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ON PORTMANTEAU TESTS OF RANDOMNESS

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Contents

Introduction

Testing randomness of a given univariate time series usually stands at the beginning of any further in-depth analysis and helps to answer several important questions such as the economic hypotheses of the life cycle-permanent income (see [Hall, 1978]), market efficiency (see [Kantor, 1979]) or rational expectations (see [Fama, 1970] or [Fama, 1991]). That is why this topic still attracts attention of many researchers and why a large number of testing procedures have already been developed for this purpose, see [Kuan, 2003] for a selective introductory overview.

Uncomplicated portmanteau tests (see [Arranz, 2005]) play a dominant role among them, especially in practical applications. They have also been used for testing seasonality (by considering only proper autocorrelations) or goodness-of-fit (see [McLeod and Li, 1983]), for statistical process control (see [Atienza et al., 2002]) and for inference in time series of counts (see [Jung and Tremayne, 2003]), among others.

Roughly speaking, such tests are usually somehow related to the popular one introduced in [Box and Pierce, 1970] and they only concentrate on the dependence among the observations not too distant in time, at least by the definition considered here. As practitioners strongly prefer versatile, simple, quick, and easy to implement diagnostic tools, we further consider only those portmanteau tests based on autocorrelation-like coefficients, computed in time domain and not requiring any simulations. We review them extensively in Chapter 1.

Then we focus attention on the portmanteau tests based on ranks due to their wide range of possible applications even in the frequent cases of some data distortion or distributional uncertainty.

In Chapter 2, we summarize some common features of most Monte Carlo experiments conducted in this work, including the null hypothesis, alternatives, software and realization details.

In Chapter 3, we introduce weighted serial modifications of the well known sign and turning point test statistics to any positive lag, explore their asymptotic properties as well as finite sample moment characteristics, show their benefits and use them successfully for portmanteau testing against some common alternatives. We also propose their orthonormal versions and illustrate their advantages in the case of shorter time series and also in Chapter 4 where we employ them in a newly developed methodology for assessing random number generators (RNGs). We then test the new evaluating approach in a large Monte Carlo study involving five popular RNGs.

In Chapter 5, we extend the theory regarding Kendall's rank autocorrelations by computing their exact variances at higher lags. This allows us to use these coefficients in correctly sized portmanteau tests whose application is also discussed and illustrated by Monte Carlo experiments.

The Kendall autocorrelations are computationally quite demanding when calculated from longer time series. This is one of the reasons why we introduce their simpler weighted versions in Chapter 6. Besides, we also find their exact means and variances, establish their joint asymptotic normality and search for some good weighting functions and trimming rules by means of simulations.

Next we deal with certain (signed-)rank autocorrelations of scores in Chapter 7. Especially, we discuss their joint asymptotic normality and find simple exact formulae for their means, variances and covariances. Besides, we also discover and analyse their surprisingly good performance in testing against conditional heteroscedasticity, not reported yet to the best of our knowledge.

Then we proceed with some recommendations on how to construct powerful portmanteau tests.

In Chapter 8, we criticise every portmanteau statistic based on a (possibly weighted) sum of some common serial correlation coefficients and show both theoretically and empirically that its finite sample and asymptotic distribution may differ significantly even if quite long time series are examined.

In Chapter 9, we suggest a unifying view of portmanteau testing, introduce original portmanteau statistics in the new spirit and show their dominance over the most widespread benchmark in many important cases.

We proceed in the same way even in Chapter 10 where we establish joint asymptotic independence of various types of rank autocorrelation coefficients and solve the problem on how to combine them optimally in a single portmanteau statistic. This combining is then shown advantageous for testing against trend alternatives in a Monte Carlo study.

Finally, we summarize some interesting results achieved, discuss their possible applications, suggest further extensions to our work and produce several related problems to be solved.

It remains to note that virtually all the results are derived under a null hypothesis. The only exception can be found in Chapter 4 and lies in providing some asymptotic relative efficiency details regarding contiguous ARMA and trend alternatives.

The accompanying CD contains an electronic hypertext copy of this publication and the directories corresponding to each chapter where all the outputs and auxiliary computations are placed to help the reader with prospective checking of all the information provided, see *readme.rtf* therein for more comments.

There is a great deal of programming and computer work hidden behind these pages. Even most theoretical derivations are performed or at least verified with the aid of a software tool (concretely MAPLE 8.00), which is not surprising, for the computer algebra systems like this are known to be very beneficial in the context of nonparametric statistics, see e.g. [Dufour and Roy, 1986] and in particular [van de Wiel et al., 1999].

We hope that this work will be found useful and that it will help the others to improve their portmanteau testing skills.

Chapter 1

Portmanteau Tests Under Review

First of all, we explain our notation and terminology used throughout this work.

1.1 Notation

Let $R_1, R_2, R_3, \dots$ and $R_1^+, R_2^+, R_3^+, \dots$ be the ranks respectively associated with a given time series $Y_1, Y_2, \dots, Y_T$ and its absolute values $|Y_1|, |Y_2|, \dots, |Y_T|$. We use the natural convention $R_k^{(+)} = R^{(+)}_{((k-1) \bmod T)+1}$ for $k = T+1, T+2, \dots$

Independent and identically distributed random variables will be called white noise. They figure in most of the null hypotheses considered:

H_0^N: Y_t's are independent and identically distributed gaussian random variables
H_0^W: Y_t's are independent and identically distributed random variables with finite variance
H_0: Y_t's are independent and identically distributed continuous random variables
H_0^S: Y_t's are independent and identically distributed symmetric continuous random variables
H_0^E: Y_t's are exchangeable random variables with a continuous distribution

As a rule, H_0^S occurs mainly in Chapter 7, H_0^E or H_0 are generally assumed for rank tests and H_0^N or H_0^W are usually supposed to hold for parametric tests. Apparently, $H_0^E \supset H_0 \supset H_0^S \supset H_0^N$. We point out that the distribution of any rank-based statistic is the same under both H_0 and H_0^E.

We sometimes write E_0, var_0 or cov_0 to highlight the fact that the means, variances or covariances are computed under a prespecified null hypothesis.

Portmanteau tests do not require their alternatives to be specified concretely and they are thus ordinarily assumed very general.

Cumulative and quantile distribution functions will be denoted by F_* and F_*^{-1} with the subscript $*$ clearly indicating the associated distribution. Besides, $F_{\mathrm{N}(0,1)}$ and its density will be sometimes symbolised by Φ and φ as usual.

Let us also recall the widespread definitions of sample ordinary and partial autocorrelations $\widehat{r}(k)$'s and $\widehat{p}(k)$'s at lag $k < T$, very useful in the context of portmanteau tests:

$$\widehat{r}(k) = \frac{\sum_{i=1}^{T-k}(Y_i - \bar{Y})(Y_{i+k} - \bar{Y})}{\sum_{i=1}^{T}(Y_i - \bar{Y})^2}, \quad \bar{Y} = \frac{1}{T}\sum_{i=1}^{T} Y_i,$$

$$\widehat{p}(k) = \frac{|\widehat{R}^*(k)|}{|\widehat{R}(k)|},$$

where $\widehat{R}(k) = (\widehat{r}(i-j))_{i,j=1}^k$ stands for the sample correlation matrix of the random vector $(Y_1, \ldots, Y_k)'$ and $\widehat{R}^*(k)$ differs from $\widehat{R}(k)$ only in the last column that is replaced by the vector of the first k sample autocorrelations

$$\widehat{\mathbf{r}}_k = \big(\widehat{r}(1), \widehat{r}(2), \ldots, \widehat{r}(k)\big)'.$$

And lastly, we refer to [Pollock et al., 1999] where accurate calculation, consistency and asymptotic moments of the sample ordinary autocorrelations are discussed, among others. We know at present that $\sqrt{T}\widehat{\mathbf{r}}_k$ is asymptotically standard normal not only under H_0^N, but also under H_0^W (see [Dufour and Roy, 1986]) which still guarantees that the function mapping $\widehat{\mathbf{r}}_k$ to

$$\widehat{\mathbf{p}}_k = \big(\widehat{p}(1), \ldots, \widehat{p}(k)\big)'$$

is continuous. As $\sqrt{T}\widehat{\mathbf{p}}_k$ is asymptotically standard normal under H_0^N (see e.g. [Monti, 1994]), it keeps the same asymptotic distribution under H_0^W as well. Although we further report the original assumptions given to ensure the asymptotic distributions of the portmanteau statistics (often primarily intended for goodness-of-fit testing), it follows that their asymptotic properties usually remain unchanged even under H_0^W.

1.2 A Brief Excursion to the Universe of Portmanteau Tests

We start in the early seventies of the last century when [Box and Pierce, 1970] proposed the portmanteau statistics

$$Q_1 = T \sum_{k=1}^{m} \widehat{r}^2(k),$$

$$Q_2 = T(T+2) \sum_{k=1}^{m} \frac{\widehat{r}^2(k)}{T-k}$$

with the so called portmanteau (= threshold) parameter m selected conveniently in advance, and proved their $\chi^2(m)$ asymptotic distribution under H_0^N. That is to say that the sample ordinary autocorrelations are then asymptotically standard normal with

$$\mathrm{E}_0\big(\widehat{r}(k)\big) \doteq 0 \quad \text{and} \quad \mathrm{var}_0\big(\widehat{r}(k)\big) \doteq \frac{T-k}{T(T+2)} \doteq \frac{1}{T}.$$

It is well known (under H_0^N and for T large enough) that the true distribution of Q_1 is always shifted to the left of the asymptotic one, while that of Q_2 is almost centered but with a slightly heavier right tail than $\chi^2(m)$ for $m > 2$. Therefore the actual significance level for Q_1 is likely to be smaller (and for Q_2 larger) than expected. Besides, this discrepancy increases with m in both cases. See [Battaglia, 1990].

Many fruitful ideas on how to modify or further improve these statistics have appeared since then. The most interesting are briefly outlined below.

1.2.1 Idea 1: to transform or rearrange data before testing

Portmanteau tests can alternatively be applied to transformed or rearranged observations, which can sometimes lead to a substantial power increase. For example, absolute values or their powers are usually successfully employed this way for testing against time series with time varying higher order moments (see e.g. [Rodríguez and Ruiz, 2005] and references therein), and [Lai, 2001] showed that portmanteau tests of H_0^N against some noisy low dimensional chaos alternatives are much more powerful if applied unchanged to rearranged data $Y^{(1)}, \dots, Y^{(T-1)}$, where $Y^{(k)} = Y_i$ if Y_{i-1} is the kth lowest observation ($i = 2, \dots, T$, $k = 1, \dots, T-1$).

In general, the choice of a suitable transform may be a delicate problem. However, both the link between ordinary autocorrelations of a generalised linear process and of its squares from [Palma and Zevallos, 2004] and the connection between autocovariance functions of two transforms of a time series established in [Abadir and Talmain, 2005] could provide some guidance in this respect.

It should perhaps be mentioned as well that suitably transformed data may meet all required moment (or other) conditions while the original data need not.

1.2.2 Idea 2: to stabilize variances of sample autocorrelations

Portmanteau tests (or at least their size) can be improved by using sample autocorrelations modified with a variance stabilizing transformation. [Kwan and Sim, 1996a, Kwan and Sim, 1996b] follow this idea, use Fisher's ($\sim\ z_1$), Jenkins' ($\sim\ z_4$) and both Hotelling's ($\sim\ z_2$ and z_3) transformations of $\widehat{r}(k)$'s:

$$
\begin{aligned}
z_{1k} &= \frac{1}{2}\ln\Big(\frac{1+\widehat{r}(k)}{1-\widehat{r}(k)}\Big) && \Big\{E_0(z_{1k}) \doteq 0, \mathrm{var}_0(z_{1k}) \doteq \frac{1}{T-k-3}\Big\},\\
z_{2k} &= z_{1k} - \frac{\big(3z_{1k}+\widehat{r}(k)\big)}{4(T-k)} && \Big\{E_0(z_{2k}) \doteq 0, \mathrm{var}_0(z_{2k}) \doteq \frac{1}{T-k-1}\Big\},\\
z_{3k} &= z_{2k} - \frac{23z_{1k}+33\widehat{r}(k)-5\widehat{r}^3(k)}{96(T-k)^2} && \Big\{E_0(z_{3k}) \doteq 0, \mathrm{var}_0(z_{3k}) \doteq \frac{1}{T-k-1}\Big\},\\
z_{4k} &= \sin^{-1}\big(\widehat{r}(k)\big) && \Big\{E_0(z_{4k}) \doteq 0, \mathrm{var}_0(z_{4k}) \doteq \frac{T-k-1}{(T-k)^2}\Big\},
\end{aligned}
$$

and show both their asymptotic normality and asymptotic independence under H_0^N. The portmanteau statistics

$$Q_{2+i} = \sum_{k=1}^{m} \frac{z_{ik}^2}{\mathrm{var}_0(z_{ik})}, \quad i = 1, 2, 3, 4,$$

are then asymptotically $\chi^2(m)$ distributed. However, the more accurate $\chi^2\big(E(Q_{2+i})\big)$ approximate distribution is suggested for their application in practice. For example, if m is small relative to the time series length T, then

$$E_0(Q_3) \doteq E_0(Q_4) \doteq E_0(Q_5) \doteq m - \frac{m(m+4)}{T}, \quad \text{and} \quad E_0(Q_6) \doteq m - \frac{m(m+1)}{T}.$$

Q_3 to Q_6 control test size with increasing m slightly better than Q_2 or Q_1. On the other hand, they are usually less powerful than Q_2. See also [Lai, 2001] and [Kwan et al., 2005], among others.

Virtually the same approach can hypothetically be applied even to other measures of serial dependence. For example, the case of rank autocorrelations and Fisher's variance-stabilizing transformation is addressed in [Kwan et al., 2004].

1.2.3 Idea 3: to employ alternative measures of dependence

In principle, portmanteau tests can be based on other than ordinary autocorrelations. For example, [Baragona and Battaglia, 2000] and [Gallagher, 2001] suggested to use sample inverse autocorrelations and sample covariations, respectively. But the portmanteau statistics based on the former coefficients are quite complicated to handle and those using the covariations do not lead to any significant improvement in the case of finite variance data, and therefore we do not discuss any of them in detail here.

On the other hand, sample partial autocorrelations are far more popular and their asymptotics is clarified much better. [Anderson, 1993] derived both asymptotic normality and asymptotic independence of $\widehat{p}(k)$'s, $k \ll T$, under H_0^N:

$$\sqrt{T}\Big(\widehat{p}(k) + \frac{(1+\delta)}{T}\Big) \sim_{\text{asympt.}} N(0,1)$$

where $\delta = 0$ for k odd and $\delta = 1$ if k is an even number. At the same time, [Monti, 1994] proposed the test statistic Q_7:

$$Q_7 = T(T+2)\sum_{k=1}^{m} \frac{\widehat{p}^2(k)}{T-k}$$

and proved its $\chi^2(m)$ asymptotic distribution under H_0^N. Finally, [Kwan, 2003] considered its finite sample modification

$$Q_8 = T^2 \sum_{k=1}^{m} \frac{\big(\widehat{p}(k) + \delta/T + (k/\delta - \delta)/T^2\big)^2}{T-k-2}$$

based on Anderson's more accurate approximations to $\mathrm{E}_0\big(\widehat{p}(k)\big)$ and $\mathrm{var}_0\big(\widehat{p}(k)\big)$, see references given ibidem.

Simulation experiments indicate that Q_8 is better than Q_7 as for the test size. However, Q_8 does not appear, on average, more efficient than Q_2 although it may be sometimes significantly more powerful, see also [Kwan and Wu, 1997] or [Peña and Rodríguez, 2002].

Rank measures of serial dependence and their use in portmanteau tests are discussed in another subsection.

1.2.4 Idea 4: to standardize the autocorrelations or statistics properly

The finite sample behaviour of portmanteau statistics could possibly be improved if the sample autocorrelations or the whole statistics were standardized conveniently, for example by their (almost) exact means and variances. This is the reason why [Ljung and Box, 1978] advocated Q_2 against Q_1 and why both [Li and McLeod, 1981] and [Kheoh and McLeod, 1992] argued for another revised statistic Q_9:

$$Q_9 = T\sum_{k=1}^{m} \widehat{r}^2(k) + \frac{m(m+1)}{2T}.$$

However, there is hardly any reason for employing Q_9 instead of Q_2, at least according to the Monte Carlo studies performed ibidem.

[Dufour and Roy, 1985], [Dufour and Roy, 1986] derived the exact means, variances and covariances of sample ordinary autocorrelations under H_0^N $(1 \le k < h < T/2, T > 3)$:

$$\begin{aligned}
E_0\big(\widehat{r}(k)\big) &= -\frac{(T-k)}{T(T-1)},\\
\mathrm{var}_0\big(\widehat{r}(k)\big) &= \frac{T^4-(k+3)T^3+3kT^2+2k(k+1)T-4k^2}{(T+1)T^2(T-1)^2},\\
\mathrm{cov}_0\big(\widehat{r}(k),\widehat{r}(h)\big) &= \frac{2\big[kh(T-1)-(T-h)(T^2-k)\big]}{(T+1)T^2(T-1)^2},
\end{aligned}$$

and used them for building portmanteau statistics:

$$\begin{aligned}
Q_{10} &= \big(\widehat{\mathbf{r}}_m - E_0(\widehat{\mathbf{r}}_m)\big)'\big(\mathrm{var}_0(\widehat{\mathbf{r}}_m)\big)^{-1}\big(\widehat{\mathbf{r}}_m - E_0(\widehat{\mathbf{r}}_m)\big),\\
Q_{11} &= \sum_{k=1}^{m}\frac{\big(\widehat{r}(k)-E_0\widehat{r}(k)\big)^2}{\mathrm{var}_0\big(\widehat{r}(k)\big)},
\end{aligned}$$

asymptotically $\chi^2(m)$ distributed under the null hypothesis. Empirical studies show that Q_{10} and Q_{11} behave similarly and slightly better than Q_2 as for the test size, and that Q_{11} usually outdoes Q_2 a little in terms of power. See also [Kwan et al., 2005], [Lai, 2001], [Kwan and Sim, 1996a], and [Kwan and Sim, 1988].

Other attempts in this spirit include using beta approximations for sample autocorrelations with lower T's (see e.g. [Hallin and Mélard, 1988] and references given there) or rewriting $\widehat{r}(k)$'s as ratios of quadratic forms and employing the corresponding approximation theory, see e.g. [Ayadi and Mélard, 2004] and also references given there for yet other approaches.

Besides, [Dufour and Roy, 1985], [Dufour and Roy, 1986] also found quite accurate bounds for $\mathrm{var}\big(\widehat{r}(k)\big)$'s and $\mathrm{cov}\big(\widehat{r}(k),\widehat{r}(h)\big)$'s, valid for any continuous white noise $(1 \le k < h < T/2, T > 3)$:

$$\begin{aligned}
\mathrm{var}_0\big(\widehat{r}(k)\big) &\le \frac{T^3-(k+5)T^2+(5k+6)T+2k(k-4)}{T(T-1)^2(T-3)},\\
\mathrm{cov}_0\big(\widehat{r}(k),\widehat{r}(h)\big) &\le \frac{(T-h)(T+k)-2kh}{T(T-1)(T-2)(T-3)} - E_0\big(\widehat{r}(k)\big)E_0\big(\widehat{r}(h)\big) = \frac{2(k+2)}{T^3}+O(T^{-4}),\\
\mathrm{cov}_0\big(\widehat{r}(k),\widehat{r}(h)\big) &\ge -\frac{(T-h)(T+k)-2kh}{T^2(T-1)(T-3)} - E_0\big(\widehat{r}(k)\big)E_0\big(\widehat{r}(h)\big) = -\frac{2(T-h+3)}{T^3}+O(T^{-4}),
\end{aligned}$$

the upper and lower bounds are interchanged for $(T-h)(T+k)-2kh < 0$. They allow us to construct conservative portmanteau tests, see ibidem.

These results are further extended by [Dufour et al., 2006] who propose several ways on how to bound the distribution of $\widehat{r}(k)$'s computed from some observations that are symmetric about known medians (but possibly not identically distributed, discontinuous or arbitrarily heavy-tailed).

1.2.5 Idea 5: not to ignore covariances between sample autocorrelations

Considering the pattern among sample autocorrelations could possibly lead to further improvements.

[Rodríguez and Ruiz, 2005] propose a solution essentially different from Q_{10}. They suggest to use the portmanteau statistics

$$Q_{12}(i) = T\sum_{k=1}^{m-i}\Big(\sum_{j=0}^{i}\widehat{r}(k+j)\Big)^2,\ i = 0, 1, 2, \ldots, m-1,$$

and to approximate them under any white noise with the finite fourth order moments as follows:

$$\frac{Q_{12}(i)}{a} \sim_{\text{asympt.}} \chi^2(b)$$

where

$$a = \frac{v}{u},\quad b = \frac{u^2}{v},\quad u = (i+1)(m-i),\quad v = (m-2i)(i+1)^2 + 2\sum_{j=1}^{i} j^2\big(m-1+i-3(j-1)\big).$$

Besides, the authors also give complicated formulae for the parameters β, μ and σ that can be used to approximate $Q_{12}(i)$ with a normal distribution:

$$\frac{1}{\sigma}\left(Q_{12}^{\beta}(i) - \mu\right) \sim N(0,1),$$

and tabularize their values for some common choices of i and m.

Monte Carlo experiments reveal that both these approximations behave similarly if $m > 10$ and T is sufficiently large. Besides, it appears that $i \doteq [\frac{m}{3}] - 1$ often proves best and that $Q_{12}(i)$ with optimum i can provide more satisfactory results than Q_1 or Q_{14} when applied to the absolute values of some time series with highly persistent volatility.

1.2.6 Idea 6: to assign higher weights to lower lags

It also seems very fruitful to assign higher weights to the autocorrelations that are expected the most significant under the alternatives. This idea dates back to [Knoke, 1977] and has been applied several times since then, see e.g. [Hong, 1996] or [Hong and Shehadeh, 1999] and references therein. Such a representative portmanteau statistic is

$$Q_{13} = T\sum_{k=1}^{m} w_k \widehat{r}^2(k)$$

where $w_k \geq 0$, $k = 1, \ldots, m$, are some suitable weights, for example those derived from the Bartlett kernel:

$$w_k = 1 - \frac{k-1}{m},\ k = 1, \ldots, m.$$

The threshold parameter m can be selected wisely in advance or perhaps determined by a convenient data-driven procedure, see e.g. [Hong, 1996].

Both the asymptotic distribution of Q_{13} under H_0^W and its upper quantiles can be approximated using the theory developed for quadratic forms in normal variables, see e.g. [Imhof, 1961], [Solomon and Stephens, 1977] or [Kuonen, 1999]. For example, if we assume at least one positive weight and define

$$\mu_i = \sum_{k=1}^{m} w_k^i, \; i = 1, 2, 3, \dots$$

then two such simple (two-moment and three-moment) approximations are as follows:

$$\frac{Q_{13}}{a} \sim_{\text{asympt.}} \chi^2(b) \quad \text{and} \quad (Q_{13} - \mu_1)\sqrt{\frac{h}{\mu_2}} + h \;\sim_{\text{asympt.}} \chi^2(h)$$

where

$$a = \frac{\mu_2}{\mu_1}, \quad b = \frac{\mu_1^2}{\mu_2}, \quad \text{and} \quad h = \frac{\mu_2^3}{\mu_3^2}.$$

Alternatively, one could possibly use a normalizing root transformation, preferably that from [Chen and Deo, 2004] which gives (for all positive w_k's):

$$\frac{Q_{13}^{\beta} - \mu_1^{\beta} - \beta(\beta-1)\mu_1^{\beta-2}\mu_2}{\beta\mu_1^{\beta-1}\sqrt{2\mu_2}} \sim_{\text{asympt.}} N(0,1) \quad \text{where} \quad \beta = 1 - \frac{2\mu_1\mu_3}{3\mu_2^2}.$$

Comparative Monte Carlo studies show that Q_{13} with the Bartlett weights clearly outperforms Q_1 and Q_2 in many respects, at least in a series of common situations considered.

[Peña and Rodríguez, 2002, Peña and Rodríguez, 2006] proposed two other portmanteau statistics interpretable in the same spirit:

$$Q_{14} = T\big(1 - |\widehat{R}(m+1)|^{\frac{1}{m}}\big),$$
$$Q_{15} = -\frac{T}{m+1}\log\big(|\widehat{R}(m+1)|\big),$$

based on the determinant of the sample autocorrelation matrix $\widehat{R}(m+1) = (\widehat{r}(i-j))_{i,j=1}^{m+1}$ where $\widehat{r}(k)$'s can be possibly replaced with $\widehat{r}_a(k)$'s:

$$\widehat{r}_a(k) = \sqrt{\frac{(T+2)}{(T-k)}}\widehat{r}(k), \; k = 1, \dots, m.$$

Besides, [Mokkadem, 1997] (see also [Dette and Spreckelsen, 2000]) introduced an unweighted portmanteau statistic based on $-\sum_{k=1}^{m}\log\big(1 - \widehat{p}^2(k)\big)$ that is closely related to Q_{15} because

$$|\widehat{R}(m+1)| = \prod_{k=1}^{m}\big(1 - \widehat{p}^2(k)\big)^{m+1-k},$$

see e.g. [Peña and Rodríguez, 2002].

In fact, the distributions of Q_{14} and Q_{15} were approximated under H_0^N the same way as Q_{13} with $w_k = (m-k+1)/m$ and $w_k = (m-k+1)/(m+1)$, respectively:

$$Q_{14}: \quad \mu_1 = \tfrac{m+1}{2}, \quad \mu_2 = \tfrac{(m+1)(2m+1)}{6m}, \quad \mu_3 = \tfrac{(m+1)^2}{4m}, \quad h = \tfrac{2(2m+1)^3}{27m(m+1)}, \quad \beta = \tfrac{m^2+m+1}{(2m+1)^2} \to \tfrac{1}{4},$$
$$Q_{15}: \quad \mu_1 = \tfrac{m}{2}, \quad \mu_2 = \tfrac{m(2m+1)}{6(m+1)}, \quad \mu_3 = \tfrac{m^2}{4(m+1)}, \quad h = \tfrac{2(2m+1)^3}{27m(m+1)}, \quad \beta = \tfrac{m^2+m+1}{(2m+1)^2} \to \tfrac{1}{4}.$$

The authors considered both the power transformation and the two-moment one. Both of them are said to produce comparable results in practice. It also appears that Q_{14} often equals or even outperforms both Q_2 and Q_7 and that Q_{15} usually beats Q_{14} and some versions of Q_{13}. Besides, Q_{15} seems less sensitive to the values of m than Q_2 or Q_7. The same holds for Q_{13} and Q_{14} to a certain extent, too.

Some portmanteau statistics of the form of weighted sums are also discussed in the next subsection.

1.2.7 Idea 7: to consider the signs of sample autocorrelations

The well-known statistics Q_1, Q_2 and Q_7 completely ignore the information present in the autocorrelation signs. That is why [Levich and Rizzo, 1998] proposed the test statistics

$$\begin{aligned} Q_{16} &= \sqrt{T}\sum_{k=1}^{m}\widehat{r}(k), \\ Q_{17} &= \sqrt{T}\sum_{k=1}^{m}\widehat{\pi}(k), \end{aligned}$$

based on the sum of the first few sample ordinary or partial autocorrelation coefficients. They were approximated by their $N(0,m)$ asymptotic distribution under the null hypothesis and shown advantageous in the presence of small (but persistent) autocorrelations of a specific sign (and for large T's only).

A few other (possibly weighted) sum-based portmanteau statistics appear in the literature ([Hong, 1997], [Hallin et al., 1985], ...); see also [Richardson and Smith, 1994] and [Daniel, 2001] for unifying approaches to such statistics and for their power investigation.

1.2.8 Idea 8: to allow for outliers or infinite variance

Robust autocorrelation estimators (see e.g. [Bustos and Yohai, 1986] and [Chan and Wei, 1992]) can be found useful in the presence of outliers and they have already been employed successfully in portmanteau tests indeed, see e.g. [Li, 1988], [Chan, 1994] and [Duchesne, 2004]. Besides, they can likewise be used even for robust goodness-of-fit portmanteau testing of estimated residuals, see also [Jiang et al., 2001] and [Allende et al., 2004]. However, all these estimators usually lead to the tests with lower power in comparison to their non-robust counterparts, which is undesirable especially if the data are not contaminated with any outlier.

In principle, if some observations are clearly identifiable as outliers, they can be treated as missing values and used in some portmanteau tests allowing for them (see [Stoffer and Toloi, 1992]).

[Runde, 1997] proposed an impractical modification of Q_1, usable (only) for infinite variance observations. [Gallagher, 2001] used the so called autocovariations in the portmanteau tests assuming only a finite mean. However, their application to finite variance time series do not result in any significant improvement in comparison with the tests based on Q_1 or Q_2.

Alternatively, some rank measures of autocorrelation can be employed. This approach seems very promising because it usually works well even in the case of some distribution uncertainty or severe data distortion (caused e.g. by outliers).

The Spearman sample rank autocorrelations $\widehat{r}_S(k)$'s:

$$\widehat{r}_S(k) = \frac{\sum_{i=1}^{T-k}(R_i - \bar{R})(R_{i+k} - \bar{R})}{\sum_{i=1}^{T}(R_i - \bar{R})^2}, \quad \bar{R} = \frac{1}{T}\sum_{i=1}^{T} R_i, \quad k = 1, \ldots, T-1,$$

(where $\bar{R}$ and $\sum_{i=1}^{T}(R_i - \bar{R})^2$ are only deterministic functions of T) are probably most frequently employed in this context. They were discussed in detail in [Dufour and Roy, 1985] and [Dufour and Roy, 1986] where their exact means, variances and covariances under H_0^E were derived ($1 \leq k < h < T/2$):

$$\begin{aligned}
E_0\big(\widehat{r}_S(k)\big) &= -\frac{(T-k)}{T(T-1)}, \\
\mathrm{var}_0\big(\widehat{r}_S(k)\big) &= \frac{5T^4 - (5k+9)T^3 + 9(k-2)T^2 + 2k(5k+8)T + 16k^2}{5(T-1)^2T^2(T+1)}, \\
\mathrm{cov}_0\big(\widehat{r}_S(k), \widehat{r}_S(h)\big) &= -\frac{2[5T^3 - (5h-6)T^2 - (5hk - k + 6h)T - 8hk]}{5(T-1)^2T^2(T+1)},
\end{aligned}$$

useful for defining the standardized versions $\widetilde{r}_S(k)$'s of $\widehat{r}_S(k)$'s:

$$\widetilde{r}_S(k) = \frac{\widehat{r}_S(k) - E_0\big(\widehat{r}_S(k)\big)}{\sqrt{\mathrm{var}_0\big(\widehat{r}_S(k)\big)}}.$$

The asymptotic multivariate standard normal distribution of $\big(\widetilde{r}_S(1), \ldots, \widetilde{r}_S(m)\big)'$ under H_0^E was proved ibidem.

Spearman's rank autocorrelations have already been employed in portmanteau statistics similar to Q_2 (see e.g. [Dufour and Roy, 1986], [Wong and Li, 1995] or [Burns, 2002]) that appear much more credible (but usually also slightly less powerful) than their parametric alternatives if applied to heavy-tailed or contaminated data.

Several other rank autocorrelation-like measures of serial dependence have already been proposed in the literature:

- autocorrelations of scores (= transformed ranks), see for example [Ferretti et al., 1995] and [Hallin et al., 1985, Hallin et al., 1987] - the sums of suchlike coefficients or of their squares have also been used for testing very successfully, see ibidem
- signed f-rank autocorrelations, tailored to H_0^S (see [Hallin et al., 1990])
- rank-based partial autocorrelations, see [Garel and Hallin, 2000]
- Kendall's rank autocorrelations, see [Ferguson et al., 2000]
- Gini's rank autocorrelations, see [Borroni, 2003a] and [Borroni, 2003b]
- sign-and-rank autocorrelations, recently proposed for testing zero median white noise (see [Hallin et al., 2006])

Sufficient conditions for asymptotic normality of general serial rank coefficients were given in [Mason and Turova, 2000], [Haeusler et al., 2000], [Turova, 2004] for continuous white noise and in [Tran, 1990], [Nieuwenhuis and Ruymgaart, 1990], [Harel and Puri, 1990] for some weakly dependent alternatives.

In general, rank tests require ranking the data first, which is usually more time and space consuming, especially for longer time series. However, the test statistics based on signs or turning points do not suffer from this drawback as they can be computed directly from the original data.

Besides, rank tests are often supposed to be behind with their power. But in fact, numerous optimal rank-based methods (including portmanteau tests) are uniformly at least as powerful as their parametric alternatives, see e.g. [Paindaveine, 2004] and references therein.

Most rank autocorrelations also have other advantages such as

- simplicity and intuitive interpretation
- simple exact asymptotics without any need of simulations or tabularized critical values
- independence from the type of the underlying continuous distribution
- extreme robustness to outliers
- easy use in one-sided testing
- straightforward multivariate generalization (several concepts of ranks and signs have already been proposed for multivariate data and a great many univariate rank methods have already been generalised this way, see e.g. [Oja and Randles, 2004], [Hallin and Paindaveine, 2005], [Oja and Paindaveine, 2005], [Taskinen et al., 2005] and references therein)

We should also mention the fact that there are several extensive surveys of older results regarding nonparametric tests in the time series context, see at least the bibliographical one in [Dufour et al., 1982].

1.3 Selection of the Threshold Parameter

It still remains to discuss the choice of the threshold parameter m that can in general significantly affect both the test size and power, see for example [Peña and Rodríguez, 2006]. We have already mentioned its recommended values for some specific portmanteau statistics. In the other cases, m is supposed not to be chosen pointlessly too high, which is in good accordance with the theoretical results achieved in [Battaglia, 1990]. Similarly, [Burns, 2002] investigated the statistic Q_2 empirically and recommended not to set the parameter m higher than 5% of the total time series length (see also [Chen, 2002]). Besides, rich empirical experience with shorter time series often favours $m \leq 10$, too.

In practice, one should also take into account the alternatives most often expected. For example, it is generally believed that portmanteau tests against seasonal alternatives of some specific period(s) should be based only on sample autocorrelations (of a convenient kind) at the critical lags.

We add that [Keenan, 1997] and [Harris et al., 2003] made the first steps to allow for the thresholds and autocorrelation lags dependent on the time series length T.

1.4 Portmanteau Testing of Dependent Data

The aforementioned portmanteau tests are insensitive to certain uncorrelated alternatives and thus tempt to be used unaltered even for testing serial uncorrelatedness, which would be however wrong and quite misleading. This is evident in the light of [Romano and Thombs, 1996] where it is clearly demonstrated that the normalized sample ordinary autocorrelation $\sqrt{T}\widehat{r}(1)$ may be both asymptotically dependent and of any positive asymptotic variance, even in the case of 1-dependent serially uncorrelated sequences. Besides, [Taylor, 1984] found that the sample variances of $\sqrt{T}\widehat{r}(k)$'s computed from approximately uncorrelated long financial time series are much higher than the asymptotic ones corresponding to a white noise process. See also [Lo and MacKinlay, 1989], [Lobato et al., 2001] and [Lobato et al., 2002], among others.

However, the zero mean asymptotic normal distribution $N(0, \Sigma)$ of $\sqrt{T}\big(\widehat{\mathbf{r}}_m - \mathrm{E}(\widehat{\mathbf{r}}_m)\big)$ was established (under some additional conditions) even for some dependent data, for example for conventional linear models (see e.g. [Anderson and Walker, 1964], [Anderson and de Gooijer, 1988], [Boshnakov, 1989], [Brockwell and Davis, 1991], [Phillips and Solo, 1992]), for some long memory linear (see e.g. [Hosking, 1996]) and nonlinear (see e.g. [Pérez and Ruiz, 2003]) processes, for linear models driven by dependent innovations (see e.g. [Hannan and Heyde, 1972] and related results from [Wu and Min, 2005]), for α-mixing time series (see e.g. [Romano and Thombs, 1996]) and even for the processes that are near epoch dependent on α- and ϕ-mixing sequences (see [de Jong and Davidson, 2000] and [Lobato et al., 2002]). See also [Davis and Mikosch, 2000] for a review of the limit theory for the sample ordinary autocorrelations of some popular processes including linear, stochastic volatility, bilinear, and ARCH time series.

Portmanteau testing for some null hypotheses of serial uncorrelatedness is thus possible and several (usually Q_1-like) portmanteau statistics have already been proposed for this problem, see e.g. [Lo and MacKinlay, 1989], [Horowitz et al., 2006], [Guo and Phillips, 2001], [Lobato et al., 2001], [Lobato, 2001] and [Lobato et al., 2002]. However, they generally require either bootstrap specification of their null distribution or using an estimator of Σ that is consistent under the concrete null hypothesis employed; see for example [Romano and Thombs, 1996] for some powerful bootstrap techniques potentially useful in this context, [den Haan and Levin, 1997] for a guide on these estimators and [Jansson, 2002], [Kiefer and Vogelsang, 2002] and references therein for some recent results on them.

The theory from [Harel and Puri, 1990], [Nieuwenhuis and Ruymgaart, 1990] and [Tran, 1990] indicates that even rank autocorrelations could be used this way in the future.

Portmanteau tests are also widely used for testing the appropriateness of a fitted model. This is most often realized by testing sample ordinary autocorrelations of (estimated) residuals or their squares for their asymptotic (usually normal) distribution under correct model specification that is known in a large variety of models, see e.g. [Box and Pierce, 1970] for ARMA models, [Pierce, 1971] for regression models with ARMA errors, [Pierce, 1972] for dynamic-disturbance models, [McLeod, 1978] for multiplicative seasonal ARMA models, [McLeod and Li, 1983] for ARMA models (and squared-residual autocorrelations), [Li and McLeod, 1986] for FARIMA models, [Söderström and Stoica, 1990] for possibly nonlinear prediction error models, [Li, 1992] for general nonlinear (e.g. threshold) models, [Monti, 1994] for ARMA models (and partial residual autocorrelations), [Ling and Li, 1997] for FARIMA-GARCH models (and both residual and squared-residual autocorrelations), [Whang, 1998] for nonlinear regression models with an unknown form of heteroscedasticity, [Berkes et al., 2003] for pure GARCH models (and squared-residual autocorrelations), [Li and Yu, 2003] for autoregressive conditional duration models, [Berkes et al., 2004b] for pure GARCH(1,1) models (and a weighted quadratic form of squared-residual autocorrelations), [Andreou and Werker, 2005] for general locally asymptotically normal models (and both residual and squared-residual autocorrelations), [Francq et al., 2005] for ARMA models with uncorrelated (possibly dependent) innovations, [Li and Li, 2005] for pure GARCH models (and both absolute residual and squared-residual autocorrelations obtained by the least absolute deviation approach) and [Wong and Ling, 2005] and [Chen, 2005] for models allowing for general conditional mean and variance (the former for the joint distribution of residual and squared-residual autocorrelations and the letter for autocorrelations between transformed residuals). Besides, even rank autocorrelations of estimated residuals have been investigated, see for example [Hallin and Puri, 1994], [Ferretti et al., 1995] and [Andreou and Werker, 2005], and some autocorrelations computed from (possibly transformed) autoregression or nonlinear rank scores (see [Hallin and Jurečková, 1999] and [Mukherjee, 1999]) are likely to be used for goodness-of-fit testing soon; see [Hallin et al., 1999] for a similar application of these rank scores in time series context.

The portmanteau statistics are then often constructed naturally as quadratic forms in the sample residual autocorrelations used. Besides, even the statistics Q_1, Q_2, Q_7, and Q_9 to Q_{15} were originally proposed for goodness-of-fit testing, see the original articles and also [Chen and Deo, 2000], [Chen and Deo, 2004] for Q_{13} and [Hong, 1997] for a weighted Q_{16}-like statistic.

Note that portmanteau tests of independence based on ordinary autocorrelations are inconsistent against all the time series constituted by squared residuals well estimated from an ARMA(p,q) process (see [McLeod and Li, 1983]) or even from a more general conditional homoscedasticity model (see [Chen, 2005]).

If a large data set is available, one could possibly use one its part for model estimation and another one for checking model validity and optimality. Considering ARMAX models, [Fassò, 2000] formally derived the asymptotic multivariate normality of the residual autocorrelations in such a validation subset.

1.5 Practical Application to Real Time Series

Portmanteau statistics based on sample ordinary autocorrelations can be seriously affected by outliers, see e.g. [Martin and Yohai, 1986], [Chan, 1995], [Lee et al., 2001] and [Burns, 2002] for exact theoretical justification and empirical illustration.

Monte Carlo studies performed in [Chen, 2002] and [Kwan et al., 2005] indicate that nonzero skewness of tested data significantly influences the finite sample distribution of both the sample ordinary autocorrelations and the portmanteau tests based on them. This observation corresponds to the empirical evidence that some portmanteau tests deteriorate with the kurtosis of the data whose squares they are applied to, see e.g. [Burns, 2002], [Chen, 2002] and [de Lima, 1997].

[Kwan et al., 2005] show considerable robustness of Q_1 to heavy tails when the data comes from a symmetric distribution. However, even this insensitivity is not absolute as [Burns, 2002] and [Chen, 2002] illustrate by the data samples from the Student distribution with only a few degrees of freedom.

Furthermore, the standardized sample ordinary autocorrelations converge in probability to zero when computed from independent, identically distributed random variables with finite mean but infinite variance, see e.g. [Runde, 1997]. Apparently, it can influence a portmanteau test dramatically as long as their asymptotic distribution is still assumed to be N(0,1) (see e.g. [de Lima, 1997], [Burns, 2002] and [Chen, 2002]).

Note that the sample partial autocorrelations are very likely to mimic all these bad properties of their ordinary counterparts.

Besides, the (possibly uncentered) sample ordinary autocorrelations computed from heavy-tailed dependent data can also be very misleading, see for example [Davis and Resnick, 1985], [Davis and Resnick, 1986], [Cohen et al., 1998], [Resnick et al., 1999], [Davis and Mikosch, 1998], [Davis and Mikosch, 2000], ... although they can still be used for correct testing even then, see e.g. [Runde, 1999] and [Resnick and van den Berg, 2000]. For example, the sample ordinary autocorrelation function of GARCH and other popular models and of their squares exhibits the standard asymptotics only if their fourth and eighth moments are finite, respectively (see e.g. [Davis and Mikosch, 1998] or [Davis and Mikosch, 2000]).

In the light of the information presented above, all common parametric portmanteau tests seem quite likely to produce unreliable results in practice, especially when applied to higher powers of real data or to their absolute values. Note that not the fourth or fifth moments of financial time series are usually finite, see [Loretan and Phillips, 1994], [de Lima, 1997] and in particular [Cont, 2001], a pedagogical overview of the statistical properties shared by the asset returns of financial markets.

Nevertheless, all these problems can be overcome easily by using rank-based autocorrelations that do not appear to suffer from any of these drawbacks, see e.g. [Burns, 2002] and [Kwan et al., 2005].

Chapter 2

Details About Monte Carlo Studies

This chapter summarizes the common characteristics of most Monte Carlo experiments conducted in this work.

2.1 Hypotheses

We always assume the null hypothesis H_0^S of continuous and symmetric white noise. We use only that with zero mean and unit variance, denote it by $\{\varepsilon_t\}_{t=1}^T$ and model it with N(0,1) or standardized t(3) distribution.

Up to five groups of time series can figure in the simulations, each with eight different nontrivial representatives and with the corresponding white noise. The groups are called TREND, SHORTTREND, LONGARMA, ARMA, and GARCH and all the results regarding them are stored in the subdirectories OutputTREND, OutputSHORTTREND, OutputARMA, OutputLONGARMA and OutputGARCH, respectively.

To be more specific, the time series for testing are always recruited from some of the undermentioned classess:

The TREND Class

These time series $\{Y_t\}_{t=1}^T$ are under consideration:

A) **No trend (H_0^S).**	$Y_t = \varepsilon_t$
B) **Linear trend.**	$Y_t = 0.015t + \varepsilon_t$
C) **Quadratic trend.**	$Y_t = 0.00015t^2 + \varepsilon_t$
D) **Exponential trend.**	$Y_t = \exp(0.01t) + \varepsilon_t$
E) **Piece-wise constant trend.**	$Y_t = 1.4\,\mathrm{I}(t > \lfloor T/5 \rfloor) + \varepsilon_t$
F) **Piece-wise linear trend.**	$Y_t = 0.09\,\mathrm{I}(t > \lfloor T/5 \rfloor)(t - \lfloor T/5 \rfloor) + \varepsilon_t$
G) **Sinusoidal trend.**	$Y_t = 0.7\cos(2\pi t/T) + \varepsilon_t$
H) **Arch-shaped trend.**	$Y_t = 0.6\cos(\pi t/T) + \varepsilon_t$
I) **Increasing curly trend.**	$Y_t = 0.015t + 0.015\cos(\pi t/2) + \varepsilon_t$

Note that the eighth label (H) is somewhat misleading here (and in the SHORTTREND class as well) because the actual trend is not arch-shaped as such but rather similar to the tilted $\sim$ or to something like that.

The SHORTTREND Class

The following time series $\{Y_t\}_{t=1}^T$ are included:

A) **No trend (H_0^S).**	$Y_t = \varepsilon_t$
B) **Linear trend.**	$Y_t = 0.09t + \varepsilon_t$
C) **Quadratic trend.**	$Y_t = 0.0035t^2 + \varepsilon_t$
D) **Exponential trend.**	$Y_t = \exp(0.05t) + \varepsilon_t$
E) **Piece-wise constant trend.**	$Y_t = 2.0\,\mathrm{I}(t > \lfloor 3T/4 \rfloor) + \varepsilon_t$
F) **Piece-wise linear trend.**	$Y_t = 0.1\,\mathrm{I}(t > \lfloor 3T/4 \rfloor)(t - \lfloor 3T/4 \rfloor) + \varepsilon_t$
G) **Sinusoidal trend.**	$Y_t = 1.0\cos(2\pi t/T) + \varepsilon_t$
H) **Arch-shaped trend.**	$Y_t = 0.9\cos(\pi t/T) + \varepsilon_t$
I) **Increasing curly trend.**	$Y_t = 0.09t + 0.09\cos(\pi t/2) + \varepsilon_t$

The ARMA Class

This group consists of weakly stationary ARMA(1,1) processes $\{Y_t\}$,

$$Y_t = a_1 Y_{t-1} + \varepsilon_t + b_1 \varepsilon_{t-1}.$$

Their parameter vectors (a_1, b_1) are equal to $(0,0)$, $(-0.3,0)$, $(0,-0.3)$, $(0.3,0)$, $(0,0.3)$, $(0.2,0.2)$, $(-0.2,-0.2)$, $(0.2,0.4)$, and $(0.4,0.2)$, respectively.

The LONGARMA Class

This class also includes weakly stationary ARMA(1,1) processes $\{Y_t\}$,

$$Y_t = a_1 Y_{t-1} + \varepsilon_t + b_1 \varepsilon_{t-1},$$

but this time determined by the parameter couples (a_1, b_1) equal to $(0,0)$, $(-0.2,0)$, $(0,-0.2)$, $(0.2,0)$, $(0,0.2)$, $(0.1,0.1)$, $(-0.1,-0.1)$, $(0.1,0.2)$, and $(0.2,0.1)$.

The GARCH Class

Weakly stationary GARCH(1,1) models $\{Y_t\}$,

$$Y_t = \varepsilon_t \sigma_t, \quad \sigma_t^2 = c + a_1 Y_{t-1}^2 + b_1 \sigma_{t-1}^2,$$

are considered here, with $c = 1$ and with parameter vectors (a_1, b_1) equal to $(0,0)$, $(0.15,0.8)$, $(0.2,0.2)$, $(0.3,0.3)$, $(0.4,0.4)$, $(0.2,0.4)$, $(0.2,0.6)$, $(0.4,0.2)$, and $(0.6,0.2)$.

General Comments on the Alternatives

There is no doubt that ARMA processes deserve our attention as they stand at the birth of most stochastic time series models ever created. The ARMA(1,1) ones with normal white noise are especially popular among them. We further refer to [Brockwell and Davis, 1991] for a transparent survey.

The pure GARCH (1,1) processes play a key role in modelling volatile time series and often outperform many other models for conditional heteroscedasticity, see e.g. [Hansen and Lunde, 2005]. They thus seem to represent the class of volatile alternatives quite well. They were proposed by [Bollerslev, 1986] and reviewed in [Giraitis et al., 2006], [Li et al., 2002], [Berkes et al., 2004a], and [Degiannakis and Xekalaki, 2004], among others.

In practice, the asymptotic distribution of the GARCH innovations ε_t's is usually assumed standard normal or the standardized Student one with three or slightly more degrees of freedom, see e.g. [Tsay, 2001]. It is because such heavy-tailed symmetric distributions of ε_t's better correspond to the empirical findings that important economic variables usually exhibit finite variances but already the fourth or fifth infinite moments (see e.g. [de Lima, 1997], [Cont, 2001] and references therein) and that there is no need to assume nonzero skewness of many such variables, see for example [Kim and White, 2004] or [Lisi, 2005].

Note that H_0^S corresponds to the choice $a_1 = b_1 = 0$ of the GARCH or ARMA parameters. We test it on the absolute values $|Y_t|$'s of the GARCH time series and on the original values Y_t's in all the other cases considered.

As far as the simulations of the ARMA, LONGARMA and GARCH time series are concerned, we generate them 300 observations longer and then trim them from the origin to the correct length in order to eliminate the influence of the automatically preset initial conditions.

2.2 Realization and Output

All tests considered in the Monte Carlo studies use the asymptotic distributions of their test statistics under H_0^S. The test T_S associated with the test statistic

$$S_S(m) = \sum_{k=1}^{m} \widetilde{r}_S^2(k)$$

is often employed as a benchmark, with the $\chi^2(m)$ asymptotic distribution.

For each time series, each length T and each distribution of white noise considered, $N = 10\ 000$ realizations of $\{Y_t\}_{t=1}^T$ are generated and the empirical frequencies of rejection of H_0^S by each test at the nominal level $\alpha = 0.10$ are computed for a number of values of the threshold parameter m. They are stored in the text files with the substring *Port* in their name. Besides, both the sample means and sample standard deviations of the standardized or orthonormalized (see Section 3.7) serial rank coefficients used for testing are stored in the files **Corr*.txt* or **OrtN*.txt*. The results are graphically illustrated in the files with the *eps* extension, too. Each file name also includes some important characteristics regarding the associated Monte Carlo experiment.

The outputs can be reproduced easily with the aid of MATLAB 6.5 [The MathWorks, 2002] by running the program **SimAllFin.m*, with the auxiliary functions **SimFin.m* (and possibly

$*$*SimFin2.m*), *rankitall.m* and *randraw.m* ($*$ always stands for a substring that differs from one case to another). The paths in the first code must be set properly in advance. In fact, the first two *m*–files are quite flexible so that the distribution of white noise, time series length and numerous other features can be changed quite easily. The last two *m*–functions can be downloaded freely from the internet, see references therein for more details. All the programs are placed in the directory reserved for the corresponding chapter.

Any departure from this scheme will be indicated in the right place, as well as the use of other codes created in MAPLE 8.00 [MapleSoft, 2002] or R 2.1.0 [R Development Core Team, 2005].

Chapter 3

New Serial Rank Coefficients

A great many rank measures of dependence have already been proposed (see e.g. [Tarsitano, 2002] for an extensive overview) and others still appear, see for example [Genest and Plante, 2003], [Blest, 2000], [Blest, 1999] or [Shieh, 1998]. Some of them have also been modified for testing serial dependence, such as that of Spearman ([Dufour and Roy, 1985], [Dufour and Roy, 1986]), Gini ([Borroni, 2003a],[Borroni, 2003b]) or Kendall ([Ferguson et al., 2000]). We are going to introduce serial versions of yet other rank coefficients and investigate both their asymptotic properties and the finite sample behaviour.

