{\frac{1}{p}} f_w^*(t)\}.\end{aligned}$$

For $1 \leq q < \infty$, using the Fubini's theorem, we obtain

$$\begin{aligned}\left(\int_0^\infty \frac{q}{p} t^{\frac{q}{p}-1}(f_w^*(t))^q dt\right)^{\frac{1}{q}} &= \left(\int_0^\infty \frac{q}{p} t^{\frac{q}{p}-1}\left(\int_0^{(f_w^*(t))^q} 1 ds\right) dt\right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty \int_0^\infty \frac{q}{p} t^{\frac{q}{p}-1} \chi_{(0,(f_w^*(t))^q)}(s) ds dt\right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty \int_0^\infty \frac{q}{p} t^{\frac{q}{p}-1} \chi_{(0,(f_w^*(t))^q)}(s) dt ds\right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty \int_0^\infty \frac{q}{p} t^{\frac{q}{p}-1} \chi_{\{t\in\mathbb{R}:(f_w^*(t))^q > s\}}(t) dt ds\right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty \int_0^\infty \frac{q}{p} t^{\frac{q}{p}-1} \chi_{\{t\in\mathbb{R}:(f_w^*(t)) > r\}}(t) dt q r^{q-1} dr\right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty \int_0^\infty \frac{q^2}{p} t^{\frac{q}{p}-1} r^{q-1} \chi_{(0,D_{f,w}(r))}(t) dt dr\right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty \int_0^{D_{f,w}(r)} \frac{q^2}{p} t^{\frac{q}{p}-1} r^{q-1} dt dr\right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty q r^{q-1} D_{f,w}(r)^{\frac{q}{p}} dr\right)^{\frac{1}{q}}.\end{aligned}$$

□

Remark: Let (X, ϱ, μ) be a space of homogeneous type and let w be a weight. Then

(i) the weighted Lebesgue norms $\|.\|_{p,w}$, $1 \le p \le \infty$, are function norms,
(ii) the weighted Lorentz norms $\|.\|_{p,q,w}$, $1 \le q \le p < \infty$, are function norms,
(iii) for every $1 < p < \infty$ and $1 \le q \le \infty$ there is another norm $\|.\|_X$ equivalent to the Lorentz norm $\|.\|_{p,q,w}$, i.e. there are two constants $C_l, C_u > 0$ satisfying $C_l \|.\|_{p,q,w} \le \|.\|_X \le C_u \|.\|_{p,q,w}$, such that the space $L_w^{p,q}$ with $\|.\|_X$ is a Banach function space,
(iv) the weighted Lorentz space $L_w^{p,p} = L_w^p$ for every $1 \le p \le \infty$ and if we considered $L_w^{\infty,q}$ for every $1 \le q < \infty$, then $L_w^{\infty,q}$ would contain only zero function.

Proof. Considering that $d\nu(x) = w(x)d\mu(x)$, $x \in X$, is also a measure we can again refer to [5]. □

Lemma 1.2.7 *Let f be a μ-measurable function on a space of homogeneous type (X, ϱ, μ) and let w be a weight. Then for $1 \le p < \infty$ we have*

$$p \int_0^\infty s^{p-1} D_{f,w}(s)ds = \int_X |f(x)|^p \, w(x)d\mu(x).$$

Proof. By the Fubini's theorem we have

$$\begin{aligned} p \int_0^\infty s^{p-1} D_{f,w}(s)ds &= p \int_0^\infty s^{p-1} \int_{\{|f|>s\}} w(x)d\mu(x)ds \\ &= p \int_0^\infty \int_{\{|f|>s\}} s^{p-1} w(x)d\mu(x)ds \\ &= p \int_X \int_0^{|f(x)|} s^{p-1} w(x)dsd\mu(x) \\ &= \int_X |f(x)|^p \, w(x)d\mu(x). \end{aligned}$$

□

1.3 Coverings

There are many versions of various covering lemmas and theorems with different conditions and assertions. In this section we specify and prove those that will be needed further. For reference see e.g. [1], [4], [10] and [14].

Lemma 1.3.1 *Let (X, ϱ, μ) be a space of homogeneous type and let E be a bounded subset of X. Moreover assume that for every $x \in E$ there exists a pair (y_x, r_x), $y_x \in X$ and $r_x > 0$, such that the ball $B(y_x, r_x)$ contains x and $\sup_{x \in E} \varrho(x, y_x)$ is finite. Then there is a sequence of disjoint balls $\{B(y_{x_i}, r_{x_i})\}_{i=1}^\infty \subset \{B(y_x, r_x)\}_{x \in E}$ such that*

$$E \subset \bigcup_{i=1}^\infty B(y_{x_i}, 5d^2 r_{x_i}).$$

Proof. The set E is bounded, hence either we can suppose that there is at least one ball from the collection $\{B(y_x, r_x)\}_{x\in E}$ containing E, which would finish the proof, or that $\sup_{x\in E} r_x$ is finite. The latter case implies the existence of a ball B such that $B(y_x, r_x) \subset B$ for every $x \in E$.

Now we inductively construct a sequence $\{x_k\}_{k\in\mathbb{N}} \subset E$ in the following way. For $k = 1$ we find $x_1 \in E$ such that

$$r_{x_1} > \frac{1}{2} \sup_{x\in E} r_x.$$

Then certainly $x_1 \in B(y_{x_1}, r_{x_1})$. For a positive integer $k > 1$ we find $x_k \in E_k = E \setminus \bigcup_{i=1}^{k-1} B(y_{x_i} 5d^2 r_{x_i})$ such that

$$r_{x_k} > \frac{1}{2} \sup_{x\in E_k} r_x.$$

Then certainly $x_k \in B(y_{x_k}, r_{x_k})$ and moreover the ball $B(y_{x_k}, r_{x_k})$ is disjoint with every ball $B(y_{x_i}, r_{x_i})$, $i < k$, otherwise we can find $i < k$ such that

$$B(y_{x_k}, r_{x_k}) \cap B(y_{x_i}, r_{x_i}) \neq \emptyset$$

and thus

$$\begin{aligned} \varrho(x_k, y_{x_i}) &\leq d(\varrho(x_k, y_{x_k}) + \varrho(y_{x_k}, y_{x_i})) < dr_{x_k} + d^2(r_{x_i} + r_{x_k}) \\ &< 2d^2 r_{x_i} + d^2 r_{x_i} + 2d^2 r_{x_i} < 5d^2 r_{x_i}, \end{aligned}$$

which is the contradiction with $x_k \in E_k$.

The construction can progress in two ways. Either for a finite k there is no point $x_k \in E_k$, i.e. E is already fully covered with the balls $B(y_{x_i}, 5d^2 r_{x_i})$, $i = 1, \ldots, k-1$, the construction stops and the proof is finished, or the construction continues infinitely. In the latter case r_{x_k} has to tend to zero, otherwise there is a constant $\varepsilon > 0$ and infinitely many disjoint balls $B(y_{x_i}, r_{x_i})$ with the radius greater than ε contained in B. Recalling the lower bound condition we have $\mu(B(y_{x_i}, r_{x_i})) \geq C_\mu \varepsilon^n$ and thus

$$\sum_{i=1}^{\infty} C_\mu \varepsilon^n \leq \sum_{i=1}^{\infty} \mu(B(y_{x_i}, r_{x_i})) = \mu(\bigcup_{i=1}^{\infty} B(y_{x_i}, r_{x_i})) \leq \mu(B),$$

which is the contradiction with the assumption that every ball has finite measure applied to B. Hence we have an infinite sequence $B(y_{x_i}, 5d^2 r_{x_i})$ of balls with the radius tending to zero. Now for the contradiction we suppose that there is an uncovered $x \in E \setminus \bigcup_{i=1}^{\infty} B(y_{x_i}, 5d^2 r_{x_i})$. Since $r_{x_i} \searrow 0$, we can find big enough $k \in \mathbb{N}$ such that $r_x > 2r_{x_k}$. Hence in the k-th step of the construction there was $x \in E_k$ such that $\frac{1}{2} r_x > r_{x_k}$ and thus

$$r_{x_k} > \sup_{x\in E_k} \frac{1}{2} r_x > r_{x_k},$$

which is a contradiction. Thus the collection $\{B(y_{x_i}, 5d^2 r_{x_i})\}_{i=1}^{\infty}$ covers E. □

Lemma 1.3.2 *Let $\mathcal{W}$ be a collection of balls in a separable quasi-metric space (X, ϱ). Then there is a maximal disjoint countable subcollection $\mathcal{W}'$, i.e. for every ball B from $\mathcal{W}$ there is a ball in $\mathcal{W}'$ with non-empty intersection with B.*

Proof. In a separable quasi-metric space there is a countable collection of balls $\{B_i\}_{i\in\mathbb{N}}$ such that for every open set U we have $U = \bigcup_{B_i\subset U} B_i$. Now we construct the subcollection $\mathcal{W}'$ from $\mathcal{W}$ in the following way.

- For B_1, either there exists a ball $V_1 \in \mathcal{W}$ such that $B_1 \subset V_1$ and we put it into $\mathcal{W}'$, or there is no such V_1 and we continue to the next step.
- For B_k, $k > 1$, $k \in \mathbb{N}$, either there exists a ball $V_k \in \mathcal{W}$ such that $B_k \subset V_k$ and V_k is disjoint with every ball $V_i \in \mathcal{W}'$, $i < k$, and we put it into $\mathcal{W}'$, or there is no such a V_k and we repeat the process with k replaced by $k + 1$.

The constructed collection $\mathcal{W}'$ is obviously disjoint and countable. If we suppose that there is a ball $V \in \mathcal{W}$ disjoint with every ball from $\mathcal{W}'$, i.e. $\mathcal{W}'$ is not maximal, then we can find k such that $B_k \subset V$. Hence either there was no suitable V_k in the construction which is the contradiction because V is suitable, or there is $V_k \in \mathcal{W}'$ added in the k-th step and $\emptyset \neq B_k \subset V \cap V_k = \emptyset$ which is also impossible. Thus $\mathcal{W}'$ is also maximal. □

Lemma 1.3.3 *Let E be a subset of a separable quasi-metric space (X, ϱ) and let $\mathcal{W}$ be a collection of balls $B(x, r)$ in X with $x \in E$ and $r > 0$ that covers E. Assume that*

$$R = \sup\{r > 0 : B(x, r) \in \mathcal{W}\} < \infty.$$

Then there is a countable disjoint subcollection $\mathcal{V} \subset \mathcal{W}$ such that

$$E \subset \bigcup_{B(x,r)\in\mathcal{V}} B(x, 5d^2 r).$$

Proof. Let $\mathcal{W}_1$ be a collection of balls from $\mathcal{W}$ such that their radius r satisfies $\frac{1}{2}R < r \leq R$. Using Lemma 1.3.2 we find a maximal disjoint countable subcollection $\mathcal{V}_1$. For a positive integer $k > 1$, let $\mathcal{W}_k$ be a collection of balls from $\mathcal{W}$ such that their radius satisfies $2^{-k} < \frac{r}{R} \leq 2^{-k+1}$ and their intersection with any ball from $\bigcup_{i=1}^{k-1} \mathcal{V}_i$ is empty. Using Lemma 1.3.2 we find a maximal disjoint countable subcollection $\mathcal{V}_k$. Then the collection of balls $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ is obviously disjoint and countable. Last property needed is

$$E \subset \bigcup_{B(x,r)\in\mathcal{V}} B(x, 5d^2 r).$$

We fix $z \in E$ and find $B(y, s) \in \mathcal{W}$ such that $z \in B(y, s)$. Then we find $k \in \mathbb{N}$ such that $2^{-k} < \frac{s}{R} \leq 2^{-k+1}$. Now either $B(y, s) \in \mathcal{V}_k \subset \mathcal{V}$ and the proof is finished or $B(y, s) \notin \mathcal{V}_k$. In the latter case either $B(y, s) \in \mathcal{W}_k$ and there is $B(x, r) \in \mathcal{V}_k$ such that $B(x, r) \cap B(y, s) \neq \emptyset$ or $B(y, s) \notin \mathcal{W}_k$ and there is $B(x, r) \in \bigcup_{i=1}^{k-1} \mathcal{V}_i$ such that $B(x, r) \cap B(y, s) \neq \emptyset$. Hence we certainly have

$$B(x, r) \in \bigcup_{i=1}^{k} \mathcal{V}_i \text{ such that } B(x, r) \cap B(y, s) \neq \emptyset.$$

