Title: Biochemical and mechanical processes in synovial fluid –
modeling, mathematical analysis and computational simulations

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Abstract: Synovial fluid is a polymeric liquid which generally behaves as a viscoelastic fluid due to the presence of polysaccharide molecules called hyaluronan. In this thesis, we study the biological and biochemical properties of synovial fluid, its complex rheology and interaction with synovial membrane during filtration process. From the mathematical point of view, we model the synovial fluid as a viscous incompressible fluid for which we develop a novel generalized power–law fluid model wherein the power–law exponent depends on the concentration of the hyaluronan. Such a model is adequate to describe the flows of synovial fluid as long as it is not subjected to instantaneous stimuli. Moreover, we try to find a suitable linear viscoelastic model which can describe the viscoelastic responses of synovial fluid during small deformation experiments, as, again, a function of concentration. Then, we consider the governing equations, namely the constraint of incompressibility, the balance of linear momentum – generalized Navier–Stokes equations and the convection–diffusion equation for the concentration of hyaluronan. The part of mathematical analysis is focused on the formulation of the stationary problem in the weak sense and the proof of the existence of the corresponding weak solution, for the case of a generalized viscous problem with concentration dependent power–law exponent. For that, we use the method of monotone operators, where the essential role plays the proof of Hölder continuity of the concentration. In the numerical part of the thesis, we consider different numerical stabilization methods which ensure better numerical solvability of the system with dominant convection, as is typical for synovial fluid flow. By their implementation into already existing code, we numerically solve for the flow of the synovial fluid in a rectangular cavity, in order to gain some insight into the response of such a fluid so that we can solve in the future the flows in more realistic geometries. We also compare the solutions obtained with different models of generalized viscosities and different stabilization techniques. As last, we propose a mathematical model for flow and transport processes of diluted solutions, and afterwards of synovial fluid, in domains separated by a leaky semipermeable membrane. We formulate transmission conditions for the flow and the solute concentration across the membrane which take into account the property of the membrane to partly reject the solute, the accumulation of rejected solute at the membrane, and the influence of the solute concentration on the volume flow, known as the osmotic effect.

Keywords: Synovial fluid, generalized viscosity, linear viscoelasticity, Navier–Stokes equations, generalized Sobolev space, $C^{0,\alpha}$–regularity, stabilized finite element methods, membrane transport.