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DIPLOMOVÁ PRÁCE

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Asymptotické chování řešení evolučních rovnic na neomezených prostorových oblastech

Katedra matematické analýzy

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V Praze dne 13.4.2004 Lukáš Poul

Asymptotic Behaviour of Solutions of Evolutionary Partial Differential Equations on Unbounded Spatial Domains

Contents

Název práce: Asymptotické chování řešení evolučních rovnic na neomezených oblastech
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Abstrakt: V práci studujeme asymptotické chování řešení kvazilineární evoluční rovnice $u_t(t,x) = F(u_x(t,x))_x - h(u(t,x)), \quad t,x > 0$ se zadanou nezápornou počáteční podmínkou a homogenní Dirichletovou okrajovou podmínkou. Dokážeme existenci řešení, které pro velké časy konverguje k prostorově lokalizované vlně stacionárního řešení cestující do nekonečna. Nejprve je za předpokladu striktní monotonie F dokázána existence, jednoznačnost a regularita řešení. Dále je zpracována existence 'ground state solution', tj. netriviálního integrovatelného řešení příslušného stacionárního problému na $\mathbb{R}$; následuje rozpracování teorie koncentrované kompaktnosti. V závěru pak dostaneme hlavní výsledek aplikací teorie nulových bodů.
Klíčová slova: kvazilineární parabolické rovnice, asymptotické chování, koncentrovaná kompaktnost, cestující vlny, teorie nulových bodů.

Title: Asymptotic Behaviour of Solutions to Evolutionary Partial Differential Equations on Unbounded Spatial Domains
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Abstract: We study the large-time behaviour of solutions to a quasilinear evolutionary equation $u_t(t,x) = F(u_x(t,x))_x - h(u(t,x)), \quad t,x > 0$ endowed with the non-negative initial datum and homogeneous Dirichlet boundary conditions. The existence of a solution converging for large times to a spatially localized wave of the stationary solution travelling to infinity is shown. Under the assumption of strict monotonicity of F, the existence, uniqueness and regularity of the solution is proved. In the sequel, the existence of a 'ground state solution', i.e. nontrivial integrable solution to the steady state problem on $\mathbb{R}$, is studied, being followed by the work on the concentrated compactness for quasilinear equations and the applications of the zero number theory.
Keywords: quasilinear parabolic equations, large-time behaviour, concentrated compactness, travelling waves, zero number theory.

1 Introduction

1.1 Notation

Throughout this work we use the following notation:

(1) $C_C(\Omega)$ denotes the space of continuous functions having compact support in Ω.

(2) $C^k(\Omega)$ denotes the space of functions which derivatives are continuous up to the order k. If $k = \infty$, then we understand by this space the intersection of all $C^l(\Omega)$ for l being positive integer. In the case of Ω be a closed set, we say that $h \in C^k(\Omega)$ if there exists a function $\tilde{h}$ and an open set $\tilde{\Omega} \supset \Omega$ such that $h \equiv \tilde{h}|_{\Omega}$ and $\tilde{h} \in C^k(\tilde{\Omega})$.

(3) $\mathcal{D}(\Omega)$ denotes the space of functions lying in $C^\infty(\Omega)$ and having compact support in Ω.

(4) $C^{k,\alpha}(\Omega)$ denotes the space of functions on Ω having continous derivatives up to the order k with the kth derivative being Hölder continuous with the exponent $\alpha \in (0, 1)$.

(5) $L^p(\Omega)$ denotes the space of Lebesgue measurable functions in Ω, integrable with the p-th power of the absolute value in Ω for $p \in [1, \infty)$, and being essentialy bounded in Ω for $p = \infty$.

(6) $W^{k,p}(\Omega)$ denotes the Sobolev space of functions having their (distributional) derivatives regular up to order k and lying in $L^p(\Omega)$.

(7) $W_0^{k,p}(\Omega)$ denotes the closure of $\mathcal{D}(\Omega)$ in the topology of $W^{k,p}(\Omega)$.

(8) $L^p(I; V)$ denotes the space of Bochner integrable functions defined on interval I with values in the Banach space V.

(9) $W^{1,p,q}(I; V_1, V_2)$ denotes the space of Bochner integrable functions lying in $L^p(I; V_1)$ which (distributional) derivatives are from $L^q(I; V_2)$, and V_1 is imbedded into V_2 .

(10) $L^p(\Omega, \varrho)$ denotes the Lebesgue space with the weight ϱ. The norm is given by $\|u\|^p_{L^p(\Omega,\varrho)} = \int_\Omega |u|^p \varrho \, dx$ for $1 \leq p < \infty$, and by $\|u\|_{L^\infty(\Omega)} = \text{ess sup}\{|u(x)|\varrho(x) : x \in \Omega\}$ for $p = \infty$.

(11) $W^{k,p}(\Omega, \varrho)$ denotes the Sobolev space of functions having their (distributional) derivatives regular up to order k and lying in $L^p(\Omega, \varrho)$.

(12) $|\Omega|$ denotes the Lebesgue measure of the (measurable) set Ω.

(13) $\langle f, \varphi \rangle$ denotes the distributional duality pairing between the distribution f and the test function φ.

(14) $\langle x^*, x \rangle_X$ denotes the duality pairing between X and X^*.

(15) $\mathbb{R}_+ \equiv (0, \infty)$.

(16) $\overline{\mathbb{R}}_+ \equiv [0, \infty)$.

(17) $f|_\Omega$ denotes the restriction of the function f to the set Ω.

(18) $f[A] = \{f(x) : x \in A\}$.

(19) χ_Ω denotes the characteristic function of the set Ω (attains 1 on Ω and 0 elsewhere).

1.2 Heat Equation and Applications

Consider an infinitely long rod with given distribution of the temperature at the initial time $t_0 = 0$; let us investigate the time evolution of the temperature of $t > 0$. It turns out, by reasons concerning both experimental observations and physical models, that we can describe the temperature u of the rod as a function of one real variable satisfying the equation

$$u_t - u_{xx} + h(u) = 0 \tag{I-1}$$

where h corresponds to some additional heat sources in the rod (for details, see, e.g., Horák et al. [10]). The equation (I-1) does not appear only in the study of the heat propagation along a rod; studying, for example, diffusion processes or reaction-diffusion processes, we can obtain the similar equation (for the case of diffusion process of neutrons with the body, one can see, e.g., Drška et al. [4]). Not only physics is influenced by the heat equation, it turns out in the theory of stochastic differential equations that the option's price at the market is driven by the so called Black–Scholes equation (awarded by the Nobel Prize in Economic Sciences, 1997) which is, in fact, a heat equation (for details, see, e.g., Dupačová et al. [5]).

So this huge applicability and importance of the heat equation in physics, economics and, consequently, everday life is a sufficient reason for studying the qualitative properties of solutions of the heat equations and its generalizations.

1.3 Known Results and Generalizations

The question of the large-time behaviour of solutions to the semilinear problem

$$u_t - \Delta u + h(u) = 0 \tag{I-2}$$

was studied by many authors. Among them, one of the first results were obtained by Zelenyak [17] and Matano [13] who studied the problem on a compact interval in one dimension. They have shown that in case of the interval $[0,1]$ every *bounded* solution of (I-2) with the homogenous Dirichlet boundary conditions converges to a stationary state as $t \to \infty$.

In case of a higher space dimension and the spatial domain being a ball the similar result was obtained by Haraux–Poláčik [9]. On the other hand, also the question of non-convergent bounded solutions' existence appeared, and the positive answer was given by Poláčik–Rybakowski [14].
In case of unbounded domains, the asymptotic behaviour of solutions, more precisely convergence to stationary states and travelling waves, was studied for example by Aronson–Weinberger [3], Feireisl–Petzeltová [7], Fašangová–Feireisl [6] and many others. In the paper by Feireisl–Petzeltová [7], the problem (I-2) is considered on $\mathbb{R}^N$, $N \geq 3$ and for special types of the nonlinearity h it is shown that the convergence to the stationary state is a threshold phenomenon. More precisely, for any non-negative, compactly supported initial datum $\tilde{u} \in W^{1,2}(\mathbb{R}_+)$, $\tilde{u} \neq 0$, there exists a number $\alpha_C > 0$ such that if u is a solution of (I-2) with $u(0) = \alpha\tilde{u}$, then for $0 \leq \alpha < \alpha_C$ the solution converges to zero in $W^{2,2}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$, for $\alpha > \alpha_C$ it converges to infinity in $W^{1,2}(\mathbb{R}^N)$, and for $\alpha = \alpha_C$ the convergence to the solution of the corresponding stationary problem for large times occurs. The result is obtained by combination of the method of concentrated compactness by Lions [12], results concerning uniqueness of solutions to the stationary problem in $\mathbb{R}^N$, and discretetness of the ω-limit set of solutions. On the other hand, in the paper by Fašangová–Feireisl [6] it is shown that in case of the problem (I-2) considered on the real half-line $\mathbb{R}_+$, for any non-negative initial datum $\tilde{u} \in W_0^{1,2}(\mathbb{R}_+)$, $\tilde{u} \neq 0$ one can obtain similar conclusions to that by Feireisl–Petzeltová [7] for case of the Neumann boundary conditions, whereas in case of homogeneous Dirichlet boundary conditions it turns out that there exists a set of 'critical' solutions converging to the travelling wave of solution to the stationary problem, moving to infinity for large times. The main tools are the concentrated compactness theory for semilinear equations by Lions [12] and the zero number theory by Angenent [2].

In this work, we deal with further generalizations of the result obtained by Fašangová–Feireisl [6]. More precisely, we are concerned with the study of the large-time behaviour of solutions to a quasilinear parabolic equation considered on a half-line $\mathbb{R}_+$.

$$u_t(t,x) - (F(u_x(t,x)))_x + h(u(t,x)) = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}_+ \tag{1.1}$$

supplemented by a non-negative initial datum

$$u(0,x) = u_0(x), x \in \mathbb{R}_+ \tag{1.2}$$

and homogeneous Dirichlet boundary conditions

$$u(t,0) = 0, \quad \lim_{x\to\infty} u(t,x) = 0, \quad t > 0. \tag{1.3}$$

We claim the following result

Theorem 1.1 (Main result). *Let $F \in C^2(\mathbb{R})$ and $h \in C^1(\overline{\mathbb{R}}_+)$. Moreover, assume that there exists $\mu > 0$ such that*

$$0 < \mu \leq F'(w) \text{ for any } w \in \mathbb{R};$$

$h'(0) > 0$ and, under notation $H(t) = \int_0^t h(s)\,ds$,

$$\zeta_0 = \inf\{s > 0 : H(s) \leq 0\} > 0, \quad h(\zeta_0) < 0.$$

Then there exists an initial datum u_0 and a function $y : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} y(t) = \infty$ such that if u is the corresponding solution of (1.1), (1.2) and (1.3), then

$$\lim_{t\to\infty} \|u(t,\cdot) - w_g(\cdot + y(t))\|_{W^{2,2}(\mathbb{R}_+)} = 0,$$

where w_g is a (unique) solution of the stationary problem

$$\begin{aligned} -F(w_x)_x + h(w) &= 0, \\ w(0) &= \max\{w(x) : x \in \mathbb{R}\} > 0, \\ \lim_{|x|\to\infty} w(x) &= 0. \end{aligned}$$

The work is structured as follows: First, we show that the problem (1.1), (1.2) and (1.3) is well-posed, i.e., the solutions are uniquely determined in a sufficiently large class of functions, for initial datum $u_0 \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ the solution exists, and the solution semigroup is a continuous mapping for any fixed time $t > 0$ (these topics correspond to Section 2, Section 3, and some parts of Section 5). Although the results concerning well-posedness are generally known to hold, it is not so easy to find them, since their parts are scattered in the literature. Thus, both for the sake of entirety and completeness of the work and reader's comfort, we present them here. In Section 4 the existence and uniqueness of a nontrivial integrable solution to the stationary problem on $\mathbb{R}$, the so called ground state solution, is studied. In Section 5 we give a proof that the set of initial data which solutions converge to zero in $W^{2,2}(\mathbb{R}_+)$ is open in $L^2(\mathbb{R})_+ \cap L^\infty(\mathbb{R}_+)$. Moreover, we present there some results of Angenent's Zero point theory in order to obtain existence of solutions which level-sets are intervals for any fixed time $t \geq 0$. Further, we deal with verification that several conclusions of the concentrated compactness theory by Lions hold also for quasilinear parabolic problems. At last, having proved all the stuff above, show that the set of initial data which solutions' $W^{1,2}(\mathbb{R}_+)$ norm converges to infinity is open. Finally, in Section 6 the proof of Theorem 1.1 is given. The last part of the work, Appendix, contains a brief list of several theorems and lemmas that we have used throughout this work.

1.4 Admissible Types of Solution

Within investigating the large-time behaviour of a solution to (1.1), (1.2), and (1.3), we shall require two different kinds of solution's properties — integrability and regularity. While integrability follows from the notion of a generalized solution, the regularity will come from the results of classical theory. Since the problem is posed on an unbounded interval, one cannot say in general that good classical properties of a solution imply some kind of integrability properties, and vice versa. However, as we shall see in the sequel, our solutions will be both regular and integrable.

We shall describe the integrability in terms of a *weak* solution which definition we adopt from the presentation of the *bounded generalized solution* by Ladyženskaja et al [11]. We shall say that u is a *weak* solution to (1.1), (1.2) and (1.3) on $(0,T) \times \mathbb{R}_+$ if

$$u \in L^2((0,T); W_0^{1,2}(\mathbb{R}_+)) \cap C([0,T]; L^2(\mathbb{R}_+)) \cap L^\infty((0,T) \times \mathbb{R}_+)$$

and satisfies the integral identity

$$\int_{\mathbb{R}_+} (u(t)\eta(t) - u(0)\eta(0))\,dx + \int_0^t \int_{\mathbb{R}_+} (-u\eta_t + F(u_x)\eta_x + h(u)\eta)\,dx\,dt = 0 \quad (1.4)$$

for any $\eta \in W^{1,2,2}((0,T); W_0^{1,2}(\mathbb{R}_+), L^2(\mathbb{R}_+))$ and $0 \leq t \leq T$. As we shall see below, we can omit the condition $u \in C([0,T]; L^2(\mathbb{R}_+))$, since we shall verify it as a consequence of the remaining ones.

On the other hand, we appreciate also the notion of a solution which satisfies the equation (1.1) pointwise. This leads us to the following definition.

We shall say that the function $u \in C([0,T) \times \overline{\mathbb{R}}_+)$ is a *classical* solution of the problem (1.1), (1.2), and (1.3) if its first time and second spatial derivative are continuous in $(0,T) \times \mathbb{R}_+$, the equation (1.1) is satisfied pointwise and so do the initial and boundary conditions (1.2) and (1.3).

Although we have stated here the concept of weak and classical solutions, it is not completely satisfactory. The reason lies in the regularization efect of a solution to an equation of a parabolic type. Through the regulatization it may occur that even in the case the initial datum is a distribution, the solution is regular on $(0,T) \times \mathbb{R}_+$ (i.e. with continous first time and second spatial derivative). That is why, we introduce, moreover, the notion of a *classical solution with generalized initial condition* in the following way: We shall say that the function $u \in C((0,T) \times \overline{\mathbb{R}}_+)$ is a classical solution with generalized initial condition u_0 if the first time and second spatial derivatives of u are continuous on $(0,T) \times \mathbb{R}_+$, (1.1) is satisfied pointwise, the boundary condition (1.3) is satisfied for any $t > 0$, and the integral identity

$$\lim_{t \to 0+} \int_{\mathbb{R}_+} u(t)\eta \, dx = \int_{\mathbb{R}_+} u_0 \eta \, dx$$

holds for any $\eta \in \mathcal{D}(\mathbb{R}_+)$.

2 Uniqueness

We shall show that under certain hypotheses imposed on functions F and h the weak solutions are determined uniquely by their initial data in quite a large class of solutions. To do this, we shall assume F to be continuously differentiable and such that there exist constants $\underline{\mu}, \overline{\mu} > 0$ so that

$$0 < \underline{\mu} \leq F'(w) \leq \overline{\mu}, \quad w \in \mathbb{R}. \tag{2.1}$$

Note that, without loss of generality, we can assume $F(0) = 0$, and, consequently, if $v, w \in W^{1,2}(\mathbb{R}_+)$, then $F(v_x), F(w_x) \in L^2(\mathbb{R}_+)$, and

$$\underline{\mu}|v_x - w_x|^2 \leq (F(v_x) - F(w_x))(v_x - w_x) \leq \overline{\mu}|v_x - w_x|^2. \tag{2.2}$$

Moreover, we shall suppose the function h to be continuously differentiable on $\overline{\mathbb{R}}_+$, and

$$h(0) = 0, \quad h'(0) > 0. \tag{2.3}$$

Theorem 2.1. *Let F, h be continuously differentiable functions such that (2.1) and (2.3) are satisfied.*
Then for any $u_0 \in L^\infty(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ there exists at most one weak solution $u : (0,T) \times \mathbb{R}_+ \to \mathbb{R}$ of (1.1), (1.2), (1.3) lying in the class

$$L^2((0,T); W_0^{1,2}(\mathbb{R}_+)) \cap L^\infty((0,T) \times \mathbb{R}_+).$$

Moreover, $u \in C([0,T]; L^2(\mathbb{R}_+))$, and, if

$$u, v \in L^2((0,T); W_0^{1,2}(\mathbb{R}_+)) \cap L^\infty((0,T) \times \mathbb{R}_+)$$

are two solutions of (1.4) with initial data $u_0, v_0 \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, then there exists a constant M depending only on $\|u_0\|_{L^\infty(\mathbb{R}_+)}$, $\|v_0\|_{L^\infty(\mathbb{R}_+)}$ and the structural properties of h such that

$$\|u(t) - v(t)\|_{L^2(\mathbb{R}_+)} \leq e^{M(t-s)} \|u(s) - v(s)\|_{L^2(\mathbb{R}_+)} \tag{2.4}$$

for all $0 \leq s \leq t \leq T$.