Moreover $r > 2^{-k}R$ and $s \leq 2^{-k+1}R$ implies $s < 2r$. Thus we have

$$\varrho(z, x) \leq d(\varrho(z, y) + \varrho(y, x)) < d(s + d(s + r)) < 5d^2 r$$

and $z \in B(x, 5d^2 r)$. □

Definition 1.3.4 Let (X, ϱ, μ) be a space of homogeneous type and let E be an open subset of X. Let $R > 0$ and $z \in E$. Then we say that $B = B(z, r)$ is a Whitney ball for E bounded by R if

$$r = \min\left\{\frac{1}{2d}\operatorname{dist}(z, X \setminus E), R\right\}.$$

A Whitney covering $\mathcal{V}$ of E bounded by R is a collection of countably many Whitney balls for E bounded by R such that

(i) $E = \bigcup_{B\in\mathcal{V}} B$,
(ii) the balls $\{B(z, r/5d^2)\}_{B(z,r)\in\mathcal{V}}$ are pairwise disjoint,
(iii) there is a positive constant P such that

$$\sum_{B\in\mathcal{V}} \chi_B(x) \leq P$$

for every $x \in E$.

As a consequence of Lemma 1.3.3 we have the following metric version of the Whitney covering lemma.

Lemma 1.3.5 *Let (X, ϱ, μ) be a space of homogeneous type and let E be an open subset of X. Let $R > 0$ and let $\mathcal{W}$ be a collection of all Whitney balls $B(z, r)$, $z \in E$, for E bounded by R. Then there is a Whitney covering $\mathcal{V} \subset \mathcal{W}$ of E bounded by R.*

Proof. Recalling the separability of X from Lemma 1.1.3 and applying Lemma 1.3.3 to the balls $B(z, \frac{r}{5d^2})$, $B(z, r) \in \mathcal{W}$, we obtain a collection of balls $\mathcal{V} = \{B(x_j, r_j)\}_{j=1}^{\infty}$ that covers E and that $B(x_j, \frac{r_j}{5d^2})$ are pairwise disjoint. Let $x \in E$. Now if $x \in B(x_j, r_j)$ for any $j \in \mathbb{N}$, then for any $y \in B(x_j, \frac{r_j}{5d^2})$ we have

$$\varrho(y, x) \leq d(\varrho(y, x_j) + \varrho(x, x_j)) \leq \frac{r_j}{5d} + dr_j \leq \frac{5d^2+1}{5d} r_j$$

and for any $y \in B(x, \frac{r_j}{5d^2})$ we have

$$\varrho(y, x_j) \leq d(\varrho(y, x) + \varrho(x_j, x)) \leq \frac{r_j}{5d} + dr_j \leq \frac{5d^2+1}{5d} r_j.$$

That implies

$$B(x_j, \frac{r_j}{5d^2}) \subset B(x, \frac{5d^2+1}{5d} r_j) \text{ and } B(x, \frac{r_j}{5d^2}) \subset B(x_j, \frac{5d^2+1}{5d} r_j).$$

Thus, using Lemma 1.1.2, we have

$$\sum_{B(x_j,r_j)\ni x} \mu(B(x_j, \frac{r_j}{5d^2})) = \mu(\bigcup_{B(x_j,r_j)\ni x} B(x_j, \frac{r_j}{5d^2})) \leq \mu(B(x, \frac{5d^2+1}{5d} r_j)),$$

$$\begin{aligned}\sum_{B(x_j,r_j)\ni x} \mu(B(x_j, \frac{r_j}{5d^2})) &\geq \frac{1}{D_{5d^3+d}} \sum_{B(x_j,r_j)\ni x} \mu(B(x_j, \frac{5d^2+1}{5d} r_j)) \\ &\geq \frac{1}{D_{5d^3+d}} \sum_{B(x_j,r_j)\ni x} \mu(B(x, \frac{r_j}{5d^2})) \\ &\geq \frac{1}{D^2_{5d^3+d}} \sum_{B(x_j,r_j)\ni x} \mu(B(x, \frac{5d^2+1}{5d} r_j)).\end{aligned}$$

Now for $r_j = R$ we have

$$\mu(B(x, \frac{5d^2+1}{5d}R)) \geq \frac{1}{D^2_{5d^3+d}} \sum_{\substack{B(x_j,r_j)\ni x \\ r_j=R}} \mu(B(x, \frac{5d^2+1}{5d}R)),$$

which implies

$$\sum_{\substack{B(x_j,r_j)\ni x \\ r_j=R}} 1 \leq D^2_{5d^3+d}.$$

For $r_j = \frac{1}{2d}\operatorname{dist}(x_j, X \setminus E)$ denote $\Delta(x) = \operatorname{dist}(x, X \setminus E)$. Then

$$2dr_j = \Delta(x_j) \leq d(\Delta(x) + \varrho(x, x_j)) < d\Delta(x) + dr_j \quad \Rightarrow \quad r_j < \Delta(x)$$

and

$$\Delta(x) \leq d(\Delta(x_j) + \varrho(x, x_j)) < 2d^2 r_j + dr_j \leq 3d^2 r_j \quad \Rightarrow \quad \frac{\Delta(x)}{3d^2} < r_j.$$

Hence, using Lemma 1.1.2, we have

$$\begin{aligned}
\mu(B(x, \frac{5d^2+1}{5d}\Delta(x))) &\geq \mu(B(x, \frac{5d^2+1}{5d}r_j)) \\
&\geq \frac{1}{D^2_{5d^3+d}} \sum_{\substack{B(x_j,r_j)\ni x \\ r_j<R}} \mu(B(x, \frac{5d^2+1}{5d}r_j)) \\
&\geq \frac{1}{D^2_{5d^3+d}} \sum_{\substack{B(x_j,r_j)\ni x \\ r_j<R}} \mu(B(x, \frac{5d^2+1}{15d^3}\Delta(x))) \\
&\geq \frac{1}{D^2_{5d^3+d}D_{3d^2}} \sum_{\substack{B(x_j,r_j)\ni x \\ r_j<R}} \mu(B(x, \frac{5d^2+1}{5d}\Delta(x)))
\end{aligned}$$

and that implies

$$\sum_{\substack{B(x_j,r_j)\ni x \\ r_j<R}} 1 \leq D^2_{5d^3+d}D_{3d^2}.$$

Thus for every $x \in E$ we have

$$\sum_{B\in\mathcal{V}} \chi_B(x) = \sum_{\substack{B(x_j,r_j)\ni x \\ r_j<R}} 1 + \sum_{\substack{B(x_j,r_j)\ni x \\ r_j=R}} 1 \leq D^2_{5d^3+d} + D^2_{5d^3+d}D_{3d^2}$$

and the claim follows with $P = D^2_{5d^3+d} + D^2_{5d^3+d}D_{3d^2}$. □

2. Good λ-inequalities in $\mathbb{R}^n$

2.1 Good λ-inequality in $\mathbb{R}^n$

The idea of the good λ-inequality was first introduced in early 1970s in the paper of D.L. Burkholder and R.F. Gundy [6] as a method for studying operators on the L^p spaces. The authors developed a technique consisting of relating a pair of operators by the distribution function on the n-dimensional Euclidean space $\mathbb{R}^n$ with the Lebesgue measure m. The "bad" operator, i.e. the complex one with unknown properties, was denoted by T and the "good" operator, i.e. the simpler one, was denoted by M. The inequality obtained by this technique allowed to transfer some of the properties of M to T. This inequality became later known as the good λ-inequality and it stated that for every $\varepsilon > 0$ there is $\gamma > 0$ such that

$$m(\{x \in \mathbb{R}^n : |Tf(x)| > 2\lambda, |Mf(x)| \leq \gamma\lambda\}) \leq \varepsilon m(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\})$$

for every $\lambda > 0$. One of the important consequences of this inequality is the desired $L^p(\mathbb{R}^n, m)$ norm estimate of the operator T in the terms of the operator M. In other words, the $L^p(\mathbb{R}^n, m)$ norm of Tf is bounded whenever the $L^p(\mathbb{R}^n, m)$ norm of Mf is bounded and that enables one to transfer the integrability of M to T. Often, the method of proof allows one to replace the Lebesgue measure by a weighted one.

Remark: We can also consider the Muckenhoupt weight classes in the Euclidean space with the Lebesgue measure. The only difference is that the balls B are usually replaced by the cubes Q with sides parallel to the axes. This replacement does not change any properties and in addition sometimes makes the calculation easier. The Muckenhoupt cube-version weight classes in $\mathbb{R}^n$ are further denoted by $A_\infty(Q)$, $A'_\infty(Q)$, $A_p(Q)$, where $1 \leq p < \infty$, and for $1 < p < q < \infty$ they satisfy

$$A_1(Q) \subsetneq A_p(Q) \subsetneq A_q(Q) \subsetneq A'_\infty(Q) = A_\infty(Q).$$

If we assume that the measure on $\mathbb{R}^n$ satisfies only the doubling condition, then

$$A_1(Q) \subsetneq A_p(Q) \subsetneq A_q(Q) \subsetneq A'_\infty(Q) \subsetneq A_\infty(Q).$$

Indeed, if we first assume that we are on $\mathbb{R}$ with the Lebesgue measure. Then $w \in A_p(Q)$ if and only if

$$w = |x|^\beta, \text{ where } \begin{cases} \beta \in (-1, p-1), & p > 1, \\ \beta \in (-1, p-1], & p = 1. \end{cases}$$

Moreover we have

$$A'_\infty(Q) = \bigcup_{1 \leq p < \infty} A_p(Q).$$

Thus really $A_1(Q) \neq A_p(Q) \neq A_q(Q) \neq A'_\infty(Q)$. For the Euclidean space $\mathbb{R}$ with the doubling measure μ let

$$d\mu(x) = \begin{cases} \frac{3}{2}\,dx, & 0 \leq x \leq 1, \\ \frac{1}{2}\,dx, & 1 < x \leq 2, \\ 1\,dx, & \text{otherwise,} \end{cases} \qquad d\nu(x) = \begin{cases} 0\,dx, & 0 < x < 1, \\ 1\,dx, & \text{otherwise.} \end{cases}$$

Then the weight w defined by $d\nu(x) = w(x)d\mu(x)$ satisfies only A_∞ condition. For the more precise argument see Strömberg and Torchinsky [18] and García-Cuerva and Rubio de Francia [9].

