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BAKALÁŘSKÁ PRÁCE

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Kompaktní a slabě kompaktní operátory v Banachových prostorech funkcí

Katedra matematické analýzy

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Název Práce: Kompaktní a slabě kompaktní operátory v Banachových prostorech funkcí

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Abstrakt: V práci jsou studovány vlastnosti slabých topologií na Banachově prostoru funkcí generovaných jistými podmnožinami jejich asociovaných prostorů. Charakterizujeme relativně sekvenciálně kompaktní podmnožiny ve slabé topologii a dokazujeme ekvivalenci relativní slabé kompaktnosti a relativní slabé sekvenciální kompaktnosti. Na závěr aplikujeme dosažené poznatky na lineární operátory a jejich asociované operátory mezi Banachovými prostory funkcí.

Klíčová slova: Banachův prostor funkcí, asociovaný prostor, slabá topologie, slabá kompaktnost, slabě kompaktní operátor, asociovaný operátor.

Title: Compact and weakly compact operators in Banach function spaces

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Abstract: We study properties of weak topologies induced on Banach function spaces by certain subsets of their associate spaces. We characterise relative sequential compactness in the weak topology and prove that the notions of relative weak compactness and relative weak sequential compactness coincide. Finally we apply the results attained to linear operators and their adjoints acting on Banach function spaces.

Keywords: Banach function space, associate space, weak topology, weak compactness, weakly compact operator, adjoint operator.

Contents

1. Introduction

Suppose that X is a linear space and $\mathcal{F}$ is a collection of semi-norms on X. We say that $\mathcal{F}$ generates a topology τ on X, if τ is the coarsest topology such that every semi-norm in $\mathcal{F}$ is continuous. In this case we write $\tau = \sigma(X, \mathcal{F})$ and also say that $\mathcal{F}$ is a topologizing family for X.

If we start with a Banach space X then every functional φ on X defines a semi-norm

$$S_\varphi(f) = |\varphi(f)|, \; f \in X,$$

hence if we identify all such semi-norms with corresponding functionals, we can consider the topology $\sigma(X, X^*)$ on X. This well-known locally convex topology is also called the weak topology and various considerable results about this topology are known; for instance, it is an example of a nonmetrizable locally convex topology (in case that X is infinite-dimensional, of course) for which the notion of sequential compactness and compactness are the same.

If one deals with Banach function space X (in the sense described in the following section) consisting of some measurable functions on a given measure space (R, μ), then it turns out that some new opportunities how to define a topologizing family appear — to a given Banach function space X, the associated space X' consists of all functions g such that the product fg is integrable over R for every f in X. Therefore every such g in X' defines a functional on X by the formula

$$L_g(f) = \int_R fg \, \mathrm{d}\mu, \; f \in X,$$

and $|L_g|$ defines a semi-norm on X. Via this identification, we can consider the topology $\sigma(X, X')$ on X. Since the associate space X' is isometrically isomorphic to a subspace of X^*, this topology is weaker than the topology $\sigma(X, X^*)$ in general.

In this thesis we study the properties of the topology $\sigma(X, X')$ and its consequences to linear operators. In Section 3 we present some facts about convergence in the topology $\sigma(X, X')$ and we also introduce topology finer than the one just presented but still coarser than weak topology and having X' as a dual.

In Section 4 we will prove that in the topology $\sigma(X, X')$ the notions of relative sequential compactness and relative compactness coincide; we also give a necessary and sufficient condition to establish a relative sequential compactness in this topology. We will also characterise the compactness of the unit ball in the topology $\sigma(X, X')$.

The final section deals with linear operators between Banach function spaces; we introduce the notion of an adjoint operator, characterise its existence and prove some basic properties. Next we put in context the results from Section 4 to $\sigma(X, X')$-compact and absolutely continuous operators. We show that a $\sigma(X, X')$-compact operator may not have $\sigma(X, X')$-compact adjoint and vice versa. However, the relationship between $\sigma(X, X')$-compactness of a linear operator and the existence of its adjoint operator remains unanswered.

2. Banach function spaces

Banach function spaces are Banach spaces of measurable functions in which the norm is appropriately related to the underlying measure. The theory of Banach function spaces can be regarded as a generalization of the theory of the Lebesgue spaces L^p, so the reader can keep these spaces in mind as a model for this theory.

In this section we recall the definitions and some basic facts about Banach function spaces which we will need in the following text. We shall not prove the well-known results; all of these can be found in the monograph by C. Bennett and R. Sharpley [BS].

Let (R, μ) be a totally σ-finite measure space, i.e., there exists a sequence $\{R_n\}$ of measurable sets of finite measure satisfying $\bigcup\{R_n;\, n \in \omega\} = R$. We suppose that such sequence was chosen once and for all. Let $\mathscr{M}^+$ be the set of all measurable functions on R whose values lie in $[0, \infty]$. Denote by χ_E the characteristic function of a measurable set E of R.

Definition. A mapping $\rho\colon \mathscr{M}^+ \to [0, \infty]$ is called a *Banach function norm* if for all f, g, f_n $(n \in \omega)$ in $\mathscr{M}^+$ for all constants $a \geq 0$ and for every measurable subset E of R the following properties hold:

$(P1)$ $\rho(f) = 0 \;\leftrightarrow\; f = 0$ μ-a.e.; $\rho(af) = a\rho(f)$; $\rho(f+g) \leq \rho(f) + \rho(g)$

$(P2)$ $0 \leq f \leq g$ μ-a.e. $\;\to\; \rho(f) \leq \rho(g)$

$(P3)$ $0 \leq f_n \uparrow f$ μ-a.e. $\;\to\; \rho(f_n) \uparrow \rho(f)$

$(P4)$ $\mu(E) < \infty \;\to\; \rho(\chi_E) < \infty$

$(P5)$ $\mu(E) < \infty \;\to\; \int_E f \, \mathrm{d}\mu \leq C_E \rho(f)$

for some constant C_E, $0 < C_E < \infty$, depending on E but independent of f.

Now let $\mathscr{M}$ denote the set of all μ-measurable scalar-valued (real or complex) functions defined on R.

Definition. Let ρ be a Banach function norm. The collection $X = X_\rho$ of all functions f in $\mathscr{M}$ for which $\rho(|f|)$ is finite is called a *Banach function space*. For each $f \in X$ we define

$$\|f\|_X = \rho(|f|).$$

It follows directly from the definition and property (P5) that every function in X is locally summable and hence finite μ-a.e. Property (P4) shows that X contains the characteristic functions of measurable sets of finite measure. By linearity every simple function belongs to X. Let us note that different authors use various kinds of definitions of the notion of a simple function. In our setting, by simple function we always mean a finite-measure supported function which has finite range.

Lemma. (Fatou lemma) *Let* $\{f_n\}$ *be a sequence in* X *such that* $f_n \to f$ μ*-a.e. and* $\liminf_{n\to\infty} \|f_n\|_X < \infty$*. Then* $f \in X$ *and*

$$\|f\|_X \le \liminf_{n\to\infty} \|f_n\|_X.$$

Given a Banach function space X, the associate space X' consists of all functions g in $\mathscr{M}$ such that fg is integrable for every f in X. The norm on X' is defined by

$$\|g\|_{X'} = \sup\left\{ \int_R |fg| \, \mathrm{d}\mu;\ f \in X, \|f\|_X \le 1 \right\}.$$

Note that X' itself is a Banach function space and $\|\cdot\|_{X'}$ is a Banach function norm (see [BS, Chapter 1, Theorem 2.2]). Moreover the norm of a function in the associate space X' is also given by (cf. [BS, Chapter 1, Lemma 2.8])

$$\|g\|_{X'} = \sup\left\{ \left| \int_R fg \, \mathrm{d}\mu \right|;\ f \in X, \|f\|_X \le 1 \right\}.$$

For every f in X and g in X' we have the Hölder inequality

$$\int_R |fg| \, \mathrm{d}\mu \le \|f\|_X \|g\|_{X'}.$$

Due to the effort G. G. Lorentz and W. A. J. Luxemburg ([BS, Chapter 1, Theorem 2.7]) we have that $X'' = (X')' = X$ and in that case $\|f\|_X = \|f\|_{X''}$.

Let $\{E_n\}$ be a sequence of measurable subsets of R. We shall write $E_n \to \emptyset$ μ-a.e. if the characteristic functions χ_{E_n} converge to the null function pointwise μ-a.e. If the sequence $\{E_n\}$ is decreasing we write $E_n \downarrow \emptyset$ μ-a.e.

Definition. A function f in a Banach function space X is said to have *absolutely continuous norm* in X if $\|f\chi_{E_n}\|_X \to 0$ for every sequence $\{E_n\}$ satisfying $E_n \to \emptyset$ μ-a.e. The set of all functions in X of absolutely continuous norm is denoted by X_a. If X_a coincides with X then the space X is said to have absolutely continuous norm.

Definition. Let X be a Banach function space. The closure in X of the set of simple functions is denoted by X_b.