Proof. As the first step, we shall show some preliminary estimates. Suppose that u belongs to $L^2((0,T); W_0^{1,2}(\mathbb{R}_+)) \cap L^\infty((0,T) \times \mathbb{R}_+)$ and is a solution of (1.4). From essential boundedness of u and continuous differentiability of h it follows that h is Lipschitz continuous with some constant H on the essential range of u and, consequently, the composed mapping $h \circ u$ lies in

$L^2((0,T); L^2(\mathbb{R}_+))$. The estimate on the norm of u_t in the sense of distributions gives us

$$\begin{aligned} |\langle u_t, \varphi\rangle| &\leq \|F(u_x)\|_{L^2((0,T)\times\mathbb{R}_+)}\|\varphi\|_{L^2((0,T);W^{1,2}(\mathbb{R}_+))} + \\ &\quad + H\|u\|_{L^2((0,T)\times\mathbb{R}_+)}\|\varphi\|_{L^2((0,T)\times\mathbb{R}_+)} \\ &\leq C\|\varphi\|_{L^2((0,T);W^{1,2}(\mathbb{R}_+))}, \quad \varphi \in \mathcal{D}((0,T)\times\mathbb{R}_+) \end{aligned}$$

with $C = (\overline{\mu} + H)\|u\|_{L^2((0,T);W_0^{1,2}(\mathbb{R}_+))}$. Thus we can conclude that u_t belongs to $L^2((0,T); W_0^{1,2}(\mathbb{R}_+))^*$ which can be identified with $L^2((0,T); W^{-1,2}(\mathbb{R}_+))$. Hence, in view of Proposition A.4,

$$u \in C([0,T]; L^2(\mathbb{R}_+)).$$

Suppose now u and v to be two solutions of (1.4), both belonging to

$$L^2((0,T); W^{1,2}(\mathbb{R}_+)) \cap L^\infty((0,T)\times\mathbb{R}_+).$$

From essential boundedness of u and v and continuous differentiability of h we obtain h is Lipschitz continuous with a constant H on the union of essential ranges of u and v.
Substracting the equations for u and v, taking $u - v \in L^2((0,T); W_0^{1,2}(\mathbb{R}_+))$ for the test function and applying (2.2) yields for $0 \leq s \leq t \leq T$

$$\begin{aligned} \frac{1}{2}\|u(t) - v(t)\|^2_{L^2(\mathbb{R}_+)} &\leq \frac{1}{2}\|u(s) - v(s)\|^2_{L^2(\mathbb{R}_+)} + H\|u - v\|^2_{L^2((s,t);L^2(\mathbb{R}_+))} + \\ &\quad - \int_s^t \int_{\mathbb{R}_+} (F(u_x) - F(v_x))(u_x - v_x)\, dx\, d\tau \\ &\leq \frac{1}{2}\|u(s) - v(s)\|^2_{L^2(\mathbb{R}_+)} + H\|u - v\|^2_{L^2((s,t);L^2(\mathbb{R}_+))}. \end{aligned}$$

Finally, with the aid of the Gronwall lemma we obtain

$$\|u(t) - v(t)\|_{L^2(\mathbb{R}_+)} \leq \|u(s) - v(s)\|_{L^2(\mathbb{R}_+)} e^{H(t-s)}$$

which completes both parts of the proof. □

3 Existence

3.1 Regularity of Weak Solutions

We report the following result (Theorem 1.1 of Chapter 5 by Ladyženskaja et al. [11])

Proposition 3.1. *Let F, h be continuously differentiable functions satisfying hypotheses (2.1) and (2.3). Let*

$$u \in L^2((0,T); W_0^{1,2}(\mathbb{R}_+)) \cap L^\infty((0,T) \times \mathbb{R}_+)$$

be a solution of (1.4) on $(0,T) \times \mathbb{R}_+$. Then u is locally Hölder continuous in $(0,T) \times \overline{\mathbb{R}}_+$ with Hölder exponent $\alpha > 0$ depending only on $\underline{\mu}, \overline{\mu}$ and

$$M = \text{ess sup}\{|u(t,x)| : (t,x) \in (0,T) \times \mathbb{R}_+\}$$

Furthermore, the Hölderian norm of u on any compact $Q \subset (0,T) \times \overline{\mathbb{R}}_+$ depends only on $\underline{\mu}, \overline{\mu}, M$ and $\inf\{t : (t,x) \in Q\}$. If, moreover, u_0 is Hölder continuous on $\mathbb{R}_+$, then u is globally Hölder continuous on $[0,T] \times \overline{\mathbb{R}}_+$ with the Hölderian norm depending only on $\underline{\mu}, \overline{\mu}, M$ and the Hölderian norm of u_0.

3.2 A priori Gradient Estimates

We present *a priori* estimates on the gradient u_x of any regular solution of (1.1), (1.2), (1.3). This result can be found as Theorem 3.1 in Chapter 5 by Ladyženskaja et al. [11].

Proposition 3.2. *Assume F, h are continuously differentiable functions such that (2.1) and (2.3) are satisfied. Furthermore, let u be a classical solution of (1.1), (1.2) and (1.3), and*

$$M = \text{ess sup}\{|u(t,x)| : (t,x) \in (0,T) \times \mathbb{R}_+\} < \infty.$$

Then u_x is locally Hölder continuous in $(0,T) \times \overline{\mathbb{R}}_+$ with Hölder exponent $\alpha > 0$ depending only on $\underline{\mu}, \overline{\mu}$ and M. Furthermore, the Hölderian norm of u_x on any compact $Q \subset (0,T) \times \overline{\mathbb{R}}_+$ depends only on $\underline{\mu}, \overline{\mu}, M$ and

$$\inf\{t : (t,x) \in Q\}.$$

If, moreover, $(u_0)_x$ is Hölder continous on $\overline{\mathbb{R}}_+$, then u_x is globally Hölder continous on $[0,T] \times \overline{\mathbb{R}}_+$ with the Hölderian norm depending only on $\underline{\mu}, \overline{\mu}, M$ and the Hölderian norm of u_0.

Remark 3.1. The difference between Propositions 3.1 and 3.2 lies in the fact that the former holds for any weak solution from $L^2((0,T);W^{1,2}(\mathbb{R}_+)) \cap L^\infty((0,T)\times\mathbb{R}_+)$ while the latter requires *a priori* the solution to be classical. However, as we shall see below, any weak solution is classical, so the Proposition 3.2 applies as well.

As a combination of Propositions 3.1 and 3.2 together with Theorem 5.4 of Chapter 5 by Ladyženskaja et al. [11] we obtain the following result:

Theorem 3.1. *Assume F and h to be continuously differentiable such that (2.1) and (2.3) are satisfied. Let, moreover, F' be locally Hölder continuous on $\mathbb{R}$. Let u be a classical solution of (1.1), (1.2) and (1.3) such that*

$$M = \text{ess sup}\{|u(t,x)| : (t,x) \in (0,T)\times\mathbb{R}_+\} < \infty.$$

Then u_t, u_x and u_{xx} are locally Hölder continous in $(0,T)\times\overline{\mathbb{R}}_+$ with the Hölder exponent $\alpha > 0$ depending only on $\underline{\mu}, \overline{\mu}$ and M.
Furthermore, the Hölderian norm of u, u_t, u_x and u_{xx} on any compact $Q \subset (0,T)\times\overline{\mathbb{R}}_+$ depends only on $\underline{\mu}, \overline{\mu}, M$ and $\inf\{t : (t,x)\in Q\}$.
If, moreover, $(u_0)_{xx}$ is Hölder continous on $\overline{\mathbb{R}}_+$, then u_t, u_x and u_{xx} are globally Hölder continous on $[0,T]\times\overline{\mathbb{R}}_+$ with the Hölderian norm depending only on $\underline{\mu}, \overline{\mu}, M$ and the Hölderian norm of $(u_0)_{xx}$.

3.3 Comparison Principle

Lemma 3.1 (Comparison principle). *Let $F, h \in C^1(\mathbb{R})$ be such that (2.1) and (2.3) are satisfied. Denote $\Omega = (a,b)$ for some $-\infty \le a < b \le \infty$. Let u, v be two (continuous) solutions of (1.1) on $(0,T)\times\Omega$, belonging to the space $L^2((0,T);W^{1,2}(\Omega)) \cap L^\infty((0,T)\times\mathbb{R}_+)$.*
If $u(t,x) \le v(t,x)$ for (t,x) from the parabolic boundary $\{0\}\times\overline{\Omega}\cup[0,T]\times\partial\Omega$, then

$$u(t,x) \le v(t,x)$$

for all $(t,x) \in (0,T)\times\Omega$.

Proof. Since $u, v \in L^2((0,T);W^{1,2}(\Omega))$, the positive part of the difference $(u-v)_+$ belongs to $L^2((0,T);W_0^{1,2}(\Omega))$ and, moreover, $((u-v)_+)_x$ is $(u-v)_x$ for $u-v>0$, zero otherwise (cf. Ziemer [18], Corollary 2.1.8). Multiplying the difference of equations for u and v by $(u-v)_+$, integrating over $(0,t)\times\Omega$, employing the local Lipschitz continuity of h we obtain

$$\begin{aligned}\frac{1}{2}\|(u(t)-v(t))_+\|^2_{L^2(\Omega)} &\leq -\int_0^t\int_\Omega (F(u_x)-F(v_x))((u-v)_+)_x\,dx\,ds+\\ &\quad -\int_0^t\int_\Omega (h(u)-h(v))(u-v)_+\,dx\,ds\\ &\leq H\int_0^t \|(u(s)-v(s))_+\|^2_{L^2(\Omega)}\,ds\end{aligned}$$

and the Gronwall lemma yields $u(t,x) \leq v(t,x)$ for all $t \in (0,T)\times\Omega$. □

Remark 3.2. Note that the statement above holds even in the case when u is a *subsolution* and v is a *supersolution*, i.e., $u_t - F(u_x)_x + h(u) \leq 0$ and $v_t - F(v_x)_x + h(v) \geq 0$.

Remark 3.3. As a consequence of Lemma 3.1 we obtain locally in time the L^∞-controllability of bounded-interval solutions arising from bounded initial data independently of the interval boundedness.
Indeed, suppose $u \in L^2((0,T);W_0^{1,2}(\Omega)) \cap L^\infty((0,T)\times\Omega)$ be a weak solution of (1.1) on $(0,T)\times\Omega$ with homogeneous Dirichlet boundary condition and initial datum $u_0 \in L^\infty(\Omega)$, where $\Omega \subset \mathbb{R}_+$ is a bounded interval. Specifically, let there exist constants $\underline{u}_0$, $\overline{u}_0$ such that

$$\underline{u}_0 \leq u_0(x) \leq \overline{u}_0, \quad \text{for a.a. } x \in \Omega$$

Denote by $\underline{u}, \overline{u}$ the solutions of the ordinary differential equation

$$w_t = -h(w(t)), \quad w(0) = w_0$$

defined out of the existence intervals by $-\infty$, resp. $+\infty$. Then

$$\underline{u}(t) \leq u(t,x) \leq \overline{u}(t), \quad \text{for a.a. } (t,x) \in (0,T)\times\Omega. \tag{3.1}$$

As we will see later, the unbounded-interval solution can be obtained as a limit of bounded-interval solutions, therefore, as the estimate (3.1) is stable under the limit passage, the L^∞-controllability of an unbounded-interval solution also holds.
If we do an additional assumption

$$h(s) \geq 0 \text{ for any } s \geq s_0, \tag{3.2}$$

then for any non-negative L^∞-bounded inital datum u_0 we obtain the uniform L^∞-boundedness of the corresponding solution (we will get even more, for details, see Theorem 3.2).

3.4 Integrability of Solutions

In the next part we are going to show the existence of an unbounded-interval solution of (1.1), (1.2) and (1.3). This solution will be the classical with generalized initial condition and also the weak one lying in the space $L^2((0,T);W_0^{1,2}(\mathbb{R}_+))$. Although we will get even Hölder continuity of derivatives of u, the natural question whether these nice classical properties imply also some kind of better integrability properties, cannot be answered in the positive way for a general function because the unboundedness of the domain plays a significant role, as the example

$$f(t,x) = \frac{1}{x+1}\max\left\{0, \frac{1}{x+1} - |t|\right\}, \quad (t,x) \in [-1,1] \times [0,\infty),$$

shows. Thought $f(t,x)$ is even Lipschitz continuous on $[-1,1]\times[0,\infty)$, since $g(t) = \int_0^\infty f(t,x)\,dx = -\ln|t| - 1$ we have $f \in L^1((-1,1)\times(0,\infty))$, but g is *not* bounded at zero and thus $f \notin C([-1,1];L^1(\mathbb{R}_+))$.
However, the aim of this part is to prove some kind of integral-norm estimates of bounded-interval solutions independently of the lenght of the interval. Later, this will enable us to do a limit passage with a certain subsequence of bounded-interval solutions, and, by virtue of the weak lower semicontinuity of the norm, to prove the same boundedness for the unbounded-interval solution.

Lemma 3.2. *Suppose that $F \in C^2(\mathbb{R})$ and $h \in C^1(\overline{\mathbb{R}}_+)$ satisfy hypotheses (2.1), (2.3) and (3.2). Let u^N be a classical solution of (1.1) on $(0,T)\times\Omega_N$ with $\Omega_N = (0,N)$ and $0 < T < \infty$, endowed with a non-negative initial datum $u_0 \in \mathcal{D}(\Omega_N)$ and homogeneous Dirichlet boundary conditions $u^N(t,0) = u^N(t,N) = 0$ for all $t \in (0,T)$.*
Then, under extension of u^N, u_t^N, u_x^N and u_{xx}^N by zero outside Ω_N,

$$u^N \in C^1((0,T);L^2(\mathbb{R}_+)) \cap C((0,T);W_0^{1,2}(\mathbb{R}_+))$$

with the norm over any compact time-interval $[s,t] \subset (0,T)$ bounded in terms of $\frac{1}{s}$, t, $\|u_0\|_{L^2(\mathbb{R}_+)}$, $\|u_0\|_{L^\infty(\mathbb{R}_+)}$, $\overline{\mu}$, $\underline{\mu}$ and h.
Moreover, the following estimates hold

(i)

$$\|u^N(t)\|_{L^2(\mathbb{R}_+)} \le \|u^N(s)\|_{L^2(\mathbb{R}_+)} e^{H(t-s)}, \quad 0 \le s \le t < T, \tag{3.3}$$

where H is a constant depending on h and $\|u^N(s)\|_{L^\infty(\mathbb{R}_+)}$.

(ii)

$$\|u_x^N\|_{L^2((0,T);W_0^{1,2}(\mathbb{R}_+))} \le \frac{e^{HT}}{\sqrt{2\underline{\mu}}}\|u_0\|_{L^2(\mathbb{R}_+)}, \tag{3.4}$$

where H is a constant depending only on $\|u_0\|_{L^\infty(\mathbb{R}_+)}$ and h.

(iii)

$$\|u_x^N(t)\|_{L^2(\mathbb{R}_+)} \leq C_1 \|u_x^N(s)\|_{L^2(\mathbb{R}_+)} e^{C_2(t-s)}, \quad 0 \leq s \leq t < T, \tag{3.5}$$

where $C_1 = \frac{\overline{\mu}}{\underline{\mu}}$ and $C_2 = H\frac{\overline{\mu}}{\underline{\mu}}$ with H depending only on $\|u^N(s)\|_{L^\infty(\mathbb{R}_+)}$ and h.

(iv)

$$\|u_x^N(s)\|_{L^2(\mathbb{R}_+)} \leq \frac{C_1 e^{C_2 s}}{\sqrt{s}} \|u_0\|_{L^2(\mathbb{R}_+)}, \quad 0 < s \leq T, \tag{3.6}$$

where $C_1 = \frac{\overline{\mu}}{\underline{\mu}\sqrt{\underline{\mu}}}$ and $C_2 = H(\frac{\overline{\mu}}{\underline{\mu}}+1)$ and H depends only on $\|u_0^N\|_{L^\infty(\mathbb{R}_+)}$ and h.

(v)

$$\|u_{xx}^N\|_{L^2((s,t);L^2(\mathbb{R}_+))} \leq C_1 \frac{1}{\sqrt{s}} e^{C_2 t} \|u_0\|_{L^2(\mathbb{R}_+)}, \tag{3.7}$$

where $C_1 = \sqrt{\frac{\overline{\mu}^3}{2\underline{\mu}^3}}$ and $C_2 = H(\frac{\overline{\mu}}{\underline{\mu}} + 1)$ with H depending solely on h and $\|u_0\|_{L^\infty(\mathbb{R}_+)}$.

(vi)

$$\|u_t^N(t)\|_{L^2(\mathbb{R}_+)} \leq \|u_t^N(s)\|_{L^2(\mathbb{R}_+)} e^{H(t-s)}, \quad 0 \leq s \leq t < T, \tag{3.8}$$

where H depends solely on $\|u^N(s)\|_{L^\infty(\mathbb{R}_+)}$ and h.

Proof. (i) Assume u^N to be a classical solution of the problem, as described above. Then u^N, u_t^N, u_x^N and u_{xx}^N are continuous, therefore integrable in the bounded interval Ω_N. Suppose that u^N, u_t^N, u_x^N and u_{xx}^N are extended to be zero outside Ω_N. Multiplying (1.1) on u^N, integrating by parts in the space variable, and employing (2.1) and the Lipschitz continuity of h on the range of $u^N(t)$ yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u^N(t)\|_{L^2(\mathbb{R}_+)}^2 &\leq \int_{\Omega_N} F(u_x^N(t))u_x^N(t)\,dx - \int_{\Omega_N} h(u^N(t))u^N(t)\,dx \\ &\leq H\int_{\mathbb{R}_+} |u^N(t)|^2\,dx \end{aligned}$$

and, by virtue of the Gronwall lemma, we obtain

$$\|u^N(t)\|_{L^2(\mathbb{R}_+)} \leq \|u^N(s)\|_{L^2(\mathbb{R}_+)} e^{H(t-s)}, \quad 0 \leq s \leq t < T.$$

(ii) Similarly, we estimate the derivative.

$$\int_0^t \|u_x^N(s)\|^2_{L^2(\Omega_N)}\,ds \leq \frac{1}{2\underline{\mu}}\|u_0\|^2_{L^2(\Omega_N)} + \frac{H}{\underline{\mu}}\int_0^t \|u^N(s)\|^2_{L^2(\Omega_N)};$$

whence extending u^N by zero on the complement of Ω_N and employing (3.3) we can write

$$\|u_x^N\|_{L^2((0,T);W_0^{1,2}(\mathbb{R}_+))} \leq C\|u_0\|_{L^2(\mathbb{R}_+)},$$

where $C = \frac{e^{HT}}{\sqrt{2\underline{\mu}}}$, with H depending only on $\|u_0\|_{L^\infty(\mathbb{R}_+)}$ and the structural properties of h.

(iii) Multiplying (1.1) on $F(u_x(t))_x$ and integrating in the space and time variable over $\Omega_N \times (s,t)$, $0 \leq s \leq t \leq T$ yields

$$\begin{aligned}\int_s^t\int_{\Omega_N} u_t^N F(u_x^N)_x\,dx\,d\tau &= \int_s^t\int_{\Omega_N} (F'(u_x^N))^2(u_{xx}^N)^2\,dx\,d\tau - \\ &- \int_s^t\int_{\Omega_N} h(u^N)F(u_x^N)_x\,dx\,d\tau.\end{aligned}$$

Under notation $I(x) = \int_0^x F(y)\,dy$, we can apply the by parts integration on the first term. Moreover, by virtue of the inequality $\underline{\mu}x^2 \leq 2I(x) \leq \overline{\mu}x^2$, and Lipschitz continuity of h on the range of u^N, we obtain

$$\begin{aligned}\int_{\Omega_N} I(u_x^N(t))\,dx &\leq \int_{\Omega_N} I(u_x^N(s))\,dx + H\int_s^t\int_{\Omega_N} \underbrace{F(u_x^N)u_x^N}_{\leq\overline{\mu}(u_x^N)^2}\,dx\,d\tau \\ &\leq \int_{\Omega_N} I(u_x^N(s))\,dx + \int_s^t\int_{\Omega_N} 2H\frac{\overline{\mu}}{\underline{\mu}}I(u_x^N)\,dx\,d\tau\end{aligned}$$

and, with the aid of the Gronwall inequality, we can write

$$\|u_x^N(t)\|_{L^2(\mathbb{R}_+)} \leq C_1\|u_x^N(s)\|_{L^2(\mathbb{R}_+)}e^{C_2(t-s)}$$

where $C_1 = \frac{\overline{\mu}}{\underline{\mu}}$ and $C_2 = H\frac{\overline{\mu}}{\underline{\mu}}$.

(iv) At this stage, we can employ the Chebyshev inequality

$$|\{\tau \in (0,s) : \|u_x^N(\tau)\|^2_{L^2(\mathbb{R}_+)} \geq K\}| \leq \frac{1}{K}\int_0^s \|u_x^N(\sigma)\|^2_{L^2(\mathbb{R}_+)}\,d\sigma$$

to obtain the existence of $\xi \in (0,s)$ such that

$$\|u_x^N(\xi)\|_{L^2(\mathbb{R}_+)} \leq \sqrt{\frac{2}{s}}\|u_x^N\|_{L^2((0,s);L^2(\mathbb{R}_+))}, \quad 0 < s < T.$$

Thus we can use (3.5) to conclude that

$$\|u_x^N(t)\|_{L^2(\mathbb{R}_+)} \leq C_1 e^{C_2 t} \frac{1}{\sqrt{t}} \|u_0\|_{L^2(\mathbb{R}_+)}, \quad 0 < t < T,$$

where $C_1 = \frac{\overline{\mu}}{\underline{\mu}\sqrt{\underline{\mu}}}$ and $C_2 = H(\frac{\overline{\mu}}{\underline{\mu}} + 1)$ with a constant H depending only on $\|u_0^N\|_{L^\infty(\mathbb{R}_+)}$ and h.