Due to the generality of the technique for the operators T and M, this kind of result can be improved for suitable-chosen M and T. First improvement of this type was introduced by R.R. Coifman and C. Fefferman [7]. They considered Mf to be the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{Q \ni x} \frac{1}{m(Q)} \int_Q |f(y)| \, dy, \tag{2.1}$$

where Q denote a cube in $\mathbb{R}^n$ with sides parallel to the axes, and Tf as the general singular integral operator, i.e. such an operator $T : f \to K * f$ in $\mathbb{R}^n$, where $*$ denotes the convolution, with the convolution kernel K that satisfies the standard conditions:

(i) $||\widehat{K}||_\infty \leq C$,
(ii) $|K(x)| \leq \frac{C}{|x|^n}$,
(iii) $|K(x) - K(x-y)| \leq \frac{C|y|}{|x|^{n+1}}$ for $|y| < \frac{|x|}{2}$,

where $x, y \in \mathbb{R}^n$ and $\widehat{K}$ denotes the Fourier transform of K. For the maximal operator

$$T^* f(x) = \sup_{Q_x} \left| \int_{\mathbb{R}^n \setminus Q_x} K(x-y) f(y) dy \right|,$$

where the supremum ranges over all cubes Q_x centered at x with sides parallel to the axes, they derived that for every $w \in A'_\infty(Q)$ there is a constant $C > 0$ such that

$$w(\{x \in \mathbb{R}^n : T^* f(x) > 2\lambda, Mf(x) \leq \gamma\lambda\}) \leq C\gamma^\delta w(\{x \in \mathbb{R}^n : T^* f(x) > \lambda\})$$

for every $\lambda > 0$, every $\gamma > 0$ and δ from $A'_\infty(Q)$ condition. With this so-called "good λ-inequality" as the heart of the proof they proved

Theorem 2.1.1 *Let f be a Lebesgue measurable function on $\mathbb{R}^n$. Suppose that the weight function w satisfies $A'_\infty(Q)$ condition, then*

$$\int_{\mathbb{R}^n} |Tf(x)|^p \, w(x) dx \leq \frac{2^p}{\gamma^p (1 - 2^p \gamma^\delta)} \int_{\mathbb{R}^n} |Mf(x)|^p \, w(x) dx$$

for every $1 < p < \infty$ and γ small enough such that $2^p \gamma^\delta < 1$. Moreover, if $w \in A_p(Q)$, then there is a constant $C \geq 1$ such that

$$\int_{\mathbb{R}^n} |Tf(x)|^p \, w(x) dx \leq \frac{2^p C^p}{\gamma^p (1 - 2^p \gamma^\delta)} \int_{\mathbb{R}^n} |f(x)|^p \, w(x) dx$$

for all $f \in L^p_w(\mathbb{R}^n, m)$.

Taking the p-th root, this can be rewritten as

$$\|Tf\|_{p,w} \leq \frac{2C}{\gamma (1 - 2^p \gamma^\delta)^{\frac{1}{p}}} \, \|f\|_{p,w} = C(p, \gamma) \, \|f\|_{p,w} \, .$$

This result has two main disadvantages. The first is that no single $\gamma > 0$ works for all values of p since γ has to satisfy $2^p\gamma^\delta < 1$ in order to $C(p,\gamma) > 0$. The other is that due to the relation between p and γ, the expression $\frac{2}{\gamma(1-2^p\gamma^\delta)^{1/p}}$ is of the order of 2^p for $p \to \infty$. Thus the operator norm of T,

$$\|T\| = \sup_{\|f\|_{p,w}\le 1} \|Tf\|_{p,w} \le \sup_{\|f\|_{p,w}\le 1} C(p,\gamma)\,\|f\|_{p,w} \le \frac{2C}{\gamma(1-2^p\gamma^\delta)^{\frac{1}{p}}},$$

is also of the order of 2^p. However while operators such a T should have operator norm of the order of p (see [17]) when $p \to \infty$. Both problems are caused by the constant 2 in the good λ-inequality. With enough precision this constant can be lowered to any $\beta > 1$, but never to 1, since it would imply boundedness of the norm of T for large p. Unfortunately the improvement $\beta > 1$ does not solve any of the problems, because the estimate then still yields only the exponential growth.

2.2 Rearranged good λ-inequality in $\mathbb{R}^n$

Next improvement was introduced by R.J. Bagby and D.S. Kurtz (see [2]). In their joint work from 1986 they sharpened the good λ-inequality by reformulating it in the terms of rearrangement rather than the distribution and proved an inequality, in which the relation between p and γ is needed no more. Unlike Fefferman and Coifman they worked with the Calderón-Zygmund kernel K, i.e. such a function $K(x)$, homogeneous of the degree $-n$, that satisfies the conditions:

(i) $|K(x)| \le \frac{C}{|x|^n}$,
(ii) $\int_{\{a<|x|<b\}} K(x)dx = 0$, $0 < a < b$,
(iii) $|K(x) - K(x-y)| \le \frac{C|y|}{|x|^{n+1}}$ for $|x| \ge 2\,|y|$

for every $x, y \in \mathbb{R}^n$. Then, in order to study the Calderón-Zygmund singular integral operator

$$Kf(x) = \lim_{\varepsilon\to 0} \int_{\{y\in\mathbb{R}^n : |x-y|>\varepsilon\}} K(x-y)f(y)dy,$$

they considered the maximal singular integral operator

$$Tf(x) = \sup_{\varepsilon>0} \left| \int_{\{y\in\mathbb{R}^n : |x-y|>\varepsilon\}} K(x-y)f(y)dy \right|. \tag{2.2}$$

In the rest of this section T and M are defined by (2.2) and (2.1) respectively and for all proofs see [2].

Lemma 2.2.1 *Let $w \in A'_\infty(Q)$, then for every $0 < \gamma < 1$ there is a constant $C = C(\gamma) > 0$ such that*

$$(Tf)^*_w(t) \le C(Mf)^*_w(\gamma t) + (Tf)^*_w(2t) \tag{2.3}$$

for every Lebesgue measurable function f on $\mathbb{R}^n$ and every $t > 0$.

Iterating (2.3), with $\gamma = \frac{1}{2}$, one can get

Theorem 2.2.2 *Let $w \in A'_\infty(Q)$, then there are two constants $C_1, C_2 > 0$ such that*

$$(Tf)^*_w(t) \le C_1 (Mf)^*_w\left(\frac{t}{2}\right) + C_1 \int_t^\infty (Mf)^*_w(s)\frac{ds}{s}$$

and

$$(Tf)^{**}_w(t) \le (Tf)^*_w(t) + C_2 (Mf)^{**}_w(t)$$

for every Lebesgue measurable function f on $\mathbb{R}^n$ and every $t > 0$.

A few applications of the previous theorem follow.

Corollary 2.2.3 *Let $w \in A'_\infty(Q)$ and let f be a Lebesgue measurable function on $\mathbb{R}^n$. If $(Mf)^*_w$ is finite-valued and Tf is bounded except on a set of finite w-measure, then Tf is finite almost everywhere.*

Corollary 2.2.4 *If $w \in A'_\infty(Q)$, then there is a constant $C > 0$, independent of f and p, such that*

$$\left(\int_{\mathbb{R}^n} |Tf(x)|^p\, w(x)dx\right)^{\frac{1}{p}} \le Cp \left(\int_{\mathbb{R}^n} |Mf(x)|^p\, w(x)dx\right)^{\frac{1}{p}}$$

for every Lebesgue measurable function f on $\mathbb{R}^n$ and every $1 \le p < \infty$.

Note that in Corollary 2.2.4 the expression Cp yields a linear growth in p compared to Theorem 2.1.1, where the growth is exponential. Note also that due to the rearrangement approach it is easy to derive the same result for the Lorentz spaces $L^{p,q}_w(\mathbb{R}^n, m)$.

The following Corollary concerns the space weak-L^∞ introduced by Bennett, DeVore and Sharpley in [3]. This space consists of w-measurable functions f such that f^*_w is finite for $t > 0$ and $f^{**}_w(t) - f^*_w(t)$ is a bounded function of t. The "norm" in the space Weak $-L^\infty$ is defined by

$$\|f\|_{\text{Weak}\,-L^\infty} = \sup_{t>0}(f^{**}_w(t) - f^*_w(t)).$$

Remark: Function $\|.\|_{\text{Weak}\,-L^\infty}$ is not actually a function norm because it does not satisfy (BF_1). Indeed, if we take $f \equiv c$, $c \in \mathbb{R}$, then also $f^* = c = f^{**}$ and $\|f\|_{\text{Weak}\,-L^\infty} = 0$.

Corollary 2.2.5 *Let $w \in A'_\infty(Q)$ and suppose that Tf is bounded except on a set of finite w-measure. If $f \in L^\infty_w(\mathbb{R}^n, m)$, then Tf is in the space* Weak $-L^\infty$ *and*

$$\|Tf\|_{\text{Weak}\,-L^\infty} \le C\,\|f\|_{\infty,w}\,.$$

If there is $\alpha > 1$ such that $(Mf)^{**}_w(t) \le C(\log(\frac{2}{t}))^{\alpha-1}$ for every $0 < t \le 1$, then there is $\varepsilon > 0$ such that $\exp(\varepsilon(Tf)^{\frac{1}{\alpha}})$ is w-integrable over sets of finite w-measure.

Lemma 2.2.6 *Let $w \in A_1(Q)$, then there is a constant $C > 0$ such that*

$$(Mf)^*_w(t) \leq Cf^{**}_w(t)$$

for every Lebesgue measurable function f on $\mathbb{R}^n$ and every $t > 0$.

Corollary 2.2.7 *Let $w \in A_1(Q)$, then there is a constant $C > 0$ such that*

$$(Tf)^*_w(t) \leq Cf^{**}_w(t) + C\int_t^\infty f^*_w(s)\frac{ds}{s}$$

and

$$\begin{aligned}(Tf)^*_w(t) &\leq C\frac{1}{t}\int_0^t f^{**}_w(s)ds + C\int_t^\infty f^*_w(s)\frac{ds}{s} \\ &\leq C\frac{1}{t}\int_0^t f^{**}_w(s)ds + C\int_t^\infty f^{**}_w(s)\frac{ds}{s}\end{aligned}$$

for every Lebesgue measurable function f on $\mathbb{R}^n$ and every $t > 0$.

2.3 Better good λ-inequality in $\mathbb{R}^n$

A year later D.S. Kurtz introduced in [13] a slightly different approach. His idea was to eliminate the requirement that allow the good λ-inequality to hold only for Mf relatively small and replace it with a pointwise estimate between Mf and Tf. Working with this idea and T and M defined by (2.2) and (2.1) respectively he obtained

Theorem 2.3.1 *Let $w \in A_\infty(Q)$ and let $0 < \varepsilon < 1$, then there is a constant $C > 0$ such that*

$$w(\{x \in \mathbb{R}^n : Tf(x) > CMf(x) + \lambda\}) \leq \varepsilon w(\{x \in \mathbb{R}^n : Tf(x) > \lambda\})$$

for every Lebesgue measurable function f on $\mathbb{R}^n$ and every $\lambda > 0$.

Notice that this "better good λ-inequality" immediately implies the classical one with $\gamma = \frac{1}{C}$. It is also easy to see that the assertion of Theorem 2.3.1 is equivalent to the assertion of the Lemma 2.2.1 and thus has also the same applications.

3. Better good λ-inequality

3.1 Fractional maximal operators

In this section we consider two versions of the fractional maximal operator that are variants of the Hardy-Littlewood maximal operator suitable for our purpose.

