Charles University Prague<br>Faculty of Mathematics and Physics

## DOCTORAL THESIS



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# Numerical Solution of a Fredholm Integral Equation of the Second Kind Related to Induction Heating 

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I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.
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Název práce: Numerické řešení Fredholmovy integrální rovnice druhého druhu související s indukčním ohřevem.

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Abstrakt: Tato práce se zabývá numerickým řešením integrálních rovnic druhého druhu se singulárním jádrem popisujícím indukční ohřev. Numerické řešení využívá kolokační a Nyströmovy metody. V případě kolokačních metod je neznámá funkce aproximována lineární kombinací bázových funkcí (nejčastěji polynomů určitého stupně) tak, aby na předem zvolených bodech odpovídala přesnému řešení. Nyströmova metoda je založena na nahrazení integrálu v integrální rovnici numerickou kvadraturou nebo kubaturovu. Tato práce popisuje obě metody. V této práci jsou odvozeny odhady chyb. Odhady chyb jsou v jednoduchých příkladech srovnány s přesným řešením.

Klíčová slova: integrální rovnice druhého druhu, indukční ohřev, numerické řešení, kolokační metody, Nyströmovy metody

Title: Numerical solution of a Fredholm integral equation of the second kind related to induction heating

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Abstract: This thesis deals with numerical solution of an integral equation of the second kind with special singular kernel function related to induction heating. The numerical solution is based on collocation and Nyström methods. The idea of collocation methods is to choose a finite-dimensional space of candidate solutions (usually polynomials up to a certain degree). The Nyström methods are based on approximation of the integral in equation by numerical integration rule. This thesis describes and gives error estimates of both methods. Error estimates are compared to the exact solutions in simple cases.

Keywords: integral equation of the second kind, induction heating, numerical solution, collocation methods, Nyström methods

## Contents

Introduction ..... 2
1 Induction heating model ..... 3
2 Basics of functional analysis ..... 7
2.1 Normed linear and Banach spaces ..... 7
2.2 Linear operators ..... 8
2.3 Compact operators ..... 11
2.4 Collectively compact operators ..... 13
2.5 Metric spaces and totally bounded sets ..... 14
2.6 Eigenvalues and invertibility of operator $(\lambda \mathcal{I}-\mathcal{T})$ ..... 15
3 Existence and uniqueness of solution of induction heating model ..... 17
4 Integral equation of the second kind with diagonal singularity ..... 23
5 Collocation methods ..... 24
5.1 General theory ..... 24
5.2 Convergence condition ..... 27
5.3 Piecewise constant collocation ..... 28
6 Nyström method ..... 34
6.1 Integration rule and kernel function conditions ..... 35
6.2 Convergence of Nyström method ..... 38
7 Special Numerical integration rules ..... 55
7.1 One dimensional case ..... 55
7.2 Integration rule in higher dimensions ..... 61
8 Example in one dimensional case ..... 88
8.1 Collocation method ..... 89
8.1.1 Piecewise constant collocation ..... 89
8.1.2 Piecewise linear collocation ..... 91
8.2 Nyström method ..... 96
8.2.1 Nyström method 1 ..... 98
8.2.2 Nyström method 2 ..... 104
8.3 Method comparison ..... 112
9 Numerical Solution of Induction Heating Model ..... 114
9.1 Collocation method ..... 114
9.2 Nyström method ..... 117
9.3 Example of induction heating ..... 119
Conclusion ..... 127
Appendix ..... 128
List of literature ..... 130
Attachments ..... 133

## Introduction

This thesis deals with solution of a Fredholm integral equation of the second kind with a singular kernel function. The kernel has a special type of singularity called diagonal singularity. Many numerical methods were developed for solving integral equations of the second kind. In this thesis is examined application of collocation and modified Nyström method. The main goal of this thesis is the proof of convergence, error estimation of modified Nyström method and comparing error behavior of both methods in case of diagonal singular kernel function.

Fredholm integral equations of the second kind describe many physical problems. One example is the integral equation for computation of the eddy currents of density in induction heating problem, which are needed to compute the specific Joule losses in the heated body (explained in [9]). The Joule losses are necessary for calculating temperature distribution in the body.

Another purpose of this thesis is the proof of existence and uniqueness of the solution of the eddy currents of density, applying collocation and Nyström method for the induction heating problem, showing its convergence and computation of illustrative example.

## 1. Induction heating model



Figure 1.1: Heated body and coil.
A bounded metal body $\Omega_{1}$ with a Lipschitz-continuous boundary is heated by external electromagnetic field produced by inductor $\Omega_{2}$ (see fig. 1.1). The inductor is formed by a conductor of general shape and position that carries harmonic current. Due to absence of ferromagnetic parts all electromagnetic quantities may be expressed in terms of their phasors.

The derivation of mathematical model was showed in [9]. Let us here rewrite most important facts. Let's choose a point $x=\left(x_{1}, x_{2}, x_{3}\right)$ within the metal body. Phasor $\vec{A}$ of the potential $\overrightarrow{A(x)}$ at this point is given by superposition of two components generated by the field current $I_{\text {ext }}$ in the inductor $\overrightarrow{A(s x)}$ and eddy currents of density $J_{\text {eddy }}$ produced within the body $\overrightarrow{A(t x)}$. The symbol $t=\left(t_{1}, t_{2}, t_{3}\right)$ means another point in the body (different from $\left.x\right)$ and $s=\left(s_{1}, s_{2}, s_{3}\right)$ is a point at the inductor.

$$
\begin{equation*}
\overrightarrow{A(x)}=\overrightarrow{A(t x)}+\overrightarrow{A(s x)} \tag{1.1}
\end{equation*}
$$

where

$$
\begin{aligned}
\overrightarrow{A(s x)} & =\frac{\mu_{0} I_{e x t}}{4 \pi} \int_{\Omega_{2}} \frac{d l(s)}{\sqrt{\left(x_{1}-s_{1}\right)^{2}+\left(x_{2}-s_{2}\right)^{2}+\left(x_{3}-s_{3}\right)^{2}}}, \\
\overrightarrow{A(t x)} & =\frac{\mu_{0}}{4 \pi} \int_{\Omega_{1}} \frac{J_{e d d y}(t)}{\sqrt{\left(x_{1}-t_{1}\right)^{2}+\left(x_{2}-t_{2}\right)^{2}+\left(x_{3}-t_{3}\right)^{2}}} d t_{1} d t_{2} d t_{3} .
\end{aligned}
$$

Here $\mu_{0}$ is the permeability of vacuum, $d l(s)$ a length element of the inductor. All remaining quantities follow from fig. 1. For future use let us sign

$$
r(x, t)=\sqrt{\left(x_{1}-t_{1}\right)^{2}+\left(x_{2}-t_{2}\right)^{2}+\left(x_{3}-t_{3}\right)^{2}}
$$

and

$$
r(x, s)=\sqrt{\left(x_{1}-s_{1}\right)^{2}+\left(x_{2}-s_{2}\right)^{2}+\left(x_{3}-s_{3}\right)^{2}} .
$$

The Maxwell equations [8] yield that

$$
\begin{equation*}
\operatorname{rot} E=-\frac{\partial B}{\partial t} \tag{1.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{div}(B)=0 \tag{1.3}
\end{equation*}
$$

where $E$ is called electric field intensity and $B$ is called magnetic induction. Operators rot and div for $F=\left(F_{x_{1}}, F_{x_{2}}, F_{x_{3}}\right)$ expressed in cartesian coordinates are

$$
\operatorname{rot} F=\nabla \times F=\left(\frac{\partial F_{x_{3}}}{\partial x_{2}}-\frac{\partial F_{x_{2}}}{\partial x_{3}}, \frac{\partial F_{x_{1}}}{\partial x_{3}}-\frac{\partial F_{x_{3}}}{\partial x_{1}}, \frac{\partial F_{x_{2}}}{\partial x_{1}}-\frac{\partial F_{x_{1}}}{\partial x_{2}}\right)
$$

and

$$
\operatorname{div} F=\nabla \cdot F=\frac{\partial F_{x_{1}}}{\partial x_{1}}+\frac{\partial F_{x_{2}}}{\partial x_{2}}+\frac{\partial F_{x_{3}}}{\partial x_{3}} .
$$

Due to (1.3) we can use that

$$
B=\operatorname{rot} A
$$

and by (1.2) we have

$$
\begin{equation*}
\operatorname{rot} E=-\frac{\partial \operatorname{rot} A}{\partial t} \tag{1.4}
\end{equation*}
$$

Interchanging the order of operators rot and $\partial / \partial t$ in (1.4) (this can be done, because all derivations exist) we have

$$
\begin{equation*}
\operatorname{rot}\left(E+\frac{\partial A}{\partial t}\right)=0 . \tag{1.5}
\end{equation*}
$$

This means that

$$
E+\frac{\partial A}{\partial t}
$$

is a gradient of a potential function. We can use scalar potential $\varphi$ (for details see [8]) to rewrite (1.5) into the form

$$
\begin{equation*}
E=-\operatorname{grad}(\varphi)-\frac{\partial A}{\partial t} . \tag{1.6}
\end{equation*}
$$

Applying (1.6) to body that is not connected to any external source of voltage we have that

$$
\operatorname{grad}(\varphi)=0 .
$$

Then

$$
E=-\iota \omega A
$$

where $\omega$ denotes angular frequency of the field current and $\iota$ is complex unit. $J_{\text {eddy }}=\gamma E$, where $\gamma$ denotes the temperature dependent electrical conductivity of the metal. From this we have

$$
\begin{equation*}
\overrightarrow{A(x)}=\frac{\iota}{\omega \gamma} J_{e d d y}(x) \tag{1.7}
\end{equation*}
$$

Substituting (1.7) to (1.1) we obtain Fredholm integral equation for $J_{\text {eddy }}$ :

$$
\begin{equation*}
\iota J_{e d d y}(x)-\kappa(x) \int_{\Omega_{1}} \frac{J_{e d d y}(t)}{r(x, t)} d t_{1} d t_{2} d t_{3}=\kappa(x) I_{e x t} \int_{\Omega_{2}} \frac{d l(s)}{r(x, s)} \tag{1.8}
\end{equation*}
$$

where

$$
\begin{equation*}
\kappa(x)=\frac{\omega \gamma(T(x)) \mu_{0}}{4 \pi} . \tag{1.9}
\end{equation*}
$$

For each bounded and continuous temperature distribution $T, \kappa(x)$ is a real, positive, bounded and continuous function. It holds

$$
\begin{equation*}
0<\kappa_{d} \leq \kappa(x) \leq \kappa_{u}<\infty, \text { for all } x \in \Omega_{1} . \tag{1.10}
\end{equation*}
$$

Since $J_{\text {eddy }}$ is a phasor

$$
J_{e d d y}=\left(J_{e d d y, x_{1}}, J_{e d d y, x_{2}}, J_{e d d y, x_{3}}\right)
$$

we can rewrite (1.8) into three complex integral equations

$$
\begin{align*}
& \iota J_{e d d y, x_{1}}(x)-\kappa(x) \int_{\Omega_{1}} \frac{J_{e d d y, x_{1}}(t)}{r(x, t)} d t_{1} d t_{2} d t_{3}=\kappa(x) I_{e x t} \int_{\Omega_{2}} \frac{d l(s) \cdot e_{x_{1}}}{r(x, s)}  \tag{1.11}\\
& \iota J_{e d d y, x_{2}}(x)-\kappa(x) \int_{\Omega_{1}} \frac{J_{e d d y, x_{2}}(t)}{r(x, t)} d t_{1} d t_{2} d t_{3}=\kappa(x) I_{e x t} \int_{\Omega_{2}} \frac{d l(s) \cdot e_{x_{2}}}{r(x, s)}  \tag{1.12}\\
& \iota J_{e d d y, x_{3}}(x)-\kappa(x) \int_{\Omega_{1}} \frac{J_{e d d y, x_{3}}(t)}{r(x, t)} d t_{1} d t_{2} d t_{3}=\kappa(x) I_{e x t} \int_{\Omega_{2}} \frac{d l(s) \cdot e_{x_{3}}}{r(x, s)} \tag{1.13}
\end{align*}
$$

where $e_{x_{1}}$ is unite vector $e_{x_{1}}=(1,0,0), e_{x_{2}}$ is unite vector $e_{x_{2}}=(0,1,0)$ and $e_{x_{3}}$ is unite vector $e_{x_{3}}=(0,0,1)$.

Since equations (1.11), (1.12) and (1.13) have only change in index we will work only on equation (1.11). With the notation

$$
\begin{equation*}
F(x)=\kappa(x) \int_{\Omega_{2}} \frac{d l(s) \cdot e_{x_{1}}}{r(x, s)} \tag{1.14}
\end{equation*}
$$

the equation (1.11) has the form

$$
\begin{equation*}
\iota J_{e d d y, x_{1}}(x)-\kappa(x) \int_{\Omega_{1}} \frac{J_{e d d y, x_{1}}(t)}{r(x, t)} d t_{1} d t_{2} d t_{3}=I_{e x t} F(x) . \tag{1.15}
\end{equation*}
$$

$J_{e d d y, x_{1}}: \Omega_{1} \subset R^{3} \rightarrow C$ is a complex function of real variable. Then $J_{\text {eddy }, x_{1}}$ can be rewritten to

$$
\begin{equation*}
J_{e d d y, x_{1}}(x)=J_{R}(x)+\iota J_{I}(x) \tag{1.16}
\end{equation*}
$$

where

$$
J_{R}(x)=\operatorname{Re} J_{e d d y, x_{1}}(x)
$$

and

$$
J_{I}(x)=\operatorname{Im} J_{e d d y, x_{1}}(x) .
$$

$I_{\text {ext }}$ is a complex number. Then we can define

$$
I_{R}=\operatorname{Re}\left(I_{e x t}\right), I_{I}=\operatorname{Im}\left(I_{e x t}\right) .
$$

From the definition of $\kappa(x)$ we have that $F(x)$ defined by (1.14) is a real function. With this notation we can rewrite (1.15) into

$$
\begin{equation*}
\iota\left(J_{R}(x)+\iota J_{I}(x)\right)-\kappa(x) \int_{\Omega_{1}} \frac{J_{R}(t)+\iota J_{I}(t)}{r(x, t)} d t_{1} d t_{2} d t_{3}=\left(I_{R}+\iota I_{I}\right) F(x) . \tag{1.17}
\end{equation*}
$$

By splitting complex equation (1.17) into two real equations we have

$$
\begin{align*}
J_{R}(x)-\kappa(x) \int_{\Omega_{1}} \frac{J_{I}(t)}{r(x, t)} d t & =I_{I} F(x), \\
-J_{I}(x)-\kappa(x) \int_{\Omega_{1}} \frac{J_{R}(t)}{r(x, t)} d t & =I_{R} F(x) . \tag{1.18}
\end{align*}
$$

Real equations (1.18) are equivalent to complex equation (1.15).
The specific Joule losses which are needed to compute temperature distribution are given by

$$
\begin{equation*}
\omega_{J}(x)=\frac{1}{\gamma} J_{e}(x) \overline{J_{e}(x)} \tag{1.19}
\end{equation*}
$$

where

$$
\begin{aligned}
& J_{e}(x)=\sqrt{\left[\operatorname{Re} J_{e d d y, x_{1}}(x)\right]^{2}+\left[\operatorname{Re} J_{e d d y, x_{2}}(x)\right]^{2}+\left[\operatorname{Re} J_{e d d y, x_{3}}(x)\right]^{2}}+ \\
& \quad+\iota \sqrt{\left[\operatorname{Im} J_{e d d y, x_{1}}(x)\right]^{2}+\left[\operatorname{ImJ}_{e d d y, x_{2}}(x)\right]^{2}+\left[\operatorname{Im} J_{e d d y, x_{3}}(x)\right]^{2}}
\end{aligned}
$$

and $\overline{J_{e}(x)}$ is complex conjugate to $J_{e}(x)$.
In following text we will show existence and uniqueness of (1.15) and derive numerical schemes.

## 2. Basics of functional analysis

Before showing existence and uniqueness of solution of (1.15) let us remember some basic facts from functional analysis. They can be found in many literature for example in [1].

### 2.1 Normed linear and Banach spaces

Normed linear and Banach spaces are the standard setting for studying and solving a large proportion of the problems in integral equations. In this section, we gather some important facts and describe most important Banach spaces.

Definition 2.1 (Norm). Let $\mathcal{V}$ be a vector space over the body $\mathbb{K}$, where $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Norm is a function $\|$.$\| defined from \mathcal{V}$ to $\mathbb{R}$ with following properties

1. $\|v\| \geq 0$ for all $v \in \mathcal{V}$
2. $\|v\|=0$ if and only if $v=0$
3. $\|\alpha v\|=|\alpha|\|v\|$ for all $v \in \mathcal{V}$ and $\alpha \in \mathbb{K}$
4. $\|u+v\| \leq\|u\|+\|v\|$ for all $u, v \in \mathcal{V}$

Definition 2.2 (Normed linear space). Let $\mathcal{X}$ be a vector space and let ||.\| be norm on $\mathcal{X}$. Then $(\mathcal{X},\|\|$.$) is called a normed linear space.$

Definition 2.3 (Cauchy sequence). Let $\mathcal{X}$ be a normed linear space. A sequence $\left\{x_{n}\right\} \subset \mathcal{X}$ is called Cauchy if

$$
\forall_{\varepsilon>0} \exists_{N>0} \forall_{n, m>N}\left\|x_{n}-x_{m}\right\|<\varepsilon .
$$

Definition 2.4 (Banach space). Let $\mathcal{X}$ be a normed linear space. $\mathcal{X}$ is called Banach space if every Cauchy sequence from $\mathcal{X}$ converges.

Now let's write some examples of Banach spaces. Proof that spaces below with appropriate norm are Banach spaces can be found for example in [1]. Let $D \subset \mathbb{R}^{n}$ be closed and bounded set. First example is space $\mathcal{C}(D)$ - a space of continuous functions on $D$ with the $\|\cdot\|_{\infty}$ norm defined as

$$
\begin{equation*}
\|f\|_{\infty}=\max _{x \in D}|f(x)| . \tag{2.1}
\end{equation*}
$$

Another example is space $L^{\infty}(D)$. Let $v$ be a Lebesgue measurable function on $D$. For later we will write measurable instead of Lebesgue measurable. Let's define a class

$$
\begin{equation*}
[v]=\{w, w \text { is measurable on } D \text { and } v=w(\text { a.e. })\} . \tag{2.2}
\end{equation*}
$$

The symbol (a.e.) means almost everywhere. Note that Lebesgue integrals of elements of class $[v]$ are identical. To define upper bound for functions $v$ from class $[v]$ is used essential supremum.

$$
\begin{equation*}
\operatorname{ess}_{x \in D} \sup v(x)=\inf \{C \in \mathbb{R}: v(x) \leq C \text { for almost every } x \in D\} \tag{2.3}
\end{equation*}
$$

From the definition it is clear that

$$
\begin{equation*}
\operatorname{ess}_{x \in D} \sup _{x} v(x) \leq \sup _{x \in D} v(x) \tag{2.4}
\end{equation*}
$$

For a measurable function $v$ on $D$ we can define a norm

$$
\begin{equation*}
\|v\|_{L^{\infty}(D)}=\operatorname{ess} \sup _{x \in D}|v(x)| . \tag{2.5}
\end{equation*}
$$

This norm is known as essential supremum norm. The space $L^{\infty}(D)$ is then defined by

$$
\begin{equation*}
L^{\infty}(D)=\left\{[v], v \text { is measurable on } D \text { and }\|v\|_{L^{\infty}(D)}<\infty\right\} . \tag{2.6}
\end{equation*}
$$

To handle with essential supremum is needed following inequality, which can be derived from Hölder's inequality.

Proposition 2.1. Let

$$
\int_{D}|g(t)| d t<\infty
$$

and let $f \in L^{\infty}(D)$. Then

$$
\int_{D}|f(t) g(t) d t| \leq\|f\|_{E_{(D)}^{\infty}} \int_{D}|g(t)| d t .
$$

Proof. It follows immediately from Hölder's inequality which can be found for example in [2].

### 2.2 Linear operators

Integral equations can be described by linear operators. To examine the theoretical solvability of a mathematical problem, to develop numerical methods for its solution and to prove convergence of numerical solution, we must know additional properties about the operators involved in our problem. In this section let's write some known important facts.

Definition 2.5 (Operator). Let $\mathcal{V}$ and $\mathcal{W}$ be sets. Operator $\mathcal{T}$ from $\mathcal{V}$ to $\mathcal{W}$ is a rule that assigns to each element in a subset of $\mathcal{V}$ an unique element in $\mathcal{W}$. The domain $\mathcal{D}(\mathcal{T})$ of operator $\mathcal{T}$ is the subset of $\mathcal{V}$, where $\mathcal{T}$ is defined. The range $\mathcal{R}(\mathcal{T})$ of operator $\mathcal{T}$ is defined as following set

$$
\mathcal{R}(\mathcal{T})=\{w \in \mathcal{W}, \exists v \in \mathcal{D}(\mathcal{T}), w=\mathcal{T} v\} .
$$

The null set of operator $\mathcal{T}$ is defined as

$$
\mathcal{N}(\mathcal{T})=\{v \in \mathcal{V}, \mathcal{T}(v)=0\} .
$$

From now assume that for operator $\mathcal{T}: \mathcal{V} \rightarrow \mathcal{W}$ is $\mathcal{D}(\mathcal{T})=\mathcal{V}$, unless it is stated to be otherwise.

Definition 2.6 (One-to-one operator). Let $\mathcal{T}$ be an operator from a set $\mathcal{V}$ to a set $\mathcal{W}$. Operator $\mathcal{T}$ is one-to-one (also called injective) if for all $v_{1}, v_{2} \in \mathcal{V}$ it holds

$$
v_{1} \neq v_{2} \Rightarrow \mathcal{T}\left(v_{1}\right) \neq \mathcal{T}\left(v_{2}\right)
$$

Definition 2.7 (Surjective Operator). Let $\mathcal{T}$ be an operator from a set $\mathcal{V}$ to a set $\mathcal{W}$. Operator $\mathcal{T}$ is surjective if

$$
\mathcal{R}(\mathcal{T})=\mathcal{W}
$$

Definition 2.8 (Bijective Operator). Let $\mathcal{T}$ be an operator from a set $\mathcal{V}$ to a set $\mathcal{W}$. Operator $\mathcal{T}$ is bijective (also called bijection) if it is one-to-one and surjective.

When operator is bijective we can define inverse operator.
Definition 2.9 (Inverse operator). Let $\mathcal{V}$ and $\mathcal{W}$ be sets. Let operator $\mathcal{T}: \mathcal{V} \rightarrow \mathcal{W}$ be bijective. Then operator $\mathcal{T}^{-1}: \mathcal{W} \rightarrow \mathcal{V}$ defined as

$$
v=\mathcal{T}^{-1}(w) \Leftrightarrow w=\mathcal{T}(v)
$$

is an inverse of operator $\mathcal{T}$. If operator $\mathcal{T}$ is one-to-one we can define inverse $\mathcal{T}^{-1}$ from $\mathcal{R}(\mathcal{T}) \subset \mathcal{W}$ to $\mathcal{V}$.

Definition 2.10 (Identical operator). Let $\mathcal{V}$ be a set. The identical operator $\mathcal{I}$ is defined by

$$
\mathcal{I} y=y \text { for all } y \in \mathcal{V}
$$

Definition 2.11 (Continuous Operator). Let $\mathcal{T}$ be an operator from $\mathcal{V}$ to $\mathcal{W}$, where $\mathcal{V}$ and $\mathcal{W}$ are normed linear spaces. Operator $\mathcal{T}$ is continuous at $v \in \mathcal{D}(\mathcal{T})$ if

$$
\left\{v_{n}\right\} \subset \mathcal{D}(\mathcal{T}) \text { and } v_{n} \rightarrow v \text { in } \mathcal{V} \Rightarrow \mathcal{T}\left(v_{n}\right) \rightarrow \mathcal{T}(v) \text { in } \mathcal{W} .
$$

Operator $\mathcal{T}$ is continuous if it continuous at all $v \in \mathcal{D}(\mathcal{T})$.
Definition 2.12 (Bounded Operator). Let $\mathcal{V}$ and $\mathcal{W}$ be normed linear spaces. Operator $\mathcal{T}$ from $\mathcal{V}$ to $\mathcal{W}$ is bounded if for any $0<r<\infty$ there exists $0<R<\infty$ such that

$$
v \in \mathcal{D}(\mathcal{T}) \text { and }\|v\|_{\mathcal{V}} \leq r \Rightarrow\|\mathcal{T}(v)\|_{\mathcal{W}} \leq R
$$

Definition 2.13 (Linear Operator). Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{T}$ be an operator from $\mathcal{V}$ to $\mathcal{W}$ with $\mathcal{D}(\mathcal{T})=\mathcal{V}$, where $\mathcal{V}$ and $\mathcal{W}$ are vector spaces. Operator $\mathcal{T}: \mathcal{V} \rightarrow \mathcal{W}$ is linear operator if

$$
\mathcal{T}\left(v_{1}+v_{2}\right)=\mathcal{T}\left(v_{1}\right)+\mathcal{T}\left(v_{2}\right), \forall_{v_{1}, v_{2} \in \mathcal{V}}
$$

and

$$
\mathcal{T}(\alpha v)=\alpha \mathcal{T}(v), \forall_{v \in \mathcal{V}, \alpha \in \mathbb{K}} .
$$

For linear operators we will write $\mathcal{T} v$ instead of $\mathcal{T}(v)$.
For linear operator $\mathcal{T}: \mathcal{V} \rightarrow \mathcal{W}$ the null set $\mathcal{N}(\mathcal{T})$ is a subspace of $\mathcal{V}$. If operator is one-to-one we have following corollary.

Corollary 2.2. Let $\mathcal{V}$ and $\mathcal{W}$ be normed linear spaces. Let $\mathcal{T}$ be continuous linear operator from $\mathcal{V}$ to $\mathcal{W}$. Then $\mathcal{T}$ is one-to-one if and only if $\mathcal{N}(\mathcal{T})=\{0\}$.

Definition 2.14 (Operator Norm). Let $\mathcal{T}: \mathcal{V} \rightarrow \mathcal{W}$ be continuous linear operator. The norm is defined by

$$
\|\mathcal{T}\|_{\mathcal{V}, \mathcal{W}}=\sup _{0 \neq v \in \mathcal{V}} \frac{\|\mathcal{T} v\|_{\mathcal{W}}}{\|v\|_{\mathcal{V}}}
$$

or equivalently

$$
\|\mathcal{T}\|_{\mathcal{V}, \mathcal{W}}=\sup _{v \in \mathcal{V},\|v\|_{\mathcal{V}}=1}\|\mathcal{T} v\|_{\mathcal{W}} .
$$

Theorem 2.3. Let $\mathcal{V}, \mathcal{W}$ be normed linear spaces and let $\mathcal{T}: \mathcal{V} \rightarrow \mathcal{W}$ be linear operator. Then $\mathcal{T}$ is continuous if and only if it is bounded.

Definition 2.15 (Finite-rank operator). Let $\mathcal{V}$ and $\mathcal{W}$ be normed linear spaces. The linear operator $\mathcal{T}: \mathcal{V} \rightarrow \mathcal{W}$ is a finite-rank operator if $\mathcal{R}(\mathcal{T})$ is finite dimensional.

Now let's formulate one important theorem used in numerical analysis. The theorem is used when we want to analyze the solvability of problems that are "close" to another problem known to be uniquely solvable. The proof of the theorem can be found for example in [1] and also in many other books.

Theorem 2.4 (Geometric series theorem). Let $\mathcal{X}$ be a Banach space and let $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{X}$ be a continuous linear operator such that $\|\mathcal{T}\|<1$. Then operator $\mathcal{I}-\mathcal{T}$ is invertible and

$$
\begin{equation*}
(\mathcal{I}-\mathcal{T})^{-1}=\sum_{n=0}^{\infty} \mathcal{T}^{n} \tag{2.7}
\end{equation*}
$$

At the end of this section let us write two important theorems for a sequence of operators. They can be found for example in [1].

Theorem 2.5 (Principle of uniform boundedness). Let $\mathcal{T}_{n}$ be a sequence of bounded linear operators from a Banach space $\mathcal{X}$ to a normed linear space $\mathcal{Y}$. Assume

$$
\lim _{n \rightarrow \infty} \mathcal{T}_{n} y
$$

exists in $\mathcal{Y}$ for every $y \in \mathcal{X}$. Then

$$
\begin{equation*}
\sup _{n}\left\|\mathcal{T}_{n}\right\| \leq T<\infty \tag{2.8}
\end{equation*}
$$

Theorem 2.6 (Banach-Steinhaus theorem). Let $\mathcal{X}$ be Banach space and let $\mathcal{Y}$ be normed linear space. Let $\left\{\mathcal{T}_{n}\right\}$ be a sequence of continuous linear operators from $\mathcal{X}$ to $\mathcal{Y}$ such that

$$
\lim _{n \rightarrow \infty} \mathcal{T}_{n} y
$$

exists for all $y \in \mathcal{X}$ and let

$$
\mathcal{T} y=\lim _{n \rightarrow \infty} \mathcal{T}_{n} y
$$

Then $\mathcal{T}$ is continuous linear operator from $\mathcal{X}$ to $\mathcal{Y}$.

### 2.3 Compact operators

When a vector space $\mathcal{V}$ is finite-dimensional, the equation

$$
A y=f
$$

has a well-developed solvability theory. To extend these results to infinitedimensional spaces, it is used theory of compact operators.

Definition 2.16 (Compact set). Let $S$ be a subset of a vector space $\mathcal{V}$. $S$ is compact if every sequence $\left\{x_{j}\right\} \subset S$ contains a subsequence $\left\{x_{j_{k}}\right\}$ that converges to some $x \in S$.

Definition 2.17 (Compact operator). Let $\mathcal{X}$ and $\mathcal{Y}$ be normed linear spaces and let $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{Y}$ be a linear operator. $\mathcal{T}$ is compact operator if the set $\left\{\mathcal{T} y,\|y\|_{\mathcal{X}} \leq 1\right\}$ has compact closure in $\mathcal{Y}$. It means that for every bounded sequence $\left\{y_{k}\right\} \subset \mathcal{X}$, the sequence $\left\{\mathcal{T} y_{k}\right\}$ contains a subsequence which converges in $\mathcal{Y}$.