(v) On the other hand, estimating the term $\int_s^t \int_{\mathbb{R}_+} (F'(u_x^N))^2 (u_{xx}^N)^2 \, dx \, d\tau$, $0 \leq s \leq t \leq T$, together with the results obtained in the previous parts of this proof yields

$$\begin{aligned}
\underline{\mu}^2 \int_s^t \int_{\Omega_N} (u_{xx}^N)^2 \, dx \, d\tau &\leq \int_{\Omega_N} I((u^N(s))_x) \, dx + H \int_s^t \int_{\Omega_N} F(u_x^N) u_x^N \, dx \, d\tau \\
&\leq \int_{\Omega_N} I(u_x^N(s)) \, dx + 2H \frac{\overline{\mu}}{\underline{\mu}} \int_s^t \int_{\Omega_N} I(u_x^N) \, dx \, d\tau \\
&\leq e^{2H \frac{\overline{\mu}}{\underline{\mu}}(t-s)} \int_{\Omega_N} I(u_x^N(s)) \, dx \\
&\leq \frac{\overline{\mu}}{2} e^{2H \frac{\overline{\mu}}{\underline{\mu}}(t-s)} \int_{\Omega_N} |u_x^N(s)|^2 \, dx \\
&\leq C_1 e^{C_2 t} \frac{1}{s} \|u_0\|^2_{L^2(\mathbb{R}_+)},
\end{aligned}$$

where $C_1 = \sqrt{\frac{\overline{\mu}^3}{2\underline{\mu}^3}}$ and $C_2 = H(\frac{\overline{\mu}}{\underline{\mu}} + 1)$ with H depending solely on $\|u_0\|_{L^\infty(\mathbb{R}_+)}$ and the structural properties of h. This estimate means the boundedness of u_{xx} in $L^2((s,T) \times \Omega_N)$ for $0 < s < T$ in terms of $\overline{\mu}$, $\underline{\mu}$, $\|u_0\|_{L^\infty(\mathbb{R}_+)}$ and the structural properties of h.

Recalling back the equation (1.1), we conclude

$$\begin{aligned}
\|u_t\|_{L^2((s,T);L^2(\Omega_N))} &\leq \overline{\mu}^2 \|u_{xx}\|_{L^2((s,T);L^2(\Omega_N))} + H \|u\|_{L^2((s,T);L^2(\Omega_N))} \\
&\leq C(\frac{1}{s}, T, \overline{\mu}, \underline{\mu}, \|u_0\|_{L^\infty(\mathbb{R}_+)}) \|u_0^N\|_{W^{1,2}(\Omega_N)};
\end{aligned}$$

whence extending u^N and u_0^N by zero on $\mathbb{R}_+ \setminus \Omega_N$, we can conclude

$$\|u^N\|_{W^{1,2}((s,T);L^2(\mathbb{R}_+))} \leq C(\frac{1}{s}, T, \overline{\mu}, \underline{\mu}, \|u_0\|_{L^\infty(\mathbb{R}_+)}) \|u_0\|_{L^2(\mathbb{R}_+)}.$$

(vi) Now consider two classical solutions u^N and v^N of the bounded-interval problem (1.1) on $(0,T) \times \Omega_N$ with initial data u_0 and v_0 and homogeneous Dirichlet boundary conditions. Subtraction of the equations for u^N and v^N,

multiplication the difference on $u^N - v^N$, and integration both in the space and time variable yields

$$\begin{aligned}\int_s^t \int_{\Omega_N} (F(u_x^N) - F(v_x^N))(u_x^N - v_x^N)\,dx\,d\tau &\le \frac{1}{2}\|u^N(s) - v^N(s)\|^2_{L^2(\Omega_N)} + \\ &\quad + H\int_s^t \int_{\Omega_N} |u^N - v^N|^2\,dx\,d\tau \\ &\le C\|u^N(s) - v^N(s)\|^2_{L^2(\Omega_N)}\end{aligned}$$

where $C = C(\frac{1}{s}, T, \overline{\mu}, \underline{\mu}, \|u_0\|_{L^\infty(\mathbb{R}_+)})$ (we shall agree on having on mind this identity whenever we write constant C in this proof). Consequently,

$$\int_s^t \int_{\Omega_N} |u_x^N - v_x^N|^2\,dx\,d\tau \le C\|u^N(s) - v^N(s)\|^2_{L^2(\Omega_N)}, 0 \le s \le t \le T. \quad (3.9)$$

Setting $v_0^N = u^N(\eta)$ for $\eta > 0$ sufficiently small, dividing the equation by η and letting η to zero one obtains that boundedness of $\|u_t^N(s)\|_{L^2(\Omega_N)}$ implies boundedness of u_t^N in $L^2((s,t); W_0^{1,2}(\Omega_N))$ for $0 \le s < t \le T$, more precisely

$$\int_s^t \int_{\Omega_N} |u_{tx}^N|^2\,dx\,d\tau \le C\|u_t^N(s)\|^2_{L^2(\Omega_N)}.$$

Because of $u_t \in L^2((s,T); L^2(\mathbb{R}_+))$ for any $0 < s < T$, we can apply the Chebyshev inequality

$$|\{\tau \in (\frac{s}{2}, s) : \|u_t^N(\tau)\|^2_{L^2(\mathbb{R}_+)} \ge K\}| \le \frac{1}{K}\int_{\frac{s}{2}}^s \|u_t^N(\tau)\|^2_{L^2(\mathbb{R}_+)}\,d\tau$$

to obtain existence of $\xi \in (\frac{s}{2}, s)$ such that

$$\|u_t^N(\xi)\|_{L^2(\mathbb{R}_+)} \le \sqrt{\frac{4}{s}}\|u_t^N\|_{L^2((\frac{s}{2},T);L^2(\mathbb{R}_+))} \le C\|u_0\|_{L^2(\mathbb{R}_+)}.$$

Thus we have

$$\int_s^t \|u_{tx}^N\|^2_{L^2(\mathbb{R}_+)}\,dt \le \int_\xi^t \|u_{tx}^N\|^2_{L^2(\mathbb{R}_+)}\,dt \le C\|u_0\|^2_{L^2(\mathbb{R}_+)}.$$

and the boundedness of u^N in $W^{1,2}((s,t); W_0^{1,2}(\mathbb{R}_+))$ for $0 < s < t \le T$ follows.

Now consider the problem (1.1) differentiated with respect to the time variable. Writing v in place of u_t^N we obtain

$$\left.\begin{aligned} v_t - F''(u_x^N)u_{xx}^N v_x - F'(u_x^N)v_{xx} + h'(u^N)v = 0 \\ v(t,0) = v(t,N) = 0, 0 < t < T \\ v(0,x) = u_t^N(0,x) := F'((u_0)_x)(u_0)_{xx} - h(u_0) \end{aligned}\right\} \quad (3.10)$$

Multiplying (3.10) on $\varphi \in \mathcal{D}((s,T) \times \Omega_N)$ and integrating both in the space and time variable we obtain, by virtue of the estimates obtained above, the estimate on v_t in the sense of distributions

$$\begin{aligned} |\langle v_t, \varphi\rangle| &= \left| \int_s^t \int_{\Omega_N} (F''(u_x)u_{xx}v_x\varphi + F'(u_x)v_{xx}\varphi + h'(u)v\varphi)\, dx\, d\tau \right| \\ &\le \int_s^t \int_{\Omega_N} F'(u_x)|v_x\varphi_x|\, dx\, d\tau + H \int_s^t \int_{\Omega_N} |v| \cdot |\varphi|\, dx\, d\tau \\ &\le \overline{\mu} \|v\|_{W^{1,2}((s,t);W_0^{1,2}(\Omega_N))} \|\varphi\|_{L^2((s,t);W^{1,2}(\Omega_N))} + \\ &\quad + H\|v\|_{L^2((s,t);L^2(\Omega_N))} |\varphi\|_{L^2((s,t);L^2(\Omega_N))} \\ &\le C\|u_0\|_{L^2(\Omega_N)} \|\varphi\|_{L^2((s,t);W_0^{1,2}(\Omega_N))}. \end{aligned}$$

This means that v_t is a bounded linear form on the space $L^2((s,t);W_0^{1,2}(\Omega_N))$ for any $0 < s < t \le T$, therefore, it can be identified with an element belonging to the class $L^2((s,t);W^{-1,2}(\Omega_N))$.
Multiplying the equation in (3.10) on $u_t^N = v$ and integrating over $(s,t)\times\Omega_N$ yields

$$\begin{aligned} \|v(t)\|^2_{L^2(\mathbb{R}_+)} - \|v(s)\|^2_{L^2(\Omega_N)} &= -2\int_s^t \int_{\Omega_N} F'(u_x^N)v_x^2\, dx\, d\tau - \\ &\quad -2\int_s^t \int_{\Omega_N} h'(u^N)v^2\, dx\, d\tau, \end{aligned}$$

and thus

$$\|v(t)\|^2_{L^2(\mathbb{R}_+)} \le \|v(s)\|^2_{L^2(\Omega_N)} + 2H \int_s^t \|v\|^2_{L^2(\mathbb{R}_+)}\, d\tau.$$

Consequently, by virtue of the Gronwall lemma, it follows

$$\|u_t^N(t)\|_{L^2(\mathbb{R}_+)} \le \|u_t^N(s)\|_{L^2(\mathbb{R}_+)} e^{H(t-s)}, \quad 0 < s < t < T.$$

By virtue of boundedness of u_{tt}^N in $L^2((s,t);W^{-1,2}(\Omega_N))$, and boundedness of u^N in $W^{1,2}((s,t);W_0^{1,2}(\mathbb{R}_+))$, we conclude, with the aid of Proposition A.4, that $u^N \in C^1([s,t]; L^2(\mathbb{R}_+)) \cap C([s,t]; W_0^{1,2}(\mathbb{R}_+))$ for any $0 < s < t < T$; therefore,

$$u^N \in C^1((0,T); L^2(\mathbb{R}_+)) \cap C((0,T); W_0^{1,2}(\mathbb{R}_+)).$$

□

Remark 3.4. Note that we have proved the lemma above under rather restrictive assumption (3.2) that makes the solutions starting from L^∞-bounded initial data uniformly bounded with respect to the time variable. However, if we do not assume (3.2), then the only thing that changes is more significant dependence of the Lipschitz constant H on the structural properties of h. More precisely, the solution with a blow-up at finite time may occur. Thus we have to take into account also the possible explosion of H to infinity. However, denoting the maximum existence interval of the solution u by T_{max}, we can see that the conclusions of the lemma above hold for u when they are considered over any (fixed) interval $(0,T] \subset (0, T_{max})$.

3.5 Existence of Solutions

In this part, we are going to establish the existence of a solution to (1.1), (1.2) and (1.3). The idea is to start with a solution u^N of

$$\left.\begin{aligned} u_t^N(t,x) - (F(u_x^N(t,x))_x + h(u^N(t,x)) &= 0,\ t \in (0,T) \times \Omega_N \\ u^N(0,x) &= u_0(x),\ x \in \Omega_N \\ u^N(t,x) &= 0,\ (t,x) \in (0,T) \times \partial\Omega_N \end{aligned}\right\} \tag{3.11}$$

where the initial datum $u_0 \in \mathcal{D}(\Omega_N)$ for certain N and Ω_N is a bounded interval, say $\Omega_N = (0, N)$. While letting N to infinity we show the convergence of a certain subsequence of $\{u^N\}_N$ to some function u and verify that it solves (1.1), (1.2) and (1.3). Consequently, we show existence of a solution for a general initial datum by passing to the limit while approximating the given initial datum by smooth functions. In the end we show that this solution is classical with generalized initial condition.

The whole proof can be done in several steps:

Step 1

Suppose $\operatorname{supp} u_0 \in \mathcal{D}(\Omega_N)$ for certain N. By virtue of Remark 3.3 and Theorem 6.1 of Chapter 5 by Ladyženskaja et al. [11], the problem (3.11) posseses a unique classical solution u^N defined on some maximal time interval $[0, T_{max}^N)$, the length of which can be estimated from below by a (positive) constant depending only on $\|u_0\|_{L^\infty(\Omega_N)}$ and the structural properties of h.

Moreover, employing the notation of Remark 3.3, we recall that

$$\underline{u}(t) \le u^N(t,x) \le \overline{u}(t) \quad \text{for all } t \in [0,T],\ T < T^N_{max},$$

which means that under the assumption (3.2) we have $T^N_{max} = \infty$ and u^N is uniformly bounded on $[0,\infty)$ by a constant depending only on the structural properties of h and $\|u_0\|_{L^\infty(\Omega_N)}$ whenever u_0 is non-negative.

Step 2

Extend the function u^N, u^N_x, u^N_{xx} and u^N_t to be zero outside Ω_N. Since the initial datum $u_0 \in \mathcal{D}(\Omega_N)$ is sufficiently regular, we obtain, by virtue of Proposition 3.2 and Theorem 3.1, uniform boundedness of $\{u^N\}$, $\{u^N_x\}$, $\{u^N_{xx}\}$ and $\{u^N_t\}$ in the space of Hölder continuous functions with some exponent $\alpha > 0$. By virtue of the Arzelá–Ascoli theorem, this yields a subsequence (not relabeled) converging uniformly over any compact $[0,T] \times Q \subset [0,T] \times \overline{\mathbb{R}}_+$. Denote the limit of u^N by u; since the convergence of all the sequences mentioned above is locally uniform, the derivatives of u match with the limits of the derivatives and we can write

$$\underbrace{u^N_t(t,x)}_{\to u_t(t,x)} - \underbrace{F'(u^N_x(t,x))u^N_{xx}(t,x)}_{\to F'(u_x(t,x))u_{xx}(t,x)} + \underbrace{h(u^N(t,x))}_{\to h(u(t,x))} = 0.$$

So, u satisfies (1.1) pointwise and is the classical solution of (1.1), (1.2) and (1.3).

At this stage, we have to verify that u is also integrable. Indeed, since $\{u^N\}_N$ is uniformly bounded in $L^2((0,T);W^{1,2}_0(\mathbb{R}_+))$ by virtue of Lemma 3.2, and so does $\{u^N_t\}_N$ in $L^2((0,T);W^{-1,2}(\mathbb{R}_+))$, existence of a subsequence converging weakly in $L^2((0,T);W^{1,2}_0(\mathbb{R}_+))$ to some function $\tilde{u}$ follows. Moreover, Proposition A.4 yields a subsequence of $\{u^N\}_N$ (not relabeled) converging in $L^2((0,T) \times Q)$ to $\tilde{u}$ for any compact $Q \subset \overline{\mathbb{R}}_+$. Since the convergence in L^p spaces implies convergence almost everywhere, we obtain that u and $\tilde{u}$ coincide and thus u is both regular and integrable.

Step 3

In the previous step we have shown existence of an unbounded-interval solution to the initial datum u_0 lying in $\mathcal{D}(\mathbb{R}_+)$. Now, we are interested in in the existence of solutions to the initial data from a wider class, namely $L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$.

Consider a function $u_0 \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$. By the density argument, there exists a sequence of functions u^N_0 lying in $\mathcal{D}(\mathbb{R}_+)$ such that u^N_0 converge to u_0 in $L^2(\mathbb{R}_+)$ and $\|u^N_0\|_{L^\infty(\Omega_N)}$ is bounded. Denote the solutions corresponding to u^N_0 by u^N. Uniform boundedness of u^N in $L^2((0,T);W^{1,2}_0(\mathbb{R}_+))$ implies boundedness of u^N_t is $L^2((0,T);W^{-1,2}(\mathbb{R}_+))$ and thus there exists a subsequence of $\{u^N_t\}_N$ (not relabeled) converging weakly to u_t. Observe that the estimate (3.9) can be obtained by the same techniques also for the case of $\mathbb{R}_+$. Taking into account also (2.4) we obtain that $\{u^N\}$ is a Cauchy sequence in $L^2((0,T);W_0^{1,2}(\mathbb{R}_+))$ and also in $C([0,T];L^2(\mathbb{R}_+))$. Denote the limit by u, then the limit passage writes for $\varphi \in \mathcal{D}([0,T]\times\mathbb{R}_+)$ as follows

$$
\begin{aligned}
0 \;=\; & -\underbrace{\int_0^T\int_{\mathbb{R}_+} u^N\varphi_t\,dx\,dt}_{\to\int_0^T\int_{\mathbb{R}_+} u\varphi_t\,dx\,dt} + \\
& + \underbrace{\int_0^T u^N(T)\varphi(T)\,dx}_{\to\int_{\mathbb{R}_+} u(T)\varphi(T)\,dx} - \underbrace{\int_0^T u_0\varphi(0)\,dx}_{\to\int_{\mathbb{R}_+} u_0\varphi(0)\,dx} + \\
& + \underbrace{\int_0^T\int_{\mathbb{R}_+} F(u_x^N)\varphi_x\,dx\,dt}_{\to\int_0^T\int_{\mathbb{R}_+} F(u_x)_x\varphi_x\,dx\,dt} + \underbrace{\int_0^T\int_{\mathbb{R}_+} h(u^N)\varphi\,dx\,dt}_{\to\int_0^T\int_{\mathbb{R}_+} h(u)\varphi\,dx\,dt}
\end{aligned}
$$

Thus u is a weak solution corresponding to $u_0 \in L^2(\mathbb{R}_+)\cap L^\infty(\mathbb{R}_+)$. Moreover, we claim that u is classical with generalized initial condition. Indeed, consider any $\varepsilon > 0$; by virtue of Remark 3.3 and Theorem 3.1 we obtain that $\{u^N(\varepsilon)\}_N$ is uniformly bounded in the Hölderian norm with some exponent α, and so do the first time and second spatial derivative. Employing the Arzelá–Ascoli theorem, similarly as it was done in *Step 2*, yields a subsequence converging to function $\tilde{u}$. By similar arguments as those used in *Step 2*, u and $\tilde{u}$ coincide, u solves (1.1) pointwise and is locally Hölder continuous along with its derivatives u_t, u_x and u_{xx}.

Step 4

In the previous steps we have infered existence of an unbounded-interval solution and shown the regularity concerning classical properties such as Hölder continuity of the solution and its derivatives. The question is, whether also some integral-type estimates for bounded-interval solutions in Lemma 3.2 preserve the limit passage. Sure, the estimates concerning norms in reflexive spaces are preserved simply by virtue of the weak lower semicontinuity. However, we have also infered some estimates concerning norms in spaces of Bochner integrable functions, where the 'outer' space is not reflexive, e.g., the estimate (3.6) for controllability of $\|u(t)\|_{W_0^{1,2}(\mathbb{R}_+)}$ in terms of t, $1/t$ and $\|u_0\|_{L^2(\mathbb{R}_+)}$ for $0 < t < \infty$, which means, in fact, the boundedness in $L^\infty((s,t);W_0^{1,2}(\mathbb{R}_+))$ for any $0 < s < t < \infty$. A simple argument shows that the estimates of this type preserve the limit passage too. We shall show the technique on one example, e.g. the one recalled above, since the others

can be maintained by the similar procedure.
Consider the sequence of solutions u^N converging to u weakly in the space $L^2((0,T);W_0^{1,2}(\mathbb{R}_+))$. By virtue of the Aubin lemma (Proposition A.1), we obtain that u^N converges to u in $L^2((0,T)\times Q)$ for any compact $Q\subset\mathbb{R}_+$. In view of the fact that convergence in L^2 implies pointwise convergence almost everywhere, and the Fubini theorem, we obtain for almost every $t\in(0,T)$ that the sequence $u^N(t)$ converges to $u(t)$ in $L^2(Q)$. Take such a $t\in(0,T)$, then from boundedness in the sense of (3.6) and reflexivity of $W_0^{1,2}(\mathbb{R}_+)$, we obtain existence of a subsequence u^{N_k} converging weakly in $W_0^{1,2}(\mathbb{R}_+)$ to some function $\tilde{u}\in W_0^{1,2}(\mathbb{R}_+)$. The weak lower semicontinuity of the norm yields the estimate (3.6) is valid for $\tilde{u}$ too. As u^N already converges to u, functions $\tilde{u}$ and u must coincide, thus the estimate (3.6) preserves the limit passage.

Step 5

As we have claimed before, the assumption (3.2) makes the solution with L^∞-bounded initial datum to stay L^∞-bounded uniformly on the whole existence interval. We claim even more, if u is a solution starting from $u_0\in L^2(\mathbb{R}_+)\cap L^\infty(\mathbb{R}_+)$, then

$$\limsup_{t\to\infty}\|u(t)\|_{L^\infty(\mathbb{R}_+)}\le s_0. \tag{3.12}$$

Note that, without loss of generality, we can assume that $h(s_0)=0$.