3.1 Definitions and Notation

Let us recall that the MacMahon rank correlation, Fechner index and Moore–Wallis rank coefficient are based, respectively, on the sums

$$\sum_{i=1}^{T-1} i^2\, \mathrm{I}(R_i > R_{i+1}), \qquad \sum_{i=1}^{T-1} \big[\mathrm{I}(R_i < R_{i+1}) - \mathrm{I}(R_i > R_{i+1})\big], \qquad \sum_{i=1}^{T-1} \mathrm{I}(R_i < R_{i+1}),$$

see e.g. [Tarsitano, 2002] and references therein. These coefficients were also investigated in a large comparison experiment documented ibidem and they proved quite good in relevant situations. The following definition introduces two useful and natural serial generalizations of such correlations and of other tests of independence based on runs, signs or turning points, see for instance [Jolliffe, 1981], [Goodman and Grunfeld, 1961], [Cox and Stuart, 1955], [Moore and Wallis, 1943], and [Wallis and Moore, 1941].

Definition 1. Let the sample weighted (noncircular) Moore and Wallis serial rank coefficients $\widehat{r}_{M,w}(k)$'s and $\widehat{r}_{W,w}(k)$'s, $k = 1, 2, \ldots$, be defined as follows:

$$\widehat{r}_{M,w}(k) = \sum_{i=1}^{T-k} w(i)\, \mathrm{I}(R_i > R_{i+k}),$$

$$\widehat{r}_{W,w}(k) = \sum_{i=1}^{T-2k} w(i)\big[\mathrm{I}(R_i > R_{i+k}, R_{i+k} < R_{i+2k}) + \mathrm{I}(R_i < R_{i+k}, R_{i+k} > R_{i+2k})\big],$$

where $w(\cdot)$ is an arbitrary weighting function, possibly dependent on k. Their circular versions $\widehat{r}^{\circ}_{M,w}(k)$'s and $\widehat{r}^{\circ}_{W,w}(k)$'s are defined in the same way but with T replaced by $T+k$. The unweighted coefficients $\widehat{r}_M(k)$'s, $\widehat{r}_W(k)$'s, $\widehat{r}^{\circ}_M(k)$'s and $\widehat{r}^{\circ}_W(k)$'s correspond to the case of constant unit weights. If $w(i) = i^j$, $j = 0, 1, 2\dots$, we will sometimes replace the subscript w with j.

Note. The coefficients $\widehat{r}_W(1)$ and $\widehat{r}_M(1)$ express the total number of turning points or negative differences, respectively. Both such characteristics have already been suggested for testing in the time series context, see e.g. [Moore and Wallis, 1943] and [Wallis and Moore, 1941]. Besides, [Cox and Stuart, 1955] investigated a class of sign statistics including $\widehat{r}_{M,w}(k)$'s for $k > T/2$ and proposed a few such coefficients, and [Noether, 1956] considered a sequential test virtually based on $\widehat{r}_M(k)$'s. Furthermore, $\widehat{r}_W(k)$'s are in spirit similar to the autocorrelation of two signs, considered without any deeper statistical theory in [Christoffersen and Diebold, 2002] and for some (multivariate) time series problems in [Hallin and Paindaveine, 2005] and in the articles cited there.

3.2 Moment Structure

The means and variances of the Moore and Wallis serial rank coefficients are given by the following theorem.

Theorem 2 (Means and variances)**.** *If H_0^E holds, then*

$$\begin{aligned}
\mathrm{E}\big(\widehat{r}_{M,w}(k)\big) &= \frac{1}{2}\sum_{i=1}^{T-k} w(i), && (1 \le k < T)\\
\operatorname{var}\big(\widehat{r}_{M,w}(k)\big) &= \frac{1}{2}a_M + \frac{1}{3}b_M + \frac{1}{4}c_M - \big[\mathrm{E}\big(\widehat{r}_{M,w}(k)\big)\big]^2 = \frac{1}{4}a_M - \frac{1}{6}b_M, && (1 \le k < T)\\
\mathrm{E}\big(\widehat{r}_{W,w}(k)\big) &= \frac{2}{3}\sum_{i=1}^{T-2k} w(i), && (1 \le k < T/2)\\
\operatorname{var}\big(\widehat{r}_{W,w}(k)\big) &= \frac{2}{3}a_W + \frac{5}{6}b_W + \frac{9}{10}c_W + \frac{4}{9}d_W - \big[\mathrm{E}\big(\widehat{r}_{W,w}(k)\big)\big]^2\\
&= \frac{2}{9}a_W - \frac{1}{18}b_W + \frac{1}{90}c_W, && (1 \le k < T/2)
\end{aligned}$$

where

$$a_M = \sum_{i=1}^{T-k} w^2(i), \qquad b_M = \sum_{i=1}^{T-2k} w(i)w(i+k), \qquad c_M = \sum_{i=1}^{T-k}\sum_{j=1}^{T-k} w(i)w(j) - a_M - 2b_M,$$

and

$$\begin{aligned}
a_W &= \sum_{i=1}^{T-2k} w^2(i), & b_W &= \sum_{i=1}^{T-3k} w(i)w(i+k),\\
c_W &= \sum_{i=1}^{T-4k} w(i)w(i+2k), & d_W &= \sum_{i=1}^{T-2k}\sum_{j=1}^{T-2k} w(i)w(j) - a_W - 2b_W - 2c_W.
\end{aligned}$$

Furthermore, replacing T with $T+k$ in these formulae leads directly to the expressions for the means and variances of the circular coefficients $\widehat{r}^{\circ}_{M,w}(k)$'s and $\widehat{r}^{\circ}_{W,w}(k)$'s.

Note. Each sum is always treated as zero in this chapter if its lower bound on the summing index exceeds the higher.

Proof. The ranks R_i's are exchangeable under H_0^E and consequently

$$\begin{aligned}
\mathrm{E}\big(\widehat{r}_{M,w}(k)\big) &= \sum_{i=1}^{T-k} w(i)P(R_1 > R_2) = \frac{1}{2}\sum_{i=1}^{T-k} w(i),\\
\mathrm{E}\big(\widehat{r}_{W,w}(k)\big) &= \sum_{i=1}^{T-2k} w(i)\big[P(R_1 > R_2, R_2 < R_3) + P(R_1 < R_2, R_2 > R_3)\big] = \frac{2}{3}\sum_{i=1}^{T-2k} w(i),\\
\mathrm{E}\big(\widehat{r}^2_{M,w}(k)\big) &= a_M p_a^M + 2b_M p_b^M + c_M p_c^M,\\
\mathrm{E}\big(\widehat{r}^2_{W,w}(k)\big) &= a_W p_a^W + 2b_W p_b^W + 2c_W p_c^W + d_W p_d^W,
\end{aligned}$$

where

$$\begin{aligned}
p_a^M &= P(R_1 > R_2) = \frac{1}{2},\\
p_b^M &= P(R_1 > R_2, R_2 > R_3) = \frac{1}{6},\\
p_c^M &= P(R_1 > R_2, R_3 > R_4) = \frac{1}{4},
\end{aligned}$$

and

$$\begin{aligned}
p_a^W &= 0 + 0 + P(R_1 > R_2, R_2 < R_3) + P(R_1 < R_2, R_2 > R_3)\\
&= 2\cdot\frac{1}{3} = \frac{2}{3},\\
p_b^W &= 0 + 0 + P(R_1 > R_2, R_2 < R_3, R_3 > R_4) + P(R_1 < R_2, R_2 > R_3, R_3 < R_4)\\
&= 2\cdot\frac{5}{24} = \frac{5}{12},\\
p_c^W &= P(R_1 > R_2, R_2 < R_3, R_3 > R_4, R_4 < R_5)\\
&\quad + P(R_1 > R_2, R_2 < R_3, R_3 < R_4, R_4 > R_5)\\
&\quad + P(R_1 < R_2, R_2 > R_3, R_3 < R_4, R_4 > R_5)\\
&\quad + P(R_1 < R_2, R_2 > R_3, R_3 > R_4, R_4 < R_5)\\
&= 2\cdot\frac{16}{120} + 2\cdot\frac{11}{120} = \frac{9}{20},\\
p_d^W &= P(R_1 > R_2, R_2 < R_3, R_4 > R_5, R_5 < R_6)\\
&\quad + P(R_1 < R_2, R_2 > R_3, R_4 > R_5, R_5 < R_6)\\
&\quad + P(R_1 > R_2, R_2 < R_3, R_4 < R_5, R_5 > R_6)\\
&\quad + P(R_1 < R_2, R_2 > R_3, R_4 < R_5, R_5 > R_6)\\
&= 4\cdot\frac{1}{9} = \frac{4}{9}.
\end{aligned}$$

The probabilities p_c^W and p_d^W can be checked with the R code *MWProbs.r* and the rest is abundantly obvious. $\square$

Corollary. The means and variances of $\widehat{r}_{M,w}(k)$'s, $1 \le k < T$, and $\widehat{r}_{W,w}(k)$'s, $1 \le k < T/2$, can be obtained under H_0^E for any polynomial weights with the aid of the two MAPLE programs *MWEsVarsM.mws* and *MWEsVarsW.mws*. The exact formulae for $\mathrm{E}\big(\widehat{r}_{M,w}(k)\big)$, $1 \le k < T$, $\mathrm{var}\big(\widehat{r}_{M,w}(k)\big)$, $1 \le k < T/2$, $\mathrm{E}\big(\widehat{r}_{W,w}(k)\big)$, $1 \le k < T/2$, and $\mathrm{var}\big(\widehat{r}_{W,w}(k)\big)$, $1 \le k < T/4$, are given below for some simple choices of weights. They can be checked with the R code *MWExactMom.r*.

$\boxed{w(i) = 1}$

$$\begin{aligned}
\mathrm{E}\big(\widehat{r}_M(k)\big) &= \frac{1}{2}(T-k) \\
\mathrm{var}\big(\widehat{r}_M(k)\big) &= \frac{1}{12}(T+k) \\
\mathrm{E}\big(\widehat{r}_W(k)\big) &= \frac{2}{3}(T-2k) \\
\mathrm{var}\big(\widehat{r}_W(k)\big) &= \frac{8}{45}(T-2k) + \frac{k}{30}
\end{aligned}$$

$\boxed{w(i) = i}$

$$\begin{aligned}
\mathrm{E}\big(\widehat{r}_{M,1}(k)\big) &= \frac{1}{4}(T-k)^2 + \frac{1}{4}(T-k) \\
\mathrm{var}\big(\widehat{r}_{M,1}(k)\big) &= \frac{1}{36}(T-k)^3 + \frac{2k+1}{24}(T-k)^2 + \frac{6k+1}{72}(T-k) - \frac{1}{36}(k^3-k) \\
\mathrm{E}\big(\widehat{r}_{W,1}(k)\big) &= \frac{1}{3}(T-2k)^2 + \frac{1}{3}(T-2k) \\
\mathrm{var}\big(\widehat{r}_{W,1}(k)\big) &= \frac{8}{135}(T-2k)^3 + \frac{3k+16}{180}(T-2k)^2 + \frac{9k+16}{540}(T-2k) + \frac{1}{180}(k^3+k)
\end{aligned}$$

$\boxed{w(i) = i^2}$

$$\begin{aligned}
\mathrm{E}\big(\widehat{r}_{M,2}(k)\big) &= \frac{1}{6}(T-k)^3 + \frac{1}{4}(T-k)^2 + \frac{1}{12}(T-k) \\
\mathrm{var}\big(\widehat{r}_{M,2}(k)\big) &= \frac{1}{60}(T-k)^5 + \frac{2k+1}{24}(T-k)^4 - \frac{(2k^2-6k-1)}{36}(T-k)^3 \\
&\quad - \frac{(k^2-k)}{12}(T-k)^2 - \frac{(10k^2+1)}{360}(T-k) + \frac{1}{180}(k^5-k) \\
\mathrm{E}\big(\widehat{r}_{W,2}(k)\big) &= \frac{2}{9}(T-2k)^3 + \frac{1}{3}(T-2k)^2 + \frac{1}{9}(T-2k) \\
\mathrm{var}\big(\widehat{r}_{W,2}(k)\big) &= \frac{8}{225}(T-2k)^5 + \frac{3k+16}{180}(T-2k)^4 - \frac{(k^2-9k-16)}{270}(T-2k)^3 \\
&\quad - \frac{(k^2-3k)}{180}(T-2k)^2 - \frac{(5k^2+16)}{2700}(T-2k) - \frac{1}{900}(9k^5+k)
\end{aligned}$$

Note. These results for $\widehat{r}_M(1)$ and $\widehat{r}_W(1)$ agree with those from [Moore and Wallis, 1943] and [Wallis and Moore, 1941].

In general, circular serial coefficients can be found useful for some processes defined on a circle, for example in the case of 24 measurements (at hourly intervals) of a daily cycle. However, we are not interested in such time series and that is why we will focus on the noncircular coefficients in the following text.

Theorem 3 (Covariances between $\widehat{r}_{M,w}(k)$'s)**.** *Under the null hypothesis H_0^E, the covariances between $\widehat{r}_{M,w}(k)$'s can be expressed in the following way ($1 \le h < k < T$):*

$$\begin{aligned}\operatorname{cov}\bigl(\widehat{r}_{M,w_k}(k), \widehat{r}_{M,w_h}(h)\bigr) &= p_M\pi_p^M + q_M\pi_q^M + r_M\pi_r^M + s_M\pi_s^M + t_M\pi_t^M \\ &\quad - \mathrm{E}\bigl(\widehat{r}_{M,w_k}(k)\bigr)\,\mathrm{E}\bigl(\widehat{r}_{M,w_h}(h)\bigr) \\ &= \frac{1}{12}(p_M - q_M - r_M + s_M),\end{aligned}$$

where

$$p_M = \sum_{i=1}^{T-k} w_k(i)w_h(i), \qquad q_M = \sum_{i=1}^{T-h-k} w_k(i)w_h(i+k),$$

$$r_M = \sum_{j=1}^{T-h-k} w_k(j+h)w_h(j), \qquad s_M = \sum_{i=1}^{T-k} w_k(i)w_h(i+k-h),$$

$$t_M = \sum_{i=1}^{T-k}\sum_{j=1}^{T-h} w_k(i)w_h(j) - p_M - q_M - r_M - s_M,$$

$$\pi_p^M = P(R_1 > R_2, R_1 > R_3) = \frac{1}{3}, \qquad \pi_q^M = P(R_1 > R_2, R_2 > R_3) = \frac{1}{6},$$

$$\pi_r^M = P(R_1 > R_2, R_3 > R_1) = \frac{1}{6}, \qquad \pi_s^M = P(R_1 > R_2, R_3 > R_2) = \frac{1}{3},$$

$$\pi_t^M = P(R_1 > R_2, R_3 > R_4) = \frac{1}{4}.$$

Proof. All that follows directly from the definition of covariances and from the exchangeability of the ranks under H_0^E. The ranks may coincide and each such case must be treated separately. □

Theorem 4 (Covariances between $\widehat{r}_{W,w}(k)$'s)**.** *Under the null hypothesis H_0^E, the covariances between $\widehat{r}_{W,w}(k)$'s can be expressed in the following way ($1 \le h < k < T/2$):*

$$\begin{aligned}&\operatorname{cov}\bigl(\widehat{r}_{W,w_k}(k), \widehat{r}_{W,w_h}(h)\bigr) \\ &\quad = p_W\pi_p^W + q_W\pi_q^W + r_W\pi_r^W + s_W\pi_s^W + t_W\pi_t^W + u_W\pi_u^W + v_W\pi_v^W + w_W\pi_w^W + x_W\pi_x^W \\ &\qquad + z_W\pi_z^W - \mathrm{E}\bigl(\widehat{r}_{W,w_k}(k)\bigr)\,\mathrm{E}\bigl(\widehat{r}_{W,w_h}(h)\bigr) \\ &\quad = \frac{1}{180}(p_W - 2q_W - 2r_W + s_W + t_W + 4u_W + v_W - 2w_W - 2x_W) \quad \text{if} \quad k \neq 2h, \\ &\quad = \frac{1}{180}(-5p_W - 5q_W - 2r_W + s_W + t_W + 4u_W + v_W - 2w_W - 2x_W) \quad \text{if} \quad k = 2h,\end{aligned}$$

where

$$p_W = \sum_{i=1}^{T-2k} w_k(i)w_h(i),$$

$$q_W = \sum_{i=1}^{T-2k} w_k(i)w_h(i+k) \ \textit{if } k > 2h, \\ = \sum_{i=1}^{T-2h-k} w_k(i)w_h(i+k) \ \textit{if } k \le 2h,$$

$$r_W = \sum_{j=1}^{T-2k-h} w_k(j+h)w_h(j),$$

$$s_W = \sum_{i=1}^{T-2h-2k} w_k(i)w_h(i+2k),$$

$$t_W = \sum_{j=1}^{T-2k-2h} w_k(j+2h)w_h(j),$$

$$u_W = \sum_{i=1}^{T-2k} w_k(i)w_h(i+k-h),$$

$$v_W = \sum_{i=1}^{T-2k} w_k(i)w_h(i+2k-2h) \ \textit{if } k \neq 2h, \\ = 0 \ \textit{if } k = 2h,$$

$$w_W = \sum_{i=1}^{T-h-2k} w_k(i)w_h(i+2k-h),$$

$$x_W = \sum_{i=1}^{T-2k} w_k(i)w_h(i+k-2h) \ \textit{if } k > 2h, \\ = 0 \ \textit{if } k = 2h, \\ = \sum_{j=1}^{T-k-2h} w_k(j-k+2h)w_h(j) \ \textit{if } k < 2h,$$

$$z_W = \sum_{i=1}^{T-2k}\sum_{j=1}^{T-2h} w_k(i)w_h(j) - p_W - q_W \\ - r_W - s_W - t_W - u_W - v_W - w_W - x_W,$$

$$\begin{aligned}
\pi_p^W &= 2P(R_1 > R_2, R_2 < R_3, R_1 > R_4, R_4 < R_5) + 2P(R_1 > R_2, R_2 < R_3, R_1 < R_4, R_4 > R_5) \\
&= 2\cdot\frac{16}{120} + 2\cdot\frac{11}{120} = \frac{9}{20} \ \textit{if } k \neq 2h, \\
&= 2P(R_1 > R_2, R_2 < R_3, R_1 > R_4, R_4 < R_2) + 2P(R_1 > R_2, R_2 < R_3, R_1 < R_4, R_4 > R_2) \\
&= 2\cdot\frac{2}{24} + 2\cdot\frac{3}{24} = \frac{5}{12} \ \textit{if } k = 2h, \\
\pi_q^W &= 2P(R_1 > R_2, R_2 < R_3, R_2 > R_4, R_4 < R_5) + 2P(R_1 > R_2, R_2 < R_3, R_2 < R_4, R_4 > R_5) \\
&= 2\cdot\frac{8}{120} + 2\cdot\frac{18}{120} = \frac{13}{30} \ \textit{if } k \neq 2h, \\
&= 2P(R_1 > R_2, R_2 < R_3, R_2 > R_4, R_4 < R_3) + 2P(R_1 > R_2, R_2 < R_3, R_2 < R_4, R_4 > R_3) \\
&= 2\cdot\frac{2}{24} + 2\cdot\frac{3}{24} = \frac{5}{12} \ \textit{if } k = 2h, \\
\pi_r^W &= 2P(R_1 > R_2, R_2 < R_3, R_4 > R_1, R_1 < R_5) + 2P(R_1 > R_2, R_2 < R_3, R_4 < R_1, R_1 > R_5) \\
&= 2\cdot\frac{8}{120} + 2\cdot\frac{18}{120} = \frac{13}{30}, \\
\pi_s^W &= 2P(R_1 > R_2, R_2 < R_3, R_3 > R_4, R_4 < R_5) + 2P(R_1 > R_2, R_2 < R_3, R_3 < R_4, R_4 > R_5) \\
&= 2\cdot\frac{16}{120} + 2\cdot\frac{11}{120} = \frac{9}{20},
\end{aligned}$$

$$
\begin{aligned}
\pi_t^W &= 2P(R_1 > R_2, R_2 < R_3, R_4 > R_5, R_5 < R_1) + 2P(R_1 > R_2, R_2 < R_3, R_4 < R_5, R_5 > R_1) \\
&= 2 \cdot \frac{16}{120} + 2 \cdot \frac{11}{120} = \frac{9}{20}, \\
\pi_u^W &= 2P(R_1 > R_2, R_2 < R_3, R_4 > R_2, R_2 < R_5) + 2P(R_1 > R_2, R_2 < R_3, R_4 < R_2, R_2 > R_5) \\
&= 2 \cdot \frac{24}{120} + 2 \cdot \frac{4}{120} = \frac{7}{15}, \\
\pi_v^W &= 2P(R_1 > R_2, R_2 < R_3, R_4 > R_5, R_5 < R_3) + 2P(R_1 > R_2, R_2 < R_3, R_4 < R_5, R_5 > R_3) \\
&= 2 \cdot \frac{16}{120} + 2 \cdot \frac{11}{120} = \frac{9}{20}, \\
\pi_w^W &= 2P(R_1 > R_2, R_2 < R_3, R_4 > R_3, R_3 < R_5) + 2P(R_1 > R_2, R_2 < R_3, R_4 < R_3, R_3 > R_5) \\
&= 2 \cdot \frac{8}{120} + 2 \cdot \frac{18}{120} = \frac{13}{30}, \\
\pi_x^W &= 2P(R_1 > R_2, R_2 < R_3, R_4 > R_5, R_5 < R_2) + 2P(R_1 > R_2, R_2 < R_3, R_4 < R_5, R_5 > R_2) \\
&= 2 \cdot \frac{8}{120} + 2 \cdot \frac{18}{120} = \frac{13}{30}, \\
\pi_z^W &= 4P(R_1 > R_2, R_2 < R_3, R_4 > R_5, R_5 < R_6) \\
&= 4 \cdot \frac{1}{9} = \frac{4}{9}.
\end{aligned}
$$

Proof. All that follows immediately from the definition of covariances and from the exchangeability of the ranks under H_0^E. The ranks may coincide and each such case must be treated separately. Some details can be verified by means of the R code *MWProbs.r*. $\square$

Note. We mark w with the h or k to denote the proper lag to be used in case the weights are lag-dependent.

Corollary. The exact formulae for the covariances $\operatorname{cov}\big(\widehat{r}_{M,w_h}(h), \widehat{r}_{M,w_k}(k)\big)$'s, $1 \le h < k < T$, and $\operatorname{cov}\big(\widehat{r}_{W,w_h}(h), \widehat{r}_{W,w_k}(k)\big)$'s, $1 \le h < k < T/2$, can be obtained under H_0^E for any polynomial weighting function with the aid of the Maple codes *MWCovsM.mws* and *MWCovsW.mws*, respectively. They are listed below for some simple choices of weights and under the additional assumptions $h + k < T$ in the former case and $h + k < T/2$ in the latter. They can be verified with the program *MWExactMom.r*.

$\boxed{w(i) = 1}$

$\operatorname{cov}\big(\widehat{r}_M(k), \widehat{r}_M(h)\big) =$

$$= \frac{1}{6} h$$

$\mathrm{cov}\big(\widehat{r}_W(k), \widehat{r}_W(h)\big) =$

$$\begin{aligned}
&= -\frac{1}{45}k + \frac{2}{45}h \text{ if } k < 2h \\
&= -\frac{2}{45}T + \frac{11}{180}k + \frac{1}{18}h \text{ if } k = 2h \\
&= 0 \text{ if } k > 2h
\end{aligned}$$

$\boxed{w(i) = i}$

$\mathrm{cov}\big(\widehat{r}_{M,1}(k), \widehat{r}_{M,1}(h)\big) =$

$$= \frac{1}{12}hT^2 + \Big(-\frac{1}{12}kh - \frac{1}{12}h^2 + \frac{1}{12}h\Big)T + \frac{1}{24}kh^2 + \frac{1}{72}h^3 - \frac{1}{24}kh - \frac{1}{24}h^2 + \frac{1}{36}h$$

$\mathrm{cov}\big(\widehat{r}_{W,1}(k), \widehat{r}_{W,1}(h)\big) =$

$$\begin{aligned}
&= \Big(-\frac{1}{90}k + \frac{1}{45}h\Big)T^2 + \Big(\frac{1}{45}k^2 - \frac{1}{45}kh - \frac{2}{45}h^2 - \frac{1}{90}k + \frac{1}{45}h\Big)T \\
&\quad - \frac{1}{135}k^3 - \frac{1}{90}k^2h + \frac{1}{18}kh^2 + \frac{2}{135}h^3 + \frac{1}{90}k^2 - \frac{1}{90}kh - \frac{1}{45}h^2 - \frac{1}{270}k + \frac{1}{135}h \text{ if } k < 2h \\
&= -\frac{2}{135}T^3 + \Big(\frac{19}{360}k + \frac{1}{20}h - \frac{1}{45}\Big)T^2 + \Big(-\frac{4}{45}k^2 - \frac{1}{30}kh - \frac{1}{9}h^2 + \frac{19}{360}k + \frac{1}{20}h - \frac{1}{135}\Big)T \\
&\quad + \frac{67}{1080}k^3 - \frac{1}{45}k^2h + \frac{1}{15}kh^2 + \frac{2}{27}h^3 - \frac{2}{45}k^2 - \frac{1}{60}kh - \frac{1}{18}h^2 + \frac{11}{1080}k + \frac{1}{108}h \text{ if } k = 2h \\
&= \frac{1}{90}kh^2 \text{ if } k > 2h
\end{aligned}$$

$\boxed{w(i) = i^2}$

$\mathrm{cov}\big(\widehat{r}_{M,2}(k), \widehat{r}_{M,2}(h)\big) =$

$$\begin{aligned}
&= \frac{1}{12}hT^4 + \Big(-\frac{2}{9}kh - \frac{1}{6}h^2 + \frac{1}{6}h\Big)T^3 + \Big(\frac{1}{4}k^2h + \frac{1}{4}kh^2 + \frac{1}{6}h^3 - \frac{1}{3}kh - \frac{1}{4}h^2 + \frac{1}{12}h\Big)T^2 \\
&\quad + \Big(-\frac{1}{6}k^3h - \frac{1}{12}k^2h^2 - \frac{1}{6}kh^3 - \frac{1}{12}h^4 + \frac{1}{4}k^2h + \frac{1}{4}kh^2 + \frac{1}{6}h^3 - \frac{1}{9}kh - \frac{1}{12}h^2\Big)T \\
&\quad + \frac{1}{18}k^4h + \frac{1}{36}k^2h^3 + \frac{1}{24}kh^4 + \frac{7}{360}h^5 - \frac{1}{12}k^3h - \frac{1}{24}k^2h^2 - \frac{1}{12}kh^3 - \frac{1}{24}h^4 + \frac{1}{24}k^2h \\
&\quad + \frac{1}{24}kh^2 + \frac{1}{36}h^3 - \frac{1}{180}h
\end{aligned}$$

$\operatorname{cov}\big(\widehat{r}_{W,2}(k), \widehat{r}_{W,2}(h)\big) =$

$$
\begin{aligned}
&= \Big(-\frac{1}{90}k + \frac{1}{45}h\Big)T^4 + \Big(\frac{2}{45}k^2 - \frac{2}{45}kh - \frac{4}{45}h^2 - \frac{1}{45}k + \frac{2}{45}h\Big)T^3 \\
&\quad + \Big(-\frac{1}{15}k^3 + \frac{1}{45}k^2h + \frac{7}{45}kh^2 + \frac{7}{45}h^3 + \frac{1}{15}k^2 - \frac{1}{15}kh - \frac{2}{15}h^2 - \frac{1}{90}k + \frac{1}{45}h\Big)T^2 \\
&\quad + \Big(\frac{2}{45}k^4 - \frac{4}{45}k^2h^2 - \frac{8}{45}kh^3 - \frac{2}{15}h^4 - \frac{1}{15}k^3 + \frac{1}{45}k^2h + \frac{7}{45}kh^2 + \frac{7}{45}h^3 + \frac{1}{45}k^2 - \frac{1}{45}kh \\
&\quad - \frac{2}{45}h^2\Big)T - \frac{8}{675}k^5 + \frac{1}{270}k^4h - \frac{2}{135}k^3h^2 + \frac{14}{135}k^2h^3 + \frac{1}{30}kh^4 + \frac{4}{75}h^5 + \frac{1}{45}k^4 - \frac{2}{45}k^2h^2 \\
&\quad - \frac{4}{45}kh^3 - \frac{1}{15}h^4 - \frac{1}{90}k^3 + \frac{1}{270}k^2h + \frac{7}{270}kh^2 + \frac{7}{270}h^3 + \frac{1}{1350}k - \frac{1}{675}h \text{ if } k < 2h \\
&= -\frac{2}{225}T^5 + \Big(\frac{19}{360}k + \frac{1}{20}h - \frac{1}{45}\Big)T^4 + \Big(-\frac{101}{540}k^2 - \frac{1}{15}kh - \frac{29}{135}h^2 + \frac{19}{180}k + \frac{1}{10}h - \frac{2}{135}\Big)T^3 \\
&\quad + \Big(\frac{2}{5}k^3 - \frac{7}{90}k^2h + \frac{14}{45}kh^2 + \frac{19}{45}h^3 - \frac{101}{360}k^2 - \frac{1}{10}kh - \frac{29}{90}h^2 + \frac{19}{360}k + \frac{1}{20}h\Big)T^2 \\
&\quad + \Big(-\frac{4}{9}k^4 + \frac{8}{45}k^3h - \frac{1}{15}k^2h^2 - \frac{4}{9}kh^3 - \frac{2}{5}h^4 + \frac{2}{5}k^3 - \frac{7}{90}k^2h + \frac{14}{45}kh^2 + \frac{19}{45}h^3 - \frac{101}{1080}k^2 \\
&\quad - \frac{1}{30}kh - \frac{29}{270}h^2 + \frac{1}{675}\Big)T + \frac{1061}{5400}k^5 - \frac{4}{45}k^4h - \frac{8}{135}k^3h^2 + \frac{16}{135}k^2h^3 + \frac{53}{270}kh^4 + \frac{4}{27}h^5 \\
&\quad - \frac{2}{9}k^4 + \frac{4}{45}k^3h - \frac{1}{30}k^2h^2 - \frac{2}{9}kh^3 - \frac{1}{5}h^4 + \frac{1}{15}k^3 - \frac{7}{540}k^2h + \frac{7}{135}kh^2 + \frac{19}{270}h^3 \\
&\quad - \frac{11}{5400}k - \frac{1}{540}h \text{ if } k = 2h \\
&= \Big(\frac{1}{45}kh^2 - \frac{1}{45}h^3\Big)T^2 + \Big(-\frac{2}{45}k^2h^2 + \frac{2}{45}h^4 + \frac{1}{45}kh^2 - \frac{1}{45}h^3\Big)T \\
&\quad + \frac{2}{45}k^2h^3 - \frac{7}{270}kh^4 - \frac{4}{135}h^5 - \frac{1}{45}k^2h^2 + \frac{1}{45}h^4 + \frac{1}{270}kh^2 - \frac{1}{270}h^3 \text{ if } k > 2h
\end{aligned}
$$

Note. In general, the assumption of polynomial weights is not too restrictive because any sufficiently smooth function can be approximated satisfactorily by its Taylor polynomial.

Theorem 5 (Covariances between $\widehat{r}_{M,w_M}(h)$'s and $\widehat{r}_{W,w_W}(k)$'s)**.** *If H_0^E holds, then the coefficients $\widehat{r}_{M,w_M}(h)$'s and $\widehat{r}_{W,w_W}(k)$'s are mutually uncorrelated for every possible choice of weighting functions w_M and w_W, i.e.*

$$\operatorname{cov}\big(\widehat{r}_{M,w_M}(h), \widehat{r}_{W,w_W}(k)\big) = 0, \qquad 1 \le h < T,\ 1 \le k < T/2. \tag{3.1}$$

Proof. It follows directly from the definition of covariances and from the exchangeability of the ranks under H_0^E that

$$
\begin{aligned}
\operatorname{cov}\big(\widehat{r}_{M,w_M}(h), \widehat{r}_{W,w_W}(k)\big) &= A\pi_A + B\pi_B + C\pi_C + D\pi_D + E\pi_E + F\pi_F + G\pi_G \\
&\quad - \operatorname{E}\big(\widehat{r}_{M,w_M}(h)\big)\operatorname{E}\big(\widehat{r}_{W,w_W}(k)\big),
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \sum_{i=1}^{T-h} w_W(i)w_M(i) \text{ if } h > 2k,\\
&= \sum_{i=1}^{T-2k} w_W(i)w_M(i) \text{ if } h \leq 2k,
\end{aligned}
$$

$$
B = \sum_{j=1}^{T-2k-h} w_W(j+h)w_M(j),
$$

$$
\begin{aligned}
C &= \sum_{i=1}^{T-h-k} w_W(i)w_M(i+k) \text{ if } h > k,\\
&= \sum_{i=1}^{T-2k} w_W(i)w_M(i+k) \text{ if } h \leq k,
\end{aligned}
$$

$$
\begin{aligned}
D &= \sum_{j=1}^{T-h-k} w_W(j+h-k)w_M(j) \text{ if } h > k,\\
&= 0 \text{ if } h = k,\\
&= \sum_{i=1}^{T-2k} w_W(i)w_M(i+k-h) \text{ if } h \leq k,
\end{aligned}
$$

$$
E = \sum_{i=1}^{T-2h-k} w_W(i)w_M(i+2k),
$$

$$
\begin{aligned}
F &= \sum_{j=1}^{T-h} w_W(j+h-2k)w_M(j) \text{ if } h > 2k,\\
&= 0 \text{ if } h = k \text{ or } h = 2k,\\
&= \sum_{i=1}^{T-2k} w_W(i)w_M(j+2k-h) \text{ otherwise},
\end{aligned}
$$

$$
\begin{aligned}
G = &\sum_{i=1}^{T-2k}\sum_{j=1}^{T-h} w_W(i)w_M(j) - A - B\\
&- C - D - E - F - G,
\end{aligned}
$$

$$
\begin{aligned}
\pi_A &= P(R_1 > R_2, R_2 < R_3)\\
&= \frac{1}{3} \text{ if } h = k,\\
&= P(R_1 > R_2, R_2 < R_3, R_1 > R_3) + P(R_1 < R_2, R_2 > R_3, R_1 > R_3)\\
&= \frac{1}{6} + \frac{1}{6} = \frac{1}{3} \text{ if } h = 2k,\\
&= P(R_1 > R_2, R_2 < R_3, R_1 > R_4) + P(R_1 < R_2, R_2 > R_3, R_1 > R_4)\\
&= \frac{5}{24} + \frac{3}{24} = \frac{1}{3} \text{ otherwise},\\
\pi_B &= P(R_1 > R_2, R_2 < R_3, R_4 > R_1) + P(R_1 < R_2, R_2 > R_3, R_4 > R_1)\\
&= \frac{3}{24} + \frac{5}{24} = \frac{1}{3} \text{ if } k \neq 2h,
\end{aligned}
$$

$$
\begin{aligned}
\pi_C &= P(R_1 < R_2, R_2 > R_3) \\
&= \frac{1}{3} \text{ if } h = k, \\
&= P(R_1 > R_2, R_2 < R_3, R_2 > R_4) + P(R_1 < R_2, R_2 > R_3, R_2 > R_4) \\
&= \frac{2}{24} + \frac{6}{24} = \frac{1}{3} \text{ if } h \neq k, \\
\pi_D &= P(R_1 > R_2, R_2 < R_3, R_4 > R_2) + P(R_1 < R_2, R_2 > R_3, R_4 > R_2) \\
&= \frac{6}{24} + \frac{2}{24} = \frac{1}{3}, \\
\pi_E &= P(R_1 > R_2, R_2 < R_3, R_3 > R_4) + P(R_1 < R_2, R_2 > R_3, R_3 > R_4) \\
&= \frac{5}{24} + \frac{3}{24} = \frac{1}{3}, \\
\pi_F &= P(R_1 > R_2, R_2 < R_3, R_4 > R_3) + P(R_1 < R_2, R_2 > R_3, R_4 > R_3) \\
&= \frac{3}{24} + \frac{5}{24} = \frac{1}{3}, \\
\pi_G &= P(R_1 > R_2, R_2 < R_3, R_4 > R_5) + P(R_1 < R_2, R_2 > R_3, R_4 > R_5) \\
&= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.
\end{aligned}
$$

Therefore

$$
\operatorname{cov}(\widehat{r}_{M,w_M}(h), \widehat{r}_{W,w_W}(k)) = \frac{1}{3}(A+B+C+D+E+F+G) - \frac{2}{3} \cdot \frac{1}{2} \sum_{i=1}^{T-2k} \sum_{j=1}^{T-h} w_W(i) w_M(j) = 0.
$$

Some details regarding this proof can be verified by means of the programs *MWProbs.r* and *MWCovsMW.mws*. □

Corollary. The well known sign and turning point statistics $\widehat{r}_M(1)$ and $\widehat{r}_W(1)$ are thus uncorrelated under H_0^E. Later we shall see that this implies their asymptotic independence in that case.

3.3 Standardization

Apparently, too small/high values of $\widehat{r}_{M,w}(k)$'s or $\widehat{r}_{W,w}(k)$'s agree with strong positive/negative serial dependence at lag k. However, they should be standardized in some way that would allow their better interpretation, for example by rescaling them to be just covered by the $[-1, 1]$ interval. We prefer to normalize them with their true means and variances under H_0^E, which leads to another useful definition:

Definition 6. We define the standardized Moore and Wallis serial rank coefficients $\widetilde{r}_{M,w}(k)$'s and $\widetilde{r}_{W,w}(k)$'s as follows:

$$
\widetilde{r}_{M,w}(k) = \frac{\mathrm{E}_0\big(\widehat{r}_{M,w}(k)\big) - \widehat{r}_{M,w}(k)}{\sqrt{\operatorname{var}_0\big(\widehat{r}_{M,w}(k)\big)}}, \qquad (1 \le k < T)
$$

$$\widetilde{r}_{W,w}(k) = \frac{\mathrm{E}_0\big(\widehat{r}_{W,w}(k)\big) - \widehat{r}_{W,w}(k)}{\sqrt{\mathrm{var}_0\big(\widehat{r}_{W,w}(k)\big)}}, \qquad (1 \le k < T/2)$$

and we define their standardized circular modifications analogously. All the subscripts of r remain unchanged and always keep their original meaning.

Note. Higher values of these standardized coefficients correspond to positive serial dependence as usual. Their evaluation under H_0^E is made possible by the following theorem.

3.4 Asymptotic Distribution

Theorem 7 (Asymptotic distribution under H_0^E). *Let us assume that H_0^E holds and that $p, q \ge 1$ are two integers.*

1. *If*

$$\frac{\max_{1\le i\le T-1} w_M(i)}{\min_{1\le k\le p} \sqrt{\mathrm{var}\big(\widehat{r}_{M,w_M}(k)\big)}} = o(1) \quad \textit{for} \quad T \to \infty, \tag{3.2}$$

then

$$\widetilde{\mathbf{r}}_M = \widetilde{\mathbf{r}}_{M,w_M,p} = \big(\widetilde{r}_{M,w_M}(1), \ldots, \widetilde{r}_{M,w_M}(p)\big)' \sim_{asympt.} N(0, U_{p\times p}).$$

2. *If*

$$\frac{\max_{1\le i\le T-2} w_W(i)}{\min_{1\le k\le q} \sqrt{\mathrm{var}\big(\widehat{r}_{W,w_W}(k)\big)}} = o(1) \quad \textit{for} \quad T \to \infty, \tag{3.3}$$

then

$$\widetilde{\mathbf{r}}_W = \widetilde{\mathbf{r}}_{W,w_W,q} = \big(\widetilde{r}_{W,w_W}(1), \ldots, \widetilde{r}_{W,w_W}(q)\big)' \sim_{asympt.} N(0, V_{q\times q}).$$

3. *If both (3.2) and (3.3) hold, then even the vector $\widetilde{\mathbf{r}}_{MW} = (\widetilde{\mathbf{r}}'_{M,w_M}, \widetilde{\mathbf{r}}'_{W,w_W})'$ is asymptotically zero mean normal with the block diagonal variance matrix $W = \left(\begin{smallmatrix} U_{p\times p} & 0 \\ 0 & V_{q\times q}\end{smallmatrix}\right)$.*

The matrices $U_{p\times p}$ and $V_{q\times q}$ can be inferred from the theory in Section 3.2.

Note. Analogous statements can be formulated even for the circular standardized coefficients $\widetilde{r}^{\circ}_{M,w_M}(k)$'s and $\widetilde{r}^{\circ}_{W,w_W}(k)$'s.

Proof. We only focus on (3) because (1) and (2) can be treated in the same way.

If H_0 holds, then the asymptotic distribution of $\widetilde{\mathbf{r}}_{MW}$ does not depend on $\mathcal{L}(Y_t)$, hence we can assume that Y_t's are uniformly distributed on the $[0, 1]$ interval.

If we set $z = \max(p, 2q)$ and define random vectors

$$\mathbf{X}(i) = \big(X_1(i), \ldots, X_{p+q}(i)\big)', \ i = 1, 2, \ldots, T - z,$$

such that

$$X_k(i) = \frac{w_M(i)\big[\frac{1}{2} - \mathrm{I}(Y_i > Y_{i+k})\big]}{\sqrt{\mathrm{var}\big(\widehat{r}_{M,w_M}(k)\big)}}, \ 1 \le k \le p,$$
$$X_{p+m}(i) = \frac{w_W(i)\big[\frac{2}{3} - \mathrm{I}(Y_i > Y_{i+m}, Y_{i+m} < Y_{i+2m}) - \mathrm{I}(Y_i < Y_{i+m}, Y_{i+m} > Y_{i+2m})\big]}{\sqrt{\mathrm{var}\big(\widehat{r}_{W,w_W}(m)\big)}}, \ 1 \le m \le q,$$

then the assumptions (3.2) and (3.3) guarantee that $\widetilde{\mathbf{r}}_{MW}$ is asymptotically equivalent to the sum $\mathbf{S}_T$ of the zero mean z-dependent random vectors $\mathbf{X}(i)$'s,

$$\mathbf{S}_T = \sum_{i=1}^{T-z} \mathbf{X}(i).$$

The asymptotic zero mean normal distribution thus follows easily from (3.1) and from any suitable central limit theorem for m-dependent random vectors. Alternatively, we could use the Cramér-Wold device and a univariate central limit theorem of this type.

The joint distribution of R_i's is identical under both H_0 and H_0^E (see [Dufour and Roy, 1986]), which implies that the same holds even for the asymptotic distribution of $\widetilde{\mathbf{r}}_{MW}$. □

Corollary. If $w_M(i) = i^{j_M}, \ j_M = 0, 1, 2, \ w_W(i) = i^{j_W}, \ j_W = 0, 1, 2,$ and $1 \le h < k < T/4$, then

$$\begin{aligned}
\mathrm{cov}\big(\widetilde{r}_{M,w_M}(k), \widetilde{r}_{M,w_M}(h)\big) &= O(T^{-1}), \\
\mathrm{cov}\big(\widetilde{r}_{W,w_W}(k), \widetilde{r}_{W,w_W}(h)\big) &= O(T^{-1}) \text{ if } k < 2h, \\
&= -\frac{1}{4} \text{ if } k = 2h, \\
&= 0 \text{ or } O(T^{-3}) \text{ if } k > 2h,
\end{aligned}$$

and the asymptotic variance matrix $W = (W_{kh})_{k,h=1}^{p+q}$ of $\widetilde{\mathbf{r}}_{MW}$ has then the following elements:

$$\begin{aligned}
W_{kh} &= 1 \text{ if } k = h \text{ and } k, h = 1, 2, \ldots, p + q, \\
&= -\frac{1}{4} \text{ if } k - p = 2(h - p) \text{ or } h - p = 2(k - p), \ p < k, h \le p + q, \\
&= 0 \text{ otherwise.}
\end{aligned}$$

In other words, certain components of $\widetilde{\mathbf{r}}_{MW}$ are then asymptotically jointly zero mean normal and independent, for example $\widetilde{\mathbf{r}}_M$ and $\widetilde{\mathbf{r}}_W$, or all the Moore serial rank coefficients together with all the Wallis ones at odd lags. This fact has strong implications for any statistical inference based on $\widetilde{\mathbf{r}}_M$ and $\widetilde{\mathbf{r}}_W$.

The proof of Theorem 7 indicates that $\widetilde{\mathbf{r}}_{MW}$ remains asymptotically normal even under a large number of mixing alternatives as far as a proper central limit theorem can be applied, see [Bradley, 2005] for a survey of strong mixing conditions and their properties. We recall that many important models (such as numerous Markov, ARMA, GARCH or ACD processes) have been proved α-mixing, see [Bradley, 2005], [Meitz and Saikkonen, 2004], [Carrasco and Chen, 2000] and references therein.

Besides, both $\widetilde{r}_{M,w}(k)$'s and $\widetilde{r}_{W,w}(k)$'s are closely related to the weighted serial rank statistics considered in [Harel and Puri, 1990] and the $\widetilde{r}_{M,w}(k)$'s roughly correspond to those investigated in [Jogdeo, 1968].

More theoretical results can be applied in the special case of constant weights. For example, $\widetilde{r}_{W}(k)$'s have much in common with U-statistics, e.g. with the following incomplete ones of degree two:

$$U_T(k) = \frac{2}{(T-k)(T-k-1)} \sum_{i=1}^{T-k} \sum_{j=i+1}^{T-k} u_{i,j} g(\mathbf{Z}_i, \mathbf{Z}_j),$$

where

$$\begin{aligned} \mathbf{Z}_i &= (Y_i, Y_{i+k}),\ \mathbf{Z}_j = (Y_j, Y_{j+k}), \\ g(\mathbf{Z}_i, \mathbf{Z}_j) &= \mathrm{I}(Y_i > Y_j, Y_{i+k} < Y_{j+k}) + \mathrm{I}(Y_i < Y_j, Y_{i+k} > Y_{j+k}), \end{aligned}$$

and

$$\begin{aligned} u_{i,j} &= 1 \text{ if } j = i + k, \\ &= 0 \text{ otherwise.} \end{aligned}$$

See [Hsing and Wu, 2004] and references therein for current knowledge in this field. Another relation of $\widetilde{r}_{W}(k)$'s (and $\widetilde{r}_{M}(k)$'s) to U-statistics follows from [Hallin et al., 1985].

Another theory suitable for individual $\widetilde{r}_{M}(k)$'s and $\widetilde{r}_{W}(k)$'s can be found in [Turova, 2004] and includes the Berry-Esseen and central limit theorems. Besides, all the results proved for serial linear rank statistics (see, for example, [Hallin et al., 1985] and [Hallin et al., 1987]) may be applied to $\widetilde{r}_{M}(k)$'s and $\widetilde{r}_{W}(k)$'s as well. For example, useful asymptotic relative efficiency details can be obtained this way in the context of testing randomness against contiguous ARMA alternatives. We illustrate this in the next section.

Although there exists a good deal of related theory, we believe that Theorem 7 still remains useful because it allows for general weights, its proof is easy, its weak assumptions can be easily verified and some of its statements for inconstant w_M or w_W do not follow directly from any other available results, at least to the best of our knowledge.

3.5 Asymptotic Relative Efficiency (ARMA Alternatives)

First, let us consider the statistics $S_M(k)$'s and $S_W(k)$'s:

$$S_M(k) = \frac{1}{T-k} \sum_{i=1}^{T-k} \mathrm{I}(R_i > R_{i+k}),$$

$$S_W(k) = \frac{1}{T-2k} \sum_{i=1}^{T-2k} \left[\mathrm{I}(R_i > R_{i+k}, R_{i+k} < R_{i+2k}) + \mathrm{I}(R_i < R_{i+k}, R_{i+k} > R_{i+2k}) \right],$$

with the associated score-generating functions $J_{M,k}(u_1, \ldots, u_{k+1})$ and $J_{W,k}(u_1, \ldots, u_{2k+1})$ defined in $[0,1] \times \cdots \times [0,1]$:

$$\begin{aligned} J_{M,k}(u_1, \ldots, u_{k+1}) &= \mathrm{I}(u_1 > u_{k+1}), \\ J_{W,k}(u_1, \ldots, u_{2k+1}) &= \mathrm{I}(u_1 > u_{k+1}, u_{k+1} < u_{2k+1}) + \mathrm{I}(u_1 < u_{k+1}, u_{k+1} > u_{2k+1}). \end{aligned}$$

In particular, we focus on their centered versions $S_M^c(k)$ and $S_W^c(k)$ that arise from replacing $J_{M,k}$ and $J_{W,k}$ with $J^\star_{M,k}$ and $J^\star_{W,k}$:

$$\begin{aligned} J^\star_{M,k}(u_1, \ldots, u_{k+1}) &= \mathrm{I}(u_1 > u_{k+1}) - u_1 + u_{k+1} - \frac{1}{2}, \\ J^\star_{W,k}(u_1, \ldots, u_{2k+1}) &= \mathrm{I}(u_1 > u_{k+1}, u_{k+1} < u_{2k+1}) + \mathrm{I}(u_1 < u_{k+1}, u_{k+1} > u_{2k+1}) \\ &\qquad + u_1^2 - u_1 + u_{2k+1}^2 - u_{2k+1} + 2u_{k+1} - 2u_{k+1}^2 - \frac{2}{3}. \end{aligned}$$

[Hallin et al., 1985] developed a theory for such statistics that can be applied directly in this context. Their results relevant for us are summarized in the following theorems.

Theorem 8 (ARMA alternatives contiguous to H_0)**.** *Let ε_t's be independent and identically distributed zero mean random variables with the finite third moments, density $f(x)$, distribution function $F(x)$ and the quantile function $F^{-1}(u) = \inf\{x | F(x) \geq u\},\ 0 < u < 1$, such that*

1. *$f(x)$ is almost everywhere (a.e.) derivable and its derivative $f'(x)$ satisfies*

$$\int_{-\infty}^{\infty} |f'(x)| dx < \infty.$$

2. *$f(x)$ is absolutely continuous on finite intervals and*

$$0 < I(f) = \int_{-\infty}^{\infty} \left(\frac{f'(x)}{f(x)} \right)^2 f(x) dx < \infty.$$

3. *The function ϕ,*

$$\phi\big(F^{-1}(u)\big) = -\frac{f'\big(F^{-1}(u)\big)}{f\big(F^{-1}(u)\big)},\ 0 < u < 1,\ \Big(\text{i.e. } \phi(x) = -\frac{f'(x)}{f(x)} \text{ a.e.}\Big),$$

is derivable (a.e.), and its derivative $\phi'(x)$ satisfies (a.e.) a Lipschitz condition

$$|\phi'(x) - \phi'(y)| < A|x - y|.$$

Then the ARMA(p,q) alternative $H_1(T)$ with its associated vector $d = (d_1, \dots, d_{\max(p,q)})$:

$$Y_t - T^{-1/2}\sum_{i=1}^{p} a_i Y_{t-i} = \varepsilon_t + T^{-1/2}\sum_{i=1}^{q} b_i \varepsilon_{t-i}$$

where $a_1, \dots, a_p, b_1, \dots, b_q$ are arbitrary real numbers and

$$\begin{aligned} d_i &= a_i + b_i \text{ if } 1 \le i < \min(p,q), \\ &= a_i \text{ if } i \le p \text{ and } i > q, \\ &= b_i \text{ if } i \le q \text{ and } i > p, \end{aligned}$$

is contiguous to $H_0 = H_0(T)$.