Now let's describe properties of compact operators. The proof of following theorems can be found in [1] and in many other literature of functional analysis.

Theorem 2.7. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed linear spaces. Let operator $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{Y}$ be a bounded, linear and finite-rank operator. Then $\mathcal{T}$ is compact operator.

Theorem 2.8. Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ be normed linear spaces. Let $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{Y}$ and $\mathcal{S}: \mathcal{Y} \rightarrow \mathcal{Z}$ be continuous linear operators. Let either $\mathcal{T}$ or $\mathcal{S}$ be compact operator. Then operator $\mathcal{S} \mathcal{T}$ is compact operator from $\mathcal{X}$ to $\mathcal{Z}$.

Theorem 2.9. Let $\mathcal{X}$ be normed linear space and let $\mathcal{Y}$ be Banach space. Let $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{Y}$ be a continuous linear operator. Let $\left\{\mathcal{T}_{n}\right\}$ be a sequence of compact linear operators from $\mathcal{X}$ to $\mathcal{Y}$ such that $\mathcal{T}_{n} \rightarrow \mathcal{T}$. Then $\mathcal{T}$ is compact operator.

Corollary 2.10. Let $\mathcal{X}$ be normed linear space and let $\mathcal{Y}$ be Banach space. Let linear operator $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{Y}$ be a limit of finite rank operators. Then $\mathcal{T}$ is compact operator.

For characterization of compact operators on $\mathcal{C}(D)$ we need to characterize compact sets on $\mathcal{C}(D)$. This is done by Arzela - Ascoli theorem. From now let $r(x, y)$ denote Euclidean distance between $x$ and $y$.

Definition 2.18 (Uniformly bounded function). A family of functions $\mathcal{F} \subset \mathcal{C}(D)$ is uniformly bounded if there is a number $M<\infty$ such that

$$
|f(x)| \leq M, \forall_{f \in \mathcal{F}} \forall_{x \in D}
$$

Definition 2.19 (Equicontinuous function). A family of functions $\mathcal{F} \subset \mathcal{C}(D)$ is equicontinuous on $D$ if

$$
\forall_{\epsilon>0} \exists_{\delta>0}, \forall_{x, x^{\prime} \in D} r\left(x, x^{\prime}\right)<\delta \Rightarrow\left|f(x)-f\left(x^{\prime}\right)\right|<\epsilon, \forall_{f \in \mathcal{F}} .
$$

Theorem 2.11 (Arzela - Ascoli). Let $S \subset \mathcal{C}(D)$ and let $D$ be closed and bounded set. Suppose that functions from $S$ are uniformly bounded and equicontinuous. Then $\bar{S}$ is compact in $C(D)$.

This theoretical theorem makes characterization of compact integral operators on $\mathcal{C}(D)$. Important conditions for compactness of operator $\mathcal{T}$ were showed in [3] in the following theorem.

Theorem 2.12. Let $D \subset \mathbb{R}^{n}$ be closed and bounded set. Assume that operator $\mathcal{T}$ of the form

$$
\begin{equation*}
\mathcal{T} y(x)=\int_{D} k(x, t) y(t) d t \tag{2.9}
\end{equation*}
$$

satisfies following conditions:

$$
\begin{equation*}
\max _{x \in D} \int_{D}|k(x, t)| d t \leq M_{1}<\infty \tag{2.10}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{h \rightarrow 0} \omega(h)=0 \tag{2.11}
\end{equation*}
$$

where

$$
\begin{equation*}
\omega(h) \equiv \sup _{x, x^{\prime} \in D, r\left(x, x^{\prime}\right) \leq h} \int_{D}\left|k(x, t)-k\left(x^{\prime}, t\right)\right| d t . \tag{2.12}
\end{equation*}
$$

Then $\mathcal{T}: \mathcal{C}(D) \rightarrow \mathcal{C}(D)$ is compact operator.
To prove compactness of given integral operator we need to verify conditions (2.10) and (2.11). By following lemmas we will describe classes of functions that satisfy these conditions. Proof of following lemmas can be found in [4].

Lemma 2.13. Let $D \subset R^{n}$ be closed and bounded set and let integral operator $\mathcal{T}$ be of the form

$$
\mathcal{T} y(x)=\int_{D} k(x, t) y(t) d t
$$

where $k(x, t)$ is continuous function. Then $\mathcal{T}: \mathcal{C}(D) \rightarrow \mathcal{C}(D)$ is compact operator.
Thus, integral operator $\mathcal{T}$ is compact for all continuous kernel functions. Other generalization for non-continuous functions can be done by using theorem 2.9 .

Lemma 2.14. Let $D \subset \mathbb{R}^{n}$ be closed and bounded set. Assume integral operator $\mathcal{T}: \mathcal{C}(D) \rightarrow \mathcal{C}(D)$ of the form

$$
\mathcal{T} y(x)=\int_{D} k(x, t) y(t) d t
$$

and assume that there we can define a sequence of continuous functions $k_{n}(x, t)$ such that

$$
\begin{equation*}
\max _{x \in D} \int_{D}\left|k(x, t)-k_{n}(x, t)\right| d t \rightarrow 0 \tag{2.13}
\end{equation*}
$$

Then $\mathcal{T}$ is compact linear operator on $\mathcal{C}(D)$.

### 2.4 Collectively compact operators

Now we will describe important group of operators in approximation theory. Let $\mathcal{X}$ be a Banach space and let $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{X}$. The theoretical solution of

$$
\mathcal{T} y=f
$$

involves operator approximation by a sequence of operators $\left\{\mathcal{T}_{n}, n \geq 1\right\}$ such that the equation

$$
\mathcal{T}_{n} y_{n}=f
$$

can be solved by some means (for example system of linear equations). We want to show that approximation $y_{n}$ converges to $y$. There are two kind of operator convergence. Pointwise convergence $\mathcal{T}_{n} \rightarrow \mathcal{T}$, which means that for each $y \in \mathcal{X}$

$$
\left\|\mathcal{T}_{n} y-\mathcal{T} y\right\|_{\mathcal{X}} \rightarrow 0, \text { as } n \rightarrow \infty
$$

and norm convergence, which means that

$$
\left\|\mathcal{T}_{n}-\mathcal{T}\right\| \rightarrow 0, \text { as } n \rightarrow \infty
$$

where $\|$.$\| is an operator norm. The norm convergence is equivalent to$

$$
\sup _{\substack{y \in \mathcal{X} \\\|y\|_{\mathcal{X}}=1}}\left\|\mathcal{T}_{n} y-\mathcal{T} y\right\|_{\mathcal{X}} \rightarrow 0, \text { as } n \rightarrow \infty
$$

From above we can see that operators $\mathcal{T}_{n}$ converge to operator $\mathcal{T}$ in norm if the sequence $\mathcal{T}_{n} y$ converges to $\mathcal{T} y$ uniformly for all $y \in \mathcal{X}$.

The sequence of operators converges pointwise rather than norm. To compensate the discrepancy between the pointwise and norm convergence can be used theory of collectively compact operators described by Anselone in [5].
Definition 2.20 (Colectivelly compact operators). Let $\mathcal{X}$ be a Banach space. Let $\left\{\mathcal{T}_{n}, n \geq 1\right\}$ be a family of linear operators on $\mathcal{X}$ into $\mathcal{X}$. Assume that the set

$$
\mathcal{S}=\left\{\mathcal{T}_{n} y, n \geq 1,\|y\| \leq 1\right\}
$$

has compact closure in $\mathcal{X}$. Then $\left\{\mathcal{T}_{n}, n \geq 1\right\}$ is called collectively compact family of operators on $\mathcal{X}$.

Most important properties of collectively compact operators related to integral equations of the second kind are summarized in following theorem
Theorem 2.15. Let $\mathcal{X}$ be a Banach space and let $\mathcal{T}$ be a linear operator defined on $\mathcal{X}$ into $\mathcal{X}$. Assume that $\left\{\mathcal{T}_{n}, n \geq 1\right\}$ is collectively compact family of operators on $\mathcal{X}$. Finally assume that

$$
\begin{equation*}
\mathcal{T}_{n} y \rightarrow \mathcal{T} y \text { as } n \rightarrow \infty \text { for all } y \in \mathcal{X} \tag{2.14}
\end{equation*}
$$

Then $\mathcal{T}$ is compact operator, for any compact operator $\mathcal{M}: \mathcal{X} \rightarrow \mathcal{X}$ it holds

$$
\begin{equation*}
\left\|\left(\mathcal{T}-\mathcal{T}_{n}\right) \mathcal{M}\right\| \rightarrow 0 \text { as } n \rightarrow \infty \tag{2.15}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\left(\mathcal{T}-\mathcal{T}_{n}\right) \mathcal{T}_{n}\right\| \rightarrow 0 \text { as } n \rightarrow \infty \tag{2.16}
\end{equation*}
$$

Proof. Proof can be found in chapter 1 in [5] - proposition 1.8 and corollary 1.9.

### 2.5 Metric spaces and totally bounded sets

Another relation between pointwise and uniform convergence is described for special sets in metric spaces. Let us write here some basic facts of metric spaces. Definitions and properties can be found for example in [11].

Definition 2.21 (Metric space). A metric space is an ordered pair $(M, d)$ where $M$ is a set and d is a function $M \times M \rightarrow \mathbb{R}$ called metric such that for any $x, y, z \in M$ holds:

1. $d(x, y) \geq 0$
2. $d(x, y)=0$ if and only if $x=y$
3. $d(x, y)=d(y, x)$
4. $d(x, z) \leq d(x, y)+d(y, z)$

A metric space $M$ is called complete if every Cauchy sequence in $M$ converges in $M$.

From the definition of metric 2.21 and norm 2.1 we can see that every normed linear space is a metric space - we can define $d(x, y)=\|x-y\|$. Let's define open ball in $M$

$$
\begin{equation*}
\mathcal{B}_{r}(x)=\{y \in M, d(x, y)<r\} . \tag{2.17}
\end{equation*}
$$

Definition 2.22 (Totally bounded set). Let $(M, d)$ be a metric space and let $A \subset M . A$ is totally bounded if for any real number $\varepsilon>0$ there exists a $\varepsilon$-mesh $\left\{a_{1}, \ldots, a_{m}\right\} \subset A$ such that

$$
A \subset \bigcup_{j=1}^{m} \mathcal{B}_{\varepsilon}\left(a_{j}\right)
$$

Definition 2.23 (Relatively compact). Let $(M, d)$ be a metric space and let $A \subset M . A$ is relatively compact if it has compact closure (i.e. $\bar{A}$ is compact).

Properties of totally bounded set is given by following theorem. A similar theorem can be found in [5].

Theorem 2.16. Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. Let $\mathcal{T}$ be bounded linear operator defined from $\mathcal{X}$ to $\mathcal{Y}$ and let $\left\{\mathcal{T}_{n}, n \geq 1\right\}$ be a sequence of bounded linear operators defined from $\mathcal{X}$ to $\mathcal{Y}$. Assume that for all $y \in \mathcal{X}$ the sequence $\mathcal{T}_{n} y$ converges to $\mathcal{T} y$. Let $S \subset \mathcal{X}$ be totally bounded set. Then

$$
\begin{equation*}
\sup _{y \in S}\left\|\mathcal{T}_{n} y-\mathcal{T} y\right\|_{\mathcal{Y}} \rightarrow 0 \text { as } n \rightarrow \infty \tag{2.18}
\end{equation*}
$$

Proof. Take $\varepsilon>0$. Since $S$ is totally bounded set there exist $y_{1}, \ldots, y_{m} \in S$ such that

$$
\begin{equation*}
\sup _{y \in S}\left\|y-y_{s}\right\|_{\mathcal{X}} \leq \frac{\varepsilon}{2\left(\left\|\mathcal{T}_{n}\right\|+\|\mathcal{T}\|\right)} \tag{2.19}
\end{equation*}
$$

where

$$
s=\arg \min _{j=1, \ldots, m}\left\|y-y_{j}\right\|_{\mathcal{X}}
$$

and $\arg \min$ is the argument of minimum. Since $\mathcal{T}_{n} y \rightarrow \mathcal{T} y$ for each $y$ we have that there exists $n_{0}$ such that for $n \geq n_{0}$

$$
\left\|\left(\mathcal{T}_{n}-\mathcal{T}\right) y_{j}\right\|_{\mathcal{Y}}<\frac{\varepsilon}{2}, \text { for all } j=1, \ldots, m
$$

From here and (2.19)

$$
\begin{gathered}
\sup _{y \in S}\left\|\left(\mathcal{T}_{n}-\mathcal{T}\right) y\right\|_{\mathcal{Y}} \leq \sup _{y \in S}\left\|\left(\mathcal{T}_{n}-\mathcal{T}\right)\left(y-y_{s}\right)\right\|_{\mathcal{Y}}+\left\|\left(\mathcal{T}_{n}-\mathcal{T}\right) y_{s}\right\|_{\mathcal{Y}} \leq \\
\leq\left\|\left(\mathcal{T}_{n}-\mathcal{T}\right)\right\| \sup _{y \in S}\left\|\left(y-y_{s}\right)\right\|_{\mathcal{X}}+\left\|\left(\mathcal{T}_{n}-\mathcal{T}\right) y_{s}\right\|_{\mathcal{Y}} \leq \varepsilon
\end{gathered}
$$

The last theorem shows that pointwise convergence $\mathcal{T}_{n} \rightarrow \mathcal{T}$ is uniform on totally bounded set. Every Banach space is complete metric space. In complete metric spaces the the terms totally bounded set and relative compact set coincide. The relatively compact sets are described by Arzela-Ascoli theorem 2.11. Hence we have following corollary.

Corollary 2.17. Let $\mathcal{T}$ be bounded linear operator defined from space $\mathcal{C}(D)$ to Banach space $\mathcal{Y}$ and let $\left\{\mathcal{T}_{n}, n \geq 1\right\}$ be a sequence of bounded linear operators defined from space $\mathcal{C}(D)$ to Banach space $\mathcal{Y}$. Assume that for all $y \in \mathcal{C}(D)$ the sequence $\mathcal{T}_{n} y$ converges to $\mathcal{T} y$. Let the functions from $S \subset \mathcal{C}(D)$ be equicontinuous and uniformly bounded. Then

$$
\begin{equation*}
\sup _{y \in S}\left\|\mathcal{T}_{n} y-\mathcal{T} y\right\|_{\mathcal{Y}} \rightarrow 0 \text { as } n \rightarrow \infty \tag{2.20}
\end{equation*}
$$

### 2.6 Eigenvalues and invertibility of operator $(\lambda \mathcal{I}-\mathcal{T})$

In this chapter we will show conditions needed for existence of operator

$$
(\lambda \mathcal{I}-\mathcal{T})^{-1}
$$

This operator relates to integral equation of the second kind. First way is to use geometric series theorem 2.4. We obtain following proposition. The proof can be found in [4].

Proposition 2.18 (Inverse Theorem proposition). Let $\mathcal{X}$ be a Banach space and let $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{X}$ be continuous linear operator. Assume that

$$
\begin{equation*}
\frac{1}{|\lambda|}\|\mathcal{T}\|<1 \tag{2.21}
\end{equation*}
$$

Then $(\lambda \mathcal{I}-\mathcal{T})$ has inverse, which means that $(\lambda \mathcal{I}-\mathcal{T})^{-1}: \mathcal{X} \rightarrow \mathcal{X}$ is a bounded linear operator.

To use this proposition is needed the bound of operator $\mathcal{T}$. Other way to show existence of inverse is to use Fredholm alternative theorem.

Definition 2.24 (Eigenvalue). Let $\mathcal{X}$ be a normed linear and let $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{X}$ be continuous linear operator. Then $\lambda$ is eigenvalue of $\mathcal{T}$ if there exists some $y \neq 0$ such that

$$
\begin{equation*}
\mathcal{T} y=\lambda y . \tag{2.22}
\end{equation*}
$$

It means that operator $\lambda \mathcal{I}-\mathcal{T}$ is not one-to-one. All vectors $y$ satisfying (2.22) are eigenvectors appropriate to eigenvalue $\lambda$.

In finite dimensional spaces is linear operator one-to-one if and only if it is surjective. In infinite spaces we need following theorem.

Theorem 2.19 (Fredholm alternative). Let $\mathcal{X}$ be a Banach space, let $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{X}$ be compact linear operator and let $\lambda \neq 0$. Then either the equation $(\lambda \mathcal{I}-\mathcal{T}) y=0$ has a non-trivial solution, or the equation $(\lambda \mathcal{I}-\mathcal{T}) y=f$ has a solution for all $f$. In such case the operator $(\lambda \mathcal{I}-\mathcal{T})$ has a bounded inverse.

The last theorem gives us important condition for invertibility of $(\lambda \mathcal{I}-\mathcal{T})$.
Proposition 2.20. Let $\mathcal{X}$ be a Banach space and let $\mathcal{T}: \mathcal{X} \rightarrow \mathcal{X}$ be compact linear operator. Assume that $\lambda \neq 0$ is not eigenvalue of operator $\mathcal{T}$. Then $(\lambda \mathcal{I}-\mathcal{T})$ has bounded inverse

Proof. By the last theorem is $(\lambda \mathcal{I}-\mathcal{T})$ is invertible if $\lambda \neq 0$ is not eigenvalue of $\mathcal{T}$.

## 3. Existence and uniqueness of solution of induction heating model

Now we can use theory written in previous chapter to prove existence and uniqueness of solution of integral equation (1.15). First we need to rewrite (1.15) into an operator form. By multiplying (1.15) with $-\iota$ we obtain

$$
J_{e d d y, x_{1}}(x)+\iota \kappa(x) \int_{\Omega_{1}} \frac{J_{e d d y, x_{1}}(t)}{r(x, t)} d t_{1} d t_{2} d t_{3}=-\iota I_{e x t} F(x) .
$$

Hence
$J_{e d d y, x_{1}}(x)-\kappa(x) \int_{\Omega_{1}} \frac{-\iota\left[\operatorname{Re}\left(J_{e d d y, x_{1}}\right)(t)+\iota \operatorname{Im}\left(J_{e d d y, x_{1}}\right)(t)\right]}{r(x, t)} d t_{1} d t_{2} d t_{3}=-\iota I_{e x t} F(x)$.
If we define

$$
\begin{equation*}
\widetilde{F}(x)=-\iota I_{e x t} F(x) \tag{3.1}
\end{equation*}
$$

we can rewrite (1.15) into an operator form

$$
\begin{equation*}
(I-\mathcal{K}) J_{e d d y, x_{1}}=\widetilde{F} \tag{3.2}
\end{equation*}
$$

where operator $\mathcal{K}$ is defined by

$$
\begin{equation*}
\mathcal{K} J_{e d d y, x_{1}}(x)=\kappa(x) \int_{\Omega_{1}} \frac{-\iota \operatorname{Re}\left(J_{e d d y, x_{1}}\right)(t)+\operatorname{Im}\left(J_{e d d y, x_{1}}\right)(t)}{r(x, t)} d t \tag{3.3}
\end{equation*}
$$

and where $\widetilde{F}$ is defined in (3.1). We need to show that operator $\mathcal{K}$ is compact linear operator on space $\mathcal{C}\left(\Omega_{1}\right)$ and 1 is not its eigenvalue. First let's prove following lemma.

Lemma 3.1. Let $x=\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{3}$. Let $B_{R}(x)$ be a ball with its center at point $x$ and radius $R$. Note that $B_{r}(x)$ is defined as

$$
\begin{equation*}
B_{R}(x)=\left\{t=\left(t_{1}, t_{2}, t_{3}\right) \in \mathbb{R}^{3},\left(x_{1}-t_{1}\right)^{2}+\left(x_{2}-t_{2}\right)^{2}+\left(x_{3}-t_{3}\right)^{2} \leq R^{2}\right\} \tag{3.4}
\end{equation*}
$$

Then

$$
\begin{equation*}
I_{R}(x)=\int_{B_{R}(x)} \frac{1}{r(x, t)} d t=2 \pi R^{2} \tag{3.5}
\end{equation*}
$$

Proof. Let's make a transformation to the sphere coordinates

$$
\begin{aligned}
\phi & :(\rho, \phi, \theta) \rightarrow\left(t_{1}, t_{2}, t_{3}\right) \\
t_{1} & =x_{1}+\rho \sin (\theta) \cos (\varphi) \\
t_{2} & =x_{2}+\rho \sin (\theta) \sin (\varphi) \\
t_{3} & =x_{3}+\rho \cos (\theta)
\end{aligned}
$$

$$
\rho \in[0, r], \varphi \in[0,2 \pi), \theta \in[0, \pi] .
$$

Note that $\rho=r(x, t)$. The Jacobian of this transformation is

$$
J_{\phi}=\rho^{2} \sin (\theta) .
$$

By substitution theorem A-5 we obtain

$$
\begin{aligned}
I_{R}(x) & =\int_{B_{R}(x)} \frac{1}{r(x, t)} d t_{1} d t_{2} d t_{3}=\int_{0}^{R} \int_{0}^{2 \pi} \int_{0}^{\pi} \rho \sin \theta d \theta d \varphi d \rho= \\
& =2 \pi \int_{0}^{R} \int_{0}^{\pi} \rho \sin \theta d \theta d \rho=2 \pi \int_{0}^{R} \rho d \rho \int_{0}^{\pi} \sin \theta d \theta=2 \pi R^{2} .
\end{aligned}
$$

Corollary 3.2. There exists a constant $C_{1}<\infty$ such that

$$
\begin{equation*}
\max _{x \in \Omega_{1}} \int_{\Omega_{1}} \frac{1}{r(x, t)} d t \leq C_{1} \tag{3.6}
\end{equation*}
$$

Proof. Let us define

$$
\begin{equation*}
d\left(\Omega_{1}\right)=\max _{x, t \in \Omega_{1}} r(x, t) . \tag{3.7}
\end{equation*}
$$

Since $\Omega_{1}$ is closed and bounded set it is $d\left(\Omega_{1}\right)<\infty$. In fact $d\left(\Omega_{1}\right)$ is a diameter of $\Omega_{1}$. For each $x \in \Omega_{1}$ is

$$
\begin{equation*}
\Omega_{1} \subset B_{d\left(\Omega_{1}\right)}(x) \tag{3.8}
\end{equation*}
$$

By lemma 3.1 and (3.8) we have

$$
\int_{\Omega_{1}} \frac{1}{r(x, t)} d t \leq \int_{B_{d\left(\Omega_{1}\right)}(x)} \frac{1}{r(x, t)} d t=2 \pi d\left(\Omega_{1}\right)^{2}<\infty
$$

Hence

$$
C_{1}=2 \pi d\left(\Omega_{1}\right)^{2}
$$

Now let's prove that our operator from $\mathcal{K}$ defined by (3.3) is compact operator. Theorem 3.3. Let $\Omega_{1} \subset \mathbb{R}^{3}$ be closed and bounded set. Operator $\mathcal{K}$ defined by the formula

$$
\begin{equation*}
\mathcal{K} J_{e d d y, x_{1}}(x)=\kappa(x) \int_{\Omega_{1}} \frac{-\iota \operatorname{Re}\left(J_{e d d y, x_{1}}\right)(t)+\operatorname{Im}\left(J_{e d d y, x_{1}}\right)(t)}{r(x, t)} d t \tag{3.9}
\end{equation*}
$$

is compact operator on $\mathcal{C}\left(\Omega_{1}\right)$.
Proof. Let's define following operators: $\mathcal{K}_{1}: \mathcal{C}\left(\Omega_{1}\right) \rightarrow \mathcal{C}\left(\Omega_{1}\right), \mathcal{M}_{1}: \mathcal{C}\left(\Omega_{1}\right) \rightarrow$ $\mathcal{C}\left(\Omega_{1}\right)$ and $\mathcal{N}: \mathcal{C}\left(\Omega_{1}\right) \rightarrow \mathcal{C}\left(\Omega_{1}\right)$.

$$
\begin{gather*}
\mathcal{K}_{1}: \mathcal{K}_{1} y(x)=\int_{\Omega_{1}} \frac{y(t)}{r(x, t)} d t  \tag{3.10}\\
\mathcal{M}_{1}: \mathcal{M}_{1} y(x)=\kappa(x) y(x) \tag{3.11}
\end{gather*}
$$

$$
\begin{equation*}
\mathcal{N}: \mathcal{N} y(x)=\operatorname{Im}(y(x))-\iota \operatorname{Re}(y(x)) . \tag{3.12}
\end{equation*}
$$

Then

$$
\mathcal{K} y(x)=\mathcal{M}_{1} \mathcal{K}_{1} \mathcal{N} y(x)
$$

By corollary 3.2 is function

$$
\frac{1}{r(x, t)}
$$

integrable for all $x \in \Omega_{1}$ and $\mathcal{K}_{1}$ is linear operator. By corollary 3.2 we have

$$
\left\|\mathcal{K}_{1} y\right\|_{\infty} \leq\|y\|_{\infty} \max _{x \in \Omega_{1}} \int_{\Omega_{1}} \frac{1}{r(x, t)} d t \leq C_{1}\|y\|_{\infty}<\infty
$$

and $\mathcal{K}_{1}$ is bounded operator. By theorem 2.3 is $\mathcal{K}_{1}$ continuous linear operator. To prove compactness of $\mathcal{K}_{1}$ we will use lemma 2.14. We need to fulfill condition (2.13). Let us define approximation of $r(x, t)$ by continuous function

$$
r_{n}(x, t)=\left\{\begin{array}{l}
r_{n}(x, t)=r(x, t), \text { when } r(x, t) \geq \frac{1}{n}  \tag{3.13}\\
r_{n}(x, t)=\frac{1}{n}, \text { when } r(x, t)<\frac{1}{n} .
\end{array}\right.
$$

It is easy to see that

$$
r_{n}(x, t) \geq r(x, t), \text { for all } n \text { and } x, t \in \mathbb{R}^{3}
$$

and

$$
\frac{1}{r_{n}(x, t)} \leq \frac{1}{r(x, t)}, \text { for all } n \text { and } x, t \in \mathbb{R}^{3} .
$$

Hence

$$
\begin{gathered}
\int_{\Omega_{1}}\left|\frac{1}{r(x, t)}-\frac{1}{r_{n}(x, t)}\right| d t=\int_{\Omega_{1}}\left(\frac{1}{r(x, t)}-\frac{1}{r_{n}(x, t)}\right) d t= \\
=\int_{\Omega_{1} \cap B_{n^{-1}(x)}}\left(\frac{1}{r(x, t)}-\frac{1}{r_{n}(x, t)}\right) d t \leq \int_{B_{n^{-1}}(x)}\left(\frac{1}{r(x, t)}-\frac{1}{r_{n}(x, t)}\right) d t= \\
\int_{B_{n^{-1}}(x)} \frac{1}{r(x, t)} d t-\int_{B_{n^{-1}}} n d t=I_{n^{-1}}(x)-\frac{4}{3} \pi\left(\frac{1}{n}\right)^{3} n=2 \pi \frac{1}{n^{2}}-\frac{4}{3} \pi \frac{1}{n^{2}}=\frac{2 \pi}{3 n^{2}}
\end{gathered}
$$

where $B_{n^{-1}}(x)$ is defined by (3.4) and $I_{n^{-1}}(x)$ by 3.5 in lemma 3.1. Hence

$$
\begin{equation*}
\max _{x \in \Omega_{1}} \int_{\Omega_{1}}\left|\frac{1}{r(x, t)}-\frac{1}{r_{n}(x, t)}\right| d t \leq \frac{2 \pi}{3 n^{2}} \rightarrow 0 \text { as } n \rightarrow \infty . \tag{3.14}
\end{equation*}
$$

By theorem 2.14 is $\mathcal{K}_{1}$ compact operator on $\mathcal{C}\left(\Omega_{1}\right)$. It's easy to see that operator $\mathcal{M}_{1}$ is a linear operator.

$$
\begin{equation*}
\left\|\mathcal{M}_{1} y\right\|_{\infty}=\max _{x \in \Omega_{1}}|\kappa(x) y(x)| \tag{3.15}
\end{equation*}
$$

From (1.10)

$$
\begin{equation*}
|\kappa(x)|<\kappa_{u}<\infty, \text { for all } x \in \Omega_{1} . \tag{3.16}
\end{equation*}
$$

From (3.16) and (3.15) is

$$
\left\|\mathcal{M}_{1} y\right\|_{\infty} \leq \kappa_{u}\|y\|_{\infty}
$$

and $\mathcal{M}_{1}$ is bounded linear operator. By theorem 2.3 is $\mathcal{M}_{1}$ continuous linear operator. By theorem 2.8 is operator $\mathcal{M}_{1} \mathcal{K}_{1}$ compact operator on $\mathcal{C}\left(\Omega_{1}\right)$. For future use let's define operator

$$
\mathcal{M}=\mathcal{M}_{1} \mathcal{K}_{1}
$$

To verify compactness of $\mathcal{K}=\mathcal{M} \mathcal{N}$ we will use theorem 2.8. We need to show that $\mathcal{N}$ is continuous linear operator. For the linearity let's choose some $y_{2}, y_{2} \in \mathcal{C}\left(\Omega_{1}\right)$ and $\lambda \in \mathbb{C}$.

$$
\begin{aligned}
& y_{1}=a_{1}+\iota b_{1} \\
& y_{2}=a_{2}+\iota b_{2} \\
& \lambda=\lambda_{1}+\iota \lambda_{2}
\end{aligned}
$$

where $a_{1}, b_{1}, a_{2}$ and $b_{2}$ are real functions and $\lambda_{1}, \lambda_{2} \in \mathbb{R}$

$$
\begin{gathered}
\mathcal{N}\left(y_{1}+y_{2}\right)=\mathcal{N}\left(a_{1}+\iota b_{1}+a_{2}+\iota b_{2}\right)=-\iota\left(a_{1}+a_{2}\right)+b_{1}+b_{2}= \\
=\left(-\iota a_{1}+b_{1}\right)+\left(-\iota a_{2}+b_{2}\right)=\mathcal{N}\left(a_{1}+\iota b_{1}\right)+\mathcal{N}\left(a_{2}+\iota b_{2}\right)=\mathcal{N}\left(y_{1}\right)+\mathcal{N}\left(y_{2}\right)
\end{gathered}
$$

and

$$
\begin{gathered}
\mathcal{N}\left(\lambda y_{1}\right)=\mathcal{N}\left[\left(\lambda_{1}+\iota \lambda_{2}\right)\left(a_{1}+\iota b_{1}\right)\right]=\mathcal{N}\left[\lambda_{1} a_{1}-\lambda_{2} b_{1}+\iota\left(\lambda_{1} b_{1}+\lambda_{2} a_{1}\right)\right] \\
=-\iota\left(\lambda_{1} a_{1}-\lambda_{2} b_{1}\right)+\left(\lambda_{1} b_{1}+\lambda_{2} a_{1}\right)=-\iota \lambda_{1} a_{1}+\iota \lambda_{2} b_{1}+\lambda_{1} b_{1}+\lambda_{2} a_{1} \\
=\left(-\iota \lambda_{1} a_{1}+\lambda_{1} b_{1}\right)+\left(\iota \lambda_{2} b_{1}+\lambda_{2} a_{1}\right)=\lambda_{1}\left(b_{1}-\iota a_{1}\right)+\iota \lambda_{2}\left(b_{1}-\iota a_{1}\right)= \\
=\left(\lambda_{1}+\iota \lambda_{2}\right)\left(b_{1}-\iota a_{1}\right)=\lambda \mathcal{N}\left(a_{1}+\iota b_{1}\right)=\lambda \mathcal{N} y_{1}
\end{gathered}
$$

and we have shown that $\mathcal{N}$ is linear operator. For continuity of operator $\mathcal{N}$ we will show that $\mathcal{N}$ is bounded and use theorem 2.3. Let's choose $y \in \mathcal{C}\left(\Omega_{1}\right)$. Then

$$
\|\mathcal{N} y\|_{\infty}=\max _{x \in \Omega_{1}}|-\iota \operatorname{Rey}(x)+\operatorname{Imy}(x)| \leq 2 \max _{x \in \Omega_{1}}|y(x)| \leq 2\|y\|_{\infty} .
$$

Hence $\mathcal{N}$ is continuous linear operator and by theorem 2.8 is $\mathcal{K}=\mathcal{M} \mathcal{N}$ compact operator.