Definition 3.1.1 Let (X, ϱ, μ) be a space of homogeneous type and let E be a μ-measurable subset of X. Let f be a μ-measurable function on X, then

$$\fint_E f(x)d\mu(x) = \frac{1}{\mu(E)} \int_E f(x)d\mu(x).$$

Definition 3.1.2 Let (X, ϱ, μ) be a space of homogeneous type, $0 \leq \alpha < n$, $R > 0$, $x \in X$ and $f \in L^1_{\text{loc}}$, then we define the fractional maximal operators

$$M_\alpha^R f(x) = \sup_{0<r<R} \fint_{B(x,r)} r^\alpha \left| f(y) \right| d\mu(y),$$

$$M_\alpha f(x) = \sup_{B(z,r)\ni x} \frac{1}{\mu(B(z,r))^{1-\frac{\alpha}{n}}} \int_{B(z,r)} \left| f(y) \right| d\mu(y).$$

It is easy to see that for every $x \in X$ we have

$$M_\alpha^R f(x) \leq C_\mu^{-\frac{\alpha}{n}} M_\alpha f(x). \tag{3.1}$$

Lemma 3.1.3 *Let (X, ϱ, μ) be a space of homogeneous type and $0 \leq \alpha < n$, then for the constant D_{5d^2} from Lemma 1.1.2 we have*

$$\mu(\{x \in X : M_\alpha f(x) > \lambda\}) \leq D_{5d^2} \lambda^{\frac{n}{\alpha - n}} \left(\int_X \left| f(y) \right| d\mu(y) \right)^{\frac{n}{n-\alpha}}$$

for every $f \in L^1$ and $\lambda > 0$.

Proof. Fix $\lambda > 0$ and $f \in L^1$. Denote the set $\{x \in X : M_\alpha f(x) > \lambda\}$ by Ω_λ and for every $x \in \Omega_\lambda$ let $B_x = B(y, r)$ be a ball containing x such that

$$\frac{1}{\mu(B_x)^{1-\frac{\alpha}{n}}} \int_{B_x} \left| f(y) \right| d\mu(y) > \lambda. \tag{3.2}$$

Such a ball has to exist by the definition of Ω_λ and M_α. Obviously, the collection of balls $\{B_x\}_{x\in\Omega_\lambda}$ covers Ω_λ and due to the integrability of f we have

$$\int_X \left| f(y) \right| d\mu(y) \leq C$$

for some $C > 0$. Now we see that balls B_x, $x \in \Omega_\lambda$, with radius greater than $C_\mu^{-1/n} \left(\frac{C}{\lambda}\right)^{1/(n-\alpha)}$ cannot exist, because, by applying the lower bound condition, we would have

$$\frac{1}{\mu(B_x)^{1-\frac{\alpha}{n}}} \int_{B_x} \left| f(y) \right| d\mu(y) \leq \frac{C}{(C_\mu r^n)^{1-\frac{\alpha}{n}}} < \frac{C}{\left(\left(\frac{C}{\lambda}\right)^{\frac{n}{n-\alpha}}\right)^{\frac{n-\alpha}{n}}} = \lambda,$$

which is the contradiction with (3.2). Therefore B_x, $x \in \Omega_\lambda$, have uniformly bounded diameter. Recalling the separability of X from Lemma 1.1.3 we apply Lemma 1.3.3 to $\{B_x\}_{x\in\Omega_\lambda}$ to obtain a countable disjoint subcollection $\{B_{x_k} = B(y_k, r_k)\}_{k=1}^{\infty} \subset \{B_x\}_{x\in\Omega_\lambda}$ such that

$$\mu(\Omega_\lambda) \leq \sum_{k=1}^{\infty} \mu(B(y_k, 5d^2 r_k)). \tag{3.3}$$

Therefore, combining (3.2), (3.3) and Lemma 1.1.2, we get

$$\begin{aligned}
\mu(\Omega_\lambda) &\leq \sum_{k=1}^{\infty} \mu(B(y_k, 5d^2 r_k)) \leq D_{5d^2} \sum_{k=1}^{\infty} \mu(B(y_k, r_k)) \\
&\leq D_{5d^2} \sum_{k=1}^{\infty} \lambda^{\frac{n}{\alpha-n}} \left(\int_{B_{x_k}} |f(y)| \, d\mu(y) \right)^{\frac{n}{n-\alpha}} \\
&\leq D_{5d^2} \lambda^{\frac{n}{\alpha-n}} \left(\sum_{k=1}^{\infty} \int_{B_{x_k}} |f(y)| \, d\mu(y) \right)^{\frac{n}{n-\alpha}} \leq D_{5d^2} \lambda^{\frac{n}{\alpha-n}} \|f\|_{L^1}^{\frac{n}{n-\alpha}} .
\end{aligned}$$

□

3.2 Better good λ-inequality

The aim of this section is to prove the better good λ-inequality for the Riesz potential and a few of its corollaries, which create the base for deriving the weighted Lebesgue and Lorentz norm estimates.

Definition 3.2.1 Let (X, ϱ, μ) be a space of homogeneous type, $0 \leq \alpha < n$, $R > 0$, $x \in X$ and $f \in L^1_{\mathrm{loc}}$, then we define the Riesz potential

$$I_\alpha^R f(x) = \int_0^R \left(\fint_{B(x,t)} f(y) d\mu(y) \right) dt^\alpha.$$

The sharp Lorentz-norm estimates for this version of the Riesz potential are thoroughly studied by Malý and Pick in [15] and a very similar version with a few results also appear in [14].

Lemma 3.2.2 *Let (X, ϱ, μ) be a space of homogeneous type, $R > 0$, $0 \leq \alpha < n$ and suppose that f, g are μ-measurable functions on X. Then for the constant D_{2d} from Lemma 1.1.2 we have*

$$\int_X I_\alpha^R f(x) g(x) d\mu(x) \leq D_{2d} \int_X I_\alpha^R g(y) f(y) d\mu(y).$$

Proof. For $y \in B(x, r)$ we have $B(y, r) \subset B(x, 2dr)$ and thus, using Lemma 1.1.2, we have

$$\mu(B(y, r)) \leq \mu(B(x, 2dr)) \leq D_{2d} \mu(B(x, r)).$$

Hence

$$
\begin{aligned}
\int_X I_\alpha^R f(x)g(x)d\mu(x) &= \int_X \left(\int_0^R \left(\fint_{B(x,t)} f(y)d\mu(y)\right) dt^\alpha\right) g(x)d\mu(x)\\
&= \int_0^R \left(\int_X \left(\int_{B(x,t)} \frac{f(y)g(x)}{\mu(B(x,t))} d\mu(y)\right) d\mu(x)\right) dt^\alpha\\
&= \int_0^R \left(\int_X \left(\int_X \frac{f(y)\chi_{B(x,t)}(y)g(x)}{\mu(B(x,t))} d\mu(y)\right) d\mu(x)\right) dt^\alpha\\
&= \int_0^R \left(\int_X \left(\int_X \frac{f(y)\chi_{B(x,t)}(y)g(x)}{\mu(B(x,t))} d\mu(x)\right) d\mu(y)\right) dt^\alpha\\
&= \int_0^R \left(\int_X \left(\int_X \frac{f(y)\chi_{B(y,t)}(x)g(x)}{\mu(B(x,t))} d\mu(x)\right) d\mu(y)\right) dt^\alpha\\
&\leq D_{2d} \int_0^R \left(\int_X f(y) \left(\int_X \frac{\chi_{B(y,t)}(x)g(x)}{\mu(B(y,t))} d\mu(x)\right) d\mu(y)\right) dt^\alpha\\
&= D_{2d} \int_X \left(\int_0^R \left(\fint_{B(y,t)} g(x)d\mu(x)\right) dt^\alpha\right) f(y)d\mu(y)\\
&= D_{2d} \int_X I_\alpha^R g(y) f(y) d\mu(y).
\end{aligned}
$$

□

Theorem 3.2.3 (Better good λ-inequality) *Let (X, ϱ, μ) be a space of homogeneous type, $0 < \varepsilon < 1$, $0 \leq \alpha < n$, $R > 0$ and $w \in A_\infty$. Then there is a constant C such that for every μ-measurable function f on X we have*

$$w(\{x \in X : I_\alpha^R f(x) > CM_\alpha^R f(x) + C'\lambda\}) \leq \varepsilon w(\{x \in X : I_\alpha^R f(x) > \lambda\}) \quad (3.4)$$

for $C' = \max\{1, \frac{D_{4d3}}{(2d^2)^\alpha}\}$ and every $\lambda > 0$.

Proof. Fix $\lambda > 0$, μ-measurable function f on X and let $C > 0$ be a constant with the exact value to be specified later. Set

$$G = \{x \in X : I_\alpha^R f(x) > \lambda\},\ G^C = \{x \in X : I_\alpha^R f(x) > CM_\alpha^R f(x) + C'\lambda\}.$$

Obviously G, G^C are open. Let $z \in G$ and $r = \min\left\{\frac{1}{2d}\operatorname{dist}(z, X \setminus E), \frac{R}{3}\right\}$, then $B = B(z, r)$ is a Whitney ball for G bounded by $\frac{R}{3}$. At first we show that for every $0 < \varepsilon' < 1$ the constant C can be set such that

$$\mu\left(B \cap G^C\right) \leq \varepsilon' \mu(B). \quad (3.5)$$

Fix arbitrary $z' \in B$ and set

$$a = \frac{r^\alpha}{\mu(B(z', 3d^2 r))} \int_{B(z',3d^2 r)} f(y)d\mu(y).$$

Now we define $\delta = \delta(\varepsilon, D) \in (0, 1)$ with the exact value to be specified later and set

$$E = B \cap \left\{x \in X : I_\alpha^{\delta r} f(x) > a\right\}.$$

Since $\delta \in (0,1)$, for $y \in B(x,r)$, $y' \in B(z', 3d^2r)$ and $x \in B(z,r)$ we have

$$\varrho(y,z') \le d(\varrho(y,x) + \varrho(x,z')) \le dr + d^2(\varrho(x,z) + \varrho(z,z')) \le 3d^2 r$$

and

$$\varrho(y',z) \le d(\varrho(y',z') + \varrho(z',z)) \le 3d^3 r + dr \le 4d^3 r.$$

Thus $B(x,\delta r) \subset B(x,r) \subset B(z', 3d^2r) \subset B(z, 4d^3r)$ for every $x \in B(z,r)$. We also notice that for $y \notin B(z', 3d^2r)$, $t \in (0, \delta r)$, the intersection of $B(y,t)$ and $B(z,r)$ is empty, otherwise there is $y'' \in B(y,t) \cap B(z,r)$ and

$$\varrho(z',y) \le d(\varrho(z',z) + \varrho(z,y)) < dr + d^2(\varrho(z,y'') + \varrho(y'',y)) < 3d^2 r,$$

which is the contradiction with $y \notin B(z', 3d^2r)$. Thus for $y \notin B(z', 3d^2r)$ we have

$$I_\alpha^{\delta r}\chi_B(y) = \int_0^{\delta r} \fint_{B(y,t)} \chi_B(x)d\mu(x)dt^\alpha = \int_0^{\delta r} \frac{\mu(B(y,t) \cap B(z,r))}{\mu(B(y,t))} dt^\alpha = 0$$

and for $y \in B(z', 3d^2r)$ we have

$$I_\alpha^{\delta r}\chi_B(y) = \int_0^{\delta r} \frac{\mu(B(y,t) \cap B(z,r))}{\mu(B(y,t))} dt^\alpha \le \delta^\alpha r^\alpha.$$