Proof. See [Hallin et al., 1985], in particular Proposition 3.1. □

Theorem 9 (Asymptotic distribution under contiguous ARMA alternatives)**.** *If a contiguous ARMA alternative from the preceding theorem holds, then*

$$S_M^c(k) \sim_{asympt.} N\Big(\sum_{i=1}^{k} d_i C_{i,k}^M(F), V_{M,k}\Big) \quad \text{and} \quad S_W^c(k) \sim_{asympt.} N\Big(\sum_{i=1}^{2k} d_i C_{i,k}^W(F), V_{W,k}\Big)$$

$$\left(\text{i.e. } \widetilde{r}_M(k) \sim_{asympt.} N\Big(\frac{-1}{\sqrt{V_{M,k}}}\sum_{i=1}^{k} d_i C_{i,k}^M(F), 1\Big),\ \widetilde{r}_W(k) \sim_{asympt.} N\Big(\frac{-1}{\sqrt{V_{W,k}}}\sum_{i=1}^{2k} d_i C_{i,k}^W(F), 1\Big)\right)$$

where

$$\begin{aligned} V_{M,k} &= \int_{[0,1]^{k+1}} {J_{M,k}^{\star}}^2(u_1, \dots, u_{k+1}) du_1 \dots du_{k+1} \\ &\quad + 2\sum_{j=1}^{k} \int_{[0,1]^{j+k+1}} J_{M,k}^{\star}(u_1, \dots, u_{k+1}) J_{M,k}^{\star}(u_{j+1}, \dots, u_{j+k+1}) du_1 \dots du_{j+k+1} \\ &= \frac{1}{12}, \text{ (independent of } k \text{ and } F\text{)} \\ V_{W,k} &= \int_{[0,1]^{2k+1}} {J_{W,k}^{\star}}^2(u_1, \dots, u_{2k+1}) du_1 \dots du_{2k+1} \\ &\quad + 2\sum_{j=1}^{2k} \int_{[0,1]^{j+2k+1}} J_{W,k}^{\star}(u_1, \dots, u_{2k+1}) J_{W,k}^{\star}(u_{j+1}, \dots, u_{j+2k+1}) du_1 \dots du_{j+2k+1} \\ &= \frac{8}{45}, \text{ (independent of } k \text{ and } F\text{)} \\ C_{i,k}^M(F) &= \int_{[0,1]^{k+1}} J_{M,k}^{\star}(u_1, \dots, u_{k+1}) \sum_{j=0}^{k-i} \phi\big(F^{-1}(u_{k+1-j})\big) F^{-1}(u_{k+1-j-i}) du_1 \dots du_{k+1} \\ &= 0, \text{ (independent of } k, i \text{ or } F\text{)} \end{aligned}$$

$$\begin{aligned}
C_{i,k}^{W}(F) &= \int_{[0,1]^{2k+1}} J_{W,k}^{\star}(u_1,\ldots,u_{2k+1}) \sum_{j=0}^{2k-i} \phi\big(F^{-1}(u_{2k+1-j})\big) F^{-1}(u_{2k+1-j-i}) du_1 \ldots du_{2k+1} \\
&= 0 \ \textit{if } i \neq k \textit{ and } i \neq 2k, \\
&= \textit{a number independent of } k \textit{ and } i \textit{ otherwise.}
\end{aligned}$$

Besides, $C_{k,k}^{W}(F) = -C_{2k,k}^{W}(F)$, *and*

$$C_{k,k}^{W}(F_{Norm}) \doteq -0.184, \quad C_{k,k}^{W}(F_{Log}) = -\frac{7}{36} \doteq -0.194, \quad C_{k,k}^{W}(F_{Lap}) = -\frac{2}{9} \doteq -0.222$$

for the N(0,1), standard logistic and standard Laplace white noise distribution, respectively.

Proof. It is a direct application of general Propositions 4.1 and 4.2 from [Hallin et al., 1985] to the special case of $\widetilde{r}_M(k)$'s and $\widetilde{r}_W(k)$'s addressed here. The simple score-generating functions $J_{M,k}^{\star}$ and $J_{W,k}^{\star}$ lead to some simplification, together with the fact that

$$\int_0^1 \phi\big(F^{-1}(u)\big) du = 0 \quad \text{and} \quad \int_0^1 F^{-1}(u) du = 0.$$

Some computational details can be checked with the programs *MWAREM.mws*, *MWAREW.mws* and *MWAREW.r*. □

Note. The variances $V_{M,k}$ and $V_{W,k}$ should be the same even under the null hypothesis H_0 and therefore they could be inferred directly from Theorem 2.

Corollary. The asymptotic distribution of $\widetilde{r}_M(k)$'s or $\widetilde{r}_W(k)$'s does not distinguish the type of linear process (AR, MA and ARMA) used as a contiguous alternative. Besides, the Moore autocorrelations $\widetilde{r}_M(k)$'s are completely insensitive to such alternatives while the Wallis serial rank coefficients $\widetilde{r}_W(k)$'s exhibit the same drawback only for $d_k = d_{2k}$. Apparently, the contiguous ARMA alternatives driven by the Laplace white noise are more easily detectable by means of $\widetilde{r}_W(k)$'s than if they were based on normal or logistic innovations. Furthermore, we can deduce that $C_{k,k}^{W}$ remains invariant under scale transformations of f. See [Hallin et al., 1985] for more details.

Now the asymptotic relative efficiency (ARE) of $\widetilde{r}_W(k)$'s with respect to other statistics relevant to our work can be computed easily. For comparison, we will consider the sample standardized ordinary autocorrelations $\widetilde{r}(k)$'s as well as the standardized van der Waerden, Wilcoxon, Laplace and Spearman serial rank coefficients $\widetilde{r}_{\mathrm{vdW}}(k)$'s, $\widetilde{r}_{\mathrm{Wil}}(k)$'s, $\widetilde{r}_{\mathrm{Lap}}(k)$'s and $\widetilde{r}_S(k)$'s, associated with the score-generating functions

$$\begin{aligned}
J_{\mathrm{vdW}}(u_1,\ldots,u_{k+1}) &= \phi_{\mathrm{Norm}}\big(F_{\mathrm{Norm}}^{-1}(u_1)\big) F_{\mathrm{Norm}}^{-1}(u_{k+1}), \\
J_{\mathrm{Wil}}(u_1,\ldots,u_{k+1}) &= \phi_{\mathrm{Log}}\big(F_{\mathrm{Log}}^{-1}(u_1)\big) F_{\mathrm{Log}}^{-1}(u_{k+1}), \\
J_{\mathrm{Lap}}(u_1,\ldots,u_{k+1}) &= \phi_{\mathrm{Lap}}\big(F_{\mathrm{Lap}}^{-1}(u_1)\big) F_{\mathrm{Lap}}^{-1}(u_{k+1}), \\
J_{\mathrm{S}}(u_1,\ldots,u_{k+1}) &= u_1 u_{k+1}.
\end{aligned}$$

Some tests based on $\widetilde{r}_{\mathrm{vdW}}(k)$'s, $\widetilde{r}_{\mathrm{Wil}}(k)$'s and $\widetilde{r}_{\mathrm{Lap}}(k)$'s are known to have certain optimal properties in the case of contiguous ARMA alternatives with normal, logistic and Laplace white noise, respectively (see for example [Hallin et al., 1985], [Hallin et al., 1987], [Hallin and Puri, 1988a] and [Hallin and Puri, 1988b]).

Let us recall the fact that if $\widetilde{r}_A$, $\widetilde{r}_B$ and $\widetilde{r}_C$ are three sample standardized autocorrelation coefficients with $N(m_A, V_A)$, $N(m_B, V_B)$ and $N(m_C, V_C)$ asymptotic distributions under a specific contiguous alternative, $m_A, m_B, m_C, V_A, V_B, V_C \neq 0$, then the asymptotic relative efficiency $e(\widetilde{r}_A, \widetilde{r}_B)$ of $\widetilde{r}_A$ with respect to $\widetilde{r}_B$ is defined as

$$e(\widetilde{r}_A, \widetilde{r}_B) = \frac{m_A^2/V_A}{m_B^2/V_B}, \quad \text{i.e.} \quad e(\widetilde{r}_A, \widetilde{r}_B) = \frac{e(\widetilde{r}_A, \widetilde{r}_C)}{e(\widetilde{r}_B, \widetilde{r}_C)}.$$

The coefficient $\widetilde{r}_W(k)$ is somewhat exceptional as its asymptotic distribution depends not only on d_k, but also on d_{2k}. Therefore its direct comparison to the other autocorrelations seems problematic. However, the following theorem could still be found useful.

Theorem 10 (Asymptotic relative efficiency of $\widetilde{r}_W(k)$'s)**.** *Let the assumptions of Theorem 9 hold and let us consider two nontrivial cases with $d_k \neq 0$: $d_{2k} = 0$ and $d_{2k} = -d_k$. Then*

		Normal white noise		*Logistic white noise*		*Laplace white noise*	
		$d_{2k}=0$	$d_{2k}=-d_k$	$d_{2k}=0$	$d_{2k}=-d_k$	$d_{2k}=0$	$d_{2k}=-d_k$
$e\big(\widetilde{r}_W(k), \widetilde{r}(k)\big)$	$\doteq$	*0.19*	*0.76*	*0.20*	*0.81*	*0.23*	*0.91*
$e\big(\widetilde{r}_W(k), \widetilde{r}_{vdW}(k)\big)$	$\doteq$	*0.19*	*0.76*	*0.20*	*0.81*	*0.23*	*0.91*
$e\big(\widetilde{r}_W(k), \widetilde{r}_{Wil}(k)\big)$	$\doteq$	*0.20*	*0.80*	*0.19*	*0.78*	*0.19*	*0.75*
$e\big(\widetilde{r}_W(k), \widetilde{r}_{Lap}(k)\big)$	$\doteq$	*0.31*	*1.24*	*0.26*	*1.05*	*0.14*	*0.56*
$e\big(\widetilde{r}_W(k), \widetilde{r}_S(k)\big)$	$\doteq$	*0.21*	*0.83*	*0.21*	*0.85*	*0.22*	*0.88*

Proof. It follows from the preceding comment, Theorem 9 and from Corollary 5.1. with Table 1 in [Hallin et al., 1985]. Computational details can be checked with *MWAREComp.mws*. □

Note. These results can be checked thanks to [Knoke, 1977] and [Aiyar, 1981], where some ARE's regarding the turning point test, $\widehat{r}_{\mathrm{S}}(1)$ and $\widehat{r}_{\mathrm{vdW}}(1)$ are computed in a similar context.

In fact, the ARE's of various sums of these coefficients or of their squares could be calculated virtually in the same way; see also [Hallin et al., 1987] where Theorem 9 is generalised to the joint distribution of the (unweighted) serial rank statistics.

Unfortunately, the assumptions adopted in [Hallin et al., 1985] and here rule out any white noise coming from the Cauchy, Student, (centered) uniform and other useful distributions. Besides, these results are only of asymptotic nature. Although the Wallis coefficients appear clearly worse than the others in our ARE comparison, there is still some hope that they could prove better under some contiguous ARMA alternatives with other white noise distributions or under fixed ones. Besides, both $\widetilde{r}_M(k)$'s and $\widetilde{r}_W(k)$'s might be found useful in other context, e.g. for evaluating random number generators or for testing against other types of alternatives.

3.6 Asymptotic Relative Efficiency (Trend Alternatives)

For short, we focus in detail only on the most frequent case of the increasing linear trend alternatives with N(0,1) white noise $\{\varepsilon_t\}$:

$$Y_t = \alpha + \beta t + \varepsilon_t, \quad \beta \geq 0, \qquad (t = 1, \ldots, T).$$

The null hypothesis then corresponds to $\beta = 0$.

Now we recall the theory from [Stuart, 1954], [Noether, 1955] and [Stuart, 1956] that can be employed in this situation. Let S_1, S_2 be two consistent and asymptotically normal test statistics computed from Y_t's and let m_i be the least integer such that

$$E_0^{(m_i)}(S_i) \equiv \left[\frac{\partial^{m_i}}{\partial \beta^{m_i}} \mathrm{E}(S_i)\right]_{\beta=0} \neq 0, \quad i = 1, 2.$$

If we further define $\delta_i > 0$ and $c_i^2 > 0$ by

$$\lim_{T\to\infty} \frac{\left(E_0^{(m_i)}(S_i)\right)^2}{T^{(2m_i\delta_i)} \operatorname{var}_0(S_i)} = c_i^2, \quad i = 1, 2$$

(assuming that the limit exists), then the asymptotic relative efficiency (ARE) $e(S_1, S_2)$ of S_1 compared to S_2 is

$$\begin{aligned} e(S_1, S_2) &= 0 && \text{if } \delta_1 < \delta_2 \\ &= \left(\frac{c_1^2}{c_2^2}\right)^{1/(2m\delta)} && \text{if } \delta_1 = \delta_2 = \delta \text{ and } m_1 = m_2 = m. \end{aligned}$$

The values of m, δ, and c^2 have already been determined for a large number of statistics including $\widehat{r}_S(1)$, $\widehat{r}_M(1)$, $\widehat{r}_W(1)$ and also the ordinary least squares estimator $\widehat{r}_{\text{OLS}}$ of the slope β (see [Stuart, 1954] and [Stuart, 1956]). Moreover, we computed these characteristics for the Moore and Wallis autocorrelations at any fixed lag:

	m	δ	c^2
$\widehat{r}_{\text{OLS}}$	1	$\frac{3}{2}$	$\frac{1}{12}$
$\widehat{r}_S(1)$	2	$\frac{5}{4}$	$\frac{1}{4\pi^2}$
$\widehat{r}_M(k)$ $(k \leq \frac{T}{2})$	1	$\frac{1}{2}$	$\frac{3k^2}{\pi}$
$\widehat{r}_W(k)$ $(k \leq \frac{T}{4})$	2	$\frac{1}{4}$	$\frac{135k^4}{8\pi^2}$

It appears that $\widehat{r}_M(k)$'s and $\widehat{r}_W(k)$'s are inefficient with respect to both $\widehat{r}_{\text{OLS}}$ and $\widehat{r}_S(1)$ for any fixed k, at least from this point of view. However, we are going to show that these conclusions need not hold if we choose k proportionately to T. In fact, anyway one cannot distinguish between the fixed and proportionate choice of k in practical applications. For example, there is no difference between $k = 25$ and $k = T/4$ in the case of time series with $T = 100$ observations.

It turns out that $\widehat{r}_M(k)$'s maximize their contribution to ARE for $k_{\text{opt}} = 2T/3$ and $\widehat{r}_W(k)$'s do so for $k_{\text{opt}} = 2T/5$. Besides, if we choose $k = aT$, $0 < a < 1$, then m remains the same and δ increases by one, which leads to nonzero $e(\widehat{r}_M(k), \widehat{r}_{\mathrm{OLS}})$ and $e\big(\widehat{r}_W(k), \widehat{r}_S(1)\big)$, see Tables 3.1 and 3.2. These findings significantly extend the results from [Cox and Stuart, 1955] where the optimality of $\widehat{r}_M(2T/3)$ among all $\widehat{r}_M(k)$'s, $k > T/2$, is shown.

We also considered $\widehat{r}_{M,w}(k)$'s and $\widehat{r}_{W,w}(k)$'s with $w(i) = i^p$ for several reasonable power exponents p's. We achieved the same m's and δ's for $p > -1/2$ and calculated the ARE's $e(\widehat{r}_{M,w}(k), \widehat{r}_{\mathrm{OLS}})$ and $e\big(\widehat{r}_{W,w}(k), \widehat{r}_S(1)\big)$ for selected k's, see Tables 3.1 and 3.2. As the dependence of these ARE's on k looks similarly for all p's considered, we illustrate it only for $p = 0$ that seems optimal in this respect (see Figures 3.1 and 3.2).

As far as all the new statements in this section and their proofs are concerned, they follow directly from the theory mentioned above and from the fact that

$$(Y_t - Y_{t+k}) \sim N(-k\beta, 2),$$

$$(Y_t - Y_{t+k}, Y_{t+k} - Y_{t+2k}) \sim N\left(\begin{pmatrix} -k\beta \\ -k\beta \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\right)$$

under the alternative. See *MWTrendARE1.mws* and *MWTrendARE2.mws* for more details, outputs and for all the computations.

Finally, we would like to point out that [Aiyar et al., 1979] calculated the ARE's for a large number of statistics (including $\widehat{r}_M(k)$'s, $k > T/2$) in the case of quite general increasing trends with possibly non-normal errors. Their results indicate that k_{opt} for $\widehat{r}_M(k)$'s, $k > T/2$, often lies in the interval $[2T/3, 3T/4]$ and that the asymptotic relative efficiency of $\widehat{r}_M(k_{\mathrm{opt}})$ with respect to the best linear rank test grows with the trend rate of increase. The authors themselves suggest to use $\widehat{r}_M(k)$'s with $k_{\mathrm{opt}} = 3T/4$ if nothing is known about the trend a priori.

	Asymptotic relative efficiencies of $\widehat{r}_{M,w}(k)$'s									
$p = \backslash k =$	0.1T	0.2T	0.3T	0.4T	0.5T	0.6T	2T/3	0.7T	0.8T	0.9T
$-1/4$	?	?	?	?	0.752	0.788	0.795	0.793	0.757	0.650
$-1/8$	?	?	?	?	0.776	0.814	0.821	0.819	0.782	0.672
0	0.439	0.625	0.730	0.778	0.782	0.819	0.827	0.825	0.788	0.676
1/8	?	?	?	?	0.778	0.816	0.824	0.822	0.785	0.674
1/4	?	?	?	?	0.771	0.808	0.816	0.814	0.777	0.667
1/2	0.420	0.593	0.685	0.729	0.752	0.788	0.795	0.793	0.757	0.650
1	0.387	0.541	0.626	0.673	0.710	0.744	0.752	0.750	0.716	0.615
3/2	0.359	0.498	0.578	0.629	0.674	0.706	0.713	0.711	0.679	0.583
2	0.336	0.464	0.541	0.595	0.643	0.674	0.680	0.678	0.648	0.556
5/2	0.316	0.436	0.511	0.568	0.616	0.646	0.652	0.650	0.621	0.533

Table 3.1: The asymptotic relative efficiencies of $\widehat{r}_{M,w}(k)$'s, $w(i) = i^p$, with respect to $\widehat{r}_{\mathrm{OLS}}$ in the case of the linear trend alternatives with N(0,1) errors. The symbol ? stands for the values that could not have been computed explicitly with the aid of MAPLE.

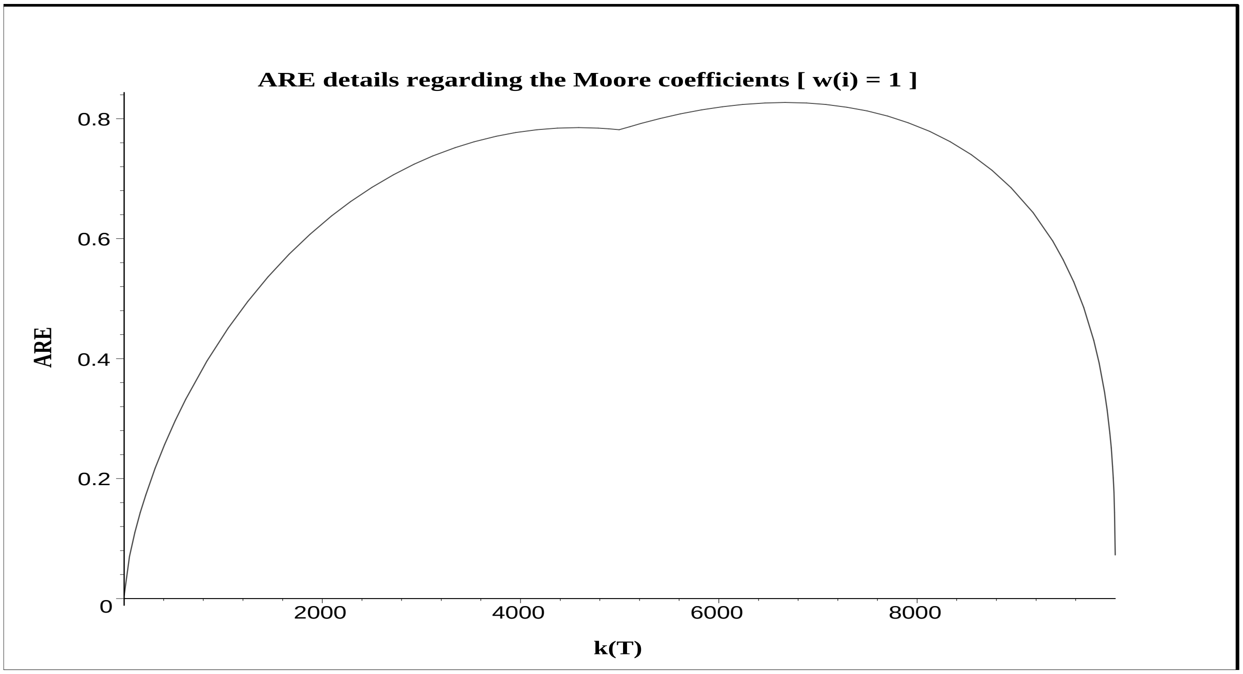

Figure 3.1: The asymptotic relative efficiency of $\widehat{r}_M(k)$'s with respect to $\widehat{r}_{\mathrm{OLS}}$ in the case of the linear trend alternatives with N(0,1) errors.

	Asymptotic relative efficiencies of $\widehat{r}_{W,w}(k)$'s							
$p = \backslash k =$	0.10T	0.15T	0.20T	0.25T	0.30T	0.35T	0.40T	0.45T
$-1/4$	?	?	?	?	?	0.736	0.755	0.722
$-1/8$	?	?	?	?	?	0.751	0.770	0.737
0	0.350	0.470	0.572	0.655	0.715	0.754	0.773	0.740
1/8	?	?	?	?	?	0.752	0.771	0.738
1/4	?	?	?	?	?	0.748	0.767	0.734
1/2	0.342	0.459	0.557	0.637	0.695	0.736	0.755	0.722
1	0.330	0.442	0.536	0.612	0.669	0.711	0.730	0.698
3/2	0.319	0.427	0.517	0.590	0.647	0.689	0.707	0.677
2	0.309	0.414	0.501	0.572	0.628	0.670	0.688	0.658
5/2	0.301	0.402	0.487	0.556	0.612	0.653	0.670	0.641

Table 3.2: The asymptotic relative efficiencies of $\widehat{r}_{W,w}(k)$'s, $w(i) = i^p$, with respect to $\widehat{r}_{\mathrm{S}}(1)$ in the case of the linear trend alternatives with N(0,1) errors. The symbol ? stands for the values that could not have been computed explicitly with the aid of MAPLE.

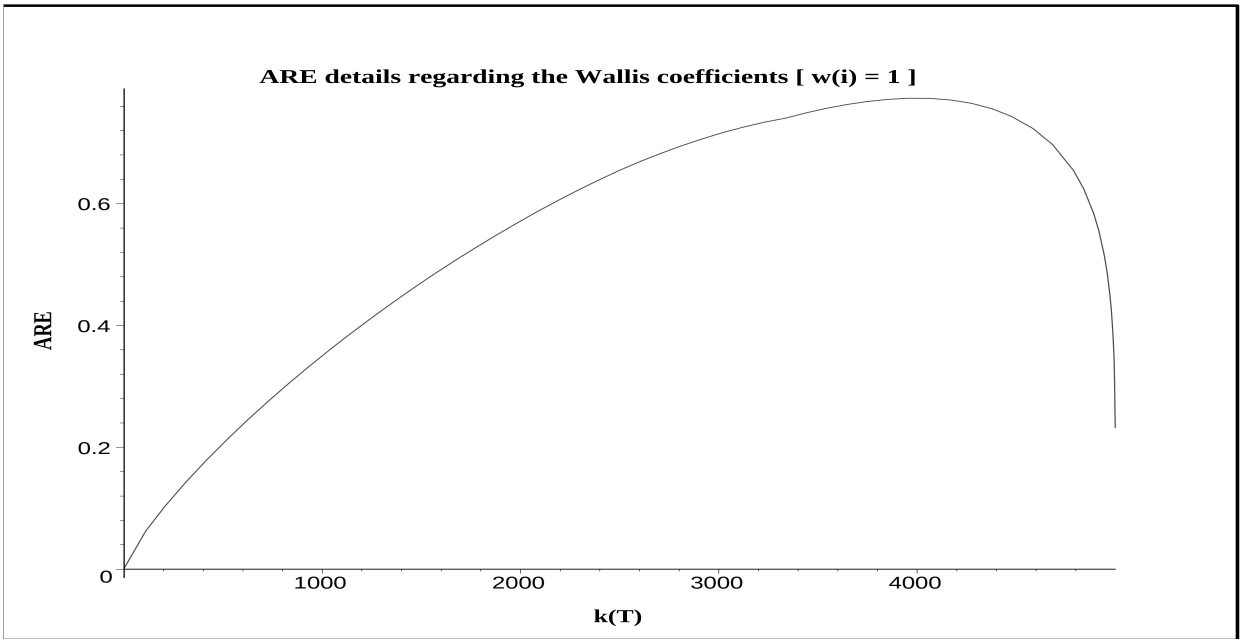

Figure 3.2: The asymptotic relative efficiency of $\widehat{r}_W(k)$'s with respect to $\widehat{r}_S(1)$ in the case of the linear trend alternatives with N(0,1) errors.

3.7 Orthonormalization

The joint asymptotic normality of $\widetilde{r}_{M,w}(k)$'s and $\widetilde{r}_{W,w}(k)$'s under H_0^E allows us to use these coefficients together for statistical inference (e.g. in correlogram-based tools) instead of the ordinary autocorrelations. However, the nonzero covariances between them could complicate the decision making in finite samples and even asymptotically. This problem can partly be solved by using only selected $\widetilde{r}_{W,w}(k)$'s (e.g. only those at odd lags). Nevertheless, a complex solution of this problem also exists and lies in the use of some asymptotically independent modifications of the coefficients. Prime candidates for this application are defined below.

Definition 11 (Orthonormal(ized) coefficients)**.** Let the orthonormal(ized) coefficients $r^{\perp}_{M,w}(k)$'s and $r^{\perp}_{W,w}(k)$'s be defined with the aid of the Gram-Schmidt orthogonalization in the following recursive way:

$$
\begin{aligned}
r^{\perp}_{M,w}(1) &= \widetilde{r}_{M,w}(1),\\
r^{\perp}_{M,w}(k) &= \frac{\widetilde{r}_{M,w}(k) - \sum_{i=1}^{k-1} \operatorname{cov}_0\big(\widetilde{r}_{M,w}(k), r^{\perp}_{M,w}(i)\big) r^{\perp}_{M,w}(i)}{\sqrt{1 - \sum_{i=1}^{k-1} \operatorname{cov}_0^2\big(\widetilde{r}_{M,w}(k), r^{\perp}_{M,w}(i)\big)}},\ k = 2, 3, \ldots,\\
r^{\perp}_{W,w}(1) &= \widetilde{r}_{W,w}(1),\\
r^{\perp}_{W,w}(k) &= \frac{\widetilde{r}_{W,w}(k) - \sum_{i=1}^{k-1} \operatorname{cov}_0\big(\widetilde{r}_{W,w}(k), r^{\perp}_{W,w}(i)\big) r^{\perp}_{W,w}(i)}{\sqrt{1 - \sum_{i=1}^{k-1} \operatorname{cov}_0^2\big(\widetilde{r}_{W,w}(k), r^{\perp}_{W,w}(i)\big)}},\ k = 2, 3, \ldots,
\end{aligned}
$$

and if, say,

$$r^{\perp}_{M,w}(i) = \sum_{j=1}^{i} a_j \widetilde{r}_{M,w}(j) \quad \text{and} \quad r^{\perp}_{W,w}(i) = \sum_{j=1}^{i} b_j \widetilde{r}_{W,w}(j),$$

then

$$\operatorname{cov}_0\big(\widetilde{r}_{M,w}(k), r^{\perp}_{M,w}(i)\big) = \sum_{j=1}^{i} a_j \operatorname{cov}_0\big(\widetilde{r}_{M,w}(k), \widetilde{r}_{M,w}(j)\big),$$

$$\operatorname{cov}_0\big(\widetilde{r}_{W,w}(k), r^{\perp}_{W,w}(i)\big) = \sum_{j=1}^{i} b_j \operatorname{cov}_0\big(\widetilde{r}_{W,w}(k), \widetilde{r}_{W,w}(j)\big).$$

Corollary. The orthonormal coefficients $r^{\perp}_{M,w}(k)$'s and $r^{\perp}_{W,w}(k)$'s are mutually uncorrelated and, under the assumptions of Theorem 7, also asymptotically independent standard normal.

Note. These orthonormal coefficients have at least two important advantages: they can be used straightforwardly in correlograms and they can be combined in a single portmanteau statistic quite simply. Both these benefits will be found important further in this work. We will also show that this concept of orthonormalization leads to significant improvements in statistical inference even if applied to other standardized rank autocorrelations. This holds especially in the case of short time series when one cannot rely on any asymptotic results too much. The importance of considering the finite sample covariance structure of sample autocorrelations will be stressed again in Chapter 8.

In practice, the vectors of some sample autocorrelation coefficients are often orthonormalized by multiplying by the square root of their inverse variance matrix. However, such approach may lead to the orthonormal vectors whose elements are difficult to interpret meaningfully. Our orthonormalization procedure thus seems more reasonable even from this point of view, as its output vectors consist again of some measures of autocorrelation at the same individual lags. Apparently, either of the two orthonormal vectors resulting from the two orthonormalization procedures can be multiplied with a suitable orthonormal matrix to coincide with the other.

Our concept of the orthonormal autocorrelations is likely to be new as we are not aware of its practical application of any kind.

3.8 Advantages and Disadvantages

The sample Moore and Wallis serial rank coefficients possess all the advantages common to all rank autocorrelations and mentioned in Subsection 1.2.8, and also another three:

- speed and other computational benefits. This may be the key merit in some financial applications when a large number of long sequences (typically with thousands of observations) have to be processed simultaneously. These coefficients can be computed directly from the original series, do not need any time consuming ranking of the data, and their calculation requires little computer memory space and is quite resistant to numerical inaccuracies.
- forecast interpretation. Obviously, $\widetilde{r}_M(k)$'s and $\widetilde{r}_W(k)$'s can be interpreted easily as measures of predictability of the forecast direction in the first one or two periods of length k.

- quick update. The coefficients $\widehat{r}_{M,w}(k)$'s and $\widehat{r}_{W,w}(k)$'s can be updated quickly with every new observation because there is no need to rank the data again or to employ all the time series history. Their unweighted versions are thus especially convenient for sequential testing, see e.g. [Noether, 1956] for such a modification of $\widehat{r}_M(k)$'s.

On the other hand, these coefficients also exhibit several drawbacks. For example, they completely ignore the magnitudes of all observations, which often weakens their power in testing. Besides, their unweighted versions take no account of the location of the signs or turning points in time and even the weighted modifications do not solve this problem completely. However, unequal weighting could be found useful if we wanted to lay emphasis on the time series behaviour in certain time spans (typically if we intended to favour the most topical observations) or if we expected some alternatives of a special type.

3.9 Monte Carlo Simulations

We have conducted many Monte Carlo experiments to find possible applications of the Moore and Wallis serial rank coefficients and to justify their existence that way. Some interesting results regarding their use in portmanteau tests will be presented in this section.

To be more specific, we will focus on the portmanteau tests $T_S, T_{M,j}, T_{W,j}, T_{MW,j}$, $j = 0, 1, 2$, and their $^\perp$ versions, based on the statistics listed below:

$$\begin{aligned} S_S(m) &= \sum_{k=1}^{m} \widetilde{r}_S^2(k), & S_S^\perp(m) &= \sum_{k=1}^{m} r_S^{\perp 2}(k), \\ S_{M,j}(m) &= \widehat{\mathbf{r}}'_{M,j,m} V_{M,j,m}^{-1} \widehat{\mathbf{r}}_{M,j,m}, & S_{M,j}^\perp(m) &= \sum_{k=1}^{m} r_{M,j}^{\perp 2}(k), \\ S_{W,j}(m) &= \widehat{\mathbf{r}}'_{W,j,m} V_{W,j,m}^{-1} \widehat{\mathbf{r}}_{W,j,m}, & S_{W,j}^\perp(m) &= \sum_{k=1}^{m} r_{W,j}^{\perp 2}(k), \\ S_{MW,j}(m) &= S_{M,j}(m) + S_{W,j}(m), & S_{MW,j}^\perp(m) &= S_{M,j}^\perp(m) + S_{W,j}^\perp(m), \end{aligned}$$

where $V_{\bullet,j,m}$ is the finite sample variance matrix of $\widehat{\mathbf{r}}_{\bullet,j,m} = \left(\widehat{r}_{\bullet,j}(1), \ldots, \widehat{r}_{\bullet,j}(m)\right)'$ under the null hypothesis H_0^S ($j = 0, 1, 2$, and $\bullet$ substitutes either M or W). Obviously, $S_S(m), S_S^\perp(m), S_{M,j}(m), S_{M,j}^\perp(m), S_{W,j}(m), S_{W,j}^\perp(m) \sim_{\text{asympt.}} \chi^2(m)$ and $S_{MW,j}(m), S_{MW,j}^\perp(m) \sim_{\text{asympt.}} \chi^2(2m)$ under the null hypothesis. $^\perp$ versions are included in order to investigate the advantages and disadvantages of the orthonormal coefficients.

The power of the tests will be investigated separately in each category SHORTTREND + TREND, ARMA, and GARCH. It is good to keep in mind that $T_{M,0}(1)$ corresponds to the sign test while $T_{W,0}(1)$ virtually coincides with the turning point one.

The TREND Class ($T = 100, 200$) + the SHORTTREND Class ($T = 25$)

As far as the test power is concerned, all the simulations support the conclusions listed below and agree quite well with the ARE considerations in Section 3.6:

- The Moore coefficients are more suitable for testing here than the Wallis ones, at least in most situations considered. However, the reverse can be true in some special cases. See Figures 3.3, 3.4, 3.7, 3.8 and 3.11.

- $T_{M,j}(1)$ and $T_{W,j}(1)$ are far less powerful than the tests $T_{M,j}(m)$ and $T_{W,j}(m)$ with optimum values of m, $j = 0, 1, 2$ (see Figures 3.3, 3.4 and 3.9). In fact, this striking rise in power with increasing m can by itself fully justify the introduction of the Moore and Wallis autocorrelations at higher lags.

- Each of the tests $T_{M,0}(m)$, $T_{M,1}(m)$ and $T_{M,2}(m)$ can outperform the other two in some situations, which justifies the existence of the weighted modifications of the Moore coefficients. To be more specific, $T_{M,0}(m)$ appears the most generally applicable while $T_{M,1}(m)$ and $T_{M,2}(m)$ seem useful only in certain special cases (e.g. for some alternatives with a structure change) that are fortunately often identifiable a priori, see Figure 3.3. The same holds even for the tests $T_{W,0}(m)$, $T_{W,1}(m)$ and $T_{W,2}(m)$, see Figure 3.4.

- The portmanteau tests based on the Moore autocorrelations can often almost achieve the maximum power of the benchmark test T_S and even exceed it, see Figures 3.3 and 3.9.

- The overall performance of $T_{MW,j}(m)$ with an optimum m is better than that of the individual tests $T_{M,j}(m)$ and $T_{W,j}(m)$, $j = 0, 1, 2$, (see Figures 3.7, 3.8 and 3.11). Such tests can also outdistance the benchmark, see Figure 3.10.

- The sums of the squared orthonormal coefficients achieve virtually the same power as the alternative tests based on the quadratic forms, compare Figures 3.5 and 3.6 with Figures 3.3 and 3.4. However, they are much better than the sums of the squared standardized coefficients (compare the performance of T_S and $T_S^{\perp}$ in Figures 3.11 and 3.12). This feature will be illustrated in Chapter 7, too.

  We further present the average values of the standardized and ortonormalized Spearman and Moore coefficients resulting from our Monte Carlo testing under the alternative

$$Y_t = 0.09t + \varepsilon_t, \quad \varepsilon_t \sim N(0,1), \quad t = 1, \ldots, 25 :$$

$k=$	1	2	3	4	5	6	7	8
$\widetilde{r}_S(k)$	1.406	1.260	1.105	0.944	0.761	0.586	0.406	0.231
$r_S^{\perp}(k)$	1.406	1.390	1.372	1.353	1.317	1.292	1.265	1.243
$\widetilde{r}_{M,0}(k)$	0.414	0.786	1.093	1.376	1.580	1.763	1.907	2.005
$r_{M,0}^{\perp}(k)$	0.414	0.757	0.968	1.093	1.088	1.050	0.973	0.859

  They speak for themselves and also confirm our claims that the covariances between sample autocorrelation coefficients may play an important role in the case of short time series.

- If the null hypothesis does not hold, some coefficients typically testify against it more than the others. Apparently, the tests based solely on them would be even more powerful than those considered here. For example, leaving the first few Moore or Wallis coefficients out in the tests against a monotone trend alternative would probably lead to further power increase.

The ARMA Class with $T = 200$ and $T = 500$

The test power investigation strongly suggests the following conclusions:

- All tests based on the Moore coefficients are virtually insensitive to all the ARMA(1,1) alternatives considered, see Figure 3.16. This is in good accordance with the results regarding their ARE, see Theorem 9.

- $T_{W,0}(1)$ beats all the other examined tests based on the Moore or Wallis rank autocorrelations and it can be quite powerful even for relatively short time series, see Figures 3.13 to 3.15.

In other words, there is hardly any advantage in using the Moore and Wallis coefficients at higher lags in this context. However, there is still some hope that they could be found useful in the case of some higher order (S)AR(I)MA alternatives (with possibly unconventional innovations).

The GARCH Class with $T = 1\ 000$ and $T = 5\ 000$

As for the test power, the logical inferences drawn from the simulations are as follows:

- In general, the Moore serial rank coefficients do not seem successfully applicable in this context, at least in the case of symmetric GARCH time series with only several thousands of observations at most. See Figures 3.17 and 3.19. However, they could possibly prove better if applied to some GARCH alternatives with asymmetric innovations and/or with other formulae for the conditional variance σ_t^2.

- On the other hand, the tests $T_{W,1}$, $T_{W,2}$ and especially $T_{W,0}$ appear much more promising, although all of them are still less powerful than the benchmark T_S. See Figures 3.18 and 3.20. $T_{W,0}(m)$ can therefore be recommended for quick testing of extremely long time series hypothetically driven by a GARCH process. Such series are quite common (not only) in finance.

- Besides, $T_{W,0}(m)$ often achieves its maximum power for $m > 1$, which again shows the great importance of the Wallis serial rank coefficients at higher lags (see Figure 3.18 or 3.20).

Test Size Comparison

The most interesting findings regarding the test size can be summarized in the following items:

- Generally, the tests $T_{M,i}(m)$, $T_{W,i}(m)$, and $T_{MW,i}(m)$, $i = 0, 1, 2$, seem correctly sized for almost all considered values of m, contrary to the benchmark test $T_S(m)$ (see Figures 3.3 and 3.4). The small discrepancies for $m = 1$ might be attributed to the discrete nature of the test statistic, although the pseudorandom number generator could be to blame, too (see Chapter 4). This drawback diminishes with increasing T or m (see the upper left subplot in Figures 3.8 to 3.20) and some proposals for its elimination are given in the last chapter. Note that the Moore serial rank coefficients at higher lags are useful for portmanteau testing even from this point of view.

- These tests reach virtually the same size as their alternatives based on the orthonormal coefficients, see also Figures 3.5 and 3.6.

3.10 Accompanying Figures

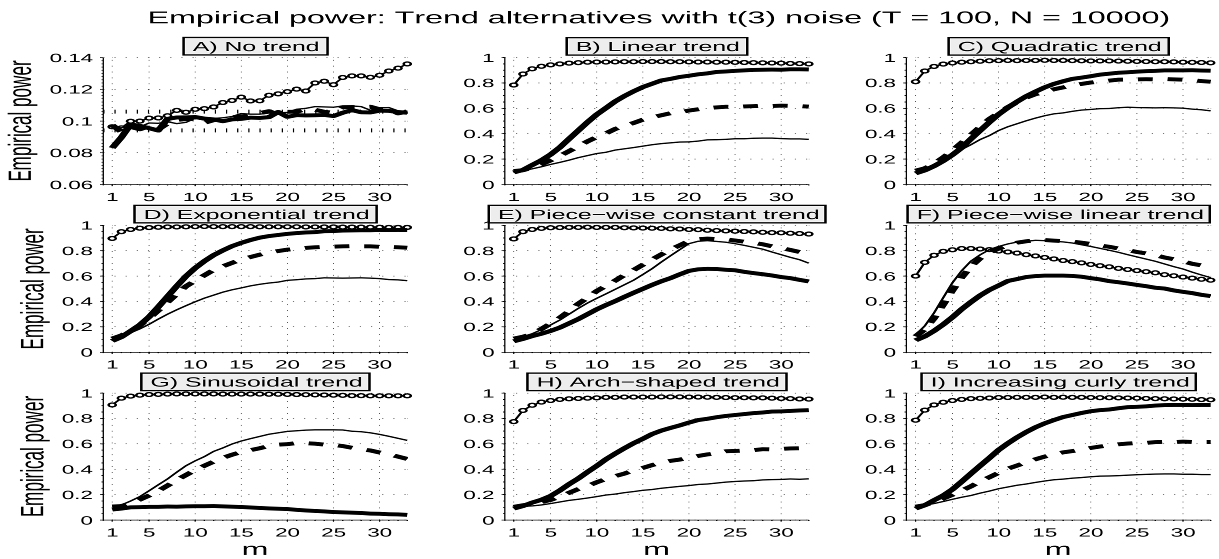

Figure 3.3: Behaviour of the tests based on $S_S(m)$ (ooooo), $S_{M,0}(m)$ (▬▬), $S_{M,1}(m)$ (▬ ▬ ▬) and $S_{M,2}(m)$ (——) when applied to a trend plus standardized t(3) white noise ($T = 100$ ... time series length, $N = 10\,000$ ... number of replications, ıııııııı ... bounds of 95% confidence intervals for the empirical size).

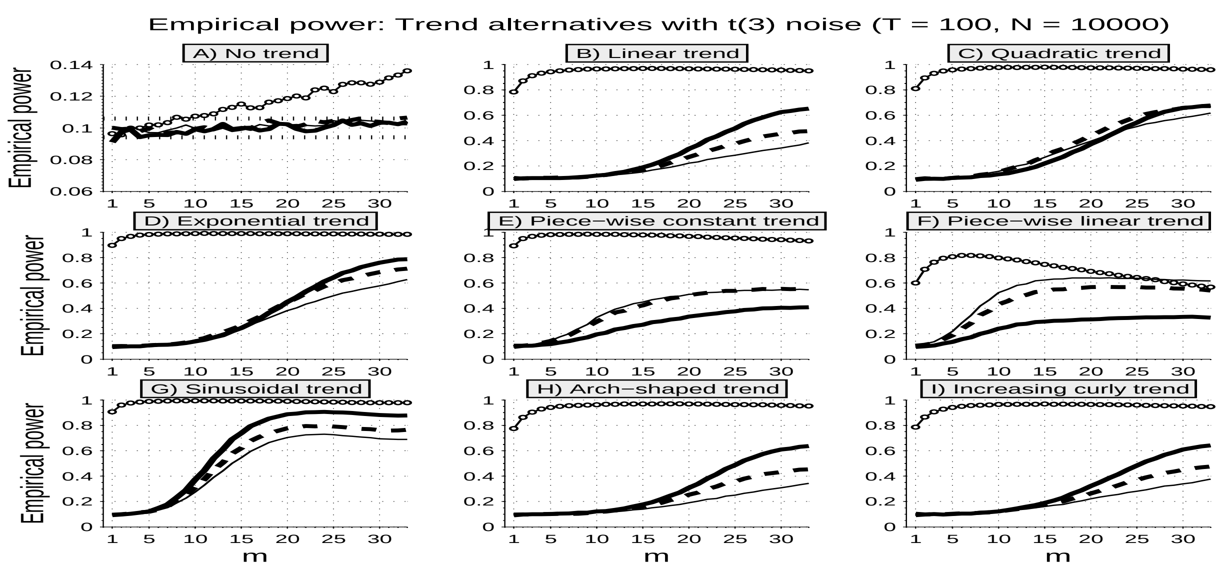

Figure 3.4: Behaviour of the tests based on $S_S(m)$ (ooooo), $S_{W,0}(m)$ (▬▬), $S_{W,1}(m)$ (▬ ▬ ▬) and $S_{W,2}(m)$ (——) when applied to a trend plus standardized t(3) white noise ($T = 100$ ... time series length, $N = 10\,000$ ... number of replications, ıııııııı ... bounds of 95% confidence intervals for the empirical size).

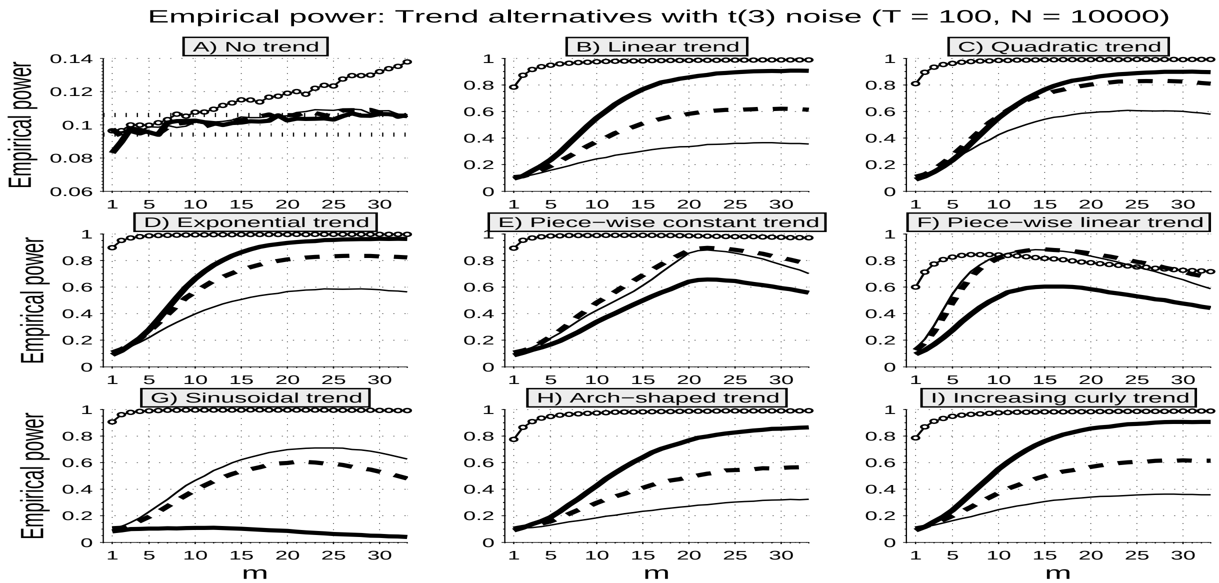

Figure 3.5: Behaviour of the tests based on $S_S^\perp(m)$ (∞∞∞), $S_{M,0}^\perp(m)$ (▬▬), $S_{M,1}^\perp(m)$ (▬ ▬ ▬) and $S_{M,2}^\perp(m)$ (——) when applied to a trend plus standardized t(3) white noise ($T = 100$ … time series length, $N = 10\,000$ … number of replications, ⁞⁞⁞⁞⁞⁞ … bounds of 95% confidence intervals for the empirical size).

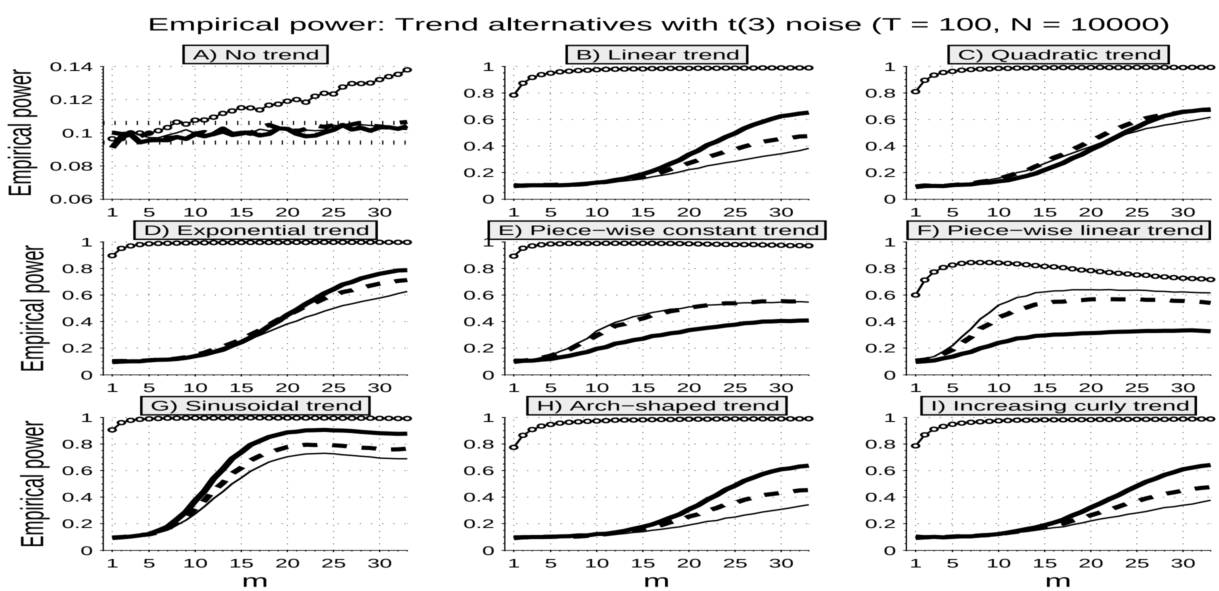

Figure 3.6: Behaviour of the tests based on $S_S^\perp(m)$ (∞∞∞), $S_{W,0}^\perp(m)$ (▬▬), $S_{W,1}^\perp(m)$ (▬ ▬ ▬) and $S_{W,2}^\perp(m)$ (——) when applied to a trend plus standardized t(3) white noise ($T = 100$ … time series length, $N = 10\,000$ … number of replications, ⁞⁞⁞⁞⁞⁞ … bounds of 95% confidence intervals for the empirical size).

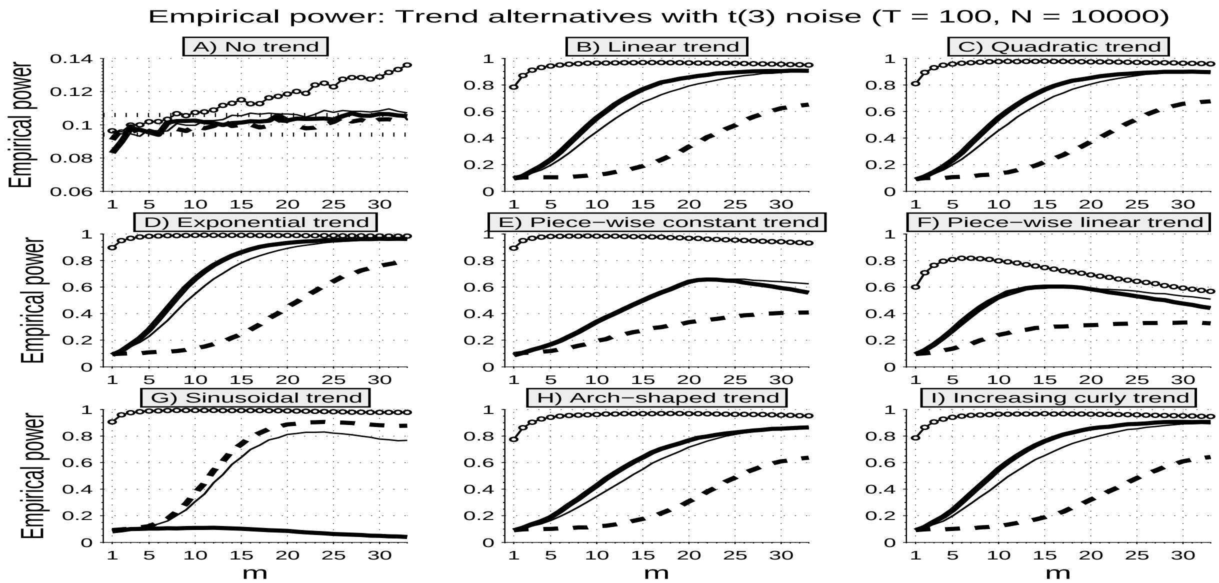

Figure 3.7: Behaviour of the tests based on $S_S(m)$ (∞∞∞), $S_{M,0}(m)$ (▬▬), $S_{W,0}(m)$ (▬ ▬ ▬) and $S_{MW,0}(m)$ (——) when applied to a trend plus standardized t(3) white noise ($T = 100$ …time series length, $N = 10\,000$ …number of replications, ▪▪▪▪▪▪ …bounds of 95% confidence intervals for the empirical size).