Consider the situation at time $t=1$. By virtue of the previous parts, the function $u(t)\in C_0(\mathbb{R}_+)$, and thus there exist $0<a<b<\infty$ such that $u(1,x)\le s_0$ for $x\notin(a,b)$. From boundedness of $u(1)$ in $L^\infty(\mathbb{R}_+)$ there exists a smooth function v_1 such that v_1-s_0 has a compact support in $\mathbb{R}_+$ and $u(1,x)\le v_1(x)$ for any $x\in\mathbb{R}_+$. Denote by $\tilde{v}$ the solution of (1.1) satisfying Dirichlet boundary conditions $\tilde{v}(t,0)=s_0$ and $\lim_{x\to\infty}\tilde{v}(t,x)=s_0$, and initial condition at time $t=1$ $\tilde{v}(1)=v_1$. Clearly, the function $\tilde{v}$ is a lifting by s_0 of a solution to $\dot{v}^*(t)-F(v^*_x)_x+h^*(v^*)=0$ where $h^*(y)=h(y+s_0)$, thus, by virtue of previous parts, the function $\tilde{v}-s_0$ lies in $C((1,\infty);W_0^{1,2}(\mathbb{R}_+))$. Moreover, one can observe that the conclusion of Comparison principle in Lemma 3.1 holds for comparation of $\tilde{v}$ and u, and thus $\tilde{v}\ge u$ on $(1,\infty)\times\mathbb{R}_+$. Non-negativity of h on $[s_0,\infty)$, together with boundedness of $\tilde{v}$ from below by s_0 yields $\tilde{v}_t-F(\tilde{v}_x)_x\le 0$, i.e. $\tilde{v}$ is a subsolution. Denote by v the solution of $v_t-F(v_x)_x=0$ with the same initial and boundary conditions as those imposed on $\tilde{v}$; then the same procedure as that done on $\tilde{v}$ follows that $v-s_0\in C((1,\infty);W_0^{1,2}(\mathbb{R}_+))$. Observing the conclusion of Lemma 3.1 holds for comparing $\tilde{v}$ and v, we obtain that it suffices to show

$$\limsup_{t\to\infty}\|v(t)\|_{L^\infty(\mathbb{R}_+)}\le s_0.$$

Multiplying the equation for v on $v - s_0$, integrating by parts in the space variable, employing (2.1) and using the estimate on the norm of $L^2(\mathbb{R}_+)$ in terms of the norm of $W_0^{1,2}(\mathbb{R}_+)$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|v(t) - s_0\|^2_{L^2(\mathbb{R}_+)} \leq -\underline{\mu}\|v_x(t)\|^2_{L^2(\mathbb{R}_+)} \leq -C\|v(t) - s_0\|^2_{L^2(\mathbb{R}_+)}, C > 0.$$

Thus

$$\limsup_{t\to\infty} \|v(t) - s_0\|_{L^2(\mathbb{R}_+)} = 0.$$

By virtue of

$$\frac{1}{2}\left(\|v(0) - s_0\|^2_{L^2(\mathbb{R}_+)} - \|v(t) - s_0\|^2_{L^2(\mathbb{R}_+)}\right) \geq \underline{\mu}\int_0^t \|v_x(t)\|^2_{L^2(\mathbb{R}_+)}\,dx$$

we infer that there exists a sequence τ^n converging to infinity such that $|\tau^n - \tau^{n+1}| < 2$ and

$$\lim_{n\to\infty} \|v_x(\tau^n)\|_{L^2(\mathbb{R}_+)} = 0.$$

Multiplying the equation for v by $F(v_x)_x$, integrating it both in the space and time variable, and using (2.1) we derive

$$\|v_x(t)\|_{L^2(\mathbb{R}_+)} \leq C\|v_x(s)\|_{L^2(\mathbb{R}_+)},$$

where C depends solely on $\underline{\mu}$ and $\overline{\mu}$. Consequently, we obtain the convergence of $\|v_x(t)\|_{L^2(\mathbb{R}_+)}$ to zero for $t \to \infty$, and thus, due to the imbedding of $W^{1,2}(\mathbb{R}_+)$ into $L^\infty(\mathbb{R}_+)$, we have that

$$\lim_{t\to\infty} \|v(t) - s_0\|_{L^\infty(\mathbb{R}_+)} = 0.$$

Collecting together all the steps done above we obtain the following theorem.

Theorem 3.2. *Let $F \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$ and suppose that (2.1), (2.3) and (3.2) hold. Furthermore, suppose that*

$$u_0 \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+), u_0 \geq 0.$$

Then the following holds

(i) *The problem (1.1), (1.2) and (1.3) possesses a unique weak solution in the class*

$$u \in L^2((0,T); W^{1,2}(\mathbb{R}_+)) \cap L^\infty((0,T) \times \mathbb{R}_+),$$

for any $0 < T < \infty$, defined on time interval $(0, \infty)$.

(ii) *Moreover, the solution belongs to*

$$C([0,\infty); L^2(\mathbb{R}_+)) \cap C^1((0,\infty); L^2(\mathbb{R}_+)) \cap W^{1,2}((s,t); W_0^{1,2}(\mathbb{R}_+))$$

for any $0 < s < t < \infty$.

(iii) *The solution is regular on any compact* $Q \subset (0,\infty) \times \overline{\mathbb{R}}_+$, *more specifically,* u_t, u_x *and* u_{xx} *are Hölder continous on* Q *with the Hölderian norm bounded in terms of*

$$\|u_0\|_{L^\infty(\mathbb{R}_+)}, \text{ and } \inf\{t : (t,x) \in Q\}.$$

(iv) *If, moreover,* u_0, u_{0x} *and* u_{0xx} *are Hölder continuous on* $\overline{\mathbb{R}}_+$, *then* u_t, u_x *and* u_{xx} *are Hölder continuous on* $[0,T] \times \overline{\mathbb{R}}_+$ *for any* $0 < T < \infty$ *with the corresponding Hölder norm depending solely on that of* u_0 *and the structural properties of* h.

(v) *Furthermore, the following estimates hold*

$$\|u(t)\|_{L^2(\mathbb{R}_+)} \leq e^{C_2(t-s)}\|u(s)\|_{L^2(\mathbb{R}_+)}, \tag{3.13}$$

$$\|u(t)\|_{W_0^{1,2}(\mathbb{R}_+)} \leq C_1 e^{C_2(t-s)}\frac{1}{\sqrt{t}}\|u(s)\|_{W_0^{1,2}(\mathbb{R}_+)}, \tag{3.14}$$

$$\|u_t(t)\|_{L^2(\mathbb{R}_+)} \leq C_1 e^{C_2(t-s)}\|u_t(s)\|_{L^2(\mathbb{R}_+)}, \tag{3.15}$$

$$\|u_t(t)\|_{L^2(\mathbb{R}_+)} \leq C_1 e^{C_2 t}\frac{1}{t}\|u_0\|_{L^2(\mathbb{R}_+)}, \tag{3.16}$$

where $0 < s < t < \infty$ *and* C_1 *and* C_2 *are constants depending only on* $\overline{\mu}$, $\underline{\mu}$, $\|u_0\|_{L^\infty(\mathbb{R}_+)}$, *and* h.

(vi) $\limsup_{t\to\infty} \|u(t)\|_{L^\infty(\mathbb{R}_+)} \leq s_0$

Remark 3.5. If we do not assume (3.2), then the theorem above applies as well under a slight modification that the solutions have guaranteed their existence only on some time interval $(0, T_{max})$ which (positive) length can be estimated from below by virtue of Remark 3.3. In this case, all the statements of the theorem above (except for *(vi)*) stay valid in the sense that they are considered only for time intervals $(0,T) \subset [0,T] \subset [0,T_{max})$. If the situation $T_{max} < \infty$ occurs, then we have, moreover,

$$\lim_{t\to T_{max}-} \|u(t)\|_{L^\infty(\mathbb{R}_+)} = \infty.$$

Indeed, suppose the contrary, i.e., there exists a sequence $\{t_n\}$, $t_n < T_{max}$ and $\lim t_n = T_{max}$ such that $\|u(t_n)\|_{L^\infty(\mathbb{R}_+)} \leq c < \infty$. Let $\Delta > 0$ be an estimate from below on the length of the existence interval for a solution with inital datum u_0, $\|u_0\|_{L^\infty(\mathbb{R}_+)} \leq c$. Since $t_n \to T_{max}$, there exists t_n such that $T_{max} - t_n < \Delta/2$, $u(t_n) \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, so the solution u exists at least till time $t_n + \Delta > T_{max}$ which contradicts the definition of T_{max}.

3.6 Energy

In the study of the asymptotic behaviour of dynamical systems it comes useful to have a function that characterizes (in some sense) the behaviour of some of system's variables and has certain known (or, rather, prescribed) properties. It turns out that in the case of a dynamical system describing some physical model with dissipation of energy, the right choice is just the function corresponding to the energy. This function is non-increasing along any trajectory and expresses the relation between the basic variables characterizing the system's state. This observation of relation between the behaviour of a dynamical system describing a physical model, and the energy leads to a notion of a *Lyapunov function* (sometimes called energy) of the system, i.e. a continuous function which is non-increasing along any trajectory of the dynamical system.
The right question at this stage is, whether the dynamical system corresponding to (1.1), (1.2) and (1.3) admits such a Lyapunov function, and, if it has some additional interesting properties. The following lemma answers to both these questions.

Lemma 3.3. *Let F be twice continuously differentiable and let h be continuously differentiable on $\mathbb{R}$ such that (2.1), (2.3) and (3.2) hold.*
Then the problem (1.1), (1.2), (1.3) admits a Lyapunov function (energy functional)

$$E(v) = \int_{\mathbb{R}_+} (I(v_x) + H(v))\, dx, \quad v \in W_0^{1,2}(\mathbb{R}_+), \tag{3.17}$$

where $I(t) = \int_0^t F(s)\, dx$ and $H(t) = \int_0^t h(s)\, ds$.
Moreover, for any non-negative initial datum $u_0 \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ with u being the corresponding solution it holds

$$\frac{d}{dt} E(u(t)) = -\|u_t(t)\|^2_{L^2(\mathbb{R}_+)}, \quad t > 0. \tag{3.18}$$

Proof. Since, by virtue of Theorem 3.2, $u_t(t) \in L^2((s,t); W_0^{1,2}(\mathbb{R}_+))$ for any $0 < s < t < \infty$, we can consider it as a test function. Multiplying u_t on (1.1), integrating over $(s,t) \times \mathbb{R}_+$ and employing the Fubini theorem together with the by parts integration formula yields

$$\begin{aligned}
\int_s^t \int_{\mathbb{R}_+} |u_t|^2 \, d\tau \, dx &= -\int_s^t \int_{\mathbb{R}_+} F(u_x) u_{xt} \, dx \, d\tau - \int_s^t \int_{\mathbb{R}_+} h(u) u_t \, dx \, d\tau \\
&= -\int_{\mathbb{R}_+} (I(u_x(t)) - I(u_x(s))) \, dx - \\
&\quad - \int_{\mathbb{R}_+} (H(u(t)) - H(u(s))) \, dx \\
&= -(E(u(t)) - E(u(s))),
\end{aligned}$$

where $I(t) = \int_0^t F(s) \, ds$, $H(t) = \int_0^t h(s) \, ds$. Accordingly, by $\|u_t\|_{L^2(\mathbb{R}_+)} \in L^2((s,t))$ for any $0 < s < t < \infty$, the function $t \mapsto E(u(t))$ is absolutely continuous, hence it is differentiable almost everywhere, and

$$\frac{d}{dt} E(u(t)) = -\|u_t(t)\|^2_{L^2(\mathbb{R}_+)} \text{ for a.a. } 0 < t < \infty.$$

However, the continuity of the right-hand side yields continuous differentiability of the mapping

$$t \mapsto E(u(t))$$

at any point $t > 0$. □

4 Ground State Solutions

In this section we study classical solutions of the stationary problem

$$\left.\begin{aligned} -F(w_x)_x + h(w) &= 0, \ x \in \mathbb{R} \\ w(0) &= \max_{x\in\mathbb{R}} w(x) > 0 \\ \lim_{|x|\to\infty} w(x) &= 0. \end{aligned}\right\} \tag{4.1}$$

The (unique) solution to this problem will be called the *ground state solution.* As we shall see bellow, the necessary and sufficient condition for the ground state solution's existence is the following assumption on h:

$$\zeta_0 := \inf\{s > 0 : H(s) \leq 0\} > 0, \quad h(\zeta_0) < 0 \tag{4.2}$$

where $H(s) = \int_0^s h(t)\, dt$.

Theorem 4.1. *Let $F \in C^2(\mathbb{R})$ and $h \in C^1(\overline{\mathbb{R}}_+)$ be such that hypotheses (2.1), (2.3) and (4.2) are satisfied. Then the problem (4.1) admits a unique classical solution w_g. Moreover,*

$$\begin{aligned} w_g(0) &= \zeta_0 && (4.3)\\ w_g(x) &> 0, \quad x \in \mathbb{R} && (4.4)\\ w_g'(x) &> 0, \quad x < 0 && (4.5)\\ w_g'(x) &< 0, \quad x > 0 && (4.6)\\ w_g &\in W^{2,2}(\mathbb{R}) && (4.7) \end{aligned}$$

Furthermore, if F is odd, *then*

$$w_g(x) = w_g(-x), \quad x \in \mathbb{R} \tag{4.8}$$

Proof. Without loss of generality, we can consider that the function h is extended to be continuously differentiable on $\mathbb{R}$. Multiplying the equation in (4.1) by w_x, integrating from 0 to x and denoting $S(t) := \int_0^t F'(s)s\, ds$ we obtain

$$S(w_x(x)) - S(w_x(0)) = H(w(x)) - H(w(0)), \quad x \in \mathbb{R}.$$

Because of $H(w(0)) = H(\zeta_0) = 0$ and $S(w_x(0)) = S(0) = 0$, the equation takes the form

$$S(w_x(x)) = H(w(x)), \quad x \in \mathbb{R}. \tag{4.9}$$

Since $S'(x) = F'(x)x$ is positive for positive arguments and negative for the negative ones, we obtain the restrictions $S_+ := S|_{[0,\infty)}$ and $S_- := S|_{(-\infty,0]}$ are injective, therefore invertible. Moreover, the estimates

$$\frac{\underline{\mu}}{2}t^2 \leq S(t) \leq \frac{\overline{\mu}}{2}t^2, \quad t \in \mathbb{R} \tag{4.10}$$

imply

$$\left.\begin{array}{rcl} \sqrt{\frac{2}{\overline{\mu}}s} & \leq S_+^{-1}(s) \leq & \sqrt{\frac{2}{\underline{\mu}}s} \\ -\sqrt{\frac{2}{\underline{\mu}}s} & \leq S_-^{-1}(s) \leq & -\sqrt{\frac{2}{\overline{\mu}}s} \end{array}, \quad s \in [0,\infty) \right\} \tag{4.11}$$

First, let us treat the solution for $x > 0$. From (4.9) we obtain

$$w_x(x) = S_-^{-1}(H(w(x)))$$

and so

$$\frac{w_x}{S_-^{-1}(H(w))} = 1, \ H(w) \neq 0; \quad w_x = 0, \ H(w) = 0. \tag{4.12}$$

As $w_1 = \zeta_0$ and $w_2 = 0$ are stationary solutions of (4.12), the proof of the third statement of (4.1) will be finished if we are able to prove the existence of a trajectory connecting w_1 and w_2 such that the connection to w_2 is realized 'at infinity'. For this purpose, it suffices to show that $\frac{1}{S_-^{-1}(H(s))}$ is integrable on the left neighbourhood of ζ_0 and is not integrable on the right neighbourhood of 0.

Let $x < \zeta_0$ be sufficiently close to ζ_0 such that $h(x) < -\varepsilon < 0$ still holds for some ε. Consequently, (4.11) yields

$$\begin{aligned} \frac{1}{|S_-^{-1}(H(x))|} &\leq \sqrt{\frac{\overline{\mu}}{2(H(x) - H(\zeta_0))}} \\ &\leq \sqrt{\frac{\overline{\mu}}{2\varepsilon}}\frac{1}{\sqrt{\zeta_0 - x}} \end{aligned}$$

which is integrable at ζ_0.

On the other hand, since h is continuously differentiable and $h'(0) > 0$, we can take $x > 0$ close to 0 such that $0 < h'(x) < \varepsilon$ still holds for some $\varepsilon > 0$. Thus we obtain

$$0 \leq H(x) \leq \frac{\varepsilon}{2}x^2,$$

and in combination with (4.11) we get similarly to the previous part

$$\begin{aligned} \frac{1}{|S_-^{-1}(H(x))|} &\geq \sqrt{\frac{\underline{\mu}}{2H(x)}} \\ &\geq \frac{1}{x}\sqrt{\frac{\underline{\mu}}{\varepsilon}} \end{aligned}$$

which is *not* integrable at zero.
The construction of a solution for $x < 0$ can be done analogously. What remains is to prove that the constructed solution is classical, i.e. $w \in C^2(\mathbb{R})$, and that it belongs to $W^{2,2}(\mathbb{R})$. However, continuity of the second derivative of w follows from continuity of w, w_x and (2.1). To show that $w \in W^{2,2}(\mathbb{R})$, it suffices to prove $w, w_x \in L^2(\mathbb{R})$; then the integrability of $|w|^2$ on $\mathbb{R}$ and continuous differentiability of h yield $w_{xx} \in L^2(\mathbb{R})$ which ends the proof of the existence and qualitative part of the statement. Since $h'(0) > 0$, there exist $\varepsilon_+, \varepsilon_-$ and δ positive such that $\varepsilon_- \geq h'(x) \geq \varepsilon_+$ for $0 \leq x \leq \delta$. Consequently,

$$\frac{\varepsilon_-}{2}s^2 \geq H(s) = \int_0^s h(t)\,dt \geq \frac{\varepsilon_+}{2}s^2, \quad s \leq \delta.$$

By $\lim_{|x|\to\infty} w(x) = 0$, we obtain existence of $R > 0$ such that $w(x) < \delta$ for any $|x| \geq R$, which yields, with the aid of (4.11),

$$\left.\begin{array}{rlll} -\sqrt{\frac{\varepsilon_-}{\mu}}w(x) & \leq w_x(x) \leq & -\sqrt{\frac{\varepsilon_+}{\mu}}w(x), & x > R \\ \sqrt{\frac{\varepsilon_+}{\mu}}w(x) & \leq w_x(x) \leq & \sqrt{\frac{\varepsilon_-}{\mu}}w(x), & x < -R. \end{array}\right\} \tag{4.13}$$

Consequently,

$$\begin{array}{rcl} w(x) & \leq & C_0 e^{-C_+ x}, \quad x \geq R, \\ w(x) & \leq & C_0 e^{C_- x}, \quad x \leq -R, \\ w(x) & \leq & C_0, \quad |x| \leq R, \end{array}$$

and this implies boundedness of w in $L^2(\mathbb{R})$, and, due to (4.13), we have also $w_x \in L^2(\mathbb{R})$.
If we consider, moreover, F to be odd function, i.e. $F(-x) = -F(x)$, then the statement (4.8) follows from $S_-(-x) = S_+(x), x \geq 0$.

Assume now that functions u, v satisfy (4.1). First, let us show that

$$u(0) = v(0) = \zeta_0.$$

Indeed, if this is not the case, i.e. the function u solving (4.1) has $u(0) \neq \zeta_0$, we can distinguish the following cases:

- $h(u(0)) > 0$: continuity of h implies $u_{xx}(x) = \frac{h(u(x))}{F'(u_x(x))} > 0$ for x small enough, thus the function u is strictly convex on the neighbourhood of zero which is a contradiction with $u(0) = \max_{x\in\mathbb{R}} u(x)$.

- $h(u(0)) = 0$: from continuous differentiability of h we obtain, likewise in the case of the zero stationary solution, that $u(t) = u(0)$ is a stationary solution which is unattainable by any trajectory of a different solution. This contradicts $u(x) \to_{|x|\to\infty} 0$.