Using Lemma 3.2.2, Lemma 1.1.2 and the definition of E, we get

$$\begin{aligned} a\mu(E) &= \int_E a d\mu(x) < \int_B I_\alpha^{\delta r} f(x) d\mu(x) \\ &= \int_X I_\alpha^{\delta r} f(x)\chi_B(x) d\mu(x) \le D_{2d} \int_X I_\alpha^{\delta r}\chi_B(y) f(y) d\mu(y) \\ &\le D_{2d} \int_{B(z',3d^2r)} \delta^\alpha r^\alpha f(y) d\mu(y) \le D_{2d}\delta^\alpha a\mu(B(z',3d^2r)) \\ &\le D_{2d}\delta^\alpha a\mu(B(z,4d^3r)) \le D_{2d}D_{4d^3}\delta^\alpha a\mu(B(z,r)), \end{aligned}$$

which gives us $\mu(E) \le D_{2d}D_{4d^3}\delta^\alpha \mu(B)$ and thus for δ satisfying $D_{2d}D_{4d^3}\delta^\alpha < \varepsilon'$ we get

$$\mu(E) \le \varepsilon' \mu(B). \tag{3.6}$$

To finish the proof of (3.5) we will show that we can set the constant C such that

$$B \cap G^C \setminus E = \emptyset \text{ and thus } \mu\left(B \cap G^C \setminus E\right) = 0.$$

Fix $x \in B \cap G^C \setminus E$ and suppose $4r < \frac{R}{2d^2}$. Then we decompose $I_\alpha^R f(x)$ as follows:

$$\begin{aligned} I_\alpha^R f(x) = I_\alpha^{\delta r} f(x) & \\ &+ \int_{\delta r}^{4r} \fint_{B(x,t)} f(y)d\mu(y)dt^\alpha \\ &+ \int_{4r}^{\frac{R}{2d^2}} \fint_{B(x,t)} f(y)d\mu(y)dt^\alpha \\ &+ \int_{\frac{R}{2d^2}}^{R} \fint_{B(x,t)} f(y)d\mu(y)dt^\alpha. \end{aligned}$$

For the case $4r \geq \frac{R}{2d^2}$ we consider the same decomposition only, the second integral is treated as zero. Now we can easily see that

$$\int_{\delta r}^{4r} \fint_{B(x,t)} f(y) d\mu(y) dt^\alpha \leq M_\alpha^R f(x) \int_{\delta r}^{4r} \frac{dt^\alpha}{t^\alpha}$$
$$= M_\alpha^R f(x) \int_{(\delta r)^\alpha}^{(4r)^\alpha} \frac{ds}{s} = \alpha M_\alpha^R f(x) \log\left(\frac{4}{\delta}\right) \tag{3.7}$$

and analogously

$$\int_{\frac{R}{2d^2}}^{R} \fint_{B(x,t)} f(y) d\mu(y) dt^\alpha \leq \alpha M_\alpha^R f(x) \log(2d^2). \tag{3.8}$$

For the integration from $4r$ to $\frac{R}{2d^2}$ we observe that for a non-trivial case we have $4r < \frac{R}{2d^2}$ and thus, from the definition of the Whitney balls, we can find $z'' \in B(z, 3dr) \smallsetminus G$. Due to $z'' \notin G$ we obtain

$$I_\alpha^R f(z'') \leq \lambda \tag{3.9}$$

and since $t > 4r$, we have the following set of inclusions

$$B(x,t) \subset B(z'', 2d^2 t) \subset B(x, 4d^3 t). \tag{3.10}$$

From (3.9) and (3.10) now follows

$$\int_{4r}^{\frac{R}{2d^2}} \fint_{B(x,t)} f(y) d\mu(y) dt^\alpha \leq \int_{4r}^{\frac{R}{2d^2}} \frac{1}{\mu(B(x,t))} \int_{B(z'',2d^2t)} f(y) d\mu(y) dt^\alpha$$
$$\leq \int_{4r}^{\frac{R}{2d^2}} \frac{D_{4d^3}}{\mu(B(x,4d^3t))} \int_{B(z'',2d^2t)} f(y) d\mu(y) dt^\alpha$$
$$\leq \int_{4r}^{\frac{R}{2d^2}} D_{4d^3} \fint_{B(z'',2d^2t)} f(y) d\mu(y) dt^\alpha$$
$$= \int_{8d^2r}^{R} \frac{D_{4d^3}}{(2d^2)^\alpha} \fint_{B(z'',t)} f(y) d\mu(y) dt^\alpha$$
$$\leq \frac{D_{4d^3}}{(2d^2)^\alpha} I_\alpha^R f(z'') \leq C'\lambda. \tag{3.11}$$

To estimate $I_\alpha^{\delta r} f(x)$ we recall that z' was chosen arbitrarily. Therefore we can set $z' = x$ and since $x \notin E$, we get

$$I_\alpha^{\delta r} f(x) \leq a = \frac{r^\alpha}{\mu(B(z', 3d^2 r))} \int_{B(z',3d^2r)} f(y) d\mu(y)$$
$$= \frac{r^\alpha}{\mu(B(x, 3d^2 r))} \int_{B(x,3d^2r)} f(y) d\mu(y)$$
$$\leq \frac{1}{(3d^2)^\alpha} M_\alpha^R f(x). \tag{3.12}$$

From (3.12), (3.7), (3.8) and (3.11) we see that for $x \in B \cap G^C \setminus E$ we have

$$I_\alpha^R f(x) \le \left(\frac{1}{(3d^2)^\alpha} + \log(\frac{4}{\delta}) + \log(2d^2) \right) M_\alpha^R f(x) + C'\lambda.$$

Thus by setting $C = (\frac{1}{(3d^2)^\alpha} + \log(\frac{4}{\delta}) + \log(2d^2))$ and because $x \in G^C$ we also have

$$I_\alpha^R f(x) > \left(\frac{1}{(3d^2)^\alpha} + \log(\frac{4}{\delta}) + \log(2d^2) \right) M_\alpha^R f(x) + C'\lambda.$$

Therefore $\mu\left(B \cap G^C \setminus E\right) = 0$ and

$$\mu(B \cap G^C) \le \mu\left(B \cap G^C \setminus E\right) + \mu(E) \le \varepsilon' \mu(B),$$

which proves (3.5). Now for any $0 < \varepsilon < 1$ and P from Lemma 1.3.5, applying A_∞ condition, we can set ε' small enough to get

$$w(B \cap G^C) \le \frac{\varepsilon}{P} w(B).$$

Then, using Lemma 1.3.5 and summing over the balls in the Whitney covering $\mathcal{V}$, we obtain

$$w(G^C) = w(\bigcup_{B \in \mathcal{V}} B \cap G^C) \le \sum_{B \in \mathcal{V}} w(B \cap G^C) \le \sum_{B \in \mathcal{V}} \frac{\varepsilon}{P} w(B) \le \varepsilon w(G),$$

which is the assertion of the theorem. □

Because we have chosen the better good λ-inequality approach, we obtain the rearranged good λ-inequality as an easy consequence.

Corollary 3.2.4 *Let (X, ϱ, μ) be a space of homogeneous type, $w \in A_\infty$, $R > 0$ and $0 \le \alpha < n$. Then for $0 < \gamma < 1$ and a μ-measurable function f there is a constant $C > 0$ such that*

$$(I_\alpha^R f)_w^*(t) \le C(M_\alpha^R f)_w^*(\gamma t) + C'(I_\alpha^R f)_w^*(2t) \tag{3.13}$$

for $C' = \max\{1, \frac{D_{4d^3}}{(2d^2)^\alpha}\}$ and every $t > 0$.

Proof. Setting $\lambda = (I_\alpha^R f)_w^*(2t)$ in Theorem 3.2.3 we get

$$\begin{aligned} w(\{x \in X : I_\alpha^R f(x) > C M_\alpha^R f(x) + C'(I_\alpha^R f)_w^*(2t)\}) \\ \le \varepsilon w(\{x \in X : I_\alpha^R f(x) > (I_\alpha^R f)_w^*(2t)\}). \end{aligned} \tag{3.14}$$

Fix $0 < \gamma < 1$ and set $\varepsilon = \frac{1-\gamma}{2}$. By the definition of f_w^* we have

$$D_{f,w}(f_w^*(t)) = D_{f,w}(\inf\{s > 0 : D_{f,w}(s) \le t\}) \le t.$$

Hence, using (3.14), we obtain

$$\begin{aligned} w(\{x \in X : I_\alpha^R f(x) > C(M_\alpha^R f)_w^*(\gamma t) + C'(I_\alpha^R f)_w^*(2t)\}) \\ \le w(\{x \in X : I_\alpha^R f(x) > C M_\alpha^R f(x) + C'(I_\alpha^R f)_w^*(2t)\}) \\ + w(\{x \in X : M_\alpha^R f(x) > C(M_\alpha^R f)_w^*(\gamma t)\}) \\ \le \varepsilon w(\{x \in X : I_\alpha^R f(x) > (I_\alpha^R f)_w^*(2t)\}) + D_{M_\alpha^R f,w}((M_\alpha^R f)_w^*(\gamma t)) \\ \le \varepsilon D_{I_\alpha^R f,w}((I_\alpha^R f)_w^*(2t)) + D_{M_\alpha^R f,w}((M_\alpha^R f)_w^*(\gamma t)) \le 2\varepsilon t + \gamma t = t. \end{aligned}$$

Thus

$$D_{I_\alpha^R f,w}(C(M_\alpha^R f)_w^*(\gamma t) + C'(I_\alpha^R f)_w^*(2t)) \leq t$$

and that implies

$$(I_\alpha^R f)_w^*(t) = \inf\{\lambda > 0 : D_{I_\alpha^R f,w}(\lambda) \leq t\} \leq C(M_\alpha^R f)_w^*(\gamma t) + C'(I_\alpha^R f)_w^*(2t).$$

□

Next easy consequence of the better good λ-inequality is the good λ-inequality, which is a bit weaker but on the other hand it is easier to work with.

Corollary 3.2.5 *Let (X, ϱ, μ) be a space of homogeneous type and $w \in A_\infty$. Let $0 < \varepsilon < 1$, $1 \leq p \leq \infty$, $R > 0$ and $a > C' = \max\{1, \frac{D_{4d^3}}{(2d^2)^\alpha}\}$, $0 \leq \alpha < n$. Then there is a constant $\sigma = \sigma(\varepsilon, D)$, where D is the doubling constant, such that*

$$w(\{x \in X : I_\alpha^R f(x) > a\lambda, M_\alpha^R f(x) \leq \sigma\lambda\}) \leq \varepsilon w(\{x \in X : I_\alpha^R f(x) > \lambda\}) \quad (3.15)$$

for every $\lambda > 0$ and every μ-measurable function f on X.

Proof. For fixed $\varepsilon > 0$ and $a > C'$ we obtain $C > 0$ from Theorem 3.2.3. By setting $\sigma = C^{-1}(a - C')$ we have

$$M_\alpha^R f(x) \leq \sigma\lambda = C^{-1}(a - C')\lambda \quad \Rightarrow \quad a\lambda \geq CM_\alpha^R f(x) + C'\lambda \quad (3.16)$$

for any $x \in X$ and thus, using (3.16) and (3.4), we obtain

$$\begin{aligned} w(\{x \in X : I_\alpha^R f(x) > a\lambda, M_\alpha^R f(x) \leq \sigma\lambda\}) \\ &\leq w(\{x \in X : I_\alpha^R f(x) > CM_\alpha^R f(x) + C'\lambda, M_\alpha^R f(x) \leq \sigma\lambda\}) \\ &\leq w(\{x \in X : I_\alpha^R f(x) > CM_\alpha^R f(x) + C'\lambda\}) \\ &\leq \varepsilon w(\{x \in X : I_\alpha^R f(x) > \lambda\}). \end{aligned}$$

□

Corollary 3.2.6 *Let (X, ϱ, μ) be a space of homogeneous type. Let $w \in A_\infty$, $R > 0$ and $a > \max\{1, \frac{D_{4d^3}}{(2d^2)^\alpha}\}$, $0 \leq \alpha < n$. Then for every $1 \leq p < \infty$ and every μ-measurable function f on X we have*

$$\left\|I_\alpha^R f\right\|_{p,w} \leq \frac{a}{\sigma(1 - a^p\varepsilon)^{1/p}} \left\|M_\alpha^R f\right\|_{p,w}$$

as long as $\varepsilon < a^{-p}$.