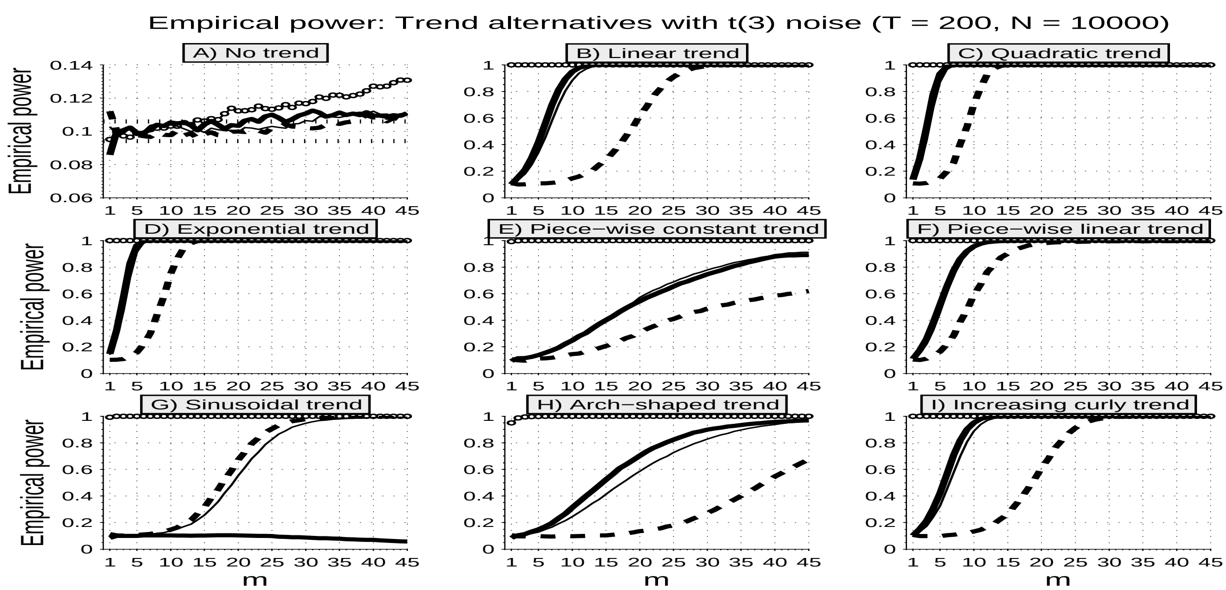

Figure 3.8: Behaviour of the tests based on $S_S(m)$ (∞∞∞), $S_{M,0}(m)$ (▬▬), $S_{W,0}(m)$ (▬ ▬ ▬) and $S_{MW,0}(m)$ (——) when applied to a trend plus standardized t(3) white noise ($T = 200$ …time series length, $N = 10\,000$ …number of replications, ▪▪▪▪▪▪ …bounds of 95% confidence intervals for the empirical size).

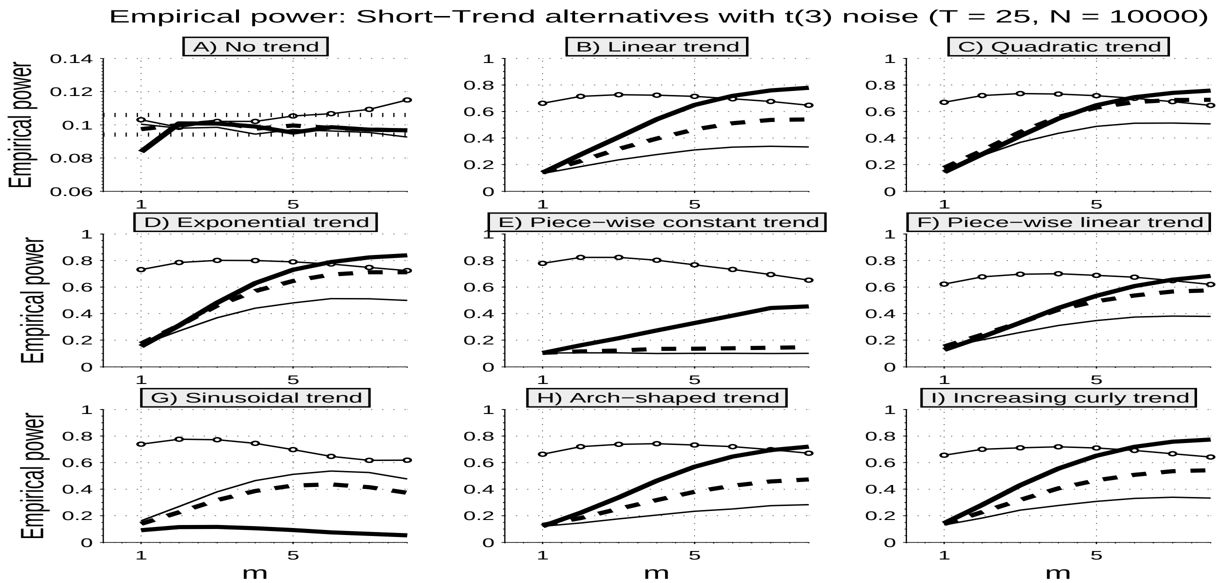

Figure 3.9: Behaviour of the tests based on $S_S(m)$ (ooooo), $S_{M,0}(m)$ (▬▬), $S_{M,1}(m)$ (▬ ▬ ▬) and $S_{M,2}(m)$ (——) when applied to a short trend plus standardized t(3) white noise ($T = 25$ … time series length, $N = 10\ 000$ … number of replications, ⁞⁞⁞⁞⁞⁞ … bounds of 95% confidence intervals for the empirical size).

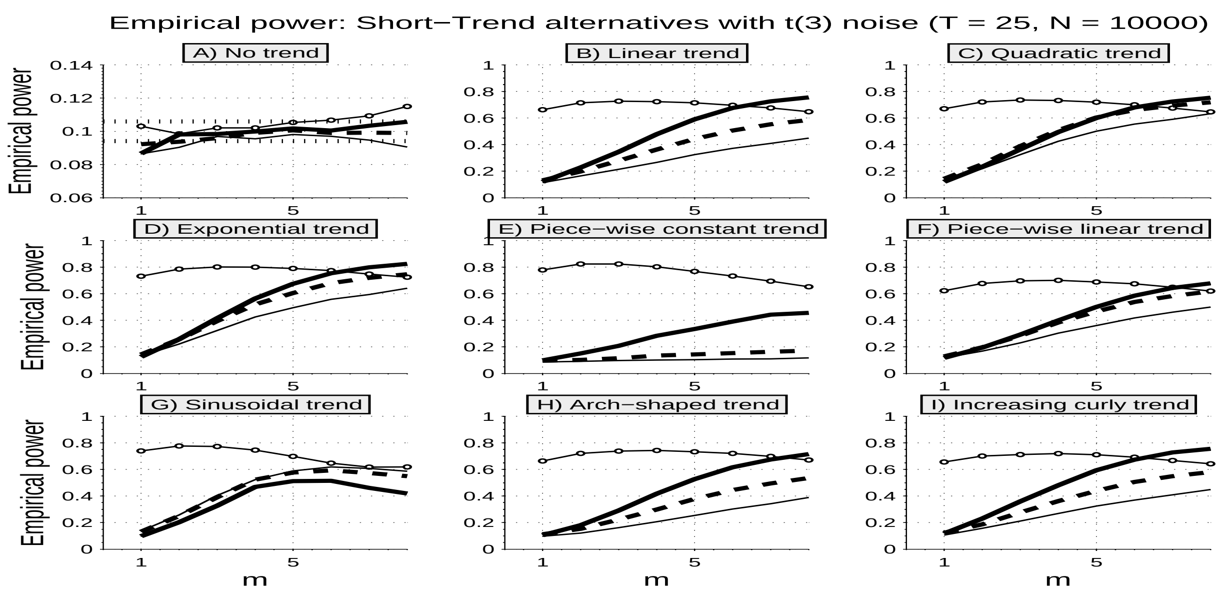

Figure 3.10: Behaviour of the tests based on $S_S(m)$ (ooooo), $S_{MW,0}(m)$ (▬▬), $S_{MW,1}(m)$ (▬ ▬ ▬) and $S_{MW,2}(m)$ (——) when applied to a short trend plus standardized t(3) white noise ($T = 25$ … time series length, $N = 10\ 000$ … number of replications, ⁞⁞⁞⁞⁞⁞ … bounds of 95% confidence intervals for the empirical size).

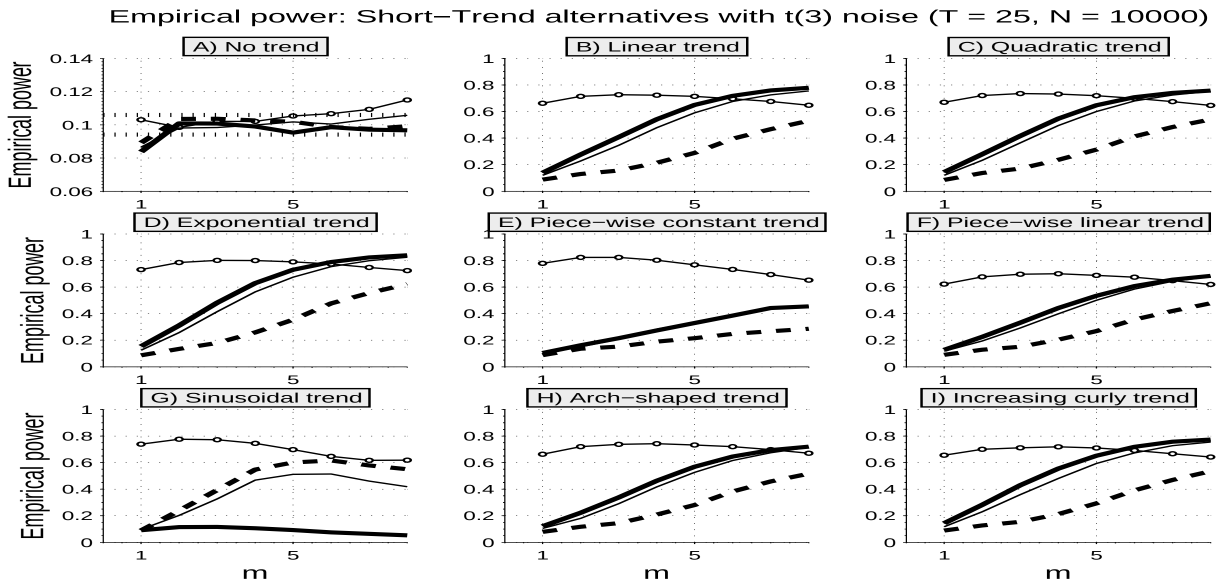

Figure 3.11: Behaviour of the tests based on $S_S(m)$ ( ), $S_{M,0}(m)$ ( ), $S_{W,0}(m)$ ( ) and $S_{MW,0}(m)$ ( ) when applied to a short trend plus standardized t(3) white noise ($T = 25$ . . . time series length, $N = 10\ 000$ . . . number of replications, . . . bounds of 95% confidence intervals for the empirical size).

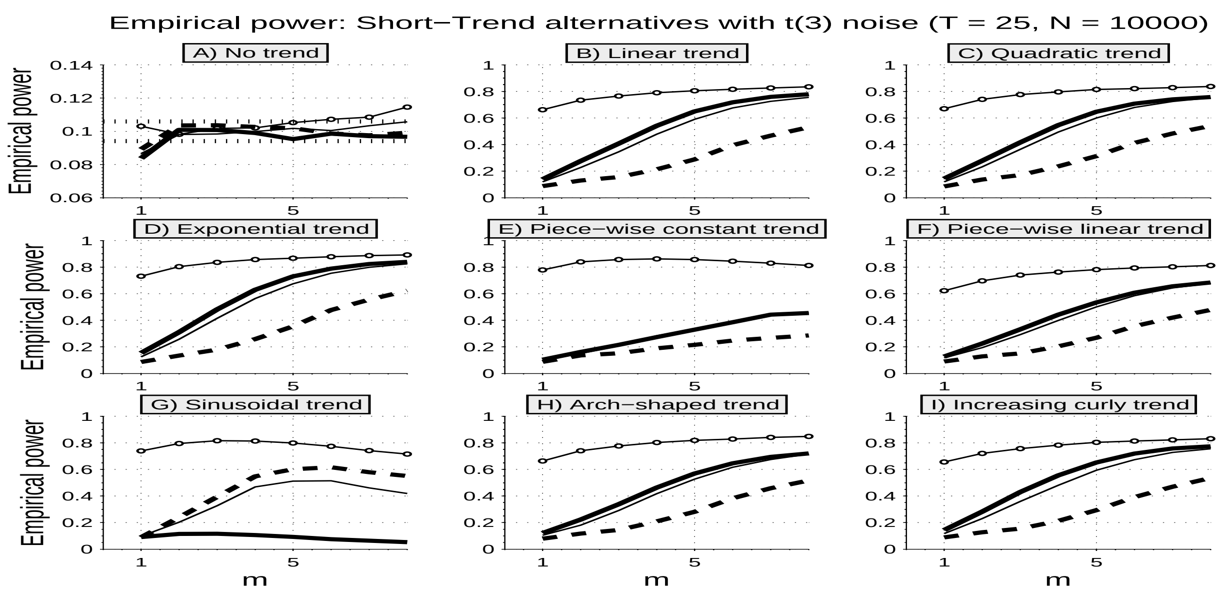

Figure 3.12: Behaviour of the tests based on $S_S^{\perp}(m)$ ( ), $S_{M,0}^{\perp}(m)$ ( ), $S_{W,0}^{\perp}(m)$ ( ) and $S_{MW,0}^{\perp}(m)$ ( ) when applied to a short trend plus standardized t(3) white noise ($T = 25$ . . . time series length, $N = 10\ 000$ . . . number of replications, . . . bounds of 95% confidence intervals for the empirical size).

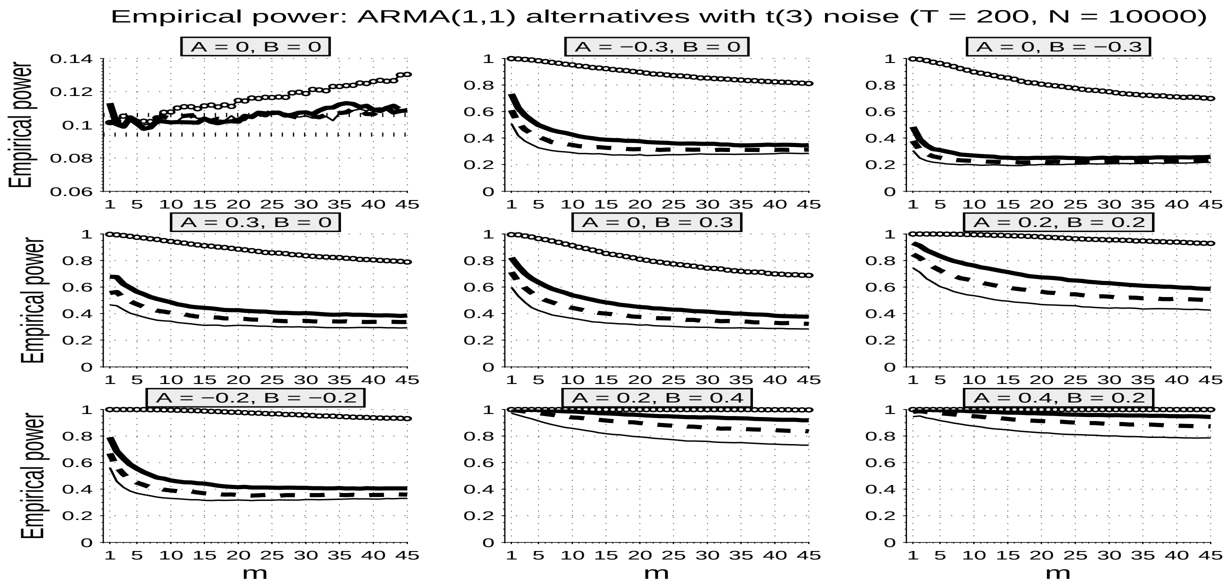

Figure 3.13: Behaviour of the tests based on $S_S(m)$ (ooooo), $S_{W,0}(m)$ (▬▬), $S_{W,1}(m)$ (▬ ▬ ▬) and $S_{W,2}(m)$ (——) when applied to ARMA(1,1) processes with standardized t(3) white noise ($T = 200$ ... time series length, $N = 10\ 000$ ... number of replications, ıııııııı ... bounds of 95% confidence intervals for the empirical size, A,B ... ARMA parameters a_1, b_1).

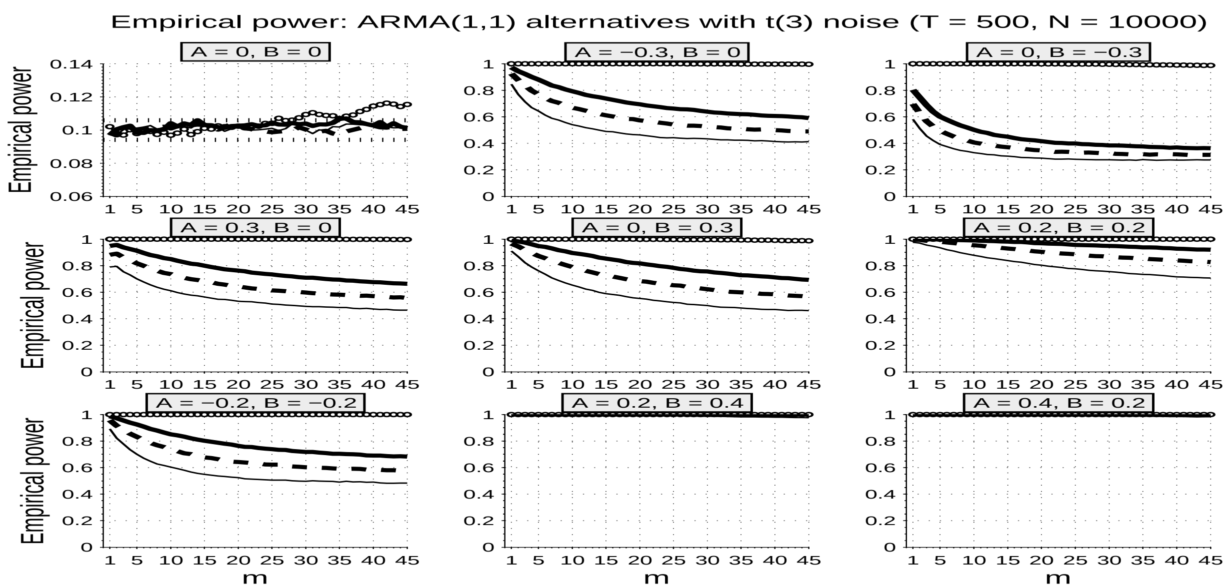

Figure 3.14: Behaviour of the tests based on $S_S(m)$ (ooooo), $S_{W,0}(m)$ (▬▬), $S_{W,1}(m)$ (▬ ▬ ▬) and $S_{W,2}(m)$ (——) when applied to ARMA(1,1) processes with standardized t(3) white noise ($T = 500$ ... time series length, $N = 10\ 000$ ... number of replications, ıııııııı ... bounds of 95% confidence intervals for the empirical size, A,B ... ARMA parameters a_1, b_1).

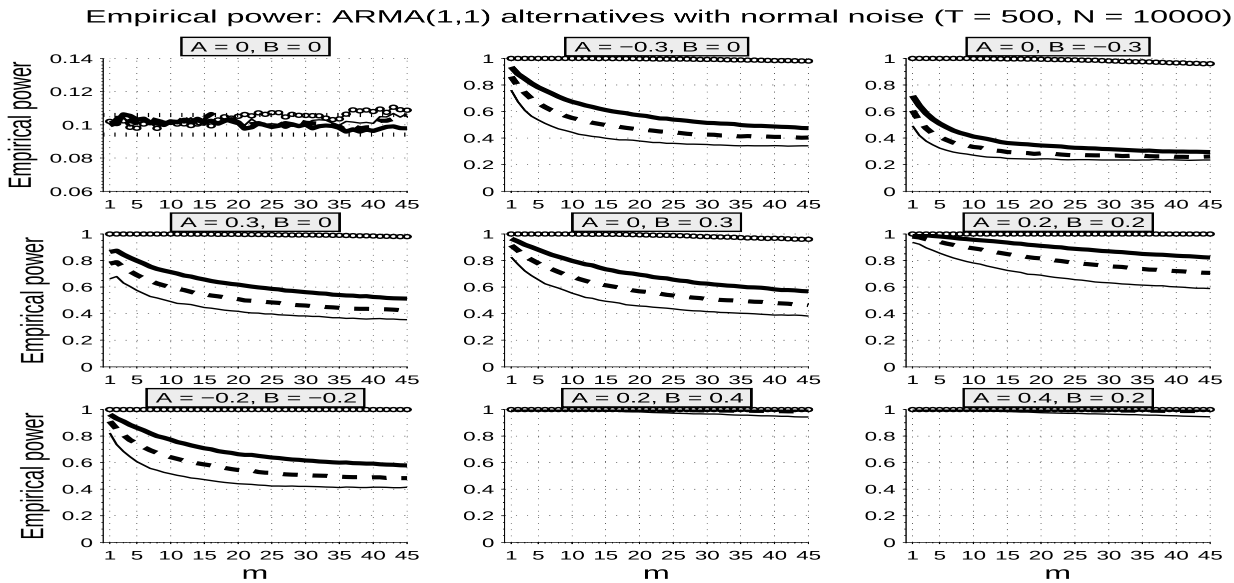

Figure 3.15: Behaviour of the tests based on $S_S(m)$ (∞∞∞), $S_{W,0}(m)$ (▬▬), $S_{W,1}(m)$ (▬ ▬ ▬) and $S_{W,2}(m)$ (——) when applied to ARMA(1,1) processes with standard normal white noise ($T = 500$ ... time series length, $N = 10\ 000$ ... number of replications, ıııııııı ... bounds of 95% confidence intervals for the empirical size, A,B ... ARMA parameters a_1, b_1).

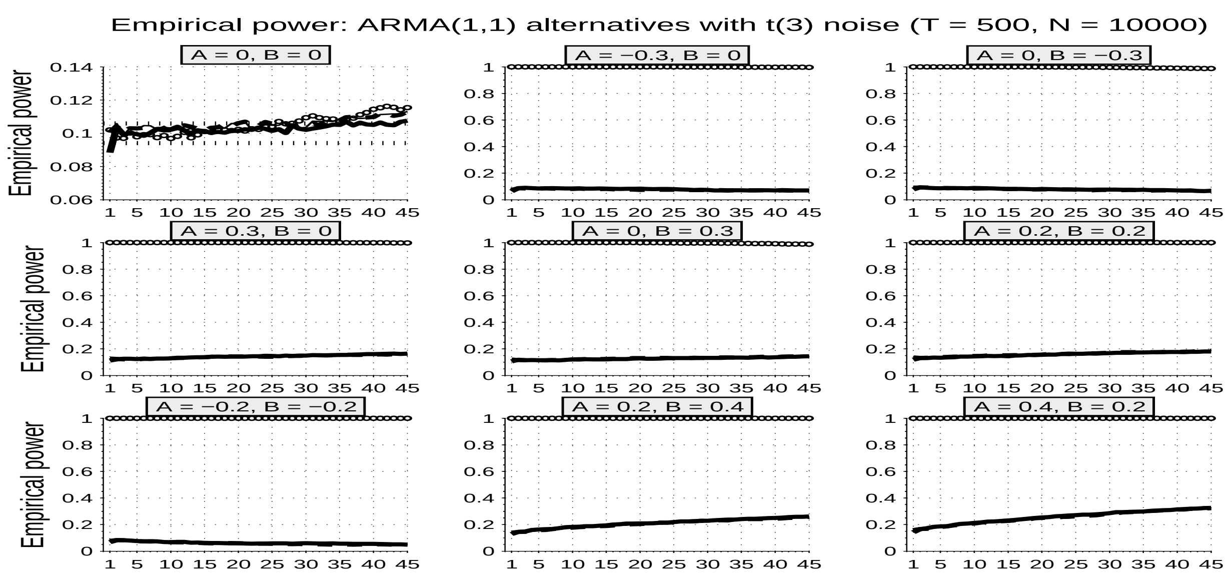

Figure 3.16: Behaviour of the tests based on $S_S(m)$ (∞∞∞), $S_{M,0}(m)$ (▬▬), $S_{M,1}(m)$ (▬ ▬ ▬) and $S_{M,2}(m)$ (——) when applied to ARMA(1,1) processes with standardized t(3) white noise ($T = 500$ ... time series length, $N = 10\ 000$ ... number of replications, ıııııııı ... bounds of 95% confidence intervals for the empirical size, A,B ... ARMA parameters a_1, b_1).

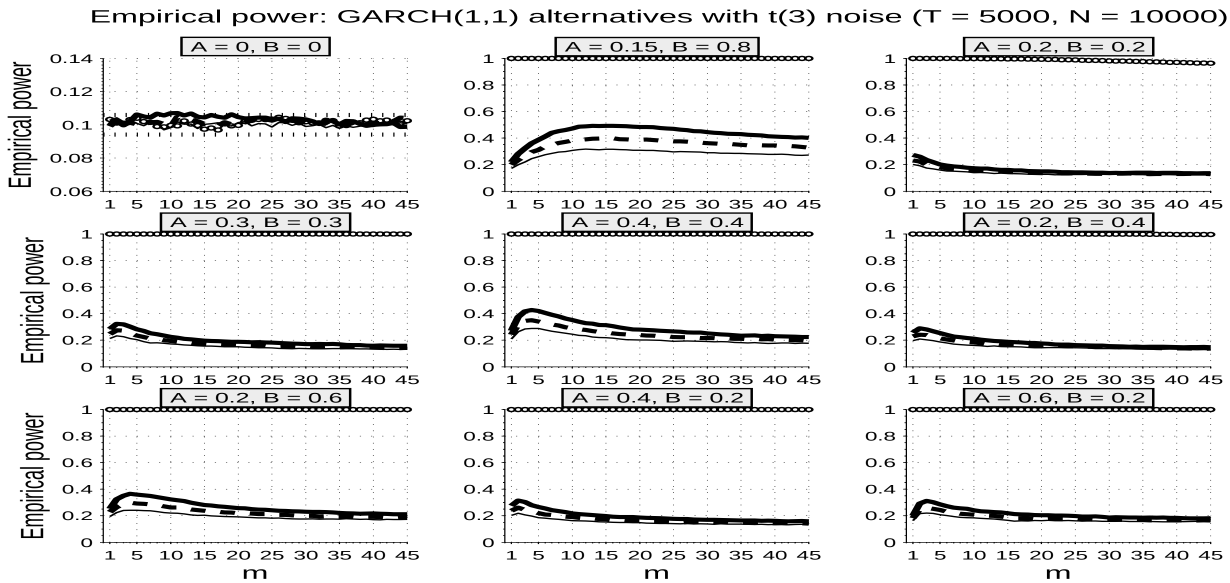

Figure 3.17: Behaviour of the tests based on $S_S(m)$ (∞∞∞), $S_{M,0}(m)$ (▬▬), $S_{M,1}(m)$ (▬ ▬ ▬) and $S_{M,2}(m)$ (——) when applied to GARCH(1,1) models with standardized t(3) innovations ($T = 5\ 000$ ... time series length, $N = 10\ 000$ ... number of replications, ıııııııı ... bounds of 95% confidence intervals for the empirical size, A,B ... GARCH parameters a_1, b_1).

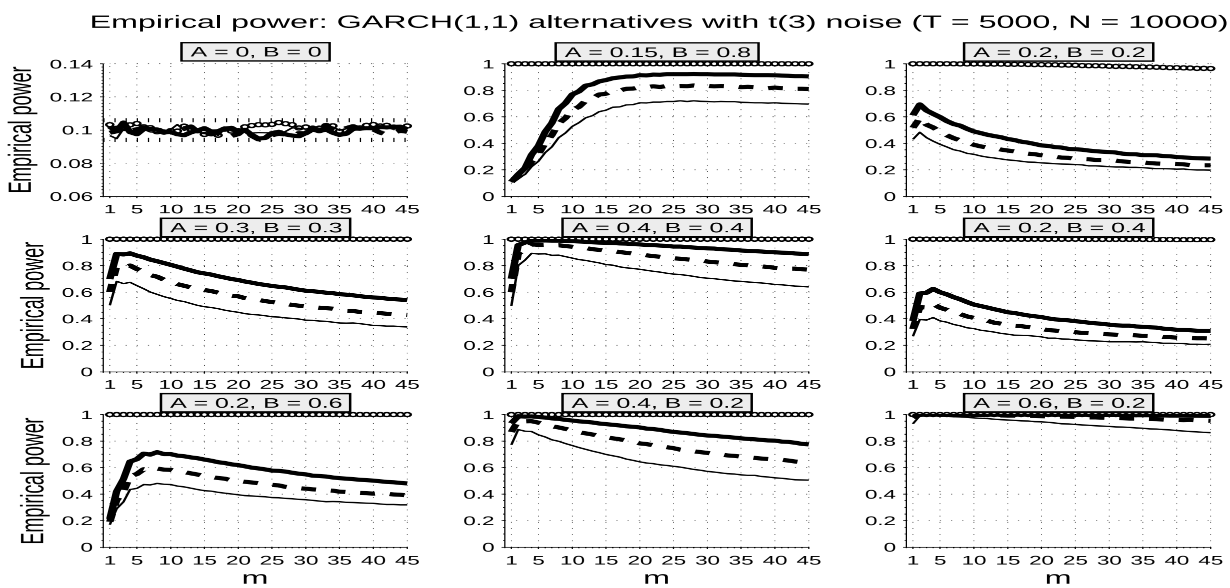

Figure 3.18: Behaviour of the tests based on $S_S(m)$ (∞∞∞), $S_{W,0}(m)$ (▬▬), $S_{W,1}(m)$ (▬ ▬ ▬) and $S_{W,2}(m)$ (——) when applied to GARCH(1,1) models with standardized t(3) innovations ($T = 5\ 000$ ... time series length, $N = 10\ 000$ ... number of replications, ıııııııı ... bounds of 95% confidence intervals for the empirical size, A,B ... GARCH parameters a_1, b_1).

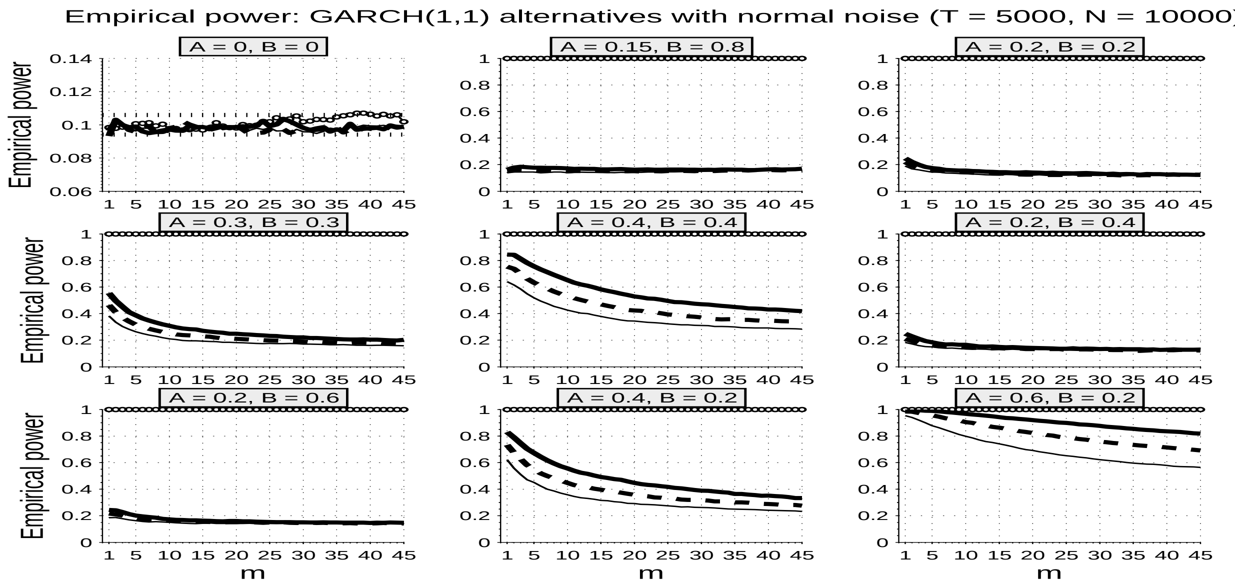

Figure 3.19: Behaviour of the tests based on $S_S(m)$ (ooooo), $S_{M,0}(m)$ (▬▬), $S_{M,1}(m)$ (▬ ▬ ▬) and $S_{M,2}(m)$ (——) when applied to GARCH(1,1) models with standard normal innovations ($T = 5\ 000$ … time series length, $N = 10\ 000$ … number of replications, ııııııı … bounds of 95% confidence intervals for the empirical size, A,B … GARCH parameters a_1, b_1).

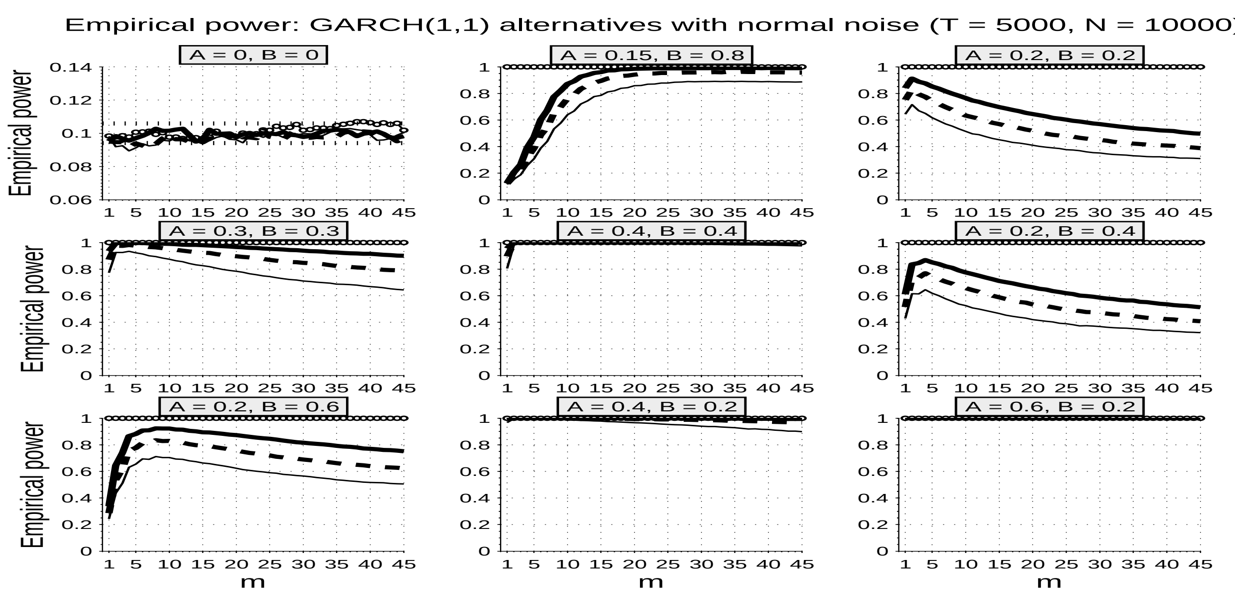

Figure 3.20: Behaviour of the tests based on $S_S(m)$ (ooooo), $S_{W,0}(m)$ (▬▬), $S_{W,1}(m)$ (▬ ▬ ▬) and $S_{W,2}(m)$ (——) when applied to GARCH(1,1) models with standard normal innovations ($T = 5\ 000$ … time series length, $N = 10\ 000$ … number of replications, ııııııı … bounds of 95% confidence intervals for the empirical size, A,B … GARCH parameters a_1, b_1).

Chapter 4

Testing Random Number Generators

The means, variances and covariances of the Moore and Wallis serial rank coefficients should be the same for both independent and pseudo-random sequences, which might be used for testing the quality of (pseudo-)random number generators (RNGs). Such general tests do not figure in any of the recent summary papers covering this area, see e.g. [Soto, 1999] and [L'Ecuyer and Simard, 2005], and this is why we investigate this possibility here. Besides, we would like to show that the Moore and Wallis orthonormal autocorrelations can be helpful even in this context.

4.1 Methodology and Realization

We focus on the following five RNGs:

RNG1:	the Wichmann-Hill generator,
RNG2:	Marsaglia's multiply-with-carry generator,
RNG3:	Marsaglia's Super-Duper generator,
RNG4:	the Mersenne-Twister generator,
RNG5:	the Knuth-TAOCP generator (the 2002 version),

that are implemented and referenced in R, see [R Development Core Team, 2005]. We use each of them with three different seed initializations ($-1, 0$, and 1). For comparison, we also analyse Marsaglia's allegedly perfect random numbers from his well known CD [Marsaglia, 1995], stored in the subdirectory TrueRandNum.

We always get $r = 200$ time series with $T = 500\,000$ observations, use each of them to compute the orthonormal Moore and Wallis serial rank coefficients at lags 1 to $c = 500$ and calculate the one-sided (upper) p–values from their squares by means of the $\chi^2(1)$ inverse distribution function. Finally, we analyse the matrices $P^M_{200\times 500} = (p^M_{i,j})$ and $P^W_{200\times 500} = (p^W_{i,j})$ of the p–values associated with the Moore and Wallis orthonormal rank autocorrelations, respectively. (Generally, we use M and W to distinguish anything related to these two kinds of coefficients.)

Suppose we are just processing such a matrix $P = (p_{i,j})$. First, we employ the Fisher and Tippett combining methods, used and described in Chapter 9 in a different context. To be more specific, we use Fisher's method to combine all p–values from the column Tippett ones, which leads to the final p–value p_F from the test T_F based on the statistic

$$S_F = -2 \sum_{j=1}^{c} \ln\Big(1 - \big(1 - \min_{i=1,\dots,r} (p_{i,j})\big)^r\Big)$$

with the $\chi^2(2c)$ asymptotic distribution under the null hypothesis H_0 of ideal randomness.

And second, we apply Pearson's χ^2 multinomial goodness-of-fit test to all $p_{i,j}$'s and compute both its p–values p_n for any reasonably small integer number n of isometric interval cells and the overall p–values

$$\bar{p}_N = (N-1) \min_{n=2,\dots,N} p_n$$

resulting from Bonferroni's crude approach to combining possibly dependent tests. The choice of just the Pearson test can be justified easily, see e.g. [Cressie and Read, 1984], [Haberman, 1988] and references therein. By the way, it also seems promising to combine the p–values $p_{i,j}$'s into m–tuples ($m = 2, 3, \dots$) in a reasonable way and then to test their m–dimensional uniformity, e.g. again with a goodness-of-fit test.

Our primary intention is to show that the Moore and Wallis orthonormal autocorrelations can be useful in this context and that even common RNGs can significantly influence some Monte Carlo studies based on them. Even these elementary statistical tools are then sufficient for this purpose. That is to say there is a common belief that perfect RNGs should pass any simple statistical test. Besides, the Moore and Wallis coefficients can be interpreted as measures of the forecast direction predictability and that is another reason why any good RNG should not fail in the tests based on them.

The generation of (pseudo-)random numbers and computation of the standardized Moore and Wallis autocorrelations can be reproduced with *RNGDat.r* and *RNTrueRan.r*, their orthonormalization with *RNGDatOrt.m* and *RNTrueRanOrt.m* and the statistical analysis with *RNEval.r*, *RNEvalAdd.r*, *RNEvalAdd2.r* and *RNExam.r*. All the final results are stored in the RNGAnalysis subdirectory and some of them are also illustrated in Tables 4.1 to 4.6.

It remains to assess the results and provide them with a few comments. We consider several evaluation schemes for the purpose of greater objectivity. All of them are based on p_F^M, p_F^W, $\bar{p}_{N_M}^M$ and $\bar{p}_{N_W}^W$ and differ from one another only in the choice of N_M and N_W. We reject H_0 at a level α if at least one of these p–values is less than $\alpha/4$.

It is evident that RNG2(-1) leads to rejecting H_0 at the level $\alpha = 0.003$, and both RNG2(0) and RNG2(1) allow us to reject H_0 even for α as low as 0.0001. Besides, the p_n^M's and $\bar{p}_N^M$'s observed also indicate some discrepancies regarding this generator. We thus conclude that there is something wrong with RNG2 and we do not consider it hereafter. It is quite surprising because RNG2 is said to pass all the Diehard tests (see [Marsaglia, 1995]), at least according to the R documentation.

We can get very interesting results, depending on N_M and N_W:

- ($N_M = 500, N_W = 500$) This is the choice where the benchmark Marsaglia's numbers clearly look as perfectly random from all our points of view. Nevertheless, both RNG3(1) and RNG5(0) would reject H_0 at a significance level lower than 0.05.

- ($N_M = 700, N_W = 700$) Although there is no apparent change in the behaviour of Marsaglia's numbers, we could reject H_0 in the case of RNG3(1), RNG4(0), and RNG5(0) at the levels $0.05, 0.04$, and 0.02, respectively.

- ($N_M = 800, N_W = 800$) The benchmark still seems all right but RNG1(0), RNG1(1), RNG3(-1), RNG3(0), RNG3(1), RNG4(0), RNG5(0) and RNG5(1) reject H_0 for $\alpha = 0.05$ and RNG1(0), RNG3(-1), RNG4(0), RNG5(0) do so even for $\alpha = 0.003$.

- ($N_M = 700, N_W = 1400$) Marsaglia's numbers and all the generators except for RNG4(-1), RNG4(1) and RNG5(-1) reject H_0 at a level lower than 0.03, with RNG1(0), RNG1(1), RNG3(-1), RNG3(0), RNG4(0) and RNG5(0) rejecting at a level lower than 0.002.

- ($N_M = 1200$) We can reject H_0 due to $\bar{p}_{N_M}$ at a level lower than 0.0002 in all the cases.

Note that RNG1, RNG3, RNG4, and RNG5 would appear even much worse if we based their evaluation only on $\bar{p}^W_{N_W}$'s and if we computed these overall p–values more naturally, i.e. only from the individual p–values p_n's with sufficiently high n's.

You can also consult the files *RNGOrtCoefMsExam.txt* and *RNGOrtCoefWsExam.txt* where the individual contributions of all cells to the Pearson χ^2 statistic are reported for several problematic n's in each case. They further confirm the peculiar behaviour of some generators and give deeper insight into why their randomness is rejected.

In principle, these results need not be infallible due to several factors including round-off errors, numerical inaccuracies, bad implementation or asymptotic approximations to the finite sample distributions. All of them might deteriorate (but almost never improve) the output. We neither plan to investigate all these factors in detail, nor do we intend to conduct similar experiments with other T's, r's, and c's because it would lie far beyond the scope of this work. Therefore we are not going to render any definitive judgments. In fact, we are not even competent enough to draw irrefutable conclusions regarding RNGs and related topics.

However, it seems reasonable to assume the results quite reliable at least when Marsaglia's numbers appear correct (although they need not be absolutely faultless either, see [Davies, 1999]). The implications are then straightforward:

- the outputs of Monte Carlo studies should be judged very reservedly and with extreme care,

- the way RNGs are initialised matters considerably (and thus it should not be chosen at random),

- RNG4(1) appears the best of all the possibilities investigated.

Nevertheless, much more investigation would be necessary to assess the RNGs at least a little objectively. Some proposals for further improvements are made in the last chapter.

We can conclude that the Moore and Wallis orthonormal autocorrelations can be useful for testing the quality of RNGs and that all the Monte Carlo studies in this work may be somewhat biased due to the RNG employed.

4.2 Accompanying Tables

Random Number Generator No. 1

	Seed = −1				Seed = 0				Seed = 1			
	M		W		M		W		M		W	
p_F	0.11381		0.32835		0.82310		0.42284		0.14625		0.23693	
$\bar{p}_2$	0.68565	(2)	0.04170	(2)	0.74225	(2)	0.36578	(2)	0.29669	(2)	0.77111	(2)
$\bar{p}_{100}$	1	(95)	1	(2)	1	(74)	1	(76)	1	(92)	1	(80)
$\bar{p}_{200}$	1	(155)	1	(2)	1	(161)	1	(114)	1	(153)	1	(105)
$\bar{p}_{300}$	1	(238)	1	(2)	1	(239)	1	(243)	1	(248)	1	(105)
$\bar{p}_{400}$	1	(395)	1	(2)	0.24703	(366)	1	(243)	0.52437	(397)	1	(105)
$\bar{p}_{500}$	0.90174	(405)	1	(478)	0.30894	(366)	1	(486)	0.02084	(459)	1	(478)
$\bar{p}_{600}$	1	(405)	1	(478)	0.37085	(366)	1	(502)	0.02502	(459)	0.14984	(536)
$\bar{p}_{700}$	0.84657	(682)	1	(671)	0.43276	(366)	1	(676)	0.02920	(459)	0.17485	(536)
$\bar{p}_{800}$	0.03247	(797)	0.12670	(778)	0.01413	(751)	0.00054	(782)	0.00282	(727)	0.00229	(778)
$\bar{p}_{900}$	0.00169	(882)	0.00693	(877)	0.00076	(881)	0.00012	(813)	0.00318	(727)	0.00019	(847)
$\bar{p}_{1000}$	1.5E-06	(983)	0.00379	(909)	0.00003	(995)	0.00013	(813)	0.00017	(986)	0.00021	(847)
$\bar{p}_{1100}$	9.8E-09	(1078)	0.00417	(909)	1.5E-06	(1069)	0.00015	(813)	4.6E-06	(1061)	0.00023	(847)
$\bar{p}_{1200}$	6.1E-14	(1189)	0.00455	(909)	1.5E-11	(1194)	0.00016	(813)	1.5E-08	(1146)	0.00025	(847)
$\bar{p}_{1300}$	–	–	0.00493	(909)	–	–	0.00017	(813)	–	–	0.00027	(847)
$\bar{p}_{1400}$	–	–	0.00531	(909)	–	–	0.00019	(813)	–	–	0.00029	(847)

Table 4.1: Selected outputs regarding RNG1, including both Fisher's p–values p_F's and Pearson's overall p–values $\bar{p}_N$'s, $N = 2, 100, \ldots, 1\,400$, each with the most influential number of cells in parenthesis. The letters M, W denote the relation to the Moore or Wallis orthonormal autocorrelations.

Random Number Generator No. 2

	Seed = −1				Seed = 0				Seed = 1			
	M		W		M		W		M		W	
p_F	0.00052		0.81083		0.00001		0.22623		0.00002		0.65180	
$\bar{p}_2$	0.00461	(2)	0.06208	(2)	0.65797	(2)	0.84952	(2)	0.01682	(2)	1	(2)
$\bar{p}_{100}$	0.45595	(2)	1	(2)	0.01379	(77)	1	(18)	0.01341	(4)	1	(18)
$\bar{p}_{200}$	0.91651	(2)	1	(2)	0.01328	(160)	1	(166)	0.02695	(4)	1	(18)
$\bar{p}_{300}$	1	(2)	1	(2)	0.01996	(160)	1	(274)	0.04050	(4)	1	(18)
$\bar{p}_{400}$	1	(2)	1	(2)	0.02663	(160)	1	(332)	0.00649	(379)	1	(18)
$\bar{p}_{500}$	1	(2)	1	(2)	0.03331	(160)	1	(332)	0.00812	(379)	1	(18)
$\bar{p}_{600}$	1	(2)	1	(2)	0.03999	(160)	1	(332)	0.00975	(379)	1	(18)
$\bar{p}_{700}$	1	(629)	1	(671)	0.04666	(160)	1	(699)	0.01137	(379)	1	(646)
$\bar{p}_{800}$	1	(727)	1	(722)	0.00519	(777)	0.00974	(776)	0.00167	(755)	0.09831	(797)
$\bar{p}_{900}$	0.00184	(825)	0.59274	(898)	0.00448	(851)	0.00402	(870)	0.00001	(861)	0.04151	(875)
$\bar{p}_{1000}$	7.5E-07	(957)	0.44242	(935)	0.00011	(989)	0.00446	(870)	3.8E-07	(974)	0.04613	(875)
$\bar{p}_{1100}$	1.6E-08	(1080)	0.48671	(935)	1.4E-06	(1099)	0.00491	(870)	5.0E-10	(1097)	0.05075	(875)
$\bar{p}_{1200}$	4.8E-10	(1178)	0.53099	(935)	2.1E-09	(1167)	0.00536	(870)	6.2E-13	(1199)	0.05537	(875)
$\bar{p}_{1300}$	–	–	0.57528	(935)	–	–	0.00581	(870)	–	–	0.05998	(875)
$\bar{p}_{1400}$	–	–	0.61956	(935)	–	–	0.00625	(870)	–	–	0.06460	(875)

Table 4.2: Selected outputs regarding RNG2, including both Fisher's p–values p_F's and Pearson's overall p–values $\bar{p}_N$'s, $N = 2, 100, \ldots, 1\,400$, each with the most influential number of cells in parenthesis. The letters M, W denote the relation to the Moore or Wallis orthonormal autocorrelations.

Random Number Generator No. 3

	Seed = −1		Seed = 0		Seed = 1	
	M	W	M	W	M	W
p_F	0.85716	0.57444	0.55431	0.56818	0.38730	0.48162
$\bar{p}_2$	0.14230 (2)	0.57783 (2)	0.64884 (2)	1 (2)	0.31461 (2)	0.33007 (2)
$\bar{p}_{100}$	1 (92)	0.47135 (97)	1 (96)	1 (96)	0.85076 (94)	1 (22)
$\bar{p}_{200}$	0.14577 (169)	0.33606 (184)	1 (158)	1 (96)	1 (94)	1 (166)
$\bar{p}_{300}$	0.21902 (169)	0.50493 (184)	1 (201)	1 (253)	1 (279)	1 (166)
$\bar{p}_{400}$	0.04604 (368)	0.25077 (327)	1 (336)	1 (253)	1 (391)	1 (166)
$\bar{p}_{500}$	0.04850 (434)	0.10158 (487)	1 (420)	1 (470)	0.29736 (431)	0.00812 (498)
$\bar{p}_{600}$	0.05822 (434)	0.12194 (487)	1 (420)	1 (470)	0.35695 (431)	0.00975 (498)
$\bar{p}_{700}$	0.06794 (434)	0.14229 (487)	1 (691)	0.20928 (682)	0.13842 (695)	0.01137 (498)
$\bar{p}_{800}$	0.07766 (434)	0.00066 (790)	1 (764)	0.00213 (768)	0.03489 (794)	0.00751 (772)
$\bar{p}_{900}$	0.00012 (899)	0.00002 (864)	0.00381 (900)	9.7E-07 (825)	0.00063 (867)	0.00057 (817)
$\bar{p}_{1000}$	3.5E-06 (992)	0.00002 (864)	0.00005 (958)	1.1E-06 (825)	0.00001 (997)	0.00063 (817)
$\bar{p}_{1100}$	1.8E-08 (1030)	0.00002 (864)	0.00006 (958)	1.2E-06 (825)	8.9E-08 (1100)	0.00070 (817)
$\bar{p}_{1200}$	2.7E-11 (1171)	0.00002 (864)	1.8E-08 (1114)	1.3E-06 (825)	4.4E-11 (1162)	0.00076 (817)
$\bar{p}_{1300}$	– –	0.00003 (864)	– –	1.4E-06 (825)	– –	0.00082 (817)
$\bar{p}_{1400}$	– –	0.00003 (864)	– –	1.5E-06 (825)	– –	0.00089 (817)

Table 4.3: Selected outputs regarding RNG3, including both Fisher's p–values p_F's and Pearson's overall p–values $\bar{p}_N$'s, $N = 2, 100, \dots, 1\ 400$, each with the most influential number of cells in parenthesis. The letters M, W denote the relation to the Moore or Wallis orthonormal autocorrelations.