Equivalently X_b is the closure in X of the set of bounded functions supported in sets of finite measure (see [BS, Chapter 1, Proposition 3.10]).

Definition. A closed linear subspace Y of a Banach function space X is called an *order ideal* of X if it has the following property:

$$f \in Y \text{ and } |g| \le |f| \ \mu\text{-a.e.} \ \to \ g \in Y.$$

Definition. A closed linear subspace Y of the dual space X^* of a Banach function space X is said to be *norm-fundamental* if

$$\|f\|_X = \sup\big\{|\varphi(f)|;\ \varphi \in Y, \|\varphi\|_{X^*} \leq 1\big\}$$

for every f in X.

The subspaces X_a and X_b are order ideals of X and $X_a \subseteq X_b \subseteq X$, moreover the subspace X_b is always relatively large in the sense that it is isometrically isomorphic to a norm-fundamental subspace of the dual space $(X')^*$. By contrast it can happen that X_a contains only the null function and in general all inclusions in

$$\{0\} \subseteq X_a \subseteq X_b \subseteq X$$

may be proper (cf. the example at the end of the section).

As we mentioned at the beginning, an important example of Banach function space is the Lebesgue space $L^p = L^p(R,\mu)$, $(1 \leq p \leq \infty)$ for which the Banach function norm is given by

$$\rho_{L^p}(f) = \begin{cases} \left(\int_R f^p \,\mathrm{d}\mu\right)^{1/p}, & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_R f, & p = \infty. \end{cases}$$

Let ℓ_p stand for the Lebesgue space over ω with counting measure.

To introduce another function space let (R,μ) be an interval $(0,\infty)$ with Lebesgue measure. For each $f \in \mathscr{M}^+$ let

$$\rho(f) = \int_0^1 f(x)\,\mathrm{d}x + \operatorname*{ess\,sup}_{1 \leq x < \infty} f(x).$$

Then ρ defines a Banach function norm, the subspace X_a consists of all functions of X_ρ which vanish on the interval $(1,\infty)$ and $X_b = L^1(0,\infty)$.

3. Weak topologies

In Banach spaces the most known topologies except the norm topology are so-called weak and weak* topologies generated by some semi-norms on the given space. If one deals with a Banach function space, some new natural possibilities how to generate a topology appear. In this section we will focus on two such concepts and we will derive some auxiliary propositions which will be helpful to prove some results in Section 4 similar as for weak or weak* topologies, in particular the equivalence of the notions of compactness and sequential compactness.

Let us recall a few definitions of some basic notions of general topology. We start with a simple concept of generating new topologies on a given set.

Definition. Let X be a set. Denote $\mathcal{F} = \{f_\alpha\colon X \to X_\alpha;\ \alpha \in A\}$ a collection of maps where all X_α are topological spaces. The coarsest topology on X such that every map from $\mathcal{F}$ is continuous is called *the topology induced by the family* $\mathcal{F}$ and it is denoted by $\sigma(X, \mathcal{F})$.

Note that such topology always exists and its subbase consists of sets $f_\alpha^{-1}[O]$, where $\alpha \in A$ and O is open in X_α. The topology $\sigma(X, \mathcal{F})$ is also called the weak topology of X induced by $\mathcal{F}$ and the set $\mathcal{F}$ is said to be a topologizing family for X.

The following lemma characterises continuity of a map into X with the topology induced by some $\mathcal{F}$.

Lemma 3.1. *Let X be a set and let $\mathcal{F}$ be a topologizing family for X. Suppose that (Y, τ) is a topological space and $g\colon (Y, \tau) \to (X, \sigma(X, \mathcal{F}))$. Then g is continuous if and only if $f \circ g$ is continuous for every f in $\mathcal{F}$.*

Proof. If g is continuous then $f \circ g$ is a composition of continuous maps hence itself continuous. Conversely let $f \circ g$ be continuous for every $f \in \mathcal{F}$. To establish the continuity of g it suffices to show that $g^{-1}[G] \in \tau$ for every subbase set G. Since every subbase set G is of the form $f_\alpha^{-1}[O]$, where O is open in X_α, we have to verify that $g^{-1}\big[f_\alpha^{-1}[O]\big]$ is open in τ, which follows directly by continuity of $f_\alpha \circ g$. □

If one deals with a vector space X, it is natural to choose the topologizing family $\mathcal{F}$ from the collection of semi-norms on X. Recall that a map S from X into the set of real numbers is called a semi-norm if $0 \le S(f)$, $S(f+g) \le S(f) + S(g)$ and $S(af) = |a|S(f)$ for every pair f, g in X and every scalar a.

Such topologizing family defines a locally convex linear topology. If the equality $S(f) = 0$ for all $S \in \mathcal{F}$ implies $f = 0$, then such topology is Hausdorff.

We will also need the notions of weak-type boundedness and convergence.

Definition. Let X be a vector space and let $\mathcal{F}$ be a collection of semi-norms on X. The subset Y of X is said to be $\sigma(X, \mathcal{F})$-*bounded* whenever for each $S \in \mathcal{F}$ the set $S[Y]$ is bounded in $\mathbb{R}$. The sequence $\{f_n\}$ in X is said to be $\sigma(X, \mathcal{F})$-*Cauchy* if for every $\varepsilon > 0$ there exists a corresponding $n_0 \in \omega$ such that $S(f_n - f_m) < \varepsilon$ for all $n, m \in \omega$, $m, n \geq n_0$ and is said to be $\sigma(X, \mathcal{F})$-*convergent* to $f \in X$ if $S(f_n - f)$ tends to zero for every $S \in \mathcal{F}$.

If we start with a Banach function space X, then every functional φ on X defines a semi-norm by the formula

$$S_\varphi(f) = |\varphi(f)|,$$

so it is natural to consider the topologizing family for X as a subcollection of these semi-norms. The well-known topology induced by all these semi-norms is called the weak topology and is denoted by $\sigma(X, X^*)$ or just w.

On the dual space X^* we can also consider the topology $\sigma(X^*, X^{**})$ or the weaker one $\sigma(X^*, \varepsilon X)$, where $\varepsilon\colon X \to X^{**}$ denotes the canonical embedding. Such topology is also denoted by $\sigma(X^*, X)$, weak* or just w^*. Recall that weak and weak* topologies coincide if and only if $\varepsilon X = X^{**}$, i.e., if X is reflexive.

3.1 The $\sigma(X, X')$ topology

In a function space let us introduce a yet weaker topology than the topology $\sigma(X, X^*)$. Suppose that X is a Banach function space and X' is its associate space. Since every g in X' defines a linear functional L_g on X by the formula

$$L_g(f) = \int_R fg \, \mathrm{d}\mu,$$

we can identify X' with a subspace of X^*. Moreover X' is isometrically isomorphic to a norm-fundamental subspace of the dual X^*, so $\|L_g\|_{X^*} = \|g\|_{X'}$ (see [BS, Chapter 1, Theorem 2.9]). Hence we can consider the topology induced by all such semi-norms $|L_g|$ where $g \in X'$ and we will denote such topology by $\sigma(X, X')$. For a subset M of X' we define a topology $\sigma(X, M)$ analogously. If we also denote $\sigma(X, \|\cdot\|)$ the original norm topology, we have

$$\sigma(X, X') \subseteq \sigma(X, X^*) \subseteq \sigma(X, \|\cdot\|).$$

Note that since X' is canonically isometrically isomorphic to X^* if and only if X has absolutely continuous norm (see [BS, Chapter 1, Corollary 4.3]), then $\sigma(X, X')$ and $\sigma(X, X^*)$ coincide if and only if $X = X_a$. Nevertheless the $\sigma(X, X')$ topology has in general some similar properties as the topology $\sigma(X, X^*)$ as we will see in Section 4.

Since have two different natural posibilities how to define "weak" topology on the dual space X^*, one could attempt to mimic this property and define the notion of $\sigma(X', X)$ topology. However, this turns out to be futile since X and X'' coincide we do not obtain anything new. This account therefore implies that any proposition stated about the topology $\sigma(X, X')$ also holds for the topology $\sigma(X', X)$.

Our first purpose is to study the convergence in $\sigma(X, X')$ topology or more general in $\sigma(X, M)$ topology where the set M is some order ideal in X'.

Let us note that some of the presented results (Theorems 3.4 and 3.7) are evolved by the virtue of the paper by W. A. J. Luxemburg and A. C. Zaanen (cf. [LZ, Chapter 3]). In our thesis the notion of a Banach function space is a little different, hence the proofs are also adjusted to our setting. We also add a few examples. Theorem 3.6 is also known (cf. [BS, Chapter 1, Theorem 5.2]) and the proof is presented for the continuity of the text.

The following theorem is a trivial application of the Hölder's inequality and the uniform boundedness principle.