Now its time to prove that 1 is not eigenvalue of operator $\mathcal{K}$. In this case it can be done with using inner product.

Definition 3.1 (Inner product). Let $\mathcal{V}$ be a vector space. Inner product is a function (.,.): $\mathcal{V} \times \mathcal{V} \rightarrow \mathbb{K}, \mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ with following properties

1. $(x, x) \geq 0,(x, x)=0$ if and only if $x=0$
2. $(x, y)=\overline{(y, x)}$
3. $(\lambda x, y)=\lambda(x, y),(x, \lambda y)=\bar{\lambda}(x, y)$
4. $(x+y, z)=(x, z)+(y, z)$ and $(x, y+z)=(x, y)+(x, z)$
$x, y, z \in \mathcal{V}$ and $\lambda \in \mathbb{K}$. The space $\mathcal{V}$ with inner product is called inner product space. For $v \in \mathcal{V}$

$$
\begin{equation*}
\|v\|=\sqrt{(v, v)} \tag{3.17}
\end{equation*}
$$

In literature [1] is proved that (3.17) is norm.

Theorem 3.4 (Weighted inner product). Let $\gamma(t)$ be positive, piecewise continuous function defined on whole $D$. Then

$$
\begin{equation*}
(f, g)_{\gamma}=\int_{D} \gamma(t) f(t) \overline{g(t)} d t \tag{3.18}
\end{equation*}
$$

is an inner product on $\mathcal{C}(D)$.
Proof. The proof can be found in many books of functional analysis for example in [1].

Definition 3.2. The inner product defined in (3.18) is called $\gamma$-weighted inner product.

Definition 3.3 (Antisymmetric operator). Let $D \subset \mathbb{R}^{n}$ be closed and bounded set. Operator $\mathcal{K}: \mathcal{C}(D) \rightarrow \mathcal{C}(D)$ is antisymmetric in respect to $\gamma$-weighted inner product if there exists positive, bounded and piecewise continuous function $\gamma$ defined on whole $D$ such that

$$
\begin{equation*}
\left(\mathcal{K} y_{1}, y_{2}\right)_{\gamma}=-\left(y_{1}, \mathcal{K} y_{2}\right)_{\gamma}, \text { for all } y_{1}, y_{2} \in \mathcal{C}(D) \tag{3.19}
\end{equation*}
$$

(., .) $)_{\gamma}$ denotes $\gamma$-weighted inner product defined in (3.18).

Theorem 3.5. Let operator $\mathcal{K}: \mathcal{C}(D) \rightarrow \mathcal{C}(D)$ be antisymmetric operator in respect to $\gamma$-weighted inner product. Let $\lambda$ be eigenvalue of $\mathcal{K}$. Then

$$
\operatorname{Re} \lambda=0 .
$$

Proof. Let $\mathcal{K}$ be an antisymmetric operator and let $\lambda$ be its eigenvalue and let $u \neq 0$ be appropriate eigenvector. Then using properties of inner product we have

$$
0=(u, \mathcal{K} u)_{\gamma}+(\mathcal{K} u, u)_{\gamma}=(u, \lambda u)_{\gamma}+(\lambda u, u)_{\gamma}=(\bar{\lambda}+\lambda)(u, u)_{\gamma}
$$

Since $(u, u)_{\gamma} \neq 0$ we get $\lambda+\bar{\lambda}=0$. This implies that $\operatorname{Re} \lambda=0$.
Theorem 3.6. Operator $\mathcal{K}$ from (3.9) is an antisymmetric operator in respect to $1 / \kappa$-weighted inner product.

Proof. From (1.10)

$$
0<\frac{1}{\kappa_{u}} \leq \frac{1}{\kappa(x)} \leq \frac{1}{\kappa_{d}}<\infty, \text { for all } x \in \Omega_{1}
$$

and

$$
\frac{1}{\kappa(x)}
$$

is positive, bounded and piecewise continuous function on $\Omega_{1}$. So we can define $(1 / \kappa)$-weighted inner product. First let's show that for real functions $u, v \in \mathcal{C}\left(\Omega_{1}\right)$ is

$$
\begin{gather*}
M=(u, \mathcal{M} v)_{\frac{1}{\kappa}}-(\mathcal{M} u, v)_{\frac{1}{\kappa}}=0, \text { where } \mathcal{M}=\mathcal{M}_{1} \mathcal{K}_{1}  \tag{3.20}\\
(u, \mathcal{M} v)_{\frac{1}{\kappa}}-(\mathcal{M} u, v)_{\frac{1}{\kappa}}=\int_{\Omega_{1}} \frac{1}{\kappa(t)} u(t) \mathcal{M} v(t) d t-\int_{\Omega_{1}} \frac{1}{\kappa(\tau)} \mathcal{M} u(\tau) v(\tau) d \tau=
\end{gather*}
$$

$$
\begin{gather*}
=\int_{\Omega_{1}} \frac{1}{\kappa(t)} \kappa(t) u(t) \int_{\Omega_{1}} \frac{v(\tau)}{r(t, \tau)} d \tau d t-\int_{\Omega_{1}} \frac{1}{\kappa(\tau)} \kappa(\tau) \int_{\Omega_{1}} \frac{u(t)}{r(t, \tau)} d t v(\tau) d \tau= \\
=\int_{\Omega_{1}} \int_{\Omega_{1}} \frac{u(t) v(\tau)}{r(\tau, t)} d \tau d t-\int_{\Omega_{1}} \int_{\Omega_{1}} \frac{u(t) v(\tau)}{r(t, \tau)} d t d \tau \tag{3.21}
\end{gather*}
$$

For second integral we have by (3.6) in corollary 3.2

$$
\begin{gathered}
\int_{\Omega_{1}} \int_{\Omega_{1}}\left|\frac{u(t) v(\tau)}{r(t, \tau)}\right| d t d \tau \leq\|u\|_{\infty}\|v\|_{\infty} \int_{\Omega_{1}} \int_{\Omega_{1}} \frac{1}{r(t, \tau)} d t d \tau \leq \\
\leq\|u\|_{\infty}\|v\|_{\infty} \int_{\Omega_{1}}\left(\max _{\tau \in \Omega_{1}} \int_{\Omega_{1}} \frac{1}{r(t, \tau)} d t\right) d \tau= \\
=\|u\|_{\infty}\|v\|_{\infty}\left(\max _{\tau \in \Omega_{1}} \int_{\Omega_{1}} \frac{1}{r(t, \tau)} d t\right) \int_{\Omega_{1}} d \tau=\|u\|_{\infty}\|v\|_{\infty} C_{1} \int_{\Omega_{1}} 1 d \tau<\infty .
\end{gathered}
$$

So the order of integration in the second integral in (3.21) can be changed by Fubini's theorem A-3 and we have $M=0$. Now let's verify (3.19). First write complex function $y_{1}$ and $y_{2}$ onto the form

$$
y_{1}(x)=u_{1}(x)+\iota v_{1}(x)
$$

and

$$
y_{2}(x)=u_{2}(x)+\iota v_{2}(x) .
$$

From

$$
\mathcal{K}=\mathcal{M} \mathcal{N}
$$

we have

$$
\begin{align*}
& \left(\mathcal{K} y_{1}, y_{2}\right)_{\frac{1}{\kappa}}=\left(\mathcal{M}\left(-\iota u_{1}+v_{1}\right), u_{2}+\iota v_{2}\right)_{\frac{1}{\kappa}}=\left(-\iota \mathcal{M} u_{1}+\mathcal{M} v_{1}, u_{2}+\iota v_{2}\right)_{\frac{1}{\kappa}}= \\
& =-\iota\left(\mathcal{M} u_{1}, u_{2}\right)_{\frac{1}{\kappa}}-\iota\left(\mathcal{M} v_{1}, v_{2}\right)_{\frac{1}{\kappa}}+\left(\mathcal{M} v_{1}, u_{2}\right)_{\frac{1}{\kappa}}-\left(\mathcal{M} u_{1}, v_{2}\right)_{\frac{1}{\kappa}}  \tag{3.22}\\
& \left(y_{1}, \mathcal{K} y_{2}\right)_{\frac{1}{\kappa}}=\left(u_{1}+\iota v_{1}, \mathcal{M}\left(-\iota u_{2}+v_{2}\right)\right)_{\frac{1}{\kappa}}=\left(u_{1}+\iota v_{1},-\iota \mathcal{M} u_{2}+\mathcal{M} v_{2}\right)_{\frac{1}{\kappa}}= \\
& =\iota\left(u_{1}, \mathcal{M} u_{2}\right)_{\frac{1}{\kappa}}+\iota\left(v_{1}, \mathcal{M} v_{2}\right)_{\frac{1}{\kappa}}-\left(v_{1}, \mathcal{M} u_{2}\right)_{\frac{1}{\kappa}}+\left(u_{1}, \mathcal{M} v_{2}\right)_{\frac{1}{\kappa}} . \tag{3.23}
\end{align*}
$$

By (3.20), (3.22) and (3.23) is

$$
\left(\mathcal{K} y_{1}, y_{2}\right)_{\frac{1}{\kappa}}+\left(y_{1}, \mathcal{K} y_{2}\right)_{\frac{1}{\kappa}}=0
$$

(3.19) holds and the proof is complete.

By the last theorem is operator $\mathcal{K}$ antisymmetric with respect to the $1 / \kappa$-weighted inner product. According to theorem 3.5 number 1 is not eigenvalue of operator $\mathcal{K}$. Since $\mathcal{K}$ is compact operator, we have by proposition 2.20 that operator $\mathcal{I}-\mathcal{K}$ is invertible on space $\mathcal{C}\left(\Omega_{1}\right)$ and (1.15) is solvable on space $\mathcal{C}\left(\Omega_{1}\right)$.

## 4. Integral equation of the second kind with diagonal singularity

Now we will research for solution of more general integral equation of the second kind of the following form

$$
\begin{equation*}
\lambda y(x)-\int_{D} k(x, t) y(t) d t=f(x), x \in D, \lambda \neq 0 . \tag{4.1}
\end{equation*}
$$

where $D \subset R^{m}(m \geq 1)$ is closed, bounded and connected set. A set is connected if it is not a subset of a disjoint union of two open sets. The assumption for the set $D$ is not restrictive. In the definition of the original problem (1.15) is $D$ substituted with $\Omega_{1}$. $\Omega_{1}$ is a metal body. Metal body should be closed, bounded and connected set.

Assume that the kernel function $k(x, t)$ is singular when $x=t$. This type of singularity is called diagonal singularity. As we saw existence of solution is done by operator calculus. Operator calculus is also used in proof of convergence of numerical methods. The equation (4.1) can be rewritten into operator form:

$$
\begin{equation*}
(\lambda \mathcal{I}-\mathcal{K}) y=f \tag{4.2}
\end{equation*}
$$

where integral operator $\mathcal{K}$ is defined by

$$
\begin{equation*}
\mathcal{K} y(x)=\int_{D} k(x, t) y(t) d t \tag{4.3}
\end{equation*}
$$

The integral operator $\mathcal{K}$ is assumed to be compact on $\mathcal{C}(D)$. Assume that (4.1) has for every $f(x)$ unique solution. It means that $\lambda \neq 0$ is not eigenvalue of operator $\mathcal{K}$.

Many methods were developed for integral equations of the second kind. Almost all of them are described in [3]. First kind of methods are degenerate kernel methods. The idea is to approximate the kernel function. Other methods are called projection methods. The equation (4.1) is solved approximately. The idea is to choose a finite dimensional family of functions that contains a function $y_{n}$ close to the exact solution $y$. There are various senses in which $y_{n}$ can be said to "satisfy approximately". We obtain different methods. The most popular of these are collocation methods and Galerkin methods. These methods lead to a solution of a system of linear equations. In this thesis we will examine collocation methods.

Other way is to substitute the integral by a numerical integration rule. Such methods are called Nyström methods. Original Nyström method is not suitable for solution of equations with diagonal singular kernel. However there are some modifications of the original method which can be used. These methods also lead to a solution of a system of linear equations.

## 5. Collocation methods

### 5.1 General theory

Let us choose a sequence of finite dimensional subspaces $\mathcal{X}_{n} \subset \mathcal{C}(D)$. Let $\mathcal{X}_{n}$ have a basis $\left\{\phi_{1}, \ldots, \phi_{n}\right\}$. We will find approximate solution $y_{n} \in \mathcal{X}_{n}$ of the form

$$
\begin{equation*}
y_{n}(x)=\sum_{j=1}^{n} c_{j} \phi_{j}(x) . \tag{5.1}
\end{equation*}
$$

By substituting $y_{n}$ defined in (5.1) into (4.1) we obtain

$$
\sum_{j=1}^{n} c_{j}\left[\lambda \phi_{j}(x)-\int_{D} k(x, t) \phi_{j}(t) d t\right]=f(x)
$$

Now let's define residual by

$$
\begin{equation*}
r_{n}=\sum_{j=1}^{n} c_{j}\left[\lambda \phi_{j}(x)-\int_{D} k(x, t) \phi_{j}(t) d t\right]-f(x) . \tag{5.2}
\end{equation*}
$$

In operator form

$$
r_{n}=(\lambda \mathcal{I}-\mathcal{K}) y_{n}-f .
$$

Now let's pick up distinct approximation points $x_{1}, \ldots, x_{n} \in D$. Since want the solution to be exact at the approximation points we get

$$
\begin{equation*}
r_{n}\left(x_{i}\right)=0, i=1, . ., n . \tag{5.3}
\end{equation*}
$$

This leads to the problem to find $c_{1}, \ldots, c_{n}$ as a solution of linear system

$$
\begin{equation*}
\sum_{j=1}^{n} c_{j}\left[\lambda \phi_{j}\left(x_{i}\right)-\int_{D} k\left(x_{i}, t\right) \phi_{j}(t) d t\right]=f\left(x_{i}\right), i=1, \ldots, n \tag{5.4}
\end{equation*}
$$

To write (5.4) into more abstract form let's define projection operator. Let's define $\mathcal{P}_{n} y$ to be an element of $\mathcal{X}_{n}$ that interpolates $y$ at the approximation points $\left\{x_{1}, \ldots, x_{n}\right\}$ as

$$
\begin{equation*}
\mathcal{P}_{n} y(x)=\sum_{j=1}^{n} c_{j} \phi_{j}(x) \tag{5.5}
\end{equation*}
$$

where coefficients $c_{j}$ are determined as solution of system of linear equations

$$
\sum_{j=1}^{n} c_{j} \phi_{j}\left(x_{i}\right)=y\left(x_{i}\right), i=1, \ldots, n
$$

The system has unique solution if

$$
\begin{equation*}
\operatorname{det}\left[\phi_{j}\left(x_{i}\right)\right]_{k, i=1}^{n} \neq 0 \tag{5.6}
\end{equation*}
$$

Therefor (5.6) is assumed to be satisfied in this chapter and whenever the collocation method is discussed. Projection operator $\mathcal{P}_{n}$ maps $\mathcal{C}(D)$ onto $\mathcal{X}_{n}$ and

$$
\mathcal{P}_{n} y=y, y \in \mathcal{X}_{n} .
$$

This implies that $\mathcal{P}_{n}^{2}=\mathcal{P}_{n}$ and

$$
\left\|\mathcal{P}_{n}\right\|=\left\|\mathcal{P}_{n}^{2}\right\| \leq\left\|\mathcal{P}_{n}\right\|^{2}
$$

and

$$
\begin{equation*}
\left\|\mathcal{P}_{n}\right\| \geq 1 \tag{5.7}
\end{equation*}
$$

Note that

$$
\mathcal{P}_{n} z=0 \text { if and only if } z\left(x_{i}\right)=0, i=1, \ldots, n
$$

With the definition of projection operator the condition (5.3) can be can be now rewritten as

$$
\begin{equation*}
\mathcal{P}_{n} r_{n}=0 \tag{5.8}
\end{equation*}
$$

or equivalently as

$$
\begin{equation*}
\mathcal{P}_{n}(\lambda \mathcal{I}-\mathcal{K}) y_{n}=\mathcal{P}_{n} f, y_{n} \in \mathcal{X}_{n} . \tag{5.9}
\end{equation*}
$$

Form (5.9) is not suitable for for error analysis. If $y_{n}$ is a solution of (5.9), then (5.9) is equivalent to the expression

$$
\begin{equation*}
\left(\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right) y_{n}=\mathcal{P}_{n} f . \tag{5.10}
\end{equation*}
$$

Now let's formulate theorem from [4], which give us relation between (4.2) and (5.10).

Theorem 5.1. Let $\mathcal{X}$ be a Banach space. Assume that operator $\mathcal{K}: \mathcal{X} \rightarrow \mathcal{X}$ is bounded and assume that operator $\lambda \mathcal{I}-\mathcal{K}$ is a bijection on $\mathcal{X}$ (it means that $\lambda$ is not eigenvalue of operator $\mathcal{K})$. Further assume that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|\mathcal{K}-\mathcal{P}_{n} \mathcal{K}\right\|=0 \tag{5.11}
\end{equation*}
$$

Then operators $\left(\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right)^{-1}$ exist mapping $\mathcal{X}$ onto $\mathcal{X}$ for all sufficiently large $n \geq N$ and are uniformly bounded:

$$
\begin{equation*}
\sup _{n \geq N}\left\|\left(\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right)^{-1}\right\|<C_{N}<\infty \tag{5.12}
\end{equation*}
$$

For the solution of equations $(\lambda \mathcal{I}-\mathcal{K}) y=f$ and $\left(\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right) y_{n}=\mathcal{P}_{n} f$ we have

$$
\begin{equation*}
y-y_{n}=\lambda\left(\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right)^{-1}\left(y-\mathcal{P}_{n} y\right) \text { for all } n \geq N \tag{5.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{|\lambda|}{\left\|\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right\|}\left\|y-\mathcal{P}_{n} y\right\| \leq\left\|y-y_{n}\right\| \leq|\lambda|\left\|\left(\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right)^{-1}\right\|\left\|y-\mathcal{P}_{n} y\right\| . \tag{5.14}
\end{equation*}
$$

Moreover, $y_{n}$ converges to $y$ if and only if $\mathcal{P}_{n} y$ converges to $y$. If the convergence occurs then $\left\|y-y_{n}\right\|$ and $\left\|P_{n} y-y\right\|$ converges by the same speed.

Proof. From the identity

$$
\begin{equation*}
\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}=\lambda \mathcal{I}-\mathcal{K}+\left(\mathcal{K}-\mathcal{P}_{n} \mathcal{K}\right)=(\lambda \mathcal{I}-\mathcal{K})\left(\mathcal{I}+(\lambda \mathcal{I}-\mathcal{K})^{-1}\left(\mathcal{K}-\mathcal{P}_{n} \mathcal{K}\right)\right) \tag{5.15}
\end{equation*}
$$

we get that if the operator $\left(\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right)^{-1}$ exists, is bounded and has following form:

$$
\begin{equation*}
\left(\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right)^{-1}=\left(\mathcal{I}+(\lambda \mathcal{I}-\mathcal{K})^{-1}\left(\mathcal{K}-\mathcal{P}_{n} \mathcal{K}\right)\right)^{-1}(\lambda \mathcal{I}-\mathcal{K})^{-1} . \tag{5.16}
\end{equation*}
$$

Since $\lambda \mathcal{I}-\mathcal{K}$ is a bijection on $\mathcal{X}$ operator $(\lambda \mathcal{I}-\mathcal{K})^{-1}$ exists. To show existence of $\left(\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right)^{-1}$ we need to show that operator

$$
\begin{equation*}
\mathcal{L}=\left(\mathcal{I}+(\lambda \mathcal{I}-\mathcal{K})^{-1}\left(\mathcal{K}-\mathcal{P}_{n} \mathcal{K}\right)\right)^{-1} \tag{5.17}
\end{equation*}
$$

exists and is bounded. By assumption (5.11) we can choose sufficiently large $N$ such that

$$
\begin{equation*}
\varepsilon_{N}=\sup _{n \geq N}\left\|\mathcal{K}-\mathcal{P}_{n} \mathcal{K}\right\|<\frac{1}{\left\|(\lambda \mathcal{I}-\mathcal{K})^{-1}\right\|} \tag{5.18}
\end{equation*}
$$

Then

$$
\left\|(\lambda \mathcal{I}-\mathcal{K})^{-1}\left(\mathcal{K}-\mathcal{P}_{n} \mathcal{K}\right)\right\|<1, n \geq N
$$

and by inverse theorem proposition 2.18 operator $\mathcal{L}$ exists and is bounded. Hence operator $\left(\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right)^{-1}$ exists and is continuous for all $n \geq N$. First let's bound $\|\mathcal{L}\|$.

$$
\begin{aligned}
\|\mathcal{L}\| & =\left\|\left(\mathcal{I}+(\lambda \mathcal{I}-\mathcal{K})^{-1}\left(\mathcal{K}-\mathcal{P}_{n} \mathcal{K}\right)\right)^{-1}\right\|=\left\|\sum_{n=0}^{\infty}\left[(-1)^{n}(\lambda \mathcal{I}-\mathcal{K})^{-1}\left(\mathcal{K}-\mathcal{P}_{n} \mathcal{K}\right)\right]^{n}\right\| \leq \\
& \leq \sum_{n=0}^{\infty}\left(\left\|(\lambda \mathcal{I}-\mathcal{K})^{-1}\right\|\left\|\left(\mathcal{K}-\mathcal{P}_{n} \mathcal{K}\right)\right\|\right)^{n}=\frac{1}{1-\left\|(\lambda \mathcal{I}-\mathcal{K})^{-1}\right\|\left\|\left(\mathcal{K}-\mathcal{P}_{n} \mathcal{K}\right)\right\|}
\end{aligned}
$$

By (5.18) and (5.16) we get

$$
\begin{equation*}
\left\|\left(\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right)^{-1}\right\| \leq \frac{\left\|(\lambda \mathcal{I}-\mathcal{K})^{-1}\right\|}{1-\varepsilon_{N}\left\|(\lambda \mathcal{I}-\mathcal{K})^{-1}\right\|}=C_{N}, n \geq N \tag{5.19}
\end{equation*}
$$

where

$$
C_{N} \rightarrow\left\|(\lambda \mathcal{I}-\mathcal{K})^{-1}\right\| \text { as } N \rightarrow \infty
$$

and we have proved (5.12). To proof (5.13) let's apply $\mathcal{P}_{n}$ to the equation $(\lambda \mathcal{I}-\mathcal{K}) y=f:$

$$
\begin{gather*}
\mathcal{P}_{n}(\lambda \mathcal{I}-\mathcal{K}) y=\mathcal{P}_{n} f \\
\lambda \mathcal{P}_{n} y-\mathcal{P}_{n} \mathcal{K} y=\mathcal{P}_{n} f \\
\lambda \mathcal{P}_{n} y-\lambda y+\lambda y-\mathcal{P}_{n} \mathcal{K} y=\mathcal{P}_{n} f \\
\lambda y-\mathcal{P}_{n} \mathcal{K} y=\mathcal{P}_{n} f-\lambda \mathcal{P}_{n} y+\lambda y \\
\left(\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right) y=\mathcal{P}_{n} f+\lambda\left(y-\mathcal{P}_{n} y\right) . \tag{5.20}
\end{gather*}
$$

For $y_{n}$ we have equation (5.10). Subtracting equation (5.10) from the equation (5.20) we get

$$
\left(\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right) y-\left(\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right) y_{n}=\mathcal{P}_{n} f+\lambda\left(y-\mathcal{P}_{n} y\right)-\mathcal{P}_{n} f
$$

$$
\begin{equation*}
\left(\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right)\left(y-y_{n}\right)=\lambda\left(y-\mathcal{P}_{n} y\right) . \tag{5.21}
\end{equation*}
$$

By applying $\left(\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right)^{-1}$ we have

$$
y-y_{n}=\lambda\left(\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right)^{-1}\left(y-\mathcal{P}_{n} y\right), \text { for } n \geq N
$$

and we have proved (5.13). The upper bound in (5.14) follows immediately from (5.13). For the lower bound in (5.14) let's use (5.21). Taking norms we have

$$
\begin{equation*}
|\lambda|\left\|y-\mathcal{P}_{n} y\right\| \leq\left\|\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right\|\left\|y-y_{n}\right\| . \tag{5.22}
\end{equation*}
$$

By dividing (5.22) with $\left\|\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right\|$ we have proved (5.14). For lower bound uniform in $n$ note that for $n \geq N$

$$
\left\|\lambda \mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right\| \leq\|\lambda \mathcal{I}-\mathcal{K}\|+\left\|\mathcal{K}-\mathcal{P}_{n} \mathcal{K}\right\| \leq\|\lambda \mathcal{I}-\mathcal{K}\|+\varepsilon_{N} .
$$

The inequality (5.14) can be replaced by

$$
\begin{equation*}
\frac{|\lambda|}{\|\lambda \mathcal{I}-\mathcal{K}\|+\varepsilon_{N}}\left\|y-\mathcal{P}_{n} y\right\| \leq\left\|y-y_{n}\right\| \leq|\lambda| C_{N}\left\|y-\mathcal{P}_{n} y\right\|, n \geq N \tag{5.23}
\end{equation*}
$$

where

$$
C_{N} \rightarrow\left\|(\lambda \mathcal{I}-\mathcal{K})^{-1}\right\|, \text { as } N \rightarrow \infty
$$

This shows that $y_{n}$ converges to $y$ if and only if $\mathcal{P}_{n} y$ converges to $y$ for all $y \in \mathcal{C}(D)$. If the convergence occurs, then $\left\|y-y_{n}\right\|$ converges by the same speed as $\left\|y-\mathcal{P}_{n} y\right\|$. One way on finding numerical solution is to decompose the integration region into elements and approximate function $y$ by low degree polynomial.

### 5.2 Convergence condition

The theorem 5.1 has important condition (5.11). One way is to prove it directly. But in several cases it is not necessarily. Sufficient condition assuring (5.11) is released by following lemma.
Lemma 5.2. Let $\mathcal{X}$ be a Banach space and let $\left\{\mathcal{P}_{n}\right\}$ be a family of bounded projections on $\mathcal{X}$ satisfying

$$
\begin{equation*}
P_{n} u \rightarrow u \text { as } n \rightarrow \infty \text { for all } u \in \mathcal{X} . \tag{5.24}
\end{equation*}
$$

If the operator $\mathcal{K}: \mathcal{X} \rightarrow \mathcal{X}$ is compact operator then it holds

$$
\begin{equation*}
\left\|\mathcal{K}-\mathcal{P}_{n} \mathcal{K}\right\| \rightarrow 0 \text { as } n \rightarrow \infty \tag{5.25}
\end{equation*}
$$

Proof. From the definition of operator norm we have

$$
\left\|\mathcal{K}-\mathcal{P}_{n} \mathcal{K}\right\|=\sup _{\|u\| \leq 1}\left\|\mathcal{K} u-\mathcal{P}_{n} \mathcal{K} u\right\|=\sup _{z \in \mathcal{K}(U)}\left\|z-\mathcal{P}_{n} z\right\|,
$$

where

$$
\mathcal{K}(U)=\{\mathcal{K} u,\|u\| \leq 1\} .
$$

Since $\mathcal{K}$ is compact operator is the set $\mathcal{K}(U)$ relatively compact. Since $\mathcal{X}$ is Banach space is $\mathcal{K}(U)$ totally bounded set. Therefore by assumption (5.24) and theorem 2.16 with $\mathcal{T}=\mathcal{I}$ and $\mathcal{T}_{n}=\mathcal{P}_{n}$ we get

$$
\sup _{z \in \mathcal{K}(U)}\left\|z-\mathcal{P}_{n} z\right\| \rightarrow 0 \text { as } n \rightarrow \infty
$$

and the proof is complete.

### 5.3 Piecewise constant collocation

In case of multidimensional problem is sometimes needed to use piecewise constant collocation. Suppose that we have a family $\left\{D_{1}, \ldots, D_{n}\right\}$ of disjoint nonempty connected sets such that $\cup_{i=1}^{n} D_{i}=D$. Connected set means that it cannot be represented as the union of two or more disjoint nonempty open subsets. For each $D_{i}$ let's choose an approximation point $x_{i}$ at the interior of $D_{i}$. Now we can approximate $y(x)$ by $y_{n}(x)$, where

$$
y_{n}(x)=\widetilde{y}_{i}, \text { for } x \in D_{i} .
$$

Note that the basis functions in this case are the characteristic functions $\left\{\chi_{D_{i}}, \ldots, \chi_{D_{n}}\right\}$ defined as

$$
\chi_{D_{i}}(x)=\left\{\begin{array}{l}
1 \text { when } x \in D_{i}  \tag{5.26}\\
0 \text { when } x \notin D_{i} .
\end{array}\right.
$$

In this case for all $i \chi_{D_{i}} \notin \mathcal{C}(D)$. For simplicity let's use notation $\chi_{D_{i}}=\chi_{i}$. The approximation $y_{n}$ is of the form

$$
\begin{equation*}
y_{n}(x)=\sum_{i=1}^{n} \chi_{i}(x) \widetilde{y}_{i} . \tag{5.27}
\end{equation*}
$$

Note that for the basis functions defined by (5.26) and the approximation points $x_{i} \in D$ it holds

$$
\begin{equation*}
\chi_{i}\left(x_{j}\right)=\delta_{i j} \tag{5.28}
\end{equation*}
$$

where $\delta_{i j}$ is Kronecker delta. The approximation $y_{n}$ restricted to $D_{i}$ is constant function. Hence it is uniform continuous (each $D_{i}$ is closed and bounded set).