Random Number Generator No. 4

	Seed = −1		Seed = 0		Seed = 1	
	M	W	M	W	M	W
p_F	0.55659	0.41500	0.87013	0.58007	0.91034	0.97978
$\bar{p}_2$	0.36914 (2)	0.63977 (2)	0.75663 (2)	0.72795 (2)	0.14402 (2)	0.64884 (2)
$\bar{p}_{100}$	1 (94)	1 (58)	1 (24)	1 (86)	1 (72)	1 (5)
$\bar{p}_{200}$	1 (136)	1 (136)	1 (190)	0.96578 (170)	1 (72)	1 (5)
$\bar{p}_{300}$	1 (292)	1 (210)	1 (230)	1 (170)	1 (72)	1 (5)
$\bar{p}_{400}$	1 (292)	1 (210)	1 (345)	0.74362 (340)	1 (72)	1 (5)
$\bar{p}_{500}$	1 (431)	1 (485)	1 (345)	0.19897 (474)	1 (72)	1 (467)
$\bar{p}_{600}$	1 (431)	1 (485)	1 (345)	0.23885 (474)	1 (72)	1 (467)
$\bar{p}_{700}$	1 (431)	1 (696)	1 (690)	0.00787 (691)	1 (649)	1 (649)
$\bar{p}_{800}$	0.02559 (792)	0.20206 (758)	0.13542 (799)	0.00069 (749)	1 (781)	1 (761)
$\bar{p}_{900}$	0.01051 (881)	0.22735 (758)	0.00085 (872)	0.00017 (808)	0.03882 (875)	0.40773 (832)
$\bar{p}_{1000}$	1.5E-06 (978)	0.25264 (758)	3.6E-07 (1000)	0.00019 (808)	0.00230 (983)	0.45308 (832)
$\bar{p}_{1100}$	9.6E-07 (1019)	0.27793 (758)	4.2E-11 (1084)	0.00021 (808)	1.1E-06 (1096)	0.49843 (832)
$\bar{p}_{1200}$	2.4E-08 (1191)	0.30322 (758)	4.5E-11 (1084)	0.00023 (808)	3.5E-09 (1113)	0.54379 (832)
$\bar{p}_{1300}$	– –	0.32851 (758)	– –	0.00024 (808)	– –	0.58914 (832)
$\bar{p}_{1400}$	– –	0.35380 (758)	– –	0.00026 (808)	– –	0.63449 (832)

Table 4.4: Selected outputs regarding RNG4, including both Fisher's p–values p_F's and Pearson's overall p–values $\bar{p}_N$'s, $N = 2, 100, \dots, 1\ 400$, each with the most influential number of cells in parenthesis. The letters M, W denote the relation to the Moore or Wallis orthonormal autocorrelations.

Random Number Generator No. 5

	Seed $=-1$				Seed $=0$				Seed $=1$			
	M		W		M		W		M		W	
p_F	0.66292		0.74477		0.43675		0.87417		0.11573		0.87557	
$\bar{p}_2$	0.37252	(2)	1	(2)	0.19698	(2)	0.64884	(2)	0.55217	(2)	1	(2)
$\bar{p}_{100}$	1	(98)	1	(91)	1	(8)	1	(91)	1	(74)	1	(4)
$\bar{p}_{200}$	1	(196)	1	(125)	1	(8)	1	(193)	1	(146)	1	(4)
$\bar{p}_{300}$	1	(196)	1	(125)	1	(8)	1	(272)	1	(146)	1	(261)
$\bar{p}_{400}$	1	(392)	1	(339)	1	(371)	0.35297	(332)	0.50148	(379)	1	(314)
$\bar{p}_{500}$	1	(392)	1	(471)	1	(371)	0.01218	(477)	0.62718	(379)	1	(435)
$\bar{p}_{600}$	1	(392)	1	(471)	1	(371)	0.01215	(506)	0.75286	(379)	1	(435)
$\bar{p}_{700}$	1	(392)	1	(471)	1	(371)	0.00404	(685)	0.87855	(379)	0.88785	(696)
$\bar{p}_{800}$	1	(780)	0.70505	(792)	1	(757)	0.00029	(794)	0.32536	(776)	0.00652	(776)
$\bar{p}_{900}$	0.17347	(890)	0.09737	(815)	0.70935	(898)	0.00001	(849)	0.00399	(876)	0.00322	(823)
$\bar{p}_{1000}$	0.00199	(924)	0.10820	(815)	0.02111	(1000)	0.00001	(849)	0.00006	(992)	0.00358	(823)
$\bar{p}_{1100}$	0.00001	(1020)	0.11903	(815)	0.00049	(1093)	0.00001	(849)	2.5E-06	(1091)	0.00394	(823)
$\bar{p}_{1200}$	2.5E-06	(1175)	0.12986	(815)	0.00004	(1174)	0.00001	(849)	2.0E-08	(1154)	0.00430	(823)
$\bar{p}_{1300}$	–	–	0.14069	(815)	–	–	0.00001	(849)	–	–	0.00465	(823)
$\bar{p}_{1400}$	–	–	0.15152	(815)	–	–	0.00001	(849)	–	–	0.00501	(823)

Table 4.5: Selected outputs regarding RNG5, including both Fisher's p–values p_F's and Pearson's overall p–values $\bar{p}_N$'s, $N = 2, 100, \ldots, 1\,400$, each with the most influential number of cells in parenthesis. The letters M, W denote the relation to the Moore or Wallis orthonormal autocorrelations.

Marsaglia's Random Numbers

	M		W	
p_F	0.61085		0.81276	
$\bar{p}_2$	0.90435	(2)	0.09250	(2)
$\bar{p}_{100}$	1	(4)	1	(52)
$\bar{p}_{200}$	1	(4)	1	(52)
$\bar{p}_{300}$	1	(255)	1	(52)
$\bar{p}_{400}$	0.86766	(370)	1	(52)
$\bar{p}_{500}$	1	(370)	1	(429)
$\bar{p}_{600}$	1	(370)	1	(429)
$\bar{p}_{700}$	0.75951	(699)	1	(429)
$\bar{p}_{800}$	0.04278	(772)	0.19087	(707)
$\bar{p}_{900}$	0.00740	(862)	0.00231	(898)
$\bar{p}_{1000}$	0.00014	(989)	0.00257	(898)
$\bar{p}_{1100}$	0.00001	(1051)	0.00283	(898)
$\bar{p}_{1200}$	4.3E-08	(1138)	0.00309	(898)
$\bar{p}_{1300}$	–	–	0.00334	(898)
$\bar{p}_{1400}$	–	–	0.00360	(898)

Table 4.6: Selected outputs regarding Marsaglia's numbers, including Fisher's p–values p_F's and Pearson's overall p–values $\bar{p}_N$'s, $N = 2, 100, \ldots, 1\,400$, each with the most influential number of cells in parenthesis. The letters M, W denote the relation to the Moore or Wallis orthonormal autocorrelations.

Chapter 5

Kendall's Rank Autocorrelations

In this chapter, we are going to extend the results regarding the noncircular Kendall rank autocorrelations (from [Ferguson et al., 2000]) and to judge their usefulness and possible application in portmanteau tests.

5.1 Theory

We start with the definition from [Ferguson et al., 2000]:

Definition 12. The Kendall rank autocorrelations $\widehat{r}_K(k)$'s, $k = 1, 2, \dots$, are defined by

$$\widehat{r}_K(k) = 1 - \frac{4N_{k,T}}{(T-k)(T-k-1)} \qquad \text{where} \quad N_{k,T} = \sum_{i=1}^{T-k}\sum_{j=1}^{T-k} \mathrm{I}(R_i < R_j, R_{i+k} > R_{j+k}).$$

Here we will also consider their exactly and roughly standardized versions $\widetilde{r}_K(k)$'s and $\bar{r}_K(k)$'s:

$$\widetilde{r}_K(k) = \frac{\widehat{r}_K(k) - \mathrm{E}_0\big(\widehat{r}_K(k)\big)}{\sqrt{\mathrm{var}_0\big(\widehat{r}_K(k)\big)}}, \qquad \bar{r}_K(k) = \frac{\widehat{r}_K(k) - \mathrm{E}_0\big(\widehat{r}_K(k)\big)}{2/(3\sqrt{T})}.$$

The following theorem summarizes some relevant results proved ibidem.

Theorem 13. *If we assume the null hypothesis H_0^E to be true, then*

$$\begin{aligned}
\mathrm{E}_0\big(N_{k,T}\big) &= \frac{(3T-3k-1)(T-k)}{12} - \frac{k}{6} && \textit{for } 1 \le k < \tfrac{T}{2},\\
&= \frac{(T-k)(T-k-1)}{4} && \textit{for } \tfrac{T}{2} \le k < T-1,\\
\mathrm{var}_0(N_{1,T}) &= \frac{10T^3 - 37T^2 + 27T + 74}{360} && \textit{for } T \ge 4,\\
\sqrt{T}\widehat{r}_K(k) &\sim_{asympt.} N(0, \tfrac{4}{9}) && \textit{for } k \ge 1.
\end{aligned}$$

Besides, it also holds asymptotically that $\widetilde{r}_K(k)$'s are then equivalent to $\widetilde{r}_S(k)$'s, jointly standard normal and independent.

Proof. See [Ferguson et al., 2000]. □

Unfortunately, there are no formulae for $\mathrm{var}_0\big(\widehat{r}_K(k)\big)$'s at lags $k > 1$ given there (because of their extremely tedious computation), although the authors indicated that they could be found essentially in the same way. But such formulae are needed for the accurate standardization of $\widehat{r}_K(k)$'s and that is why we derive them now.

Theorem 14 (Variances of $\widehat{r}_K(k)$'s)**.** *If H_0^E holds, $T > k > 0$, and $T \geq 8$, then*

$$
\begin{aligned}
\mathrm{var}_0\big(\widehat{r}_K(k)\big) = & \\
&= \frac{2[10T^3 + (-30k+15)T^2 + (30k^2 - 30k - 25)T + (-10k^3 + 15k^2 + 25k)]}{45(T-k)^2(T-k-1)^2}, \quad k < T < 2k, \\
&= \frac{2[10T^3 + (-30k+13)T^2 + (30k^2 - 34k - 21)T + (-10k^3 + 31k^2 + 17k)]}{45(T-k)^2(T-k-1)^2}, \quad 2k \leq T < 3k, \\
&= \frac{2[10T^3 + (-30k-7)T^2 + (30k^2 + 46k - 37)T + (-10k^3 - 29k^2 + 65k)]}{45(T-k)^2(T-k-1)^2}, \quad 3k \leq T < 4k, \\
&= \frac{2[10T^3 + (-30k-7)T^2 + (30k^2 + 46k - 49)T + (-10k^3 - 29k^2 + 113k)]}{45(T-k)^2(T-k-1)^2}, \quad T \geq 4k.
\end{aligned}
$$

Proof. It is sufficient to get $\mathrm{var}_0(N_{k,T})$. The second moment

$$
\mathrm{E}_0(N_{k,T}^2) = \sum_i^{T-k}\sum_j^{T-k}\sum_p^{T-k}\sum_q^{T-k} P(R_i < R_j, R_{i+k} > R_{j+k}, R_p < R_q, R_{p+k} > R_{q+k})
$$

can be decomposed as in [Ferguson et al., 2000]:

$$
\begin{aligned}
\mathrm{E}_0(N_{k,T}^2) &= A + 2B + 2C + D, \\
A &= \sum_{i,j} P(R_i < R_j, R_{i+k} > R_{j+k}), \\
B &= \sum_{i,j,q\neq j} P(R_i < R_j, R_{i+k} > R_{j+k}, R_i < R_q, R_{i+k} > R_{q+k}), \\
C &= \sum_{i,j\neq p,p\neq i} P(R_i < R_j, R_{i+k} > R_{j+k}, R_p < R_i, R_{p+k} > R_{i+k}), \\
D &= \sum_{i,j,p,q \text{ distinct}} P(R_i < R_j, R_{i+k} > R_{j+k}, R_p < R_q, R_{p+k} > R_{q+k}),
\end{aligned}
$$

where $A = \mathrm{E}_0(N_{k,T})$ is already known.

As far as the terms B, C, D are concerned, the indices of R's may mutually coincide and thus numerous special cases must be considered separately, see [Ferguson et al., 2000] for some details. All the complicated computation was realized by means of the MAPLE program *KMomComp.mws* (with almost 20 pages of a dense code). The auxiliary R files *KProbs.r* and *KExactMom.r* compute empirically the probabilities $P(R_i < R_j, R_{i+k} > R_{j+k}, R_p < R_q, R_{p+k} > R_{q+k})$ and the variances $\mathrm{var}(N_{k,T})$ for small T's (and under H_0^E). □

Note. These formulae agree exactly with the results obtained for $k = 1$ in [Ferguson et al., 2000].

5.2 Monte Carlo Simulations

Both the coefficients $\widetilde{r}_K(k)$'s and $\bar{r}_K(k)$'s are asymptotically independent and thus they seem to be prime candidates for using in the portmanteau tests. The empirical study in [Ferguson et al., 2000] indicates that some improvement could really be achieved this way in practice (despite the asymptotic equivalence of $\widetilde{r}_K(k)$'s and $\widetilde{r}_S(k)$'s under H_0^E).

We investigate such possibility in this small comparative Monte Carlo study where the benchmark test $T_S(m)$ and the tests $T_{K_A}(m)$ and $T_{K_E}(m)$ based on the statistics $S_{K_A}(m)$ and $S_{K_E}(m)$ are judged,

$$S_{K_A}(m) = \sum_{k=1}^{m} \bar{r}_K^2(k) \sim_{\text{asympt.}} \chi^2(m), \qquad S_{K_E}(m) = \sum_{k=1}^{m} \widetilde{r}_K^2(k) \sim_{\text{asympt.}} \chi^2(m).$$

We take account of all the time series from the classes TREND ($T = 100$), SHORTTREND ($T = 25$), ARMA ($T = 100$) and GARCH ($T = 200$). The lines —— , - - - and ∘∘∘∘∘ correspond to the tests T_S, T_{K_A} and T_{K_E}, respectively.

The results lead to the following conclusions:

- The test $T_{K_A}(m)$ is badly sized even for very small m's and quite long time series with $T = 200$ observations, see Figure 5.4. It clearly demonstrates that the standardization by means of the asymptotic variance is inappropriate, and it also justifies our effort spent on the derivation of the exact formulae for $\text{var}_0\big(\widehat{r}_K(k)\big)$'s.

- In general, the tests T_S and T_{K_E} always exhibit virtually the same size. Besides, they also have almost the same power when longer time series are investigated, see Figures 5.3 and 5.4. Therefore we cannot recommend to use T_{K_E} for longer time series, as the computation of S_{K_E} is then usually more demanding than the evaluation of S_S.

- However, $T_{K_E}(m)$ clearly outperforms $T_S(m)$ in the case of short trend alternatives when it is much less sensitive to the misspecification of m and its maximum power may exceed that of $T_S(m)$ even by more than 5 percentage points, see Figures 5.1 and 5.2.

- $T_{K_E}(1)$ is sometimes much less powerful than $T_{K_E}(m)$ with a higher value of m (see e.g. Figures 5.1, 5.2 and 5.4), which speaks in favour of the use and existence of the precisely standardized Kendall rank autocorrelations at higher lags.

5.3 Accompanying Figures

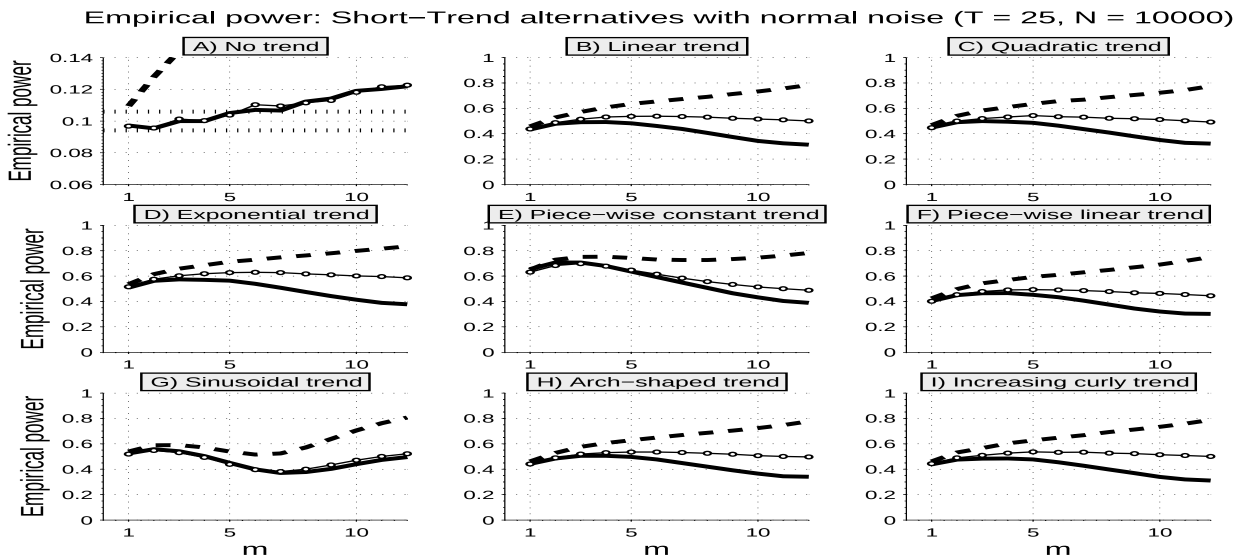

Figure 5.1: Behaviour of the tests based on $S_S(m)$ (▬▬), $S_{K_A}(m)$ (▬ ▬ ▬) and $S_{K_E}(m)$ (○○○○○) when applied to a short trend plus N(0,1) white noise ($T = 25$ … time series length, $N = 10\,000$ … number of replications, ⋯⋯ … bounds of 95% confidence intervals for the empirical size).

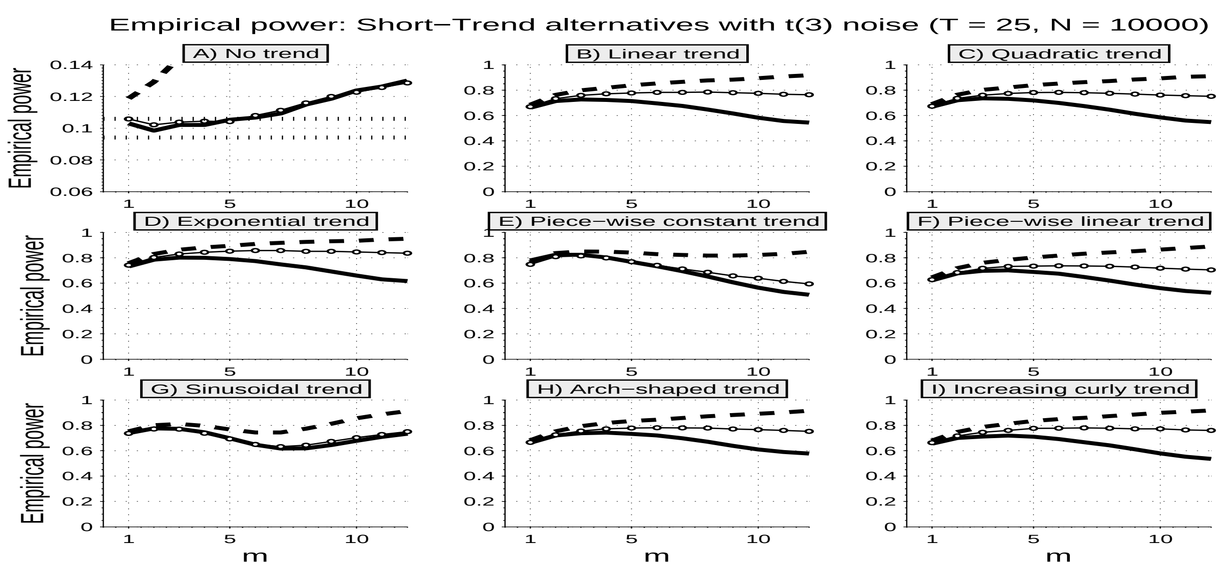

Figure 5.2: Behaviour of the tests based on $S_S(m)$ (▬▬), $S_{K_A}(m)$ (▬ ▬ ▬) and $S_{K_E}(m)$ (○○○○○) when applied to a short trend plus standardized t(3) white noise ($T = 25$ … time series length, $N = 10\,000$ … number of replications, ⋯⋯ … bounds of 95% confidence intervals for the empirical size).

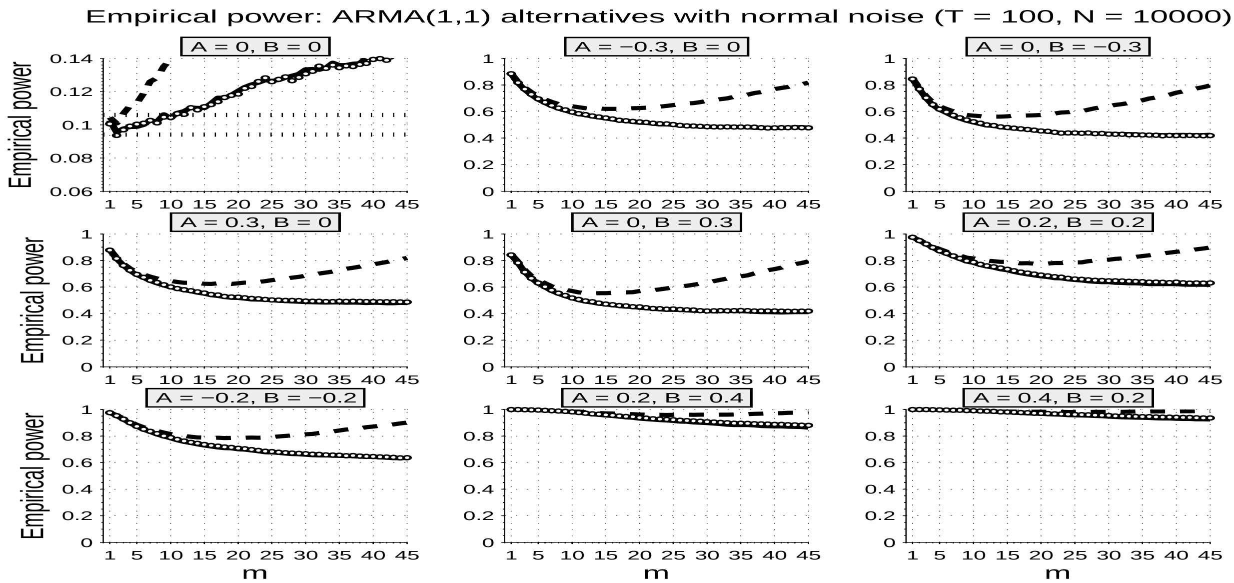

Figure 5.3: Behaviour of the tests based on $S_S(m)$ (—), $S_{K_A}(m)$ (- - -) and $S_{K_E}(m)$ (∘∘∘∘∘) when applied to ARMA(1,1) processes with N(0,1) white noise ($T = 100$ …time series length, $N = 10\,000$ …number of replications, ⋮⋮⋮⋮ …bounds of 95% confidence intervals for the empirical size, A,B …ARMA parameters a_1, b_1).

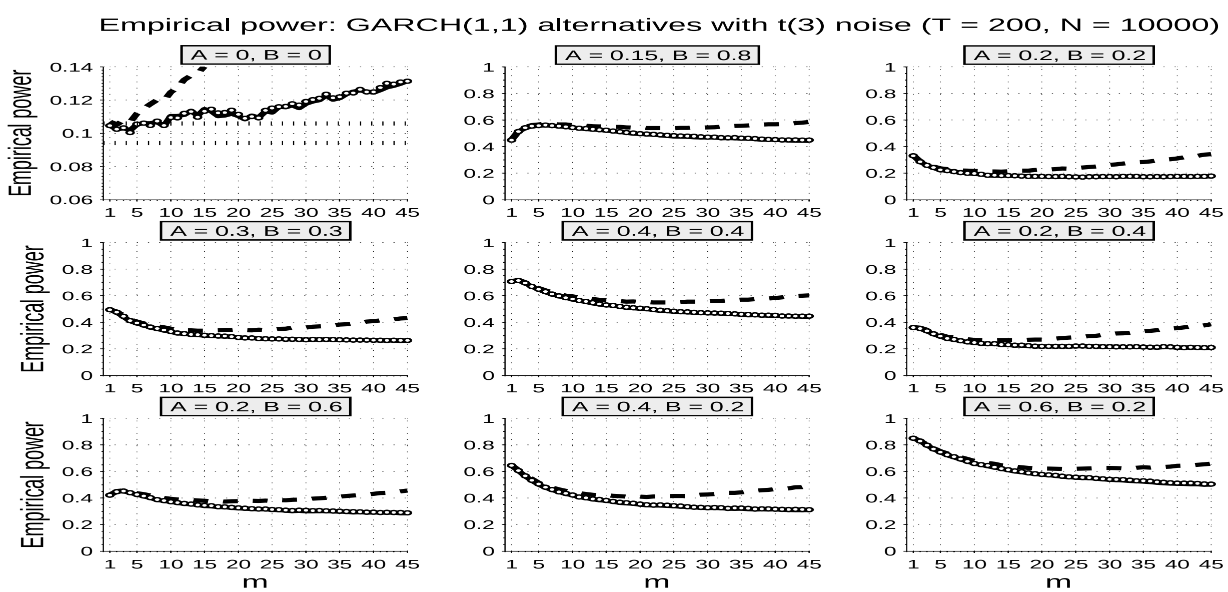

Figure 5.4: Behaviour of the tests based on $S_S(m)$ (—), $S_{K_A}(m)$ (- - -) and $S_{K_E}(m)$ (∘∘∘∘∘) when applied to GARCH(1,1) models with standardized t(3) innovations ($T = 200$ …time series length, $N = 10\,000$ …number of replications, ⋮⋮⋮⋮ …bounds of 95% confidence intervals for the empirical size, A,B …GARCH parameters a_1, b_1).

Chapter 6

Weighted Kendall's Serial Rank Coefficients

Kendall's rank autocorrelations are quite slow to compute, especially in the case of long time series. We show that this drawback can be overcome by introducing their weighted modifications that can be obtained easily as the most natural generalization of both those serial unweighted [Ferguson et al., 2000] and nonserial weighted [Shieh, 1998] Kendall's serial rank coefficients.

6.1 Theory

For simplicity, we only introduce the standardized versions of the weighted Kendall autocorrelations.

Definition 15. The standardized weighted Kendall serial rank coefficients $\widetilde{r}_{K,w}(k)$'s, $k = 1, 2, \dots$, are defined in the following way:

$$\widetilde{r}_{K,w}(k) = \frac{N_{k,w,T} - \mathrm{E}_0(N_{k,w,T})}{\sqrt{\mathrm{var}_0(N_{k,w,T})}}, \quad \text{where} \quad N_{k,w,T} = \sum_{i=1}^{T-k} \sum_{j=1}^{T-k} w(i,j)\, \mathrm{I}(R_i < R_j, R_{i+k} > R_{j+k})$$

and w is a real function symmetric in its arguments and possibly dependent on k.

As we want to speed up the computation, we focus only on the trimmed weighting functions satisfying $w(i,j) = 0$ for $|i-j| > m$, $m < T$, namely on those listed below:

$$w_1(i,j) = \mathrm{I}(|i-j| \le m), \qquad w_2(i,j) = |i-j|\, \mathrm{I}(|i-j| \le m),$$
$$w_3(i,j) = (m + 1 - |i-j|)\, \mathrm{I}(|i-j| \le m), \qquad w_4(i,j) = (i+j)\, \mathrm{I}(|i-j| \le m).$$

The means, variances and joint asymptotic normality of $\widetilde{r}_{K,w}(k)$'s can be obtained easily in all these cases.

Theorem 16. *If the null hypothesis H_0^E holds, $w = w_1, w_2, w_3$ or w_4, $T \ge 8$, $T \ge 5k + m$, and $m \ge 3k$, then the means and variances of $N_{k,w,T}$'s are as follows:*

$$\mathrm{E}_0(N_{k,w_1,T}) = \Big(\frac{1}{2}m + \frac{1}{6}\Big)T - \frac{1}{4}m^2 - \frac{1}{2}km - \frac{1}{4}m - \frac{1}{3}k,$$

$$
\begin{aligned}
\mathrm{var}_0(N_{k,w_1,T}) &= \Big(\frac{1}{9}m^2 + \frac{13}{180}m - \frac{1}{15}k - \frac{23}{180}\Big)T - \frac{5}{54}m^3 - \frac{43}{360}m^2 - \frac{1}{9}km^2 - \frac{29}{1080}m + \frac{1}{12}km \\
&\quad + \frac{31}{180}k^2 + \frac{23}{60}k, \\
\mathrm{E}_0(N_{k,w_2,T}) &= \Big(\frac{1}{4}m^2 + \frac{1}{4}m + \frac{1}{6}k\Big)T - \frac{1}{6}m^3 - \frac{1}{4}m^2 - \frac{1}{4}km^2 - \frac{1}{4}km - \frac{1}{12}m - \frac{1}{3}k^2, \\
\mathrm{var}_0(N_{k,w_2,T}) &= \Big(\frac{1}{36}m^4 + \frac{79}{540}m^3 + \frac{59}{360}m^2 - \frac{2}{9}km^2 - \frac{2}{9}km + \frac{49}{1080}m - \frac{19}{90}k^2 + \frac{7}{60}k^3 - \frac{1}{30}k\Big)T \\
&\quad - \frac{4}{135}m^5 - \frac{1}{36}km^4 - \frac{307}{2160}m^4 - \frac{23}{120}m^3 + \frac{1}{540}km^3 + \frac{16}{45}k^2m^2 - \frac{167}{2160}m^2 \\
&\quad + \frac{7}{120}km^2 + \frac{16}{45}k^2m + \frac{31}{1080}km + \frac{1}{540}m - \frac{217}{1080}k^4 + \frac{4}{5}k^3 + \frac{37}{1080}k^2, \\
\mathrm{E}_0(N_{k,w_3,T}) &= \Big(\frac{1}{4}m^2 + \frac{5}{12}m - \frac{1}{6}k + \frac{1}{6}\Big)T - \frac{1}{12}m^3 - \frac{1}{4}m^2 - \frac{1}{4}km^2 - \frac{7}{12}km - \frac{1}{6}m + \frac{1}{3}k^2 - \frac{1}{3}k, \\
\mathrm{var}_0(N_{k,w_3,T}) &= \Big(\frac{1}{36}m^4 + \frac{5}{108}m^3 - \frac{59}{360}m^2 + \frac{7}{45}km^2 + \frac{8}{15}km - \frac{11}{90}k^2m - \frac{67}{216}m + \frac{7}{60}k^3 - \frac{1}{3}k^2 \\
&\quad + \frac{31}{90}k - \frac{23}{180}\Big)T - \frac{17}{1080}m^5 - \frac{107}{2160}m^4 - \frac{1}{36}km^4 - \frac{23}{540}km^3 - \frac{29}{1080}m^3 + \frac{13}{432}m^2 \\
&\quad + \frac{127}{360}km^2 - \frac{11}{60}k^2m^2 + \frac{17}{270}k^3m + \frac{5}{216}m + \frac{791}{1080}km - \frac{107}{90}k^2m - \frac{217}{1080}k^4 \\
&\quad + \frac{233}{270}k^3 - \frac{1049}{1080}k^2 + \frac{197}{540}k, \\
\mathrm{E}_0(N_{k,w_4,T}) &= \Big(\frac{1}{2}m + \frac{1}{6}\Big)T^2 + \Big(-\frac{1}{4}m^2 + \frac{1}{4}m - km - \frac{1}{2}k + \frac{1}{6}\Big)T + \frac{1}{4}km^2 - \frac{1}{4}m^2 - \frac{1}{4}km \\
&\quad + \frac{1}{2}k^2m - \frac{1}{4}m + \frac{1}{3}k^2 - \frac{1}{3}k, \\
\mathrm{var}_0(N_{k,w_4,T}) &= \Big(\frac{4}{27}m^2 + \frac{13}{135}m - \frac{4}{45}k - \frac{23}{135}\Big)T^3 \\
&\quad + \Big(-\frac{5}{27}m^3 - \frac{4}{9}km^2 - \frac{1}{60}m^2 + \frac{49}{540}m + \frac{1}{45}km + \frac{8}{9}k + \frac{43}{90}k^2 - \frac{23}{90}\Big)T^2 \\
&\quad + \Big(\frac{17}{216}m^4 - \frac{11}{270}m^3 + \frac{10}{27}km^3 + \frac{4}{9}k^2m^2 - \frac{13}{135}m^2 - \frac{2}{45}km^2 - \frac{3}{5}k^2m + \frac{7}{135}km \\
&\quad - \frac{1}{360}m - \frac{467}{540}k^3 - \frac{71}{45}k^2 + \frac{134}{135}k - \frac{23}{270}\Big)T - \frac{1}{60}m^5 - \frac{83}{2160}m^4 - \frac{17}{216}km^4 \\
&\quad - \frac{5}{27}k^2m^3 - \frac{31}{360}m^3 + \frac{11}{270}km^3 + \frac{7}{60}k^2m^2 - \frac{4}{27}k^3m^2 + \frac{13}{135}km^2 - \frac{35}{432}m^2 \\
&\quad - \frac{47}{540}k^2m + \frac{29}{45}k^3m + \frac{23}{216}km - \frac{1}{60}m + \frac{23}{40}k^4 + \frac{107}{90}k^3 - \frac{27}{40}k^2 + \frac{23}{90}k.
\end{aligned}
$$

Proof. This proof is similar to that of Theorem 14 but the resulting formulae obtained directly that way are too difficult to simplify. Fortunately, the weighting functions $w_1, \ldots, w_4$ lead to the means and variances in a polynomial form whose maximum possible powers of T, k, and m can be determined easily in advance. We therefore compute the values of $\mathrm{E}_0(N_{k,w,T})$ and $\mathrm{var}_0(N_{k,w,T})$ for numerous choices of T, k, and m and then fit them with an adequate polynomial.

All the computations can be reproduced by means of the MAPLE codes *KWMomCompW1.mws*, *KWMomCompW2.mws*, *KWMomCompW3.mws*, and *KWMomCompW4.mws* that also allow us to calculate the moment characteristics for any weights if T and k are given. The results can be checked with *KWExactMom.r* and *KWMeanVarCheck.mws*.

We discuss only the case when $T \geq 5k + m$ and $m \geq 3k$, which is usually the most relevant in the time series context. Nevertheless, the other possibilities could be treated analogously with the same programs. □

Note. Apparently, the exact formulae for $\mathrm{E}_0(N_{k,w,T})$ and $\mathrm{var}_0(N_{k,w,T})$ can be derived the same way even for any other weighting functions in a polynomial (or other similar) form. Recall that a large number of weighting functions may be approximated by a polynomial thanks to the Taylor formula.

Note. Unfortunately, the covariances $\mathrm{cov}_0\big(\widetilde{r}_{K,w}(h), \widetilde{r}_{K,w}(k)\big)$'s, $1 \leq h < k < T$, still remain unknown because their computation seems tedious even for us. The use of $\widetilde{r}_{K,w}(k)$'s in portmanteau tests thus remains problematic. However, these autocorrelations can still be employed individually, especially the coefficient $\widetilde{r}_{K,w}(1)$. Its asymptotic standard normal distribution is guaranteed by the next theorem.

Theorem 17 (Asymptotic distribution under H_0^E)**.** *Let us assume that H_0^E holds and that w is an arbitrary symmetric weighting function such that $w(i,j) = 0$ for $|i-j| > m$. Then the individual coefficient $\widetilde{r}_{K,w}(k)$ is asymptotically standard normal for any $k \geq 1$.*

Proof. If H_0 holds, then the asymptotic distribution of $\widetilde{r}_{K,w}(k)$ does not depend on $\mathcal{L}(Y_t)$ and hence Y_t's can be treated as uniformly distributed in $[0,1]$ without any loss of generality. As $N_{k,w,T}$ can be rewritten in the following form:

$$\begin{aligned} N_{k,w,T} &= \sum_{i=1}^{T-k} \sum_{j=1}^{T-k} w(i,j)\, \mathrm{I}(R_i < R_j, R_{i+k} > R_{j+k}) \\ &= \sum_{i=1}^{T-k} \sum_{j=i+1}^{T-k} w(i,j) \big[\mathrm{I}(Y_i < Y_j, Y_{i+k} > Y_{j+k}) + \mathrm{I}(Y_i > Y_j, Y_{i+k} < Y_{j+k})\big] \\ &= \sum_{i=1}^{T-k} S_{i,k}, \end{aligned}$$

where

$$S_{i,k} = \sum_{j=i+1}^{i+m} w(i,j) \big[\mathrm{I}(Y_i < Y_j, Y_{i+k} > Y_{j+k}) + \mathrm{I}(Y_i > Y_j, Y_{i+k} < Y_{j+k})\big]$$

are $(m+k)$-dependent random variables and

$$\mathrm{E}(N_{k,w,T}) = \sum_{i=1}^{T-k} \mathrm{E}(S_{i,k}),$$

it suffices to apply any suitable central limit theorem for m-dependent random variables, just as in the proof of Theorem 7.

The joint distribution of R_i's is identical under both H_0 and H_0^E (see [Dufour and Roy, 1986]), which implies that the same holds even for the asymptotic distribution of $\widetilde{r}_{K,w}(k)$. □

Note. This theorem deserves a few more comments. First, the joint asymptotic zero mean normal distribution of the first $\widetilde{r}_{K,w}(k)$'s (possibly together with other such coefficients at the same lags but with different weighting functions) could be proved virtually the same way as in Theorem 7.

Second, $N_{k,w,T}$ can be treated as a serial linear rank statistic if the trimmed weights $w(i,j)$'s depend only on the difference $j-i$. Then the relevant theory might be applied, see e.g. [Turova, 2004], [Harel and Puri, 1990], [Hallin et al., 1987], and [Hallin et al., 1985].

And third, $N_{k,w,T}$'s have much in common with the weighted U-statistics $U_T(k)$'s of degree two:

$$U_T(k) = \frac{2}{(T-k)(T-k-1)} \sum_{i=1}^{T-k} \sum_{j=i+1}^{T-k} w(i,j) g(\mathbf{Z}_i, \mathbf{Z}_j),$$

where

$$\mathbf{Z}_i = (Y_i, Y_{i+k}),\ \ \mathbf{Z}_j = (Y_j, Y_{j+k}),$$
$$g(\mathbf{Z}_i, \mathbf{Z}_j) = \mathrm{I}(Y_i > Y_j, Y_{i+k} < Y_{j+k}) + \mathrm{I}(Y_i < Y_j, Y_{i+k} > Y_{j+k}).$$

See [Hsing and Wu, 2004] and references therein for current knowledge on them.

Now we are going to proceed with the empirical investigation of $\widetilde{r}_{K,w}(1)$.

6.2 Monte Carlo Simulations

We investigate the performance of the newly introduced weighted Kendall serial rank coefficients $\widetilde{r}_{K,w_1}(1), \widetilde{r}_{K,w_2}(1), \widetilde{r}_{K,w_3}(1)$, and $\widetilde{r}_{K,w_4}(1)$ in dependence on the trimming parameter m. To be more specific, we use the two-sided tests T_K, $T_{K,w_1}(m)$, $T_{K,w_2}(m)$, $T_{K,w_3}(m)$, and $T_{K,w_4}(m)$ based on the statistics S_K, $S_{K,w_1}(m)$, $S_{K,w_2}(m)$, $S_{K,w_3}(m)$, and $S_{K,w_4}(m)$:

$$S_K = \widetilde{r}_K(1),$$
$$S_{K,w_i}(m) = \widetilde{r}_{K,w_i}(1),\ \ i = 1,2,3,4,$$

that are all asymptotically standard normal under H_0^S. T_K is included only as a benchmark.

We consider the classes ARMA ($T = 50$), LONGARMA ($T = 100$) and GARCH ($T = 200$) and all the time series therein. In fact, it is quite natural to aim at longer time series where the differences in the computational costs are more significant and where the Moore and Wallis serial rank coefficients cannot be applied too successfully. The lines – · – ·, ▬▬▬, ∘∘∘∘∘, ■ ■ ■ and —— correspond to the tests T_K, $T_{K,w_1}(m)$, $T_{K,w_2}(m)$, $T_{K,w_3}(m)$, and $T_{K,w_4}(m)$, respectively.

All the results (illustrated in Figures 6.1 to 6.4) speak for the following conclusions:

- All the tests are correctly sized for all the time series lengths considered ($T = 50, 100, 200$). The only exception regards the tests T_{K,w_1} and T_{K,w_3} for small values of T and m ($T = 50$ and $m \leq 5$).

- The test T_{K,w_4} clearly exhibits the lowest power. However, it might still be reasonable to apply in some special cases, e.g. if we decided to prefer more topical observations.

- On the other hand, T_{K,w_1} proves the best in the case of ARMA processes and T_{K,w_2} appears so for GARCH(1,1) alternatives, especially for those with persistent volatility. As far as their overall behaviour is concerned, T_{K,w_2} seems to be the winner.
- The tests $T_{w_i},\ i = 1, 2, 3, 4$, never beat T_K significantly, at least in all the situations investigated.
- There is hardly any reason for setting $m > T/3$. In fact, satisfactory results can often be achieved even with much lower values of m.

It seems that the newly proposed tests could really be found useful for quick testing against extremely long ARMA or GARCH time series alternatives.

6.3 Accompanying Figures

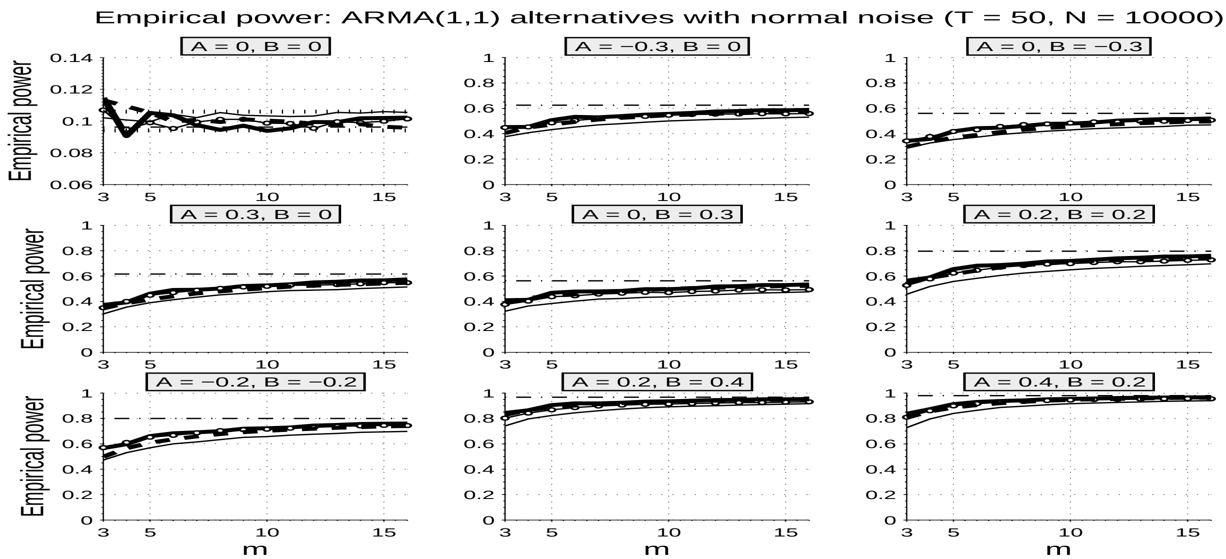

Figure 6.1: Behaviour of the tests based on S_K (—.—.), $S_{K,w_1}(m)$ (▬▬), $S_{K,w_2}(m)$ (ooooo), $S_{K,w_3}(m)$ (▬ ▬ ▬) and $S_{K,w_4}(m)$ (——) when applied to ARMA(1,1) processes with N(0,1) white noise ($T = 50$ ... time series length, $N = 10\ 000$ ... number of replications, ıııııııı ... bounds of 95% confidence intervals for the empirical size, A,B ... ARMA parameters a_1, b_1).

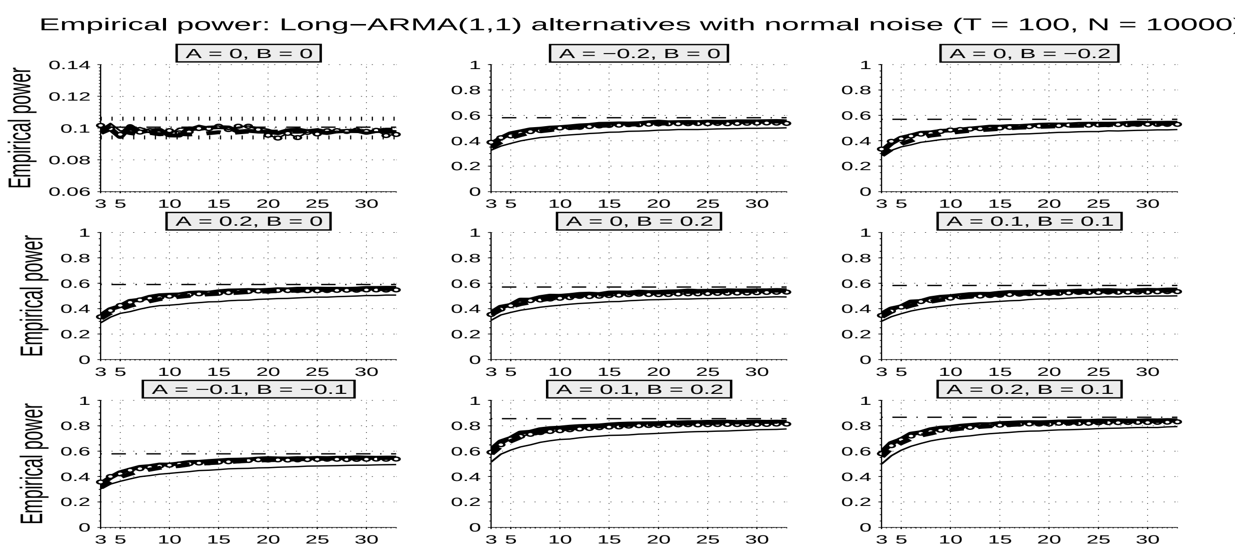

Figure 6.2: Behaviour of the tests based on S_K (—.—.), $S_{K,w_1}(m)$ (▬▬), $S_{K,w_2}(m)$ (ooooo), $S_{K,w_3}(m)$ (▬ ▬ ▬) and $S_{K,w_4}(m)$ (——) when applied to long ARMA(1,1) processes with N(0,1) white noise ($T = 100$ ... time series length, $N = 10\ 000$ ... number of replications, ıııııııı ... bounds of 95% confidence intervals for the empirical size, A,B ... ARMA parameters a_1, b_1).

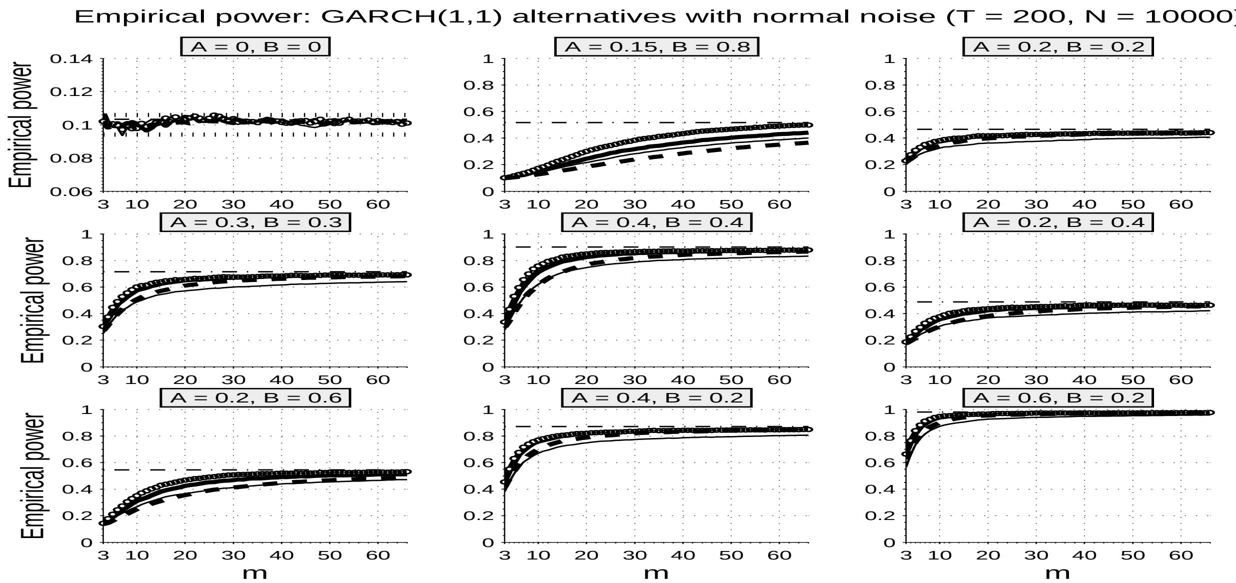

Figure 6.3: Behaviour of the tests based on S_K (–·–·), $S_{K,w_1}(m)$ (▬▬), $S_{K,w_2}(m)$ (∘∘∘∘∘), $S_{K,w_3}(m)$ (▬ ▬ ▬) and $S_{K,w_4}(m)$ (——) when applied to GARCH(1,1) models with N(0,1) innovations ($T = 200$ … time series length, $N = 10\ 000$ … number of replications, ⋮⋮⋮⋮⋮ … bounds of 95% confidence intervals for the empirical size, A,B … GARCH parameters a_1, b_1).

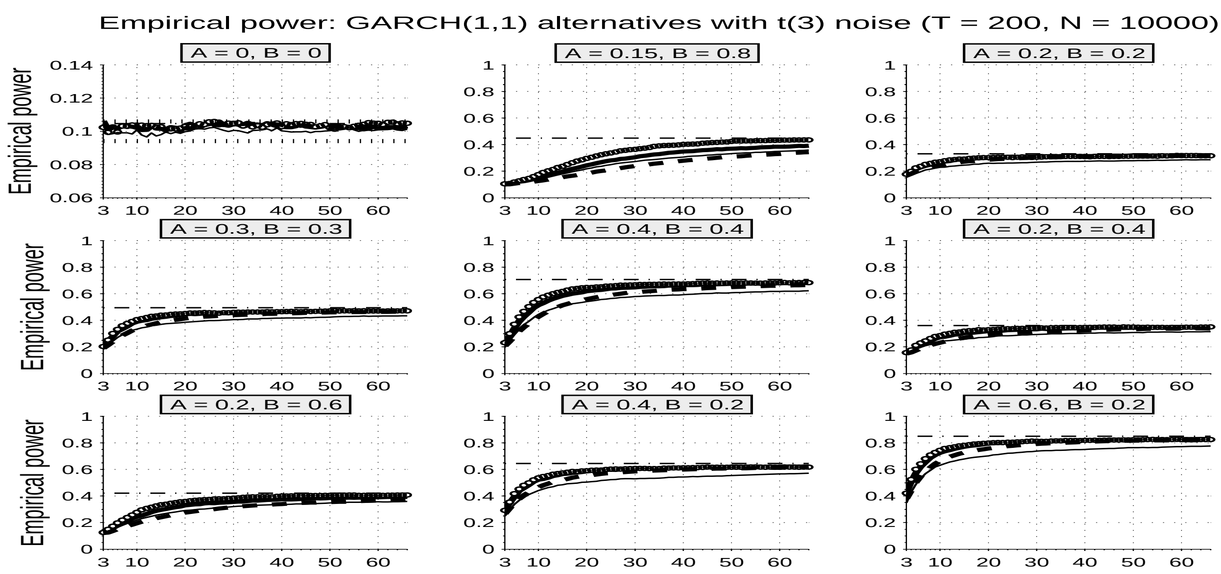

Figure 6.4: Behaviour of the tests based on S_K (–·–·), $S_{K,w_1}(m)$ (▬▬), $S_{K,w_2}(m)$ (∘∘∘∘∘), $S_{K,w_3}(m)$ (▬ ▬ ▬) and $S_{K,w_4}(m)$ (——) when applied to GARCH(1,1) models with standardized t(3) innovations ($T = 200$ … time series length, $N = 10\ 000$ … number of replications, ⋮⋮⋮⋮⋮ … bounds of 95% confidence intervals for the empirical size, A,B … GARCH parameters a_1, b_1).