Theorem 3.2. *Let X be a Banach function space over measure space (R, μ) and suppose that M is an order ideal of X' which is norm-fundamental in X^*. Then a subset Y of X is $\sigma(X, M)$-bounded if and only if it is norm-bounded in X.*

Proof. The "if" part follows directly from the Hölder inequality. Now suppose Y is $\sigma(X, M)$-bounded, so $\sup\{|L_g(f)|;\ f \in Y\}$ is finite for every $g \in M$. Every $f \in Y$ determines a linear functional L_f on M^* by the formula $L_f(g) = L_g(f) = \int_R fg \,\mathrm{d}\mu$. By the uniform boundedness principle we obtain that $\sup\{\|L_f\|_{M^*};\ f \in Y\}$ is finite. Since M is norm-fundamental in X^* we have

$$\|L_f\|_{M^*} = \sup\{|L_g(f)|;\ g \in M, \|g\|_{X'} \le 1\} = \|f\|_X$$

and therefore Y is norm-bounded. □

The following result is due to G. Vitali, H. Hahn and S. Saks and it will be crucial to the further results. The proof of this theorem can be found for example in the book by E. Hewitt and K. Stromberg [HS].

Theorem. (Hahn-Saks) *Let (R, μ) be a measure space, $\{f_n\}$ a sequence of summable scalar-valued functions over R. Suppose that $\nu_n(E) = \int_E f_n \,\mathrm{d}\mu$ converges to a finite number $\nu(E)$ for every measurable set E. Then there is a unique summable function f satisfying $\nu(E) = \int_E f \,\mathrm{d}\mu$ for every measurable E. Furthermore the measures $\widehat{\nu}_n(E) = \int_E |f_n| \,\mathrm{d}\mu$ are uniformly absolutely continuous with respect to μ, i.e., for every $E_k \downarrow \emptyset$ and $\varepsilon > 0$ there is an index k_0 such that for every $k > k_0$ is $\widehat{\nu}_n(E_k) < \varepsilon$ for every positive integer n, in particular for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\widehat{\nu}_n(E) < \varepsilon$ whenever E is measurable set satisfying $\mu(E) < \delta$ and $n \in \omega$.*

Note that since $|\nu_n(E)| \le \widehat{\nu}_n(E)$ for all $n \in \omega$ and every measurable set E, the measures ν_n are also uniformly absolutely continuous with respect to μ.

Let $\{f_n\}$ and f be measurable functions over (R, μ) as in the Hahn-Saks theorem. For $E = R$ we conclude that $\int_R f_n \,\mathrm{d}\mu \to \int_R f \,\mathrm{d}\mu$. If s denotes a simple function over R we immediately have $\int_R f_n s \,\mathrm{d}\mu \to \int_R f s \,\mathrm{d}\mu$. Naturally, our aim is to establish the convergence for more than just simple functions.

Lemma 3.3. *Let (R, μ) be a measure space, f_n, f be locally summable functions. Assume that $\int_E f_n \,\mathrm{d}\mu \to \int_E f \,\mathrm{d}\mu$ for every measurable set E of finite measure and $\sup\{\int_E |f_n| \,\mathrm{d}\mu;\ n \in \omega\} < \infty$ for all such E. Then for every bounded finite-measure supported function g we have*

$$\int_R fg \,\mathrm{d}\mu = \lim_{n\to\infty} \int_R f_n g \,\mathrm{d}\mu.$$

Proof. The conclusion clearly holds for every simple function. Denote $E = \operatorname{supp} g$. Since g is bounded and $\mu(E) < \infty$, we can uniformly approximate g by simple functions, say g_k. Then

$$\left| \int_R (f_n - f) g \,\mathrm{d}\mu \right| \le \left| \int_E (f_n - f) g_k \,\mathrm{d}\mu \right| + \left| \int_E (f_n - f)(g - g_k) \,\mathrm{d}\mu \right|$$

and for the second term we have

$$\left| \int_E (f_n - f)(g - g_k) \,\mathrm{d}\mu \right| \le \int_E \big(|f_n| + |f|\big) |g - g_k| \,\mathrm{d}\mu \le \|g - g_k\|_\infty \int_E \big(|f_n| + |f|\big) \,\mathrm{d}\mu.$$

Now for arbitrary $\varepsilon > 0$ we can first take k so large that the second term is smaller than ε for all n because the last integral is finite. Thus since g_k is simple, the first term tends to zero as $n \to \infty$. $\square$

Theorem 3.4. *Let X be a Banach function space over a totally σ-finite measure space (R, μ). Suppose that $\{f_n\}$ is a bounded sequence in X such that $\int_E f_n$ converges to a finite number as $n \to \infty$ for every measurable set E of finite measure. Then there exists some function f in X such that $f_n \to f$ in the topology $\sigma(X, (X')_b)$.*

Proof. Let $\{R_N\}$ be the sequence of sets of finite measure satisfying $R_N \uparrow R$. Suppose that N is fixed. By assumption every $\int_E f_n \,\mathrm{d}\mu$ converges to a finite number for every measurable set $E \subseteq R_N$, thus by the Hahn-Saks theorem we have unique f_N, summable over R_N, satisfying $\int_E f_n \,\mathrm{d}\mu \to \int_E f_N \,\mathrm{d}\mu$ as $n \to \infty$. Repeating this process for all $N \in \omega$ we obtain a sequence $\{f_N\}$ of functions such that every f_N is summable over R_N. By the uniqueness of such f_N on R_N we have $f_{N+1} \supseteq f_N$, that is, $\operatorname{Dom}(f_N) \subseteq \operatorname{Dom}(f_{N+1})$ and $f_{N+1} \restriction \operatorname{Dom}(f_N) = f_N$, therefore $\bigcup_N f_N$ defines a locally summable function, say f.

Let us show that f belongs to $X'' = X$. Denote K the constant such that $\|f_n\|_X \le K$ for all $n \in \omega$. We will show that $\sup\{\int |fg| \,\mathrm{d}\mu;\ \|g\|_{X'} \le 1\} \le K$. Choose an arbitrary $g \in X'$ such that $\|g\|_{X'} \le 1$ and define $g_N = \min\{N\chi_{R_N}, |g|\}$ for all $N \in \omega$. Clearly all g_N are bounded, have support of finite measure and

$g_N(x) \uparrow |g(x)|$ μ-a.e. on R. Hence $\|g_N\|_{X'} \le \|g\|_{X'} \le 1$ and by Lemma 3.3 and Hölder inequality we have the following

$$\int_R |fg_N| \,\mathrm{d}\mu = \lim_{n\to\infty} \int_R |f_n g_N| \,\mathrm{d}\mu \le \limsup_{n\to\infty} \|f_n\|_X \|g_N\|_{X'} \le K\|g\|_{X'} \le K.$$

Hence by the monotone convergence theorem

$$\int_R |fg| \,\mathrm{d}\mu = \lim_{N\to\infty} \int_R |fg_N| \,\mathrm{d}\mu \le K,$$

therefore $f \in X$.

Finally we have to establish the $\sigma(X, (X')_b)$-convergence, i.e., show that $\int_R f_n g \to \int_R fg$ for all $g \in (X')_b$. By the definition of $(X')_b$ there is a sequence of simple functions g_k converging to g in X'. Now we can continue similarly as in Lemma 3.3. We get

$$\left| \int_R (f_n - f) g \,\mathrm{d}\mu \right| \le \left| \int_R (f_n - f) g_k \,\mathrm{d}\mu \right| + \left| \int_R (f_n - f)(g - g_k) \,\mathrm{d}\mu \right|$$

with the difference that the estimate for the second term reads now by the Hölder inequality as

$$\left| \int_R (f_n - f)(g - g_k) \,\mathrm{d}\mu \right| \le \int_R \big(|f_n| + |f|\big)|g - g_k| \,\mathrm{d}\mu \le \|g - g_k\|_{X'} \big(K + \|f\|_X\big).$$

□

Notice that we actually needed the convergence of $\int_E f_n \,\mathrm{d}\mu$ only for the measurable sets contained in some R_n. We will use this fact in the proof of Theorem 4.7 below.

Let us observe that the set $(X')_b$ cannot be essentially enlarged in Theorem 3.4. Indeed, suppose $X = \ell_1$ and $f_n = e_n = \langle 0, \dots, 0, 1, 0, \dots \rangle \in X$ where the 1 is on the n^{th} position. Then $X' = \ell_\infty$, $(X')_b = c_0$ and clearly f_n is $\sigma(\ell_\infty, c_0)$ convergent to 0. Now let $g \in \ell_\infty \setminus c_0$. Then

$$\limsup_{n\to\infty} \left| \int_\omega f_n g \,\mathrm{d}\mu \right| = \limsup_{n\to\infty} |g(n)| > 0$$

and thus $L_g(f_n)$ does not tend to zero as $n \to \infty$.

On the other hand in some special cases the assumption of convergence is satisfied automatically as the next example shows.

Example 3.5. Suppose that X is Banach function space over a totally σ-finite completely atomic measure space (R, μ) such that every atom has measure larger than some $\varepsilon > 0$. Then any bounded sequence $\{f_n\}$ in X has a $\sigma(X, (X')_b)$-convergent subsequence.