Proof. Let $1 \leq p < \infty$, $\varepsilon < a^{-p}$, $z \in X$, $m \in \mathbb{N}$ and set

$$f_m(x) = \chi_{B(z,2^m)}(x) \min\{f(x), 2^m\}.$$

Then, using (3.15), we have

$$\begin{aligned} D_{I_\alpha^R f_m,w}(a\lambda) - D_{M_\alpha^R f_m,w}(\sigma\lambda) &= w(\{x \in X : I_\alpha^R f_m(x) > a\lambda\}) \\ &\quad - w(\{x \in X : M_\alpha^R f_m(x) > \sigma\lambda\}) \\ &\leq w(\{x \in X : I_\alpha^R f_m(x) > a\lambda, M_\alpha^R f_m(x) \leq \sigma\lambda\}) \\ &\leq \varepsilon w(\{x \in X : I_\alpha^R f_m(x) > \lambda\}) \\ &= \varepsilon D_{I_\alpha^R f_m,w}(\lambda) \end{aligned}$$

and thus

$$D_{I_\alpha^R f_m,w}(a\lambda) \le \varepsilon D_{I_\alpha^R f_m,w}(\lambda) + D_{M_\alpha^R f_m,w}(\sigma\lambda). \tag{3.17}$$

Now, using (3.17), Lemma 1.2.7 and several changes of variables, we get

$$\begin{aligned}\int_X I_\alpha^R f_m(x)^p w(x)d\mu(x) &= p\int_0^\infty \lambda^{p-1} D_{I_\alpha^R f_m,w}(\lambda)d\lambda \\ &= ap\int_0^\infty (a\lambda)^{p-1} D_{I_\alpha^R f_m,w}(a\lambda)d\lambda \\ &\le ap\int_0^\infty (a\lambda)^{p-1}\varepsilon D_{I_\alpha^R f_m,w}(\lambda)d\lambda \\ &\quad + ap\int_0^\infty (a\lambda)^{p-1} D_{M_\alpha^R f_m,w}(\sigma\lambda)d\lambda \\ &= a^p p\varepsilon\int_0^\infty \lambda^{p-1} D_{I_\alpha^R f_m,w}(\lambda)d\lambda \\ &\quad + p\frac{a^p}{\sigma^{p-1}}\int_0^\infty (\sigma\lambda)^{p-1} D_{M_\alpha^R f_m,w}(\sigma\lambda)d\lambda \\ &= a^p\varepsilon\int_X I_\alpha^R f_m(x)^p w(x)d\mu(x) \\ &\quad + \left(\frac{a}{\sigma}\right)^p\int_X M_\alpha^R f_m(x)^p w(x)d\mu(x).\end{aligned}$$

Since $\varepsilon < a^{-p}$, from the last inequality we derive

$$\int_X I_\alpha^R f_m(x)^p w(x)d\mu(x) \le \frac{a^p}{\sigma^p(1-a^p\varepsilon)}\int_X M_\alpha^R f_m(x)^p w(x)d\mu(x).$$

Then letting $m\to\infty$, using Levi's theorem and taking the p-th roots we obtain

$$\left\|I_\alpha^R f\right\|_{p,w} \le \frac{a}{\sigma(1-a^p\varepsilon)^{1/p}}\left\|M_\alpha^R f\right\|_{p,w}.$$

□

Corollary 3.2.7 *Let (X,ϱ,μ) be a space of homogeneous type. Let $w\in A_\infty$, $R>0$ and $0\le\alpha<n$. Then for $p>1$ there is a constant $C>0$ such that*

$$\left\|I_\alpha^R f\right\|_{p,\infty,w} \le C\left\|M_\alpha^R f\right\|_{p,\infty,w}$$

for every μ-measurable function f on X.

Proof. Let $p>1$, $a>\max\{1,\frac{D_{4d^3}}{(2d^2)^\alpha}\}$, $z\in X$, $m\in\mathbb{N}$ and set

$$f_m(x) = \chi_{B(z,2^m)}(x)\min\{f(x),2^m\}.$$

Now, using Corollary 3.2.5, we get

$$\begin{aligned}(a\lambda)^p w(\{x\in X: I_\alpha^R f_m(x)\ge a\lambda\}) &\le \varepsilon a^p\lambda^p w(\{x\in X: I_\alpha^R f_m(x)\ge\lambda\}) \\ &\quad + a^p\lambda^p w(\{x\in X: M_\alpha^R f_m(x)\ge\sigma\lambda\}) \\ &\le \varepsilon a^p\left\|I_\alpha^R f_m\right\|_{p,\infty,w}^p + C\left\|M_\alpha^R f_m\right\|_{p,\infty,w}^p\end{aligned}$$

for every $\lambda>0$. Taking the supremum over λ, letting $m\to\infty$ and using Levi's theorem we obtain the required inequality whenever ε is chosen such that

$$\varepsilon a^p < 1.$$

□

3.3 Norm estimates

In this section we extend the results of the last two corollaries in the way that we prove the norm estimates between Mf and f on the weighted Lebesgue and Lorentz spaces.

First we introduce some notation which will be used in the following two lemmas and theorem inspired by [1] and [4]. Let $B = B(x,r)$, $x \in X$, $r > 0$, be a ball. Let $0 \le \alpha < n$, $A = D_{15d^5}$, $b = 2A^2+1$ and $k \in \mathbb{Z}$. Then set $\widetilde{B} = B(x, 5d^2r)$, $\widehat{B} = B(x, 15d^5r)$ and $\Omega_k = \{x \in X : b^{k+1} \ge M_\alpha f(x) > b^k\}$. If f is a non-negative L^1 function and E is a μ-measurable set, we denote $\frac{1}{\mu(E)^{1-\alpha/n}} \int_E |f(y)| d\mu(y)$ by $m_E f$. Note that if $\mu(X) < \infty$, then $m_X f \le M_\alpha f(x)$ for all $x \in X$. In this case, for each f, we denote by k_0 the integer such that $b^{k_0+1} \ge m_X f > b^{k_0}$. Then clearly $\Omega_k = \emptyset$ for every $k < k_0$.

Lemma 3.3.1 *Let (X, ϱ, μ) be a space of homogeneous type and let A, b, k_0, k and Ω_k be as above. Then for any non-negative L^1 function f with bounded support and any $k \in \mathbb{Z}$ such that $\Omega_k \neq \emptyset$ there is a sequence $\{B_i^k\}_{i\in\mathbb{N}}$ of balls satisfying:*

(i) $\Omega_k \subset \bigcup_{i=1}^{\infty} \widetilde{B}_i^k$.

(ii) $B_i^k \cap B_j^k = \emptyset$ *if* $i \neq j$.

(iii) *If $\mu(X) = \infty$, then for every B_i^k there is $x_i^k \in B_i^k$ such that if r_i^k is the radius of B_i^k, $r \ge 5d^2 r_i^k$ and $x_i^k \in B = B(y,r)$, then*

$$b^{k+1} \ge M_\alpha f(x_i^k) \ge m_{B_i^k} f > b^k \ge m_B f.$$

(iv) *If $\mu(X) < \infty$, then* (iii) *still holds for $k > k_0$, but if $k = k_0$ we only have one ball $B_1^{k_0}$ such that $\Omega_{k_0} \subset B_1^{k_0} = X$ and*

$$b^{k_0+1} \ge M_\alpha f(x_1^{k_0}) \ge m_{B_1^{k_0}} f > b^{k_0}$$

for some $x_1^{k_0} \in B_1^{k_0}$.

(v) *If $x \notin \bigcup_{j=k}^{\infty} \bigcup_{i=1}^{\infty} \widetilde{B}_i^j$ and $M_\alpha f(x) < \infty$, then $M_\alpha f(x) < b^k$.*

Proof. In order to obtain (i) – (iv) we first assume that $\mu(X) = \infty$. If $x \in \Omega_k$, then the integrability of f implies that there is a constant $C > 0$ such that $\int_X f(y) d\mu(y) < C$. Thus, by the lower bound condition, we can take $r > 0$ big enough such that for the ball with radius r, $B(x,r)$, we have $m_{B(x,r)} f \le b^k$ for every $x \in X$. Therefore, for fixed k, the sets

$$R_k(x) = \{r > 0 : \text{there is a ball } B = B(y,r) \ni x,\ y \in X, \text{ such that } m_B f > b^k\}$$

are uniformly bounded. Hence we can choose $r_x \in R_k(x)$ in such a way that if $r \ge 5d^2 r_x$, then $r \notin R_k(x)$. Thus there is a point $y_x \in X$ such that

$$b^{k+1} \ge M_\alpha f(x) \ge m_{B(y_x, r_x)} f > b^k \ge m_{B(y,r)} f \tag{3.18}$$

for every ball $B(y,r) \ni x$ whenever $r \ge 5d^2 r_x$. Thus for every $x \in \Omega_k$ we have a ball $B(y_x, r_x)$ and the collection $\{B(y_x, r_x)\}_{x \in \Omega_k}$ is suitable for applying Lemma 1.3.1. Moreover, the boundedness of the support of f implies that Ω_k is also bounded and therefore Lemma 1.3.1 can be used to obtain a countable sequence $\{B_i^k\}_{i=1}^{\infty} \subset \{B(y_x, r_x)\}_{x\in\Omega_k}$ satisfying (i) – (iii). If $\mu(X) < \infty$ and $k > k_0$, it is easy to see that we can still find $r_x \in R_k(x)$ and $y_x \in X$ such that (3.18) holds. Then, using Lemma 1.3.1 again, we obtain (i), (ii) and the first part of (iv). If $k = k_0$, we notice that $\mu(X) < \infty$ implies the finiteness of the diameter of X, otherwise, due to the lower bound condition, we could find a ball B in X with the radius big enough such that $\mu(B) > \mu(X)$ and that would be a contradiction. Thus we are able to choose $x \in \Omega_k$ and $r > 0$ such that $B(x, r) = X$. Then with $x_1^{k_0} = x$ and $r_1^{k_0} = r$ we have the last part of (iv). To finish the proof we notice that (v) follows easily from (i) – (iv). □

Now we add some more notation valid for the next lemma and theorem. For every $k \in \mathbb{Z}$, let $\{B_i^k\}_{i\in\mathbb{N}}$ be a collection of balls satisfying (i) – (v) from the previous lemma. Then set

$$I_j^k = \{(l, m) \in \mathbb{Z} \times \mathbb{N} : l \geq k + 2, \widetilde{B}_m^l \cap \widetilde{B}_j^k \neq \emptyset\},$$

$$A_j^k = \bigcup_{(l,m)\in I_j^k} \widetilde{B}_m^l, \qquad E_j^k = \widetilde{B}_j^k \setminus A_j^k, \qquad F_j^k = B_j^k \setminus A_j^k. \tag{3.19}$$

Lemma 3.3.2 *Let (X, ϱ, μ) be a space of homogeneous type. Let A, b, k_0, k, Ω_k, I_j^k, A_j^k, E_j^k, F_j^k be as above, let f be a non-negative L^1 function with bounded support and for every $k \in \mathbb{Z}$, such that $\Omega_k \neq \emptyset$, let $\{B_i^k\}_{i\in\mathbb{N}}$ be a collection of balls satisfying* (i) – (v) *from Lemma 3.3.1. Then we have*

(i) $2\mu(A_j^k) \leq \mu(B_j^k)$,

(ii) $2\mu(E_j^k) \geq \mu(\widetilde{B}_j^k)$, $\mu(X \setminus \bigcup_{j=1}^{\infty} \bigcup_{k=-\infty}^{\infty} E_j^k) = 0$ *and if* $x \in E_j^k$ *and* $M_\alpha f(x) < \infty$, *then* $M_\alpha f(x) \leq b^{k+2}$,

(iii) $\mu(F_j^k) \geq \frac{\mu(\widetilde{B}_j^k)}{2D_{5d^2}}$ *and*

$$\sum_{k=-\infty}^{\infty} \sum_{j=1}^{\infty} \chi_{F_j^k}(x) \leq 2$$

for any $x \in X$.

