Some results in convexity and in Banach space theory

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The thesis consists of four independent chapters, each comprising a research paper. The first three of these papers have already been published and the fourth is under review.

Chapter 1 is concerned with points of simpliciality. If K is a compact Hausdorff space and $\mathcal{H} \subseteq C(K)$ is a function space, then $x \in K$ is a *point of simpliciality* if there exists a unique maximal measure ν (with respect to the Choquet order), such that $\nu(f) = f(x)$ for all $f \in \mathcal{H}$. The set of such points is denoted $\mathrm{Sim}_{\mathcal{H}}(K)$. If K is metrisable, then $\mathrm{Sim}_{\mathcal{H}}(K)$ possesses a number of good properties, such as being a G_δ. It is natural to ask whether any of these properties are preserved in the non-metrisable case. The candidate presents a series of examples to show that this is not so. There are two sets of examples. In the first set, the compact spaces K are subsets of $\mathbb{R}^3$, suitably topologised, and in the second K is the convex state space of a function space from the first set, with $\mathcal{H}$ being the natural function space $A(K)$ of affine functions on K. The second set of examples shows that the good properties alluded to above are not preserved even if convexity is assumed.

In Chapter 2, the candidate addresses the notion of remotality. A subset E of a real Banach space X is *remotal* from $x \in X$, if there exists $e \in E$ such that $\|x - e\| \geq \|x - z\|$ whenever $z \in E$. The candidate reproves a result of Martín and Rao. Specifically, if X is not reflexive, then there exists a closed bounded convex subset of X that is not remotal. The argument given is swift and clean. In addition, Martín and Rao asked, given a weakly closed and bounded subset $E \subseteq X$, whether the remotality of $\overline{\mathrm{conv}}(E)$ from some $x \in X$ implies the remotality of E from x. The candidate provides a straightforward counterexample to this question when $X = c_0$.

In the third chapter, the candidate explores linear identities and their connection to polynomials, and develops an abstract framework for their study, centred around the idea of compatibility. If $a_i \in \mathbb{R}$, $x_i \in \mathbb{R}^m$, $1 \leq i \leq k$, and $f : X \longrightarrow Y$ is a continuous map between real Banach spaces, then the candidate says that f is *compatible* with $\mathbf{x} = \sum_{i=1}^k a_i \delta_{x_i}$ if $\sum_{i=1}^k a_i f(L(x_i)) = 0$ whenever $L : \mathbb{R}^m \longrightarrow X$ is bounded and linear. Here, the $\delta_{x_i} \in C(\mathbb{R}^m)^*$ are the standard evaluations. Using the generalised Lagrange formula, a method of generating linear identities is developed from the notion of compatibility, which can be used to characterise polynomials of a certain degree. A result resembling Hilbert's lemma is proved. Under natural assumptions on $\mathbf{x}$, it is shown that polynomials are the only continuous maps that are compatible with $\mathbf{x}$. The work in this chapter was conducted jointly with P. Hájek.

The final chapter is devoted to coarse and uniform embeddings between Orlicz spaces. The general theory of such embeddings has received considerable attention from well-known mathematicians in recent years. The candidate is able to determine whether or not h_M coarsely or uniformly embeds into h_N when their upper Matuszewska-Orlicz indices differ (in most cases). Examples are given which show that these indices are sometimes not sufficient to determine such embeddability relations.

Listed below are a few specific comments. The first three are minor remarks. The last one is a query that I would like the candidate to address. Subject to the satisfactory resolution of this query, I recommend that the candidate be awarded the PhD. He draws upon a variety of techniques from general topology, measure theory, Banach space theory, linear algebra, convexity and real analysis, and has clearly demonstrated that he is able to produce creative scientific work in a number of independent fields.

Also enclosed are some minor corrections and typographical errors. I should stress that the overall standard of English in the thesis is high.

Comments/Queries

1. Perhaps the argument in Example 2.0.1, showing that E is weakly closed, could be tightened a little. If $x \in \overline{E}^w \setminus E$, then $E \cap U$ is infinite whenever $U \ni x$ is weakly open. In particular, if $m \in \mathbb{N}$ then $x_n \in U$ for some $n \geq m$. Hence, given $k \geq 2$, if $|x^k| \neq 1$, then by considering the weakly open set $\{z \in c_0 \ : \ |z^k| \neq 1\}$, we see that $|x_n^k| \neq 1$ for some $n \geq k$, which is clearly not so. Therefore $|x^k| = 1$ whenever $k \geq 2$, contradicting the fact that $x \in c_0$.

2. This comment concerns notation. In Chapter 3, I would be inclined to distinguish elements $x \in \mathbb{R}^n$ from their counterparts in $\mathcal{F}(\mathbb{R}^n)$ by writing e.g. δ_x (or ε_x, following the notation of Chapter 1). In my opinion, the distinction between $a\delta_x$ and δ_{ax} is clearer, and this approach means that $\boxplus$ can be avoided altogether.

3. On page 29, the relation $\prec$ looks like a preorder, rather than an order. For instance, if x and y are non-zero elements of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, then it is always possible to choose linear $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ satisfying $Lx = y$, and thus $\delta_x \succ \delta_y$.

4. I am not sure how one passes from equation (3.5) to the equation at the top of page 34. If we replace x by $x - \frac{s_1}{r_1}x$ and y by $y + x$, then

$$\begin{aligned} r_i(x - \tfrac{s_1}{r_1}x) + s_i(y+x) + \Delta_{0,i}(x - \tfrac{s_1}{r_1}x) &= r_i x + s_i y + \Delta_{0,i}x - r_i\tfrac{s_1}{r_1}x + s_i x - \Delta_{0,i}\tfrac{s_1}{r_1}x \\ &= r_i x + s_i y + (\Delta_{1,i} + \Delta_{0,i})x - \Delta_{0,i}\tfrac{s_1}{r_1}x \end{aligned}$$

   So I get an extra term $-\Delta_{0,i}\frac{s_1}{r_1}x$, which does not appear in the equation at the top of page 34.