Chapter 7

Autocorrelations of Scores

Contrary to the Moore and Wallis serial rank coefficients, the Spearman ones are not invariant to all possible increasing transformations of the ranks. This chapter contains a few comments and suggestions on this issue.

7.1 Introduction

Correlation of scores is a common tool for statistical analysis of ordered nonserial data, see e.g. [Tarsitano, 2002] for a review. It has been successfully generalised to the time series context in a string of articles including [Hallin et al., 1985], [Hallin et al., 1987], [Hallin and Puri, 1988a], and [Hallin and Puri, 1988b] where the f-rank (sample) autocorrelation coefficients $\widehat{r}_f(k)$'s and their standardized versions $\widetilde{r}_f(k)$'s are introduced:

$$\widehat{r}_f(k) = \frac{1}{T-k}\sum_{i=1}^{T-k} G\big(\tfrac{R_{i+k}}{T+1}\big)F^{-1}\big(\tfrac{R_i}{T+1}\big), \quad \widetilde{r}_f(k) = \frac{\widehat{r}_f(k) - \mathrm{E}_0\big(\widehat{r}_f(k)\big)}{\sqrt{\mathrm{var}_0\big(\widehat{r}_f(k)\big)}},$$

where $G(\cdot) = -f'\big(F^{-1}(\cdot)\big)/f\big(F^{-1}(\cdot)\big)$ and $F^{-1}(\cdot)$ is the quantile function associated with the density type f. For example, the van der Waerden rank autocorrelations $\widehat{r}_\varphi(k)$'s,

$$\widehat{r}_\varphi(k) = \frac{1}{T-k}\sum_{i=1}^{T-k} \Phi^{-1}\big(\tfrac{R_{i+k}}{T+1}\big)\Phi^{-1}\big(\tfrac{R_i}{T+1}\big),$$

correspond to the standard normal density φ and beat the sample ordinary ones in many respects.

Subject to certain regularity conditions, $\widehat{r}_f(k)$'s are known asymptotically jointly normal under H_0^E and optimal from several points of view for testing against ARMA alternatives with the underlying white noise density f (same references). Unfortunately, these conditions rule out several important distributions such as the Cauchy, Student, and uniform ones. Besides, the original formulae for $\mathrm{E}_0\big(\widehat{r}_f(k)\big)$ and $\mathrm{var}_0\big(\widehat{r}_f(k)\big)$ are very complicated and unusable in practice for higher T's as they include complicated double and quadruple sums over almost all the ranks, see ibidem. Fortunately, their simplification in [Hallin and Mélard, 1988] (see also [Hallin and Puri, 1994]) weakens the last mentioned drawback significantly.

If the null hypothesis H_0^S is assumed, the signed-rank modifications $\widehat{r_f^-}(k)$'s with much simpler means and variances may be employed as well, see e.g. [Hallin et al., 1990] and references therein. They are claimed to yield substantially better performance in some cases, although they are known to be asymptotically equivalent to $\widehat{r}_f(k)$'s under H_0^S (and therefore also under all local alternatives). For example, the signed-rank van der Waerden autocorrelations $\widehat{r_\varphi^-}(k)$'s,

$$\widehat{r_\varphi^-}(k) = \frac{1}{T-k}\sum_{i=1}^{T-k} \operatorname{sign}(Y_iY_{i+k})\Phi^{-1}\big(\tfrac{1}{2}+\tfrac{R_{i+k}^+}{2(T+1)}\big)\Phi^{-1}\big(\tfrac{1}{2}+\tfrac{R_i^+}{2(T+1)}\big),$$

are tailored to normal white noise and strongly encouraged to be used in practice, see ibidem.

Other related references are probably also worth mentioning. First, the signed-and-rank autocorrelations (suitable for testing the zero median white noise null hypothesis) are introduced and investigated in [Hallin et al., 2006]. Second, the application of (possibly signed) f-rank autocorrelations to the estimated residuals from ARMA(X) models is considered in [Hallin and Puri, 1994] and [Ferretti et al., 1995]. And third, multivariate generalizations of these coefficients also exist, see [Hallin and Paindaveine, 2006], [Oja and Paindaveine, 2005], [Hallin and Paindaveine, 2005], [Hallin and Paindaveine, 2004a], [Hallin and Paindaveine, 2004b], [Hallin and Paindaveine, 2002].

7.2 Theory

First, we specify the autocorrelation coefficients of our current interest.

Definition 18. Let us define the transformed ranks $P_i(F)$'s, the transformed ranks of absolute values $P_i^+(F)$'s and their signed versions $P_i^-(F)$'s in the following way:

$$P_i(F) = F^{-1}\big(\tfrac{R_i}{T+1}\big),\quad P_i^+(F) = F^{-1}\big(\xi + (1-\xi)\tfrac{R_i^+}{(T+1)}\big),\quad P_i^-(F) = \operatorname{sign}(Y_i)P_i^+(F),$$

where F^{-1} stands for any increasing quantile function and ξ is the number satisfying $F^{-1}(\xi) = 0$. Their sample (F-rank) autocorrelations will be further denoted by $\widehat{r}_F(k)$'s, $\widehat{r_F^+}(k)$'s and $\widehat{r_F^-}(k)$'s, respectively. For example,

$$\widehat{r}_F(k) = \frac{\sum_{i=1}^{T-k}\big(P_i(F)-\bar{P}(F)\big)\big(P_{i+k}(F)-\bar{P}(F)\big)}{\sum_{i=1}^{T}\big(P_i(F)-\bar{P}(F)\big)^2},\quad \bar{P}(F) = \frac{1}{T}\sum_{i=1}^{T}P_i(F).$$

Besides, the symbol $\widehat{r}_F^*(k)$ will be used to replace all the coefficients $\widehat{r}_F(k)$, $\widehat{r_F^+}(k)$, $\widehat{r_F^-}(k)$.

Basic properties of these coefficients are established by the next theorems.

Theorem 19. *If $1 \le k < h < T/2$, $T > 3$ and F^{-1} is an arbitrary increasing quantile function, then:*

$$\begin{aligned} |\widehat{r}_F^*(k)| &\le 1, \\ \mathrm{E}_0\big(\widehat{r}_F^*(k)\big) &= -\frac{T-k}{T(T-1)}, \end{aligned}$$

$$\operatorname{var}_0\big(\widehat{r}_F^*(k)\big) \leq VC = \frac{T^3-(k+5)T^2+(5k+6)T+2k(k-4)}{T(T-1)^2(T-3)},$$
$$\mathrm{E}_0\big(\widehat{r}_F^*(k)\widehat{r}_F^*(h)\big) = \frac{\big((T-h)(T+k)-2kh\big)\big(2Q^*(F)-1\big)}{T(T-1)(T-2)(T-3)},$$

and

$$\mathrm{E}_0\big(\widehat{r}_F^*(k)\big)^2 = \frac{(-T^3+(k+3)T^2-kT-6k^2)Q^*(F)+T^3-T^2(k+4)+3T(k+1)+3k(k-1)}{T(T-1)(T-2)(T-3)},$$

where

$$\frac{1}{T} \leq Q^*(F) = \frac{\sum_{i=1}^T \big(P_i^*(F)-\bar{P}^*(F)\big)^4}{\big(\sum_{i=1}^T \big(P_i^*(F)-\bar{P}^*(F)\big)^2\big)^2} \leq 1,$$
$$\bar{P}^*(F) = \frac{1}{T}\sum_{i=1}^T P_i^*(F).$$

The subscript 0 refers to H_0^S if $*$ stands for $-$ (and to H_0^E otherwise). This convention holds everywhere in this chapter.

Note. The variances and covariances follow immediately.

Proof. It suffices to realize that

- both $P_i(F)$'s and $P_i^+(F)$'s are exchangeable under H_0^E and the same holds even for $P_i^-(F)$'s under H_0^S,
- $Q^*(F)$ is a deterministic factor depending solely on F and T in this context,
- $P\big(P_1^*(F) = P_2^*(F) = \cdots = P_T^*(F)\big) = 0$ due to (strictly) increasing F^{-1}.

The rest follows from the theory developed in [Dufour and Roy, 1985], [Dufour and Roy, 1986] for the autocorrelations of exchangeable variables. Partial verification of the results is possible by means of *SCMoms.r*. □

These moment characteristics can naturally be used for standardization.

Definition 20. Let us define the standardized coefficients $\widetilde{r}_F^*(k)$'s:

$$\widetilde{r}_F^*(k) = \frac{\widehat{r}_F^*(k) - \mathrm{E}_0\big(\widehat{r}_F^*(k)\big)}{\sqrt{\operatorname{var}_0\big(\widehat{r}_F^*(k)\big)}},$$

and both their orthonormal versions $r_F^{*\perp}(k)$'s (see Section 3.7) and conservative modifications

$$\ddot{r}_F^*(k) = \frac{\widehat{r}_F^*(k) - \mathrm{E}_0\big(\widehat{r}_F^*(k)\big)}{\sqrt{VC}}, \quad k = 1, 2, \ldots$$

see e.g. [Dufour and Roy, 1986] for another example of the use of conservative variances of some autocorrelations in portmanteau tests.

Although the coefficients like $\widetilde{r}^*_F(k)$'s are already known and their exact means and variances follow easily from [Hallin and Mélard, 1988], we are not aware of any use of their conservative variances or finite sample covariances and therefore we present the formulae here.

Note. Apparently, $\operatorname{cov}\big(\widetilde{r}^*_F(k), \widetilde{r}^*_F(h)\big) = O(T^{-1})$, $1 \le k < h < T/2$, independently of F.

Theorem 21. *Let F^{-1} be an increasing quantile function continuously differentiable in $(0,1)$ that corresponds to a finite variance distribution. If H_0^E holds and $m > 0$ is a fixed integer, then*

$$\widetilde{r}_F(k) \sim_{asympt.} N(0,1), \;\; k = 1, \dots, m,$$

and these coefficients are moreover asymptotically both jointly normal and independent.

The same holds even for $\widetilde{r}^+_F(k)$'s when it suffices to consider the differentiability condition only in $(\xi, 1)$.

Proof. We can focus on $\widetilde{r}_F(k)$'s without any loss of generality. The only important stochastic term in the definition of $\widetilde{r}_F(k)$ is the product $\sum_{i=1}^{T-k} P_i(F)P_{i+k}(F)$ whose asymptotic normal distribution under H_0 results from [Harel and Puri, 1990]. The extension to H_0^E is straightforward and the statement regarding the joint asymptotic normality and independence follows directly from the Cramér-Wold device and the preceding note. □

Note. The asymptotic normality of $\widetilde{r}_F(k)$-like coefficients is also discussed elsewhere, see e.g. [Nieuwenhuis and Ruymgaart, 1990], [Tran, 1990], [Haeusler et al., 2000], and [Turova, 2004]. Besides, the case of $\widetilde{r}_\Phi(k)$'s under H_0^E is also covered by [Hallin et al., 1987].

As far as $\widetilde{r}^-_F(k)$'s are concerned, it is quite natural to assume them asymptotically jointly normal as well (see e.g. [Hallin et al., 1990]) and this normal approximation indeed works well in our Monte Carlo experiments. We do not analyse their asymptotic normality and independence here because our simulation results indicate that it is not worth any effort, and we rather refer to [Hallin and Puri, 1994], [Hallin and Paindaveine, 2006] and references therein.

Note. All these coefficients $\widetilde{r}^*_F(k)$'s have a strong advantage over the f-rank ones: their standardization is very simple and their finite sample covariances are also known and easy to compute. Furthermore, we show in Chapter 10 that $\widetilde{r}_F(k)$'s (and $\widetilde{r}^+_F(k)$'s) are uncorrelated with and asymptotically independent of $\widetilde{r}_M(k)$'s and $\widetilde{r}_W(k)$'s under H_0^E, contrary to the f-rank autocorrelations with unequal score functions. Besides, $\widetilde{r}_\Phi(k)$ is asymptotically equivalent (under H_0^E) to the most promising f-rank autocorrelation coefficient $\widetilde{r}_\varphi(k)$ and the same holds for $\widetilde{r}^-_\Phi(k)$ and $\widetilde{r}^-_\varphi(k)$ under H_0^S. It also explains why we leave the (possibly signed) f-rank autocorrelations out in further considerations.

Now we proceed with a few Monte Carlo experiments to reveal some pros and cons of $\widetilde{r}^*_F(k)$'s.

7.3 Monte Carlo Simulations

We investigate the portmanteau tests $\widetilde{T}^*_\bullet(m)$, $T^{*\perp}_\bullet(m)$ and $\ddot{T}^*_\bullet(m)$ associated with the statistics

$$\widetilde{S}^*_\bullet(m) = \sum_{k=1}^{m} \big(\widetilde{r}^*_\bullet(k)\big)^2, \quad S^{*\perp}_\bullet(m) = \sum_{k=1}^{m} \big(r^{*\perp}_\bullet(k)\big)^2, \quad \ddot{S}^*_\bullet(m) = \sum_{k=1}^{m} \big(\ddot{r}^*_\bullet(k)\big)^2;$$

$\bullet$ always stands for the cumulative distribution functions of the standard normal, logistic, Laplace and t(5) distributions. We use the subscripts t(5), Norm, Log, and Lap for short when referring to them individually. Besides, we consider the benchmark tests $\widetilde{T}_S(m)$, $\widetilde{T}_S^+(m)$, and $\widetilde{T}_S^-(m)$, based on the sum of the first m squared standardized Spearman coefficients computed from the ranks, from the ranks of absolute values and from the signed ranks, respectively. They roughly correspond to $\widetilde{T}_\bullet^*(m)$'s with a uniform distribution used. The orthonormal and conservative modifications of such Spearman coefficients are employed in $\ddot{}$ and $^\perp$ versions of $\widetilde{T}_S^*$. It remains to note that we assume the $\chi^2(m)$ null asymptotic distribution for all the tests based on the first m autocorrelations of any kind.

Simulation results have been obtained for all the classes SHORTTREND (T = 25), TREND (T = 100), ARMA (T = 50), LONGARMA (T = 100) and GARCH (T = 200). They speak for themselves and need little comment. The most important conclusions are illustrated in Figures 7.1 to 7.10:

- All the tests $\widetilde{T}_\bullet^*(m)$'s and $T_\bullet^{*\perp}(m)$'s exhibit roughly the same size behaviour as the benchmark $\widetilde{T}_S(m)$. $\ddot{T}_\bullet^*(m)$'s seem convenient for application only if the other tests are oversized.

- Not surprisingly, the tests with $*$ equal to blank space are in general the most suitable for testing against SHORTTREND, TREND, ARMA, and LONGARMA alternatives while the $+$ tests are the most advisable against the GARCH ones. There is hardly any reason to employ the $-$ tests for testing against TREND or SHORTTREND alternatives and besides, they never lead to a power growth higher than one or two percentage points even if the time series from ARMA and LONGARMA classes are considered.

- The tests based on the orthonormal coefficients should generally be preferred, especially for shorter time series when their use may cause a power increase higher than 12 percentage points. This shows the importance of the covariances between the single coefficients.

- As far as the TREND and SHORTTREND alternatives are concerned, $T_{\text{Norm}}^\perp(m)$ slightly dominates the other tests $T_\bullet^\perp(m)$ in the case of N(0,1) white noise while $T_S^\perp(m)$ clearly leads the area when the t(3) distribution is used instead. But $T_{\text{Norm}}^\perp(m)$ is never significantly outperformed by any other test $T_\bullet^\perp(m)$ if applied to the ARMA and LONGARMA alternatives.

- As for the GARCH alternatives, $T_{\text{Lap}}^{+\perp}(m)$ appears the best and sometimes beats the benchmark $T_S^{+\perp}(m)$ and the test $T_{\text{Norm}}^{+\perp}(m)$ even by more than 24 and 5 percentage points, respectively. This conclusively confirms the qualities of the score autocorrelation coefficients considered here and gives reasons for their existence. By the way, it is still quite possible that there exists a function F leading to even better results.

Given a white noise or an ARMA time series, [Chan, 1995] convincingly demonstrates (both theoretically and empirically) that two additive outliers of the same sign, say at times t and $t+k$, crucially influence the sample ordinary autocorrelation at lag k and force its value to 0.5 as their magnitude and time series length increase. Besides, [Burns, 2002] further supports these results. In light of this, the conclusions regarding GARCH alternatives are not as surprising as they could seem at first sight because one typical feature of any volatile series is that the observations with high absolute values (and thus with high transformed ranks) occur close to one another in clusters. This heuristics also partly explains why $T^{+\perp}_{\mathrm{Log}}(m)$ and $T^{+\perp}_{\mathrm{t(5)}}(m)$ achieve almost the same power as the winner $T^{+\perp}_{\mathrm{Lap}}(m)$ and clearly beat the benchmark $T^{+\perp}_{S}(m)$ in the GARCH context.

We should perhaps mention as well that [Abadir and Talmain, 2005] show an important case when a simple logarithmic transformation results in much more autocorrelated data. Therefore the success of F^{-1}_{Lap} is not quite unexpected even from this point of view.

7.4 Accompanying Figures

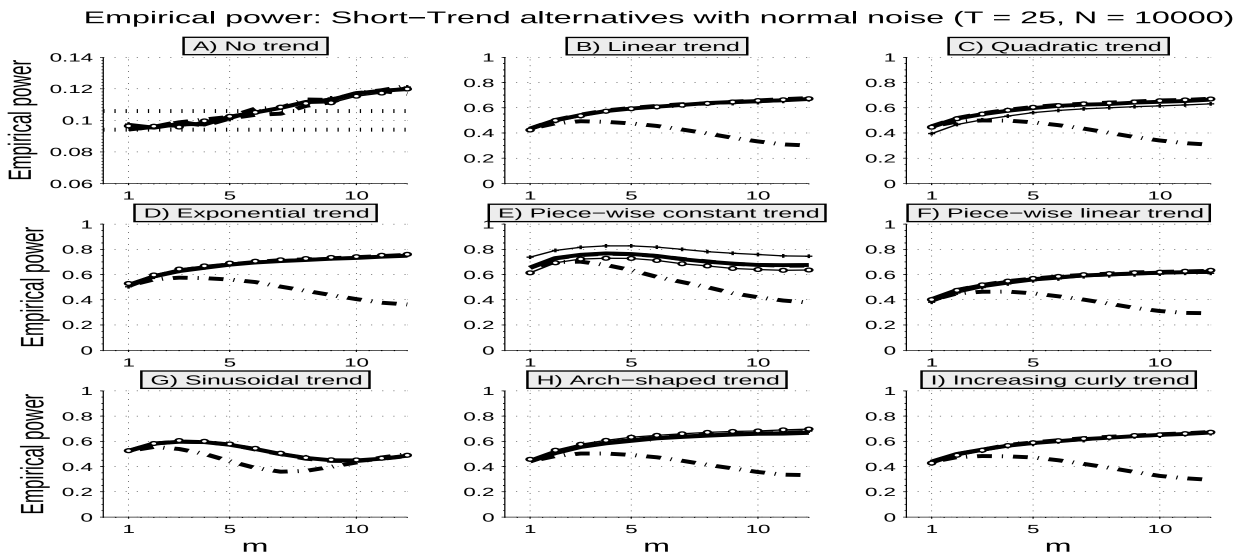

Figure 7.1: Behaviour of the tests $T_S^{\perp}(m)$ (), $T_{\text{Norm}}^{\perp}(m)$ (), $\widetilde{T}_{\text{Norm}}(m)$ (), $T_S^{-\perp}(m)$ () and $T_{\text{Norm}}^{-\perp}(m)$ () when applied to a short trend plus N(0,1) white noise ($T = 25$ … time series length, $N = 10\ 000$ … number of replications, … bounds of 95% confidence intervals for the empirical size).

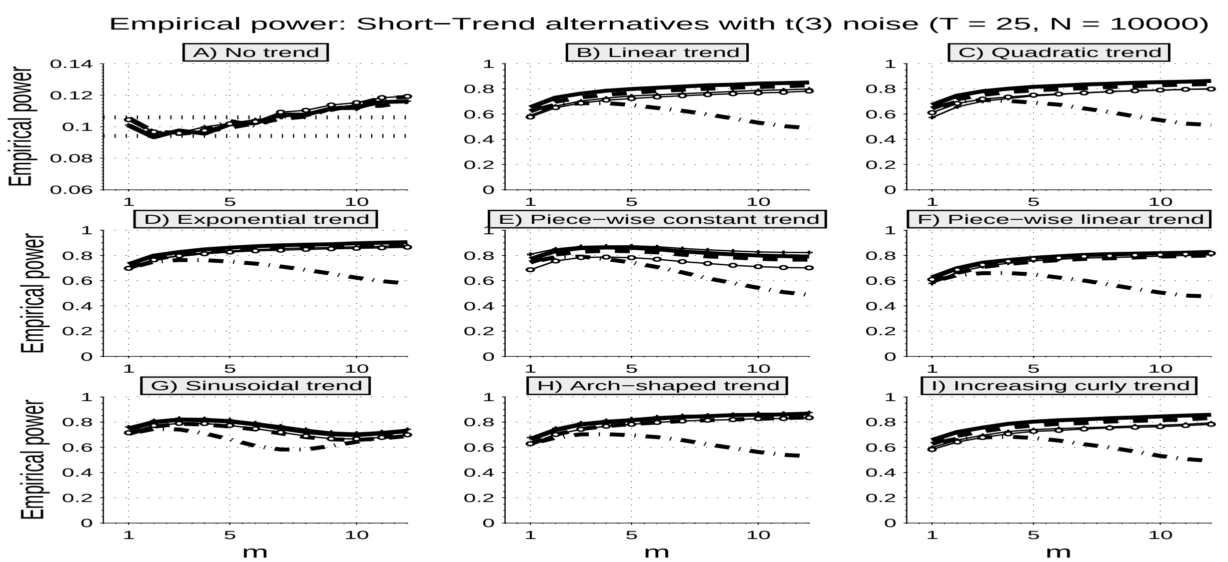

Figure 7.2: Behaviour of the tests $T_S^{\perp}(m)$ (), $T_{\text{Norm}}^{\perp}(m)$ (), $\widetilde{T}_{\text{Norm}}(m)$ (), $T_S^{-\perp}(m)$ () and $T_{\text{Norm}}^{-\perp}(m)$ () when applied to a short trend plus standardized t(3) white noise ($T = 25$ … time series length, $N = 10\ 000$ … number of replications, … bounds of 95% confidence intervals for the empirical size).

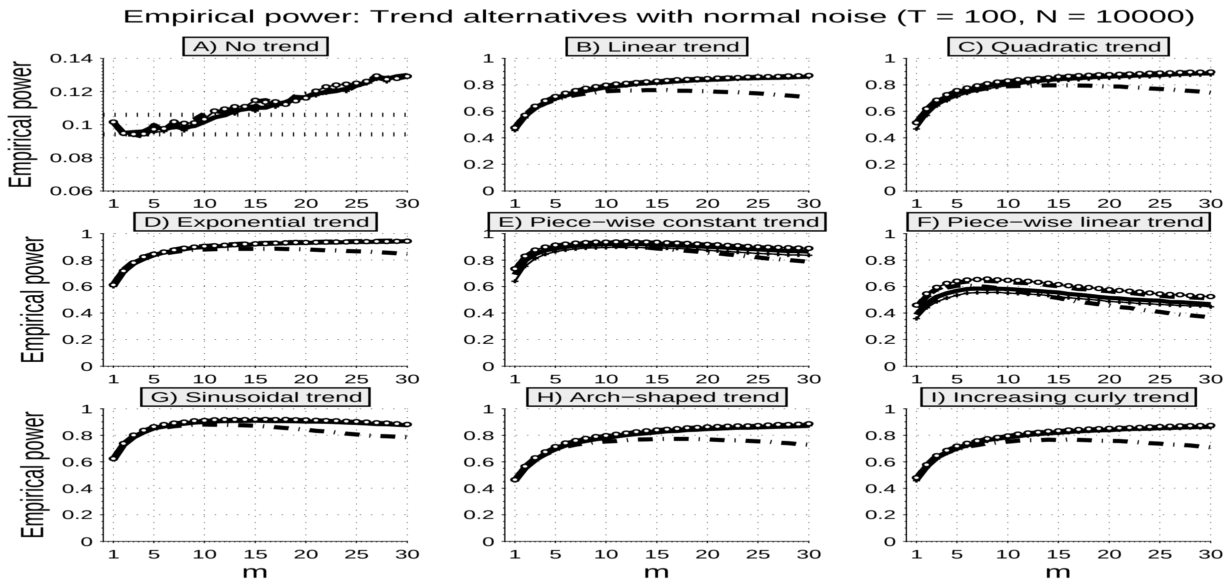

Figure 7.3: Behaviour of the tests $T_S^{\perp}(m)$, $T^{\perp}_{\mathrm{Norm}}(m)$, $\widetilde{T}_{\mathrm{Norm}}(m)$, $T_S^{-\perp}(m)$ and $T^{-\perp}_{\mathrm{Norm}}(m)$ when applied to a trend plus N(0,1) white noise ($T = 100$ … time series length, $N = 10\,000$ … number of replications, … bounds of 95% confidence intervals for the empirical size).

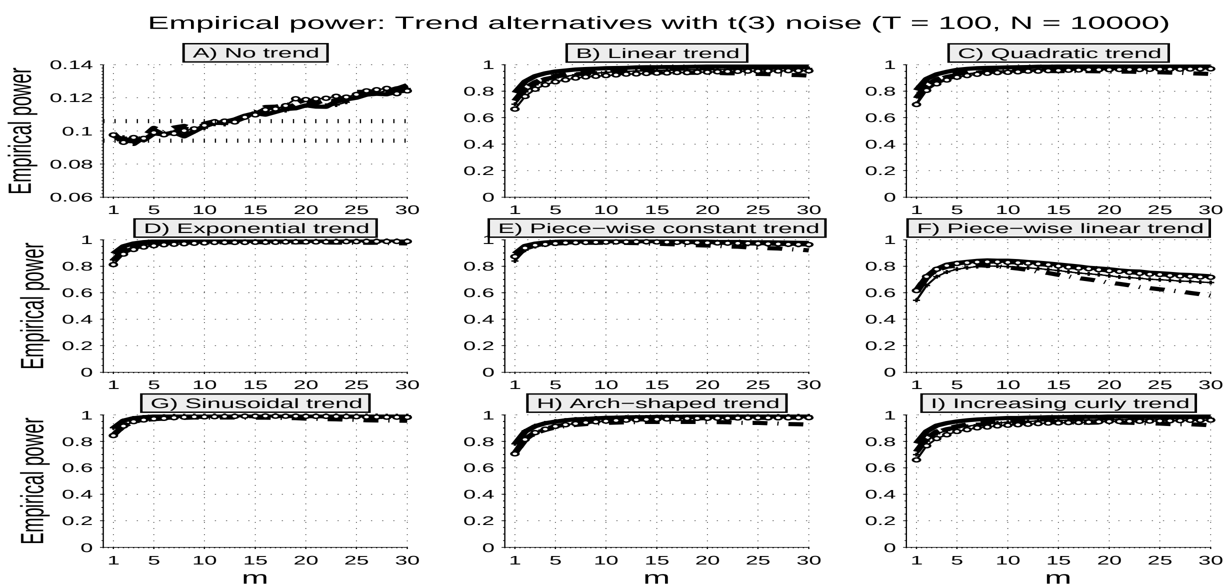

Figure 7.4: Behaviour of the tests $T_S^{\perp}(m)$, $T^{\perp}_{\mathrm{Norm}}(m)$, $\widetilde{T}_{\mathrm{Norm}}(m)$, $T_S^{-\perp}(m)$ and $T^{-\perp}_{\mathrm{Norm}}(m)$ when applied to a trend plus standardized t(3) white noise ($T = 100$ … time series length, $N = 10\,000$ … number of replications, … bounds of 95% confidence intervals for the empirical size).

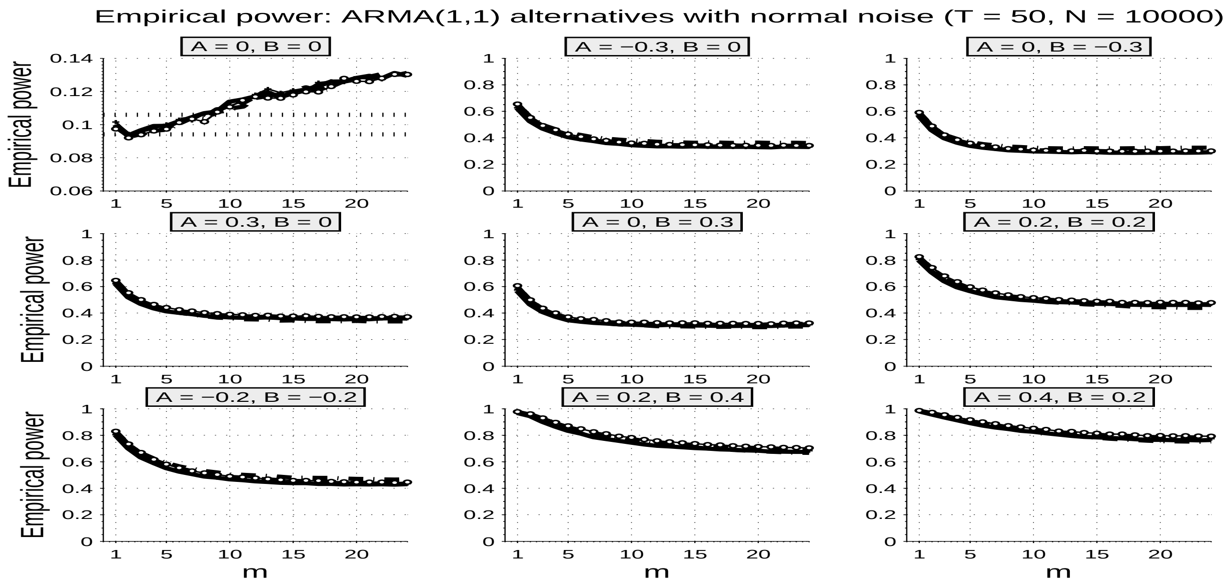

Figure 7.5: Behaviour of the tests $T_S^{\perp}(m)$ (▬▬), $T_{\text{Norm}}^{\perp}(m)$ (▬ ▬ ▬), $\widetilde{T}_{\text{Norm}}(m)$ (▬ · ▬), $T_S^{-\perp}(m)$ (+++) and $T_{\text{Norm}}^{-\perp}(m)$ (∘∘∘∘∘) when applied to ARMA(1,1) processes with N(0,1) white noise ($T = 50$ …time series length, $N = 10\,000$ …number of replications, ⋮⋮⋮⋮ …bounds of 95% confidence intervals for the empirical size, A,B …ARMA parameters a_1, b_1).

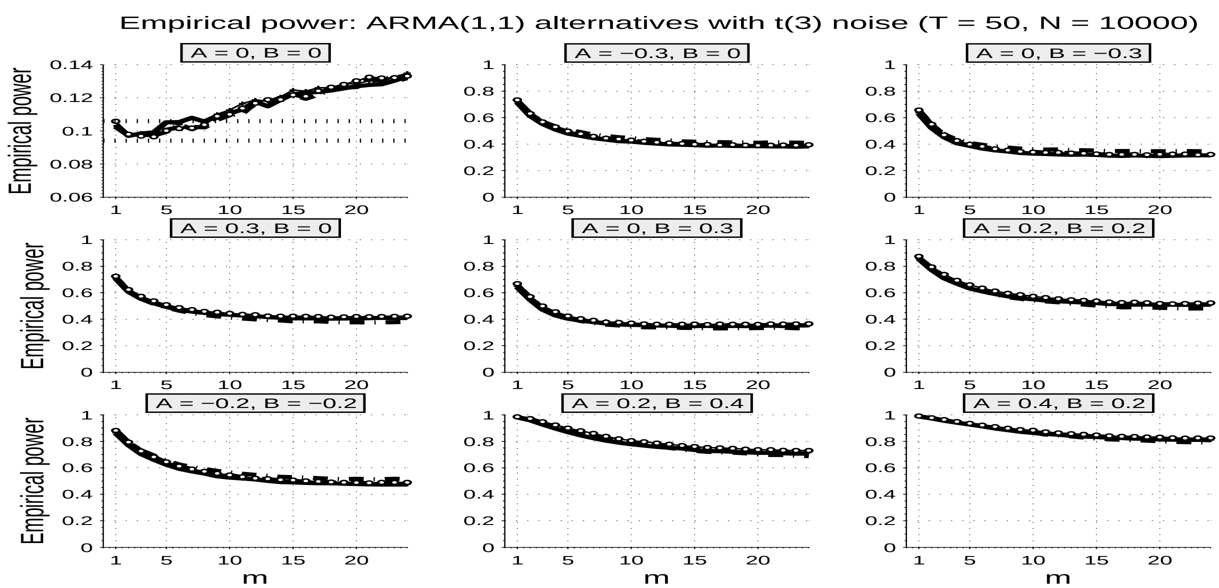

Figure 7.6: Behaviour of the tests $T_S^{\perp}(m)$ (▬▬), $T_{\text{Norm}}^{\perp}(m)$ (▬ ▬ ▬), $\widetilde{T}_{\text{Norm}}(m)$ (▬ · ▬), $T_S^{-\perp}(m)$ (+++) and $T_{\text{Norm}}^{-\perp}(m)$ (∘∘∘∘∘) when applied to ARMA(1,1) processes with standardized t(3) white noise ($T = 50$ …time series length, $N = 10\,000$ …number of replications, ⋮⋮⋮⋮ …bounds of 95% confidence intervals for the empirical size, A,B …ARMA parameters a_1, b_1).

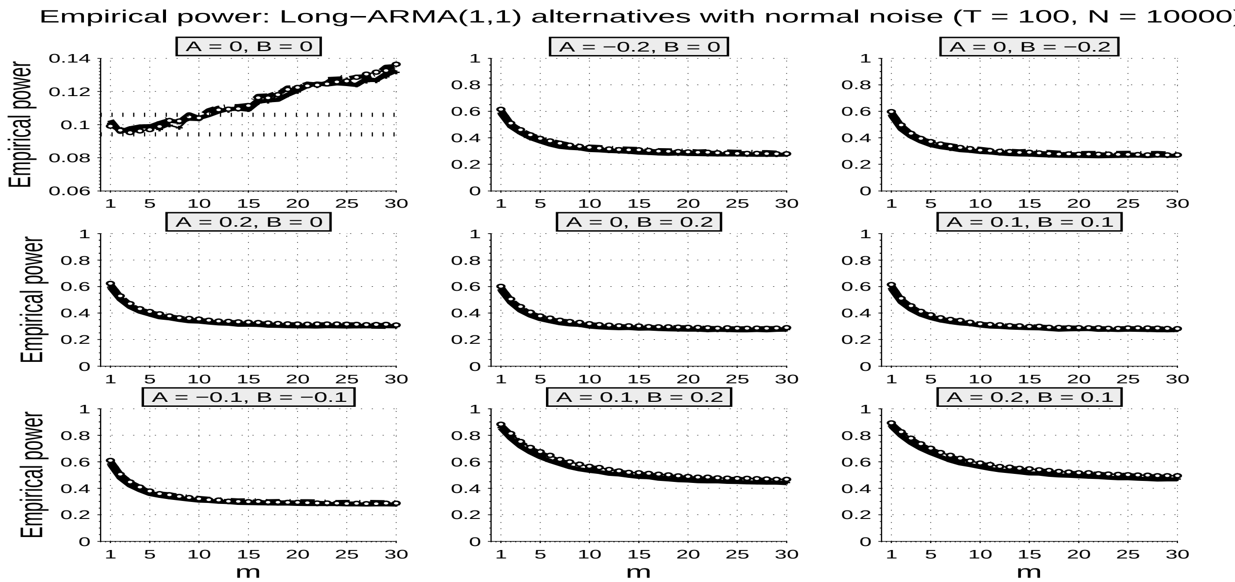

Figure 7.7: Behaviour of the tests $T_S^{\perp}(m)$ (▬▬), $T_{\text{Norm}}^{\perp}(m)$ (▬ ▬ ▬), $\widetilde{T}_{\text{Norm}}(m)$ (▬·▬·), $T_S^{-\perp}(m)$ (+++) and $T_{\text{Norm}}^{-\perp}(m)$ (∘∘∘∘∘) when applied to long ARMA(1,1) processes with N(0,1) white noise ($T = 100$ ... time series length, $N = 10\ 000$ ... number of replications, ıııııııı ... bounds of 95% confidence intervals for the empirical size, A,B ... ARMA parameters a_1, b_1).

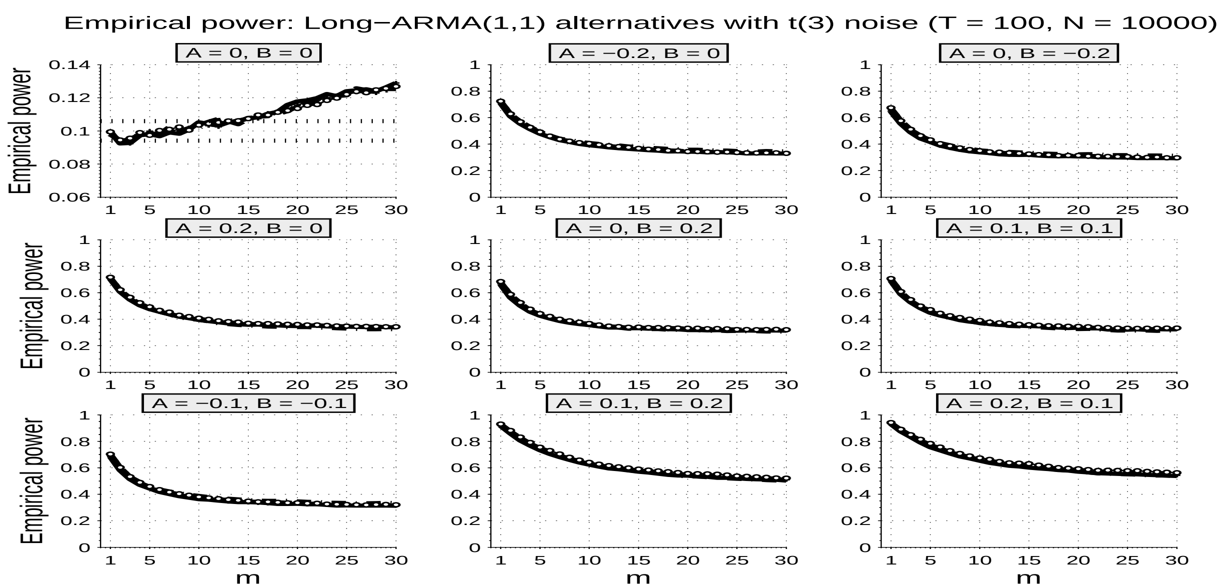

Figure 7.8: Behaviour of the tests $T_S^{\perp}(m)$ (▬▬), $T_{\text{Norm}}^{\perp}(m)$ (▬ ▬ ▬), $\widetilde{T}_{\text{Norm}}(m)$ (▬·▬·), $T_S^{-\perp}(m)$ (+++) and $T_{\text{Norm}}^{-\perp}(m)$ (∘∘∘∘∘) when applied to long ARMA(1,1) processes with standardized t(3) white noise ($T = 100$ ... time series length, $N = 10\ 000$ ... number of replications, ıııııııı ... bounds of 95% confidence intervals for the empirical size, A,B ... ARMA parameters a_1, b_1).

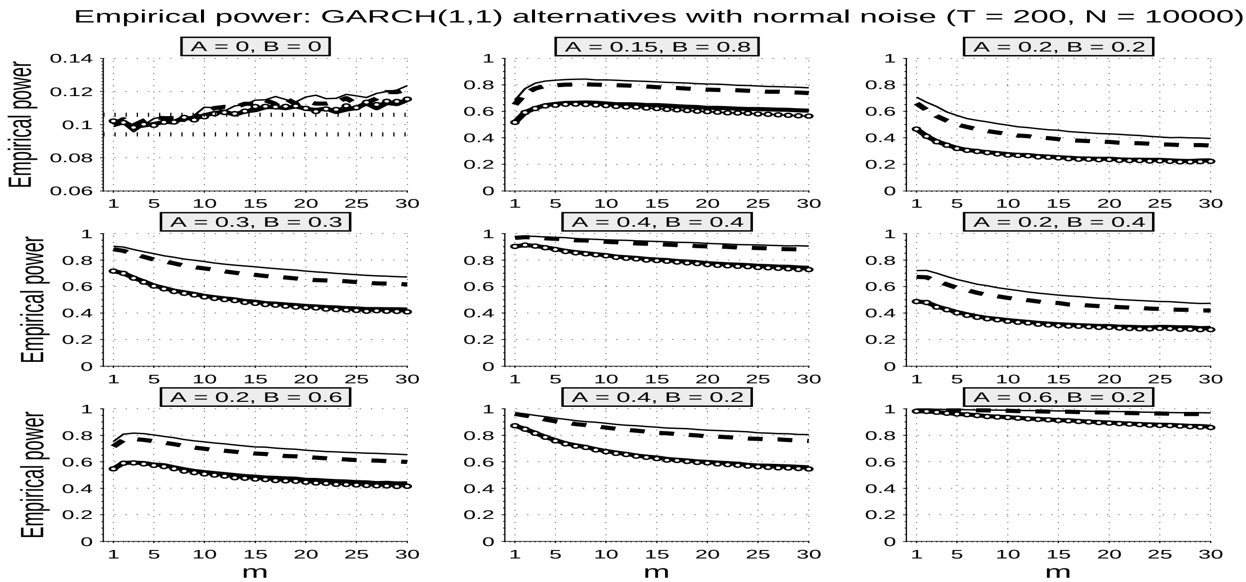

Figure 7.9: Behaviour of the tests $T_S^{+\perp}(m)$ (▬▬), $\widetilde{T}_S^{+}(m)$ (∘∘∘∘∘), $T_{\mathrm{Norm}}^{+\perp}(m)$ (▬ ▬ ▬) and $T_{\mathrm{Lap}}^{+\perp}(m)$ (——) when applied to GARCH(1,1) models with standard normal innovations ($T = 200$ … time series length, $N = 10\,000$ … number of replications, ıııııııı … bounds of 95% confidence intervals for the empirical size, A,B … GARCH parameters a_1, b_1).

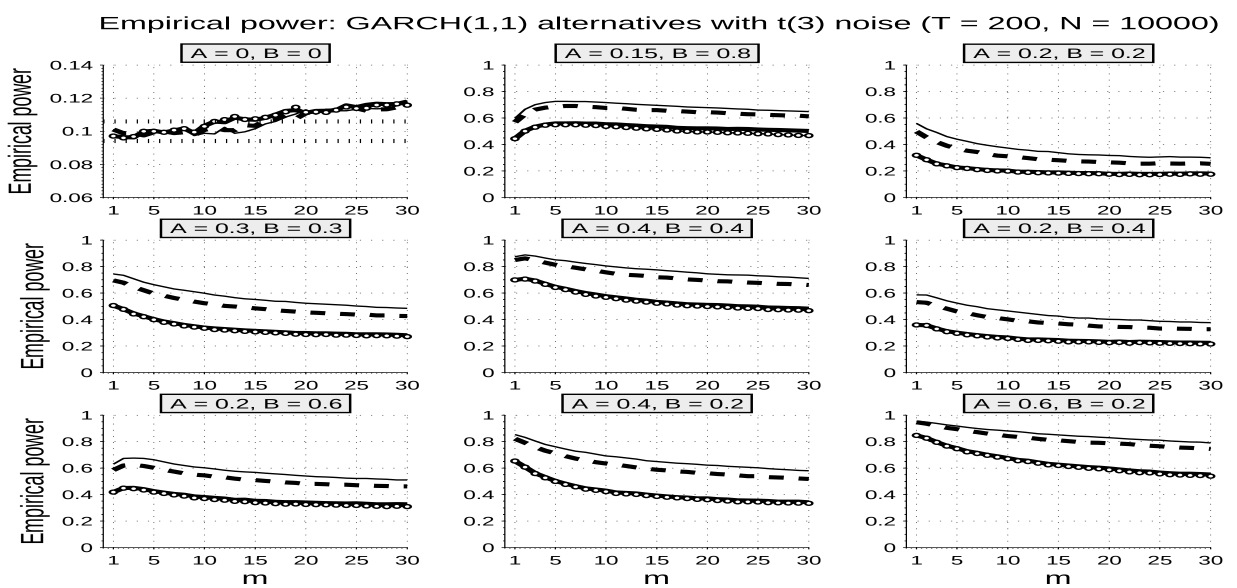

Figure 7.10: Behaviour of the tests $T_S^{+\perp}(m)$ (▬▬), $\widetilde{T}_S^{+}(m)$ (∘∘∘∘∘), $T_{\mathrm{Norm}}^{+\perp}(m)$ (▬ ▬ ▬) and $T_{\mathrm{Lap}}^{+\perp}(m)$ (——) when applied to GARCH(1,1) models with standardized t(3) innovations ($T = 200$ … time series length, $N = 10\,000$ … number of replications, ıııııııı … bounds of 95% confidence intervals for the empirical size, A,B … GARCH parameters a_1, b_1).

Chapter 8

A Cautionary Note

The asymptotic independence of some serial rank coefficients under a null hypothesis of randomness tempts into the wrong belief that their mutual covariances can always be ignored completely. But this conviction may lead to very badly sized tests (i.e. to very misleading results), which is demonstrated in this chapter.

For example, let us consider the statistic

$$S_\Sigma(m) = \sum_{k=1}^{m} \widetilde{r}_S(k).$$

It is quite representative as the tests based on (possibly weighted) sums of some autocorrelations play an important role in statistical inference, see [Hallin et al., 1985], [Hallin and Puri, 1988a], [Hallin and Puri, 1988b], [Richardson and Smith, 1994], [Hong, 1997], [Levich and Rizzo, 1998], and [Daniel, 2001], among others.

$S_\Sigma(m)$ is asymptotically zero mean normal under H_0^E, see Subsection 1.2.8. If we neglect the influence of $\mathrm{cov}_0\big(\widetilde{r}_S(k), \widetilde{r}_S(h)\big)$'s, its variance is $\mathrm{var}_0\big(S_\Sigma(m)\big) = m$ and does not depend on T. The trouble is that all the covariances $\mathrm{cov}_0\big(\widetilde{r}_S(k), \widetilde{r}_S(h)\big)$'s are asymptotically $O(-2/T)$ and therefore of the same sign for sufficiently large T's, see Subsection 1.2.8 or [Dufour and Roy, 1986]. If we take account of them, we get $\mathrm{var}_0\big(S_\Sigma(m)\big)$ far from m even for high T's:

$\mathrm{var}_0\big(S_\Sigma(m)\big)$	$T = 25$	$T = 50$	$T = 100$	$T = 200$	$T = 500$	$T = 1\,000$
$m = 5$	3.29	4.17	4.59	4.80	4.92	4.96
$m = 10$	3.13	6.38	8.19	9.10	9.64	9.82
$m = 12$	2.63	6.77	9.35	10.68	11.47	11.74
$m = 15$	–	6.91	10.82	12.90	14.16	14.58
$m = 20$	–	6.21	12.56	16.22	18.48	19.24
$m = 24$	–	5.25	13.36	18.54	21.80	22.90
$m = 25$	–	–	13.48	19.08	22.61	23.80
$m = 30$	–	–	13.67	21.47	26.54	28.26

These results can be checked by means of the Maple code *CIVars.mws*.

Fortunately, this discrepancy regarding the impact of $\mathrm{cov}_0\big(\widetilde{r}_S(k), \widetilde{r}_S(h)\big)$'s can be wrestled with easily. For example, we could simply consider their influence when computing $\mathrm{var}_0\big(S_\Sigma(m)\big)$, or we could use the orthonormal autocorrelations $r_S^{\perp}(k)$'s (see Section 3.7) instead of $\widetilde{r}_S(k)$'s. However, the dependence and nonnormal null distribution of $\widetilde{r}_S(k)$'s or $r_S^{\perp}(k)$'s in finite samples may still give rise to some inaccuracies.

Although this problem is illustrated only by $\widetilde{r}_S(k)$'s here, the results are of much more general validity. For example, the standardized ordinary autocorrelations generally suffer from the same drawback (see [Dufour and Roy, 1985]), e.g. their covariances are also asymptotically $O(-2/T)$ if computed from normal white noise. The Moore serial rank coefficients $\widetilde{r}_M(k)$'s, $\widetilde{r}_{M,1}(k)$'s and $\widetilde{r}_{M,2}(k)$'s or the standardized score autocorrelations $\widetilde{r}_F(k)$'s also have their covariances under H_0^E of the same sign for T sufficiently large (see Section 3.2 and Chapter 7) and so on. This partly explains why the sum-based tests are often empirically investigated only with very high T's (see for example [Levich and Rizzo, 1998]) or not at all.

Chapter 9

Autocorrelation Signs Matter

The benchmark portmanteau test based on the first few squared autocorrelations totally ignores their signs even when they are likely to play an important role. In this chapter, we introduce its competitors not suffering from this drawback and show their clear dominance over the benchmark in many important cases.

9.1 Theory

In principle, rank portmanteau tests check the joint asymptotic multivariate standard normal distribution of some sample serial coefficients $r_\bullet(k)$'s standardized under H_0^E. However, they are asymptotically gaussian under a great many alternatives as well (see e.g. Section 3.4, [Tran, 1990], [Nieuwenhuis and Ruymgaart, 1990], and [Harel and Puri, 1990]) and thus this normal asymptotic distribution seems less characteristic of the null hypothesis than its parameters themselves.

It therefore appears quite natural to check H_0^E with the aid of some tests based only on the minimal sufficient statistic $(\bar{r}_\bullet, \bar{s}_\bullet^2)$,

$$\bar{r}_\bullet = \frac{1}{m}\sum_{k=1}^{m} r_\bullet(k), \quad \bar{s}_\bullet^2 = \frac{1}{m}\sum_{k=1}^{m}(r_\bullet(k) - \bar{r}_\bullet)^2,$$

for unknown parameters of the univariate normal distribution of $r_\bullet(k)$'s, $k = 1, \ldots, m$. Besides, such tests have the intuitive appeal that they can be sensitive even to the alternatives with small nonzero autocorrelations of a dominant sign. As $\sqrt{m}\bar{r}_\bullet$ and $m\bar{s}_\bullet^2$ are continuous functions of $\big(r_\bullet(1), \ldots, r_\bullet(m)\big)$, they are asymptotically independent under H_0^E with asymptotic distributions N(0,1) and $\chi^2(m-1)$, respectively.