- $h(u(0)) < 0$: Denote

$$\tilde{w} = \inf\{s : H(t) \geq H(u(0)), t \in [s, u(0)]\}.$$

  In view of $h(u(0)) < 0$, it follows that $\tilde{w} < u(0)$. If $\tilde{w} = -\infty$, we have that the function u is decreasing and converges to $-\infty$, which is a contradiction. On the other hand, if $\tilde{w} \in \mathbb{R}$, continuity of h implies $h(\tilde{w}) \geq 0$; whence we can distinguish two more situations:

  - $h(\tilde{w}) = 0$: continuous differentiability of h implies, by similar arguments as in the part dealing with the unattainability of the zero stationary solution, that the function u converges for large x to $\tilde{w}$ and, since the function $x \mapsto \frac{1}{S_-^{-1}(H(x)-H(u(0))}$ is *not* integrable on the right neighbourhood of $\tilde{w}$, u cannot attain $\tilde{w}$. If $H(u(0)) \neq 0$, then $\tilde{w} \neq 0$ and this is a contradiction. On the other hand, from $H(u(0)) = 0$ we infer, by virtue of $u(0) > \zeta_0$ and $h(\tilde{w}) = 0$, that $\tilde{w} > \zeta_0 > 0$ which is a contradiction too.
  - $h(\tilde{w}) > 0$: similarly to the part dealing with the attainability of the stationary solution ζ_0 we have that the function

$$x \mapsto \frac{1}{S_-^{-1}(H(x) - H(u(0))}$$

    is integrable on the right neighbourhood of $\tilde{w}$ and so u attains $\tilde{w}$ for some $\tilde{x}$. From $u_{xx}(x) = \frac{h(u(x))}{F'(u_x(x))}$ we obtain strict convexity of u on the neighbourhood of $\tilde{x}$, and, consequently, u is increasing on the right neigbourhood of $\tilde{x}$. Finally, we obtain that u is oscillating between $u(0)$ and $\tilde{w}$ and so it cannot satisfy (4.1).

As we have just proved $u(0) = \zeta_0 = v(0)$ for any u, v solutions to (4.1), the only remaing part is to show that the solutions to the initial value problem

$$u_{xx} = \frac{h(u)}{F'(u_x)} \tag{4.14}$$

are determined uniquely by $u(0)$ and $u_x(0)$. To do this, it suffices to verify that the function $L : (r, s) \mapsto \frac{h(r)}{F'(s)}$ is locally Lipschitz continuous, and to recall on the basic ODE's theorems on existence and uniqueness of solutions. However, the local Lipschitz continuity of L follows from continuous differentiability and (2.1). Indeed

$$\frac{\partial L}{\partial s} = \frac{h'(s)}{F'(t)}\ ; \ \frac{\partial L}{\partial t} = -\frac{h(s)F''(t)}{(F'(t))^2}$$

are continuous functions, hence L is locally Lipschitz continuous and the solutions to (4.14) are determined uniquely by their initial data. $\square$

Example 4.1. Consider $F(x) = x$, $h(x) = x(1-x)$, then the straightforward computation yields $\zeta_0 = \frac{3}{2}$ and the ground state is given by

$$w_g(x) = \frac{3}{2}(1 - \tanh^2(\frac{x}{2\sqrt{2}})).$$

The graph of the concerned ground state is shown in Figure 1.

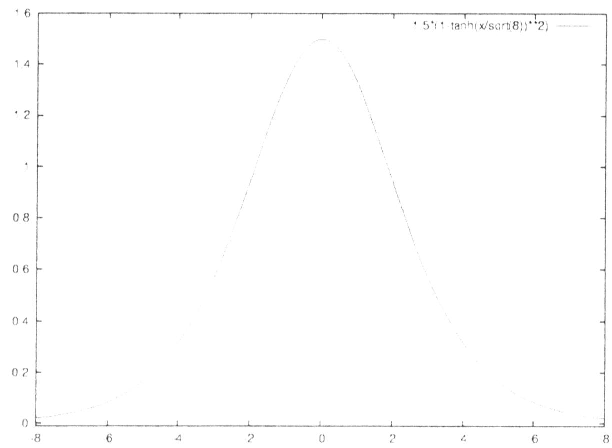

Figure 1: Ground state solution for $u_t - u_{xx} + u(1-u) = 0$

Remark 4.1. In Theorem 4.1 we have stated the sufficiency of the assumption (4.2) for the ground state's existence. Moreover, as it follows from the proof, this condition is also necessary. If this is not the case, i.e. $h(\zeta_0) \geq 0$, we can restrict ourselves to the case $h(\zeta_0) = 0$, as $h(\zeta_0) > 0$ contradicts, by continuity of h, the definition of ζ_0. In the case of $h(\zeta_0) = 0$ we obtain existence of a solution $u(x) = \zeta_0, x \in \mathbb{R}$ and, by virtue of the uniqueness of solutions to (4.14), no solution can satisfy (4.1).

Lemma 4.1. *Let $F \in C^2(\mathbb{R})$ and $h \in C^1(\overline{\mathbb{R}}_+)$ be such that the hypotheses (2.1), (2.3) and (4.2) are satisfied.*
Then for any $\varepsilon > 0$ sufficiently small and the solution w of

$$\left.\begin{aligned} -F(w_x)_x + h(w) &= 0 \\ w(0) &= \zeta_0 + \varepsilon \\ w_x(0) &= 0 \end{aligned}\right\} \tag{4.15}$$

there exist constants $L_- < 0 < L_+$ such that $w(L_-) = 0 = w(L_+)$.
Furthermore, if F is odd, *then $L_- = -L_+$.*

Proof. Multiplying the equation with w_x and integrating in the space variable yields, similarly to the first steps in the proof of Theorem 4.1,

$$S(w_x(x)) = H(w(x)) - H(\zeta_0 + \varepsilon).$$

Since $h(\zeta_0) < 0$, we can choose $\varepsilon > 0$ sufficiently small such that $h < 0$ on $[\zeta_0 - \varepsilon, \zeta_0 + \varepsilon]$. Consequently, we have $C_0 := -H(\zeta_0 + \varepsilon) > 0$.
Introducing the same definition of S_+ and S_- as in the proof of Theorem 4.1 we can write

$$\left.\begin{aligned} w_x(x) &= S_+^{-1}(H(w(x)) + C_0), \ x > 0 \\ w_x(x) &= S_-^{-1}(H(w(x)) + C_0), \ x < 0. \end{aligned}\right\} \tag{4.16}$$

Since this problem admits under the condition $w(0) = \zeta_0 + \varepsilon$ one stationary solution (which, in fact, does not solve (4.15)) and one solution increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. Because of $H(w) + C_0$ is positive for $w \in (-\delta, \zeta_0 + \varepsilon)$ with some $\delta > 0$, the estimates (4.11) estabilish integrability of $w \mapsto 1/S_\alpha^{-1}(H(w))$, $\alpha \in \{+, -\}$ on $[0, \zeta_0 + \varepsilon]$, so the solution *must* reach zero at finite x, i.e., there exist constants $L_- < 0 < L_+$ such that $w(L_-) = 0 = w(L_+)$. At last, if F is odd, the symetry of functions S_+ and S_- implies $L_- = -L_+$. □

Remark 4.2. The assumption (2.1) on F in Theorem 4.1 and in Lemma 4.1 can be waived in the sense that only existence of $\mu > 0$ such that

$$0 < \mu < F'(w), \quad w \in \mathbb{R}$$

is necessary. Indeed, as it follows from the construction of the solution to (4.1), the derivative of w_g can be estimated by virtue of (4.12) and (4.11) in the form

$$
\begin{aligned}
|w_x(x)| &\leq \sqrt{\frac{2}{\underline{\mu}}(H(w(x)) - H(w(0)))} \\
&\leq \sqrt{\frac{2}{\underline{\mu}}\sup\{H(s) - H(w(0)) : s \in [0, w(0)]\}}.
\end{aligned}
$$

Thus boundedness of w_x and continuous differentiability of F yields a constant ν such that F is Lipschitz continuous on the range of w_x with constant ν and the rest of the procedures in the proof follows.

5 Stability and Continuity

5.1 Stability of the Zero Solution

In this part we shall prove that the zero solution is stable. The crucial assumption will be

$$h'(0) > 0.$$

First of all, we introduce an auxiliary lemma.

Lemma 5.1. *Let φ be a nonnegative continuous function on $\overline{\mathbb{R}}_+$ such that*

$$\varphi(t) \leq \varphi(s) + b \int_s^t \varphi(\sigma)\, d\sigma \tag{5.1}$$

holds for some $b < 0$ and any $0 \leq s \leq t < \infty$.
Then

$$\varphi(t) \leq \varphi(s) e^{b(t-s)}, \quad 0 \leq s \leq t < \infty.$$

Proof. First of all, we observe that the function φ is non-increasing. Indeed, since $b < 0$ and $\varphi(\sigma) \geq 0$, the integral term in (5.1) is non-positive, and thus $\varphi(t) \leq \varphi(s)$ for any $0 \leq s \leq t < \infty$.
Moving the term $\varphi(s)$ to the left side of (5.1), dividing both sides by $t - s$, and letting $s \to t-$ yields

$$\varphi'(t) \leq b\varphi(t) \text{ for almost all } t > 0.$$

If $\varphi(t_0) = 0$, then, by (5.1), it follows that $\varphi(t) = 0$ for any $t > t_0$, and so the statement holds. Thus we can restrict ourselves to the case when $\varphi(t) > 0$. Dividing by $\varphi(t)$ one obtains

$$\frac{d}{dt} \ln(\varphi(t)) \leq b \quad \text{for almost all } t > 0;$$

whence from continuity of φ we finally obtain

$$\varphi(t) \leq \varphi(s) e^{b(t-s)} \quad \text{for any } 0 \leq s \leq t < \infty.$$

□

Theorem 5.1. *Suppose $F \in C^2(\mathbb{R})$ and $h \in C^1(\overline{\mathbb{R}}_+)$. Moreover, let the assumptions (2.1) and (2.3) be satisfied.*
Then the zero solution is locally asymptotically stable, i.e., there exists U – a $L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$-neighbourhood of $w_0 \equiv 0$ in the set of non-negative functions such that for any initial datum w belonging to U the corresponding solution to (1.1), (1.2) and (1.3) converges to zero in $W_0^{1,2}(\mathbb{R}_+)$ as $t \to \infty$.

Proof. From $h'(0) > 0$, and continuous differentiability of h, there exist $\varepsilon > 0$, and $\delta > 0$ such that $0 < \delta \leq h'(x)$ for any $x \in [0, \varepsilon]$. Let u be a solution to (1.1), (1.2), and (1.3) with the initial datum $u_0 \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ such that $\|u_0\|_{L^\infty(\mathbb{R}_+)} < \varepsilon$. By virtue of Comparison principle (Remark 3.3), $\|u(t)\|_{L^\infty(\mathbb{R}_+)} < \varepsilon$ for all $t \geq 0$. Multiplying (1.1) on $F(u_x)_x$, integrating over $(s,T) \times \mathbb{R}_+$, and employing the by parts integration formula for the space variable yields, by virtue of (2.1),

$$\begin{aligned} \int_{\mathbb{R}_+} I(u_x(T)) &\leq \int_{\mathbb{R}_+} I(u_x(s)) - \int_s^T \int_{\mathbb{R}_+} h'(u) F(u_x) u_x \, dx \, dt \\ &\leq \int_{\mathbb{R}_+} I(u_x(s)) - 2\delta \frac{\underline{\mu}}{\overline{\mu}} \int_s^T \int_{\mathbb{R}_+} I(u_x(t)) \, dt, \end{aligned}$$

and Lemma 5.1 implies with the aid of (3.14)

$$\|u_x(T)\|_{L^2(\mathbb{R}_+)} \leq C_1 \|u_0\|_{L^2(\mathbb{R}_+)} \frac{1}{\sqrt{s}} e^{C_2 s} e^{-C_3 T}, \quad 0 < s < T$$

where C_1, C_2 and C_3 are positive and depend only on $\overline{\mu}$, $\underline{\mu}$, h and $\|u_0^N\|_{L^\infty(\mathbb{R}_+)}$. Thus, by virtue of the imbedding of $W_0^{1,2}(\mathbb{R}_+)$ into $L^2(\mathbb{R}_+)$ we obtain

$$\|u(T)\|_{W_0^{1,2}(\mathbb{R}_+)} \leq C \|u_0\|_{L^2(\mathbb{R}_+)} e^{-C_3 T}, 1 < T < \infty; \tag{5.2}$$

whence the solutions starting from sufficiently small initial data converge to zero in $W_0^{1,2}(\mathbb{R}_+)$, and thus the zero solution is locally asymptotically stable. □

5.2 Continuous Dependence on the Initial Data

In this part we are going to prove that the solution semigroup of the problem (1.1) is a continuous mapping for any fixed time $t \geq 0$. Note that we have already shown the continuity of the semigroup in case of a mapping

$$L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$$

in Theorem 2.1 under the assumption that the solutions are locally bounded and exist. However, it is possible to enlarge this result, but before, we have to state an auxiliary lemma

Lemma 5.2. *Let $\varrho : \mathbb{R}_+ \to \mathbb{R}_+$ be a bounded, strictly positive, locally Lipschitz continuous function satisfying $|\varrho_x(x)| \leq C_\varrho \varrho$ for some constant $C_\varrho > 0$ and almost every $x \in \mathbb{R}_+$.*
Then the weighted spaces $L^2(\mathbb{R}_+, \varrho)$ and $W_0^{1,2}(\mathbb{R}_+, \varrho)$ are reflexive and separable.

Proof. To prove the lemma we have to modify several steps in paragraphs 2.15 and 3.14 by Adams [1].
First of all, we shall show that functions with compact support in $\mathbb{R}_+$ are dense in $L^2(\mathbb{R}_+, \varrho)$. Consider a function $u \in L^2(\mathbb{R}_+, \varrho)$. For given $\varepsilon > 0$ there exists $R > 0$ such that, under notation $I_R = (\frac{1}{R}, R)$ and $\overline{I}_R = [\frac{1}{R}, R]$, $||u||_{L^2(I_R,\varrho)} > ||u||_{L^2(\mathbb{R}_+,\varrho)} - \frac{\varepsilon}{2}$. Further, since $\overline{I}_R$ is a compact set, we have that there exist $\Delta > \delta > 0$ such that $\Delta \geq \varrho(x) \geq \delta$ for any $x \in \overline{I}_R$. Thus we have that the norms of $L^2(I_R, \varrho)$ and $L^2(I_R)$ are equivalent, and, by density of continuous functions with compact support in L^p spaces, there exists a function $\varphi \in C_C(I_R)$ such that

$$||u - \varphi||_{L^2(I_R,\varrho)} \leq \Delta ||u - \varphi||_{L^2(I_R)} \leq \frac{\varepsilon}{2}. \tag{5.3}$$

Consider the set of all polynomials with rational coefficients and denote it by $\mathcal{P}$. This set is countable and, by virtue of the Weierstrass theorem, the set $\mathcal{P}_R = \{\chi_{\overline{I}_R} u : u \in \mathcal{P}\}$ is dense in $C_C(\overline{I}_R)$. Thus, for given $\varphi \in C_C(\overline{I}_R)$ and $\varepsilon > 0$ there exists $p \in \mathcal{P}_R$ such that

$$||p - \varphi||_{L^2(I_R,\varrho)} \leq \Delta R ||p - \varphi||_{L^\infty(I_R)} \leq \frac{\varepsilon}{2}. \tag{5.4}$$

Combining (5.3) and (5.4) we get

$$||u - p|| \leq \varepsilon.$$

Denoting $\mathcal{S} = \bigcup_{R=1}^{\infty} \mathcal{P}_R$ we see that the set $\mathcal{S}$ is countable and dense in $L^2(\mathbb{R}_+, \varrho)$. On the other hand, from boundedness of ϱ we can introduce a σ-finite measure ν, $\nu(E) = \int_E \varrho \, dx$ for any E Lebesgue measurable. Thus the space $L^2(\mathbb{R}_+, \varrho)$ is reflexive. To show that $W_0^{1,2}(\mathbb{R}_+, \varrho)$ and $W^{1,2}(\mathbb{R}_+, \varrho)$ are separable and reflexive, it suffices to consider the following mapping:

$$I : u \in W^{1,2}(\mathbb{R}_+, \varrho) \mapsto (u, u_x) \in (L^2(\mathbb{R}_+, \varrho))^2 =: L_2^2(\mathbb{R}_+, \varrho). \tag{5.5}$$

Under notation $W := I[W^{1,2}(\mathbb{R}_+, \varrho)]$, $W_0 := I[W_0^{1,2}(\mathbb{R}_+)]$, we have that the sets W and W_0 are closed subspaces in $L_2^2(\mathbb{R}_+, \varrho)$, and the mapping I is an isometric isomorphism between $W^{1,2}(\mathbb{R}_+, \varrho)$ and W, $W_0^{1,2}(\mathbb{R}_+, \varrho)$ and W_0 respectively. Thus, as we show that the spaces W and W_0 are separable and reflexive, we obtain the same conclusion for $W^{1,2}(\mathbb{R}_+)$ and $W_0^{1,2}(\mathbb{R}_+)$. However, as $L_2^2(\mathbb{R}_+, \varrho)$ is separable, and separability in metric spaces preserves for subsets, the sets W and W_0 are separable. Moreover, W and W_0 are both closed and convex, thus weakly closed; weak compactness of the unit ball in $L_2^2(\mathbb{R}_+, \varrho)$ then yields that the unit balls of W and W_0 are weakly compact, and so W and W_0 are reflexive. Here, we have employed the Banach–Bourbaki characterization of reflexivity. □

At this stage, we can state the desired lemma on continuous dependence on initial data.

Lemma 5.3. *Suppose $F \in C^2(\mathbb{R})$ and $h \in C^1(\overline{\mathbb{R}})$ satisfy (2.1), (2.3) and (3.2). Moreover, assume that $\varrho : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded, locally Lipschitz continuous function satisfying $|\varrho_x(x)| \leq C_\varrho \varrho(x)$ for some constant $C_\varrho > 0$ and almost every $x \in \mathbb{R}_+$. For any $t \geq 0$ consider the mapping*

$$\Phi_t : u_0 \quad \mapsto \quad u(t)$$

where u is a solution of (1.1), (1.2) and (1.3). Then, under notation

$$P = \{u : \mathbb{R}_+ \to \mathbb{R} : u \geq 0\},$$

the following holds

(i) for any $t > 0$, Φ_t is continuous as a mapping

$$\Phi_t : L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+) \cap P \to L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+).$$

Moreover, for any u, v solutions to (1.1), (1.2) and (1.3) with initial data $u_0, v_0 \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+) \cap P$ there exist constants C_1 and C_2 depending solely on $\underline{\mu}, \overline{\mu}$, $\|u_0\|_{L^\infty(\mathbb{R}_+)}$, $\|v_0\|_{L^\infty(\mathbb{R}_+)}$ and the structural properties of h such that

$$\|u(t) - v(t)\|_{L^2(\mathbb{R}_+)} \leq C_1 e^{C_2 t} \|u_0 - v_0\|_{L^2(\mathbb{R}_+)} \tag{5.6}$$

$$\begin{aligned} \|u(t) - v(t)\|_{L^\infty(\mathbb{R}_+)} &\leq C_1 e^{C_2 t} \frac{1}{\sqrt[4]{t}} \|u_0 - v_0\|^{1/2}_{L^2(\mathbb{R}_+)} \times \\ &\quad \times \left(\|u_0\|_{L^2(\mathbb{R}_+)} + \|v_0\|_{L^2(\mathbb{R}_+)}\right)^{1/2}. \end{aligned} \tag{5.7}$$

(ii) for any $t > 0$, Φ_t is continuous as a mapping

$$\Phi_t : L^2(\mathbb{R}_+, \varrho) \cap L^\infty(\mathbb{R}_+) \cap P \to L^2(\mathbb{R}_+, \varrho).$$

Moreover, for any u, v solutions to (1.1), (1.2) and (1.3) with initial data $u_0, v_0 \in L^2(\mathbb{R}_+, \varrho) \cap L^\infty(\mathbb{R}_+) \cap P$ there exists a constant C depending on $\underline{\mu}$, $\overline{\mu}$, $\|u_0\|_{L^\infty(\mathbb{R}_+)}$, $\|v_0\|_{L^\infty(\mathbb{R}_+)}$ and the structural properties of h such that

$$\|u(t) - v(t)\|_{L^2(\mathbb{R}_+, \varrho)} \leq e^{Ct} \|u_0 - v_0\|_{L^2(\mathbb{R}_+, \varrho)}. \tag{5.8}$$

Proof. *(i)* By virtue of Theorem 2.1, estimate (2.4), we conclude that (5.6) holds. What remains to do is the verification of (5.7). To do this, we shall employ the Interpolation inequality in the from

$$\|u(t) - v(t)\|_{L^\infty} \leq C\|u(t) - v(t)\|^{1/2}_{W^{1,2}(\mathbb{R}_+)}\|u(t) - v(t)\|^{1/2}_{L^2(\mathbb{R}_+)}.$$

Accordingly, all we need to do is to estimate the term $\|u(t) - v(t)\|_{W_0^{1,2}(\mathbb{R}_+)}$. Employing the integrability estimate (3.14), we obtain existence of constants C_1 and C_2 depending solely on $\overline{\mu}$, $\underline{\mu}$, $\|u_0\|_{L^\infty(\mathbb{R}_+)}$, $\|v_0\|_{L^2(\mathbb{R}_+)}$ and the structural properties of h such that

$$\begin{aligned}\|u(t) - v(t)\|_{W_0^{1,2}(\mathbb{R}_+)} &\leq \|u(t)\|_{W_0^{1,2}(\mathbb{R}_+)} + \|v(t)\|_{W^{1,2}(\mathbb{R}_+)} \\ &\leq C_1 e^{C_2 t}\frac{1}{\sqrt{t}}(\|u_0\|_{L^2(\mathbb{R}_+)} + \|v_0\|_{L^2(\mathbb{R}_+)}),\end{aligned}$$

and (5.7) follows.