Proof. In order to get (i), let us first show that if $l \geq k+2, m \in \mathbb{N}$ and $\widetilde{B}_m^l \cap \widetilde{B}_j^k \neq \emptyset$, then

$$\widetilde{B}_m^l \subset \widehat{B}_j^k, \tag{3.20}$$

or even more, that $r_m^l \leq r_j^k$. To prove it, we suppose that $r_m^l > r_j^k$ and show that it leads to a contradiction. From our assumption we get $\widetilde{B}_j^k \subset \widehat{B}_m^l$ and from inequalities in Lemma 3.3.1 (iii), (iv) (applying $B(y, r) = \widehat{B}_m^l$ in (iii) and in the part of (iv), where $k > k_0$, and setting $B_1^{k_0} = \widetilde{B}_j^k = \widehat{B}_m^l$ in the other part) we have

$$b^{k+1} \geq m_{\widehat{B}_m^l} f \geq \left(\frac{\mu(B_m^l)}{\mu(\widehat{B}_m^l)}\right)^{1-\frac{\alpha}{n}} m_{B_m^l} f \geq \frac{1}{A} m_{B_m^l} f.$$

Now the third inequality in Lemma 3.3.1 (iii) and (iv), applied to the pair (l, m) and definition of b, gives

$$b^{k+1} \geq \frac{1}{A} m_{B^l_m} f > \frac{1}{A} b^l \geq \frac{1}{A} b^{k+2} \geq \left(\frac{2}{b-1}\right)^{\frac{1}{2}} b^{k+2} \geq b^{k+2-\frac{1}{2}},$$

which is a contradiction.

Now, using again the third inequality in Lemma 3.3.1 (iii) and (iv), we obtain

$$\begin{aligned} \mu(A^k_j) &\leq \sum_{(l,m)\in I^k_j} \mu(\widetilde{B}^l_m) \leq A \sum_{(l,m)\in I^k_j} \mu(B^l_m) \\ &\leq A \sum_{(l,m)\in I^k_j} \left(b^{-l} \int_{B^l_m} f(x) d\mu(x)\right)^{\frac{n}{n-\alpha}}. \end{aligned}$$

Since B^l_m are for fixed l pairwise disjoint over index m and (3.20) holds, we have

$$\begin{aligned} \mu(A^k_j) &\leq A \left(\sum_{l=k+2}^{\infty} b^{-\frac{nl}{n-\alpha}}\right) \left(\int_{\widehat{B}^k_j} f(x) d\mu(x)\right)^{\frac{n}{n-\alpha}} \\ &\leq A \left(b^{-\frac{n(k+1)}{n-\alpha}} \left(b^{\frac{n}{n-\alpha}} - 1\right)^{-1}\right) \mu(\widehat{B}^k_j) \left(m_{\widehat{B}^k_j} f\right)^{\frac{n}{n-\alpha}} \\ &\leq A^2 \left(b^{-\frac{n(k+1)}{n-\alpha}} \left(b^{\frac{n}{n-\alpha}} - 1\right)^{-1}\right) b^{\frac{(k+1)n}{n-\alpha}} \mu(B^k_j) \\ &\leq \frac{A^2}{b^{\frac{n}{n-\alpha}} - 1} \mu(B^k_j) \leq \frac{A^2}{b-1} \mu(B^k_j) \\ &\leq \frac{A^2}{2A^2+1-1} \mu(B^k_j) = \frac{\mu(B^k_j)}{2}. \end{aligned}$$

In order to prove (ii), let x be a point such that $M_\alpha f(x) < \infty$. Then $x \in \Omega_k$ for some $k \in \mathbb{Z}$. By Lemma 3.3.1 (i), $x \in \widetilde{B}^k_j$ for some $j \in \mathbb{N}$. Assume that $x \in A^k_j$, then there exists $(l, m) \in I^k_j$ such that $x \in \widetilde{B}^l_m$ and from Lemma 3.3.1 (iii) and (iv) we obtain

$$M_\alpha f(x) \geq m_{\widetilde{B}^l_m} f \geq A^{-1} m_{B^l_m} f > A^{-1} b^{k+2} > b^{k+1},$$

which is a contradiction. Thus $x \notin A^k_j$ and the sequence $\{E^k_j\}$ is a covering of $\{x \in X : M_\alpha f(x) < \infty\}$. Moreover, on account of Lemma 3.1.3, we have $\mu(\{x \in X : M_\alpha f(x) = \infty\}) = 0$. Thus we also have $\mu(X \setminus \bigcup_{j=1}^\infty \bigcup_{k=-\infty}^\infty E^k_j) = 0$. The inequality $2\mu(E^k_j) \geq \mu(\widetilde{B}^k_j)$ now follows easily from $2\mu(A^k_j) \leq \mu(B^k_j)$:

$$2\mu(E^k_j) \geq 2\mu(\widetilde{B}^k_j) - 2\mu(A^k_j) \geq 2\mu(\widetilde{B}^k_j) - \mu(B^k_j) \geq 2\mu(\widetilde{B}^k_j) - \mu(\widetilde{B}^k_j) = \mu(\widetilde{B}^k_j).$$

And the estimate $M_\alpha f(x) \leq b^{k+2}$ for $x \in E^k_j$ is the consequence of the definition of the set E^k_j because $x \in E^k_j$ implies $x \notin \Omega^l$, $l \geq k+2$.

To prove (iii) we notice that due to Lemma 3.3.1 (ii) we have

$$\sum_{j=1}^{\infty} \chi_{F^k_j}(x) = \chi_{\bigcup_{j=1}^\infty F^k_j}(x) \leq \chi_{\bigcup_{j=1}^\infty E^k_j}(x)$$

for any $k \in \mathbb{Z}$. By the definition of E_j^k (3.19) it follows that no point of X belongs to more than two of the sets $E^k = \bigcup_{j=1}^{\infty} E_j^k$. Indeed, if $x \in E^k$, then there is $j \in \mathbb{Z}$ such that $x \in \widetilde{B}_j^k$ and $x \notin A_j^k$, which implies that $x \notin \widetilde{B}_j^l$ for $l \geq k+2$ because if $x \in \widetilde{B}_j^l$, then $\widetilde{B}_j^l \cap \widetilde{B}_j^k \neq \emptyset$ and $x \in A_j^k$, which is a contradiction. Thus also $x \notin E^l$ for $l \geq k+2$ and x can only belongs to one additional set and that is E^{k+1} or E^{k-1}. Using this fact we get

$$\sum_{k=-\infty}^{\infty} \sum_{j=1}^{\infty} \chi_{F_j^k}(x) \leq \sum_{k=-\infty}^{\infty} \chi_{\bigcup_{j=1}^{\infty} E_j^k}(x) \leq 2.$$

Again the inequality $\mu(F_j^k) \geq \mu(\widetilde{B}_j^k)/(2A)$ follows easily from $2\mu(A_j^k) \leq \mu(B_j^k)$:

$$\mu(F_j^k) \geq \mu(B_j^k) - \mu(A_j^k) \geq \mu(B_j^k) - \frac{\mu(B_j^k)}{2} = \frac{\mu(B_j^k)}{2} \geq \frac{\mu(\widetilde{B}_j^k)}{2D_{5d^2}}.$$

That finishes the proof of (iii) and also the Lemma. $\square$

Remark: In Lemma 3.3.2 the items (i) and (ii) are rather auxiliary, the main result is the uniformly bounded overlapping of the sets F_j^k over the both parameters.

Theorem 3.3.3 *Let (X, ϱ, μ) be a space of homogeneous type, $0 \leq \alpha < n$ and $1 < q \leq p < \infty$. Let w, v be a pair of weights with $w, u = v^{-\frac{1}{q-1}} \in A'_\infty$. Then*

$$\|M_\alpha f\|_{p,w} \leq C \|f\|_{q,v} \tag{3.21}$$

for all $f \in L_v^q$ if and only if

$$\frac{w(B)^{\frac{q}{p}} u(B)^{q-1}}{\mu(B)^{(1-\frac{\alpha}{n})q}} \leq C < \infty \tag{3.22}$$

for every ball $B \subset X$.

Proof. The inequality (3.22) follows easily from (3.21) by taking $f = v^{-\frac{1}{q-1}} \chi_B$ for a fixed ball B.

Now assume that (3.22) holds and let b, Ω_k and A be as above. Let f be a function in L_v^q. First, in order to prove (3.21), we suppose that f has bounded support. Then, from Lemma 3.3.1 (i) – (iv), we have

$$\begin{aligned}(M_\alpha f(x))^p &= \sum_{k=-\infty}^{\infty} (M_\alpha f(x))^p \chi_{\Omega_k}(x) \leq b^p \sum_{k=-\infty}^{\infty} b^{kp} \chi_{\Omega_k}(x) \\ &\leq b^p \sum_{k=-\infty}^{\infty} \sum_{i=1}^{\infty} b^{kp} \chi_{\widetilde{B}_i^k}(x) \leq b^p \sum_{k=-\infty}^{\infty} \sum_{i=1}^{\infty} \left(m_{B_i^k} f\right)^p \chi_{\widetilde{B}_i^k}(x).\end{aligned}$$

Now, from $w \in A'_\infty$ and Lemma 1.1.2, we also have

$$w(\widetilde{B}_i^k) \leq C_{A'_\infty}^{\frac{1}{\delta}} \left(\frac{\mu(\widetilde{B}_i^k)}{\mu(B_i^k)}\right)^{\frac{1}{\delta}} w(B_i^k) \leq C D_{5d^2}^{\frac{1}{\delta}} w(B_i^k) \leq C w(B_i^k)$$

and analogously, from $u \in A'_\infty$ and Lemma 3.3.2 (iii), we have

$$u(\widetilde{B}_i^k) \leq C_{A'_\infty}^{\frac{1}{\delta}} \left(\frac{\mu(\widetilde{B}_i^k)}{\mu(F_i^k)} \right)^{\frac{1}{\delta}} u(F_i^k) \leq C(2A)^{\frac{1}{\delta}} u(F_i^k).$$