Problem is to choose a space for error analysis. We need suitable Banach space that carries piecewise continuous functions. One way is to use space $L^{\infty}(D)$. Problem is that in space $L^{\infty}(D)$ is $y\left(x_{i}\right)$ not well defined. One way to define it is showed in [7]. Let rewrite here some important facts. Let's define space $\mathfrak{C}(D)$ to be closed subspace of $L^{\infty}(D)$ that consists of functions that are almost everywhere equal to an element of $\mathcal{C}(D)$. The norm in $\mathfrak{C}(D)$ is the essential supremum norm defined in (2.5). The point evaluation functional is then defined by

$$
\begin{equation*}
\delta_{a}(f)=f(a), a \in D, f \in \mathfrak{C}(D) \tag{5.29}
\end{equation*}
$$

where $f(a)$ is defined by taking the representative function $f \in \mathfrak{C}(D)$ to be continuous. $\delta_{a}$ is bounded linear functional on $\mathfrak{C}(D)$ with norm $\left\|\delta_{a}\right\|=1$. The functional $\delta_{a}$ is needed to be extended to functional $d_{a}$ defined on the whole space $L^{\infty}(D)$ with property

$$
\begin{equation*}
d_{a}(f)=\delta_{a}(f)=f(a), \text { when } f \in \mathfrak{C}(D) \tag{5.30}
\end{equation*}
$$

and with norm the $\left\|d_{a}\right\|=1$. This can be done by following theorem, which is proved in almost every literature of functional analysis, for example in [1].

Theorem 5.3 (Hahn-Banach Theorem). Let $\mathbb{K}$ denote $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{V}_{0}$ be a subspace of a normed linear space $\mathcal{V}$, and let $l: \mathcal{V}_{0} \rightarrow \mathbb{K}$ be linear and bounded. Then there exists an extension $\widehat{l} \in \mathcal{V}$ of $l$ such that

$$
\widehat{l}(v)=l(v), \text { for all } v \in \mathcal{V}_{0}
$$

and

$$
\|l\|=\|\widehat{l}\| .
$$

Let's note one important fact. If $\mathcal{V}_{0}$ is not dense in $\mathcal{V}$ the extension need not to be unique. By theorem 5.3 we may extend functional $\delta_{a}$ from $\mathfrak{C}(D)$ to $L^{\infty}(D)$. The main goal of $[7]$ are following two theorems.

Theorem 5.4. Let $f \in L^{\infty}(D)$. Assume that $m \leq f(x) \leq M$ for all $x$ in a neighbourhood of $a \in D$. Then it holds

$$
m \leq d_{a}(f) \leq M
$$

Theorem 5.5. Let $f \in L^{\infty}(D)$. Assume that $a \in D$ is a point of continuity of $f$. Then

1. $d_{a}(f)=f(a)$
2. $d_{t}(f) \rightarrow f(a)$ as $t \rightarrow a$.

Now we can define projection operator $\mathcal{P}_{n}$ by

$$
\begin{equation*}
\mathcal{P}_{n} y(x)=d_{x_{i}}(y) \text { when } x \in D_{i} . \tag{5.31}
\end{equation*}
$$

or equivalently with the basis function

$$
\begin{equation*}
\mathcal{P}_{n} y(x)=\sum_{i=1}^{n} \chi_{i}(x) d_{x_{i}}(y) \tag{5.32}
\end{equation*}
$$

The domain of $\mathcal{P}_{n}$ is now $L^{\infty}(D)$. Let $x_{i} \in D_{i}$ be approximation point. From (5.28) we have that

$$
\mathcal{P}_{n}^{2} y(x)=\sum_{j=1}^{n} \chi_{j}(x) \sum_{i=1}^{n} \chi_{i}\left(x_{j}\right) d_{x_{i}}(y)=\sum_{j=1}^{n} \chi_{j}(x) d_{x_{j}}(y)
$$

and

$$
\begin{equation*}
\mathcal{P}_{n}^{2}=\mathcal{P}_{n} . \tag{5.33}
\end{equation*}
$$

Because the norm $\left\|d_{x_{i}}\right\|=1$ we have

$$
\begin{equation*}
\left\|\mathcal{P}_{n}\right\|=\operatorname{ess} \sup _{t \in D} \sum_{i=1}^{n}\left|\chi_{i}(t)\right|=1 \tag{5.34}
\end{equation*}
$$

From (5.34) and (5.33) is $\mathcal{P}_{n}$ bounded projection.
Then we assumed that $x_{i}$ is in the interior of $D_{i}$. Hence $x_{i}$ is point of continuity of $\chi_{i}$ and $y_{n}$. From theorem 5.5 we have that

$$
\begin{equation*}
\mathcal{P}_{n} y(x)=d_{x_{i}}(y)=y\left(x_{i}\right) \text { when } x \in D_{i} \tag{5.35}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\mathcal{P}_{n} y(x)=\sum_{i=1}^{n} \chi_{i}(x) y\left(x_{i}\right) . \tag{5.36}
\end{equation*}
$$

Now we can put $c_{i}=\widetilde{y}_{i}$ and $\phi_{i}=\chi_{i}$ into (5.4) and (5.1). We get

$$
\sum_{j=1}^{n} \widetilde{y}_{j}\left[\lambda \chi_{j}\left(x_{i}\right)-\int_{D} k\left(x_{i}, t\right) \chi_{j}(t) d t\right]=f\left(x_{i}\right), i=1, \ldots, n
$$

and

$$
\begin{equation*}
\lambda \widetilde{y}_{i}-\sum_{j=1}^{n} \widetilde{y}_{j} \int_{D_{j}} k\left(x_{i}, t\right) d t=f\left(x_{i}\right), i=1, \ldots, n . \tag{5.37}
\end{equation*}
$$

The approximate solution is

$$
\begin{equation*}
y_{n}(x)=\sum_{j=1}^{n} \widetilde{y}_{j} \chi_{j}(x) . \tag{5.38}
\end{equation*}
$$

Let's define

$$
\begin{equation*}
\rho_{i}=\max _{x, t \in D_{i}} r(x, t) . \tag{5.39}
\end{equation*}
$$

It is the diameter of $D_{i}$. Let

$$
\begin{equation*}
\tau_{n}=\max _{i=1, \ldots, n} \rho_{i} . \tag{5.40}
\end{equation*}
$$

Assume that

$$
\begin{equation*}
\tau_{n} \rightarrow 0 \text { as } n \rightarrow \infty . \tag{5.41}
\end{equation*}
$$

We want to proof that $y_{n}$ from (5.38) is good approximation of the exact solution $y$. We will use theorem 5.1 with space $\mathcal{X}=L^{\infty}(D)$. We need to fulfill condition (5.11). It can be done by following lemma.

Lemma 5.6. Let operator $\mathcal{K}: L_{\infty}(D) \rightarrow C(D)$ be compact operator. Let projection $\mathcal{P}_{n}$ on $L_{\infty}(D)$ be defined as in (5.31), where approximation points $x_{i}$ are in the interior of $D_{i}$. Then if (5.41) is then

$$
\lim _{n \rightarrow \infty}\left\|\mathcal{K}-\mathcal{P}_{n} \mathcal{K}\right\|=0
$$

Proof. Let us take $z \in \mathcal{C}(D)$. Since the approximation points $x_{i}$ are in the interior of $D_{i}$ is every $x_{i}$ a point of continuity of $\chi_{i}$ and from (5.35) we have

$$
\begin{gather*}
\left\|\mathcal{P}_{n} z-z\right\|_{\infty}=\max _{x \in D}\left|\mathcal{P}_{n} z(x)-z(x)\right|= \\
=\max _{i=1, \ldots, n} \sup _{x \in D_{i}}\left|\mathcal{P}_{n} z(x)-z(x)\right|=\max _{i=1, \ldots, n} \sup _{x \in D_{i}}\left|z\left(x_{i}\right)-z(x)\right| \leq \\
\leq \max _{i=1, \ldots, n} \sup _{x, t \in D_{i}}|z(t)-z(x)| \leq \sup _{r(x, t) \leq \tau_{n}}|z(t)-z(x)| \rightarrow 0 \text { as } n \rightarrow \infty \tag{5.42}
\end{gather*}
$$

by (5.41). Let's define $\mathcal{K}(Y)$ as

$$
\mathcal{K}(Y)=\left\{\mathcal{K} y,\|y\|_{L^{\infty}(D)} \leq 1\right\} .
$$

Since $\mathcal{K}: L_{\infty}(D) \rightarrow C(D)$ is compact operator, is the set $\mathcal{K}(Y) \subset \mathcal{C}(D)$ relatively compact and hence totally bounded. From (5.42) and totally boundedness of $\mathcal{K}(Y)$ we have

$$
\left\|\mathcal{K}-\mathcal{P}_{n} \mathcal{K}\right\|=\sup _{\|y\|_{L^{\infty}(D)} \leq 1}\left\|\mathcal{K} y-\mathcal{P}_{n} \mathcal{K} y\right\|_{L^{\infty}(D)}=\sup _{z \in \mathcal{K}(Y)}\left\|z-\mathcal{P}_{n} z\right\|_{\infty} \rightarrow 0 \text { as } n \rightarrow \infty .
$$

The error estimate of $\left\|y_{n}-y\right\|$ depends of the estimate of $\left\|\mathcal{P}_{n} y-y\right\|$. When $y$ is continuous function we have following corollary.

Corollary 5.7. Let $\left\{D_{1}, \ldots, D_{n}\right\}$ be a family of disjoint non-empty sets such that $\bigcup_{i=1}^{n} D_{i}=D$ and let the approximation points be at the interior of $D_{i}$. Let $\mathcal{P}_{n}$ be defined in (5.31). Let $y \in \mathcal{C}(D)$ and $\tau_{n}$ be defined as in (5.40). Then

$$
\begin{equation*}
\left\|\mathcal{P}_{n} y-y\right\|_{\infty} \leq \sup _{r(x, t) \leq \tau_{n}}|y(t)-y(x)| . \tag{5.43}
\end{equation*}
$$

Proof. Follows immediately from (5.42).
Now we need to define class of operators of the form (4.3), which are compact as operators from $L^{\infty}(D)$ into $\mathcal{C}(D)$. First let us write basic proposition. It is a generalization of theorem 2.12 and lemmas 2.13 and 2.14.

Proposition 5.8. Let operator $\mathcal{K}: L^{\infty}(D) \rightarrow L^{\infty}(D)$ be of the form as (4.3) and satisfy one of the following conditions:
(a) the function $k(x, t)$ satisfies (2.10) and (2.11)
(b) the function $k(x, t) \in \mathcal{C}(D) \times \mathcal{C}(D)$
(c) there exists continuous approximation of kernel function $k_{n}(x, t)$ such that (2.13) holds

Then $\mathcal{R}(\mathcal{K}) \subset \mathcal{C}(D)$ and $\mathcal{K}$ is compact operator from $L^{\infty}(D)$ to $C(D)$.
Proof. First let us prove (a). By proposition 2.1 we have

$$
\begin{align*}
\mid \mathcal{K} y(x) & -\mathcal{K} y\left(x^{\prime}\right)\left|=\left|\int_{D}\left[k(x, t)-k\left(x^{\prime}, t\right)\right] y(t)\right| \leq\right. \\
& \leq\|y\|_{L^{\infty}(D)} \int_{D}\left|k(x, t)-k\left(x^{\prime}, t\right)\right| d t . \tag{5.44}
\end{align*}
$$

First let us show that $\mathcal{R}(\mathcal{K}) \subset \mathcal{C}(D)$. Let us take $y \in L^{\infty}(D)$ and $\varepsilon>0$. From (2.11) there exists $\delta_{1}>0$ such that for all $x, x^{\prime} \in D$ satisfying $r\left(x, x^{\prime}\right)<\delta_{1}$ it holds

$$
\int_{D}\left|k(x, t)-k\left(x^{\prime}, t\right)\right| d t \leq \frac{\varepsilon}{\|y\|_{L^{\infty}(D)}} .
$$

From this and (5.44) we have for $x, x^{\prime} \in D$

$$
r\left(x, x^{\prime}\right)<\delta_{1} \Rightarrow\left|\mathcal{K} y(x)-\mathcal{K} y\left(x^{\prime}\right)\right| \leq \varepsilon,
$$

$\mathcal{K} y \in \mathcal{C}(D)$ and $\mathcal{R}(\mathcal{K}) \subset \mathcal{C}(D)$. Now consider the set

$$
S=\left\{\mathcal{K} y, y \in L^{\infty}(D),\|y\|_{L^{\infty}(D)} \leq 1\right\} .
$$

Let us take $\varepsilon>0$. From (2.11) we have that there exists $\delta_{2}$ such that for all $x, x^{\prime} \in D$ satisfying $r\left(x, x^{\prime}\right)<\delta_{2}$ we have

$$
\int_{D}\left|k(x, t)-k\left(x^{\prime}, t\right)\right| d t \leq \varepsilon .
$$

From here and (5.44) we have for $x, x^{\prime} \in D$

$$
r\left(x, x^{\prime}\right)<\delta_{2} \Rightarrow\left|\mathcal{K} y(x)-\mathcal{K} y\left(x^{\prime}\right)\right| \leq \varepsilon
$$

for all $y \in L^{\infty}(D)$ such that $\|y\|_{L^{\infty}(D)} \leq 1$ and $S$ is equicontinuous. From proposition 2.1 and (2.10) it follows

$$
\begin{aligned}
|\mathcal{K} y(x)| \leq & \int_{D}|k(x, t) y(t)| d t \leq\|y\|_{L^{\infty}(D)} \int_{D}|k(x, t)| d t \leq \\
& \leq \max _{x \in D} \int_{D}|k(x, t)| d t \leq M_{1}<\infty
\end{aligned}
$$

and $S$ is uniformly bounded. By Arzela-Ascoli theorem 2.11 is $\bar{S}$ compact and $\mathcal{K}$ is compact operator.

Now let us show that $(b) \Rightarrow(a)$. Since $k(x, t)$ is continuous function, function $|k(x, t)|$ is also continuous. $D$ is closed, bounded set. Hence

$$
|D|=\int_{D} 1 d t \leq \infty
$$

and

$$
\max _{x, t \in D}|k(x, t)| \leq M<\infty
$$

From here

$$
\max _{x \in D} \int_{D}|k(x, t)| d t \leq M \int_{D} 1 d t=M|D|<\infty
$$

and we have proved (2.10) with $M_{1}=M|D|$. Let us take $\varepsilon>0$. Since $D$ is closed and bounded set then $k(x, t)$ is uniformly continuous on $D \times D$. From the uniform continuity there exists $\delta_{3}>0$ such that for all $x, x^{\prime}, t \in D$

$$
r\left(x, x^{\prime}\right)<\delta_{3} \Rightarrow\left|k(x, t)-k\left(x^{\prime}, t\right)\right|<\frac{\varepsilon}{|D|}
$$

and hence

$$
r\left(x, x^{\prime}\right)<\delta_{3} \Rightarrow \sup _{\substack{x, x^{\prime}, t \in D \\ r\left(x, x^{\prime}\right)<\delta_{3}}}\left|k(x, t)-k\left(x^{\prime}, t\right)\right|<\frac{\varepsilon}{|D|} .
$$

From here

$$
\sup _{\substack{x, x^{\prime} \in D, r\left(x, x^{\prime}\right)<\delta_{3}}} \int_{D}\left|k(x, t)-k\left(x^{\prime}, t\right)\right| d t \leq \sup _{\substack{t, x, x^{\prime} \in D \\ r\left(x, x^{\prime}\right)<\delta_{3}}}\left|k(x, t)-k\left(x^{\prime}, t\right)\right||D| \leq \varepsilon
$$

and (2.11) follows.
Now assume that (c) is satisfied. Let's define the operator

$$
\mathcal{K}_{n} y(x)=\int_{D} k_{n}(x, t) y(t) d t .
$$

Since $k_{n}(x, t)$ is continuous function $\mathcal{K}_{n}$ is compact operator from $L^{\infty}(D)$ to $\mathcal{C}(D)$ due to (b). From proposition 2.1 and (2.13) we get

$$
\left\|\left(\mathcal{K}-\mathcal{K}_{n}\right) y\right\|_{\infty}=\max _{x \in D}\left|\int_{D}\left[k(x, t)-k_{n}(x, t)\right] y(t) d t\right| \leq
$$

$$
\begin{equation*}
\leq\|y\|_{L^{\infty}(D)} \max _{x \in D} \int_{D}\left|k(x, t)-k_{n}(x, t)\right| d t \rightarrow 0 \text { as } n \rightarrow \infty \tag{5.45}
\end{equation*}
$$

and

$$
\lim _{n \rightarrow \infty} \mathcal{K}_{n} y=\mathcal{K} y \text { for all } y \in L^{\infty}(D)
$$

By Banach-Steinhaus theorem 2.6 is $\mathcal{K}$ continuous linear operator from $L^{\infty}(D)$ to $\mathcal{C}(D)$. From (2.13) we have

$$
\left\|\mathcal{K}-\mathcal{K}_{n}\right\| \leq \max _{x \in D} \int_{D}\left|k(x, t)-k_{n}(x, t)\right| d t \rightarrow 0 \text { as } n \rightarrow \infty
$$

and $\mathcal{K}_{n} \rightarrow \mathcal{K}$. From here and theorem 2.9 is $\mathcal{K}$ compact operator from $L^{\infty}(D)$ to $\mathcal{C}(D)$.

## 6. Nyström method

The Nyström method is based on the approximation of the integral by numerical integration rule $Q_{n}$ :

$$
\begin{equation*}
\int_{D} v(x) d x \approx Q_{n} v=\sum_{j=1}^{n} \omega_{n, j} v\left(x_{n, j}\right) \tag{6.1}
\end{equation*}
$$

where $x_{n, j}$ are called the node points and satisfy

$$
x_{n, j} \in D
$$

In the following we will write $\omega_{j}$ instead of $\omega_{n, j}$ and $x_{j}$ instead of $x_{n, j}$. Here the integration rule can't be applied directly to (4.1) because the kernel function $k(x, t)$ was assumed to be singular when $x=t$. There are two ways to deal with such singularity. In both cases $k(x, t)$ is approximated by bounded kernel function $k_{n}(x, t), n=1,2, \ldots$, which coincide with $k(x, t)$ outside certain neighborhood of $x=t$. More details about construction of function $k_{n}(x, t)$ will be given in the next section. First way is to change kernel function. If we use function $k_{n}(x, t)$ instead of $k(x, t)$ and use numerical integration rule we get:

$$
\begin{equation*}
\left[\lambda y_{n}(x)-\sum_{j=1}^{n} \omega_{j} k_{n}\left(x, x_{j}\right) y_{n}\left(x_{j}\right)\right]=f(x) . \tag{6.2}
\end{equation*}
$$

Now we can run $x$ over the node points and we get system of linear equations for approximate solution $y_{n}\left(x_{i}\right)$

$$
\begin{equation*}
\left[\lambda y\left(x_{i}\right)-\sum_{j=1}^{n} \omega_{j} k_{n}\left(x_{i}, x_{j}\right) y\left(x_{j}\right)\right]=f\left(x_{i}\right), i=1, \ldots, n \tag{6.3}
\end{equation*}
$$

The numerical solution $y_{n}$ of modified Nyström method 1 is obtained by interpolation formula

$$
\begin{equation*}
y_{n}(x)=\frac{1}{\lambda}\left[f(x)+\sum_{j=1}^{n} \omega_{j} k_{n}\left(x, x_{j}\right) y_{n}\left(x_{j}\right)\right] . \tag{6.4}
\end{equation*}
$$

Other way is to weaken the singularity by following steps. First let's rewrite (4.1) onto the form as in [2].

$$
\begin{equation*}
\left[\lambda-\int_{D} k(x, t) d t\right] y(x)-\int_{D} k(x, t)[y(t)-y(x)] d t=f(x) \tag{6.5}
\end{equation*}
$$

By changing $k_{n}$ and $k$ in the second integral on the right hand side of (6.5) and using numerical integration rule we obtain

$$
\begin{equation*}
\left[\lambda-\int_{D} k(x, t) d t\right] \widetilde{y}_{n}(x)-\sum_{j=1}^{n} \omega_{j} k_{n}\left(x, x_{j}\right)\left[\widetilde{y}_{n}\left(x_{j}\right)-\widetilde{y}_{n}(x)\right]=f(x) . \tag{6.6}
\end{equation*}
$$

Now let's run $x$ through the node points and we get system of linear equations for $\widetilde{y}_{n}\left(x_{i}\right)$

$$
\begin{equation*}
\left[\lambda+\sum_{j=1, j \neq i}^{n} \omega_{j} k_{n}\left(x_{i}, x_{j}\right)-\int_{D} k\left(x_{i}, t\right) d t\right] \widetilde{y}_{n}\left(x_{i}\right)-\sum_{j=1, j \neq i}^{n} \omega_{j} k_{n}\left(x_{i}, x_{j}\right) \widetilde{y}_{n}\left(x_{j}\right)=f\left(x_{i}\right) \tag{6.7}
\end{equation*}
$$

The numerical solution of modified Nyström method 2 is obtained by interpolation formula

$$
\begin{equation*}
\widetilde{y}_{n}(x)=\frac{f(x)+\sum_{j=1}^{n} \omega_{j} k_{n}\left(x, x_{j}\right) \widetilde{y}_{n}\left(x_{j}\right)}{\lambda+\sum_{j=1}^{n} \omega_{j} k_{n}\left(x, x_{j}\right)-\int_{D} k(x, t) d t} . \tag{6.8}
\end{equation*}
$$

The integral can be calculated analytically or by some special numerical integration rule.

### 6.1 Integration rule and kernel function conditions

In this section we first need to define bounded approximation $k_{n}(x, t)$. To do it we need to make more assumption to the original kernel function $k(x, t)$. Let $r(x, t)$ be the Euclidean distance of points $x, t \in D$ defined by

$$
\begin{equation*}
r(x, t)=\sqrt{\sum_{j=1}^{m}\left|x_{i}-t_{i}\right|^{2}} . \tag{6.9}
\end{equation*}
$$

In the case of $D \subset \mathbb{R}$ is

$$
\begin{equation*}
r(x, t)=|x-t|, \tag{6.10}
\end{equation*}
$$

in the case of $D \subset \mathbb{R}^{2}$ is

$$
\begin{equation*}
r(x, t)=\sqrt{\left(x_{1}-t_{1}\right)^{2}+\left(x_{2}-t_{2}\right)^{2}} \tag{6.11}
\end{equation*}
$$

and in the case of $D \subset \mathbb{R}^{3}$ is

$$
\begin{equation*}
r(x, t)=\sqrt{\left(x_{1}-t_{1}\right)^{2}+\left(x_{2}-t_{2}\right)^{2}+\left(x_{3}-t_{3}\right)^{2}} . \tag{6.12}
\end{equation*}
$$

Let

$$
R_{D}=\max _{x, t \in D} r(x, t) .
$$

Assume that there exists function $h \in \mathcal{C}(D \times D)$ and positive non-increasing function $g \in \mathcal{C}(0, \infty)$ satisfying

$$
\begin{equation*}
\lim _{t \rightarrow 0+} g(t)=\infty \tag{6.13}
\end{equation*}
$$

such that the kernel function is of the form

$$
\begin{equation*}
k(x, t)=g(r(x, t)) h(x, t) \tag{6.14}
\end{equation*}
$$

Note that this is a special case of diagonal singularity defined in the chapter 4. In case of collocation methods this specification is not needed. However error
analysis of Nyström method cannot be done without it. In case of the original integral equation describing induction heating is $g(u)=1 / u$ and $h(x, t)=-\iota \kappa(x)$.

Now let's define bounded kernel approximation $k(x, t)$. Assume that $\mu_{n}$ is decreasing positive sequence such that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mu_{n}=0 \tag{6.15}
\end{equation*}
$$

Then kernel function can be approximated by function

$$
\begin{equation*}
k_{n}(x, t)=g_{\mu_{n}}(r(x, t)) h(x, t), \tag{6.16}
\end{equation*}
$$

where

$$
g_{\mu_{n}}(u)=\left\{\begin{array}{l}
g(u), \text { if } u \geq \mu_{n}  \tag{6.17}\\
g\left(\mu_{n}\right), \text { if } u<\mu_{n} .
\end{array}\right.
$$

Since $g \in \mathcal{C}(0, \infty)$ was positive non-increasing function $g_{\mu_{n}} \in \mathcal{C}[0, \infty)$ is also positive and non-increasing function for all $n$. Detailed assumption on sequence $\mu_{n}$ will be given later. Note that approximation (6.16) is bounded approximation of original $k(x, t)$. More properties are given by following lemma.

Lemma 6.1. For functions defined above the following properties are valid

$$
\begin{gather*}
r(x, t)>0 \text { for all } x, t \in D  \tag{6.18}\\
0 \leq g_{\mu_{n}}(r(x, t)) \leq g(r(x, t)) \text { for all } x, t \in D  \tag{6.19}\\
\text { if } n \geq m \Rightarrow g_{\mu_{n}}(r(x, t)) \geq g_{\mu_{m}}(r(x, t)) \text { for all } x, t \in D .  \tag{6.20}\\
k_{n}(x, t)=k(x, t), \text { when } r(x, t) \geq \mu_{n} . \tag{6.21}
\end{gather*}
$$

Proof. All items follow immediately from the definitions (6.16) and (6.17).
Since $h(x, t)$ is continuous function on $D \times D$ where $D$ is closed and bounded set we have

$$
\begin{equation*}
\max _{x, t \in D}|h(x, t)| \leq M<\infty . \tag{6.22}
\end{equation*}
$$

Now let's formulate important conditions to the kernel function and numerical integration rule.

Assume that the numerical integration rule is convergent for all continuous functions. It means that for all $v \in \mathcal{C}(D)$ it holds

$$
\begin{equation*}
\lim _{n \rightarrow \infty} Q_{n} v=\lim _{n \rightarrow \infty} \sum_{j=1}^{n} \omega_{j} v\left(x_{j}\right)=\int_{D} v(t) d t \tag{6.23}
\end{equation*}
$$

Assume that the weights of numerical integration rule $Q_{n}$ are positive, which means that

$$
\begin{equation*}
\omega_{j}>0, \text { for all } j=1, \ldots, n \tag{6.24}
\end{equation*}
$$

Let's define

$$
\begin{equation*}
\bar{\omega}_{n}=\max _{j=1, \ldots, n} \omega_{j} . \tag{6.25}
\end{equation*}
$$

For the sequence $\mu_{n}$ assume that

$$
\begin{equation*}
\mu_{n}^{m} \geq \rho^{m} \bar{\omega}_{n} \tag{6.26}
\end{equation*}
$$

where $0<\rho<\infty$ and that there exists $\bar{\mu}<\infty$ such that

$$
\begin{equation*}
g\left(\mu_{n}\right) \bar{\omega}_{n} \leq \bar{\mu} \text { for all } n . \tag{6.27}
\end{equation*}
$$

Also assume that exists $c_{D}<\infty$ such that

$$
\begin{equation*}
\max _{x \in D} \int_{\left\{t, r(x, t)<R_{D}\right\}} g(r(x, t)) d t \leq c_{D}, \tag{6.28}
\end{equation*}
$$

assume that

$$
\begin{equation*}
\lim _{\nu \rightarrow 0} \max _{x \in D} \int_{\{t, r(x, t)<\nu\}} g(r(x, t)) d t=0 . \tag{6.29}
\end{equation*}
$$

Finally assume that there exists constants $c<\infty$ such that for all positive, nonincreasing function $z \in \mathcal{C}[0, \infty)$ and $x \in D$ it holds

$$
\begin{equation*}
\sum_{j, r\left(x, x_{j}\right) \leq \xi} \omega_{j} z\left(r\left(x, x_{j}\right)\right) \leq c\left[z(0) \bar{\omega}_{n}+\int_{\{t, r(x, t) \leq \xi\}} z(r(x, t)) d t\right] . \tag{6.30}
\end{equation*}
$$

The last conditions seem to be very restrictive, however in next chapter we will see that it is satisfied for compound numerical integration rules. Very important is also to find the sequence $\mu_{n}$.

Proposition 6.2. For operator $\mathcal{K}$ defined by (4.3) it holds

$$
\begin{equation*}
\|\mathcal{K}\| \leq M C_{D} \tag{6.31}
\end{equation*}
$$

Proof. Let us take $y \in \mathcal{C}(D)$ such that $\|y\|_{\infty}=1$. Then by (6.28) we have

$$
\|\mathcal{K}\| \leq \max _{x \in D} \int_{D}|h(x, t) g(r(x, t))| d t \leq M \max _{x \in D} \int_{\left\{t, r(x, t) \leq R_{D}\right\}} g(r(x, t)) d t \leq M C_{D}
$$

At the end of this section let's write one important application of principal of uniform boundedness 2.5.

Proposition 6.3. Let $Q_{n}$ be numerical integration rule that converges for all continuous function and satisfies (6.24). Then

$$
\begin{equation*}
\sup _{n} \sum_{j=1}^{n} \omega_{j} \leq c_{I}<\infty . \tag{6.32}
\end{equation*}
$$

Proof. Let's apply theorem 2.5. The spaces are $\mathcal{X}=\mathcal{Y}=\mathcal{C}(D)$ and $\mathcal{T}_{n}=Q_{n}$. Since the numerical integration rule converges for all continuous functions we have by (6.23) that the limit $Q_{n} y$ exists for all $y \in \mathcal{C}(D)$. Let's take $y \in \mathcal{C}(D)$ such that $\|y\|_{\infty}=1$. Then by (2.8) in theorem 2.5 we have

$$
\sup _{n}\left\|Q_{n}\right\|_{\infty}=\sup _{n} \sum_{j=1}^{n} \omega_{j} \leq T<\infty .
$$

Hence $c_{I}=T$.