(ii) First of all, we observe that the mapping Φ_t is well-defined for initial data from $L^2(\mathbb{R}_+, \varrho) \cap P$ for any $t > 0$. Indeed, undergoing the procedures of approximating the unbounded-interval solution by bounded-interval ones we are able to derive similar estimates on norms of the solution in weighted space $L^2(\mathbb{R}_+, \varrho)$ as in the case of $L^2(\mathbb{R}_+)$.

Considering u^N to be a bounded-interval solution of (1.1), (1.2) and (1.3) on $(0, T) \times \Omega_N$ endowed with the initial datum $u_0 \in \mathcal{D}(\Omega_N) \cap P$ we can multiply the equation on $u^N\varrho$ and integrate over $(0, T) \times \Omega_N$ to obtain

$$\begin{aligned}\frac{1}{2}\|u^N(t)\|^2_{L^2(\Omega_N,\varrho)} - \frac{1}{2}\|u_0^N\|^2_{L^2(\Omega_N,\varrho)} &= -\int_0^t\int_{\Omega_N} F(u_x^N)u_x^N\varrho\,dx\,ds + \\ &\quad -\int_0^t\int_{\Omega_N} \varrho_x F(u_x^N)u^N\,dx\,ds + \\ &\quad +\int_0^t\int_{\Omega_N} h(u^N)u^N\varrho\,dx\,ds.\end{aligned}$$

By virtue of the Young inequality, the Lipschitz continuity of h on the range of u^N, the Gronwall lemma and the estimate $|\varrho_x| \leq C_\varrho\varrho$, we infer that for any $0 \leq s \leq t < \infty$

$$\|u^N(t)\|_{L^2(\mathbb{R}_+,\varrho)} \leq \|u^N(s)\|_{L^2(\mathbb{R}_+,\varrho)}e^{C(t-s)},$$

where C depends only on $\underline{\mu}$, $\overline{\mu}$, h, $\|u(s)\|_{L^\infty(\mathbb{R}_+)}$ and C_ϱ. Similarly, we obtain also the estimate on u_x^N in $L^2((0, T); L^2(\mathbb{R}_+, \varrho))$ for any $0 \leq s < t < \infty$

$$\|u_x^N\|_{L^2((0,T);L^2(\mathbb{R}_+,\varrho))} \leq C_1\|u_x^N(s)\|_{L^2(\mathbb{R}_+,\varrho)}e^{C_2(t-s)},$$

where C_1 and C_2 are constants depending on $\overline{\mu}$, $\underline{\mu}$, h, $\|u(s)\|_{L^\infty(\mathbb{R}_+)}$ and C_ϱ. Since, by virtue of Lemma 5.2, the space $W_0^{1,2}(\mathbb{R}_+, \varrho)$ is reflexive and separable, $L^2((0,T); W_0^{1,2}(\mathbb{R}_+))$ is reflexive and separable too. Employing the similar arguments to those used in the proof of Theorem 3.2 we obtain the existence of a solution in $L^2((0,T); W_0^{1,2}(\mathbb{R}_+, \varrho)) \cap C([0,T]; L^2(\mathbb{R}_+, \varrho))$ for any initial datum u_0 belonging to $\mathcal{D}(\mathbb{R}_+) \cap P$. Moreover, considering two solutions u and v both lying in $L^2((0,T); W_0^{1,2}(\mathbb{R}_+, \varrho))$ with initial data u_0, v_0 belonging to $L^2(\mathbb{R}_+, \varrho) \cap L^\infty(\mathbb{R}_+) \cap P$, multiplying the difference of equations for u and v by $(u-v)\varrho$ and integrating over $(0,T) \times \mathbb{R}_+$ we obtain

$$\begin{aligned}\frac{1}{2}\|u(t) - v(t)\|^2_{L^2(\mathbb{R},\varrho)} &= \frac{1}{2}\|u_0 - v_0\|^2_{L^2(\mathbb{R},\varrho)} + \\ &- \int_0^t \int_{\mathbb{R}} (F(u_x) - F(v_x))(u_x - v_x)\varrho \, dx \, ds + \\ &+ \int_0^t \int_{\mathbb{R}} \varrho_x (F(u_x) - F(v_x))(u - v) \, dx \, ds + \\ &+ \int_0^t \int_{\mathbb{R}} (h(u) - h(v))(u - v)\varrho \, dx \, ds.\end{aligned}$$

Employing the Young inequality on $\varrho_x(F(u_x) - F(v_x))(u - v)$ together with $|\varrho'| \le C_\varrho \varrho$ a.e., and continuous differentiability of h enables us to write

$$\|u(t) - v(t)\|^2_{L^2(\mathbb{R},\varrho)} \le e^{Ct}\|u_0 - v_0\|_{L^2(\mathbb{R}_+,\varrho)},$$

with C a constant depending on $\overline{\mu}$, $\underline{\mu}$, C_ϱ, $\|u_0\|_{L^\infty(\mathbb{R}_+)}$, $\|v_0\|_{L^\infty(\mathbb{R}_+)}$, and h. Similarly, we obtain the estimate on the difference of $u_x - v_x$ in the space $L^2((0,T); W_0^{1,2}(\mathbb{R}_+, \varrho))$

$$\|u_x - v_x\|_{L^2((0,T);L_0^2(\mathbb{R}_+,\varrho))} \le C_1 \|u(s) - v(s)\|_{L^2(\mathbb{R}_+,\varrho)} e^{C_2(t-s)},$$

where C_1 and C_2 depend solely on $\overline{\mu}$, $\underline{\mu}$, h, $\|u(s)\|_{L^\infty(\mathbb{R}_+)}$ and C_ϱ. These estimates enable us to conclude, by arguments similar to those used in the proof of Theorem 3.2, that the mapping Φ_t is well-defined, continuous as a mapping with values in $L^2(\mathbb{R}_+, \varrho)$ and, moreover, (5.8) holds. □

Corollary 5.1. *Let $F \in C^2(\mathbb{R})$ and $h \in C^1(\overline{\mathbb{R}}_+)$ be such that hypotheses (2.1), (2.3) hold. Consider the set*

$$\mathcal{A}_0 = \left\{ u_0 \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+) : \begin{array}{l} u_0 \ge 0 \text{ and} \\ u(0) = u_0 \implies \|u(t)\|_{W^{1,2}(\mathbb{R}_+)} \to_{t\to\infty} 0 \end{array} \right\},$$

where u denotes the solution of (1.1), (1.2) and (1.3) corresponding to the initial datum u_0. Then $\mathcal{A}_0$ is open in the space of non-negative functions in the topology of $L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$.

Proof. Let $u_0 \in \mathcal{A}_0$; if u is the solution of (1.1), (1.2) and (1.3) corresponding to u_0, then it converges to zero in $W_0^{1,2}(\mathbb{R}_+)$, in particular, in $L^\infty(\mathbb{R}_+)$; this yields existence of a time $t > 0$ such that $\|u(t)\|_{L^\infty(\mathbb{R}_+)} < \delta$ where $\delta > 0$ is small enough so that $h'(x) > 0$ for $0 \leq x \leq 2\delta$. In view of the continuous dependence on initial data (Lemma 5.3), there exists a neighbourhood U of u_0 in the topology of $L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ such that for any $v_0 \in U$, v_0 non-negative, the corresponding solution v satisfies $\|u(t) - v(t)\|_{L^\infty(\mathbb{R}_+)} < \delta$. Thus, by virtue of the local asymptotical stability of the zero solution, we obtain $v_0 \in \mathcal{A}_0$.

□

5.3 Zero Number Theory

In this part, we recall the results of the Zero number theory by Angenent [2].

Lemma 5.4. *Let $\{u^n\}_n$ be a sequence of continuous functions converging locally uniformly to u in $\overline{\mathbb{R}}_+$. Moreover, assume that for any u^n there exists $\gamma_n > 0$ such that u^n is increasing on $[0, \gamma_n]$ and decreasing on $[\gamma_n, \infty)$. Then there exists a number $0 \geq \gamma \leq \infty$ such that u is non-decreasing on $[0, \gamma]$ and non-increasing on $[\gamma, \infty)$.*

Proof. Consider the contrary, i.e., there exist numbers $0 \leq \xi_1 < \xi_2 < \xi_3 < \infty$ such that $u(\gamma_1) > u(\gamma_2)$ and $u(\gamma_2) < u(\gamma_3)$. Denote

$$\varepsilon = \min\{|u(\xi_i) - u(\xi_j)| : i, j = 1, 2, 3, i \neq j\}.$$

By virtue of the uniform convergence on $[\xi_1, \xi_3]$, there exists n_0 such that for $n \geq n_0$ $\|u^n - u\|_{L^\infty([\xi_1,\xi_3])} < \frac{\varepsilon}{3}$. If γ_n belongs to $[0, \xi_2]$, then $u^n(\xi_3) < u^n(\xi_2)$ and thus $|u^n(\xi_3) - u(\xi_3)| > \frac{2}{3}\varepsilon$. On the other hand, $|u^n(\xi_3) - u(\xi_3)| < \frac{\varepsilon}{3}$ which is a contradiction. The case $\gamma_n \in [\xi_2, \infty)$ can be treated analogously. □

Theorem 5.2. *Let $F \in C^2(\mathbb{R})$ and $h \in C^1(\overline{\mathbb{R}}_+)$ satisfy hypotheses (2.1) and (2.3). Moreover, let $u_0 \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ be non-negative such that there exists $\gamma_0 > 0$ and u_0 is non-decreasing on $[0, \gamma_0]$ and non-increasing on $[\gamma_0, \infty)$. Then for the corresponding solution u and any time $t > 0$ there exists γ_t such that $u(t)$ is non-decreasing on $[0, \gamma_t]$ and non-increasing on $[\gamma_t, \infty)$.*
Consequently, for any $t > 0$ and $\alpha > 0$ the set

$$\{x > 0 : u(t, x) > \alpha\}$$

is an (possibly empty) interval.

Proof. Consider an arbitrary initial datum $u_0 \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ satisfying the assumptions of the theorem. From the density argument there exists a sequence $\{u_0^n\}_n$ of non-negative smooth functions such that $\operatorname{supp} u_0^n = [0,n]$, u_0^n satisfy the assumption of u_0 (more precisely, we can even require $u_{0\,x}^n(x) > 0$ for $x < \gamma_0$ and $u_{0\,x}^n(x) < 0$ for $x > \gamma_0$), u_0^n converge to u_0 in $L^2(\mathbb{R}_+)$ and, moreover, u_0^n are uniformly bounded in $L^\infty(\mathbb{R}_+)$. In view of the techniques of Theorem 3.2, the solutions u^n corresponding to u_0^n are locally Hölder continuous along with u_t^n and u_{xx}^n and, by virtue of the Arzelá–Ascoli theorem, there exists a subsequence converging locally uniformly to u which is a solution corresponding to the initial datum u_0. Consequently, in view of Lemma 5.4, it suffices to verify the desired property only for functions u^n.

Consider a smooth function u_0^n with the support $[0,n]$ such that there exist $\gamma_0 \in (0,n)$ so that u_0^n is increasing on $[0,\gamma_0]$ and decreasing on $[\gamma_0,\infty)$. Denote by F_δ, h_δ smooth ($C^4(\mathbb{R})$, resp. $C^2(\overline{\mathbb{R}}_+)$) functions such that

$$\lim_{\delta\to 0+} \|F_\delta - F\|_{C^2(\mathbb{R})}, \quad \lim_{\delta\to 0+} \|h_\delta - h\|_{C^1(\overline{\mathbb{R}}_+)} = 0$$

and $0 < \frac{\mu}{2} \le F_\delta'(w) \le \frac{3}{2}\overline{\mu}, w \in \mathbb{R}$ hold. Let u^δ be the solution of modified problem

$$u_t^\delta - F_\delta'(u_x^\delta)u_{xx}^\delta + h_\delta(u^\delta) = 0$$

with initial datum u_0^n and homogeneous Dirichlet boundary conditions. Taking $\delta \in \{\frac{1}{n} : n \in \mathbb{N}\}$ we obtain, by virtue of the techniques of Theorem 3.2 and the Arzelá–Ascoli theorem, that there exists a subsequence (not relabeled) such that u^δ, u_t^δ, u_x^δ and u_{xx}^δ converge locally uniformly in $[0,T] \times \overline{\mathbb{R}}_+$, $0 < T < \infty$. Denote the limits by u, u_t, u_x and u_{xx}. Then the equation writes as follows

$$\underbrace{u_t^\delta}_{\to u_t} - \underbrace{F_\delta'(u_x^\delta)u_{xx}^\delta}_{\to F'(u_x)u_{xx}} + \underbrace{h_\delta(u^\delta)}_{\to h(u)} = 0,$$

so the solution $u^n = u$ can be obtained as a locally uniform limit of solutions to problem with smoother coefficients. Again, by virtue of Lemma 5.4 it suffices to verify the assertion only for u^δ.

The assertion for u^δ follows if we are able to show that the set of zeros of its x-derivative is a singleton for any fixed time $t > 0$. Since $u^\delta \ge 0$ and $u_0^n \not\equiv 0$, the strong maximum principle (cf, e.g., Protter–Weinberger [15]) assures that $u_x^\delta(t,0) > 0$ and $u_x^\delta(t,n) < 0$ for any $t > 0$. Considering the differentiated problem for $v = u_x^\delta$

$$v_t - a(t,x)v_{xx} + b(t,x)v_x + c(t,x)v = 0$$

where $a = F'_\delta(u^\delta_x)$, $b = F''_\delta(u^\delta_x)u^\delta_{xx}$ and $c = h'_\delta(u^\delta)$, we can apply Theorem 5.2 of Chapter 4 by Ladyženskaja [11] to do a bootstrap and, consequently, to obtain sufficient smoothness of coefficients a, b, c in order to apply the result of the Zero number theory by Angenent [2]. Thus, since u^n_{0x} has one zero γ_0, we obtain that $u^\delta(t)$ possesses exactly one zero γ_t for any $t > 0$ in accordance with Angenent [2]. □

5.4 Concentrated Compactness

In this part we claim that if u is a solution to (1.1), (1.2) and (1.3) with $u_0 \geq 0$ being nondecreasing on $[0, \gamma_0]$ and nonincreasing on $[\gamma_0, \infty)$, and there exists a sequence $\{t^n\}_n$, $t^n \to_{n\to\infty} \infty$, such that $\{u(t^n)\}$ is bounded in $W_0^{1,2}(\mathbb{R}_+)$, then either $u(t)$ converges to zero in $W^{1,2}(\mathbb{R}_+)$ as $t \to \infty$, or $u(t)$ converges to a spatially shifted hump of the ground state solution, which travels to infinity. More precisely, there exists a function $y : \mathbb{R}_+ \to \mathbb{R}_+$, $y(t) \to_{t\to\infty} \infty$ so that

$$\|u(t) - w_g(\cdot + y(t))\|_{W^{1,2}(\mathbb{R}_+)} \to 0 \text{ as } t \to \infty.$$

This assertion will be verified in several steps.

Step 1

To begin with, we point out that, since the energy functional (3.17) is a non-increasing function along $\{u(t) : t \geq 0\}$, and, under notation $u^n = u(t^n)$, $\|u^n\|_{W^{1,2}(\mathbb{R}_+)}$ is bounded, the energy is bounded too. Employing (3.18) we conclude, with the aid of

$$\int_1^t \|u_t(s)\|^2_{L^2(\mathbb{R}_+)}\, ds = E(u(1)) - E(u(t)) \leq C < \infty$$

where C depends solely on the boundedness of $\|u^n\|_{W_0^{1,2}(\mathbb{R}_+)}$, that u_t belongs to $L^2((1,\infty); L^2(\mathbb{R}_+))$. In view of

$$\int_1^\infty \|u_t(s)\|^2_{L^2(\mathbb{R}_+)}\, ds = \sum_{k=1}^\infty \int_k^{k+1} \|u_t(s)\|^2_{L^2(\mathbb{R}_+)}\, ds < \infty,$$

we obtain existence of a sequence $\{\tau^n\}$, $|\tau^n - \tau^{n+1}| < 2$, $\tau^n \to_{n\to\infty} \infty$, such that $\|u_t(\tau^n)\|_{L^2(\mathbb{R}_+)} \to_{n\to\infty} 0$. With the aid of (3.15) we finally obtain

$$\lim_{t\to\infty} \|u_t(t)\|_{L^2(\mathbb{R}_+)} = 0.$$

Particularly, $\|u^n_t\|_{L^2(\mathbb{R}_+)}$ converges to zero and thus is bounded. Moreover, as $\{u^n\}$ satisfy (1.1), we obtain that u^n_{xx} is bounded in $L^2(\mathbb{R}_+)$, hence the whole sequence u^n is bounded in $W^{2,2}(\mathbb{R}_+)$.