PROOF. Since $\{f_n\}$ is norm-bounded, it is $\sigma(X,(X')_b)$-bounded and it suffices to show that $\int_E f_n\,\mathrm{d}\mu$ converges to a finite number as $n\to\infty$ whenever E is measurable set of finite measure. Because the measure space (R,μ) is σ-finite, it has at most countably many atoms, say $\{a_n\}$. Since $\int_{\{a_1\}} f_n\,\mathrm{d}\mu = f_n(a_1)\mu(a_1)$ is bounded there is a subsequence $\{f_n^1\}$ such that the integrals converge as $n\to\infty$. Similarly $\int_{\{a_2\}} f_n^1\,\mathrm{d}\mu = f_n^1(a_2)\mu(a_2)$ is bounded and hence there is a subsequence $\{f_n^2\}$ of the previous one. If we continue in an obvious manner and take the diagonal sequence $\{f_n^n\}$ we obtain that $\int_{\{a_i\}} f_n^n\,\mathrm{d}\mu$ converges to a finite number for all $i\in\omega$. Now whenever $E\subseteq R$ is arbitrary set of finite measure, E contains only finitely many atoms and obviously $\int_E f_n^n\,\mathrm{d}\mu$ converges to a finite number. □

Let M be an order ideal of X containing the simple functions, i.e., $(X')_b\subseteq M\subseteq X'$. To establish the $\sigma(X,M)$-convergence, it is enough to provide that our sequence $\{f_n\}$ is $\sigma(X,M)$-Cauchy, i.e., that $L_g(f_n)$ is Cauchy for every g in M.

Theorem 3.6. *Let X be a Banach function space over a totally σ-finite measure space (R,μ) and let M be an order ideal of X' containing the simple functions. Then X is $\sigma(X,M)$-sequentially complete.*

PROOF. Let $\{f_n\}$ be a $\sigma(X,M)$-Cauchy sequence. Clearly $\{f_n\}$ is $\sigma(X,M)$-bounded and since M is norm-fundamental, $\{f_n\}$ is norm-bounded by Theorem 3.2. Because M contains simple functions, the sequence of integrals

$$\int_E f_n\,\mathrm{d}\mu = \int_R f_n\chi_E\,\mathrm{d}\mu$$

is Cauchy, hence convergent to a finite number for every measurable set E of finite measure. By Theorem 3.4 there exists f in X such that f_n is $\sigma(X,(X')_b)$-convergent to f, i.e., $\int_R f_n g\,\mathrm{d}\mu \to \int_R fg\,\mathrm{d}\mu$ for every $g\in(X')_b$. We want to guarantee the convergence for every g in M.

Let $g\in M$ be an arbitrary function. Since M is an order ideal, $g\chi_E$ belongs to M as well for any measurable set E. Thus $\nu_n(E)=\int_E f_n g\,\mathrm{d}\mu$ converges to a finite number by the assumption for every measurable set E and hence by the Hahn-Saks theorem the measures ν_n are uniformly absolutely continuous with respect to μ. Moreover if we put $\nu(E)=\int_E fg\,\mathrm{d}\mu$, then also $\nu\ll\mu$.

Denote $R_n^{\mathrm{c}}=R\setminus R_n$. Since $R_n^{\mathrm{c}}\downarrow\emptyset$ then for given $\varepsilon>0$ there is some $N\in\omega$ so that $|\nu(R_N^{\mathrm{c}})|<\varepsilon$ and $|\nu_n(R_N^{\mathrm{c}})|<\varepsilon$ for every positive integer n. If we define sets $E_n=\{x\in R;\,|g(x)|>n\}$ then $E_n\downarrow\emptyset$ and we can do the same as for R_n^{c}. Assume $N\in\omega$ is the same for both sequences of sets, otherwise we take the larger one. We can finally estimate

$$\begin{aligned}\left|\int_R (f_n-f)g\,\mathrm{d}\mu\right| &\le \left|\int_{R_N^{\mathrm{c}}\cup E_N}(f_n-f)g\,\mathrm{d}\mu\right| + \left|\int_{R_N\setminus E_N}(f_n-f)g\,\mathrm{d}\mu\right|\\ &\le |\nu_n(R_N^{\mathrm{c}}\cup E_N)| + |\nu(R_N^{\mathrm{c}}\cup E_N)| + \left|\int_{R_N\setminus E_N}(f_n-f)g\,\mathrm{d}\mu\right|.\end{aligned}$$

The first two terms are bounded by 4ε independently of n and the third summand tends to zero as $n \to \infty$ by Lemma 3.3 since the set $R_N \setminus E_N$ has finite measure and g is bounded there. $\square$

Theorem 3.7. *Let X be a Banach function space over a totally σ-finite measure space (R, μ) and let M be an order ideal of X' containing the simple functions. Suppose $\{f_n\}$ is a $\sigma(X, (X')_b)$-Cauchy sequence in X and for every $g \in M$ and every sequence $\{E_k\}$ of measurable subsets of R satisfying $E_k \downarrow \emptyset$, we have*

$$\lim_{k\to\infty} \sup_{n\in\omega} \int_{E_k} |f_n g| \,\mathrm{d}\mu = 0.$$

Then $\{f_n\}$ is $\sigma(X, M)$-convergent to some function of X.

Proof. In a view of the preceding theorem it suffices to show that $\{f_n\}$ is $\sigma(X, M)$-Cauchy, i.e., $\int_R f_n g \,\mathrm{d}\mu$ is Cauchy for every $g \in M$.

Let $g \in M$ and choose an $\varepsilon > 0$. Since $R_n^{\mathrm{c}} \downarrow \emptyset$ there is an index $N \in \omega$ such that

$$\sup_{n\in\omega} \int_{R_N^{\mathrm{c}}} |f_n g| \,\mathrm{d}\mu < \varepsilon.$$

For $k \in \omega$, define functions $g_k = g\chi_{\{|g|\le k\}}\chi_{R_N}$. Every g_k is bounded and has finite-measure support, thus $g_k \in (X')_b$. Clearly $g_k(x) \to g(x)$ μ-a.e. on R_N, thus $g_k \to g$ on R_N in measure, so $M_k = \{x \in R;\ |g(x) - g_k(x)| \ge \varepsilon\} \downarrow \emptyset$. Now by the assumption pick $K \in \omega$ so large that

$$\sup_{n\in\omega} \int_{M_K} |f_n g| \,\mathrm{d}\mu < \varepsilon.$$

Recall that since $\{f_n\}$ is $\sigma(X, (X')_b)$-bounded, it is also norm-bounded in X so there is a constant C such that $\sup_{n\in\omega} \|f_n\|_X \le C$. In addition $|g_k| \le |g|$ for every positive integer k. At this moment we can estimate

$$\begin{aligned}
&\left| \int_R f_n g \,\mathrm{d}\mu - \int_R f_n g_K \,\mathrm{d}\mu \right| \\
&\le \int_{R_N^{\mathrm{c}}} |f_n(g - g_K)| \,\mathrm{d}\mu + \int_{R_N\setminus M_K} |f_n(g - g_K)| \,\mathrm{d}\mu + \int_{M_K} |f_n(g - g_K)| \,\mathrm{d}\mu \\
&\le \int_{R_N^{\mathrm{c}}} |f_n g| \,\mathrm{d}\mu + \varepsilon \int_{R_N\setminus M_K} |f_n| \,\mathrm{d}\mu + \int_{M_K} |f_n g| \,\mathrm{d}\mu + \int_{M_K} |f_n g_K| \,\mathrm{d}\mu \\
&\le \varepsilon(3 + C\|\chi_{R_N\setminus M_K}\|_{X'})
\end{aligned}$$

independently of $n \in \omega$. Note that $\|\chi_{R_N\setminus M_K}\|_{X'}$ is finite since $R_N \setminus M_K$ has finite measure. By the assumption $\int_R f_n g_K \,\mathrm{d}\mu$ is Cauchy since $g_K \in (X')_b$ and therefore $\int_R f_n g \,\mathrm{d}\mu$ is also Cauchy. $\square$

3.2 The $|\sigma|(X, X')$ topology

We will now focus on another weak topology on Banach function space X. Let $g \in X'$ and consider the functional

$$|L|_g(f) = \int_R |fg| \, \mathrm{d}\mu.$$

Evidently every such $|L|_g$ defines a semi-norm on X, furthermore if $|L|_g(f) = 0$ for all $g \in X'$ then $f(x) = 0$ μ-a.e. on R. Thus the collection $\{|L|_g;\, g \in X'\}$ defines a locally convex Hausdorff topology on X, which we will denote by $|\sigma|(X, X')$.