### 6.2 Convergence of Nyström method

We need to show that (6.2) and (6.6) is a good approximation of (4.1). We will use operator calculus. Let's define operator $\mathcal{K}_{n}: \mathcal{C}(D) \rightarrow \mathcal{C}(D)$.

$$
\begin{equation*}
\mathcal{K}_{n} y(x)=\sum_{j=1}^{n} \omega_{j} k_{n}\left(x, x_{j}\right) y\left(x_{j}\right) \tag{6.33}
\end{equation*}
$$

where $k_{n}$ is defined in (6.16). We can see that $\mathcal{K}_{n}$ is for each $n$ compact linear operator (it is continuous linear operator of finite rank). From equation (6.2) and definition (6.33) we obtain operator form of Nyström method 1:

$$
\begin{equation*}
\left(\lambda \mathcal{I}-\mathcal{K}_{n}\right) y_{n}=f . \tag{6.34}
\end{equation*}
$$

For rewriting (6.6) into operator form we need to define another operator $\widetilde{\mathcal{K}}_{n}: \mathcal{C}(D) \rightarrow \mathcal{C}(D)$

$$
\begin{equation*}
\widetilde{\mathcal{K}}_{n} y(x)=\sum_{j=1}^{n} \omega_{j} k_{n}\left(x, x_{j}\right)\left[y\left(x_{j}\right)-y(x)\right]+\int_{D} k(x, t) y(x) d t . \tag{6.35}
\end{equation*}
$$

Then operator form of Nyström method 2 - equation (6.6) is

$$
\begin{equation*}
\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}\right) \widetilde{y}_{n}=f . \tag{6.36}
\end{equation*}
$$

First let's prove existence of $\left(\lambda \mathcal{I}-\mathcal{K}_{n}\right)^{-1}$ for enough large $n$. We will use following theorem. It is modified version of theorem 4.1.1 from [3].

Theorem 6.4. Let $\mathcal{X}$ be a Banach space, let operators $\mathcal{S}, \mathcal{T}$ be bounded on $\mathcal{X}$. For given $\lambda \neq 0$ let's assume that $\lambda I-\mathcal{T}$ is a bijection on $\mathcal{X}$ (which means $(\lambda I-\mathcal{T})^{-1}$ exists, is bounded and $\left.\mathcal{R}(\lambda I-\mathcal{T})=\mathcal{X}\right)$. If

$$
\begin{equation*}
\|(\mathcal{T}-\mathcal{S}) \mathcal{S}\|<\frac{|\lambda|}{\left\|(\lambda I-\mathcal{T})^{-1}\right\|} \tag{6.37}
\end{equation*}
$$

then $(\lambda I-\mathcal{S})^{-1}: \mathcal{R}(\lambda \mathcal{I}-\mathcal{S}) \rightarrow \mathcal{X}$ exists, is bounded and

$$
\begin{equation*}
\left\|(\lambda \mathcal{I}-\mathcal{S})^{-1}\right\| \leq \frac{1+\left\|(\lambda I-\mathcal{T})^{-1}\right\|\|\mathcal{S}\|}{|\lambda|-\left\|(\lambda I-\mathcal{T})^{-1}\right\|\|(\mathcal{T}-\mathcal{S}) \mathcal{S}\|} \tag{6.38}
\end{equation*}
$$

Let $f \in \mathcal{R}(\lambda \mathcal{I}-\mathcal{S})$. Let $y$ be solution of $(\lambda \mathcal{I}-\mathcal{T}) y=f$ and let $z$ be solution of $(\lambda \mathcal{I}-\mathcal{S}) z=f$. Then it holds

$$
\begin{equation*}
\|y-z\|_{\mathcal{X}} \leq\left\|(\lambda I-\mathcal{S})^{-1}\right\|\|\mathcal{T} y-\mathcal{S} y\|_{\mathcal{X}} . \tag{6.39}
\end{equation*}
$$

If $S$ is compact operator then $\mathcal{R}(\lambda \mathcal{I}-\mathcal{S})=\mathcal{X}$.
Proof. $(\lambda \mathcal{I}-\mathcal{T})$ has inverse by assumption. The inverse can be written in the form:

$$
(\lambda \mathcal{I}-\mathcal{T})^{-1}=\frac{1}{\lambda}\left[\mathcal{I}+(\lambda \mathcal{I}-\mathcal{T})^{-1} \mathcal{T}\right]
$$

This can be verified by identities

$$
\frac{1}{\lambda}\left[\mathcal{I}+(\lambda \mathcal{I}-\mathcal{T})^{-1} \mathcal{T}\right]=\frac{1}{\lambda}\left[(\lambda \mathcal{I}-\mathcal{T})^{-1}(\lambda \mathcal{I}-\mathcal{T})+(\lambda \mathcal{I}-\mathcal{T})^{-1} \mathcal{T}\right]=
$$

$$
=\frac{1}{\lambda}(\lambda \mathcal{I}-\mathcal{T})^{-1}[\lambda \mathcal{I}-\mathcal{T}+\mathcal{T}]
$$

Consider the approximation

$$
\begin{equation*}
\frac{1}{\lambda}\left[\mathcal{I}+(\lambda \mathcal{I}-\mathcal{T})^{-1} \mathcal{S}\right] \tag{6.40}
\end{equation*}
$$

Now let's check (6.40).

$$
\begin{gathered}
\frac{1}{\lambda}\left[\mathcal{I}+(\lambda \mathcal{I}-\mathcal{T})^{-1} \mathcal{S}\right](\lambda \mathcal{I}-\mathcal{S})=\frac{1}{\lambda}\left[(\lambda \mathcal{I}-\mathcal{T})^{-1}(\lambda \mathcal{I}-\mathcal{T})+(\lambda \mathcal{I}-\mathcal{T})^{-1} \mathcal{S}\right](\lambda \mathcal{I}-\mathcal{S})= \\
=\frac{1}{\lambda}(\lambda \mathcal{I}-\mathcal{T})^{-1}(\lambda \mathcal{I}-\mathcal{T}+\mathcal{S})(\lambda \mathcal{I}-\mathcal{S})= \\
=(\lambda \mathcal{I}-\mathcal{T})^{-1}(\lambda \mathcal{I}-\mathcal{T}+\mathcal{S})-\frac{1}{\lambda}(\lambda \mathcal{I}-\mathcal{T})^{-1}(\lambda \mathcal{I}-\mathcal{T}+\mathcal{S}) \mathcal{S}= \\
=(\lambda \mathcal{I}-\mathcal{T})^{-1}(\lambda \mathcal{I}-\mathcal{T})+(\lambda \mathcal{I}-\mathcal{T})^{-1} \mathcal{S}+\frac{1}{\lambda}(\lambda \mathcal{I}-\mathcal{T})^{-1}(\mathcal{T}-\mathcal{S}) \mathcal{S}-\frac{1}{\lambda}(\lambda \mathcal{I}-\mathcal{T})^{-1}(\lambda \mathcal{I}) \mathcal{S}= \\
=\mathcal{I}+(\lambda \mathcal{I}-\mathcal{T})^{-1} \mathcal{S}+\frac{1}{\lambda}(\lambda \mathcal{I}-\mathcal{T})^{-1}(\mathcal{T}-\mathcal{S}) \mathcal{S}-(\lambda \mathcal{I}-\mathcal{T})^{-1} \mathcal{S}
\end{gathered}
$$

from this we get

$$
\begin{equation*}
\frac{1}{\lambda}\left[\mathcal{I}+(\lambda \mathcal{I}-\mathcal{T})^{-1} \mathcal{S}\right](\lambda \mathcal{I}-\mathcal{S})=\mathcal{I}+\frac{1}{\lambda}(\lambda \mathcal{I}-\mathcal{T})^{-1}(\mathcal{T}-\mathcal{S}) \mathcal{S} \tag{6.41}
\end{equation*}
$$

By assumption (6.37)

$$
\begin{equation*}
\left\|\frac{1}{\lambda}(\lambda \mathcal{I}-\mathcal{T})^{-1}(\mathcal{T}-\mathcal{S}) \mathcal{S}\right\| \leq \frac{1}{|\lambda|}\left\|(\lambda \mathcal{I}-\mathcal{T})^{-1}\right\|\|(\mathcal{T}-\mathcal{S}) \mathcal{S}\|<1 \tag{6.42}
\end{equation*}
$$

by proposition 2.18 is the right hand side of (6.41) invertible and therefore it holds

$$
\begin{gather*}
\left\|\left[\mathcal{I}+\frac{1}{\lambda}(\lambda \mathcal{I}-\mathcal{T})^{-1}(\mathcal{T}-\mathcal{S}) \mathcal{S}\right]^{-1}\right\|=\left\|\sum_{n=0}^{\infty}(-1)^{n}\left[\frac{1}{\lambda}(\lambda \mathcal{I}-\mathcal{T})^{-1}(\mathcal{T}-\mathcal{S}) \mathcal{S}\right]^{n}\right\| \leq \\
\leq \sum_{n=0}^{\infty}\left\|\left[\frac{1}{\lambda}(\lambda \mathcal{I}-\mathcal{T})^{-1}(\mathcal{T}-\mathcal{S}) \mathcal{S}\right]^{n}\right\| \leq \sum_{n=0}^{\infty}\left\|\frac{1}{\lambda}(\lambda \mathcal{I}-\mathcal{T})^{-1}(\mathcal{T}-\mathcal{S}) \mathcal{S}\right\|^{n} \leq \\
\leq \sum_{n=0}^{\infty}\left(\left\|\frac{1}{\lambda}(\lambda \mathcal{I}-\mathcal{T})^{-1}\right\|\|(\mathcal{T}-\mathcal{S}) \mathcal{S}\|\right)^{n}= \\
=\frac{1}{1-\frac{1}{|\lambda|}\left\|(\lambda \mathcal{I}-\mathcal{T})^{-1}\right\|\|(\mathcal{T}-\mathcal{S}) \mathcal{S}\|} \tag{6.43}
\end{gather*}
$$

Since the hand right side of (6.41) is invertible the left hand side is also invertible. This implies that $(\lambda \mathcal{I}-\mathcal{S})$ is one-to-one. Otherwise the left hand side would not be invertible. So inverse operator $(\lambda \mathcal{I}-\mathcal{S})^{-1}$ exists as operator from $\mathcal{R}(\lambda \mathcal{I}-\mathcal{S}) \rightarrow \mathcal{X}$. Let's multiply (6.41) by $(\lambda \mathcal{I}-\mathcal{S})^{-1}$ from right

$$
\frac{1}{\lambda}\left(\mathcal{I}+(\lambda \mathcal{I}-\mathcal{T})^{-1} \mathcal{S}\right)=\left[\mathcal{I}+\frac{1}{\lambda}(\lambda \mathcal{I}-\mathcal{T})^{-1}(\mathcal{T}-\mathcal{S}) \mathcal{S}\right](\lambda \mathcal{I}-S)^{-1}
$$

and since the right hand side of (6.41) is invertible we get

$$
\begin{equation*}
(\lambda \mathcal{I}-S)^{-1}=\left[\mathcal{I}+\frac{1}{\lambda}(\lambda \mathcal{I}-\mathcal{T})^{-1}(\mathcal{T}-\mathcal{S}) \mathcal{S}\right]^{-1}\left[\frac{1}{\lambda}\left(\mathcal{I}+(\lambda \mathcal{I}-\mathcal{T})^{-1} \mathcal{S}\right)\right] \tag{6.44}
\end{equation*}
$$

Now from (6.44) and the bound (6.43) we can derive

$$
\begin{align*}
& \left\|(\lambda \mathcal{I}-S)^{-1}\right\| \leq \frac{\left\|\frac{1}{\lambda}\left(\mathcal{I}+(\lambda \mathcal{I}-\mathcal{T})^{-1} \mathcal{S}\right)\right\|}{1-\frac{1}{|\lambda|}\left\|(\lambda \mathcal{I}-\mathcal{T})^{-1}\right\|\|(\mathcal{T}-\mathcal{S}) \mathcal{S}\|}= \\
& \quad=\frac{\frac{1}{|\lambda|} \|\left(\mathcal{I}+(\lambda \mathcal{I}-\mathcal{T})^{-1} \mathcal{S} \|\right.}{1-\frac{1}{|\lambda|}\left\|(\lambda \mathcal{I}-\mathcal{T})^{-1}\right\|\|(\mathcal{T}-\mathcal{S}) \mathcal{S}\|} \leq \\
& \quad \leq \frac{1+\left\|(\lambda \mathcal{I}-\mathcal{T})^{-1}\right\|\|\mathcal{S}\|}{|\lambda|-\left\|(\lambda \mathcal{I}-\mathcal{T})^{-1}\right\|\|(\mathcal{T}-\mathcal{S}) \mathcal{S}\|} \tag{6.45}
\end{align*}
$$

and we have proved bound (6.38). For (6.39) let's take some $f \in \mathcal{R}(\lambda \mathcal{I}-\mathcal{S})$. Equation $(\lambda-\mathcal{T}) y=f$ can be rewritten as

$$
\begin{align*}
& (\lambda \mathcal{I}-\mathcal{S}) y+(\mathcal{S}-\mathcal{T}) y=f \\
& (\lambda \mathcal{I}-\mathcal{S}) y=f+(\mathcal{T}-\mathcal{S}) y \tag{6.46}
\end{align*}
$$

Since $(\lambda \mathcal{I}-\mathcal{S}) z=f$ let's subtract $(\lambda \mathcal{I}-\mathcal{S}) z$ from left hand side of (6.46) and $f$ from the right hand side. We obtain

$$
\begin{align*}
(\lambda \mathcal{I}-\mathcal{S}) y-(\lambda \mathcal{I}-\mathcal{S}) z & =f+(\mathcal{T}-\mathcal{S}) y-f \\
(\lambda \mathcal{I}-\mathcal{S})(y-z) & =(\mathcal{T}-\mathcal{S}) y \tag{6.47}
\end{align*}
$$

From invertibility of $\lambda \mathcal{I}-\mathcal{S}$ we have

$$
(y-z)=(\lambda \mathcal{I}-\mathcal{S})^{-1}(\mathcal{T}-\mathcal{S}) y
$$

By using norm we have error bound (6.39).
If $\mathcal{S}$ is compact operator we have by Fredholm alternative theorem that $\lambda \mathcal{I}-\mathcal{S}$ is bijection from $\mathcal{X}$ to $\mathcal{X}$. Hence $(\lambda \mathcal{I}-\mathcal{S})^{-1}$ exists as operator from $\mathcal{X}$ to $\mathcal{X}$.

We want to use theorem 6.4 with $\mathcal{S}=\mathcal{K}_{n}$ and $\mathcal{T}=\mathcal{K}$. Operator $\mathcal{K}$ was assumed to be a compact operator. Compact operators are bounded. We need to verify (6.37) and uniform boundedness of operators $\mathcal{K}_{n}$. We will use theory of collectively compact operator approximation. First let us define a set $W_{m}$ as

$$
\begin{equation*}
W_{m}=\left\{\mathcal{K}_{n} y, y \in \mathcal{C}(D),\|y\|_{\infty} \leq 1, n \geq m\right\} \tag{6.48}
\end{equation*}
$$

and prove next lemmas.
Lemma 6.5. Let $\xi \in\left(0, R_{D}\right)$, let $\mu_{n}$ be positive decreasing sequence that converges to 0 . Let the numerical integration rule and the sequence $\mu_{n}$ satisfy (6.26), (6.27) and (6.30). Then it holds

$$
\begin{equation*}
\max _{x \in D} \sum_{j, r\left(x, x_{j}\right)<\xi} \omega_{j} g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right) \leq c\left(1+\frac{1}{C_{m} \rho^{m}}\right) \max _{x \in D} \int_{\{t, r(x, t)<\xi\}} g(r(x, t)) d t \tag{6.49}
\end{equation*}
$$

for all $n$ such that $\mu_{n}<\xi$, where $c$ is constant from (6.30), $\rho$ from (6.26) and

$$
\begin{equation*}
C_{m}=\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} \tag{6.50}
\end{equation*}
$$

where $\Gamma$ is the Gamma function.
Proof. $g_{\mu_{n}}$ is for all $n$ continuous, positive non-increasing function. By applying (6.30) we get

$$
\begin{gather*}
\max _{x \in D} \sum_{j, r\left(x, x_{j}\right)<\xi} \omega_{j} g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right) \leq c\left[\bar{\omega}_{n} g_{\mu_{n}}(0)+\max _{x \in D} \int_{\{t, r(x, t)<\xi\}} g_{\mu_{n}}(r(x, t)) d t\right] \leq \\
\leq c\left[\bar{\omega}_{n} g_{\mu_{n}}(0)+\max _{x \in D} \int_{\{t, r(x, t)<\xi\}} g(r(x, t)) d t\right] \tag{6.51}
\end{gather*}
$$

Let's take some $x \in D$. Then for all $\nu$ it holds

$$
\begin{equation*}
\int_{\{t, r(x, t)<\nu\}} 1 d t=C_{m} \nu^{m} . \tag{6.52}
\end{equation*}
$$

From (6.26) we have (note that $m=1,2, \ldots$ )

$$
\begin{equation*}
\bar{\omega}_{n} \leq \frac{\mu_{n}^{m}}{\rho^{m}} \tag{6.53}
\end{equation*}
$$

From the boundedness of $g\left(\mu_{n}\right) \bar{\omega}_{n}$ - prop. (6.27), (6.25), (6.53) and (6.52) we get

$$
\begin{gathered}
g_{\mu_{n}}(0) \bar{\omega}_{n}=g\left(\mu_{n}\right) \bar{\omega}_{n} \leq g\left(\mu_{n}\right) \frac{\mu_{n}^{m}}{\rho^{m}}=g\left(\mu_{n}\right) \frac{\mu_{n}^{m} C_{m}}{\rho^{m} C_{m}}= \\
=\frac{1}{C_{m} \rho^{m}} \int_{\left\{t, r(x, t)<\mu_{n}\right\}} g\left(\mu_{n}\right) d t \leq \frac{1}{C_{m} \rho^{m}} \max _{x \in D} \int_{\left\{t, r(x, t)<\mu_{n}\right\}} g(r(x, t)) d t \leq \\
\leq \frac{1}{C_{m} \rho^{m}} \max _{x \in D} \int_{\{t, r(x, t)<\xi\}} g(r(x, t)) d t
\end{gathered}
$$

and (6.49) holds.
Lemma 6.6. Let operator $\mathcal{K}_{n}$ be defined as in (6.33), where $k_{n}$ is defined by (6.16) and where $\mu_{n}$ is positive decreasing sequence such that (6.15) holds. Under assumption (6.22), (6.23), (6.24), (6.26), (6.27), (6.28), (6.29) and (6.30) there exist $n_{0}$ such that is $W_{n_{0}}$ uniformly bounded.

Proof. Let us take some $y \in \mathcal{C}(D)$ such that $\|y\|_{\infty} \leq 1$ and $\xi \in\left(0, R_{D}\right)$. From (6.15) it follows that there exists $n_{0}$ such that if $n \geq n_{0}$ is

$$
\begin{equation*}
\mu_{n}<\xi . \tag{6.54}
\end{equation*}
$$

By (6.22) and (6.16) there exists $M<\infty$ such that

$$
\left\|\mathcal{K}_{n} y\right\|_{\infty}=\max _{x \in D}\left|\sum_{j=1}^{n} \omega_{j} h\left(x, x_{j}\right) g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right) y\left(x_{j}\right)\right| \leq
$$

$$
\leq M \max _{x \in D} \sum_{j=1}^{n}\left|\omega_{j} g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right)\right| .
$$

Function $g$ was assumed to be positive function and for the weights of numerical integration rule we assumed in (6.24) that $\omega_{j}>0$. Hence

$$
\begin{equation*}
\left\|\mathcal{K}_{n} y\right\|_{\infty} \leq M \max _{x \in D} \sum_{j=1}^{n} \omega_{j} g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right) . \tag{6.55}
\end{equation*}
$$

Now let's split the sum into two parts. We get

$$
\begin{gather*}
\max _{x \in D} \sum_{j=1}^{n} \omega_{j} g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right) \leq \\
\leq\left[\max _{x \in D} \sum_{j, r\left(x, x_{j}\right)<\xi} \omega_{j} g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right)+\max _{x \in D} \sum_{j, r\left(x, x_{j}\right) \geq \xi} \omega_{j} g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right)\right] . \tag{6.56}
\end{gather*}
$$

Since $g$ is non-increasing function we have

$$
\begin{equation*}
g(\xi) \geq g_{\mu_{n}}(r(x, t)) \text { when } r(x, t) \geq \xi \tag{6.57}
\end{equation*}
$$

From (6.23) and (6.24) we can use proposition 6.3. For the right sum in (6.56) we get by (6.57)

$$
\begin{gather*}
\max _{x \in D} \sum_{j, r\left(x, x_{j}\right) \geq \xi} \omega_{j} g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right) \leq \max _{\substack{x, t \in D \\
r(x, t) \geq \xi}} g(r(x, t)) \sum_{j, r\left(x, x_{j}\right) \geq \xi} \omega_{j} \leq \\
\leq g(\xi) \sum_{j, r\left(x, x_{j}\right) \geq \xi} \omega_{j} \leq g(\xi) \sum_{j=1}^{n} \omega_{j} \leq g(\xi) c_{I} \tag{6.58}
\end{gather*}
$$

where $c_{I}$ is defined by (6.32) in proposition 6.3. From (6.30) and (6.49) from lemma 6.5 (we can us the lemma - all assumptions are satisfied) we have for the left sum in (6.56) if $n \geq n_{0}$ that

$$
\begin{gathered}
\max _{x \in D} \sum_{j, r\left(x, x_{j}\right)<\xi} g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right) \leq \\
\leq c\left[1+\frac{1}{C_{m} \rho^{m}}\right] \max _{x \in D} \int_{\{t, r(x, t)<\xi\}} g(r(x, t)) d t \leq \\
\leq c\left[1+\frac{1}{C_{m} \rho^{m}}\right] \max _{x \in D} \int_{\left\{t, r(x, t)<R_{D}\right\}} g(r(x, t)) d t .
\end{gathered}
$$

Hence by (6.28) is

$$
\begin{equation*}
\max _{x \in D} \sum_{j, r\left(x, x_{j}\right)<\xi} g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right) \leq c\left[1+\frac{1}{C_{m} \rho^{m}}\right] c_{D} \tag{6.59}
\end{equation*}
$$

and from (6.56), (6.58), (6.59) we have

$$
\begin{equation*}
\max _{x \in D} \sum_{j=1}^{n} \omega_{j} g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right) \leq C \tag{6.60}
\end{equation*}
$$

where

$$
\begin{equation*}
C=\max \left\{c_{I} g(\xi), c\left[1+\frac{1}{C_{m} \rho^{m}}\right] c_{D}\right\} . \tag{6.61}
\end{equation*}
$$

Since $g \in \mathcal{C}\left[\xi, R_{D}\right] g(\xi)<\infty$. Constant $C<\infty$ is independent to $x$ and $n$ so we have from (6.55)

$$
\begin{equation*}
\left\|\mathcal{K}_{n} y\right\|_{\infty} \leq M C<\infty, \text { for } n \geq n_{0} \tag{6.62}
\end{equation*}
$$

and $W_{n_{0}}$ is uniformly bounded.
Corollary 6.7. Under the assumptions of last the lemma 6.6 we have that there exits $n_{0}$ and constant $C$ independent to $n$ such that for all $n \geq n_{0}$ for the operator $\mathcal{K}_{n}$ the following inequality is valid

$$
\begin{equation*}
\left\|\mathcal{K}_{n}\right\| \leq M C \tag{6.63}
\end{equation*}
$$

and $\mathcal{K}_{n}$ is bounded operator.
Proof. Follows immediately from (6.62).
Lemma 6.8. Let operator $\mathcal{K}_{n}$ be defined as in (6.33) where $k_{n}$ is defined by (6.16) and where $\mu_{n}$ is positive decreasing sequence such that (6.15) holds. Under assumption (6.22), (6.23), (6.24), (6.26), (6.27), (6.28), (6.29) and (6.30) is $W_{1}$ equicontinuous.

Proof. Let us take some $y \in \mathcal{C}(D)$ such that $\|y\|_{\infty} \leq 1$ and some $\varepsilon>0$. By (6.29) there exists $\xi>0$ such that

$$
\begin{equation*}
\max _{x \in D} \int_{\{t, r(x, t)<\xi\}} g(r(x, t)) d t<\frac{\varepsilon}{24 c M\left(1+\frac{1}{C_{m} \rho^{m}}\right)} \tag{6.64}
\end{equation*}
$$

where $M$ is defined by (6.22), $c$ by (6.30), $\rho$ in (6.26) and $C_{m}$ was defined in lemma 6.5. From the uniform continuity of $h(x, t)$ on $D \times D$ we have that there exists $\delta_{1}>0$ such that

$$
\text { for all } t, x, x^{\prime} \in D, r\left(x, x^{\prime}\right)<\delta_{1} \Rightarrow\left|h\left(x^{\prime}, t\right)-h(x, t)\right|<\frac{\varepsilon}{2 C}
$$

where $C$ is defined by (6.61). Hence

$$
\begin{equation*}
\text { for all } x, x^{\prime} \in D, r\left(x, x^{\prime}\right)<\delta_{1} \Rightarrow \max _{t \in D}\left|h\left(x^{\prime}, t\right)-h(x, t)\right|<\frac{\varepsilon}{2 C} \text {. } \tag{6.65}
\end{equation*}
$$

Function $g$ is uniformly continuous on $\left[\xi, R_{D}\right]$ and we have that there exists $\delta_{2}>0$ such that

$$
\begin{equation*}
\text { for all } u, u^{\prime} \in\left[\xi, R_{D}\right],\left|u-u^{\prime}\right|<\delta_{2} \Rightarrow\left|g\left(u^{\prime}\right)-g(u)\right|<\frac{\varepsilon}{12 c_{I} M} \tag{6.66}
\end{equation*}
$$

where $c_{I}$ is defined by (6.32) in proposition 6.3 and M is defined in (6.22). From triangular inequality we have

$$
\left|r(x, t)-r\left(x^{\prime}, t\right)\right| \leq r\left(x, x^{\prime}\right) .
$$

From here and (6.66) we have

$$
\begin{gathered}
\text { for all } t, x, x^{\prime} \in D \text { if } r(x, t) \geq \xi, r\left(x^{\prime}, t\right) \geq \xi, r\left(x, x^{\prime}\right)<\delta_{2} \Rightarrow \\
\Rightarrow\left|g(r(x, t))-g\left(r\left(x^{\prime}, t\right)\right)\right|<\frac{\varepsilon}{12 c_{I} M}
\end{gathered}
$$

and hence

$$
\begin{align*}
& \text { for all } x, x^{\prime} \in D \text { if } r\left(x, x^{\prime}\right)<\delta_{2} \Rightarrow \\
& \Rightarrow \max _{\substack{t \in D, r(x, t) \geq \xi, r\left(x^{\prime}, t\right) \geq \xi}}\left|g(r(x, t))-g\left(r\left(x^{\prime}, t\right)\right)\right|<\frac{\varepsilon}{12 c_{I} M} . \tag{6.67}
\end{align*}
$$

Now let us take $x, x^{\prime} \in D$ such that $r\left(x, x^{\prime}\right)<\delta_{2}$ and $t \in D$ such that $r(x, t) \geq \xi$ and $r\left(x^{\prime}, t\right)<\xi$. Then

$$
r(x, t)>r\left(x^{\prime}, t\right)
$$

and by triangular inequality we get

$$
\left|r(x, t)-r\left(x^{\prime}, t\right)\right|=r(x, t)-r\left(x^{\prime}, t\right) \leq r\left(x, x^{\prime}\right)
$$

Hence

$$
\begin{aligned}
& r(x, t)-\xi+\xi-r\left(x^{\prime}, t\right) \leq r\left(x, x^{\prime}\right) \\
& r(x, t)-\xi \leq r\left(x, x^{\prime}\right)+r\left(x^{\prime}, t\right)-\xi
\end{aligned}
$$

Since $r\left(x^{\prime}, t\right)<\xi$ we have that $r\left(x^{\prime}, t\right)-\xi<0$ and

$$
r(x, t)-\xi \leq r\left(x, x^{\prime}\right) .
$$

From here and (6.66) we obtain that

$$
\begin{gathered}
\text { for all } x, x^{\prime}, t \in D \text {, if } r(x, t) \geq \xi, r\left(x^{\prime}, t\right)<\xi \text { and } r\left(x, x^{\prime}\right)<\delta_{2} \Rightarrow \\
\Rightarrow|g(\xi)-g(r(x, t))|<\frac{\varepsilon}{12 c_{I} M}
\end{gathered}
$$

and hence

$$
\begin{gather*}
\quad \text { for all } x, x^{\prime} \in D \text { if } r\left(x, x^{\prime}\right)<\delta_{2} \Rightarrow \\
\Rightarrow \max _{\substack{t \in D, r\left(x^{\prime}, t\right)<\xi, r(x, t) \geq \xi}}|g(\xi)-g(r(x, t))|<\frac{\varepsilon}{12 c_{I} M} . \tag{6.68}
\end{gather*}
$$