Boundedness of $\{u^n\}_n$ in $W^{2,2}(\mathbb{R}_+)$ yields a subsequence (not relabeled) converging to some function $\tilde{u}$ weakly in $W^{2,2}(\mathbb{R}_+)$. Consider any compact $Q \subset [0,\infty)$. By virtue of the compact imbedding of $W^{2,2}(Q)$ into $W^{1,2}(Q)$, we obtain that u^n converges to $\tilde{u}$ in $W^{1,2}(Q)$ (passing to a subsequence, if necessary).
Since u^n is bounded in $W^{2,2}(\mathbb{R}_+)$, u^n_x is bounded in $L^\infty(\mathbb{R}_+)$; whence, as $F \in C^2(\mathbb{R})$, we can write

$$\int_Q |F'(u^n_x) - F'(\tilde{u}_x)|^2\,dx \le K^2 \int_Q |u^n_x - \tilde{u}_x|^2\,dx$$

where K denotes the Lipschitz constant of F' on the union of ranges of u^n_x and $\tilde{u}_x$. We have proved that $F'(u^n_x)$ converges to $F'(\tilde{u}_x)$ in $L^2(Q)$. At this stage, passing with n to infinity in (1.1) it writes as follows

$$\underbrace{u^n_t}_{\to 0 \text{ in } L^2(Q)} \quad = \quad \underbrace{F'(u^n_x)u^n_{xx}}_{\to F'(\tilde{u}_x)\tilde{u}_{xx} \text{ in } L^2(Q)} \quad - \quad \underbrace{h(u^n).}_{\to h(\tilde{u}) \text{ in } L^2(Q)}$$

Convergence of $F'(u^n_x)u^n_{xx}$ can be obtained by showing the weak convergence

$$\begin{aligned}\int_Q (F'(u^n_x)u^n_{xx} - F'(\tilde{u}_x)\tilde{u}_{xx})\varphi\,dx \quad &= \quad \int_Q (F'(u^n_x) - F'(\tilde{u}_x))u^n_{xx}\varphi\,dx + \\ &\quad + \int_Q (u^n_{xx} - \tilde{u}_{xx})F'(\tilde{u}_x)\varphi\,dx, \varphi \in \mathcal{D}(Q)\end{aligned}$$

where on the right-hand side the first term converges to zero due to convegence of $F'(u^n_x)$ shown above, boundedness of u^n_{xx} in $L^2(Q)$, and boundedness of φ in $L^\infty(Q)$; convergence of the second term follows from the fact that $F'(\tilde{u}_x)\varphi \in L^2(Q)$ is an admissible test function. Now, the norm convergence follows from the fact that u^n_t and $h(u^n)$ are Cauchy sequences in $L^2(\mathbb{R}_+)$, and thus $F'(u^n_x)u^n_{xx}$ is a Cauchy sequence in $L^2(\mathbb{R}_+)$ too. Moreover, subtracting the equations for u^n and u^m, and estimating $u^n_{xx} - u^m_{xx}$ yields, with the aid of (2.1),

$$\left.\begin{aligned}\underline{\mu}\|u^n_{xx} - u^m_{xx}\|_{L^2(Q)} \quad &\le \quad \|F'(u^n_x)(u^n_{xx} - u^m_{xx})\|_{L^2(Q)} \\ &\le \quad \|u^n_t - u^m_t\|_{L^2(Q)} + \|h(u^n) - h(u^m)\|_{L^2(Q)} + \\ &\quad\quad + \|(F'(u^n_x) - F'(u^m_x))u^m_{xx}\|_{L^2(Q)},\end{aligned}\right\} \tag{5.9}$$

and thus we see that $\{u^n_{xx}\}_n$ is a Cauchy sequence in $L^2(Q)$, so the convergence to $\tilde{u}$ in $W^{2,2}(Q)$ follows. We have shown that the function $\tilde{u}$ lying in $W^{2,2}(\mathbb{R}_+) \cap W^{1,2}_0(\mathbb{R}_+)$ solves the stationary problem

$$-F'(\tilde{u}_x)\tilde{u}_{xx} + h(\tilde{u}) = 0 \tag{5.10}$$

in $L^2(Q)$ for any compact $Q \subset \mathbb{R}_+$. Continuity of $\tilde{u}$, $\tilde{u}_x$, and (2.1) implies continuity of $\tilde{u}_{xx}$, thus $\tilde{u}$ satisfies the equation pointwise.

Consider now the initial value problem

$$F'(w_x)w_{xx} = h(w), \quad w(0) = 0, w_x(0) = c. \tag{5.11}$$

Multiplying both sides of the equation by w_x and integrating from zero to x we get, under notation $S(t) = \int_0^t F'(s)s\,ds$, $H(t) = \int_0^t h(s)\,ds$,

$$S(w_x(x)) = H(w(x)) + S(c).$$

Now, let us identify the values c for which $\lim_{x\to\infty} w(x) = 0$; suppose this case happens for $c \neq 0$, i.e. $d = S(c) > 0$. From continuity of $H(w(x))$ we can find $R > 0$ such that $S(w_x(x)) \geq \frac{d}{2}$. Thus, employing (4.10) we get

$$w_x^2(x) \geq \frac{2}{\mu} S(w_x(x)) \geq \frac{d}{\mu}.$$

Therefore, by virtue of continuity of w_x, we obtain that it cannot change the sign, and its absolute value is estimated from below; whence $w(x)$ is unbounded, which is a contradiction. Therefore, we conclude that $\tilde{u}$ is a (unique) solution of the initial value problem with $c = 0$, hence $\tilde{u} \equiv 0$.

We have shown that for any compact $Q \subset \mathbb{R}_+$ there exists a subsequence of $\{u^n\}$, which converges to zero in $W^{2,2}(Q)$. The natural question is, whether one can improve this result up to the whole $\mathbb{R}_+$. The problem arising at this point is, if there are or are not any 'humps' of the solution that are travelling to infinity, and whether one can approximate them in some sense.

Under the assumption that the initial datum u_0 is non-decreasing on $[0, \gamma_0]$, and non-increasing on $[\gamma_0, \infty)$ we get, by virtue of Theorem 5.2, that for any $n \in \mathbb{N}$ there exists exactly one point, denoted by y^n, at which u^n attains its (local) maximum. If there exists a subsequence n_k such that $u^{n_k}(y^{n_k})$ converges to zero, then we obtain that $\|u^{n_k}\|_{L^\infty(\mathbb{R}_+)}$ converges to zero, and, by virtue of the stability of the zero solution and (5.9), we conclude that $\|u^n\|_{W^{2,2}(\mathbb{R}_+)}$ converges to zero. That is why we can assume for further investigations that $u^n(y^n)$ is bounded away from zero; moreover, let us extend each u^n, u^n_x and u^n_{xx} to be equal to zero on $(-\infty, 0]$, and denote $v^n = u^n(\cdot - y^n)$.

Since v^n is a spatial shift of u^n, the arguments concerning uniform boundedness of u^n in $W^{2,2}(\mathbb{R}_+)$ hold also for v^n. Thus there exists a subsequence of v^n (not relabeled) converging to some function $\tilde{v}$ weakly in $W^{2,2}(\mathbb{R})$. Further, following the arguments concerning convergence of u^n for the case of convergence of v^n, we conclude that for any compact $Q \subset \mathbb{R}$ v^n converge to $\tilde{v}$ in $W^{2,2}(Q)$ and $\tilde{v}$ is a solution of the stationary problem (5.10) in $L^2(Q)$.

By continuity of $\tilde{v}$ and $\tilde{v}_x$, we obtain continuity of $\tilde{v}_{xx}$; whence, since by Theorem 4.1 there exists the only solution to (4.1), we have $\tilde{v} \equiv w_g$.

We shall prove that v^n converges to $\tilde{v} \equiv w_g$ in $W^{2,2}(\mathbb{R})$. In order to do this, we shall show convergence of v^n in $W^{1,2}(\mathbb{R})$ and then apply (5.9) for the case of v^n and $Q = \mathbb{R}$, to obtain convergence in $W^{2,2}(\mathbb{R})$. Since $h'(0) > 0$, there exists $\varepsilon > 0$ sufficiently small and $\delta > 0$ such that for any $x \in (0, \varepsilon)$ we have $h'(x) > \delta$. By $\lim_{|x|\to\infty} w_g(x) = 0$, there exists $R_\varepsilon > 0$ such that $w_g(x) < \frac{\varepsilon}{2}$ for $|x| \geq R_\varepsilon$. For given compact $Q_\varepsilon = [-R_\varepsilon, R_\varepsilon]$, we deduce, by virtue of convergence of v^n to w_g in $W^{2,2}(Q_\varepsilon)$ and imbedding of $W^{1,2}(Q_\varepsilon)$ into $L^\infty(Q_\varepsilon)$, that there exists n_0 such that for any $n \geq n_0$ we have $|v^n(x) - w_g(x)| < \frac{\varepsilon}{2}$ for any $x \in Q_\varepsilon$. In particular, we get $v^n(\pm R_\varepsilon) < \varepsilon$. As, by Theorem 5.2, v^n is non-decreasing for negative values, and non-increasing for the positive ones, we get $v^n|_{\mathbb{R}\setminus Q_\varepsilon} < \varepsilon$. Recalling the equation for v^n

$$v_t^n - F(v_x^n)_x + h(v^n) = 0,$$

we obtain, with the aid of multiplying it by $v^n \in W^{1,2}(\mathbb{R}_+)$ and integrating by parts over $[R_\varepsilon, \infty)$, that

$$\int_{R_\varepsilon}^{\infty} v_t^n v^n \, dx + \int_{R_\varepsilon}^{\infty} \underbrace{F(v_x^n) v_x^n}_{\underline{\mu}(v_x^n)^2 \leq} \, dx - [F(v_x^n) v^n]_{R_\varepsilon}^{\infty} + \int_{R_\varepsilon}^{\infty} \underbrace{h(v^n) v^n}_{\delta(v^n)^2 \leq} \, dx = 0.$$

Employing $\lim_{x\to\infty} w(x) = 0$, boundedness of v^n in $W^{2,2}(\mathbb{R}_+)$, and the Hölder inequality yield

$$\min\{\underline{\mu}, \delta\} \|v^n\|^2_{W^{1,2}((R_\varepsilon,\infty))} \leq C(\varepsilon + \delta_n)$$

where δ_n stands for $\|v_t^n\|_{L^2(\mathbb{R})}$ which converges to zero as $n \to \infty$. Undergoing the same procedure for $(-\infty, -R_\varepsilon]$ we conclude, that for any $\varepsilon > 0$ there exist a compact $Q \subset \mathbb{R}$ and a number n_0 such that if $n \geq n_0$, then

$$\|v^n\|_{W^{1,2}(Q)} \geq \|v^n\|_{W^{1,2}(\mathbb{R})} - \varepsilon.$$

At this stage, we continue in the following way: For given $\varepsilon > 0$ we find a compact $Q \subset \mathbb{R}$ sufficiently large and a number n_0 so that for $n \geq n_0$ we have $\|v^n\|_{W^{1,2}(Q)} > \|v^n\|_{W^{1,2}(\mathbb{R})} - \frac{\varepsilon}{4}$ and $\|w_g\|_{W^{1,2}(Q)} > \|w_g\|_{W^{1,2}(\mathbb{R})} - \frac{\varepsilon}{4}$. Consequently, we can write for $n \geq n_0$

$$\|v^n - w_g\|_{W^{1,2}(\mathbb{R})} \leq \|v^n - w_g\|_{W^{1,2}(Q)} + \|v^n\|_{W^{1,2}(\mathbb{R}\setminus Q)} + \|w_g\|_{W^{1,2}(\mathbb{R}\setminus Q)} < \varepsilon,$$

in other words,

$$\lim_{n\to\infty} \|v^n - w_g\|_{W^{1,2}(\mathbb{R})} = 0.$$

From convergence of v^n in $W^{1,2}(\mathbb{R})$ we conclude, with the aid of the estimate (5.9) done for the case of $Q = \mathbb{R}$ and $u^n = v^n$, that v^n_{xx} is a Cauchy sequence and thus it converges to $\tilde{v}_{xx} = w_{g_{xx}}$ in $L^2(\mathbb{R})$. This means that

$$\lim_{n\to\infty} \|u^n - w_g(\cdot + y^n)\|_{W^{2,2}(\mathbb{R}_+)} = 0.$$

Step 2

In this step, we are going to enlarge the result obtained in the previous part to the large-time behaviour for all $\{u(t)\}_{t>1}$. First of all, we shall verify that the boundedness of $\{u(t^n)\}$ in $W_0^{1,2}(\mathbb{R}_+)$ for some sequence $t^n \to \infty$ implies that the whole set $\{u(t)\}_{t>1}$ is bounded in $W_0^{1,2}(\mathbb{R}_+)$. Indeed, if this is not the case, i.e. there exists a sequence $\tau^n \to \infty$ such that $\|u(\tau^n)\|_{W^{1,2}(\mathbb{R}_+)}$ is *not* bounded, then the regularity part of Theorem 3.2, more precisely $u \in C((0,\infty); W_0^{1,2}(\mathbb{R}_+))$, yields existence of a sequence $\sigma^n \to \infty$ such that $\|u(\sigma^n)\|_{W_0^{1,2}(\mathbb{R}_+)} = 2\|w_g\|_{W^{1,2}(\mathbb{R})}$. Consequently, since $u(\sigma^n)$ is bounded in $W_0^{1,2}(\mathbb{R}_+)$, the procedure of the previous step yields a subsequence of $\{\sigma^n\}_n$ (not relabeled), and a sequence $\{y^n\}_n$ such that

$$\|u(\sigma^n) - w_g(\cdot + y^n)\|_{W^{2,2}(\mathbb{R}+)} \to_{n\to\infty} 0$$

However, this is a contradiction with $\|u^n\|_{W_0^{1,2}(\mathbb{R}_+)} = 2\|w_g\|_{W^{1,2}(\mathbb{R})}$.

At this stage, consider there exists a sequence $\tau^n \to \infty$ such that the choice of y^n, as described in the previous step, does not yield the convergence, i.e. there exists $\varepsilon_0 > 0$ such that

$$\|u(\tau^n) - w_g(\cdot + y^n)\|_{W^{1,2}(\mathbb{R}_+)} \geq \varepsilon_0.$$

Since $\|u(\tau^n)\|_{W_0^{1,2}(\mathbb{R}_+)}$ is bounded, the procedure of *Step 1* yields a subsequence (not relabeled) such that $\|u(\tau^n) - w_g(\cdot + y^n)\|_{W^{2,2}(\mathbb{R}_+)} \to 0$ as $n \to \infty$, which is a contradiction.

Step 3

Clearly, there are many possible choices of the sequence $\{y^n\}$ (in fact, it is a function and $y^n = y(t_n)$), not only the one used in *Step 1*. Nevertheless, all these sequences satisfy the same property that is a convergence to infinity for growing time. Indeed, assume for contradiction that for some sequence $\tau^n \to \infty$ there exists a bounded sequence $\tilde{y}^n$ such that

$$\lim_{n\to\infty} \|u^n - w_g(\cdot + \tilde{y}^n)\|_{W^{1,2}(\mathbb{R}_+)} = 0.$$

Then u^n does not converge to zero over the compact set $[0, \sup\{y^n : n \in \mathbb{N}\}]$, which is a contradiction with the results of *Step 1*.

We have proved the following theorem.

Theorem 5.3 (Concentrated compactness). *Let F be a twice differentiable function on $\mathbb{R}$ such that (2.1) is satisfied. Moreover, let h be a differentiable function on $\overline{\mathbb{R}}_+$ satisfying hypotheses (2.3) and (4.2). Furthermore, let $u_0 \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ be non-negative such that there exists γ_0 and u_0 is non-decreasing on $(0, \gamma_0]$, and non-increasing on $[\gamma_0, \infty)$.*
If u is a solution to (1.1), (1.2) and (1.3) such that there exists a sequence $\{t^n\}$, $t^n \to_{n\to\infty} \infty$ and $\|u(t^n)\|_{W_0^{1,2}(\mathbb{R}_+)}$ is uniformly bounded with respect to n, then either

- *$u(t)$ converges to zero in $W_0^{2,2}(\mathbb{R}_+)$ as $t \to \infty$, or*
- *there exists a function y, $\lim_{t\to\infty} y(t) = \infty$ such that*

$$\lim_{t\to\infty} \|u(t) - w_g(\cdot + y(t)\|_{W^{2,2}(\mathbb{R}_+)} = 0.$$

5.5 Stability of the Large-Norm Solutions

We already know that the solutions starting from sufficiently small initial data converge to zero and the set of such initial data is open. Similarly, we claim that also the set of initial data which solutions converge to infinity in $W_0^{1,2}(\mathbb{R}_+)$ is open.

First of all, we shall introduce a helpful Lemma.

Lemma 5.5. *Assume that $F \in C^2(\mathbb{R})$ and $h \in C^1(\overline{\mathbb{R}}_+)$ be such that hypotheses (2.1), (2.3) and (4.2) are satisfied. Then for any $\varepsilon > 0$ sufficiently small there exists a constant $L > 0$ depending on ε and the structural properties of h, such that if u is a solution of (1.1), (1.2) and (1.3) corresponding to the initial datum $u_0 \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ such that*

$$u_0 \geq 0 \text{ on } \mathbb{R}_+ \text{ and } u_0(x) \geq \zeta_0 + \varepsilon$$

for x in the interval $(x_0 - L, x_0 + L) \subset \mathbb{R}_+$, then

$$\lim_{t\to\infty} \|u(t)\|_{W^{1,2}(\mathbb{R}_+)} = \infty.$$

Proof. Following the technique of the proof of Lemma 5.2 by Fašangová–Feireisl [6], we can pass with similar arguments. Let w be a solution of the stationary problem

$$-F(w_x)_x + h(w) = 0, \quad w(0) = \zeta_0 + \varepsilon, w'(0) = 0.$$

From Lemma 4.1 we deduce that there exist constants L_-, L_+ depending on ε and h such that $w(L_-) = 0 = w(L_+)$. Put $L := \max\{L_-, L_+\}$ and define the function

$$v(t,x) = \begin{cases} w(x-x_0), & x \in [x_0 - L_-, x_0 + L_+], t \geq 0 \\ 0, & x \in (0, x_0 - L_-) \cup (x_0 + L_+, \infty). \end{cases}$$

Consider the problem (1.1) with homogeneous Dirichlet boundary conditions separately on $(0, L_-)$, (L_-, L_+), and (L_+, ∞), and take from each initial datum the restriction of the function $w(0, \cdot)$ to the appropriate interval. Consequently, we have that the function w restricted on $(0, L_-)$, (L_-, L_+), and (L_+, ∞) is a solution of the problem considered separately on each of the three mentioned intervals. Now, the Comparison principle (Lemma 3.1) applied separately on the problems on the intervals $(0, L_-)$, (L_-, L_+) and (L_+, ∞) ensures

$$v(t,x) \leq u(t,x)$$

for $t, x \geq 0$. Consequently, we can write for the ground state solution w_g

$$\sup_{x>0} w_g(x) = \zeta_0 < \zeta_0 + \varepsilon = \sup_{x>0} v(t,x) \leq \sup_{x>0} u(t,x)$$

and, since this holds for any $t > 0$, we conclude from Theorem 5.3 that $\|u(t_n)\|_{W^{1,2}(\mathbb{R}_+)}$ cannot be bounded for any sequence $t_n \to \infty$ and thus the statement follows. □

We have proved that there exists an initial datum which makes the corresponding solution's $W^{1,2}(\mathbb{R}_+)$ norm converge to infinity. We claim even more, our aim is to show that for any such a solution it is possible to find a neighbourhood of the initial datum so that all the solutions corresponding to these initial data converge in $W_0^{1,2}(\mathbb{R}_+)$ norm to infinity too. To do this, we introduce two more lemmas which proofs are slight modifications of the procedures shown in Lemma 5.4 and Lemma 5.5 by Fašangová–Feireisl [6].

Lemma 5.6. *Assume that $F \in C^2(\mathbb{R})$ and $h \in C^1(\overline{\mathbb{R}}_+)$ satisfy hypotheses (2.1), (2.3 and (4.2). Moreover, let u be a solution of (1.1), (1.2) and (1.3) with the initial datum $u_0 \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, $u_0 \geq 0$.*
Then for any $\varepsilon > 0$ sufficiently small there exists a number $L' > 0$ depending on ε and h such that if

$$u_0 \geq \zeta_0 - \varepsilon, \quad x \in [a,b] \subset \mathbb{R}_+$$

and $b - a \geq L'$, then

$$\lim_{t\to\infty} \|u(t)\|_{W^{1,2}(\mathbb{R}_+)} = \infty.$$

Proof. In view of Lemma 5.5, it suffices to verify that for any $L > 0$ and $\varepsilon > 0$ sufficiently small there exists $L' > 0$ such that if $u_0 \geq \zeta_0 - \varepsilon$ on an interval of the length L', then there exists time $t_0 > 0$ and $u(t_0) > \zeta_0 + \varepsilon$ on an interval of the length $2L$. By virtue of the Comparison principle it suffices to show this property only for solutions starting from initial datum $u_0 = (\zeta_0 - \varepsilon)\chi_{[a,b]}$, where χ_C denotes the characteristic function of the set C.
Assume $\varepsilon > 0$ is sufficiently small so that $h|[\zeta_0 - \varepsilon, \zeta_0 + 2\varepsilon] < 0$ holds. Considering v as a solution to the ordinary differential equation

$$v_t + h(v) = 0, v(0) = \zeta_0 - \varepsilon,$$

we conclude, by virtue of the negativity of h on $[\zeta_0 - \varepsilon, \zeta_0 + 2\varepsilon]$, that there exists $t_0 > 0$ such that $v(t) \geq \zeta_0 + 2\varepsilon$ for any $t \geq t_0$. The proof follows if we are able to show that the solution u stays 'near' v in some sense. To show this, we introduce a weighted space $L^2(\mathbb{R}, \varrho)$ with the weight $\varrho(x) = e^{-|x-c|}$ where $c = \frac{a+b}{2}$.