For example, we could apply some methods for combining independent tests of the same null hypothesis. In this special case, we want to combine two χ^2 tests based on $m\bar{r}_\bullet^2$ and $m\bar{s}_\bullet^2$. Available literature on this topic indicates that we can focus only on the sum, Fisher and Tippett procedures (see e.g. [Koziol and Perlman, 1978] and references therein).

Fisher's one is known superior to many others and asymptotically Bahadur optimal in the class of virtually all possible combination methods, see [Littell and Folks, 1971, Littell and Folks, 1973]. Contrary to many others, both Fisher's and Tippett's procedure are admissible in this context (see [Marden, 1982]) and neither of them can be generally preferred, see [Westberg, 1985].

In the case of combining independent χ^2 tests, both the sum and Fisher method are known to minimize the maximum shortcoming in power relative to several other procedures, respectively for all degrees of freedom equal to one and greater than one, see [Koziol and Perlman, 1978]. Their combination is recommended otherwise, see ibidem. Both these approaches may also have their natural Bayes interpretation, see [Koziol and Perlman, 1978] and [Koziol and Tuckwell, 1999].

In the general case of n independent test statistics $C_i \sim \chi^2(p_i),\ i = 1, \ldots, n$, under the same null hypothesis, the Tippett, Fisher and sum methods and the combination of the last two lead to the tests with the critical regions:

$$\begin{aligned}
&\min\bigl(1 - F_{\chi^2(p_1)}(C_1), \ldots, 1 - F_{\chi^2(p_n)}(C_n)\bigr) && \le 1 - (1-\alpha)^{1/n},\\
&\sum_{i=1}^{n} -2\ln\bigl(1 - F_{\chi^2(p_i)}(C_i)\bigr) && \ge F^{-1}_{\chi^2(2n)}(1-\alpha),\\
&\sum_{i=1}^{n} C_i && \ge F^{-1}_{\chi^2(\sum_{i=1}^n p_i)}(1-\alpha),\\
&\sum_{i,p_i=1} C_i - 2\sum_{i,p_i>1} \ln\bigl(1 - F_{\chi^2(p_i)}(C_i)\bigr) && \ge F^{-1}_{\chi^2(\sum_{i,p_i=1} p_i + 2\sum_{i,p_i>1} 1)}(1-\alpha),
\end{aligned}$$

respectively, where α is the overall significance level. Note that the Fisher and sum tests coincide here if all $p_i = 2$ (see [Koziol and Perlman, 1978]) and that [Han, 1989] also used the Fisher method for combining single (but non-serial) correlations together (but in a completely different way and context).

In the special situation considered in this chapter, these methods result in the tests $T^1_\bullet(m)$, $\ldots$, $T^4_\bullet(m)$ with the following critical regions:

$$\begin{aligned}
&T^1_\bullet: \ \min\bigl(1 - F_{\chi^2(1)}(m\bar{r}^2_\bullet), 1 - F_{\chi^2(m-1)}(m\bar{s}^2_\bullet)\bigr) && \le 1 - \sqrt{1-\alpha},\\
&T^2_\bullet: \ -2\ln\bigl(1 - F_{\chi^2(1)}(m\bar{r}^2_\bullet)\bigr) - 2\ln\bigl(1 - F_{\chi^2(m-1)}(m\bar{s}^2_\bullet)\bigr) && \ge F^{-1}_{\chi^2(4)}(1-\alpha),\\
&T^3_\bullet: \ m\bar{r}^2_\bullet + m\bar{s}^2_\bullet && \ge F^{-1}_{\chi^2(m)}(1-\alpha),\\
&T^4_\bullet: \ m\bar{r}^2_\bullet - 2\ln\bigl(1 - F_{\chi^2(m-1)}(m\bar{s}^2_\bullet)\bigr) && \ge F^{-1}_{\chi^2(3)}(1-\alpha).
\end{aligned}$$

Note that $T^3_\bullet$ is equal to the benchmark form of the portmanteau test, i.e. to the sum $\sum_{k=1}^m r^2_\bullet(k)$. However, it is likely to perform worse than $T^4_\bullet$, especially for higher values of m (see for instance [Koziol and Perlman, 1978]).

Another possibility is to use some weighted combining methods, for example those asymptotically Bahadur optimal (see e.g. [Berk and Cohen, 1979]) or those with a natural Bayes interpretation (see e.g. [Koziol and Perlman, 1978] and [Koziol and Tuckwell, 1999]). However, we do not intend to proceed this way here.

Finally, even the full specification tests of the normal distribution summarized and investigated in [Omelka, 2004], [Omelka, 2005] might come in handy. Unfortunately, they are designed (and often also optimized) for the testing problem assuming some gaussian random sample also under its alternative, which is usually not exactly the case of $r_\bullet(k)$'s obtained from the time series alternatives frequently considered in practice. However, there is still some hope that they will perform well even in our context. The most promising ones lead to the portmanteau tests $T_\bullet^5(m), \ldots, T_\bullet^9(m)$ associated with the statistics

$$
\begin{aligned}
S_\bullet^5 &= m\bar{r}_\bullet^2 + m\big(\bar{s}_\bullet^2 - 1 - \ln(\bar{s}_\bullet^2)\big) && \sim_{\text{asympt.}} \chi^2(2),\\
S_\bullet^6 &= \frac{m\bar{r}_\bullet^2}{\bar{s}_\bullet^2} + \frac{m(\bar{s}_\bullet^2 - 1)^2}{2\bar{s}_\bullet^4} && \sim_{\text{asympt.}} \chi^2(2),\\
S_\bullet^7 &= m\bar{r}_\bullet^2 + \frac{m}{2}(\bar{r}_\bullet^2 + \bar{s}_\bullet^2 - 1)^2 && \sim_{\text{asympt.}} \chi^2(2),\\
S_\bullet^8 &= m\bar{r}_\bullet^2 + \frac{m-1}{2}\left(\frac{m\bar{s}_\bullet^2}{m-1} - 1\right)^2 && \sim_{\text{asympt.}} \chi^2(2),\\
S_\bullet^9 &= -2\ln\big(2\big[1 - \Phi(|\sqrt{m}\bar{r}_\bullet|)\big]\big) - 2\ln\big[1 - F_{\chi^2(2)}\big(-2\ln(H_m)\big)\big] && \sim_{\text{asympt.}} \chi^2(4),
\end{aligned}
$$

where m is assumed greater than two and

$$H_m = 2\,\mathrm{I}\big(m\bar{s}_\bullet^2 \le F^{-1}_{\chi^2(m-1)}(0.5)\big)F_{\chi^2(m-1)}(m\bar{s}_\bullet^2) + 2\,\mathrm{I}\big(m\bar{s}_\bullet^2 > F^{-1}_{\chi^2(m-1)}(0.5)\big)[1 - F_{\chi^2(m-1)}(m\bar{s}_\bullet^2)].$$

Their asymptotic distribution under the null hypothesis is stated behind the $\sim$ sign. Note that $T_\bullet^2$ and $T_\bullet^9$ are based on the same idea, i.e. on the Fisher combination method.

Although a great number of portmanteau tests have already been proposed and some of them have even been tailored for the alternatives with autocorrelations of a dominant sign (see Chapter 1), none of them to the best of our knowledge coincides with any of the tests $T_\bullet^1$, $T_\bullet^2$, $T_\bullet^4$ to $T_\bullet^9$.

9.2 Monte Carlo Simulations

Now we investigate the tests $T_\bullet^1, \ldots, T_\bullet^9$ with some reasonable $r_\bullet(k)$'s in a small simulation experiment. Its results regarding the SHORTTREND ($T = 20, 25, 50$), TREND ($T = 75, 100, 200$), ARMA ($T = 50, 100, 200$), LONGARMA ($T = 200, 500$) and GARCH ($T = 200, 2000$) time series indicate several conclusions that are partly illustrated in Figures 9.1 to 9.14 and briefly summarized bellow:

- Only orthonormal coefficients prove good.
- $T_\bullet^5$, $T_\bullet^6$ and $T_\bullet^7$ are poorly sized even for moderate values of m and they are therefore quite inconvenient for any practical application.
- In general, $T_\bullet^2$ appears slightly superior to all its competitors as for the test size, especially if $r_S^\perp(k)$'s or $r_F^\perp(k)$'s are considered.
- $T_\bullet^1$, $T_\bullet^2$, $T_\bullet^8$, and $T_\bullet^9$ behave in much the same way as for their power. We suggest to use $T_\bullet^2$ due to its excellent overall performance and more acceptable size for higher values of m.
- (TREND and SHORTTREND classes) Using $T_\bullet^2$ instead of $T_\bullet^3$ can lead to the power increase as high as 40, 35, and 20 percentage points for $r_M^\perp(k)$'s, $r_W^\perp(k)$'s and $r_S^\perp(k)$'s (or $r_\Phi^\perp(k)$'s) employed, respectively. It seems then reasonable to set m close to $T/2$ for $r_M^\perp(k)$'s and $r_W^\perp(k)$'s and close to $\max(7, T/10)$ for $r_S^\perp(k)$'s or $r_\Phi^\perp(k)$'s, at least in the situations considered here.

- (ARMA and LONGARMA classes) We slightly prefer the test $T_{\bullet}^{2}$ although it is usually inferior to $T_{\bullet}^{3}$ for higher values of m. Nevertheless, $T_{\bullet}^{2}$ is more than comparable to $T_{\bullet}^{3}$ for the optimum threshold parameter m (~ 3) and there is still some hope that $T_{\bullet}^{2}$ is more suitable for testing against some other ARMA processes.

- (GARCH class) $T_{\bullet}^{2}$ performs the best especially against the most common alternatives with highly persistent volatility when it can easily beat the benchmark $T_{\bullet}^{3}$ even by more than 25 percentage points if $r_{W}^{\perp}(k)$'s are used.

9.3 Accompanying Figures

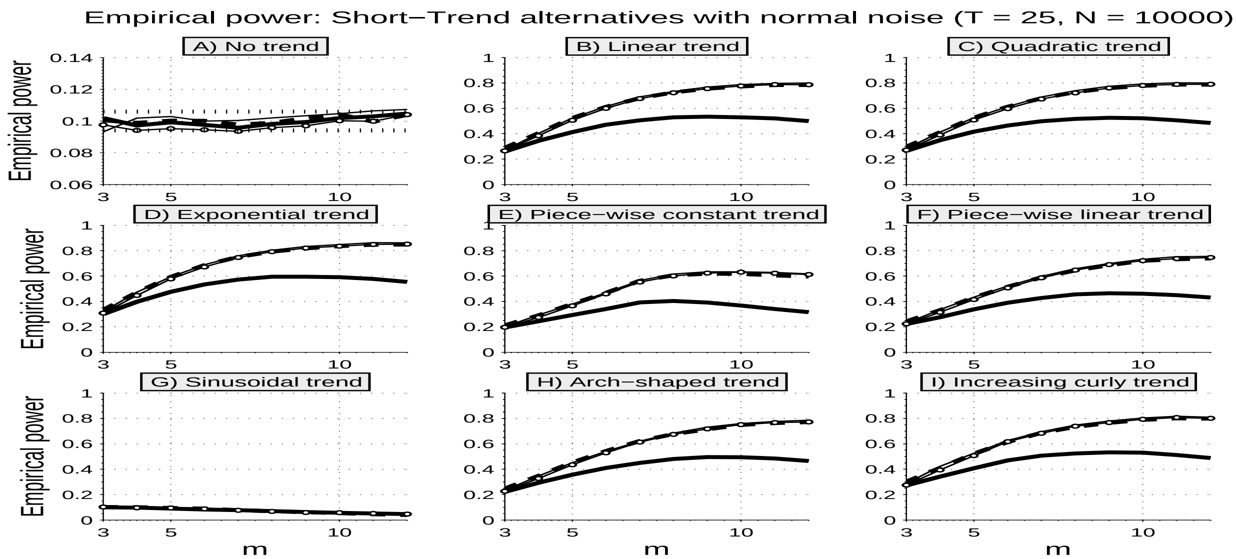

Figure 9.1: $[r_\bullet(k) = r^\perp_M(k)]$ Behaviour of the tests $T^1_\bullet(m)$ (——), $T^2_\bullet(m)$ (▬ ▬ ▬), $T^3_\bullet(m)$ (▬▬▬), and $T^8_\bullet(m)$ (∘∘∘∘∘) when applied to a short trend plus N(0,1) white noise ($T = 25$ … time series length, $N = 10\,000$ … number of replications, ıııııııı … bounds of 95% confidence intervals for the empirical size).

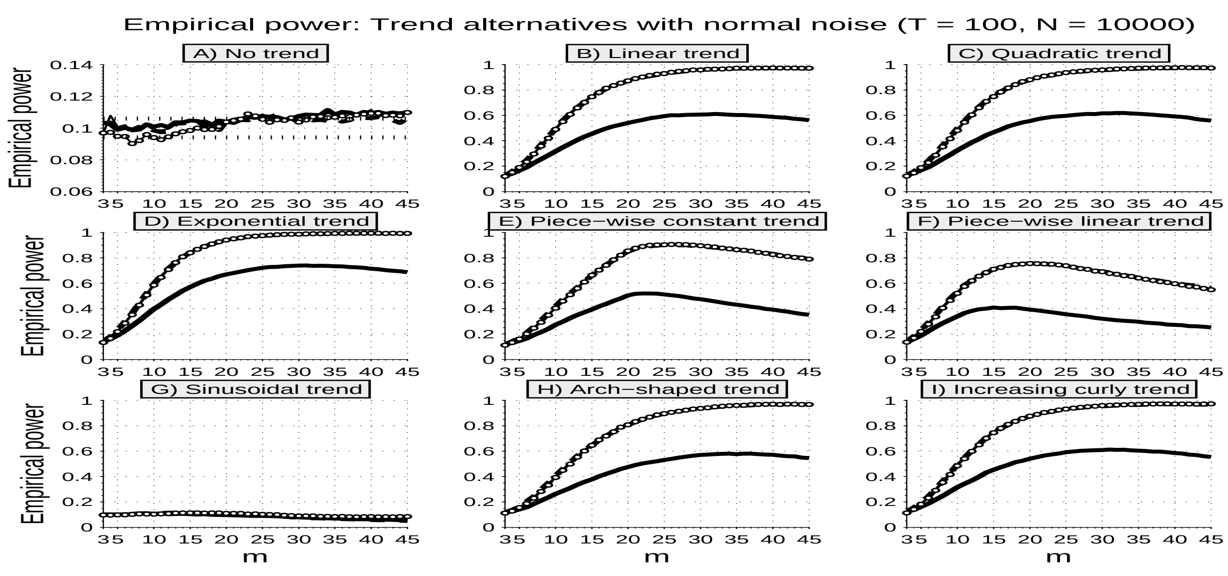

Figure 9.2: $[r_\bullet(k) = r^\perp_M(k)]$ Behaviour of the tests $T^1_\bullet(m)$ (——), $T^2_\bullet(m)$ (▬ ▬ ▬), $T^3_\bullet(m)$ (▬▬▬), and $T^8_\bullet(m)$ (∘∘∘∘∘) when applied to a trend plus N(0,1) white noise ($T = 100$ … time series length, $N = 10\,000$ … number of replications, ıııııııı … bounds of 95% confidence intervals for the empirical size).

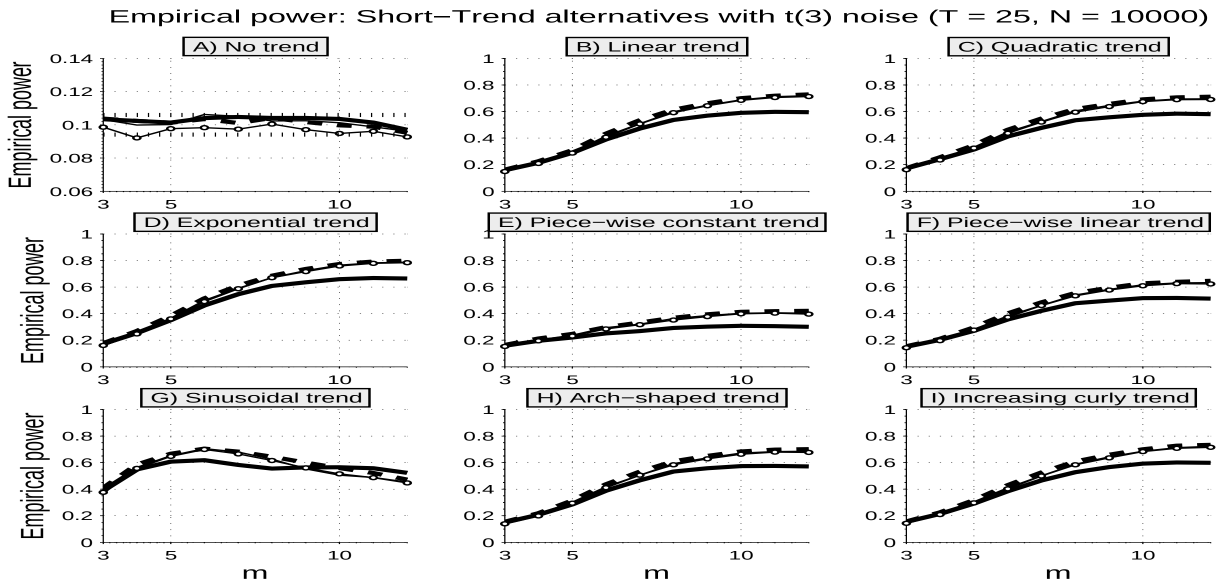

Figure 9.3: $[r_\bullet(k) = r_W^\perp(k)]$ Behaviour of the tests $T_\bullet^1(m)$ (———), $T_\bullet^2(m)$ (▬ ▬ ▬), $T_\bullet^3(m)$ (▬▬▬), and $T_\bullet^8(m)$ (∘∘∘∘∘) when applied to a short trend plus standardized t(3) white noise ($T = 25$ … time series length, $N = 10\,000$ … number of replications, ⁞⁞⁞⁞⁞⁞ … bounds of 95% confidence intervals for the empirical size).

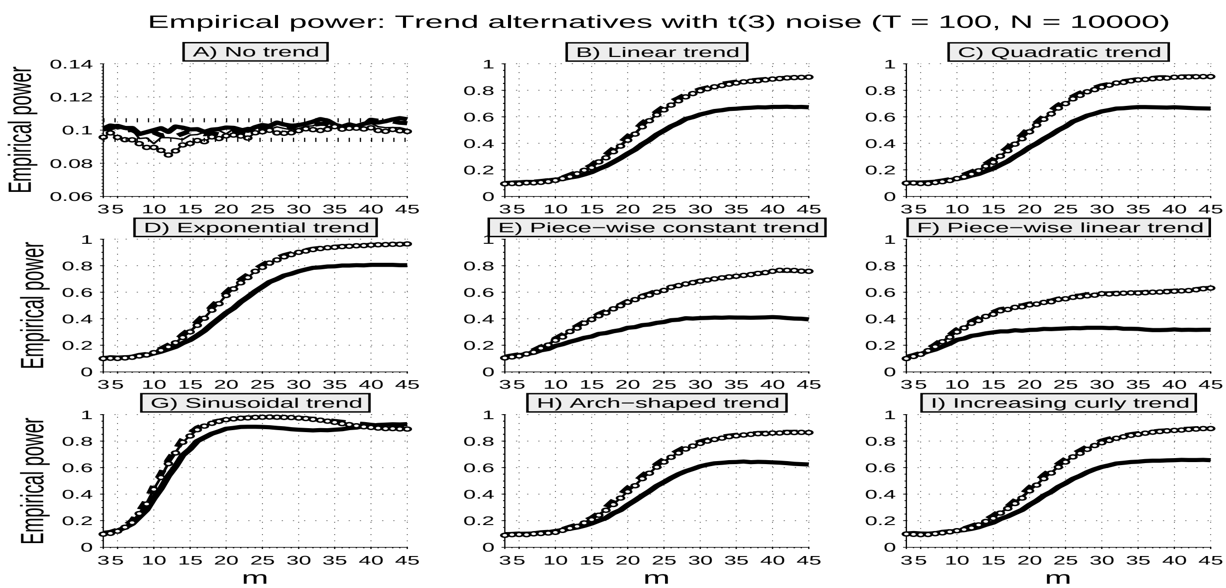

Figure 9.4: $[r_\bullet(k) = r_W^\perp(k)]$ Behaviour of the tests $T_\bullet^1(m)$ (———), $T_\bullet^2(m)$ (▬ ▬ ▬), $T_\bullet^3(m)$ (▬▬▬), and $T_\bullet^8(m)$ (∘∘∘∘∘) when applied to a trend plus standardized t(3) white noise ($T = 100$ … time series length, $N = 10\,000$ … number of replications, ⁞⁞⁞⁞⁞⁞ … bounds of 95% confidence intervals for the empirical size).

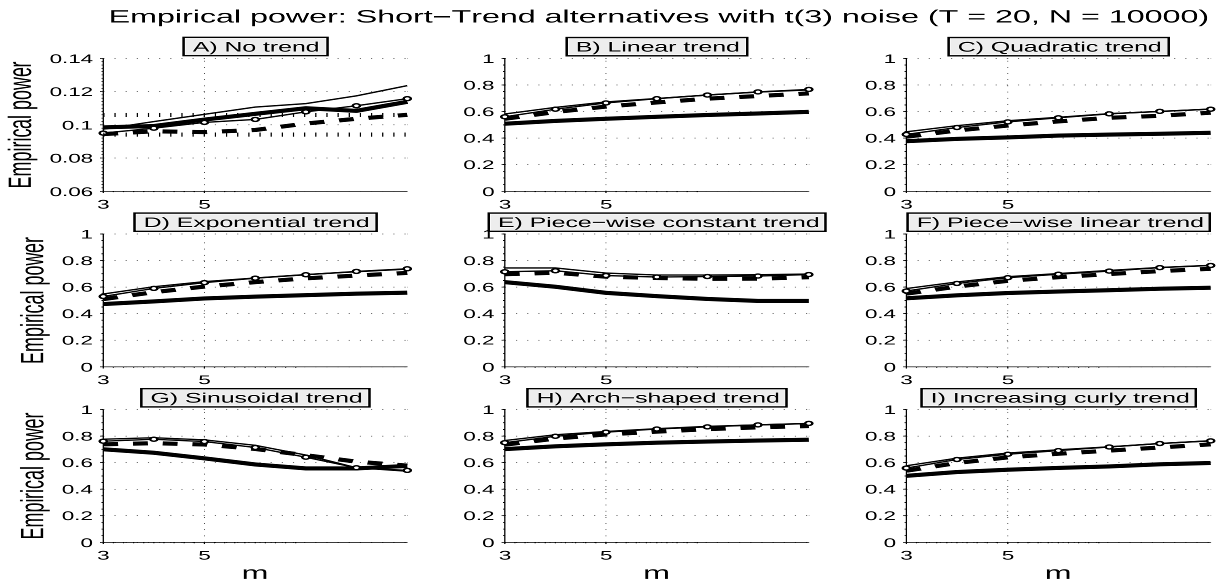

Figure 9.5: $[r_\bullet(k) = r_S^\perp(k)]$ Behaviour of the tests $T_\bullet^1(m)$ (——), $T_\bullet^2(m)$ (▬ ▬ ▬), $T_\bullet^3(m)$ (▬▬▬), and $T_\bullet^8(m)$ (∘∘∘∘∘) when applied to a short trend plus standardized t(3) white noise ($T = 20$ ... time series length, $N = 10\,000$ ... number of replications, ıııııııı ... bounds of 95% confidence intervals for the empirical size).

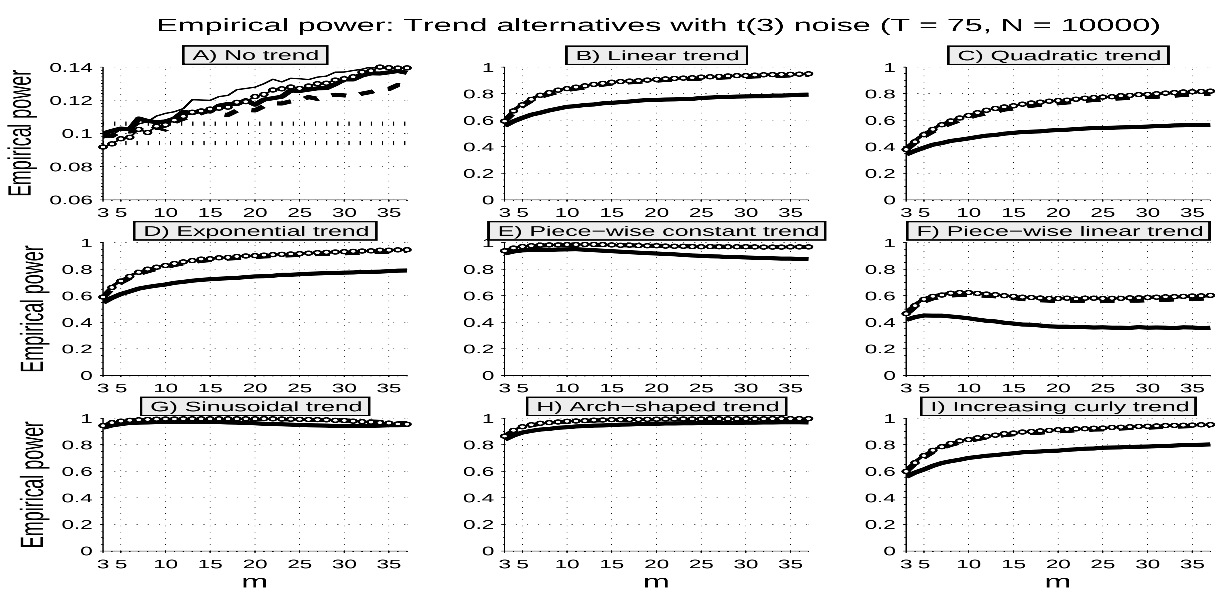

Figure 9.6: $[r_\bullet(k) = r_S^\perp(k)]$ Behaviour of the tests $T_\bullet^1(m)$ (——), $T_\bullet^2(m)$ (▬ ▬ ▬), $T_\bullet^3(m)$ (▬▬▬), and $T_\bullet^8(m)$ (∘∘∘∘∘) when applied to a trend plus standardized t(3) white noise ($T = 75$ ... time series length, $N = 10\,000$ ... number of replications, ıııııııı ... bounds of 95% confidence intervals for the empirical size).

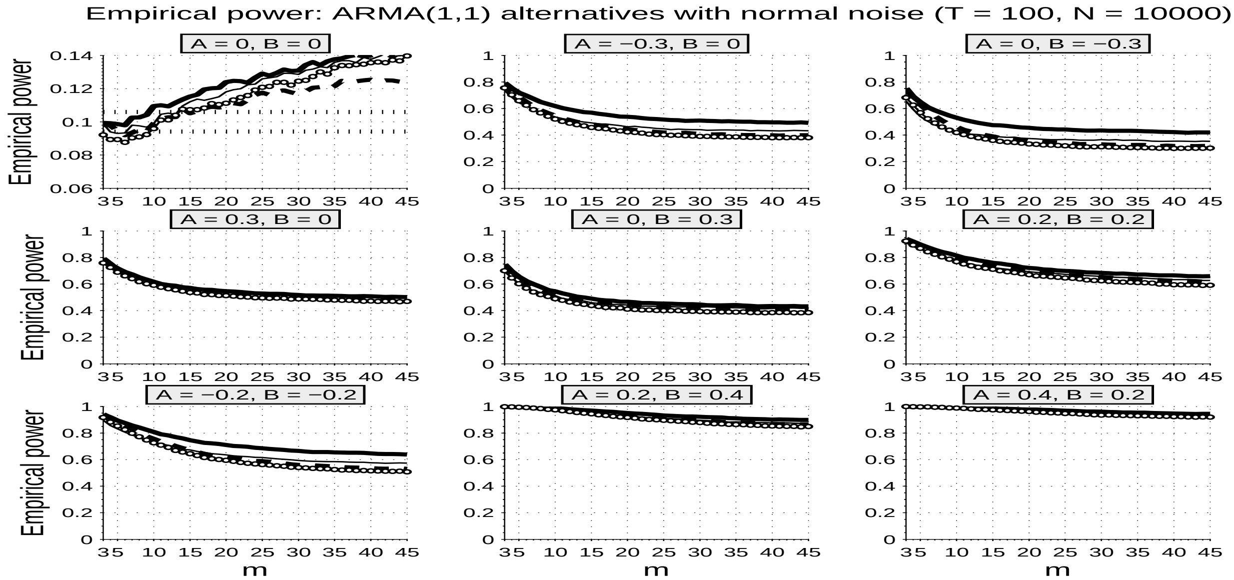

Figure 9.7: $[r_\bullet(k) = r_\Phi^\perp(k)]$ Behaviour of the tests $T_\bullet^1(m)$ (——), $T_\bullet^2(m)$ (▬ ▬ ▬), $T_\bullet^3(m)$ (▬▬▬), and $T_\bullet^8(m)$ (∘∘∘∘∘) when applied to ARMA(1,1) processes with N(0,1) white noise ($T = 100$ … time series length, $N = 10\,000$ … number of replications, ıııııııı … bounds of 95% confidence intervals for the empirical size, A,B … ARMA parameters a_1, b_1).

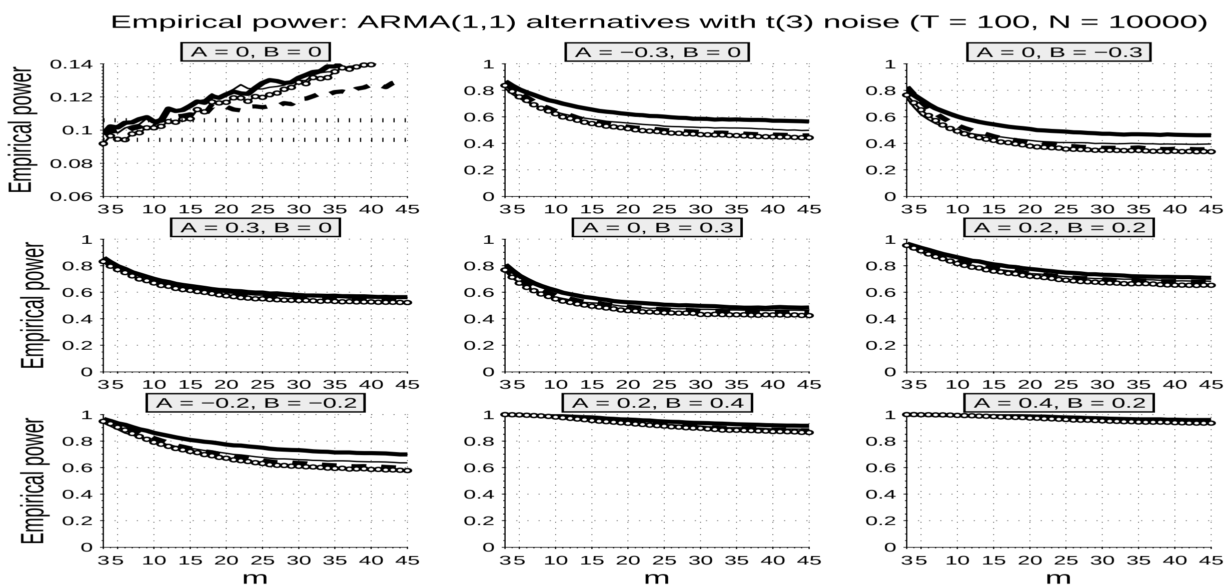

Figure 9.8: $[r_\bullet(k) = r_\Phi^\perp(k)]$ Behaviour of the tests $T_\bullet^1(m)$ (——), $T_\bullet^2(m)$ (▬ ▬ ▬), $T_\bullet^3(m)$ (▬▬▬), and $T_\bullet^8(m)$ (∘∘∘∘∘) when applied to ARMA(1,1) processes with standardized t(3) white noise ($T = 100$ … time series length, $N = 10\,000$ … number of replications, ıııııııı … bounds of 95% confidence intervals for the empirical size, A,B … ARMA parameters a_1, b_1).

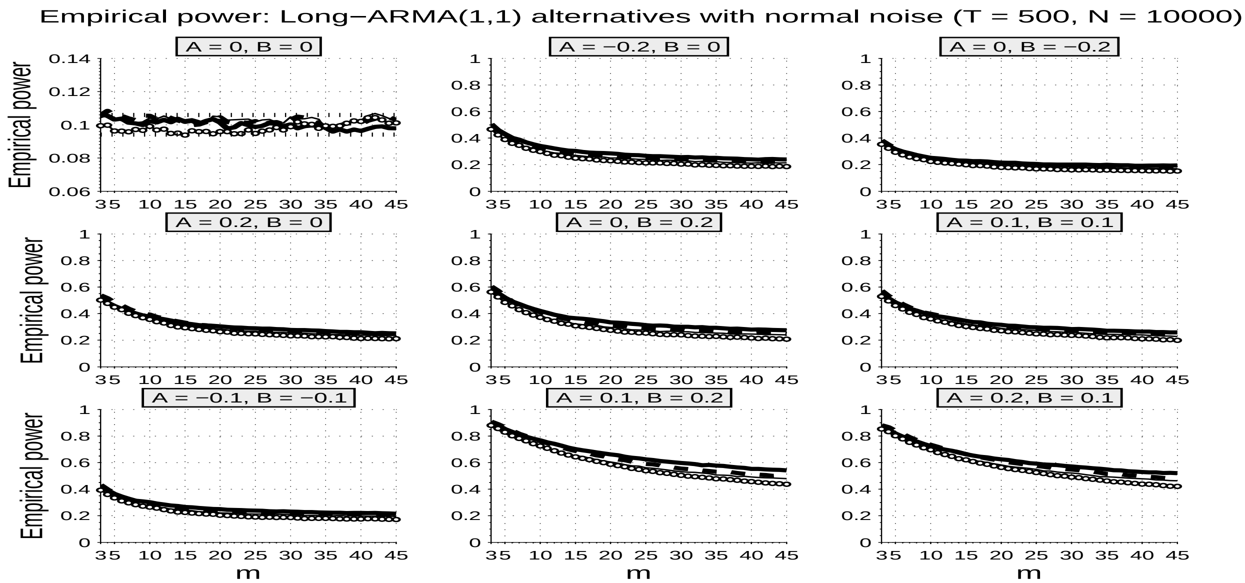

Figure 9.9: $[r_\bullet(k) = r_W^\perp(k)]$ Behaviour of the tests $T_\bullet^1(m)$ (——), $T_\bullet^2(m)$ (▬ ▬ ▬), $T_\bullet^3(m)$ (▬▬▬), and $T_\bullet^8(m)$ (∘∘∘∘∘) when applied to long ARMA(1,1) processes with N(0,1) white noise ($T = 500$ . . . time series length, $N = 10\,000$ . . . number of replications, ıııııııı . . . bounds of 95% confidence intervals for the empirical size, A,B . . . ARMA parameters a_1, b_1).

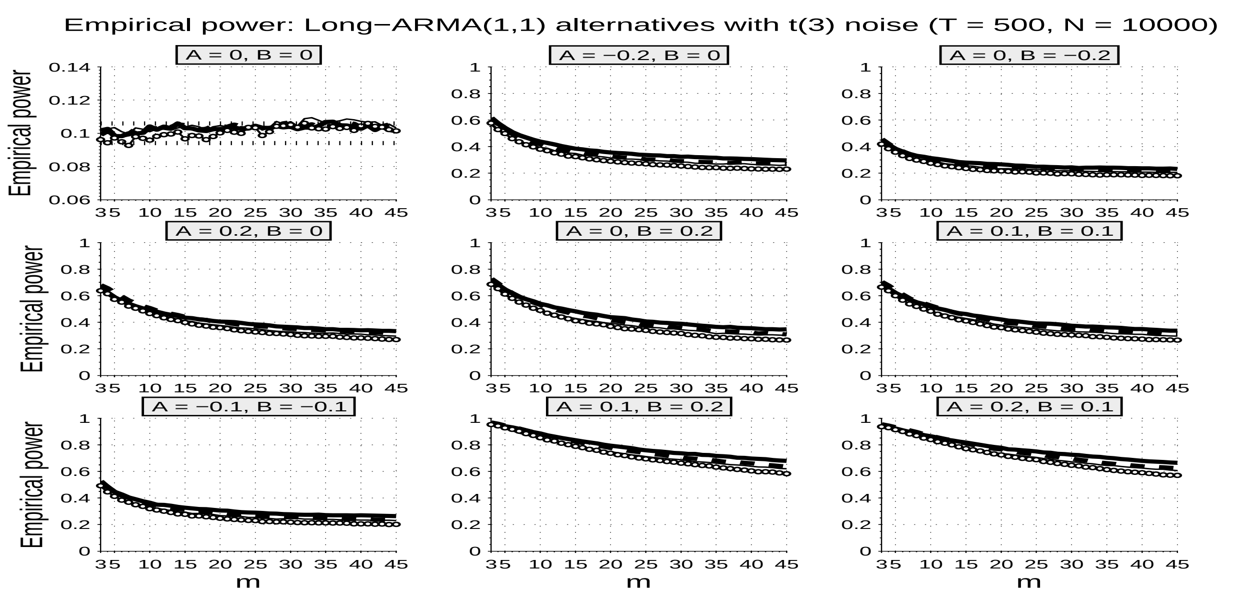

Figure 9.10: $[r_\bullet(k) = r_W^\perp(k)]$ Behaviour of the tests $T_\bullet^1(m)$ (——), $T_\bullet^2(m)$ (▬ ▬ ▬), $T_\bullet^3(m)$ (▬▬▬), and $T_\bullet^8(m)$ (∘∘∘∘∘) when applied to long ARMA(1,1) processes with standardized t(3) white noise ($T = 500$ . . . time series length, $N = 10\,000$ . . . number of replications, ıııııııı . . . bounds of 95% confidence intervals for the empirical size, A,B . . . ARMA parameters a_1, b_1).

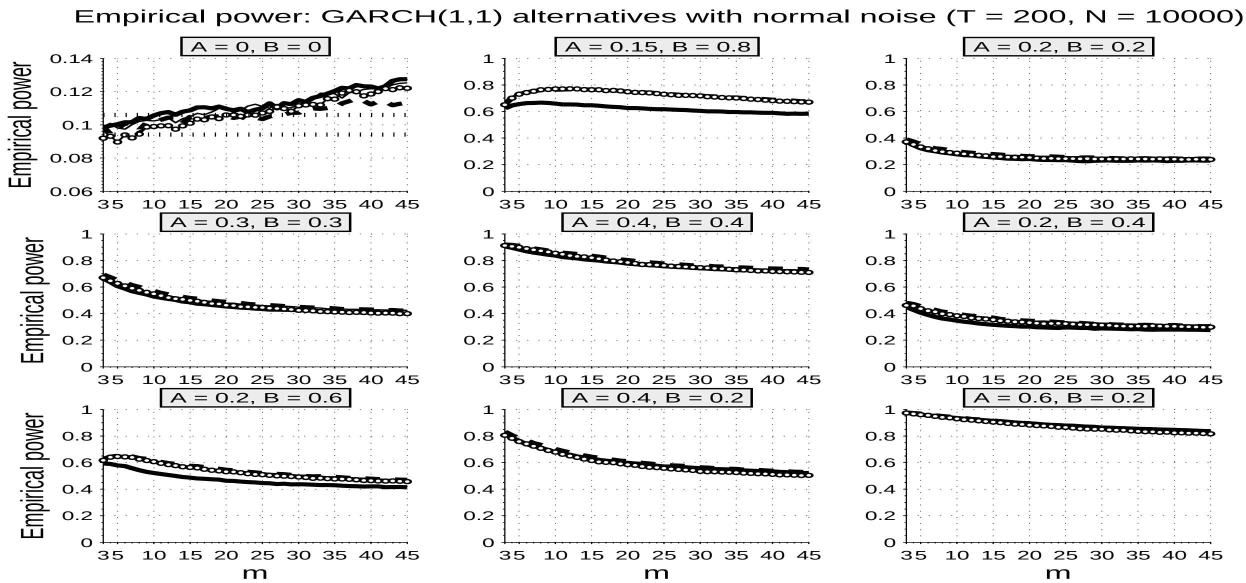

Figure 9.11: $[r_\bullet(k) = r_S^\perp(k)]$ Behaviour of the tests $T_\bullet^1(m)$ (——), $T_\bullet^2(m)$ (▬ ▬ ▬), $T_\bullet^3(m)$ (▬▬), and $T_\bullet^8(m)$ (∞∞∞) when applied to GARCH(1,1) models with N(0,1) innovations ($T = 200$ ... time series length, $N = 10\,000$ ... number of replications, ·········· ... bounds of 95% confidence intervals for the empirical size, A,B ... GARCH parameters a_1, b_1).

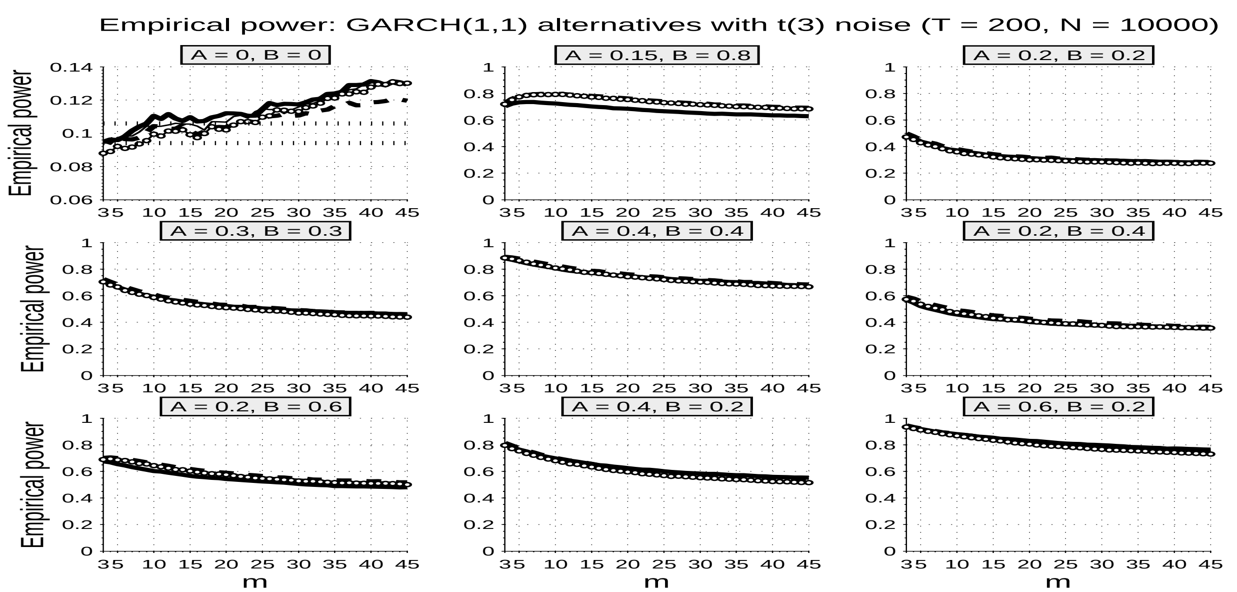

Figure 9.12: $[r_\bullet(k) = r_{F_{\mathrm{Lap}}}^{+\perp}(k)]$ Behaviour of the tests $T_\bullet^1(m)$ (——), $T_\bullet^2(m)$ (▬ ▬ ▬), $T_\bullet^3(m)$ (▬▬), and $T_\bullet^8(m)$ (∞∞∞) when applied to GARCH(1,1) models with standardized t(3) innovations ($T = 200$ ... time series length, $N = 10\,000$ ... number of replications, ·········· ... bounds of 95% confidence intervals for the empirical size, A,B ... GARCH parameters a_1, b_1).

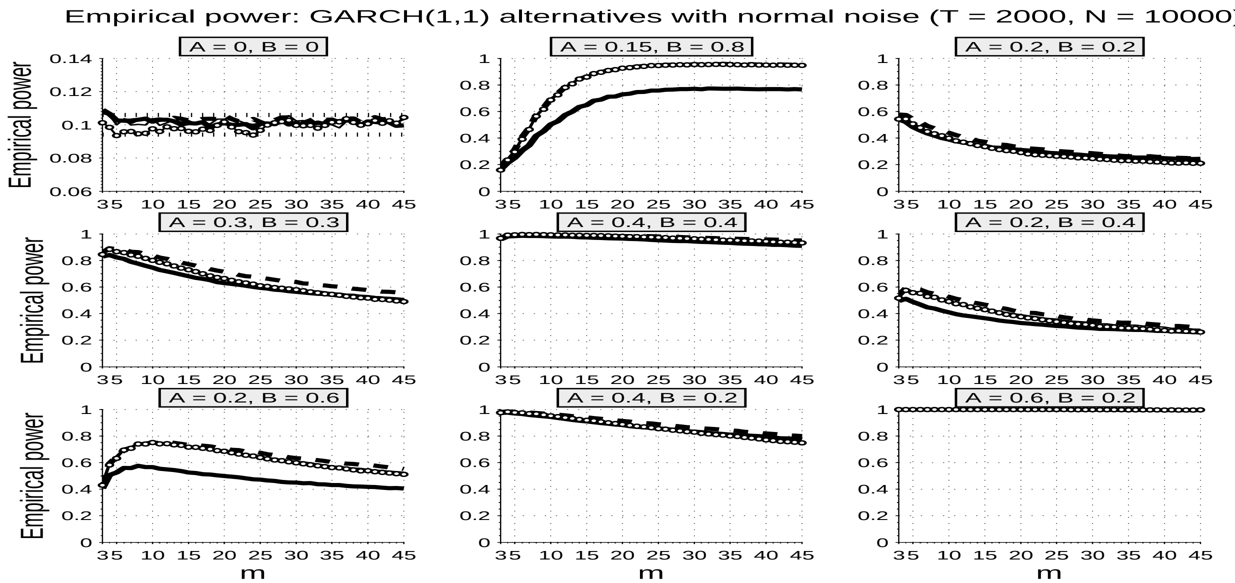

Figure 9.13: $[r_\bullet(k) = r_W^\perp(k)]$ Behaviour of the tests $T_\bullet^1(m)$ (——), $T_\bullet^2(m)$ (▬ ▬ ▬), $T_\bullet^3(m)$ (▬▬▬), and $T_\bullet^8(m)$ (∘∘∘∘∘) when applied to GARCH(1,1) models with N(0,1) innovations ($T = 2\ 000$ …time series length, $N = 10\ 000$ …number of replications, ⋯⋯ …bounds of 95% confidence intervals for the empirical size, A,B …GARCH parameters a_1, b_1).

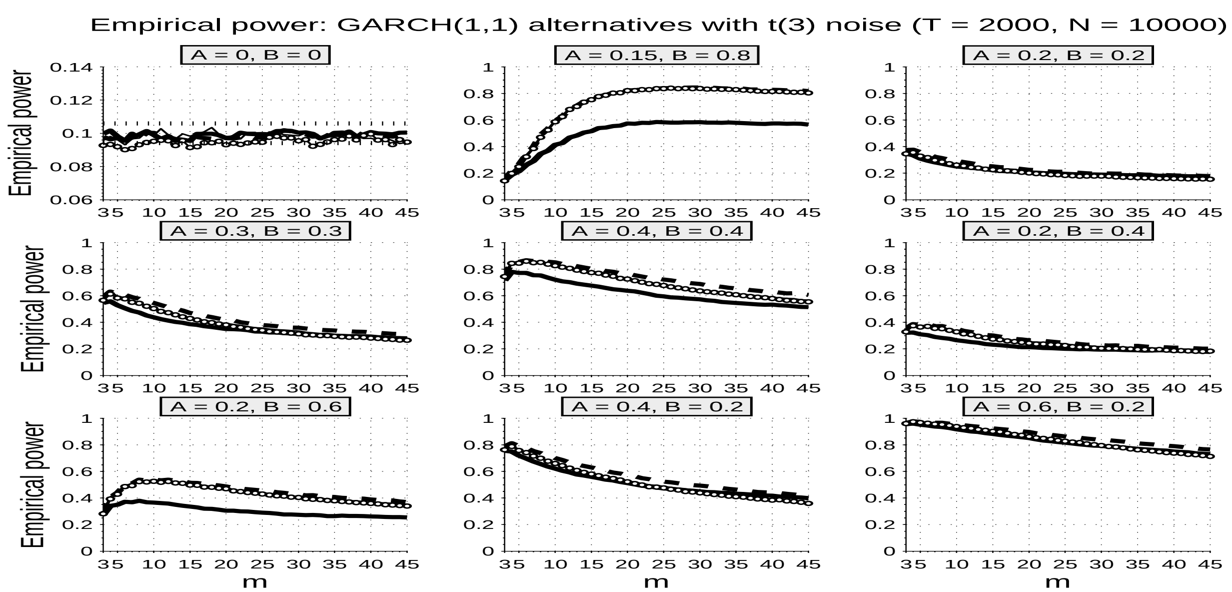

Figure 9.14: $[r_\bullet(k) = r_W^\perp(k)]$ Behaviour of the tests $T_\bullet^1(m)$ (——), $T_\bullet^2(m)$ (▬ ▬ ▬), $T_\bullet^3(m)$ (▬▬▬), and $T_\bullet^8(m)$ (∘∘∘∘∘) when applied to GARCH(1,1) models with standardized t(3) innovations ($T = 2\ 000$ …time series length, $N = 10\ 000$ …number of replications, ⋯⋯ …bounds of 95% confidence intervals for the empirical size, A,B …GARCH parameters a_1, b_1).

Chapter 10

Combining Different Types of Autocorrelations

We start this chapter with investigating joint asymptotic distributions of some types of sample autocorrelations. Then we proceed with a few suggestions in the spirit of Chapter 9 on how to combine them together in an optimal way. Finally, all such proposals will be analysed by means of Monte Carlo simulations.

10.1 Theory

First of all, we need some additional notation. To be specific, we will write $\widehat{r}_\bullet(k)$ for any sample autocorrelation at lag k and $\widehat{\mathbf{r}}_\bullet(m)$ for $\big(\widehat{r}_\bullet(1),\dots,\widehat{r}_\bullet(m)\big)'$. Their standardized and orthonormal versions (see Section 3.7) will be denoted with $\widetilde{}$ and $^\perp$ as usual. The index $\bullet$ always stands for any meaningful subscript. If F is used instead of $\bullet$, it will always be assumed to denote a quantile function meeting all the conditions of Theorem 21.

As we already know, each of the vectors $\mathbf{r}^\perp_W(m)$, $\widetilde{\mathbf{r}}_K(m)$, $\widetilde{\mathbf{r}}_M(m)$, $\widetilde{\mathbf{r}}_S(m)$, and $\widetilde{\mathbf{r}}_F(m)$ is asymptotically standard normal and the same holds even for $^\perp$ versions of the last three ones. Besides, $\widetilde{\mathbf{r}}_M(m)$ and $\mathbf{r}^\perp_W(m)$ are asymptotically jointly standard normal and mutually independent, see Theorem 7. The following statements extend this result to other types of autocorrelation coefficients.

Theorem 22. *If H_0^E holds and $T>3$, then $\operatorname{cov}_0\big(\widehat{r}_F(k),\widehat{r}_M(h)\big)=0$ and $\operatorname{cov}_0\big(\widehat{r}_S(k),\widehat{r}_M(h)\big)=0$ for any positive integers $k<T$ and $h<T$.*

Proof. The proof is based on the exchangeability of R_i's under H_0^E.

Let G be an arbitrary increasing function. It suffices to show that

$$\mathrm{E}\sum_{i=1}^{T-k} G(R_i)G(R_{i+k})\widehat{r}_M(h)=\mathrm{E}\Big(\sum_{i=1}^{T-k} G(R_i)G(R_{i+k})\Big)\,\mathrm{E}\big(\widehat{r}_M(h)\big),$$

i.e. that

$$\sum_{i=1}^{T-k}\sum_{j=1}^{T-h}\mathrm{E}\Big(G(R_i)G(R_{i+k})\,\mathrm{I}(R_j>R_{j+h})\Big)=\frac{1}{2}(T-h)(T-k)E\big(G(R_1)G(R_2)\big).$$

This will be satisfied if

$$M := \mathrm{E}\Big(G(R_i)G(R_{i+k})\,\mathrm{I}(R_j > R_{j+h})\Big) = \frac{1}{2}\,\mathrm{E}\big(G(R_1)G(R_2)\big)$$

for any allowable i, j, k, and h.