Let us take $x, x^{\prime} \in D$ such that $r\left(x, x^{\prime}\right)<\delta=\min \left\{\delta_{1}, \delta_{2}\right\}$. By (6.65) we have

$$
\begin{gathered}
\left|\mathcal{K}_{n} y\left(x^{\prime}\right)-\mathcal{K}_{n} y(x)\right|= \\
=\left|\sum_{j=1}^{n} \omega_{j}\left[g_{\mu_{n}}\left(r\left(x^{\prime}, x_{j}\right)\right) h\left(x^{\prime}, x_{j}\right)-g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right) h\left(x, x_{j}\right)\right] y\left(x_{j}\right)\right| \leq
\end{gathered}
$$

$$
\begin{align*}
& \left|\sum_{j=1}^{n} \omega_{j} g_{\mu_{n}}\left(r\left(x^{\prime}, x_{j}\right)\right)\left[h\left(x^{\prime}, x_{j}\right)-h\left(x, x_{j}\right)\right] y\left(x_{j}\right)\right|+ \\
& +\left|\sum_{j=1}^{n} \omega_{j}\left[g_{\mu_{n}}\left(r\left(x^{\prime}, x_{j}\right)\right)-g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right)\right] h\left(x, x_{j}\right) y\left(x_{j}\right)\right| \leq \\
& \quad \leq \frac{\varepsilon}{2 C}\left|\sum_{j=1}^{n} \omega_{j} g_{\mu_{n}}\left(r\left(x^{\prime}, x_{j}\right)\right)\right|+ \\
& +M\left[K_{1}\left(x, x^{\prime}\right)+K_{2}\left(x, x^{\prime}\right)+K_{3}\left(x, x^{\prime}\right)+K_{4}\left(x, x^{\prime}\right)\right] \tag{6.69}
\end{align*}
$$

where $K_{1}, K_{2}, K_{3}$ and $K_{4}$ are defined as

$$
\begin{aligned}
K_{1}\left(x, x^{\prime}\right) & =\sum_{\substack{j, r\left(x, x_{j}\right)<\xi, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left|g_{\mu_{n}}\left(r\left(x^{\prime}, x_{j}\right)\right)-g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right)\right|, \\
K_{2}\left(x, x^{\prime}\right) & =\sum_{\substack{j, r\left(x, x_{j}\right) \geq \xi, r\left(x^{\prime}, x_{j}\right) \geq \xi}} \omega_{j}\left|g_{\mu_{n}}\left(r\left(x^{\prime}, x_{j}\right)\right)-g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right)\right|, \\
K_{3}\left(x, x^{\prime}\right) & =\sum_{\substack{j, r\left(x, x_{j}\right) \geq \xi, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left|g_{\mu_{n}}\left(r\left(x^{\prime}, x_{j}\right)\right)-g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right)\right|
\end{aligned}
$$

and

$$
K_{4}\left(x, x^{\prime}\right)=\sum_{\substack{j, r\left(x, x_{j}\right)<\xi, r\left(x^{\prime}, x_{j}\right) \geq \xi}} \omega_{j}\left|g_{\mu_{n}}\left(r\left(x^{\prime}, x_{j}\right)\right)-g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right)\right| .
$$

Now let us assume that $\mu_{n}<\xi$. For $K_{1}$ we have by (6.30) and (6.49) from proposition 6.5

$$
\begin{aligned}
K_{1}\left(x, x^{\prime}\right) \leq & \sum_{j, r\left(x^{\prime}, x_{j}\right)<\xi} \omega_{j} g_{\mu_{n}}\left(r\left(x^{\prime}, x_{j}\right)\right)+\sum_{j, r\left(x, x_{j}\right)<\xi} \omega_{j} g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right) \leq \\
& \leq 2 \max _{x \in D} \sum_{j, r\left(x, x_{j}\right)<\xi} \omega_{j} g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right) \leq \\
\leq & 2 c\left[1+\frac{1}{C_{m} \rho^{m}}\right] \max _{x \in D} \int_{\{t, r(x, t)<\xi\}} g(r(x, t)) d t .
\end{aligned}
$$

From here by (6.64) we get

$$
\begin{equation*}
K_{1}\left(x, x^{\prime}\right) \leq 2 c\left[1+\frac{1}{C_{m} \rho^{m}}\right] \frac{\varepsilon}{24 c M\left(1+\frac{1}{C_{m} \rho^{m}}\right)} \leq \frac{\varepsilon}{12 M} . \tag{6.70}
\end{equation*}
$$

For $K_{2}$ we have by (6.67) and (6.32)

$$
\begin{equation*}
K_{2}\left(x, x^{\prime}\right) \leq \max _{\substack{t \in D, r(x, t) \geq \xi, r\left(x^{\prime}, t\right) \geq \xi}}\left|g\left(r\left(x^{\prime}, t\right)\right)-g(r(x, t))\right| \sum_{j=1}^{n} \omega_{j} \leq \frac{\varepsilon c_{I}}{12 c_{I} M}=\frac{\varepsilon}{12 M} . \tag{6.71}
\end{equation*}
$$

For $K_{3}$ we have from (6.30), (6.26) and (6.25)

$$
\begin{aligned}
& K_{3}\left(x, x^{\prime}\right) \leq \sum_{\substack{j, r\left(x, x_{j}\right) \geq \xi, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left|g_{\mu_{n}}\left(r\left(x^{\prime}, x_{j}\right)\right)-g(\xi)\right|+\sum_{\substack{j, r\left(x, x_{j}\right) \geq \xi, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left|g(\xi)-g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right)\right| \leq \\
& \leq \sum_{j, r\left(x^{\prime}, x_{j}\right)<\xi} \omega_{j}\left|g_{\mu_{n}}\left(r\left(x^{\prime}, x_{j}\right)\right)-g(\xi)\right|+\sum_{\substack{j, r\left(x, x_{j}\right) \geq \xi, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left|g(\xi)-g\left(r\left(x, x_{j}\right)\right)\right| \leq \\
& \leq \sum_{j, r\left(x^{\prime}, x_{j}\right)<\xi} \omega_{j} g_{\mu_{n}}\left(r\left(x^{\prime}, x_{j}\right)\right)+\sum_{j, r\left(x^{\prime}, x_{j}\right)<\xi} \omega_{j} g(\xi)+\sum_{\substack{j, r\left(x, x_{j} j\right) \geq \xi, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left|g(\xi)-g\left(r\left(x, x_{j}\right)\right)\right| \leq \\
& \quad \leq 2 \sum_{j, r\left(x^{\prime}, x_{j}\right)<\xi} \omega_{j} g_{\mu_{n}}\left(r\left(x^{\prime}, x_{j}\right)\right)+\sum_{\substack{j, r\left(x, x_{j}\right) \geq \xi, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left|g(\xi)-g\left(r\left(x, x_{j}\right)\right)\right| \leq \\
& \leq 2 \max _{x \in D} \sum_{j, r\left(x, x_{j}\right)<\xi} \omega_{j} g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right)+\sum_{\substack{j, r\left(x, x_{j}\right) \geq \xi, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left|g(\xi)-g\left(r\left(x, x_{j}\right)\right)\right| \leq \\
& \leq 2 c\left[1+\frac{1}{C_{m} \rho^{m}}\right] \max _{x \in D} \int_{\{t, r(x, t)<\xi\}} g(r(x, t)) d t+\underset{\substack{t \in, r, r(x, t) \geq \xi \\
r\left(x^{\prime}, t\right)<\xi}}{\max }|g(\xi)-g(r(x, t))| \sum_{j=1}^{n} \omega_{j} .
\end{aligned}
$$

From here, (6.64) and (6.68) we have

$$
\begin{equation*}
K_{3}\left(x, x^{\prime}\right) \leq 2 c\left[1+\frac{1}{C_{m} \rho^{m}}\right] \frac{\varepsilon}{24 c M\left(1+\frac{1}{C_{m} \rho^{m}}\right)}+\frac{\varepsilon}{12 c_{I} M} c_{I}=\frac{\varepsilon}{6 M} . \tag{6.72}
\end{equation*}
$$

Since $K_{4}\left(x, x^{\prime}\right)=K_{3}\left(x^{\prime}, x\right)$ we have by the same way

$$
\begin{equation*}
K_{4}\left(x, x^{\prime}\right) \leq \frac{\varepsilon}{6 M} \tag{6.73}
\end{equation*}
$$

From (6.69), (6.70), (6.71), (6.72) and (6.73) we have that for all $\varepsilon>0$ there exists $\delta=\min \left\{\delta_{1}, \delta_{2}\right\}$ such that for all $x, x^{\prime} \in D$ it holds

$$
r\left(x, x^{\prime}\right)<\delta \Rightarrow\left|\mathcal{K}_{n} y\left(x^{\prime}\right)-\mathcal{K}_{n} y(x)\right| \leq \varepsilon
$$

Now assume that $\mu_{n} \geq \xi$. Note that

$$
\begin{equation*}
g_{\mu_{n}}(r(x, t))=g\left(\mu_{n}\right) \text { if } r(x, t) \leq \mu_{n} . \tag{6.74}
\end{equation*}
$$

For $K_{1}$ we have by (6.74)

$$
\begin{equation*}
K_{1}\left(x, x^{\prime}\right)=\sum_{\substack{j, r\left(x, x_{j}\right)<\xi, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left|g\left(\mu_{n}\right)-g\left(\mu_{n}\right)\right|=0 . \tag{6.75}
\end{equation*}
$$

For $K_{2}$ we have by (6.67) and (6.32)

$$
\begin{gather*}
K_{2}\left(x, x^{\prime}\right)=\sum_{\substack{j, \xi \leq r\left(x, x_{j}\right)<\mu_{n}, \xi \leq r\left(x^{\prime}, x_{j}\right)<\mu_{n}}} \omega_{j}\left|g_{\mu_{n}}\left(r\left(x^{\prime}, x_{j}\right)\right)-g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right)\right|+ \\
+\sum_{\substack{j, r\left(x, x_{j}\right) \geq \mu_{n}, r\left(x^{\prime}, x_{j}\right) \geq \mu_{n}}} \omega_{j}\left|g_{\mu_{n}}\left(r\left(x^{\prime}, x_{j}\right)\right)-g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right)\right|= \\
=\sum_{\substack{j, \xi \leq r\left(x, x_{j}\right)<\mu_{n}, \xi \leq r\left(x^{\prime}, x_{j}\right)<\mu_{n}}} \omega_{j}\left|g\left(\mu_{n}\right)-g\left(\mu_{n}\right)\right|+\sum_{\substack{j, r\left(x, x_{j}\right) \geq \mu_{n}, r\left(x^{\prime}, x_{j}\right) \geq \mu_{n}}} \omega_{j}\left|g\left(r\left(x^{\prime}, x_{j}\right)\right)-g\left(r\left(x, x_{j}\right)\right)\right|= \\
=\sum_{\substack{j, r\left(x, x_{j}\right) \geq \mu_{n}, r\left(x^{\prime}, x_{j}\right) \geq \mu_{n}}} \omega_{j}\left|g\left(r\left(x^{\prime}, x_{j}\right)\right)-g\left(r\left(x, x_{j}\right)\right)\right| \leq \\
\leq \max _{\substack{t \in D, r(x, t) \geq \xi, r\left(x^{\prime}, t\right) \geq \xi}}\left|g\left(r\left(x^{\prime}, t\right)\right)-g(r(x, t))\right| \sum_{j=1}^{n} \omega_{j} \leq \frac{\varepsilon c_{I}}{12 c_{I} M}=\frac{\varepsilon}{12 M} . \tag{6.76}
\end{gather*}
$$

For $K_{3}$ we have by (6.30), (6.26) and (6.25)

$$
\begin{gathered}
K_{3}\left(x, x^{\prime}\right) \leq \sum_{\substack{j, r\left(x, x_{j}\right) \geq \xi, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left|g_{\mu_{n}}\left(r\left(x^{\prime}, x_{j}\right)\right)-g_{\mu_{n}}(\xi)\right|+ \\
+\sum_{\substack{j, r\left(x, x_{j}\right) \geq \xi, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left|g_{\mu_{n}}(\xi)-g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right)\right| \leq \\
\leq \sum_{\sum_{j, r\left(x^{\prime}, x_{j}\right)<\xi} \omega_{j}\left|g_{\mu_{n}}\left(r\left(x^{\prime}, x_{j}\right)\right)-g_{\mu_{n}}(\xi)\right|+\sum_{\substack{j, r\left(x, x_{j}\right) \geq \xi, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left|g_{\mu_{n}}(\xi)-g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right)\right|=}=\sum_{\sum_{j, r\left(x^{\prime}, x_{j}\right)<\xi} \omega_{j}\left|g\left(\mu_{n}\right)-g\left(\mu_{n}\right)\right|+\sum_{\substack{j, r\left(x, x_{j}\right) \geq \xi, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left|g_{\mu_{n}}(\xi)-g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right)\right|=}^{\sum_{\substack{j, \xi \leq r\left(x, x_{j}\right)<\mu_{n}, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left|g_{\mu_{n}}(\xi)-g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right)\right|+\sum_{\substack{j, r\left(x, x_{j}\right) \geq \mu_{n}, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left|g_{\mu_{n}}(\xi)-g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right)\right|=} \\
=\sum_{\substack{j, \xi \leq r\left(x, x_{j}\right)<\mu_{n}, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left|g\left(\mu_{n}\right)-g\left(\mu_{n}\right)\right|+\sum_{\substack{j, r\left(x, x_{j}\right) \geq \mu_{n}, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left|g\left(\mu_{n}\right)-g\left(r\left(x, x_{j}\right)\right)\right|= \\
=\sum_{\substack{j, r\left(x, x_{j}\right) \geq \mu_{n}, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left[g\left(\mu_{n}\right)-g\left(r\left(x, x_{j}\right)\right)\right] \leq \sum_{\substack{j, r\left(x, x_{j}\right) \geq \mu_{n}, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left[g(\xi)-g\left(r\left(x, x_{j}\right)\right)\right] \leq
\end{gathered}
$$

$$
\begin{gathered}
\leq \sum_{\substack{j, r\left(x, x_{j}\right) \geq \xi, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left[g(\xi)-g\left(r\left(x, x_{j}\right)\right)\right]=\sum_{\substack{j, r\left(x, x_{j}\right) \geq \xi, r\left(x^{\prime}, x_{j}\right)<\xi}} \omega_{j}\left|g(\xi)-g\left(r\left(x, x_{j}\right)\right)\right| \leq \\
\leq \max _{\substack{t \in D, r(x, t) \geq \xi \\
r\left(x^{\prime}, t\right)<\xi}}|g(\xi)-g(r(x, t))| \sum_{j=1}^{n} \omega_{j} .
\end{gathered}
$$

From here, (6.64), and (6.68) we have

$$
\begin{equation*}
K_{3}\left(x, x^{\prime}\right) \leq \frac{\varepsilon}{12 c_{I} M} c_{I}=\frac{\varepsilon}{12 M} . \tag{6.77}
\end{equation*}
$$

Since $K_{4}\left(x, x^{\prime}\right)=K_{3}\left(x^{\prime}, x\right)$ we have by the same way

$$
\begin{equation*}
K_{4}\left(x, x^{\prime}\right) \leq \frac{\varepsilon}{12 M} \tag{6.78}
\end{equation*}
$$

From (6.69), (6.75), (6.76), (6.77) and (6.78) we have that for all $\varepsilon>0$ there exists $\delta=\min \left\{\delta_{1}, \delta_{2}\right\}$ such that for all $x, x^{\prime} \in D$

$$
r\left(x, x^{\prime}\right)<\delta \Rightarrow\left|\mathcal{K}_{n} y\left(x^{\prime}\right)-\mathcal{K}_{n} y(x)\right| \leq \frac{3 \varepsilon}{4} \leq \varepsilon
$$

Hence $W_{1}$ is equicontinuous.
Now it remains to prove that for each $y \in \mathcal{C}(D)$ the sequence $\mathcal{K}_{n} y$ converges to $\mathcal{K} y$. Before doing it let us define $g_{\xi}$ as

$$
g_{\xi}(u)=\left\{\begin{array}{l}
g(u) \text { when } u \geq \xi  \tag{6.79}\\
g(\xi) \text { when } u<\xi
\end{array}\right.
$$

For all $\xi>0$ is $g(\xi)$ continuous function.
Proposition 6.9. Let $y \in \mathcal{C}(D)$. Then for all $\xi>0$ it holds

$$
\begin{equation*}
\max _{x \in D}\left|\mathcal{K}_{n} y(x)-\mathcal{K} y(x)\right| \leq L_{1}(\xi, n, y)+L_{2}(\xi, n, y)+L_{3}(\xi, y) \tag{6.80}
\end{equation*}
$$

where

$$
\begin{aligned}
& L_{1}(\xi, n, y)=\max _{x \in D}\left|\sum_{j=1}^{n} \omega_{j} h\left(x, x_{j}\right) g_{\xi}\left(r\left(x, x_{j}\right)\right) y\left(x_{j}\right)-\int_{D} h(x, t) g_{\xi}(r(x, t)) y(t) d t\right| \\
& L_{2}(\xi, n, y)=\max _{x \in D}\left|\sum_{j=1}^{n} \omega_{j} h\left(x, x_{j}\right) g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right) y\left(x_{j}\right)-\sum_{j=1}^{n} \omega_{j} h\left(x, x_{j}\right) g_{\xi}\left(r\left(x, x_{j}\right)\right) y\left(x_{j}\right)\right|
\end{aligned}
$$

and

$$
L_{3}(\xi, y)=\max _{x \in D}\left|\int_{D} h(x, t) g_{\xi}(r(x, t)) y(t) d t-\int_{D} h(x, t) g(r(x, t)) y(t) d t\right| .
$$

Proof. Follows immediately from the definition of $\mathcal{K}, \mathcal{K}_{n}$ and triangular inequality.

Lemma 6.10. Let the numerical integration rule $Q_{n}$ satisfy (6.23) and (6.24). Then for all $\xi>0$ and $y \in \mathcal{C}(D)$ it holds

$$
\begin{equation*}
\lim _{n \rightarrow \infty} L_{1}(\xi, n, y)=0 \tag{6.81}
\end{equation*}
$$

Proof. Let us take $\xi>0, y \in \mathcal{C}(D)$ and let $Q_{n}$ be the numerical integration rule. Let us define $Q$ as

$$
Q u=\int_{D} u(t) d t
$$

Then

$$
L_{1}(\xi, n, y)=\max _{x \in D}\left|Q_{n} v_{x}-Q v_{x}\right|
$$

where

$$
v_{x}(t)=h(x, t) g_{\xi}(r(x, t)) y(t)
$$

From (6.23) we have for each $u \in \mathcal{C}(D)$

$$
\left|Q_{n} u-Q u\right| \rightarrow 0 \text { as } n \rightarrow \infty .
$$

Let us define a set $S$ as

$$
S=\left\{v_{x}, x \in D\right\}
$$

and let us show that functions from $S$ are uniformly bounded and equicontinuous. From

$$
\max _{x, t \in D} v_{x}(t) \leq M g(\xi)\|y\|_{\infty}
$$

we have that $S$ is uniformly bounded. Now let us show that $S$ is equicontinuous. Let us take $\varepsilon>0$. From the uniform continuity of $y$ on $D$ we have that there exists $\delta_{1}>0$ such that

$$
\begin{equation*}
\text { for all } t, t^{\prime} \in D \text { if } r\left(t, t^{\prime}\right)<\delta_{1} \Rightarrow\left|y(t)-y\left(t^{\prime}\right)\right|<\frac{\varepsilon}{3 M g(\xi)} \tag{6.82}
\end{equation*}
$$

From the uniform continuity of $h$ on $D \times D$ we have that there exist $\delta_{2}$ such that

$$
\text { for all } t, t^{\prime}, x \in D \text { if } r\left(t, t^{\prime}\right)<\delta_{2} \Rightarrow\left|h(x, t)-h\left(x, t^{\prime}\right)\right|<\frac{\varepsilon}{3 g(\xi)\left\|_{y}\right\|_{\infty}}
$$

and hence

$$
\begin{equation*}
\text { for all } t, t^{\prime} \in D \text { if } r\left(t, t^{\prime}\right)<\delta_{2} \Rightarrow \max _{x \in D}\left|h(x, t)-h\left(x, t^{\prime}\right)\right|<\frac{\varepsilon}{3 g(\xi)\|y\|_{\infty}} . \tag{6.83}
\end{equation*}
$$

From here and uniform continuity of $g_{\xi}$ on $\left[0, R_{D}\right]$ we have that there exists $\delta_{3}$ such that

$$
\text { for all } u, u^{\prime} \in\left[0, R_{D}\right],\left|u-u^{\prime}\right|<\delta_{3} \Rightarrow\left|g(u)-g\left(u^{\prime}\right)\right|<3 M\|y\|_{\infty}
$$

From here and triangular inequality

$$
\left|r(x, t)-r\left(x, t^{\prime}\right)\right| \leq r\left(t, t^{\prime}\right) \text { for each } x, t, t^{\prime} \in D
$$

we get

$$
\text { for all } t, t^{\prime}, x \in D \text { if } r\left(t, t^{\prime}\right)<\delta_{3} \Rightarrow\left|g_{\xi}(r(x, t))-g_{\xi}\left(r\left(x, t^{\prime}\right)\right)\right|<\frac{\varepsilon}{3 M\|y\|_{\infty}}
$$ and hence

$$
\begin{equation*}
\text { for all } t, t^{\prime} \in D \text { if } r\left(t, t^{\prime}\right)<\delta_{3} \Rightarrow \max _{x \in D}\left|g_{\xi}(r(x, t))-g_{\xi}\left(r\left(x, t^{\prime}\right)\right)\right|<\frac{\varepsilon}{3 M\|y\|_{\infty}} \text {. } \tag{6.84}
\end{equation*}
$$

From (6.82), (6.83) and (6.84) we have for all $t, t^{\prime} \in D$ such that $r\left(t, t^{\prime}\right)<\delta=$ $\min \left\{\delta_{1}, \delta_{2}, \delta_{3}\right\}$

$$
\begin{aligned}
\left|v_{x}(t)-v_{x}\left(t^{\prime}\right)\right| & \leq\left|h(x, t) g_{\xi}(r(x, t)) y(t)-h\left(x, t^{\prime}\right) g_{\xi}\left(r\left(x, t^{\prime}\right)\right) y\left(t^{\prime}\right)\right| \leq \\
& \leq\left|h(x, t) g_{\xi}(r(x, t)) y(t)-h\left(x, t^{\prime}\right) g_{\xi}(r(x, t)) y(t)\right|+ \\
& +\left|h\left(x, t^{\prime}\right) g_{\xi}(r(x, t)) y(t)-h\left(x, t^{\prime}\right) g_{\xi}(r(x, t)) y\left(t^{\prime}\right)\right|+ \\
& +\left|h\left(x, t^{\prime}\right) g_{\xi}(r(x, t)) y\left(t^{\prime}\right)-h\left(x, t^{\prime}\right) g_{\xi}\left(r\left(x, t^{\prime}\right)\right) y\left(t^{\prime}\right)\right| \leq \\
& \leq\|y\|_{\infty} g(\xi) \max _{x \in D}\left|h(x, t)-h\left(x, t^{\prime}\right)\right|+M g(\xi)\left|y(t)-y\left(t^{\prime}\right)\right|+ \\
& +M\|y\|_{\infty} \max _{x \in D}\left|g_{\xi}(r(x, t))-g_{\xi}\left(r\left(x, t^{\prime}\right)\right)\right| \leq \varepsilon
\end{aligned}
$$

and $S$ is equicontinuous. From corollary 2.17 for $\mathcal{T}=Q$ and $\mathcal{T}_{n}=Q_{n}$ we have

$$
\sup _{u \in S}\left|Q_{n} u-Q u\right|=\max _{x \in D}\left|Q_{n} v_{x}-Q v_{x}\right| \rightarrow 0 \text { as } n \rightarrow \infty
$$

and lemma follows.
Lemma 6.11. Let operator $\mathcal{K}$ be defined as in (4.3) and let operator $\mathcal{K}_{n}$ be defined as in (6.33), where $k_{n}$ is defined by (6.16) and where $\mu_{n}$ is positive decreasing sequence such that (6.15) holds. Finally assume that (6.22), (6.23), (6.24), (6.26), (6.27) and (6.29) hold. Also assume that there exists constant $c$ such that (6.30) holds for all $z \in \mathcal{C}[0, \infty)$. Then for all $y \in \mathcal{C}(D)$

$$
\begin{equation*}
\mathcal{K}_{n} y \rightarrow \mathcal{K} y \text { as } n \rightarrow \infty . \tag{6.85}
\end{equation*}
$$

Proof. Let us take $y \in \mathcal{C}(D)$ and $\varepsilon>0$. Then by (6.29) there exists $\xi>0$ such that

$$
\begin{equation*}
\max _{x \in D} \int_{\{t, r(x, t)<\xi\}} g(r(x, t)) d t<\frac{\varepsilon}{4 M\|y\|_{\infty}\left(c+1+\frac{c}{C_{m} \rho^{m}}\right)} . \tag{6.86}
\end{equation*}
$$

Let us take such $\xi$. Let $L_{1}, L_{2}$ and $L_{3}$ be defined as in proposition 6.9. By (6.81) in lemma 6.10 there exists $n_{1}$ such that if $n \geq n_{1}$ is

$$
\begin{equation*}
L_{1}(\xi, n, y) \leq \frac{\varepsilon}{2} \tag{6.87}
\end{equation*}
$$

By (6.15) there exists $n_{2}$ such that if $n \geq n_{2}$ is $\mu_{n}<\xi$. Let's take $n \geq n_{2}$. Then

$$
\begin{equation*}
g(r(x, t)) \geq g_{\mu_{n}}(r(x, t)) \geq g_{\xi}(r(x, t)) . \tag{6.88}
\end{equation*}
$$

Since

$$
g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right)=g_{\xi}\left(r\left(x, x_{j}\right)\right) \text { if } r\left(x, x_{j}\right) \geq \xi
$$

we have for $L_{2}(\xi, n, y)$ by (6.49) and (6.88)

$$
\begin{align*}
& L_{2}(\xi, n, y) \leq \max _{x \in D} \sum_{j, r\left(x, x_{j}\right)<\xi} \omega_{j}\left|h\left(x, x_{j}\right)\left\|g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right)-g_{\xi}\left(r\left(x, x_{j}\right)\right)\right\| y\left(x_{j}\right)\right| \leq \\
& \leq M\|y\|_{\infty}\left(\max _{x \in D} \sum_{j, r\left(x, x_{j}\right)<\xi} \omega_{j} g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right)+\max _{x \in D} \sum_{j, r\left(x, x_{j}\right)<\xi} \omega_{j} g_{\xi}\left(r\left(x, x_{j}\right)\right)\right) \leq \\
& \quad \leq 2 M\|y\|_{\infty} \max _{x \in D} \sum_{j, r\left(x, x_{j}\right)<\xi} \omega_{j} g_{\mu_{n}}\left(r\left(x, x_{j}\right)\right) \leq \\
& \leq 2 M\|y\|_{\infty} c\left(1+\frac{1}{C_{m} \rho^{m}}\right) \max _{x \in D} \int_{\{t, r(x, t)<\xi\}} g(r(x, t)) d t . \tag{6.89}
\end{align*}
$$

Since

$$
g_{\xi}(r(x, t))=g(r(x, t)) \text { if } r(x, t) \geq \xi
$$

we have for $L_{3}(\xi, y)$

$$
\begin{align*}
& L_{3}(\xi, y) \leq M\|y\|_{\infty} \max _{x \in D} \int_{D}\left|g_{\xi}(r(x, t))-g(r(x, t))\right| d t= \\
& =M\|y\|_{\infty} \max _{x \in D} \int_{\{t, r(x, t)<\xi\}}\left|g_{\xi}(r(x, t))-g(r(x, t))\right| d t \leq \\
& \quad \leq 2 M\|y\|_{\infty} \max _{x \in D} \int_{\{t, r(x, t)<\xi\}}|g(r(x, t))| d t . \tag{6.90}
\end{align*}
$$

From (6.86), (6.89) and (6.90) if $n \geq n_{2}$ is

$$
\begin{equation*}
L_{2}(\xi, n)+L_{3}(\xi) \leq 2 M\|y\|_{\infty}\left(1+c+\frac{c}{C_{m} \rho^{m}}\right) \max _{x \in D} \int_{\{t, r(x, t)<\xi\}} g(r(x, t)) d t \leq \frac{\varepsilon}{2} \tag{6.91}
\end{equation*}
$$

From (6.91), (6.80) and (6.87) we have that if $n \geq \max \left\{n_{1}, n_{2}\right\}$ is

$$
\max _{x \in D}\left|\mathcal{K}_{n} y(x)-\mathcal{K} y(x)\right| \leq \varepsilon
$$

and (6.85) holds.
Now let us formulate main theorem about convergence of Nyström method 1.
Theorem 6.12. Let $\mu_{n}$ be decreasing positive sequence such that (6.15) holds. Let operator $\mathcal{K}$ be defined by (4.3) and operators $\mathcal{K}_{n}$ by (6.33). Assume that the numerical integration rule is convergent for all continuous function and satisfies (6.24). Also assume that (6.22), (6.26), (6.27), (6.28) and (6.29) hold and assume that there exists a constant $c<\infty$ such that (6.30) holds for all non-increasing function $z \in \mathcal{C}[0, \infty)$. Further assume that $\lambda \neq 0$ is not eigenvalue of operator $\mathcal{K}$. Then there exists $N_{1}$ such that for all $n \geq N_{1}$ the inverse $\left(\lambda \mathcal{I}-\mathcal{K}_{n}\right)^{-1}$ exists and is bounded by

$$
\begin{equation*}
\left\|\left(\lambda \mathcal{I}-\mathcal{K}_{n}\right)^{-1}\right\| \leq \frac{1+\left\|(\lambda \mathcal{I}-\mathcal{K})^{-1}\right\|\left\|\mathcal{K}_{n}\right\|}{|\lambda|-\left\|(\lambda \mathcal{I}-\mathcal{K})^{-1}\right\|\left\|\left(\mathcal{K}-\mathcal{K}_{n}\right) \mathcal{K}_{n}\right\|} \leq c_{N_{1}}<\infty . \tag{6.92}
\end{equation*}
$$

Let $y \in \mathcal{C}(D)$ be solution of $(\lambda \mathcal{I}-\mathcal{K}) y=f$ and let $y_{n}$ be solution of $\left(\lambda \mathcal{I}-\mathcal{K}_{n}\right) y_{n}=f$. Then for all $n \geq N_{1}$

$$
\begin{equation*}
\left\|y-y_{n}\right\|_{\infty} \leq\left\|\left(\lambda \mathcal{I}-\mathcal{K}_{n}\right)^{-1}\right\|\left\|\mathcal{K} y-\mathcal{K}_{n} y\right\|_{\infty} . \tag{6.93}
\end{equation*}
$$

Proof. Since the numerical integration rule converges for all continuous function the (6.23) holds. Operator $\mathcal{K}$ is bounded by proposition 6.2. From lemmas 6.6 and 6.8 there exists $n_{0}$ such that $W_{n_{0}}$ uniformly bounded and equicontinuous. From uniform boundedness of $W_{n_{0}}$ we have that operators $\mathcal{K}_{n}$ are uniformly bounded if $n \geq n_{0}$. By Arzela-Ascoli theorem is $\bar{W}_{n_{0}}$ compact in $\mathcal{C}(D)$. By lemma $6.11 \mathcal{K}_{n} y \rightarrow \mathcal{K} y$ for all $y \in \mathcal{C}(D)$. Hence $\mathcal{K}_{n}$ is a family of collectively compact operators and by theorem 2.15

$$
\begin{equation*}
\left\|\left(\mathcal{K}-\mathcal{K}_{n}\right) \mathcal{K}_{n}\right\| \rightarrow 0 \text { as } n \rightarrow \infty \tag{6.94}
\end{equation*}
$$

and there exists $\bar{N}_{1}$ such that for $n \geq \bar{N}_{1}$

$$
\left\|\left(\mathcal{K}-\mathcal{K}_{n}\right) \mathcal{K}_{n}\right\| \leq \frac{|\lambda|}{\left\|(\lambda \mathcal{I}-\mathcal{K})^{-1}\right\|}
$$

holds. Then by theorem 6.4 operators $\left(\lambda \mathcal{I}-\mathcal{K}_{n}\right)^{-1}$ exists and (6.92) holds for all $n \geq N_{1}=\max \left\{n_{0}, \bar{N}_{1}\right\}$. By applying (6.39) with $y, z=y_{n}, \mathcal{S}=\mathcal{K}_{n}$ and $\mathcal{T}=\mathcal{K}$ (6.93) follows for $n \geq N_{1}$.