Since $|L_g(f)| \leq |L|_g(f) \leq \|f\|_X \|g\|_{X'}$ for every suitable f and g, the topology $|\sigma|(X, X')$ is finer than $\sigma(X, X')$ and coarser than the norm topology. More precisely let

$$O_g = \{f \in X;\, |\textstyle\int_R fg \, \mathrm{d}\mu| < \varepsilon\}$$

be a set from basis of neighbourhood of zero element in $\sigma(X, X')$ topology. Then O_g is open in $|\sigma|(X, X')$ since

$$|O|_g = \{f \in X;\, \textstyle\int_R |fg| \, \mathrm{d}\mu < \varepsilon\} \subseteq O_g.$$

Similarly the neighbourhood of $|O|_g$ contains the norm-open set

$$\{f \in X;\, \|f\|_X \|g\|_{X'} < \varepsilon\}.$$

Therefore we can write schematically as in the previous section

$$\sigma(X, X') \subseteq |\sigma|(X, X') \subseteq \sigma(X, \|\cdot\|).$$

The just presented topology was already studied by W. A. J. Luxemburg in his Ph. D. Thesis. All these presented facts are mentioned there, however not entirely proved. Will just present two theorems which are not too difficult to prove and much-needed later. Note that the Banach function space X provided with the topology $|\sigma|(X, X')$ is complete (see [WL, Chapter 3, Theorem 4]).

The following theorem characterises when a functional belongs to associate space. This claim is stated in a quite different way than in the Luxemburg's thesis (cf. [WL, Chapter 3, Lemma 7]).

Theorem 3.8. *Let X be a Banach function space over a totally σ-finite measure space (R, μ) and suppose that $g^* \in X^*$. Then there exists a function $g \in X'$ such that $g^*(f) = L_g(f)$ for every $f \in X$ if and only if $g^*(f_n) \to 0$ for every sequence $\{f_n\} \subseteq X$ satisfying $|f_n(x)| \downarrow 0$ μ-a.e. on R.*

Proof. Let $g^* = L_g$ for some $g \in X'$ and that suppose $\{f_n\} \subseteq X$ satisfy $|f_n(x)| \downarrow 0$ μ-a.e. Then

$$|g^*(f_n)| = |L_g(f_n)| \leq \int_R |f_n g| \, \mathrm{d}\mu \to 0$$

by the dominated convergence theorem.

Conversely suppose that $g^* \in X^*$ has this property. Suppose that N is fixed and let us define $\nu(E) = g^*(\chi_E)$ for every measurable subset of R_N. Since g^* is linear, ν defines a finitely additive set function. Moreover ν is a measure on R_N. Clearly $\nu(\emptyset) = 0$. Let $\{E_n\}$ be a collection of pairwise disjoint measurable sets of R_N and $E = \bigcup_{n=1}^{\infty} E_n$. Define $F_k = E \setminus \bigcup_{n=1}^{k} E_n$. Then $\chi_{F_k} \downarrow 0$ as $k \to \infty$, every χ_{F_k} belongs to X and by the hypothesis

$$\nu(E) - \sum_{n=1}^{k} \nu(E_n) = \nu(F_k) = g^*(\chi_{F_k}) \to 0$$

as $k \to \infty$. Since $\mu(E) = 0$ implies $\nu(E) = 0$, we have that $\nu \ll \mu$ and by the Radon-Nikodym theorem there is a unique function g_N, summable over R_N, satisfying $g^*(\chi_E) = \nu(E) = \int_E g_N \,\mathrm{d}\mu$ for every measurable subset of R_N, hence for all simple functions on R_N. Repeating this process for all $N \in \omega$ we obtain a sequence $\{g_N\}$ of functions such that every g_N is summable over R_N. By the uniqueness of such g_N on R_N, we have $g_{N+1} \supseteq g_N$ and $g = \bigcup_N g_N$ defines a locally summable function on R. Moreover $g^*(f) = L_g(f)$ for every simple function f supported in some R_N. If f is nonnegative, bounded and supported in some R_N then we can choose a nonnegative simple functions f_n such that $f_n \uparrow f$ uniformly, thus $f_n \to f$ in X. Then since g^* is continuous we have that

$$g^*(f) = \lim_{n\to\infty} g^*(f_n) = \lim_{n\to\infty} \int_R f_n g \,\mathrm{d}\mu = \int_R fg \,\mathrm{d}\mu = L_g(f)$$

due to the monotone convergence theorem. Now using the dominated convergence theorem, we can do the same for any bounded f having support in some R_N.

To show that g belongs to X' we can proceed as in Theorem 3.4. Let $f \in X$ be an arbitrary function such that $\|f\|_X \leq 1$ and define

$$f_n = f\chi_{\{|f|\leq n\}}\chi_{R_n} \quad \text{and} \quad \widehat{f_n} = |f_n| / \operatorname{sgn}(g).$$

Every such f_n and $\widehat{f_n}$ is bounded, has support in R_n and $|f_n(x)| \uparrow |f(x)|$ μ-a.e. on R. By the monotone convergence theorem and preceding observation we obtain

$$\begin{aligned}\int_R |fg| \,\mathrm{d}\mu = \lim_{n\to\infty} \int_R |f_n g| \,\mathrm{d}\mu &= \lim_{n\to\infty} \int_R \widehat{f_n} g \,\mathrm{d}\mu \\ &= \lim_{n\to\infty} g^*(\widehat{f_n}) \leq \limsup_{n\to\infty} \|g^*\|_{X^*} \|\widehat{f_n}\|_X \leq \|g^*\|_{X^*}.\end{aligned}$$

Finally let $f \in X$ be an arbitrary function. Define $f_n = f\chi_{\{|f|\leq n\}}\chi_{R_n}$. Then $|f_n(x) - f(x)| \downarrow 0$ μ-a.e. on R as $n \to \infty$ and by hypothesis $g^*(f_n - f) \to 0$. Hence by the dominated convergence theorem we get

$$g^*(f) = \lim_{n\to\infty} g^*(f_n) = \lim_{n\to\infty} L_g(f_n) = L_g(f).$$

□

Theorem 3.9. *Let X be a Banach function space over a totally σ-finite measure space (R, μ). Then the dual space of X provided with the $|\sigma|(X, X')$ topology is isometrically isomorphic to X'.*

Proof. Denote by $\widehat{X}$ the dual of X in the $|\sigma|(X, X')$ topology. Since $|\sigma|(X, X')$ is stronger than the $\sigma(X, X')$ topology and weaker than the norm topology the inclusions

$$L[X'] \subseteq \widehat{X} \subseteq X^*$$

hold, where L is the isometric embedding from Section 3.1. To prove that $\widehat{X} \subseteq L[X']$, according to the previous theorem it suffices to show that for every $g^* \in \widehat{X}$ and every sequence $\{f_n\} \subseteq X$ satisfying $|f_n(x)| \downarrow 0$ μ-a.e. on R, one has $g^*(f_n) \to 0$.

To finish this it is enough to show that $f_n \to 0$ in the $|\sigma|(X, X')$ topology, i.e., that $|L|_g(f_n) \to 0$ for every $g \in X'$. However this is a trivial consequence of the dominated convergence theorem. □

4. Weak compactness

In general topology there exist several notions of compactness. Let us recall some definitions.

Definition. A subset Y of a topological space is called *compact* if every open covering of Y contains a finite subcovering. A subset Y is called *relatively compact* if the closure is compact.

A subset Y of a topological space is called *countably compact* if every countable open covering of Y contains a finite subcovering. A subset Y is called *relatively countably compact* if the closure is countably compact.

A subset Y of a topological space is called *limit-point compact* if every infinite subset of Y has at least one accumulation point that belongs to Y. A subset Y is called *relatively limit-point compact* if every infinite subset of Y has at least one accumulation point.

A subset Y of a topological space is called *sequentially compact* if every sequence in Y has converging subsequence whose limit belongs to Y. A subset Y is called *relatively sequentially compact* if every sequence in Y has convergent subsequence.

It is fairly easy to observe that in any topological space the following implications hold.

compactness
$\downarrow$
countable compactness   sequential compactness
$\searrow$ $\swarrow$
limit-point compactness

Also the same holds for the relative types of these notions and none of these implications can be reversed in general. One can verify that all just presented notions of compactness and their relative counterparts coincide in metrizable topologies. However there are examples of nonmetrizable topologies where some types of compactness are equivalent. The most known one is the weak topology of Banach space for which the relative limit-point compactness implies relative compactness due to the theorem of W. F. Eberlein and also relative limit point compactness implies relative sequential compactness according to the theorem of V. Šmulian (cf. [RM, Chapter 2.8, Theorem 6]).

To evolve a similar conclusion for $\sigma(X, X')$ topology of Banach function space we will need the following generalizations of Eberlein's and Šmulian's results proved by A. Grothendieck (first published in [AG, Propositions 2 and 6]).

Theorem 4.1. *Let X be a linear space and τ_1, τ_2 two locally convex linear Hausdorff topologies on X. Suppose that (X,τ_1) is a complete space and $(X,\tau_1)^* = (X,\tau_2)^*$. Then in the topology (X,τ_2) every relatively limit-point compact set is relatively compact.*

Theorem 4.2. *Let X be a locally convex linear Hausdorff space. Suppose that X contains a countable collection of neighbourhoods of the zero element having the origin as intersection. Then in the topology $\sigma(X,X^*)$ every relatively limit-point compact set is relatively sequentially compact.*

Theorem 4.3. *Let X be a Banach function space over a totally σ-finite measure space (R,μ) and $Y \subseteq X$. Then the following are equivalent.*

(i) *The set Y is relatively $\sigma(X,X')$-compact.*
(ii) *The set Y is relatively $\sigma(X,X')$-countably compact.*
(iii) *The set Y is relatively $\sigma(X,X')$-limit-point compact.*
(iv) *The set Y is relatively $\sigma(X,X')$-sequentially compact.*

Proof. Up to the preliminaries about different types of compactness it suffices to prove implications (iii) $\to$ (i) and (iii) $\to$ (iv).