In general, M always equals to one of the following four terms

$$A_1 = \mathrm{E}\Big(G(R_1)G(R_2)\,\mathrm{I}(R_3 > R_4)\Big), \qquad A_2 = \mathrm{E}\Big(G(R_1)G(R_2)\,\mathrm{I}(R_1 > R_3)\Big),$$
$$A_3 = \mathrm{E}\Big(G(R_1)G(R_2)\,\mathrm{I}(R_3 > R_1)\Big), \qquad A_4 = \mathrm{E}\Big(G(R_1)G(R_2)\,\mathrm{I}(R_1 > R_2)\Big).$$

Each of them can be rewritten as a sum of $T!$ summands $B(r_1, \ldots, r_T)$ over all the equiprobable permutations $(r_1, \ldots, r_T)$ of $\{1, 2, \ldots, T\}$. Let us consider any fixed integers $i, j, i \neq j$, and all such summands with $r_1 = i$ and $r_2 = j$ or with $r_1 = j$ and $r_2 = i$. It is then easy to check that exactly one half are zero and the others are $G(i)G(j)/T!$, no matter which of the terms $A_1, \ldots, A_4$ is considered. Therefore

$$A_1 = A_2 = A_3 = A_4 = \frac{1}{2}\,\mathrm{E}\big(G(R_1)G(R_2)\big)$$

and the proof is complete. □

Theorem 23. *If H_0^E holds and $T > 3$, then $\mathrm{cov}_0\big(\widehat{r}_F(k), \widehat{r}_W(h)\big) = 0$ and $\mathrm{cov}_0\big(\widehat{r}_S(k), \widehat{r}_W(h)\big) = 0$ for any positive integers $k < T$ and $h < T/2$.*

Proof. The proof is based again on the exchangeability of R_i's under H_0^E and closely mimics the previous one.

Let G be an arbitrary increasing function. It suffices to show that

$$\mathrm{E}\sum_{i=1}^{T-k} G(R_i)G(R_{i+k})\widehat{r}_W(h) = \mathrm{E}\Big(\sum_{i=1}^{T-k} G(R_i)G(R_{i+k})\Big)\,\mathrm{E}\big(\widehat{r}_W(h)\big),$$

i.e. that

$$\sum_{i=1}^{T-k}\sum_{j=1}^{T-2h} \mathrm{E}\Big(G(R_i)G(R_{i+k})\big[\mathrm{I}(R_j > R_{j+h}, R_{j+h} < R_{j+2h}) + \mathrm{I}(R_j < R_{j+h}, R_{j+h} > R_{j+2h})\big]\Big)$$

is equal to

$$\frac{2}{3}(T-2h)(T-k)E\big(G(R_1)G(R_2)\big).$$

This will be satisfied if

$$\begin{aligned} M &:= \mathrm{E}\Big(G(R_i)G(R_{i+k})\big[\mathrm{I}(R_j > R_{j+h}, R_{j+h} < R_{j+2h}) + \mathrm{I}(R_j < R_{j+h}, R_{j+h} > R_{j+2h})\big]\Big) \\ &= \frac{2}{3}\,\mathrm{E}\big(G(R_1)G(R_2)\big) \end{aligned}$$

for any allowable i, j, k, and h.

In general, M always equals to one of the following five terms:

$$\begin{aligned}
A_1 &= \mathrm{E}\Big(G(R_1)G(R_2)\big[\mathrm{I}(R_3 > R_4, R_4 < R_5) + \mathrm{I}(R_3 < R_4, R_4 > R_5)\big]\Big),\\
A_2 &= \mathrm{E}\Big(G(R_1)G(R_2)\big[\mathrm{I}(R_1 > R_3, R_3 < R_4) + \mathrm{I}(R_1 < R_3, R_3 > R_4)\big]\Big),\\
A_3 &= \mathrm{E}\Big(G(R_1)G(R_2)\big[\mathrm{I}(R_3 > R_1, R_1 < R_4) + \mathrm{I}(R_3 < R_1, R_1 > R_4)\big]\Big),\\
A_4 &= \mathrm{E}\Big(G(R_1)G(R_2)\big[\mathrm{I}(R_1 > R_2, R_2 < R_3) + \mathrm{I}(R_1 < R_2, R_2 > R_3)\big]\Big),\\
A_5 &= \mathrm{E}\Big(G(R_1)G(R_2)\big[\mathrm{I}(R_1 > R_3, R_3 < R_2) + \mathrm{I}(R_1 < R_3, R_3 > R_2)\big]\Big).
\end{aligned}$$

Each of them can be rewritten as a sum of $T!$ summands $B(r_1, \ldots, r_T)$ over all the equiprobable permutations $(r_1, \ldots, r_T)$ of $\{1, 2, \ldots, T\}$. Let us consider any fixed integers $i, j, i \neq j$, and all such summands with $r_1 = i$ and $r_2 = j$ or with $r_1 = j$ and $r_2 = i$. It is then easy to check that exactly one third are zero and the others are $G(i)G(j)/T!$, no matter which of the terms $A_1, \ldots, A_5$ is considered. Therefore

$$A_1 = A_2 = A_3 = A_4 = A_5 = \frac{2}{3}\,\mathrm{E}\big(G(R_1)G(R_2)\big)$$

and the proof is complete. □

Theorem 24. *If the null hypothesis H_0^E holds and $T \geq 3$, then $\mathrm{cov}_0\big(\widehat{r}_K(k), \widehat{r}_M(h)\big) = 0$ for $1 \leq k, h < T$ and $\lim_{T\to\infty} \mathrm{cov}_0\big(\widetilde{r}_K(k), r_W^{\perp}(h)\big) = 0$ for $1 \leq k < T$, $1 \leq h < T/2$.*

Proof. Let us focus on $\mathrm{cov}_0\big(\widehat{r}_K(k), \widehat{r}_M(h)\big)$'s first. It suffices to show that

$$\mathrm{E}\sum_{i=1}^{T-k}\sum_{j=1}^{T-k} \mathrm{I}(R_i < R_j, R_{i+k} > R_{j+k})\widehat{r}_M(h) \equiv \sum_{p=1}^{T-h}\sum_{i=1}^{T-k}\sum_{j=1}^{T-k} P\big(R_i < R_j, R_{i+k} > R_{j+k}, R_p > R_{p+h}\big)$$

is equal to

$$\sum_{i=1}^{T-k}\sum_{j=1}^{T-k} P\big(R_i < R_j, R_{i+k} > R_{j+k}\big) E\big(\widehat{r}_M(h)\big) \equiv \frac{1}{2}\sum_{p=1}^{T-h}\sum_{i=1}^{T-k}\sum_{j=1}^{T-k} P\big(R_i < R_j, R_{i+k} > R_{j+k}\big).$$

This will be satisfied if

$$M := P\Big(R_i < R_j, R_{i+k} > R_{j+k}, R_p > R_{p+h}\Big) + P\Big(R_i > R_j, R_{i+k} < R_{j+k}, R_p > R_{p+h}\Big)$$

equals to

$$N := \frac{1}{2}\Big[P\big(R_i < R_j, R_{i+k} > R_{j+k}\big) + P\big(R_i > R_j, R_{i+k} < R_{j+k}\big)\Big]$$

for any allowable i, j, p, k, h such that $i < j$. We can alternatively prove $M(a, b, c) = N(a, b, c)$ for all $a, b, c \in \{1, \ldots, 6\}$ such that $b \neq c$,

$$M(a, b, c) = P\Big(R_1 < R_2, R_a > R_3, R_b > R_c\Big) + P\Big(R_1 > R_2, R_a < R_3, R_b > R_c\Big),$$

$$N(a,b,c) = \frac{1}{2}\Big[P\big(R_1 < R_2, R_a > R_3\big) + P\big(R_1 > R_2, R_a < R_3\big)\Big],$$

due to the exchangeability of R_i's under H_0^E. But this really holds as can be verified easily by means of *CMCovsKM.r*. The first part of the proof is thus complete. The second follows directly from Theorem 23 and from the asymptotic equivalence of $\widetilde{r}_K(k)$'s and $\widetilde{r}_S(k)$'s under H_0^E:

$$\lim_{T\to\infty}\big(\widetilde{r}_K(k) - \widetilde{r}_S(k)\big) = 0 \;\text{ in probability}, \quad k = 1,2,\ldots,$$

proved in [Ferguson et al., 2000]. □

Note. All the covariances considered here can be computed exactly for $T \le 9$ with *CMCovsEx.r*. A hybrid coefficient $\widehat{r}_H(k)$ based on $\sum_{i=1}^{T-k} R_i\Phi^{-1}(R_{i+k})$ (i.e. with unequal score functions) is also included in the exact empirical calculation to indicate generally nonzero covariances between $\widehat{r}_f(k)$'s and $\widehat{r}_M(k)$'s or $\widehat{r}_W(k)$'s. Really, as everybody can check easily for small values of T, $\mathrm{cov}_0\big(\widehat{r}_f(k), \widehat{r}_M(h)\big)$'s, $\mathrm{cov}_0\big(\widehat{r}_f(k), \widehat{r}_W(h)\big)$'s (and $\mathrm{cov}_0\big(\widehat{r}_K(k), \widehat{r}_W(h)\big)$'s) need not be zero, which also shows another drawback of $\widehat{r}_f(k)$'s in comparison with $\widehat{r}_F(k)$'s.

Theorem 25. *If H_0^E holds and $T > 3$, then each of the vectors $\big(\mathbf{r}_M^{\perp}(m)', \mathbf{r}_W^{\perp}(m)', \widetilde{\mathbf{r}}_K(m)'\big)'$, $\big(\mathbf{r}_M^{\perp}(m)', \mathbf{r}_W^{\perp}(m)', \mathbf{r}_S^{\perp}(m)'\big)'$ and $\big(\mathbf{r}_M^{\perp}(m)', \mathbf{r}_W^{\perp}(m)', \mathbf{r}_F^{\perp}(m)'\big)'$ is asymptotically standard normal. The same holds even if $\widetilde{\mathbf{r}}_M(m)$, $\widetilde{\mathbf{r}}_S(m)$ and $\widetilde{\mathbf{r}}_F(m)$ are used instead of their $^{\perp}$ versions.*

Proof. It is an easy consequence of the Cramér-Wold device and Theorems 22 to 24. □

It remains to solve the question on how to combine several types of sample autocorrelations into a single portmanteau statistic in an optimal way. However, only combining the vectors $\mathbf{r}_M^{\perp}(m_M)$, $\mathbf{r}_W^{\perp}(m_W)$, and $\mathbf{r}_S^{\perp}(m_S)$ $\big($or $\mathbf{r}_\Phi^{\perp}(m_P)\big)$ in the case of TREND and SHORTTREND alternatives seems reasonable enough to be taken into consideration here.

Let $\bar{r}_{\text{All}}$ and $\bar{s}_{\text{All}}$ be the mean and standard deviation computed from all N_{All} coefficients of n_{All} types considered together and let us introduce the following notation:

$$C_{\text{All}} = N_{\text{All}}\bar{r}^2_{\text{All}}, \qquad D_{\text{All}} = N_{\text{All}}\bar{s}^2_{\text{All}},$$
$$P^C_{\text{All}} = 1 - F_{\chi^2(1)}(C_{\text{All}}), \qquad P^D_{\text{All}} = 1 - F_{\chi^2(N_{\text{All}}-1)}(D_{\text{All}}),$$

where $_{\text{All}}$ says which of the following seven possibilities:

$$\begin{array}{ll}
\big(\mathbf{r}_M^{\perp}(m_M)', \mathbf{r}_W^{\perp}(m_W)', \mathbf{r}_S^{\perp}(m_S)'\big)' & (N_{\text{All}} = m_M + m_W + m_S, n_{\text{All}} = 3, \text{All} = \{M,W,S\}),\\
\big(\mathbf{r}_M^{\perp}(m_M)', \mathbf{r}_W^{\perp}(m_W)'\big)' & (N_{\text{All}} = m_M + m_W, n_{\text{All}} = 2, \text{All} = \{M,W\}),\\
\big(\mathbf{r}_M^{\perp}(m_M)', \mathbf{r}_S^{\perp}(m_S)'\big)' & (N_{\text{All}} = m_M + m_S, n_{\text{All}} = 2, \text{All} = \{M,S\}),\\
\big(\mathbf{r}_W^{\perp}(m_W)', \mathbf{r}_S^{\perp}(m_S)'\big)' & (N_{\text{All}} = m_W + m_S, n_{\text{All}} = 2, \text{All} = \{W,S\}),\\
\big(\mathbf{r}_M^{\perp}(m_M)', \mathbf{r}_W^{\perp}(m_W)', \mathbf{r}_\Phi^{\perp}(m_\Phi)'\big)' & (N_{\text{All}} = m_M + m_W + m_P, n_{\text{All}} = 3, \text{All} = \{M,W,P\}),\\
\big(\mathbf{r}_M^{\perp}(m_M)', \mathbf{r}_\Phi^{\perp}(m_\Phi)'\big)' & (N_{\text{All}} = m_M + m_P, n_{\text{All}} = 2, \text{All} = \{M,P\}),\\
\big(\mathbf{r}_W^{\perp}(m_W)', \mathbf{r}_\Phi^{\perp}(m_\Phi)'\big)' & (N_{\text{All}} = m_W + m_P, n_{\text{All}} = 2, \text{All} = \{W,P\}),
\end{array}$$

is considered for testing. They are respectively denoted by the strings *MNWNSN*, *MNWN*, *MNSN*, *WNSN*, *MNWNPN*, *MNPN*, and *WNPN* in the output files associated with this chapter. Besides, we replace $_{\mathrm{All}}$ with the subscripts $_{M}, _{W}, _{S}$ or $_{P}$ if only $r^{\perp}_{M}(k)$'s, $r^{\perp}_{W}(k)$'s, $r^{\perp}_{S}(k)$'s or $r^{\perp}_{\Phi}(k)$'s are considered, respectively.

Naturally, we employ the theory from Chapter 9 again. To be more specific, we will investigate the tests $T_1, \ldots, T_{14}$ with the following critical regions:

$$
\begin{aligned}
T_1:&\ \min\bigl(P^C_{\mathrm{All}}, P^D_{\mathrm{All}}\bigr) && \le 1-\sqrt{1-\alpha},\\
T_2:&\ -2\ln\bigl(P^C_{\mathrm{All}}\bigr)-2\ln\bigl(P^D_{\mathrm{All}}\bigr) && \ge F^{-1}_{\chi^2(4)}(1-\alpha),\\
T_3:&\ C_{\mathrm{All}}+D_{\mathrm{All}} && \ge F^{-1}_{\chi^2(N_{\mathrm{All}})}(1-\alpha),\\
T_4:&\ C_{\mathrm{All}}-2\ln\bigl(P^D_{\mathrm{All}}\bigr) && \ge F^{-1}_{\chi^2(3)}(1-\alpha),\\
T_5:&\ C_{\mathrm{All}}+\frac{(N_{\mathrm{All}}-1)}{2}\Bigl(\frac{D_{\mathrm{All}}}{(N_{\mathrm{All}}-1)}-1\Bigr)^2 && \ge F^{-1}_{\chi^2(2)}(1-\alpha),\\
T_6:&\ \min_{*\in\mathrm{All}}\bigl(P^C_*, P^D_*\bigr) && \le 1-(1-\alpha)^{1/(2n_{\mathrm{All}})},\\
T_7:&\ \sum_{*\in\mathrm{All}}\Bigl(-2\ln\bigl(P^C_*\bigr)-2\ln\bigl(P^D_*\bigr)\Bigr) && \ge F^{-1}_{\chi^2(4n_{\mathrm{All}})}(1-\alpha),\\
T_8:&\ \sum_{*\in\mathrm{All}}\Bigl(C_*-2\ln\bigl(P^D_*\bigr)\Bigr) && \ge F^{-1}_{\chi^2(3n_{\mathrm{All}})}(1-\alpha),\\
T_9:&\ -2\ln\bigl(P^C_M\bigr)-2\ln\bigl(P^D_M\bigr) && \ge F^{-1}_{\chi^2(4)}(1-\alpha),\\
T_{10}:&\ -2\ln\bigl(P^C_W\bigr)-2\ln\bigl(P^D_W\bigr) && \ge F^{-1}_{\chi^2(4)}(1-\alpha),\\
T_{11}:&\ -2\ln\bigl(P^C_S\bigr)-2\ln\bigl(P^D_S\bigr) && \ge F^{-1}_{\chi^2(4)}(1-\alpha),\\
T_{12}:&\ C_M+D_M && \ge F^{-1}_{\chi^2(m_M)}(1-\alpha),\\
T_{13}:&\ C_W+D_W && \ge F^{-1}_{\chi^2(m_W)}(1-\alpha),\\
T_{14}:&\ C_S+D_S && \ge F^{-1}_{\chi^2(m_S)}(1-\alpha).
\end{aligned}
$$

The tests T_9 to T_{14} are included only for comparison. T_3 corresponds to the naive approach lying in summing the squares of all the coefficients considered.

We further compare these tests in a Monte Carlo study and reveal their pros and cons in the simplest case $m_M = m_W = m_S = m_P = m$.

10.2 Monte Carlo Simulations

Our simulation experiments with the tests $T_1, \ldots, T_{14}$ applied to SHORTTREND ($T = 20$, 25) and TREND ($T = 50$, 75, and 100) time series reveal several interesting facts that are illustrated in Figures 10.1 to 10.8 and listed below:

- Combining $r^{\perp}_{W}(k)$'s with $r^{\perp}_{S}(k)$'s or $r^{\perp}_{\Phi}(k)$'s leads to badly sized tests even for small values of m (probably due to their discrete nature, finite sample dependence or possibly owing to a weakness of the random number generator) and it would not be too beneficial anyway.

- Only combining $r^{\perp}_{M}(k)$'s with either $r^{\perp}_{S}(k)$'s or $r^{\perp}_{\Phi}(k)$'s gives, on average, better results than the test T_{11}. We slightly favour the tests based on $r^{\perp}_{S}(k)$'s because of their simplicity and good overall behaviour.
- T_1, T_2, and T_5 exhibit virtually the same power and clearly beat all their competitors in most cases. Especially, their power can be higher than that of T_3 and T_{11} by more than 25 and 10 percentage points, respectively. Even T_7 is still almost uniformly better than the two benchmarks T_3 and T_{11}.
- Nevertheless, we prefer T_2 to both T_1 and T_5 due to its excellent size behaviour here, acceptable for m not greater than $\max(10, T/5)$ for any $T \geq 20$.

Note that even T_{11} is still much better than T_{14}, as we already know from the previous chapter.

10.3 Accompanying Figures

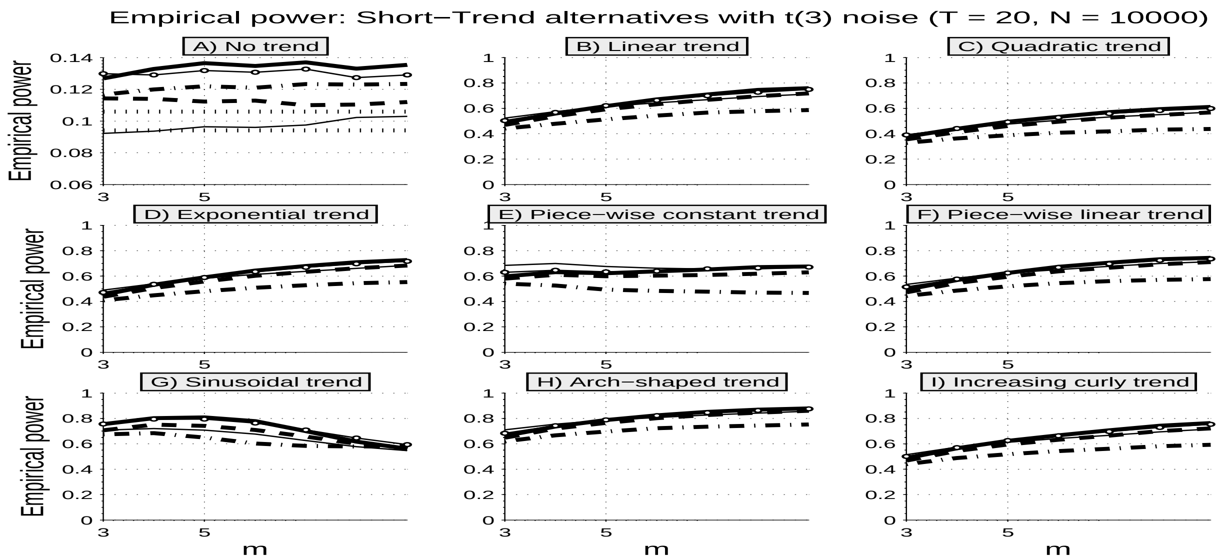

Figure 10.1: [All = $\{W, P\}$] Behaviour of the tests $T_2(m)$ (∘∘∘∘∘), $T_3(m)$ (▬ · ▬ ·), $T_5(m)$ (▬▬▬), $T_7(m)$ (▬ ▬ ▬) and $T_{11}(m)$ (——) when applied to a short trend plus standardized t(3) white noise ($T = 20$ ... time series length, $N = 10\ 000$ ... number of replications, ········ ... bounds of 95% confidence intervals for the empirical size).

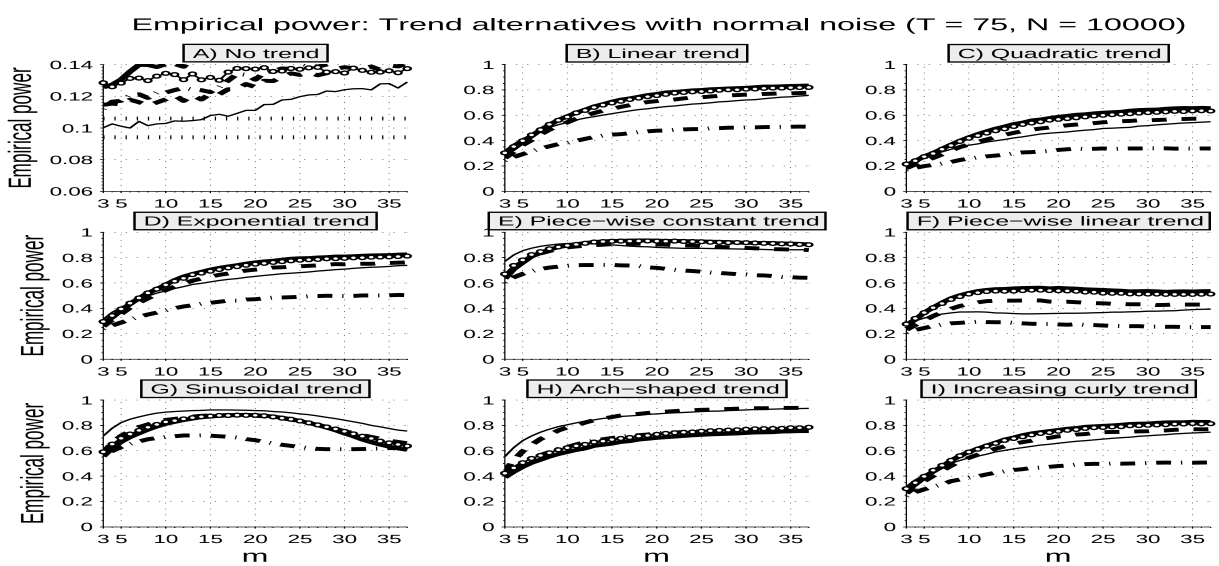

Figure 10.2: [All = $\{M, W, S\}$] Behaviour of the tests $T_2(m)$ (∘∘∘∘∘), $T_3(m)$ (▬ · ▬ ·), $T_5(m)$ (▬▬▬), $T_7(m)$ (▬ ▬ ▬) and $T_{11}(m)$ (——) when applied to a trend plus plus N(0,1) white noise ($T = 75$ ... time series length, $N = 10\ 000$ ... number of replications, ········ ... bounds of 95% confidence intervals for the empirical size).

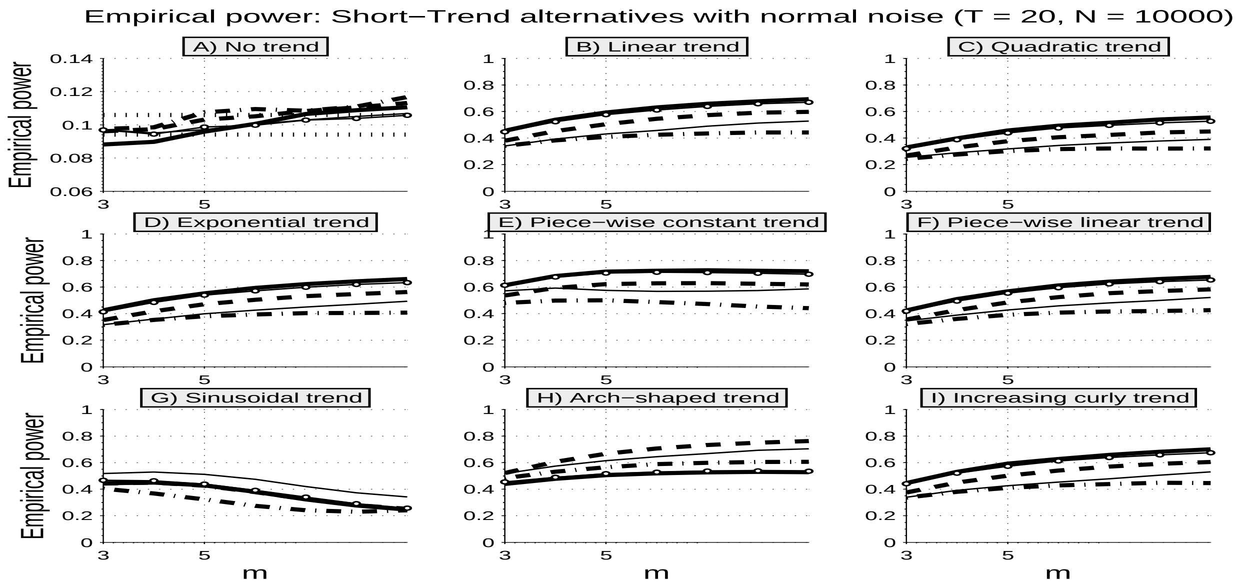

Figure 10.3: [All = $\{M, S\}$] Behaviour of the tests $T_2(m)$ (∘∘∘∘∘), $T_3(m)$ (▬ ▪ ▬), $T_5(m)$ (▬▬), $T_7(m)$ (▬ ▬ ▬) and $T_{11}(m)$ (——) when applied to a short trend plus N(0,1) white noise ($T = 20$ . . . time series length, $N = 10\ 000$ . . . number of replications, ⁞⁞⁞⁞⁞⁞⁞ . . . bounds of 95% confidence intervals for the empirical size).

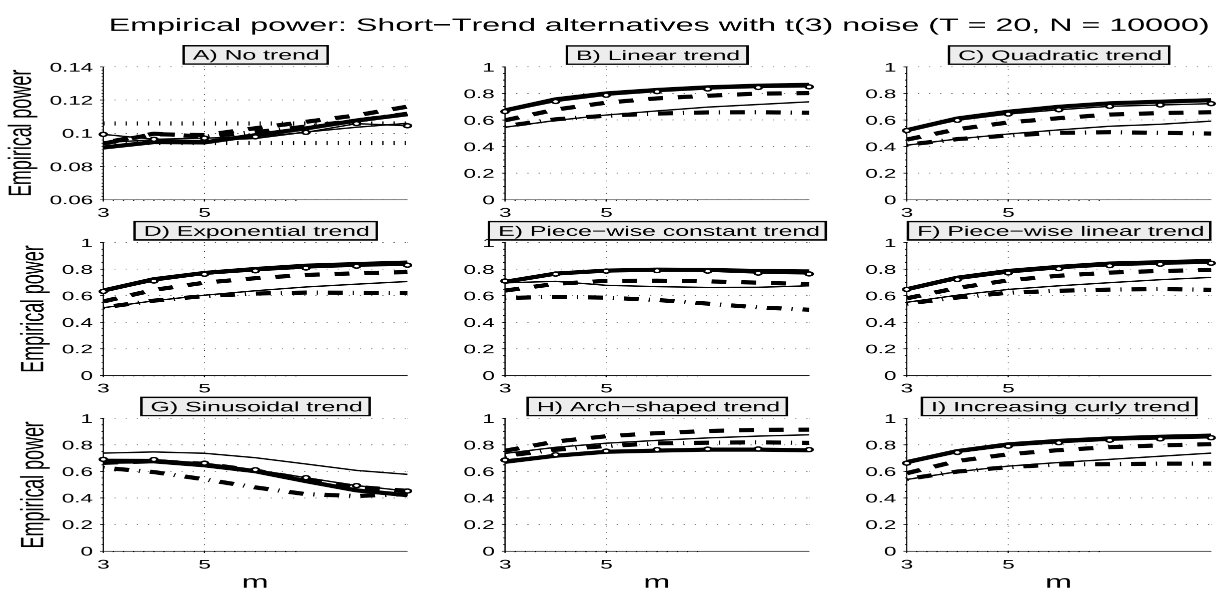

Figure 10.4: [All = $\{M, S\}$] Behaviour of the tests $T_2(m)$ (∘∘∘∘∘), $T_3(m)$ (▬ ▪ ▬), $T_5(m)$ (▬▬), $T_7(m)$ (▬ ▬ ▬) and $T_{11}(m)$ (——) when applied to a short trend plus standardized t(3) white noise ($T = 20$ . . . time series length, $N = 10\ 000$ . . . number of replications, ⁞⁞⁞⁞⁞⁞⁞ . . . bounds of 95% confidence intervals for the empirical size).

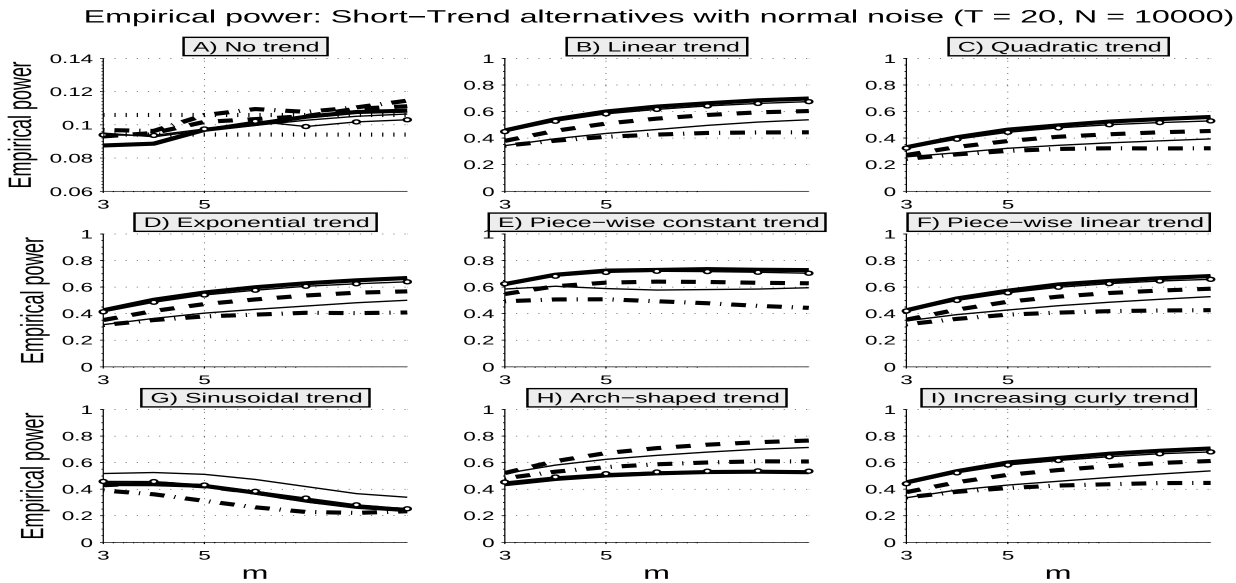

Figure 10.5: [All = $\{M, P\}$] Behaviour of the tests $T_2(m)$ ( ), $T_3(m)$ ( ), $T_5(m)$ ( ), $T_7(m)$ ( ) and $T_{11}(m)$ ( ) when applied to a short trend plus N(0,1) white noise ($T = 20$ … time series length, $N = 10\,000$ … number of replications, … bounds of 95% confidence intervals for the empirical size).

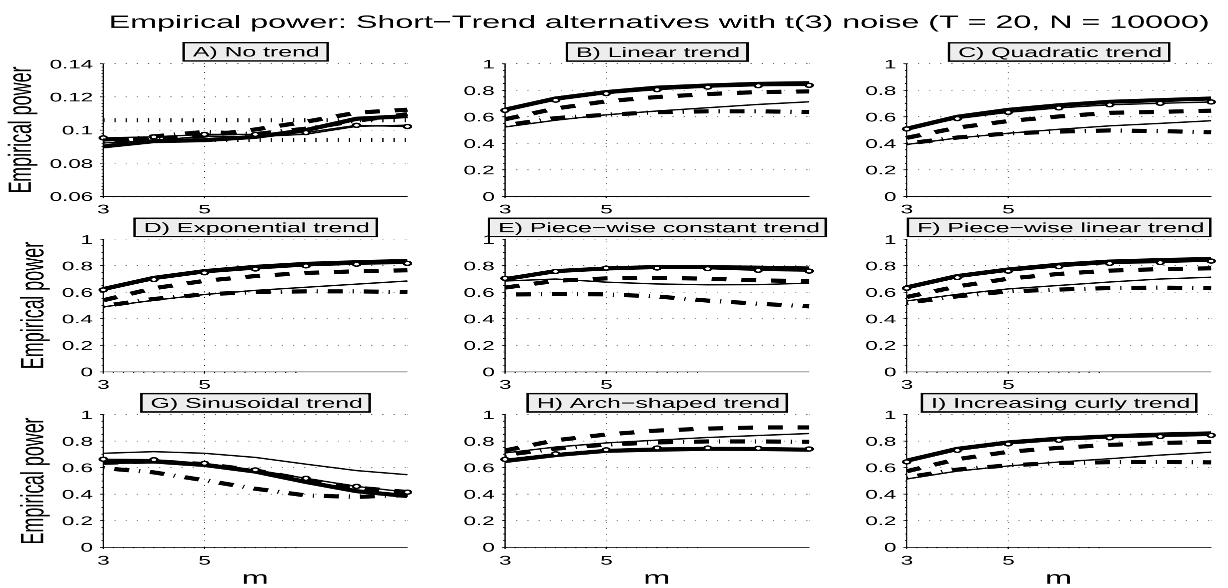

Figure 10.6: [All = $\{M, P\}$] Behaviour of the tests $T_2(m)$ ( ), $T_3(m)$ ( ), $T_5(m)$ ( ), $T_7(m)$ ( ) and $T_{11}(m)$ ( ) when applied to a short trend plus standardized t(3) white noise ($T = 20$ … time series length, $N = 10\,000$ … number of replications, … bounds of 95% confidence intervals for the empirical size).

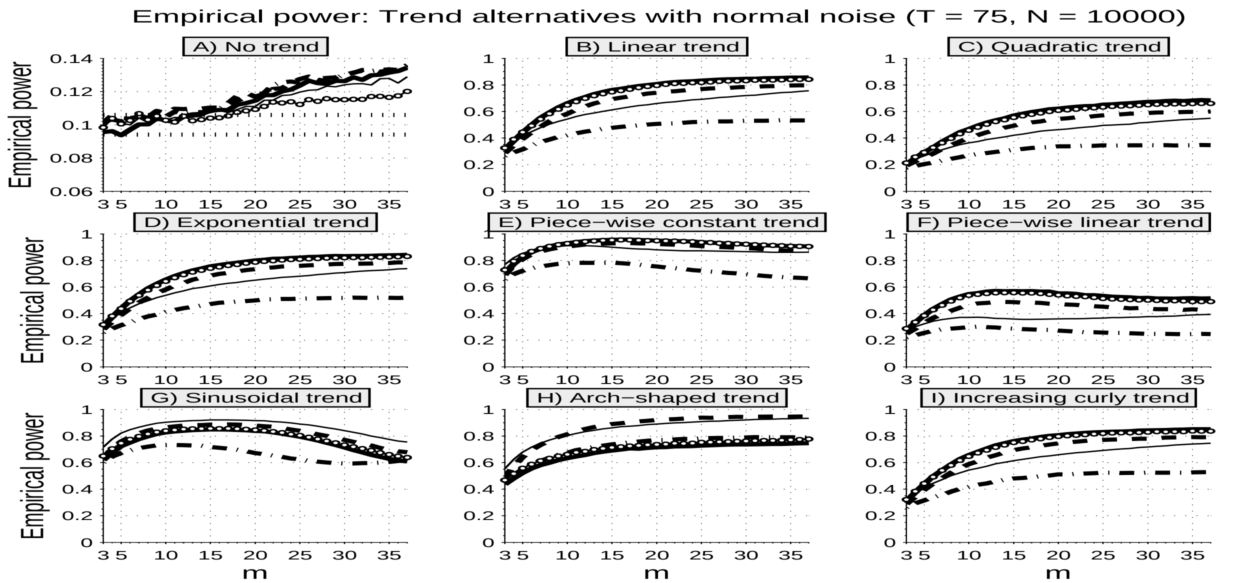

Figure 10.7: [All = $\{M, S\}$] Behaviour of the tests $T_2(m)$ (ooooo), $T_3(m)$ (▬ · ▬), $T_5(m)$ (▬▬), $T_7(m)$ (▬ ▬ ▬) and $T_{11}(m)$ (——) when applied to a trend plus N(0,1) white noise ($T = 75$ . . . time series length, $N = 10\ 000$ . . . number of replications, ıııııııı . . . bounds of 95% confidence intervals for the empirical size).

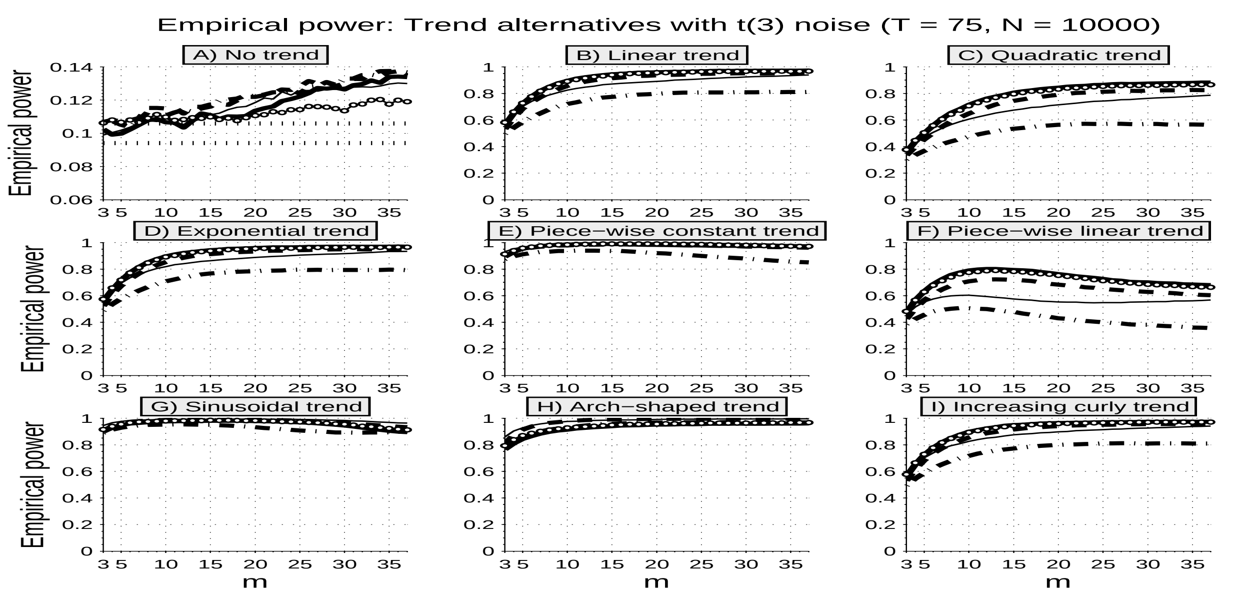

Figure 10.8: [All = $\{M, S\}$] Behaviour of the tests $T_2(m)$ (ooooo), $T_3(m)$ (▬ · ▬), $T_5(m)$ (▬▬), $T_7(m)$ (▬ ▬ ▬) and $T_{11}(m)$ (——) when applied to a trend plus standardized t(3) white noise ($T = 75$ . . . time series length, $N = 10\ 000$ . . . number of replications, ıııııııı . . . bounds of 95% confidence intervals for the empirical size).

Concluding Remarks

First, let us aggregate the improvement proposals regarding portmanteau testing against all the alternatives considered.

Generally, we suggest to orthonormalize all the sample autocorrelations used (see Section 3.7), especially in the case of shorter time series. Furthermore, it also appears that

- T_2 from Chapter 10 (based on $r_S^{\perp}(k)$'s and $r_M^{\perp}(k)$'s) should be recommended for testing against trend alternatives. However, T_9 or T_2 using the coefficients $r_M^{\perp}(k)$'s with only the most promising k's might then be even much more powerful. Apparently, such test T_9 would be more advantageous for very long time series.

- $T_\bullet^2$ or $T_\bullet^3$ from Chapter 9 (both based on $r_\Phi^{\perp}(k)$'s) should be recommended for testing against ARMA alternatives. Note that $T_\bullet^3$ (= the benchmark sum of squared autocorrelations) is optimal from a certain point of view in this case (see [Hallin et al., 1987]) and thus we can hardly expect any significant improvement from $T_\bullet^2$ in this context, at least for normal innovations.

- $T_\bullet^2$ from Chapter 9 (based on $r_{\mathrm{Lap}}^{+\perp}(k)$'s) should be recommended for testing against GARCH alternatives.

Besides, we suggest to use $T_\bullet^2$ with $r_W^{\perp}(k)$'s (and suitable k's) for quick analysis of very long stationary time series. Alternatively, we could possibly employ $\widetilde{r}_{K,w_1}(1)$ for ARMA alternatives and $\widetilde{r}_{K,w_2}(1)$ for GARCH ones (see Chapter 6) or combine both these approaches.

However, many more simulations should be performed to confirm these proposals and to create some reliable rules for optimal selection of the portmanteau parameter. For example, the portmanteau tests considered should also be investigated when applied to the trend, ARMA and GARCH alternatives with asymmetric white noise, to higher order ARMA and GARCH processes or to other (possibly nonlinear) models. It would be highly desirable to compare these tests with other relevant competitors of theirs, too.

We would also like to recall that

- Simple parametric portmanteau tests are usually inappropriate for very short time series.

- Testing against trend alternatives may be very helpful, for example in automatized computer time series analysis or in statistical process control.

- The test power need not always be the most important criterion. For example, if we have to evaluate a large collection of long time series quickly and to choose only a few of them that are the most promising for future modelling or forecasting, it may then be quite reasonable to do so just by the values of some $r_W^{\perp}(k)$-based portmanteau statistic.

Although this work answers a few important questions, many others still remain open. Some of the proposals for future research are listed below:

- Other rank correlations could possibly be serialized and then investigated in detail, see [Tarsitano, 2002] for some potential candidates. Besides, such score autocorrelations should be found that would be optimal for testing against GARCH or other volatile alternatives.

- Some corrections for continuity, ties or missing observations could be introduced for every type of rank-based autocorrelations. For example, both the Laplace correction for mean and the Sheppard one for variance seem applicable in the case of $\widehat{r}_M(k)$'s, $\widehat{r}_W(k)$'s and $\widehat{r}_K(k)$'s, see [Hájek and Šidák, 1967]. In fact, such corrections have already been employed successfully for $k = 1$, see [Ferguson et al., 2000], [Moore and Wallis, 1943], [Wallis and Moore, 1941]. The ties possibly occurring in practical applications could often be dealt with by standard methods, see [Hájek and Šidák, 1967].

- Exact higher order moments and covariances of (not only) rank-based autocorrelations and their squares could be calculated and used for finite sample corrections to each portmanteau statistic as a whole, to the autocorrelations themselves or to their (possibly joint) asymptotic distribution and its characteristics, for example by means of the orthonormalization (see Section 3.7), variance stabilizing transformations (see Section 1.2) or some sophisticated approximation techniques (see at least [Anderson et al., 1992] and recent review articles such as [Reid, 1988], [Ronchetti, 1990], [Huzurbazar, 1999], [Goutis and Casella, 1999], and [Strawderman, 2000]), among others. Such approximation methods could further be used for power investigation of the portmanteau tests and for their optimization subject to some parameters potentially involved such as the threshold parameter or weights.

  However, $\widetilde{r}_W(1)$ and $\widetilde{r}_M(1)$ (after a continuity correction) behave like normally distributed even for some $T = 13$ (see [Wallis and Moore, 1941] and [Moore and Wallis, 1943]), and therefore the Moore and Wallis serial rank coefficients are not likely to require such refinements very much.

- One could perhaps try to find some simple recurrence formulae for the probability distributions of rank-based autocorrelations, see [Moore and Wallis, 1943] for such an example regarding $\widehat{r}_M(1)$. These formulae could then serve for investigating the finite sample distributions and for establishing both their asymptotic normality and exact higher order moments, just the same way [Jirina, 1976] proposed in a non-serial case.

- The methodology for evaluating RNG's from Chapter 4 should be transferred to the C or FORTRAN code, checked carefully for numerical inaccuracies, round-off errors and other possible flaws, and investigated with different values of the parameters T, r, c and especially m. Other serial coefficients might also be used in this context, for example the weighted Moore and Wallis rank autocorrelations or the Spearman ones that are more demanding to compute but independent of both $\widehat{r}_M(k)$'s and $\widehat{r}_W(k)$'s, see Theorems 22 and 23. Besides, the comparison should be extended to encompass other RNGs and also other benchmarks because that of Marsaglia need not be perfect although it is said to pass all the Diehard tests, see [Davies, 1999].

- We note that even the current MATLAB random number generator is reported to fail some tests of randomness dramatically, see [Kahanek, 2005]. Therefore it would be interesting to know how it affected our simulations.

- Other serial rank coefficients based on the sums of some multiple products of the indicators (such as $\mathrm{I}(Y_i > Y_{i+a})$ and $\mathrm{I}(Y_j < Y_{j+b})$) could be introduced and investigated in the same way as $\widehat{r}_M(k)$'s and $\widehat{r}_W(k)$'s. There is some hope that they could be found useful for testing RNGs or elsewhere, possibly together with $\widehat{r}_M(k)$'s or $\widehat{r}_W(k)$'s.

- All the rank autocorrelations (and not only them) might be used for parameter estimation of time series models. At worst, such estimators could be employed as initial values for some better estimation procedures. First attempts in this direction have already been made, see [Allal et al., 2001] for such an application of rank-based autocorrelations in ARMA models.

- Most standardized (or orthonormalized) rank autocorrelations possess the same asymptotic joint normal distribution under H_0^E as the standardized ordinary or partial ones under $H_0^W \subset H_0^E$. Therefore the former coefficients could be employed instead of the latter alternatively, not only in all the portmanteau statistics Q_1 to Q_{17} introduced in Chapter 1, but also in other autocorrelation-based tests including spectrum-based tests (see e.g. [Milhoj, 1981], [Hong, 1996], or [Kuan, 2003]), the variance ratio test and some other tests for random walk, unit root or mean reversion (see [Lo and MacKinlay, 1989], [Richardson and Smith, 1994], and [Daniel, 2001], for example) and some tests based on the maximum likelihood approach (see e.g. [Buse, 1982] and [Newbold, 1980]), among others. Many ideas introduced for portmanteau tests (see Chapter 1) could be transferred even to these tests and vice versa.

- On the other hand, some proposals regarding rank autocorrelations could be applied even to the parametric ones, for example the orthonormalization concept (see Section 3.7) or the theory from Chapters 8, 9 and 10. By the way, the choice of involved lags, m_M's, m_S's (or m_P's) and m_W's in Chapter 10 would also deserve more attention.

- It seems very promising to extend the results from Chapters 9 and 10 in several ways. To be more specific, some good weighted or Bayesian combining methods could be employed such as those asymptotically Bahadur optimal (see e.g. [Berk and Cohen, 1979]) or those with a natural Bayes interpretation (see [Koziol and Perlman, 1978] and [Koziol and Tuckwell, 1999]). Besides, optimal tests of N(0,1) normality (say of the sample $X_1, \ldots, X_m$) against the alternatives assuming $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 1, \ldots, m$, (where $\mu_i = \beta\alpha^i, \beta + \alpha i, \beta i^\alpha, \ldots$ and $\sigma_i^2 = 1, \sigma^2, \gamma + \delta i, \ldots$) might be derived (like those from [Omelka, 2004] or [Omelka, 2005]) and applied in the spirit of Chapters 9 and 10.

- The tests considered here could also be used to detect a trend in the dispersion about a fixed location. For example, the time series could be divided into some reasonable subsets of successive observations and the tests could then be applied to a sample measure of variability within each of them. See [Cox and Stuart, 1955] and [Ury, 1966] for two illustrative examples of such approach.

- All the rank autocorrelations (and not only them) could be adapted to testing dependent data, e.g. by means of sophisticated bootstrap techniques. See [Park and Lee, 2001] for some of them applied to both the non-serial Spearman rank correlation coefficient and its Fisher transform, and consult recent review articles ([Bühlmann, 2002], [MacKinnon, 2002], [Härdle et al., 2003], [Horowitz, 2003], [Politis, 2003], [Davidson and MacKinnon, 2004], and [MacKinnon, 2006]) on the use of bootstrap methods in time series context and in econometrics at all. Note that both rank and parametric serial autocorrelations are often asymptotically normal even if computed from weakly dependent data, see Section 1.4.

- In times of powerful computers, goodness-of-fit tests are far less important than they used to be, because one can simply estimate all the reasonable models and then choose the best by means of a suitable criterion, for example by a reasonable function of their in-sample or out-of-sample prediction errors. Nevertheless, it might still be useful to adapt both the rank autocorrelations and the newly proposed portmanteau statistics to goodness-of-fit testing.

Let us focus on the last proposal in more detail. We know that the asymptotic distribution of the sample (squared-)residual ordinary autocorrelations from some true models remains asymptotically multivariate standard normal, see [Chen, 2005] and [McLeod and Li, 1983]. Therefore these autocorrelations could then be used for model validation in all portmanteau statistics (including those from Chapters 9 and 10) without any difficulties and with the same asymptotic distribution as under the null hypothesis of independence. Although most of the existing goodness-of-fit portmanteau statistics result from the gaussian likelihood-based tests against reasonable alternatives, there is still a considerable hope that the new ones from Chapters 9 and 10 might lead to some improvement in finite samples, with non-normal innovations or at least in some cases of misspecification.

The vectors of some sample (possibly rank) residual or squared-residual autocorrelations from the true model are usually asymptotically zero-mean normal with a specific variance matrix (see Section 1.4) that may be known in advance (or at least consistently estimated) and used for orthonormalizing the autocorrelations in a cautious way. Resulting orthonormal coefficients might then be employed almost as usual, especially if the asymptotic variance matrix were regular. Otherwise this approach should be corrected for singularity in a clever way. Alternatively, some sophisticated bootstrap techniques might be applied to the portmanteau statistics based on such autocorrelations of any kind.

Note that the joint distribution of any residual and squared-residual rank autocorrelations would be very helpful to know in this context as well. Unfortunately, the only similar results so far available regard the ordinary autocorrelations (see [Wong and Ling, 2005]).

At the very end, we would like to wish the reader good luck with statistics and econometrics and rosy future at all.

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