The last theorem shows that the speed of convergence of $y_{n}$ to $y$ is the same as the speed of convergence of $\left\|\mathcal{K} y-\mathcal{K}_{n} y\right\|_{\infty}$. The convergence of $\left\|\mathcal{K} y-\mathcal{K}_{n} y\right\|_{\infty} \rightarrow 0$ was proved in lemma 6.11. Now it's time to show convergence of the method 2. First let us prove one lemma.

Lemma 6.13. Let operator $\mathcal{K}_{n}: \mathcal{C}(D) \rightarrow \mathcal{C}(D)$ be defined as in (6.33) and let operator $\widetilde{\mathcal{K}}_{n}: \mathcal{C}(D) \rightarrow \mathcal{C}(D)$ be defined as in (6.35). Then

$$
\begin{equation*}
\widetilde{\mathcal{K}}_{n}=\mathcal{K}_{n}-\left(\mathcal{K}_{n} u-\mathcal{K} u\right) \mathcal{I} \tag{6.95}
\end{equation*}
$$

where $u(x)=1$ for all $x \in D$ and

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|\left(\mathcal{K}-\widetilde{\mathcal{K}}_{n}\right) \widetilde{\mathcal{K}}_{n}\right\|=0 \tag{6.96}
\end{equation*}
$$

Proof. From (6.33) and (6.35) is

$$
\begin{gathered}
\widetilde{\mathcal{K}}_{n} y(x)-\mathcal{K}_{n} y(x)=\sum_{j=1}^{n} \omega_{j} k_{n}\left(x, x_{j}\right)\left[y\left(x_{j}\right)-y(x)\right]+ \\
+\int_{D} k(x, t) y(x) d t-\sum_{j=1}^{n} \omega_{j} k_{n}\left(x, x_{j}\right) y\left(x_{j}\right)=y(x)\left[\int_{D} k(x, t) d t-\sum_{j=1}^{n} \omega_{j} k_{n}\left(x, x_{j}\right)\right]= \\
=y(x)\left[\int_{D} k(x, t) u(t) d t-\sum_{j=1}^{n} \omega_{j} k_{n}\left(x, x_{j}\right) u\left(x_{j}\right)\right]=\left(\mathcal{K} u-\mathcal{K}_{n} u\right)(x) \cdot y(x)
\end{gathered}
$$

and (6.95) follows. For (6.96) let us define operator $\mathcal{J}_{n}$ as

$$
\begin{gather*}
\mathcal{J}_{n}=\left(\mathcal{K}_{n} u-\mathcal{K} u\right) \mathcal{I} .  \tag{6.97}\\
\left\|\mathcal{J}_{n}\right\|=\left\|\mathcal{K}_{n} u-\mathcal{K} u\right\| \rightarrow 0 \text { as } n \rightarrow \infty \tag{6.98}
\end{gather*}
$$

by lemma 6.11 , where we put $y \equiv u$. Now

$$
\begin{aligned}
& \left(\mathcal{K}-\widetilde{\mathcal{K}}_{n}\right) \widetilde{\mathcal{K}}_{n}=\left(\mathcal{K}-\mathcal{K}_{n}+\mathcal{J}_{n}\right)\left(\mathcal{K}_{n}-\mathcal{J}_{n}\right)= \\
& =\left(\mathcal{K}-\mathcal{K}_{n}\right) \mathcal{K}_{n}-\left(\mathcal{K}-\mathcal{K}_{n}\right) \mathcal{J}_{n}+\mathcal{J}_{n} \mathcal{K}_{n}-\mathcal{J}_{n}^{2}
\end{aligned}
$$

Hence by (6.31) in proposition 6.2, (6.63) in corollary 6.7, (6.94) and (6.98) we have

$$
\begin{aligned}
& \left\|\left(\mathcal{K}-\widetilde{\mathcal{K}}_{n}\right) \widetilde{\mathcal{K}}_{n}\right\| \leq\left\|\left(\mathcal{K}-\mathcal{K}_{n}\right) \mathcal{K}_{n}\right\|+\left\|\mathcal{J}_{n}\right\|\left(\|\mathcal{K}\|+\left\|2 \mathcal{K}_{n}\right\|\right)+\left\|\mathcal{J}_{n}\right\|^{2} \leq \\
& \leq\left\|\left(\mathcal{K}-\mathcal{K}_{n}\right) \mathcal{K}_{n}\right\|+M C_{D}\left\|\mathcal{J}_{n}\right\|+2 M C\left\|\mathcal{J}_{n}\right\|+\left\|\mathcal{J}_{n}\right\|^{2} \rightarrow 0 \text { as } n \rightarrow \infty
\end{aligned}
$$

From above we can see that operator $\widetilde{\mathcal{K}}_{n}$ is not compact operator. We can use theorem 6.4 for $\mathcal{T}=\mathcal{K}$ and $\mathcal{S}=\widetilde{\mathcal{K}}_{n}$ but the existence the of solution of (6.36) is not guaranteed for all $f \in \mathcal{C}(D)$. However in the proof of following theorem we will see that compactness of $\widetilde{\mathcal{K}}_{n}$ is not needed. First from lemma 6.11 and (6.95) we get for all $y \in \mathcal{C}(D)$

$$
\begin{aligned}
&\left\|\mathcal{K} y-\widetilde{\mathcal{K}}_{n} y\right\|_{\infty}=\left\|\mathcal{K} y-\mathcal{K}_{n} y+\left(\mathcal{K}_{n} u-\mathcal{K} u\right) \cdot \mathcal{I} y\right\|_{\infty} \leq \\
& \leq\left\|\mathcal{K} y-\mathcal{K}_{n} y\right\|_{\infty}+\left\|\left(\mathcal{K}_{n} u-\mathcal{K} u\right)\right\|_{\infty}\|y\|_{\infty} \rightarrow 0 \text { as } n \rightarrow \infty
\end{aligned}
$$

where $\mathcal{I}$ is identical operator and $u(x)=1$. So we can write following corollary.
Corollary 6.14. Under assumptions of lemma 6.11 we have that for all $y \in \mathcal{C}(D)$

$$
\left\|\mathcal{K} y-\widetilde{\mathcal{K}}_{n} y\right\|_{\infty} \rightarrow 0 \text { as } n \rightarrow \infty
$$

All about the convergence of Nyström method 2 is written in the following theorem.

Theorem 6.15. Let $D \subset \mathbb{R}^{n}$ be closed and bounded set. Let $\mu_{n}$ be decreasing positive sequence that (6.15) holds. Let operator $\mathcal{K}$ be defined by (4.3) and operators $\widetilde{\mathcal{K}}_{n}$ by (6.35). Assume that the numerical integration rule is convergent for all continuous function and satisfies (6.24). Also assume that (6.26), (6.27), (6.28) and (6.29) hold and that there exists a constant $c<\infty$ such that (6.30) holds. Further assume that $\lambda \neq 0$ is not eigenvalue of operator $\mathcal{K}$. Then there exists $N_{2}$ such that for all $n \geq N_{2}$ the inverse $\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}\right)^{-1}$ exists and is bounded by

$$
\begin{equation*}
\left\|\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}\right)^{-1}\right\| \leq \frac{1+\left\|(\lambda \mathcal{I}-\mathcal{K})^{-1}\right\|\left\|\widetilde{\mathcal{K}}_{n}\right\|}{|\lambda|-\left\|(\lambda \mathcal{I}-\mathcal{K})^{-1}\right\|\left\|\left(\mathcal{K}-\widetilde{\mathcal{K}}_{n}\right) \widetilde{\mathcal{K}}_{n}\right\|} \leq c_{N_{2}}<\infty \tag{6.99}
\end{equation*}
$$

Let $y \in \mathcal{C}(D)$ be solution of $(\lambda \mathcal{I}-\mathcal{K}) y=f$ and let $\widetilde{y}_{n}$ be solution of $\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}\right) \widetilde{y}_{n}=f$. Then for all $n \geq N_{2}$

$$
\begin{equation*}
\left\|y-\widetilde{y}_{n}\right\|_{\infty} \leq\left\|\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}\right)^{-1}\right\|\left\|\mathcal{K} y-\widetilde{\mathcal{K}}_{n} y\right\|_{\infty} . \tag{6.100}
\end{equation*}
$$

Proof. Since the numerical integration rule converges for all continuous function the (6.23) holds. Let use theorem 6.4 for operators

$$
\mathcal{T}=\mathcal{K} \text { and } \mathcal{S}=\widetilde{\mathcal{K}}_{n} .
$$

We need to fulfill (6.37) in theorem 6.4. From (6.96) in lemma 6.13 there exists $\bar{N}_{2}$ such that

$$
\begin{equation*}
\left\|\left(\mathcal{K}-\widetilde{\mathcal{K}}_{n}\right) \widetilde{\mathcal{K}}_{n}\right\|<\frac{|\lambda|}{\left\|(\lambda \mathcal{I}-\mathcal{K})^{-1}\right\|} \text { for } n \geq n_{\bar{N}_{2}} \tag{6.101}
\end{equation*}
$$

and we have satisfied (6.37). By theorem 6.4 operator $\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}\right)^{-1}$ exists as bounded operator from $\mathcal{R}\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}\right)$ into $\mathcal{C}(D)$. Estimate (6.99) follows immediately from (6.38) by theorem 6.4. Now we will prove that $\mathcal{R}\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}\right)=\mathcal{C}(D)$. We will use following identity.

$$
\begin{equation*}
\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}=\left[\mathcal{I}-\mathcal{K}_{n}\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}+\mathcal{K}_{n}\right)^{-1}\right]\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}+\mathcal{K}_{n}\right) . \tag{6.102}
\end{equation*}
$$

From (6.95), (6.97) and (6.98) we have

$$
\left\|\widetilde{\mathcal{K}}_{n}-\mathcal{K}_{n}\right\|=\left\|\mathcal{J}_{n}\right\| \rightarrow 0 \text { as } n \rightarrow \infty
$$

and there exists $n_{0}$ such that for all $n \geq n_{0}$ is

$$
\left\|\widetilde{\mathcal{K}}_{n}-\mathcal{K}_{n}\right\|<|\lambda|
$$

and by proposition 2.18 exists $\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}+\mathcal{K}_{n}\right)^{-1}$ as as bijection from $\mathcal{C}(D)$ to $\mathcal{C}(D)$ for $n \geq n_{0}$. Now let us take $N_{2}=\max \left\{n_{0}, \bar{N}_{2}\right\}$. Since the left hand side of (6.102) is invertible then the right hand side is also invertible. From this operator [ $\left.\mathcal{I}-\mathcal{K}_{n}\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}+\mathcal{K}_{n}\right)^{-1}\right]$ is one-to-one, because otherwise the right hand side of (6.102) would not be invertible. Since $\mathcal{K}_{n}\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}+\mathcal{K}_{n}\right)^{-1}$ is compact operator it follows from Fredholm alternative theorem 2.19 that the operator $\left[\mathcal{I}-\mathcal{K}_{n}\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}+\mathcal{K}_{n}\right)^{-1}\right]^{-1}$ exists from $\mathcal{C}(D)$ into $\mathcal{C}(D)$. From this and (6.102) we get

$$
\begin{equation*}
\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}\right)^{-1}=\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}+\mathcal{K}_{n}\right)^{-1}\left[\mathcal{I}-\mathcal{K}_{n}\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}+\mathcal{K}_{n}\right)^{-1}\right]^{-1} . \tag{6.103}
\end{equation*}
$$

Both operators on the right hand side of (6.103) exist as operators from $\mathcal{C}(D)$ to $\mathcal{C}(D)$. Hence

$$
\mathcal{R}\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}\right)=\mathcal{R}\left[\mathcal{I}-\mathcal{K}_{n}\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}+\mathcal{K}_{n}\right)^{-1}\right]\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}+\mathcal{K}_{n}\right)=\mathcal{C}(D)
$$

and operator $\left(\lambda \mathcal{I}-\widetilde{\mathcal{K}}_{n}\right)^{-1}$ exists as operator from $\mathcal{C}(D)$ to $\mathcal{C}(D)$. By applying (6.39) with $y, z=\widetilde{y}_{n}, \mathcal{S}=\widetilde{\mathcal{K}}_{n}$ and $\mathcal{T}=\mathcal{K}$ (6.100) follows for $n \geq N_{2}$.

We have showed convergence for both methods. From theorems 6.12 resp. 6.15 we have that error of both methods depends on $\left\|\mathcal{K}_{n} y-\mathcal{K} y\right\|_{\infty}$ resp. $\left\|\widetilde{\mathcal{K}}_{n} y-\mathcal{K} y\right\|_{\infty}$. We can also see that the error depends on two factors. The first one is the numerical integration rule error and the second factor is kind of singularity of the kernel function $g$. In case of Nyström method 2 the singular factor is weakened by $y(t)-y(x)$ and can be simplified for a special class of functions form $\mathcal{C}(D)$ and special type of kernel singular factor $g$. Detailed error bounds depend on the kernel function and the numerical integrations rule. All will be given in the next chapters.

## 7. Special Numerical integration rules

To use of the Nyström method 1 and Nyström method 2 we need numerical integration rule that satisfies conditions (6.23), (6.24) and (6.30). In this chapter we will introduce such rules.

### 7.1 One dimensional case

In this section let $D=[a, b]$. We will take a special class of numerical integration rules. First assume that

$$
\begin{equation*}
a \leq x_{1}<x_{2}<\ldots<x_{n-1}<x_{n} \leq b . \tag{7.1}
\end{equation*}
$$

Assume that there exists $c_{1}$ such that for all $n \in \mathbb{N}$ it holds

$$
\begin{equation*}
\max \left\{\omega_{j}, \omega_{j-1}\right\} \leq c_{1}\left(x_{j}-x_{j-1}\right) \text { for } j=2, \ldots, n \tag{7.2}
\end{equation*}
$$

We need to show that numerical integration rule that satisfies (6.23), (6.24) and (7.2) also satisfies condition (6.30). All is done by following lemmas.

Lemma 7.1. Let $[a, b]$ be bounded interval and let $[\widetilde{a}, \widetilde{b}] \subset[a, b]$. Assume that $v$ is Riemann integrable function on $[a, b]$, which is positive and non-increasing on $[\widetilde{a}, \widetilde{b}]$ and equals to zero on $[a, b] \backslash[\widetilde{a}, \widetilde{b}]$. Assume that the numerical integration rule $Q_{n}$ converges for all continuous functions on $[a, b]$ and satisfies conditions (6.24), (7.1) and (7.2). Then

$$
\begin{equation*}
\sum_{j=1}^{n} \omega_{j} v\left(x_{j}\right) \leq \bar{\omega}_{n} v(\widetilde{a})+c_{1} \int_{\widetilde{a}}^{\tilde{b}} v(t) d t . \tag{7.3}
\end{equation*}
$$

Proof. Let $x_{j}$ be the node points of the numerical integration rule that satisfy (7.1). Let us define $i_{1}$ and $i_{2}$ as

$$
\begin{aligned}
i_{1} & =\min \left\{j, x_{j} \geq \widetilde{a}\right\} \\
i_{2} & =\max \left\{j, x_{j} \leq \widetilde{b}\right\}
\end{aligned}
$$

and numbers $a_{j}$

$$
a_{j}=\left\{\begin{array}{l}
x_{j} \text { if } j \neq i_{1} \\
\widetilde{a} \text { if } j=i_{1} .
\end{array}\right.
$$

From the definition of $x_{j}$ and $a_{j}$ it holds

$$
\begin{equation*}
x_{i_{1}} \geq a_{i_{1}}=\tilde{a} \tag{7.4}
\end{equation*}
$$

Since $v$ is non-increasing on $[\widetilde{a}, \widetilde{b}]$ we have

$$
\begin{equation*}
v\left(x_{i_{1}}\right) \leq v\left(a_{i_{1}}\right)=v(\widetilde{a}) . \tag{7.5}
\end{equation*}
$$

From (7.2), (7.4) and (7.5) we have

$$
\begin{gather*}
\sum_{j=1}^{n} \omega_{j} v\left(x_{j}\right)=\sum_{j, x_{j} \in[\widetilde{a}, \tilde{b}]} \omega_{j} v\left(x_{j}\right)=\sum_{j=i_{1}}^{i_{2}} \omega_{j} v\left(x_{j}\right)=\omega_{i_{1}} v\left(x_{i_{1}}\right)+\sum_{j=i_{1}+1}^{i_{2}} \omega_{j} v\left(x_{j}\right) \leq \\
\leq \bar{\omega}_{n} v(\widetilde{a})+c_{1} \sum_{j=i_{1}+1}^{i_{2}}\left(x_{j}-x_{j-1}\right) v\left(x_{j}\right)= \\
=\bar{\omega}_{n} v(\widetilde{a})+c_{1}\left(x_{i_{1}+1}-x_{i_{1}}\right) v\left(x_{i_{1}+1}\right)+c_{1} \sum_{j=i_{1}+2}^{i_{2}}\left(a_{j}-a_{j-1}\right) v\left(a_{j}\right) \leq \\
\leq \bar{\omega}_{n} v(\widetilde{a})+c_{1}\left(a_{i_{1}+1}-a_{i_{1}}\right) v\left(a_{i+1}\right)+c_{1} \sum_{j=i_{1}+2}^{i_{2}}\left(a_{j}-a_{j-1}\right) v\left(a_{j}\right)= \\
=\bar{\omega}_{n} v(\widetilde{a})+c_{1} \sum_{i=i_{1}+1}^{i_{2}}\left(a_{j}-a_{j-1}\right) v\left(a_{j}\right) . \tag{7.6}
\end{gather*}
$$

Points $a_{i_{1}}, \ldots, a_{i_{2}}$ define a partition for interval $P$ of interval $\left[\widetilde{a}, x_{i_{2}}\right]$

$$
P=\left\{\widetilde{a}=a_{i_{1}}<a_{i_{1}+1}<\ldots<a_{i_{2}}=x_{i_{2}}\right\} .
$$

Since $v$ is non-increasing the sum on the right hand side of (7.6) can be rewritten as

$$
\begin{gathered}
\sum_{j=i_{1}+1}^{i_{2}}\left(a_{j}-a_{j-1}\right) v\left(a_{j}\right)=\sum_{i=i_{1}+1}^{i_{2}}\left[a_{j}-a_{j-1}\right] \min _{x \in\left[a_{j-1}, a_{j}\right]} v(x)= \\
=\sum_{j=i_{1}+1}^{i_{2}}\left[a_{j}-a_{j-1}\right] \inf _{x \in\left[a_{j-1}, a_{j}\right]} v(x)=s(v, P)
\end{gathered}
$$

where $s(v, P)$ is the lower Riemann sum defined in A-4. Since $v(x)$ is Riemann integrable on $[a, b]$ it is also Riemann integrable on $\left[\widetilde{a}, x_{i_{2}}\right]$. From the definition of Riemann integral A-5 is

$$
\sum_{i=i_{1}+1}^{i_{2}}\left(a_{j}-a_{j-1}\right) v\left(a_{j}\right)=s(v, P) \leq \sup _{P} s(v, P)=\int_{\tilde{a}}^{x_{i_{2}}} v(t) d t \leq \int_{\tilde{a}}^{\tilde{b}} v(t) d t
$$

and the proof is complete.
Lemma 7.2. Let $[a, b]$ be bounded interval and let $[\widetilde{a}, \widetilde{b}] \subset[a, b]$. Assume that $v$ is Riemann integrable function on $[a, b]$, which is positive and non-decreasing on $[\widetilde{a}, \widetilde{b}]$ and equals to zero on $[a, b] \backslash[\widetilde{a}, \widetilde{b}]$. Assume that the numerical integration rule $Q_{n}$ converges for all continuous functions on $[a, b]$ and satisfies conditions (6.24), (7.1) and (7.2). Then

$$
\begin{equation*}
\sum_{j=1}^{n} \omega_{j} v\left(x_{j}\right) \leq \bar{\omega}_{n} v(\widetilde{b})+c_{1} \int_{\tilde{a}}^{\tilde{b}} v(t) d t . \tag{7.7}
\end{equation*}
$$

Proof. Let $x_{j}$ be node the points of the numerical integration rule that satisfy (7.1). Let us define $i_{1}$ and $i_{2}$ as

$$
\begin{aligned}
i_{1} & =\min \left\{j, x_{j} \geq \widetilde{a}\right\} \\
i_{2} & =\max \left\{j, x_{j} \leq \widetilde{b}\right\}
\end{aligned}
$$

and numbers $a_{j}$

$$
a_{j}=\left\{\begin{array}{l}
x_{j} \text { if } j \neq i_{2} \\
\widetilde{b} \text { if } j=i_{2} .
\end{array}\right.
$$

From the definition of $x_{j}$ and $a_{j}$ it holds

$$
\begin{equation*}
x_{i_{2}} \leq a_{i_{2}}=\widetilde{b} . \tag{7.8}
\end{equation*}
$$

Since $v$ is non-decreasing on $[\widetilde{a}, \widetilde{b}]$ we get

$$
\begin{equation*}
v\left(x_{i_{2}}\right) \leq v\left(a_{i_{2}}\right)=v(\widetilde{b}) . \tag{7.9}
\end{equation*}
$$

From (7.2) we get

$$
\omega_{j} \leq c_{1}\left(x_{j+1}-x_{j}\right), j=1, \ldots, n-1
$$

From here and (7.8) and (7.9) we have

$$
\begin{gather*}
\sum_{j=1}^{n} \omega_{j} v\left(x_{j}\right)=\sum_{j, x_{j} \in[\widetilde{a}, \widetilde{b}]} \omega_{j} v\left(x_{j}\right)=\sum_{j=i_{1}}^{i_{2}} \omega_{j} v\left(x_{j}\right)=\sum_{j=i_{1}}^{i_{2}-1} \omega_{j} v\left(x_{j}\right)+\omega_{i_{2}} v\left(x_{i_{2}}\right) \leq \\
\leq c_{1} \sum_{j=i_{1}}^{i_{2}-1}\left(x_{j+1}-x_{j}\right) v\left(x_{j}\right)+\bar{\omega}_{n} v(\widetilde{b})= \\
=c_{1} \sum_{j=i_{1}}^{i_{2}-2}\left(a_{j+1}-a_{j}\right) v\left(a_{j}\right)+c_{1}\left(x_{i_{2}}-x_{i_{2}-1}\right) v\left(x_{i_{2}-1}\right)+\bar{\omega}_{n} v(\widetilde{b}) \leq \\
\leq c_{1} \sum_{j=i_{1}}^{i_{2}-2}\left(a_{j+1}-a_{j}\right) v\left(a_{j}\right)+c_{1}\left(a_{i_{2}}-a_{i_{2}-1}\right) v\left(a_{i_{2}-1}\right)+\bar{\omega}_{n} v(\widetilde{b})= \\
=c_{1} \sum_{i=i_{1}}^{i_{2}-1}\left(a_{j+1}-a_{j}\right) v\left(a_{j}\right)+\bar{\omega}_{n} v(\widetilde{b}) . \tag{7.10}
\end{gather*}
$$

Points $a_{i_{1}}, \ldots, a_{i_{2}}$ define a partition for interval $P$ of interval $\left[x_{i_{1}}, \widetilde{b}\right]$

$$
P=\left\{x_{i_{1}}=a_{i_{1}}<a_{i_{1}+1}<\ldots<a_{i_{2}}=\widetilde{b}\right\} .
$$

Since $v$ is non-decreasing the sum on the right hand side of (7.10) can be rewritten as

$$
\begin{gathered}
\sum_{j=i_{1}}^{i_{2}-1}\left(a_{j+1}-a_{j}\right) v\left(a_{j}\right)=\sum_{i=i_{1}}^{i_{2}-1}\left[a_{j+1}-a_{j}\right] \min _{x \in\left[a_{j}, a_{j+1}\right]} v(x)= \\
=\sum_{j=i_{1}}^{i_{2}-1}\left[a_{j+1}-a_{j}\right] \inf _{x \in\left[a_{j}, a_{j+1}\right]} v(x)=s(v, P)
\end{gathered}
$$

where $s(v, P)$ is the lower Riemann sum defined in A-4. Since $v(x)$ is Riemann integrable on $[a, b]$ it is also Riemann integrable on $\left[x_{i_{1}}, \widetilde{b}\right]$. From the definition of Riemann integral A-5 is

$$
\sum_{i=i_{1}}^{i_{2}-1}\left(a_{j+1}-a_{j}\right) v\left(a_{j}\right)=s(v, P) \leq \sup _{P} s(v, P)=\int_{x_{i_{1}}}^{\tilde{b}} v(t) d t \leq \int_{\tilde{a}}^{\tilde{b}} v(t) d t
$$

and the proof is complete.
Lemma 7.3. Let $[a, b]$ be bounded interval. Assume that $z$ is non-increasing, Riemann integrable function on $[0, \infty)$ that satisfy $z(x) \geq 0$ for every $x$. Assume that the numerical integration rule $Q_{n}$ converges for all continuous functions and satisfies conditions (6.24) and (7.2). Then for all $x \in[a, b]$ it holds

$$
\begin{equation*}
\sum_{j,\left|x-x_{j}\right|<\xi} \omega_{j} z\left(\left|x-x_{j}\right|\right) \leq 2 \bar{\omega}_{n} z(0)+c_{1} \int_{\{t,|x-t|<\xi\}} z(|x-t|) d t \tag{7.11}
\end{equation*}
$$

for all non-increasing function $z \in C[0, \infty)$ and (6.30) is satisfied with $c=\max \left\{c_{1}, 2\right\}$.

Proof. Let us take $x \in[a, b]$ and $\xi>0$. Let us define

$$
a_{\xi}=\max \{a, x-\xi\}
$$

and

$$
b_{\xi}=\min \{b, x+\xi\} .
$$

Then

$$
\begin{equation*}
x-\xi \leq a_{\xi} \tag{7.12}
\end{equation*}
$$

and

$$
\begin{equation*}
b_{\xi} \leq x+\xi \tag{7.13}
\end{equation*}
$$

Let us define function $v_{1}$ as

$$
v_{1}(t)=\left\{\begin{array}{l}
0 \text { if } t \notin\left[a_{\xi}, x\right]  \tag{7.14}\\
z(x-t) \text { if } t \in\left[a_{\xi}, x\right] .
\end{array}\right.
$$

Then $v_{1}$ is non-decreasing on $\left[a_{\xi}, x\right]$. From here and by (7.7) in lemma 7.2, (7.12) and (7.14) we get

$$
\begin{gather*}
\sum_{j, x_{j} \in[x-\xi, x]} \omega_{j} z\left(x-x_{j}\right)=\sum_{j, x_{j} \in\left[a_{\xi}, x\right]} \omega_{j} v_{1}\left(x_{j}\right)= \\
=\sum_{j=1}^{n} \omega_{j} v_{1}\left(x_{j}\right) \leq \bar{\omega}_{n} v_{1}(x)+c_{1} \int_{a_{\xi}}^{x} v_{1}(t) d t=\bar{\omega}_{n} z(0)+c_{1} \int_{a_{\xi}}^{x} z(x-t) d t \leq \\
\leq \bar{\omega}_{n} z(0)+c_{1} \int_{x-\xi}^{x} z(x-t) d t . \tag{7.15}
\end{gather*}
$$

Let us define function $v_{2}$ as

$$
v_{2}(t)=\left\{\begin{array}{l}
z(t-x) \text { if } t \in\left[x, b_{\xi}\right]  \tag{7.16}\\
0 \text { if } t \notin\left[x, b_{\xi}\right] .
\end{array}\right.
$$

Then $v_{2}$ is non-increasing on $\left[x, b_{\xi}\right]$. From here and by (7.3) in lemma 7.1, (7.13) and (7.16) we get

$$
\begin{gather*}
\sum_{j, x_{j} \in[x, x+\xi]} \omega_{j} z\left(x_{j}-x\right)=\sum_{j, x_{j} \in\left[x, b_{\xi}\right]} \omega_{j} v_{2}\left(x_{j}\right)= \\
=\sum_{j=1}^{n} \omega_{j} v_{2}\left(x_{j}\right) \leq \bar{\omega}_{n} v_{2}(x)+c_{1} \int_{x}^{b_{\xi}} v_{2}(t) d t=\bar{\omega}_{n} z(0)+c_{1} \int_{x}^{b_{\xi}} z(t-x) d t \leq \\
=\bar{\omega}_{n} z(0)+c_{1} \int_{x}^{x+\xi} z(t-x) d t . \tag{7.17}
\end{gather*}
$$

From (7.15) and (7.17) we get

$$
\begin{gathered}
\sum_{j,\left|x-x_{j}\right|<\xi} \omega_{j} z\left(\left|x-x_{j}\right|\right) \leq \sum_{j,\left|x-x_{j}\right| \leq \xi} \omega_{j} z\left(\left|x-x_{j}\right|\right)= \\
=\sum_{j, x_{j} \in[x, x-\xi]} \omega_{j} z(x-t)+\sum_{j, x_{j} \in[x, x-\xi]} \omega_{j} z(t-x) \leq \\
\leq 2 \bar{\omega}_{n} z(0)+c_{1} \int_{x}^{x+\xi} z(t-x) d t+c_{1} \int_{x-\xi}^{x} z(x-t) d t= \\
=2 \bar{\omega}_{n} z(0)+c_{1} \int_{x-\xi}^{x+\xi} z(|x-t|) d t
\end{gathered}
$$

and (7.11) follows for all $x \in[a, b]$. Hence (6.30) follows for $c=\max \left\{c_{1}, 2\right\}$
Now let us introduce some numerical integration rules.
Definition 7.1 (Midpoint integration rule). Let $f$ be Riemann integrable function on $[a, b]$. The midpoint integration rule is defined by

$$
\begin{equation*}
\int_{a}^{b} f(x) d x \approx(b-a) f\left(\frac{a+b}{2}\right) \tag{7.18}
\end{equation*}
$$

The compound midpoint integration rule $Q_{M, n}$ is defined by

$$
\begin{equation*}
\int_{a}^{b} f(x) d x \approx Q_{M, n} f=\sum_{j=1}^{n} \frac{b-a}{n} f\left(x_{j}\right) \tag{7.19}
\end{equation*}
$$

where

$$
\begin{equation*}
x_{j}=a+j \frac{b-a}{n}-\frac{b-a}{2 n}, j=1, \ldots, n . \tag{7.20}
\end{equation*}
$$

Definition 7.2 (Simpson integration rule). Let $f$ be Riemann integrable function on $[a, b]$. The Simpson integration rule is defined by

$$
\begin{equation*}
\int_{a}^{b} f(x) d x \approx \frac{b-a}{6}\left(f(a)+4 f\left(\frac{a+b}{2}\right)+f(b)\right) . \tag{7.21}
\end{equation*}
$$

The compound Simpson integration rule $Q_{S, n}$ ( $n$ odd) is defined by

$$
\int_{a}^{b} f(x) d x \approx Q_{S, n} f=
$$

$$
\begin{equation*}
=\frac{b-a}{3(n-1)}\left[f\left(x_{1}\right)+4 \sum_{j=1}^{(n-1) / 2} f\left(x_{2 j}\right)+2 \sum_{j=1}^{(n-3) / 2} f\left(x_{2 j+1}\right)+f\left(x_{n}\right)\right] \tag{7.22}
\end{equation*}
$$

where

$$
\begin{equation*}
x_{j}=a+(j-1) \frac{b-a}{n-1}, j=1, \ldots, n . \tag{7.23}
\end{equation*}
$$

Corollary 7.4. Let

$$
h=\frac{b-a}{n-1} .
$$

Then

$$
Q_{S, n}=\sum_{j=1}^{(n-1) / 2} \frac{2 h}{6}\left[f\left(x_{2 j-1}\right)+4 f\left(x_{2 j}\right)+f\left(x_{2 j+1}\right)\right]
$$

Proof. Follows immediately from (7.21), (7.22) and (7.23).
Lemma 7.5. Compound midpoint rule satisfies condition (7.2) with constant $c_{1}=(b-a)$.