In order to prove (iii) $\to$ (i) we apply Theorem 4.1 to the space X and topologies $\tau_1 = |\sigma|(X,X')$ and $\tau_2 = \sigma(X,X')$. The dual space is X' for both cases and $|\sigma|(X,X')$ is complete as we mentioned at the beginning of the section.

The implication (iii) $\to$ (iv) follows immediately using Theorem 4.2 to the space X with the topology $|\sigma|(X,X')$ if we realise that the sets

$$\mathscr{F}_{m,n} = \{f \in X;\ \textstyle\int_R |f|\chi_{R_m}\,\mathrm{d}\mu < n^{-1}\}$$

form a countable collection of neighbourhoods of the origin satisfying

$$\bigcap_{m,n\in\omega} \mathscr{F}_{m,n} = \{0\}.$$

□

The well-known result from theory of normed spaces says that closed unit ball is weakly compact if and only if the space is reflexive. We will see that the analogue in the topology $\sigma(X,X')$ is given by a different property — absolute continuity of the norm.

To prove this theorem we need a little bit of theory of locally convex topologies. For more details see the monograph of H. H. Schaefer and M. P. Wolff [SW].

Definition. Suppose that X is a linear space and M a subspace in the algebraic dual of X. Let Y be an arbitrary subset of X. Then the *absolute polar* Y° of Y is defined by

$$Y^\circ = \{\varphi \in M;\ |\varphi(x)| \le 1, x \in Y\}.$$

Similarly we define a *polar* $^\circ N$ for any subset of M by

$$^\circ N = \{x \in X;\ |\varphi(x)| \le 1, \varphi \in N\}.$$

A symbol $Y^{\circ\circ}$ denotes a *bipolar* of a subset of X, given by $Y^{\circ\circ} = {}^\circ(Y^\circ)$.

A subset Y of X is said to be *circled* if $\lambda Y \subseteq Y$ for every $\lambda \le 1$.

The following theorem describes an important property of the bipolar. For the proof see [SW, Chapter 4, Paragraph 1.5].

Theorem 4.4. (Bipolar theorem) *Suppose that X is a linear space and M is a subspace in an algebraic dual of X. Let Y be an arbitrary subset of X. Then the bipolar $Y^{\circ\circ}$ is $\sigma(X, M)$-closed, circled convex hull of Y.*

Recall that the circled convex hull of Y is the intersection of all circled convex sets containing Y. Since that intersection is also circled and convex, such hull is the smallest set containing Y with this property.

Now we can turn back to our result for function spaces.

Theorem 4.5. *Let X be a Banach function space. Then*

$$B_{X'} = \{g \in X';\ \|g\|_{X'} \le 1\}$$

is $\sigma(X', X)$-compact if and only if X has absolutely continuous norm.

Proof. As we mentioned at the beginning of Section 3.1 the space X' is isometrically isomorphic to X^* if and only if $X = X_a$, thus if X has absolutely continuous norm then $\sigma(X', X)$ and $\sigma(X^*, X)$ could be considered as the same topologies on X'. Hence according to the Banach-Alaoglu theorem, the closed unit ball is $\sigma(X', X)$-compact.

Suppose now that $B_{X'}$ is $\sigma(X', X)$-compact. Since X' could be isometrically embedded into X^*, we can consider $B_{X'}$ as a subset in B_{X^*} with the weak* topology. Since $B_{X'}$ is circled and convex set it is its own circled convex hull. By the hypothesis, $B_{X'}$ is also weak* compact, hence weakly* closed. By the Bipolar theorem therefore $B_{X'} = (B_{X'})^{\circ\circ}$. By the definition of polar we can calculate

$$(B_{X'})^\circ = \{f \in X;\ |L_g(f)| \le 1, g \in B_{X'}\} = B_X,$$
$$^\circ(B_X) = \{\varphi \in X^*;\ |\varphi(f)| \le 1, f \in B_X\} = B_{X^*}.$$

Thus we can conclude that $B_{X'} = B_{X^*}$ and by linearity $X' = X^*$, which gives us $X = X_a$. □

As an immediate consequence we can state the following corollary.

Corollary 4.6. *Let X be a Banach function space. Then the set B_X is $\sigma(X, X')$-compact if and only if X' has absolutely continuous norm.*

Now we will return to the end of Section 3.1 and we give a necessary and sufficient condition for a subset of X to be relatively $\sigma(X, M)$-sequentially compact in the case that the underlying measure is separable. Let us start with a definition.

Let (R, μ) be a measure space. Denote by $\mathscr{R}$ the family of all measurable subsets of R of finite measure and define a map $\varrho: \mathscr{R} \times \mathscr{R} \to \mathbb{R}$ by the formula

$$\varrho(E, F) = \int_R |\chi_E - \chi_F| \, \mathrm{d}\mu = \mu(E \triangle F),$$

where $\triangle$ denotes the symmetric difference. Then up to the equivalence $E \sim F \leftrightarrow \varrho(E, F) = 0$ the map ϱ defines a complete metric on $\mathscr{R}$.

Definition. A measure μ on R is said to be *separable* if the corresponding measure space $(\mathscr{R}, \varrho)$ is separable.

Theorem 4.7. *Let X be a Banach function space over a totally σ-finite separable measure space (R, μ) and let M be an order ideal of X' containing the simple functions. Let $Y \subseteq X$. Then the following are equivalent.*

(i) *Y is relatively $\sigma(X, M)$-sequentially compact.*

(ii) *Y is bounded and whenever $g \in M$ then*

$$\lim_{n\to\infty} \sup_{f\in Y} \int_{E_n} |fg| \, \mathrm{d}\mu = 0$$

for every sequence $\{E_n\}$ of measurable subsets of R satisfying $E_n \downarrow \emptyset$.

Proof. Let Y be relatively $\sigma(X, M)$-sequentially compact. Then Y is $\sigma(X, M)$-bounded and, by Theorem 3.2, bounded in X. Suppose that there exists some $\varepsilon > 0$, some g in M and a sequence E_n of measurable subsets of R such that $E_n \downarrow \emptyset$ and $\sup_{f\in Y} \int_{E_n} |fg| \, \mathrm{d}\mu > \varepsilon$ for every $n \in \omega$. Thus we can choose a sequence $\{f_n\}$ in Y such that $\int_{E_n} |f_n g| \, \mathrm{d}\mu > \varepsilon$. In view of compactness we can assume that $\{f_n\}$ is $\sigma(X, M)$-convergent, as otherwise we can pass to a subsequence. Since M is an order ideal, the function $g\chi_E$ belongs to M for every measurable subset $E \subseteq R$, and $\int_E f_n g \, \mathrm{d}\mu = \int f_n g \chi_E \, \mathrm{d}\mu$ converges to a finite number. By the Hahn-Saks theorem the measures $\nu_n(E) = \int_E |f_n g| \, \mathrm{d}\mu$ are uniformly absolutely continuous with respect to μ and hence $\int_{E_n} |f_n g| \, \mathrm{d}\mu$ converges to zero as $n \to \infty$, which is absurd.

Suppose conversely that Y satisfies (ii) and let $\{f_k\}$ be some sequence in Y. We will show that $\int_E f_{k_l} \, \mathrm{d}\mu$ converges for some subsequence $\{f_{k_l}\}$ to a finite number for every measurable set E contained in some R_n. By the virtue of Theorem 3.4 and the remark following it, we obtain that $\{f_{k_l}\}$ is $\sigma(X, (X')_b)$-convergent and hence by Theorem 3.7, we conclude that $\{f_{k_l}\}$ is $\sigma(X, M)$ convergent to some f in X.

Since R is separable, there exists for every positive integer n a countable family $\mathcal{E}_n$ which is dense in the measurable subsets of R_n of finite measure. The union of such $\mathcal{E}_n$ contains countably many sets, say $\{E_1, E_2, \dots\}$. We will now reproduce the same process as in Example 3.5. Since Y is bounded, the sequence $\{f_k\}$ is $\sigma(X, M)$-bounded. Every characteristic function of a set of finite measure belongs to M, so $\{\int_{E_1} f_k \,\mathrm{d}\mu\}$ is bounded. We can therefore choose a subsequence, say $\{f_k^1\}$, such that the integrals converge as $k \to \infty$. By induction $\{\int_{E_{n+1}} f_k^n \,\mathrm{d}\mu\}$ is bounded and there is a subsequence $\{f_k^{n+1}\}$ such that $\int_{E_{n+1}} f_k^{n+1} \,\mathrm{d}\mu$ converges to a finite number as $k \to \infty$. We have for the diagonal sequence that $\int_{E_n} f_k^k \,\mathrm{d}\mu$ converges to a finite number for every positive integer n.