Using all the three previous inequalities, (3.22), Lemma 3.3.2 (iii) and setting

$$M^u f(x) = \sup_{B \ni x} \frac{1}{u(B)} \int_B |f(y)|\, u(y) d\mu(y),$$

we obtain

$$\begin{aligned}
\|M_\alpha f\|_{p,w}^q &\leq C \sum_{k=-\infty}^{\infty} \sum_{i=1}^{\infty} \left(m_{B_i^k} f \right)^q w(\widetilde{B}_i^k)^{\frac{q}{p}} \\
&\leq C \sum_{k=-\infty}^{\infty} \sum_{i=1}^{\infty} \left(\frac{1}{\mu(B_i^k)^{1-\frac{\alpha}{n}}} \int_{B_i^k} f(x) d\mu(x) \right)^q w(B_i^k)^{\frac{q}{p}} \\
&\leq C \sum_{k=-\infty}^{\infty} \sum_{i=1}^{\infty} \frac{w(B_i^k)^{\frac{q}{p}} u(B_i^k)^{q-1}}{\mu(B_i^k)^{(1-\frac{\alpha}{n})q}} \left(\frac{1}{u(B_i^k)} \int_{B_i^k} f(x) d\mu(x) \right)^q u(B_i^k) \\
&\leq C \sum_{k=-\infty}^{\infty} \sum_{i=1}^{\infty} \left(\frac{1}{u(B_i^k)} \int_{B_i^k} f(x) u^{-1}(x) u(x) d\mu(x) \right)^q u(F_i^k) \\
&\leq C \sum_{k=-\infty}^{\infty} \sum_{i=1}^{\infty} \int_{F_i^k} \left(M^u(fu^{-1})(y) \right)^q u(y) d\mu(y) \\
&\leq C \left\| M^u(fu^{-1}) \right\|_{q,u}^q .
\end{aligned}$$

The operator M^u is actually the Hardy-Littlewood maximal operator on the space $(X, d, ud\mu)$. Hence, from its boundedness on $(X, d, ud\mu)$ for $1 < q < \infty$ (see [8]), we get

$$\|M_\alpha f\|_{p,w} \leq C \left\| M^u(fu^{-1}) \right\|_{q,u} \leq C \left\| fu^{-1} \right\|_{q,u} = C \|f\|_{q,v},$$

which is (3.21). When f does not have bounded support, the result follows by using a density argument. □

Remark: For the implication (3.22) ⇒ (3.21) in the last theorem is the restrictive assumption $1 < q \leq p < \infty$ not necessary.

Corollary 3.3.4 *Let (X, ϱ, μ) be a space of homogeneous type. Let $0 \leq \alpha < n$, $w, v^{-\frac{1}{q-1}} \in A'_\infty$, $1 < q \leq p < \infty$, $R > 0$ and*

$$\frac{w(B)^{\frac{q}{p}} (u(B))^{q-1}}{\mu(B)^{(1-\frac{\alpha}{n})q}} < \infty$$

for every ball $B \in X$, then there is a constant $C > 0$ such that

$$\left\| I_\alpha^R f \right\|_{p,w} \leq C \|f\|_{q,v}$$

for every $f \in L_v^q$.

Proof. Combining Theorem 3.3.3 and Corollary 3.2.6 we have

$$\left\| I_\alpha^R f \right\|_{p,w} \leq C \left\| M_\alpha^R f \right\|_{p,w} \leq C \left\| M_\alpha f \right\|_{p,w} \leq C \left\| f \right\|_{q,v}.$$

□

To obtain a similar estimate in the Lorentz space we need to introduce a few facts.

Definition 3.3.5 Let (X, ϱ, μ) be a space of homogeneous type, $0 \leq \alpha < n$ and let either $1 < p < \infty$ and $1 \leq q \leq \infty$ or $p = q = 1$. Let (w, v) be a pair of weights. We say that the pair (w, v) is in $A(p, q, \alpha)$, $(w, v) \in A(p, q, \alpha)$, if there is a constant $C > 0$ such that for any ball $B \subset X$ we have

$$\left\| \chi_B \right\|_{p,q,w} \left\| \chi_B w^{-1} \right\|_{p',q',v} \leq C \mu(B)^{1-\frac{\alpha}{n}},$$

where $1 = \frac{1}{p} + \frac{1}{p'}$ and $1 = \frac{1}{q} + \frac{1}{q'}$. Note that $\left\| \chi_E \right\|_{p,q,w} = w(E)^{\frac{1}{p}}$ for every $E \subset X$.

For the proof of the following analogy of the Hölder inequality in the Lorentz space see [11] or [16].

Lemma 3.3.6 *Let (X, ϱ, μ) be a space of homogeneous type, let w be a weight and let $1 \leq p, p_1, p_2, q, q_1, q_2 \leq \infty$ with*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2},$$

then there is a constant $C > 0$ such that

$$\left\| fg \right\|_{p,q,w} \leq C \left\| f \right\|_{p_1,q_1,w} \left\| g \right\|_{p_2,q_2,w}$$

for every $f \in L_{p_1,q_1,w}$ and $g \in L_{p_2,q_2,w}$.

Lemma 3.3.7 *Let (X, ϱ, μ) be a space of homogeneous type, let w be a weight, $1 \leq q \leq p < \infty$ and let $\{E_j\}_{j=1}^\infty$ be a collection of μ-measurable sets such that there is a positive constant C so that $\sum_{j=1}^\infty \chi_{E_j}(x) \leq C$ for any $x \in X$. Then we have*

$$\sum_{j=1}^\infty \left\| \chi_{E_j} f \right\|_{p,q,w}^p \leq C \left\| f \right\|_{p,q,w}^p.$$

Proof. Let $r = \frac{p}{q} \geq 1$ and let $\|.\|_{l^r}$ be the classical norm of the sequence space. Then

$$\begin{aligned}
\left(\sum_{j=1}^\infty \left\| \chi_{E_j} f \right\|_{p,q,w}^p \right)^{\frac{q}{p}} &= \left\| q \int_0^\infty (D_{f\chi_{E_j},w}(s))^{\frac{1}{r}} s^{q-1} ds \right\|_{l^r} \\
&\leq q \int_0^\infty \left\| (D_{f\chi_{E_j},w}(s))^{\frac{1}{r}} \right\|_{l^r} s^{q-1} ds \\
&= q \int_0^\infty \left(\sum_{j=1}^\infty D_{f\chi_{E_j},w}(s) \right)^{\frac{1}{r}} s^{q-1} ds \\
&\leq q \int_0^\infty (C(D_{f,w}(s)))^{\frac{q}{p}} s^{q-1} ds = C^{\frac{q}{p}} \left\| f \right\|_{p,q,w}^q.
\end{aligned}$$

□

Lemma 3.3.8 *Let (X, ϱ, μ) be a space of homogeneous type, $1 \le q \le p < \infty$, $0 \le \alpha < n$, $(w, v) \in A(p, q, w)$ and $w \in A'_\infty$, then there is a constant $C > 0$ such that*

$$\|M_\alpha f\|_{p,\infty,w} \le C \|f\|_{p,q,v}$$

for every $f \in L^{p,q}_v$.

Proof. Let $R > 0$ and $\Omega_\lambda = \{x \in X : M_{\alpha,R} f(x) > \lambda\}$, where

$$M_{\alpha,R} f(x) = \sup_{\substack{B(z,r) \ni x \\ r \le R}} \frac{1}{\mu(B(z,r))^{1-\frac{\alpha}{n}}} \int_{B(z,r)} |f(y)|\, d\mu(y).$$

For every $x \in \Omega_\lambda$ choose a ball B_x such that $x \in B_x$, radius of B_x is less than or equal to R and

$$\frac{1}{\mu(B_x)^{1-\frac{\alpha}{n}}} \int_{B_x} |f(y)|\, d\mu(y) > \lambda.$$

Since $\Omega_\lambda \subset \bigcup_{x \in \Omega_\lambda} B_x$ and Lemma 1.1.3 ensures the separability of X, we apply Lemma 1.3.3 to obtain a countable sequence of disjoint balls $\{B_{x_i} = B(x_i, r_i)\}_{i \in \mathbb{N}}$ such that

$$\Omega_\lambda \subset \bigcup_{i \in \mathbb{N}} B(x_i, 5d^2 r_i) \text{ and } \frac{1}{\mu(B_{x_i})^{1-\frac{\alpha}{n}}} \int_{B_{x_i}} |f(y)|\, d\mu(y) > \lambda,\ i \in \mathbb{N}.$$

Since $(w, v) \in A(p, q, \alpha)$ and $w \in A'_\infty$, we apply Lemma 1.1.2 and Lemma 3.3.6 to obtain

$$\begin{aligned}
w(\Omega_\lambda) &\le \sum_{i \in \mathbb{N}} w(B(x_i, 5d^2 r_i)) \\
&\le \sum_{i \in \mathbb{N}} w(B_{x_i}) \frac{1}{C_{A'_\infty}} \left(\frac{\mu(B(x_i, 5d^2 r_i))}{\mu(B_{x_i})} \right)^{\frac{1}{\delta}} \le C \sum_{i \in \mathbb{N}} w(B_{x_i}) \\
&\le \frac{C}{\lambda^p} \sum_{i \in \mathbb{N}} w(B_{x_i}) \left(\frac{1}{\mu(B_{x_i})^{1-\frac{\alpha}{n}}} \int_{B_{x_i}} |f(y)|\, w(y)^{-1} w(y) d\mu(y) \right)^p \\
&\le \frac{C}{\lambda^p} \sum_{i \in \mathbb{N}} \left\| \chi_{B_{x_i}} \right\|^p_{p,q,w} \mu(B_{x_i})^{-p(1-\frac{\alpha}{n})} \left\| \chi_{B_{x_i}} f \right\|^p_{p,q,v} \left\| \chi_{B_{x_i}} w^{-1} \right\|^p_{p',q',v} \\
&\le \frac{C}{\lambda^p} \sum_{i \in \mathbb{N}} \left\| \chi_{B_{x_i}} f \right\|^p_{p,q,v}.
\end{aligned}$$

Since the balls B_{x_i} are disjoint, by Lemma 3.3.7, we have

$$w(\Omega_\lambda) \le \frac{C}{\lambda^p} \|f\|^p_{p,q,v} \quad \Rightarrow \quad \lambda (D_{M_{\alpha,R} f, w}(\lambda))^{\frac{1}{p}} \le C \|f\|_{p,q,v}.$$

Now, taking the supremum over λ and letting R tend to infinity we obtain the assertion of the Lemma. □

Theorem 3.3.9 *Let (X, ϱ, μ) be a space of homogeneous type, $1 \le q \le p < \infty$, $p > 1$, $0 \le \alpha < n$, $(w, v) \in A(p, q, w)$ and $w \in A'_\infty$, then there is a constant $C > 0$ such that*

$$\left\| I^R_\alpha f \right\|_{p,\infty,w} \le C \|f\|_{p,q,v}$$

for every $f \in L^{p,q}_v$.

Proof. Combining Corollary 3.2.7 and Lemma 3.3.8 we have

$$\left\| I_\alpha^R f \right\|_{p,\infty,w} \leq C \left\| M_\alpha^R f \right\|_{p,\infty,w} \leq C \left\| M_\alpha f \right\|_{p,\infty,w} \leq C \left\| f \right\|_{p,q,v}.$$

□

Remark: Unlike the case with the singular integral operator in $\mathbb{R}^n$, we are not able to derive results in the spirit of Theorem 2.2.2 and following Corollaries 2.2.3, 2.2.4, 2.2.5 and 2.2.7 because the constant $C' = \max\{1, \frac{D_{4d^3}}{(2d^2)^\alpha}\}$ in our quasi-metric rearranged good λ-inequality is in general greater than one, which does not allow us to run the iteration process of (3.13).

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