Consider, without loss of generality, that the nonlinearity h is odd. Moreover, extend the initial datum u_0 to be odd on $\mathbb{R}$. It is easy to see, that the solution of the problem on $\mathbb{R}$ is odd, thus it coincides on $\mathbb{R}_+$ with the original one. Subtracting equations for v and u, multiplying the difference by $\varrho(v - u)$, integrating over $(0, T) \times \mathbb{R}$ we obtain by virtue of the local Lipschitz continuity of h, the Young inequality, $|\varrho_x| \leq C_\varrho \varrho$ a.e., $\lim_{|x|\to\infty} \varrho(x) = 0$ and the Gronwall lemma

$$\|u(t) - v(t)\|_{L^2(\mathbb{R},\varrho)} \leq e^{\gamma t}\|u_0 - v_0\|_{L^2(\mathbb{R},\varrho)}. \tag{5.12}$$

for some $\gamma > 0$. Denote $J = \{x \geq 0 : u(t_0, x) < \zeta_0 + \varepsilon\}$. By virtue of Theorem 5.2, there exist $\alpha, \beta > 0$ such that $J = [0, \alpha) \cup (\beta, \infty)$. Estimating the left-hand side of (5.12) from below yields

$$\begin{aligned}
\int_{\mathbb{R}} |u(t_0) - v(t_0)|^2 \varrho \, dx &\geq \int_J ((\zeta_0 + 2\varepsilon) - (\zeta_0 + \varepsilon))^2 \varrho \, dx = \varepsilon^2 \int_J \varrho \, dx \\
&= \varepsilon^2 \left(\int_0^\infty e^{-|x-c|} \, dx - \int_\alpha^\beta e^{-|x-c|} \, dx \right) \\
&\geq \varepsilon^2 \left(\int_0^\infty \varrho \, dx - \int_{c-(\alpha+\beta)/2}^{c+(\alpha+\beta)/2} e^{-|x-c|} \, dx \right) \\
&= \varepsilon^2 (2e^{-(\beta-\alpha)/2} - e^{-(a+b)/2}).
\end{aligned}$$

On the other hand, the estimate of the right-hand side of (5.12) from above gives

$$\begin{aligned}
\|u_0 - v_0\|^2_{L^2(\mathbb{R},\varrho)} &\leq \int_{-\infty}^a 4(\zeta_0 - \varepsilon)^2 \varrho \, dx + \int_b^\infty (\zeta_0 - \varepsilon)^2 \varrho \, dx \\
&\leq 5(\zeta_0 - \varepsilon)^2 e^{-(b-a)/2}.
\end{aligned}$$

Combining these estimates together with (5.12) yields

$$\begin{aligned} e^{-(\beta-\alpha)/2} &\leq \frac{1}{2}\left(e^{-(a+b)/2}+5\frac{(\zeta_0-\varepsilon)^2}{\varepsilon^2}e^{2\gamma t_0}e^{-(b-a)/2}\right) \\ &\leq \frac{1}{2}\left(1+5\frac{(\zeta_0-\varepsilon)^2}{\varepsilon^2}e^{2\gamma t_0}\right)e^{-L'/2}. \end{aligned}$$

Passing with L' to infinity shows that the left-hand side must also converge to zero, i.e., L converges to infinity. Thus we have proved that for arbitrarily large L one can find L' such that for any solution starting with $u_0 \geq (\zeta_0-\varepsilon)$ on an interval of the length L' there exists $t_0 > 0$ and $u(t) \geq \zeta_0+\varepsilon$ on some interval of the length $2L$. □

Lemma 5.7. *Let $F \in C^2(\mathbb{R})$ and $h \in C^1(\overline{\mathbb{R}}_+)$ be such that (2.1), (2.3), (3.2) and (4.2) hold.*
Denote

$$\mathcal{A}_\infty := \left\{ u_0 \in L^2(\mathbb{R}_+)\cap L^\infty(\mathbb{R}_+) : \begin{array}{l} u_0 \geq 0 \text{ and} \\ u(0)=u_0 \Longrightarrow \|u(t)\|_{W^{1,2}(\mathbb{R}_+)} \to_{t\to\infty} \infty \end{array} \right\},$$

where u is a solution of (1.1), (1.2) and (1.3). Then the set $\mathcal{A}_\infty$ is open in the set of non-negative functions in the topology of $L^2(\mathbb{R}_+)\cap L^\infty(\mathbb{R}_+)$.

Proof. In view of Lemma 5.6 it suffices to show that for any $u \in \mathcal{A}_\infty$ there exists $\varepsilon > 0$ such that if $v_0 \in L^2(\mathbb{R}_+)$ and $\|u_0 - v_0\|_{L^\infty(\mathbb{R}_+)} < \varepsilon$, then $v_0 \in \mathcal{A}_\infty$. To do this, we shall show first that for any ε sufficiently small the set

$$J(t) := \{x : u(t,x) \geq \zeta_0 - \varepsilon\}$$

is an interval with the length diverging to infinity while t goes to infinity. The fact that $J(t)$ is an interval follows from Theorem 5.2. To prove the unboundedness of $|J(t)|$ let us suppose the contrary: Let there exist a constant c_0 such that $|J(t_n)| \leq c_0$ for some sequence t_n going to infinity. Since $\zeta_0 = \inf\{s : H(s) < 0\} > 0$, where $H(t) = \int_0^t h(s)\,ds$, and $h'(0) > 0$, there exists a constant c_1 such that $H(s) \geq c_1 s^2$ for any $s \in (0, \zeta_0-\varepsilon)$. Moreover, the hypothesis (3.2) implies the $L^\infty(\mathbb{R}_+)$ boundedness of the trajectory $\{u(t) : t \geq 0\}$ and thus the existence of a constant c_2 such that $\|u(t_n)\|_{L^\infty(\mathbb{R}_+)} \leq c_2$. Without loss of generality, we may assume $c_2 \geq \zeta_0$. Now, since the energy functional is a nonincreasing function along any trajectory, we can conclude with the aid of $\mu v^2 \le 2I(v), v \in \mathbb{R}$ the following

$$\begin{aligned}
E(u_0) &\ge E(u(t_n)) = \int_{\mathbb{R}_+} I(u_x(t_n))\,dx + \int_{\mathbb{R}_+} H(u(t_n))\,dx \\
&\ge \frac{\mu}{2}\int_{\mathbb{R}_+} u_x^2(t_n)\,dx + \int_{J(t_n)} H(u(t_n))\,dx + \int_{\mathbb{R}_+\setminus J(t_n)} H(u(t_n))\,dx \\
&\ge \frac{\mu}{2}\int_{\mathbb{R}_+} u_x^2(t_n)\,dx + |J(t_n)|\underbrace{\inf\{H(s) : 0 \le s \le c_2\}}_{\le 0} + \\
&\quad + c_1\left(\int_{\mathbb{R}_+} u^2(t_n)\,dx - \int_{J(t_n)} u^2(t_n)\,dx\right) \\
&\ge \min\{\frac{\mu}{2}, c_1\}\|u(t_n)\|^2_{W^{1,2}(\mathbb{R}_+)} + c_0\left(-c_1c_2^2 + \inf\{H(s) : 0 \le s \le c_2\}\right).
\end{aligned}$$

Thus we can write

$$\|u(t_n)\|^2_{W^{1,2}(\mathbb{R}_+)} \le \frac{1}{\min\{\frac{\mu}{2}, c_1\}}\left(E(u_0) + c_0c_1c_2^2 - c_0\inf\{H(s) : 0 \le s \le c_2\}\right)$$

which is a contradiction to $u \in \mathcal{A}_\infty$.

The stategy is to show that if v_0 is near u_0 in the $L^\infty(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$-norm, then $v(t)$ stays near $u(t)$ in the $L^\infty(\mathbb{R}_+)$-norm for some time t, and, consequetly, to employ Lemma 5.6.

To do this, let us take $\varepsilon > 0$ as small as 2ε is also an admissible value for Lemma 5.6. Applying this lemma with 2ε we obtain the necessary length L of the interval to ensure the divergence of the solution's trajectory in $W^{1,2}(\mathbb{R}_+)$. From the above part there exists time $t_0 > 0$ such that

$$|\{x : u(t_0, x) \ge \zeta_0 - 2\varepsilon\}| = |J(t_0)| > L.$$

Lemma 5.3 yields existence of U, a $L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$-neighbourhood of u_0, such that if $v_0 \in U$, then $\|v(t_0) - u(t_0)\|_{L^\infty(\mathbb{R}_+)} \le \varepsilon$. Thus we have

$$v(t_0, x) \ge u(t_0, x) - \varepsilon \ge \zeta_0 - 2\varepsilon, \quad x \in J(t_0),$$

hence $\|v(t)\|_{W^{1,2}(\mathbb{R}_+)}$ converges to infinity for $t \to \infty$. □

6 Proof of the Main Result

In this part of the work, we are going to collect all the results obtained before and prove Theorem 1.1. We shall do this in two steps.

Step 1

Assume first that F satisfies (2.1), and also the hypothesis (3.2) holds. Then, by virtue of Corollary 5.1, the set $\mathcal{A}_0$ of initial data which solution converge to zero in $W_0^{1,2}(\mathbb{R}_+)$ is open in the topology of $L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, and, in view of Lemma 5.7, so does the set $\mathcal{A}_\infty$. The connectedness of the set of non-negative functions lying in $L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ implies that one of these sets must be empty, or there exists a non-negative function $w_0 \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ such that neither $\mathcal{A}_0$ nor $\mathcal{A}_\infty$ contains w_0. As $\mathcal{A}_0$ is non-empty and, in view of Lemma 5.5, the set $\mathcal{A}_\infty$ does so, we obtain that the second case holds.

We infered the existence of a trajectory that is bounded in $W^{1,2}(\mathbb{R}_+)$ and does not converge to zero, thus employing Theorem 5.3 we conclude that this trajectory converges the spatially localized wave of the ground state solution, which travels to infinity.

Step 2

In the previous step, we got the existence of a trajectory converging to the travelling wave of the ground state solution under additional assumptions on F and h. Consider now that the additional assumptions on F and h are violated. Thus only the existence of $\mu > 0$ such that $0 < \mu \leq F'(w)$ for any $w \in \mathbb{R}$ holds. However, employing Remark 4.2 we get existence of the ground state solution w_g which belongs to $W^{2,2}(\mathbb{R}) \cap C^2(\mathbb{R})$. Suppose that $\|w_g\|_{C^2(\mathbb{R})} \leq C$ and define the modifications of the functions F and h in the following way: Let $\tilde{F}$ and $\tilde{h}$ coincide with F and h on $[-C-1, C+1]$; on the complement let they be extended to be sufficiently smooth and, in the case of $\tilde{F}$, to satisfty $0 < \frac{\mu}{2} \leq F'(w) \leq F'(C+1) + 1$. For $\tilde{h}$, let the extension be such that the hypothesis (3.2) is satisfied. Then the ground state for the 'tilded' problem coincides with the former one and, by virtue of the previous step, the existence of a solution $\tilde{u}$ to the 'tilded' problem and a function $\tilde{y} : \mathbb{R}_+ \to \mathbb{R}_+$, $\lim_{t\to\infty} \tilde{y}(t) = \infty$ such that

$$\lim_{t\to\infty} \|\tilde{u}(t) - w_g(\cdot + \tilde{y}(t)\|_{W^{2,2}(\mathbb{R}_+)} = 0$$

follows. By convergence in $W^{2,2}(\mathbb{R}_+)$ we obtain, in particular, convergence in $C^1(\overline{\mathbb{R}}_+)$. Thus we can find time t_0 such that

$$\|\tilde{u}(t) - w_g(\cdot + \tilde{y}(t))\|_{C^1(\overline{\mathbb{R}}_+)} < 1 \text{ for any } t \geq t_0.$$

Denoting $v_0 = \tilde{v}_0 = \tilde{u}(t_0)$ one can see that the solutions v and $\tilde{v}$ of the 'untilded' and 'tilded' problem coincide, since F and h coincide with $\tilde{F}$ and $\tilde{h}$

on $[-C-1, C+1]$. Thus we have

$$\lim_{t\to\infty} \|v(t) - w_g(\cdot + y(t))\|_{W^{2,2}(\mathbb{R}_+)} = 0$$

for $y(t) = \tilde{y}(t + t_0)$, which completes the proof of Theorem 1.1.

7 Appendix

In this section we report several results we have recalled on throughout this work.

The Aubin lemma (sometimes called also Aubin–Lions lemma or Aubin–Ehrling lemma) serves as a useful tool to show compactness of imbeddings between Bochner spaces. Here, we present the statement by Roubíček [16].

Proposition A.1 (Aubin lemma). *Let V_1, V_2 be Banach spaces and V_3 be a metrizable, locally convex space; suppose that V_1 is separable and reflexive. Moreover, assume that V_1 is imbedded compactly into V_3 and V_2 is imbedded continuously into V_3.*
Then the imbedding

$$W^{1,p,q}(I;V_1,V_3) \hookrightarrow L^p(I;V_2)$$

is compact for any $1 < p < \infty$, $1 \le q \le \infty$, $I = (0,T), 0 < T < \infty$ in the sense that bounded sets in $W^{1,p,q}(I;V_1,V_3)$ are relatively compact in $L^2(I;V_2)$.

The Gronwall lemma (sometimes also called Gronwall–Bellman lemma) plays an essential role for showing lots of properties of solutions to differential equations. One can find several, more or less general statements; here, we adopt Lemma 1.3 of Chapter 5 by Gajewski et al. [8].

Proposition A.2 (Gronwall lemma). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let g be and non-decreasing function such that*

$$f(t) \le g(t) + c\int_0^t f(s)\,ds, \quad t \in [0,T]$$

with some non-negative constant c, then

$$f(t) \le e^{ct} g(t), \quad t \in [0,T].$$

The Interpolation inequality, sometimes called the Gagliardo–Nirenberg inequality, serves for interpolations between Sobolev spaces and can be oftenly used to show a convergence with not so much efort. Here, we bring the statement of Theorem 1.8 by Roubíček [16].

Proposition A.3 (Interpolation inequality). *Let $\beta = \beta_1 + \ldots + \beta_n$ such that $\beta_1, \ldots, \beta_n \in \mathbb{N} \cup \{0\}$, moreover, let $k \in \mathbb{N}$ and r, q and p satisfy*

$$\frac{1}{r} = \frac{\beta}{n} + \lambda\left(\frac{1}{p} - \frac{k}{n}\right) + (1-\lambda)\frac{1}{q}, \quad \frac{\beta}{k} \le \lambda \le 1, 0 \le \beta \le k-1,$$

then the following holds:

$$\left\| \frac{\partial^\beta v}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}} \right\|_{L^r(\Omega)} \leq C \|v\|_{W^{k,p}(\Omega)}^{\lambda} \|v\|_{L^q(\Omega)}^{1-\lambda}$$

provided $k - \beta - \frac{n}{p}$ *is a not a negative integer (otherwise, it holds only for* $\lambda = \frac{\beta}{k}$*).*

Oftenly, one meets the case when some type of an imbedding of vector-valued Sobolev functions into the continuous ones is needed. For this purpose, we introduce here Lemma 8.4 by Roubíček [16].

Proposition A.4 (Imbeddings of vector-valued Sobolev functions). *Let* V *be a Banach space and let* H *be a Hilbert space such that the imbedding of* V *into* H *is continuous and dense. Let* q *be the conjugate exponent to* p*, i.e.* $q = \frac{p}{p-1}$*.*
Then the imbedding of $W^{1,p,q}(I; V, V^*)$ *into* $C(I; H)$ *is continuous. Moreover, the following by-parts integration formula holds for any* $u, v \in W^{1,p,q}(I; V, V^*)$ *and any* $0 \leq t_1 \leq t_2 \leq T$*:*

$$(u(t_2), v(t_2)) - (u(t_1), v(t_1)) = \int_{t_1}^{t_2} \langle \frac{d}{dt} u, v \rangle_V + \langle \frac{d}{dt} v, u \rangle_V \, dt$$

where $(\cdot, \cdot)$ *denotes the scalar product in* H*.*

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Seznam oprav k diplomové práci
Lukáš Poul, 2004

- **16**: Example: $f(t,x) = \max\{0, \frac{1}{x+1} - |t|\}$, $g(t) = 1 - |t| - \ln|t| \in L^1(-1,1)$.
- **17**: Lemma 3.2 (vi) Druhý vzorec zdola $\frac{1}{2}\frac{d}{dt}\|u^N(t)\|_{L^2}^2 \le -\int F(u_x^N(t))u_x^N(t)\,dx\ldots$
- **18**: řádek 8: Multiplying (1.1) on $F(u_x^N)_x \ldots$
- **21**: ve vzorci pro odhad $|\langle v_t, \varphi\rangle|$ má být všude u^N místo u.
- **25**: řádky 4, 5, 6: In view of the fact that convergence in L^2 implies pointwise convergence almost everywhere of a certain subsequence, and the Fubini theorem ...
- **27**: ve formulaci Theorem 3.2 *(iii)*, *(iv)* navíc 'bounded in terms of $\overline{\mu}$, $\underline{\mu}$'; ve vzorci (3.14) pak norma vpravo $L^2(\mathbb{R}_+)$.
- **35**: řádek 9 zdola: $w \mapsto 1/S_\alpha^{-1}(H(w) + C_0)$
- **38**: před řádek 13 vložit: Similarly, we obtain

  $$\|u(T)\|_{L^2(\mathbb{R}_+)}^2 \le \|u(s)\|_{L^2(\mathbb{R}_+)}^2 - 2\delta \int_0^T \|u(t)\|_{L^2(\mathbb{R}_+)}^2\,dt,$$

  and, by virtue of Lemma 5.1, $\|u(T)\|_{L^2(\mathbb{R}_+)} \le \|u_0\|_{L^2(\mathbb{R}_+)} e^{-\delta T}$ for $T > 0$.
- **41**: třetí vzorec zdola: $-\int_0^t \int_{\Omega_N} h(u^N)u^N \varrho\,dx\,ds$.
- **42**: řádek 3: $L^2((0,T); W_0^{1,2}(\mathbb{R}_+, \varrho))$, v dalších formulích pak $\mathbb{R}_+$ místo $\mathbb{R}$.
- **43**: Lemma 5.4: $0 \le \gamma \le \infty$, ve volbě $\varepsilon := \min\{|u(\xi_i - \xi_2| : i = 1, 3\}$.
- **45**: řádek 5: $u_x^\delta(t)$.
- **53**: Lemma 5.7: $\mathcal{A}_\infty$ is open in the set of non-negative functions u non-decreasing on $[0, \gamma_u]$ and non-increasing on $[\gamma_u, \infty)$ in the topology of $L^2 \cap L^\infty$.
- **54**: vzorec nahoře: $t_n > 1$, místo $E(u_0)$ psát $E(u(1))$, podobně ve druhém vzorci.
- **55**: řádek 8: The connectedness of the set of non-negative functions u from $L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ nondecreasing on $[0, \gamma_u]$ and non-increasing on $[\gamma_u, \infty)$ implies ...

Asymptotické chování řešení evolučních parciálních diferenciálních rovnic na neomezených prostorových oblastech

Lukáš Poul

Abstrakt: V práci se věnujeme studiu asymptotického chování řešení kvazilineární evoluční diferenciální rovnice parabolického typu $u_t = F(u_x)_x+h(u)$ na polopřímce. Hlavním výsledkem je důkaz existence řešení, které pro dlouhé časy konverguje k cestující vlně netriviálního řešení stacionárního problému na přímce. Hlavními nástroji jsou Teorie nulových bodů a Teorie koncentrované kompaktnosti. Výsledek je zobecněním tvrzení známého pro semilineární parabolickou rovnici.
K dosažení výsledku bylo zejména potřeba modifikovat Lionsovu metodu koncentrované kompaktnosti známou pro semilineární rovnice do kontextu kvazilineárních rovnic, dále bylo potřeba získat odhady na řešení a obejít tak metodu variace konstant užívanou v semilineárních rovnicích při zkoumání kvalitativních vlastností řešení.

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