Let E be any measurable set bounded with respect to $\{R_n\}$, i.e., $E \subseteq R_N$ for some fixed $N \in \omega$. Suppose $g = \chi_{R_N}$, then $g \in M$ and by the assumption we have that to given $\varepsilon > 0$ one can choose a δ such that

$$\sup_{k\in\omega} \int_{E\cap R_N} \left|f_k^k\right| \mathrm{d}\mu < \varepsilon$$

whenever E is measurable set such that $\mu(E) < \delta$. Since $\mathcal{E}_N$ is dense in (R_N, μ) and $E \subseteq R_N$, there is a set $E_K \in \mathcal{E}_N$ such that $\mu(E \triangle E_K) < \delta$. Hence we have

$$\left| \int_E f_k^k \,\mathrm{d}\mu - \int_{E_K} f_k^k \,\mathrm{d}\mu \right| \leq \int_{E\triangle E_K} \left|f_k^k\right| \mathrm{d}\mu < \varepsilon$$

and therefore $\int_E f_k^k \,\mathrm{d}\mu$ converges to a finite number as $k \to \infty$. □

If we put $M = X'$, we can easily prove the same result even without the assumption of separability.

Theorem 4.8. *Let X be a Banach function space over a totally σ-finite measure space (R, μ) and let Y be a subset in X. Then the following are equivalent.*

(i) *Y is relatively $\sigma(X, X')$-compact.*

(ii) *Y is bounded and whenever $g \in X'$ then*

$$\lim_{n\to\infty} \sup_{f\in Y} \int_{E_n} |fg| \,\mathrm{d}\mu = 0$$

for every sequence $\{E_n\}$ of measurable subsets of R satisfying $E_n \downarrow \emptyset$.

Proof. According to Theorem 4.3 the condition (i) implies relative $\sigma(X, X')$-compactness of Y and the rest of the proof of the implication (i)→(ii) is exactly the same as in Theorem 4.7, since the hypothesis of separability did not appear there.

Suppose conversely that (ii) holds. As $X = X''$ could be isometrically embedded into $(X')^*$, we can consider Y to be a bounded set in $(X')^*$ endowed with the weak* topology $\sigma((X')^*, X')$. To prove that Y is relatively compact it suffices to show that the closure $\overline{Y}^{w^*}$ is a part of $X'' = X$. Let $y^* \in \overline{Y}^{w^*}$. The weak* topology is the topology of pointwise convergence, hence any subbase neighbourhood of an element y^* is of the form

$$O_{g,\varepsilon} = \{\varphi \in (X')^*;\ |\varphi(g) - y^*(g)| < \varepsilon\}.$$

Since y^* belongs to the closure of Y, any such neighbourhood $O_{g,\varepsilon}$ contains an element of Y, say f. Hence we have for every $\varepsilon > 0$ and $g \in X'$ that

$$|y^*(g)| \leq |L_f(g)| + \varepsilon \leq \int_R |fg| \, \mathrm{d}\mu + \varepsilon \leq \sup_{f \in Y} \int_R |fg| \, \mathrm{d}\mu + \varepsilon$$

thus $|y^*(g)| \leq \sup_{f \in Y} \int_R |fg| \, \mathrm{d}\mu$ for every $g \in X'$. By the virtue of Theorem 3.8 it suffices to show that $y^*(g_n) \to 0$ whenever $|g_n(x)| \downarrow 0$ μ-a.e. on R. The rest of the proof is just an application of standard method also used in Theorem 3.7. Since $R_n^{\mathrm{c}} \downarrow \emptyset$, then to given $\varepsilon > 0$, take $N \in \omega$ so large that

$$\sup_{f \in Y} \int_{R_N^{\mathrm{c}}} |fg_0| \, \mathrm{d}\mu < \varepsilon.$$

Since R_N has finite measure, g_n tends to zero in the measure on R_N and hence $M_k = \{x \in R_N;\ |g_k(x)| \geq \varepsilon\} \downarrow \emptyset$. By hypothesis we can pick $K \in \omega$ so large that

$$\sup_{f \in Y} \int_{M_K} |fg_0| \, \mathrm{d}\mu < \varepsilon.$$

Since Y is norm-bounded we have some $C > 0$ such that $\|f\|_X \leq C$ for every $f \in Y$. Moreover $|g_n| \leq |g_0|$ for every $n \in \omega$. Now we have for every $n \geq K$ that

$$\begin{aligned} |y^*(g_n)| &\leq \sup_{f \in Y} \int_R |fg_n| \, \mathrm{d}\mu \\ &\leq \sup_{f \in Y} \int_{R_N^{\mathrm{c}}} |fg_n| \, \mathrm{d}\mu + \sup_{f \in Y} \int_{R_N \setminus M_K} |fg_n| \, \mathrm{d}\mu + \sup_{f \in Y} \int_{M_K} |fg_n| \, \mathrm{d}\mu \\ &\leq \varepsilon + C\varepsilon \|\chi_{R_N \setminus M_K}\|_{X'} + \varepsilon. \end{aligned}$$

and $y^*(g_n)$ tends to zero, hence $y^* \in X''$. □

The following corollary just pulls together all the presented results about compactness in the topology $\sigma(X, X')$.

Corollary 4.9. *Let X be a Banach function space over a totally σ-finite measure space (R, μ) and $Y \subseteq X$. Then the following are equivalent.*

(i) *The set Y is relatively $\sigma(X, X')$-compact.*
(ii) *The set Y is relatively $\sigma(X, X')$-countably compact.*
(iii) *The set Y is relatively $\sigma(X, X')$-limit-point compact.*
(iv) *The set Y is relatively $\sigma(X, X')$-sequentially compact.*
(v) *The set Y is bounded and*

$$\lim_{n \to \infty} \sup_{f \in Y} \int_{E_n} |fg| \, \mathrm{d}\mu = 0$$

for every $g \in X'$ and every sequence $\{E_n\}$ of measurable subsets of R satisfying $E_n \downarrow \emptyset$.

Thanks to this corollary, we can now prove Theorem 4.5 and its corollary without the Bipolar theorem. Indeed,

$$\sup_{\|f\|_X \leq 1} \int_{E_n} |fg| \, \mathrm{d}\mu = \sup_{\|f\|_X \leq 1} \int_R |fg\chi_{E_n}| \, \mathrm{d}\mu = \sup_{\|f\|_X \leq 1} \left| \int_R fg\chi_{E_n} \, \mathrm{d}\mu \right| = \|g\chi_{E_n}\|_{X'}$$

and we obtain immediately that if B_X is $\sigma(X, X')$-compact then for every $g \in X'$ $\lim_{n\to\infty} \|g\chi_{E_n}\|_{X'} = 0$ whenever $E_n \downarrow \emptyset$ i.e., X' has absolutely continuous norm. The converse still follows from the Banach-Alaoglu theorem.

5. Linear operators

In this section we will apply the results from the preceding sections to linear operators between Banach function spaces. Similarly as we have seen that the topologies $\sigma(X, X')$ and $\sigma(X, X^*)$ have some properties in common, it is natural to ask if some facts about operators between weak topologies have their counterparts in $\sigma(X, X')$ topology. We will also introduce the concept of the adjoint operator and reveal its basic properties; we will for example show that the famous Gantmacher theorem can not be restated in the words of $\sigma(X, X')$ topology in general.

To simplify the notation we will also denote the topology $\sigma(X, X')$ as weak$'$ or shortly w', similarly as w denotes the standard weak topology $\sigma(X, X^*)$.

Definition. Let X and Y be Banach function spaces over measure spaces (R, μ) and (S, ν) respectively. Given a linear operator $T: X \to Y$, define its adjoint $T': Y' \to X'$ via the identity

$$\int_S T(f) g \,\mathrm{d}\nu = \int_R T'(g) f \,\mathrm{d}\mu$$

for every $f \in X$ and $g \in Y'$, whenever the integrals converge.

Since $X'' = X$ and $Y'' = Y$ we immediately obtain that the second adjoint of T coincides with T. So this concept of adjoint operator is more "symmetric" than the notion of the classical dual operator.

Let us turn our attention to the existence of adjoint operators. Since the formula $\varphi(f) = \int_S T(f) g \,\mathrm{d}\nu$ defines a bounded functional on X, then according to Theorem 3.8 it is necessary and sufficient to $\varphi(f_n) \to 0$ whenever $|f_n(x)| \downarrow 0$ μ-a.e.

Theorem 5.1. *Let $T: X(R, \mu) \to Y(S, \nu)$ be a bounded linear operator between Banach function spaces. Then T has an adjoint operator if and only if T is weak$'$-to-weak$'$ continuous.*

Proof. Suppose that T has an adjoint operator T'. According to Lemma 3.1 $T: X \to Y$ is weak$'$-to-weak$'$ continuous if and only if $L_g \circ T$ is weak$'$ continuous for every $g \in Y'$. Therefore since $L_g \circ T = L_{T'(g)}$ is continuous if and only if it is weak$'$ continuous, we are done.

Suppose conversely that T is weak$'$-to-weak$'$ continuous and let $\{f_n\}$ be a subset of X satisfying $|f_n(x)| \downarrow 0$ μ-a.e. on R. By the dominated convergence theorem we have $L_g(f_n) \to 0$ for every $g \in X'$ so $f_n \xrightarrow{w'} 0$. Thus $T(f_n) \xrightarrow{w'} 0$

and $\varphi(f_n) = \int_S T(f_n) g \,\mathrm{d}\nu \to 0$ for every $g \in Y'$. Hence for every $g \in Y'$ there exists some $h \in X'$ such that $\varphi(f) = \int_R f h \,\mathrm{d}\mu$. To end the proof it suffices to put $T'(g) = h$. □

Definition. Let X and Y be Banach function spaces over measure spaces (R, μ) and (S, ν) respectively. Denote by $\mathcal{W}(X, Y)$ the set of weak$'$-to-weak$'$ continuous linear operators from X into Y. Let $\mathcal{B}(X, Y)$ stand for all bounded linear operators from X into Y.

We have $\mathcal{W}(X, Y) \subseteq \mathcal{B}(X, Y)$ immediately since weak$'$ continuity of $L_g \circ T$ implies norm continuity.

Notice the difference between weak and weak$'$ topology. Suppose that $T \colon X \to Y$ is bounded, i.e., norm-to-norm continuous. This happens if and only if $y^* T[B_X]$ is bounded for every $y^* \in Y^*$, i.e., $y^* \circ T \in X^*$. Since $y^* \circ T$ is weakly continuous if and only if it is continuous we obtain that T is norm-to-norm continuous if and only if T is weak-to-weak continuous.

Unfortunately such argument can not be repeated for weak$'$ topology since the boundedness of $L_g \circ T$ does not imply weak$'$ continuity in general.

Theorem 5.2. *Let X and Y be Banach function spaces over measure spaces (R, μ) and (S, ν) respectively. Suppose $T \in \mathcal{W}(X, Y)$. Then T' is bounded linear operator from Y' into X' and $\|T\| = \|T'\|$. If Z is another Banach function space and $S \in \mathcal{W}(Y, Z)$ then $ST \in \mathcal{W}(X, Z)$ and $(ST)' = T'S'$.*

Proof. According to Theorem 5.1 the adjoint T' is well-defined. The linearity of T' is evident from its definition. Since $T \in \mathcal{W}(X, Y) \subseteq \mathcal{B}(X, Y)$, we have that T is bounded, i.e.,

$$\|T\| = \sup_{\|f\|_X \le 1} \|T(f)\|_Y = \sup_{\|f\|_X \le 1} \sup_{\|g\|_{Y'} \le 1} \left| \int_S T(f) g \,\mathrm{d}\nu \right| < \infty.$$

Therefore we can calculate that

$$\begin{aligned} \|T'\| &= \sup_{\|g\|_{Y'} \le 1} \|T'(g)\|_{X'} = \sup_{\|g\|_{Y'} \le 1} \sup_{\|f\|_X \le 1} \left| \int_R T'(g) f \,\mathrm{d}\mu \right| = \\ &= \sup_{\|g\|_{Y'} \le 1} \sup_{\|f\|_X \le 1} \left| \int_S T(f) g \,\mathrm{d}\nu \right| = \sup_{\|f\|_X \le 1} \sup_{\|g\|_{Y'} \le 1} \left| \int_S T(f) g \,\mathrm{d}\nu \right| = \|T\|. \end{aligned}$$

The rest of the theorem follows immediately from the definition of the adjoint operator and we omit it. □

Definition. Let T be a linear operator from a Banach space X into a Banach function space Y. Then T is said to be $\sigma(Y, Y')$-compact or shortly weakly$'$ compact if T maps bounded sets to relatively $\sigma(Y, Y')$-compact sets.

We can also define a sequentially weakly$'$ compact operator in a similar way but according to Theorem 4.3 all such notions are equivalent.

The following results places weak$'$ compactness between weak compactness and boundedness as a property for linear operators. The former follows from the fact that $\sigma(X, X')$ is weaker than $\sigma(X, X^*)$, while the latter is a consequence of the Theorem 3.2.

Theorem 5.3. *Every weakly compact linear operator from a Banach space into a Banach function space is weakly$'$ compact.*

Theorem 5.4. *Every weakly$'$ compact linear operator from a Banach space into a Banach function space is bounded.*

Also the following result is an immediate consequence of the definition of a weakly$'$ compact operator.

Theorem 5.5. *Let X, Y and Z be Banach function spaces and suppose $T \in \mathcal{W}(X, Y)$ and $S \in \mathcal{W}(Y, Z)$. If either T or S is weakly$'$ compact, then ST is weakly$'$ compact.*

It is well known that in the weak topology a bounded linear operator T between Banach spaces X and Y is automatically weakly compact if either X or Y is reflexive. For linear operators between Banach function spaces the theory differs.

Theorem 5.6. *Let X and Y be Banach function spaces and $T \in \mathcal{W}(X, Y)$. Then T is $\sigma(Y, Y')$ compact if either X' or Y' is of absolutely continuous norm.*

Proof. This follows easily from the relative weak$'$ compactness of bounded sets established in Corollary 4.6. □

Moreover if in the previous theorem the space Y' has absolutely continuous norm, it suffices that T is weakly$'$ compact if T is bounded and X is an arbitrary Banach space.

Corollary 5.7. *Let X be a Banach space, Y be a Banach function space with separable associate space Y'. Then every bounded linear operator T from X into Y is weakly$'$ compact.*

Proof. The separability of Y' implies that $(Y')_a = Y'$ (see [BS, Chapter 1, Corollary 5.6]), hence the conclusion is obvious due to the previous theorem and the remark following it. □

Using Corollary 4.9 one can easily verify the equivalence of the following characterizations of weak$'$ compactness for linear operators.

Theorem 5.8. *Let T be a linear operator from a Banach space X into a Banach function space $Y(S, \nu)$. Then the following are equivalent.*

(i) *T is weakly$'$ compact.*

(ii) *$T[B_X]$ is relatively weakly$'$ compact.*
(iii) *Every bounded sequence $\{f_n\}$ in X has a subsequence $\{f_{n_k}\}$ such that $T(f_{n_k})$ converges in weak$'$ topology.*
(iv) *The T is bounded and*

$$\lim_{n\to\infty} \sup_{\|f\|_X\le 1} \int_{E_n} |T(f)g| \,\mathrm{d}\nu = 0$$

for every $g \in Y'$ and every sequence $\{E_n\}$ of measurable subsets of S satisfying $E_n \downarrow \emptyset$.

The very famous Gantmacher theorem (see [RM, Section 3.5, Theorem 13]) says that an operator T between Banach spaces is weakly compact if and only if its dual is. As we claimed at the beginning of the chapter no such analogy holds in function spaces in general. The trivial counterexample exploits the asymmetry of the condition in Theorem 5.6.

Example 5.9. Consider the identity operator from ℓ_∞ onto ℓ_∞ with weak$'$ topology. Since $(\ell_\infty)' = \ell_1$ and ℓ_1 is of absolutely continuous norm, we have by Theorem 5.6 that such identity is weakly compact. Clearly the adjoint operator is the identity operator from ℓ_1 onto ℓ_1. Such operator can not be $\sigma(\ell_1, \ell_\infty)$-compact since the bounded sequence $\{e_n\}$ of unit vectors does not have a convergent subsequence in w'=w topology.

At the end of the section we will put in context the notions of weakly$'$ compact operator with so-called uniformly absolutely continuous operators.

Definition. Let X be a Banach space and let Y be a Banach function space over a measurable space (S, ν). Let $T\colon X \to Y$ be a bounded operator. Then T is rumored to be uniformly absolutely continuous if

$$\lim_{n\to\infty} \sup_{\|f\|_X\le 1} \|\chi_{E_n} T(f)\|_Y = 0$$

for every sequence $\{E_n\}$ of measurable subsets of S such that $E_n \downarrow \emptyset$. In this case we write $T\colon X \overset{*}{\longrightarrow} Y$.

Theorem 5.10. *Let X be a Banach space and let Y a Banach function space over a totally σ-finite measure space (S, ν) and suppose $T\colon X \overset{*}{\longrightarrow} Y$. Then T is weakly$'$ compact.*

PROOF. Let $T\colon X \overset{*}{\longrightarrow} Y$. By the Hölder inequality, we have that

$$\lim_{n\to\infty} \sup_{\|f\|_X\le 1} \int_{E_n} |T(f)g| \,\mathrm{d}\nu \le \lim_{n\to\infty} \sup_{\|f\|_X\le 1} \|\chi_{E_n} T(f)\|_Y \|g\|_{Y'} = 0$$

whenever $g \in Y'$ and $\{E_n\}$ are measurable subsets of S satisfying $E_n \downarrow \emptyset$. Hence by Theorem 5.8, T is weakly$'$ compact. $\square$

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