Proof. From (7.19) and (7.20) for every $j=1, \ldots, n-1$ it holds for all $j=2, \ldots, n$

$$
\omega_{j}=\omega_{j-1}=\frac{b-a}{n}=x_{j}-x_{j-1} .
$$

Hence (7.2) is satisfied with $c_{1}=1$.
Lemma 7.6. Compound Simpson rule satisfies condition (7.2) with constant $c_{1}=$ 4/3.

Proof. From (7.22) and (7.23) for every $j=2, \ldots, n$ is

$$
\omega_{j}, \omega_{j-1} \leq \max \left\{\omega_{j}, j=1, \ldots, n\right\}=\frac{4}{3} \frac{b-a}{n-1}=\frac{4}{3}\left[x_{j}-x_{j-1}\right] .
$$

Hence (7.2) is satisfied with $c_{1}=4 / 3$.
Numerical integration rules errors are described in many various sources. Let us formulate estimate for compound midpoint and compound Simpson rule. Proofs of following two theorem can be found for example in [10].
Theorem 7.7. Assume that $f \in \mathcal{C}^{2}(a, b)$. Then for the midpoint rule integration error it holds

$$
\begin{equation*}
\left|\int_{a}^{b} f(x) d x-(b-a) f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^{3}}{24} \max _{\xi \in[a, b]}\left|f^{\prime \prime}(\xi)\right| . \tag{7.24}
\end{equation*}
$$

Theorem 7.8. Assume that $f \in \mathcal{C}^{4}(a, b)$. Then for the Simpson rule integration error it holds

$$
\begin{equation*}
\left|\int_{a}^{b} f(x) d x-\frac{b-a}{6}\left(f(a)+4 f\left(\frac{a+b}{2}\right)+f(b)\right)\right| \leq \frac{(b-a)^{5}}{2880} \max _{\xi \in[a, b]}\left|f^{(4)}(\xi)\right| . \tag{7.25}
\end{equation*}
$$

Last two theorems give error estimates. Error depends on the length of integration interval. In case that the function is only continuous, the best possible error bound depends on the modulus of continuity [10].

### 7.2 Integration rule in higher dimensions

In previous section we proved that the numerical integration rule in one dimensional case satisfied (6.30) under assumption (7.2). But in case of multidimensional problem we can not find similar condition. In the following two theorems we will see that for a special numerical integration rules the condition (6.30) can be satisfied.

Definition 7.3. Let the $D=\left\{\left(x_{1}, x_{2}\right), a \leq x_{1} \leq b, c \leq x_{2} \leq d\right\}$ be rectangle and let $v \in \mathcal{C}(D)$. The mid-rectangle integration rule is defined by

$$
\begin{equation*}
C_{m}(v)=(b-a)(d-c) v\left(\frac{a+b}{2}, \frac{c+d}{2}\right) . \tag{7.26}
\end{equation*}
$$

Let the rectangle $D$ be partitioned by $n_{1} n_{2}$ subrectangles ( $n_{1}$ in $x_{1}$ direction and $n_{2}$ in $x_{2}$ direction). Let

$$
\begin{equation*}
h_{1}=\frac{b-a}{n_{1}} \tag{7.27}
\end{equation*}
$$

and

$$
\begin{equation*}
h_{2}=\frac{d-c}{n_{2}} . \tag{7.28}
\end{equation*}
$$

The compound mid-rectangular integration rule is defined as

$$
\begin{equation*}
C_{m}^{n_{1} n_{2}}(v)=\sum_{k=1}^{n_{1}} \sum_{l=1}^{n_{2}} \omega v\left(x_{k}, x_{l}\right) \tag{7.29}
\end{equation*}
$$

where

$$
\begin{gather*}
\omega=h_{1} h_{2}, \\
x_{k}=a+k h_{1}-\frac{h_{1}}{2}, k=1, \ldots, n_{1} \tag{7.30}
\end{gather*}
$$

and

$$
\begin{equation*}
x_{l}=c+l h_{2}-\frac{h_{2}}{2}, l=1, \ldots, n_{2} . \tag{7.31}
\end{equation*}
$$

Theorem 7.9. Let $z \in \mathcal{C}[0, \infty)$ be non-increasing function. Let the numerical integration rule be the compound mid-rectangular rule and assume that there exists $1 \leq \alpha<\infty$ such that

$$
\begin{equation*}
h_{1}=\alpha h_{2} \tag{7.32}
\end{equation*}
$$

where $h_{1}$ and $h_{2}$ are defined by (7.27) resp. (7.28). Then there exists constant $c$ such that (6.30) is satisfied.

Proof. Let us define ball

$$
B_{x}=\left\{t \in \mathbb{R}^{2}, r(x, t)<\xi\right\}
$$

and subrectangles $D_{i j}$ as

$$
D_{i j}=\left\{\left(x_{1}^{\prime}, x_{2}^{\prime}\right), x_{i}-\frac{h_{1}}{2} \leq x_{1}^{\prime} \leq x_{i}+\frac{h_{1}}{2}, x_{j}-\frac{h_{2}}{2} \leq x_{2}^{\prime} \leq x_{j}+\frac{h_{2}}{2}\right\} .
$$

Let us choose point $x=\left(\widehat{x}_{1}, \widehat{x}_{2}\right) \in D$. Then there exist $p, q$ such that $x \in D_{p q}$. If $p$ and $q$ are not unique we can take any of them. To simplify notation let us define function

$$
z_{x}(t)=z(r(x, t))
$$

and

$$
x_{k l}=\left(x_{k}, x_{l}\right), k=1, \ldots, n_{1}, l=1, \ldots, n_{2} .
$$

Now let us define points

$$
\bar{x}_{k l}=\left\{\begin{array}{l}
\left(x_{k}-\frac{h_{1}}{2}, x_{l}-\frac{h_{2}}{2}\right), k=p+1, \ldots, n_{1}, l=q+1, \ldots, n_{2} \\
\left(x_{k}+\frac{h_{1}}{2}, x_{l}-\frac{h_{2}}{2}\right), k=1, \ldots, p-1, l=q+1, \ldots, n_{2} \\
\left(x_{k}+\frac{h_{1}}{2}, x_{l}+\frac{h_{2}}{2}\right), k=1, \ldots p-1, l=1, \ldots, q-1 \\
\left(x_{k}-\frac{h_{1}}{2}, x_{l}+\frac{h_{2}}{2}\right), k=p+1, \ldots, n_{1}, l=1, \ldots, q-1 \\
\left(\widehat{x}_{1}, x_{l}-\frac{h_{2}}{2}\right), k=p, l=q+1, \ldots, n_{2} \\
\left(x_{k}+\frac{h_{1}}{2}, \widehat{x}_{2}\right), k=1, \ldots p-1, l=q \\
\left(\widehat{x}_{1}, x_{l}+\frac{h_{2}}{2}\right), k=p l=1, \ldots, q-1 \\
\left(x_{k}-\frac{h_{1}}{2}, \widehat{x}_{2}\right), k=p+1, \ldots, n_{1}, l=q .
\end{array}\right.
$$

The relation between points $x_{k l}$ and $\bar{x}_{k l}$ is shown on the figure 7.1. The blue


Figure 7.1: Point $x$, points $x_{k l}$, points $\bar{x}_{k l}$, and sums $\widetilde{C}_{i}$
points are $x_{k l}$ and the red points are $\bar{x}_{k l}$. From their definition it holds

$$
r\left(x, x_{k l}\right) \geq r\left(x, \bar{x}_{k l}\right) \text { for all } k=1, \ldots, n_{1}, l=1, \ldots, n_{2}
$$

and since $z$ is non-increasing we have

$$
\begin{equation*}
z\left(r\left(x, x_{k l}\right)\right) \leq z\left(r\left(x, \bar{x}_{k l}\right)\right) \text { for all } k=1, \ldots, n_{1}, l=1, \ldots, n_{2} . \tag{7.33}
\end{equation*}
$$

Now let us define points

$$
\underline{x}_{k l}=\left\{\begin{array}{l}
\left(x_{k}+\frac{h_{1}}{2}, x_{l}+\frac{h_{2}}{2}\right), k=p+1, \ldots, n_{1}, l=q+1, \ldots, n_{2} \\
\left(x_{k}-\frac{h_{1}}{2}, x_{l}+\frac{h_{2}}{2}\right), k=1, \ldots, p-1, l=q+1, \ldots, n_{2} \\
\left(x_{k}-\frac{h_{1}}{2}, x_{l}-\frac{h_{2}}{2}\right), k=1, \ldots p-1, l=1, \ldots, q-1 \\
\left(x_{k}+\frac{h_{1}}{2}, x_{l}-\frac{h_{2}}{2}\right), k=p+1, \ldots, n_{1}, l=1, \ldots, q-1 \\
\left(x_{k}-\frac{h_{1}}{2}, x_{l}+\frac{h_{2}}{2}\right), k=p, l=q+1, \ldots, n_{2}, \widehat{x}_{1}>x_{p} \\
\left(x_{k}+\frac{h_{1}}{2}, x_{l}+\frac{h_{2}}{2}\right), k=p, l=q+1, \ldots, n_{2}, \widehat{x}_{1} \leq x_{p} \\
\left(x_{k}-\frac{h_{1}}{2}, x_{l}-\frac{h_{2}}{2}\right), k=1, \ldots p-1, l=q, \widehat{x}_{2}>x_{q} \\
\left(x_{k}-\frac{h_{1}}{2}, x_{l}+\frac{h_{2}}{2}\right), k=1, \ldots p-1, l=q, \widehat{x}_{2} \leq x_{q} \\
\left(x_{k}-\frac{h_{1}}{2}, x_{l}-\frac{h_{2}}{2}\right), k=p l=1, \ldots, q-1, \widehat{x}_{1}>x_{p} \\
\left(x_{k}+\frac{h_{1}}{2}, x_{l}-\frac{h_{2}}{2}\right), k=p, l=1, \ldots, q-1, \widehat{x}_{1} \leq x_{p} \\
\left(x_{k}+\frac{h_{1}}{2}, x_{l}-\frac{h_{2}}{2}\right), k=p+1, \ldots, n_{1}, l=q, \widehat{x}_{2}>x_{q} \\
\left(x_{k}+\frac{h_{1}}{2}, x_{l}+\frac{h_{2}}{2}\right), k=p+1, \ldots, n_{1}, l=q, \widehat{x}_{2} \leq x_{q} .
\end{array}\right.
$$

The relation between points $x_{k l}$ and $\underline{x}_{k l}$ is shown on the figure 7.2. The blue


Figure 7.2: Point $x$, points $x_{k l}$, points $\underline{x}_{k l}$, and sums $\widetilde{C}_{i}$
points are $x_{k l}$ and the green points are $\underline{x}_{k l}$. From their definition it holds

$$
r\left(x, x_{k l}\right) \leq r\left(x, \underline{x}_{k l}\right) \text { for all } k=1, \ldots, n_{1}, l=1, \ldots, n_{2}
$$

and since $z$ is non-increasing it holds:

$$
\begin{equation*}
z\left(r\left(x, x_{k l}\right)\right) \geq z\left(r\left(x, \underline{x}_{k l}\right)\right) \text { for all } k=1, \ldots, n_{1}, l=1, \ldots, n_{2} \tag{7.34}
\end{equation*}
$$

From figure 7.2 we can se that

$$
\begin{equation*}
\left.r\left(x, \underline{x}_{k l}\right)\right)=\sup _{t \in D_{k l}} r(x, t) \text { for all } k=1, \ldots, n_{1}, l=1, \ldots, n_{2} \tag{7.35}
\end{equation*}
$$

and hence

$$
\begin{equation*}
z\left(r\left(x, \underline{x}_{k l}\right)=\inf _{t \in D_{k l}} z(r(x, t)) \text { for all } k=1, \ldots, n_{1}, l=1, \ldots, n_{2} .\right. \tag{7.36}
\end{equation*}
$$

Let us define

$$
\begin{aligned}
& \widetilde{C}_{1}\left(z_{x}\right)=\sum_{\substack{k=p+1, \ldots, n_{1} \\
l=q+1, \ldots, n_{2} \\
r\left(x, x_{k l}\right)<\xi}} \omega z_{x}\left(x_{k l}\right), \\
& \widetilde{C}_{2}\left(z_{x}\right)=\sum_{\substack{k=1, \ldots, p-1 \\
l=q+1, \ldots, n_{2} \\
r\left(x, x_{k l}\right)<\xi}} \omega z_{x}\left(x_{k l}\right), \\
& \widetilde{C}_{3}\left(z_{x}\right)=\sum_{\substack{k=1, \ldots, p-1 \\
l=1, \ldots, q-1 \\
r\left(x, x_{k l}\right)<\xi}} \omega z_{x}\left(x_{k l}\right), \\
& \widetilde{C}_{4}\left(z_{x}\right)=\sum_{\substack{k=p+1, \ldots, n_{1} \\
l=1, \ldots,-1 \\
r\left(x, x_{k l}\right)<\xi}} \omega z_{x}\left(x_{k l}\right), \\
& \widetilde{C}_{5}\left(z_{x}\right)=\sum_{\substack{l=q+1, \ldots, n_{2} \\
r\left(x, x_{p l}\right)<\xi}} \omega z_{x}\left(x_{p l}\right), \\
& \widetilde{C}_{6}\left(z_{x}\right)=\sum_{\substack{k=1, \ldots, p-1 \\
r\left(x, x_{k q}<\xi\right.}} \omega z_{x}\left(x_{k q}\right), \\
& \widetilde{C}_{7}\left(z_{x}\right)=\sum_{\substack{l=1, \ldots, q-1 \\
r\left(x, x_{p l}\right)<\xi}} \omega z_{x}\left(x_{p l}\right)
\end{aligned}
$$

and

$$
\widetilde{C}_{8}\left(z_{x}\right)=\sum_{\substack{k=p+1, \ldots, n_{1} \\ r\left(x, x_{k q}<\xi\right.}} \omega z_{x}\left(x_{k q}\right) .
$$

Then

$$
\begin{equation*}
\sum_{\substack{k=1, \ldots, n_{1} \\ l=1, \ldots, n_{2} \\ r\left(x, x_{k l}\right)<\xi}} \omega z_{x}\left(x_{k l}\right)=\sum_{i=1}^{8} \widetilde{C}_{i}\left(z_{x}\right)+\omega z\left(r\left(x, x_{p q}\right)\right) \leq \sum_{i=1}^{8} \widetilde{C}_{i}\left(z_{x}\right)+\omega z(0) . \tag{7.37}
\end{equation*}
$$

Let us define

$$
D_{1}=\bigcup_{\substack{k=p+1, \ldots, n_{1}-1 \\ l q+1, \ldots, n_{2}-1 \\ r\left(x, x_{k+1,1+l+1}\right)<\xi}} D_{k l} .
$$

Since

$$
\bar{x}_{k+1, l+1}=\underline{x}_{k, l} \text { for all } k=p+1, \ldots, n_{1}-1, l=q+1, \ldots, n_{2}-1
$$

we get from (7.36)

$$
\begin{gather*}
\widetilde{C}_{1}\left(z_{x}\right) \leq \sum_{\substack{k=p+1, \ldots, n_{1} \\
l=q+1, \ldots, n_{2} \\
r\left(x, x_{k l}\right)<\xi}} \omega z_{x}\left(\bar{x}_{k l}\right) \leq \widetilde{C}_{5}\left(z_{x}\right)+\widetilde{C}_{8}\left(z_{x}\right)+\sum_{\substack{k=p+2, \ldots, n_{1} \\
l q+q+\ldots, n_{2} \\
r\left(x, x_{k l}\right)<\xi}} \omega z_{x}\left(\bar{x}_{k l}\right)= \\
=\widetilde{C}_{5}\left(z_{x}\right)+\widetilde{C}_{8}\left(z_{x}\right)+\sum_{\begin{array}{l}
k=p+1, \ldots, n_{1}-1 \\
l=q+1, \ldots, n_{2}-1 \\
r\left(x, x_{k+1}, l+1\right)<\xi
\end{array}} \omega z_{x}\left(\bar{x}_{k+1, l+1}\right)= \\
=\widetilde{C}_{5}\left(z_{x}\right)+\widetilde{C}_{8}\left(z_{x}\right)+\sum_{\substack{k=p+1, \ldots, n_{1}-1 \\
l=q+1, \ldots, n_{2}-1 \\
r\left(x, x_{k+1, l+1}\right)<\xi}} \omega z_{x}\left(\underline{x}_{k, l}\right) \leq \\
\leq \widetilde{C}_{5}\left(z_{x}\right)+\widetilde{C}_{8}\left(z_{x}\right)+\int_{D_{1}} \omega z_{x}(t) d t . \tag{7.38}
\end{gather*}
$$

Let us define

$$
D_{2}=\bigcup_{\substack{k=2, \ldots, p-1 \\ l=q+1, \ldots, n_{2}-1 \\ r\left(x, x_{k-1, l+1}\right)<\xi}} D_{k l}
$$

Since

$$
\bar{x}_{k-1, l+1}=\underline{x}_{k, l} \text { for all } k=2, \ldots, p-1, l=q+1, \ldots, n_{2}-1
$$

we get from (7.36)

$$
\begin{gather*}
\widetilde{C}_{2}\left(z_{x}\right) \leq \sum_{\substack{k=1, \ldots, p-1 \\
l=q+1, \ldots, n_{2} \\
r\left(x, x_{k l}\right)<\xi}} \omega z_{x}\left(\bar{x}_{k l}\right) \leq \widetilde{C}_{5}\left(z_{x}\right)+\widetilde{C}_{6}\left(z_{x}\right)+\sum_{\substack{k=1, \ldots, p-2 \\
l=q+2, \ldots, n_{2} \\
r\left(x, x_{k l}\right)<\xi}} \omega z_{x}\left(\bar{x}_{k l}\right)= \\
=\widetilde{C}_{5}\left(z_{x}\right)+\widetilde{C}_{6}\left(z_{x}\right)+\sum_{\substack{k=2, \ldots, p-1 \\
l=q+1, \ldots, n_{2}-1 \\
r\left(x, x_{k-1}, l+1\right)<\xi}} \omega z_{x}\left(\bar{x}_{k-1, l+1}\right)= \\
=\widetilde{C}_{5}\left(z_{x}\right)+\widetilde{C}_{6}\left(z_{x}\right)+\sum_{\substack{k=2, \ldots, p-1 \\
l=q+1, \ldots, n_{2}-1 \\
r\left(x, x_{k-1, l+1}<\xi\right.}} \omega z_{x}\left(x_{k, l}\right) \leq \\
\leq \widetilde{C}_{5}\left(z_{x}\right)+\widetilde{C}_{6}\left(z_{x}\right)+\int_{D_{2}} \omega z_{x}(t) d t . \tag{7.39}
\end{gather*}
$$

Let us define

$$
D_{3}=\bigcup_{\substack{k=2, \ldots, p-1 \\ l=2, \ldots, q-1 \\ r\left(x, x_{k-1, l-1}\right)<\xi}} D_{k l} .
$$

Since

$$
\bar{x}_{k-1, l-1}=\underline{x}_{k, l} \text { for all } k=1, \ldots, p-1, l=2, \ldots, q-1
$$

we get from (7.36)

$$
\widetilde{C}_{3}\left(z_{x}\right) \leq \sum_{\substack{k=1, \ldots, p-1 \\ l=1, \ldots, q-1 \\ r\left(x, x_{k l}\right)<\xi}} \omega z_{x}\left(\bar{x}_{k l}\right) \leq \widetilde{C}_{6}\left(z_{x}\right)+\widetilde{C}_{7}\left(z_{x}\right)+\sum_{\substack{k=1, \ldots, p-2 \\ l=1, \ldots, q-2 \\ r\left(x, x_{k l}\right)<\xi}} \omega z_{x}\left(\bar{x}_{k l}\right)=
$$

$$
\begin{gather*}
=\widetilde{C}_{6}\left(z_{x}\right)+\widetilde{C}_{7}\left(z_{x}\right)+\sum_{\substack{k=2, \ldots, p-1 \\
l=2, \ldots, q-1 \\
r\left(x, x_{k-1, l-1)<\xi}\right.}} \omega z_{x}\left(\bar{x}_{k-1, l-1}\right)= \\
=\widetilde{C}_{6}\left(z_{x}\right)+\widetilde{C}_{7}\left(z_{x}\right)+\sum_{\substack{k=2, \ldots, p-1 \\
l=2, \ldots, p-1 \\
r\left(x, x_{k-1, l-1}\right)<\xi}} \omega z_{x}\left(\underline{x}_{k, l}\right) \leq \\
\quad \leq \widetilde{C}_{6}\left(z_{x}\right)+\widetilde{C}_{7}\left(z_{x}\right)+\int_{D_{3}} \omega z_{x}(t) d t . \tag{7.40}
\end{gather*}
$$

Let us define

$$
D_{4}=\bigcup_{\substack{k=p+1, \ldots, n_{1}-1 \\ l=2, \ldots-1 \\ r\left(x, x_{k+1, l-1}\right)<\xi}} D_{k l} .
$$

Since

$$
\bar{x}_{k+1, l-1}=\underline{x}_{k, l} \text { for all } k=p+1, \ldots, n_{1}-1, l=2, \ldots, q-1
$$

we get from (7.36)

$$
\begin{gather*}
\widetilde{C}_{4}\left(z_{x}\right) \leq \sum_{\substack{k=p+1, \ldots, n_{1} \\
l=1, \ldots, q-1 \\
r\left(x, x_{k l}\right)<\xi}} \omega z_{x}\left(\bar{x}_{k l}\right) \leq \widetilde{C}_{7}\left(z_{x}\right)+\widetilde{C}_{8}\left(z_{x}\right)+\sum_{\substack{k=p+2, \ldots, n_{1} \\
l=1, \ldots,-2 \\
r\left(x, x_{k l}\right)<\xi}} \omega z_{x}\left(\bar{x}_{k l}\right)= \\
=\widetilde{C}_{7}\left(z_{x}\right)+\widetilde{C}_{8}\left(z_{x}\right)+\sum_{\begin{array}{l}
k=p+1, \ldots, n_{1}-1 \\
l=2, \ldots,-1 \\
r\left(x, x_{k+1}, l-1\right)<\xi
\end{array}} \omega z_{x}\left(\bar{x}_{k+1, l-1}\right)= \\
=\widetilde{C}_{7}\left(z_{x}\right)+\widetilde{C}_{8}\left(z_{x}\right)+\sum_{\substack{k=p+1, \ldots, n_{1}-1 \\
l=2, \ldots, q-1 \\
r\left(x, x_{k+1, l-1}\right)<\xi}} \omega z_{x}\left(\underline{x}_{k, l}\right) \leq \\
\leq \widetilde{C}_{7}\left(z_{x}\right)+\widetilde{C}_{8}\left(z_{x}\right)+\int_{D_{4}} \omega z_{x}(t) d t . \tag{7.41}
\end{gather*}
$$

From (7.37), (7.38), (7.39), (7.40) and (7.41) we get

$$
\begin{equation*}
\sum_{\substack{k=1, \ldots, n_{1} \\ l=1, n_{2} \\ r\left(x, x_{k}\right)<\xi}} \omega z_{x}\left(x_{k l}\right) \leq \omega z(0)+\sum_{i=1}^{4} \int_{D_{i}} z_{x}(t) d t+3 \sum_{i=5}^{8} \widetilde{C}_{i}\left(z_{x}\right) . \tag{7.42}
\end{equation*}
$$

To bound $\widetilde{C}_{5}, \widetilde{C}_{6}, \widetilde{C}_{7}$ and $\widetilde{C}_{8}$ we need to derive several inequalities. Let us define

$$
\begin{equation*}
\gamma=\left[\frac{\alpha^{2}-1}{2}\right]+1 \tag{7.43}
\end{equation*}
$$

where [] denotes the whole part of number. Let $i=0,1, \ldots$ Then from (7.43)

$$
\frac{\alpha^{2}-1}{2} \leq \gamma \leq \gamma+i
$$

Hence

$$
\alpha^{2}-1 \leq 2(\gamma+i)
$$

$$
\begin{gathered}
\alpha^{2}-1+(\gamma+i)^{2} \leq(\gamma+i)^{2}+2(\gamma+i) \\
\alpha^{2}+(\gamma+i)^{2} \leq(\gamma+i)^{2}+2(\gamma+i)+1 \\
\alpha^{2}+(\gamma+i)^{2} \leq(\gamma+i+1)^{2} \\
\alpha^{2} h_{2}^{2}+(\gamma+i)^{2} h_{2}^{2} \leq(\gamma+i+1)^{2} h_{2}^{2} \\
\sqrt{h_{1}^{2}+(\gamma+i)^{2} h_{2}^{2}} \leq(\gamma+i+1) h_{2} \\
\sqrt{h_{1}^{2}+(\gamma+i)^{2} h_{2}^{2}}+r\left(x, \bar{x}_{p, q+1}\right) \leq(\gamma+i+1) h_{2}+r\left(x, \bar{x}_{p, q+1}\right) .
\end{gathered}
$$

Since $(\gamma+i+1) h_{2}=r\left(\bar{x}_{p, q+1}, \bar{x}_{p, q+2+\gamma+i}\right)$ we get
$\sqrt{h_{1}^{2}+(\gamma+i)^{2} h_{2}^{2}}+r\left(x, \bar{x}_{p, q+1}\right) \leq r\left(\bar{x}_{p, q+1}, \bar{x}_{p, q+2+\gamma+i}\right)+r\left(x, \bar{x}_{p, q+1}\right)=r\left(x, \bar{x}_{p, q+2+\gamma+i}\right)$.
Since

$$
r\left(\bar{x}_{p, q+1}, \underline{x}_{p, q+\gamma+i}\right) \leq \sqrt{h_{1}^{2}+(\gamma+i)^{2} h_{2}^{2}}
$$

we get by triangular inequality

$$
\begin{equation*}
r\left(x, \underline{x}_{p, q+\gamma+i}\right) \leq r\left(x, \bar{x}_{p, q+2+\gamma+i}\right), i=0,1, \ldots . \tag{7.44}
\end{equation*}
$$

By similar way we get

$$
\begin{equation*}
r\left(x, \underline{x}_{p, q-\gamma-i}\right) \leq r\left(x, \bar{x}_{p, q-2-\gamma-i}\right), i=0,1, \ldots . \tag{7.45}
\end{equation*}
$$

From (7.32) is $h_{2} \leq h_{1}$. Now let us prove

$$
\begin{equation*}
h_{2}^{2}+(i+1)^{2} h_{1}^{2} \leq(i+2)^{2} h_{1}^{2}, i=0,1, \ldots . \tag{7.46}
\end{equation*}
$$

Since

$$
(i+2)^{2} h_{1}^{2}-(i+1)^{2} h_{1}^{2}-h_{2}^{2}=2(i+1) h_{1}^{2}+\left(h_{1}-h_{2}\right)\left(h_{1}+h_{2}\right) \geq 0
$$

is (7.46) proved. Hence

$$
\sqrt{h_{2}^{2}+(i+1)^{2} h_{1}^{2}}+r\left(x, \bar{x}_{p+1, q}\right) \leq(i+2) h_{1}+r\left(x, \bar{x}_{p+1, q}\right)=r\left(x, \bar{x}_{p+3+i, q}\right) .
$$

Since

$$
r\left(\bar{x}_{p+1, q}, \underline{x}_{p+i+1, q}\right) \leq \sqrt{\xi_{2}+(i+1)^{2} h_{1}^{2}}
$$

we get by triangular inequality

$$
\begin{equation*}
r\left(x, \underline{x}_{p+i+1, q}\right) \leq r\left(x, \bar{x}_{p+3+i, q}\right), i=0,1, \ldots . \tag{7.47}
\end{equation*}
$$

By similar way we get

$$
\begin{equation*}
r\left(x, \underline{x}_{p-i-1, q}\right) \leq r\left(x, \bar{x}_{p-3-i, q}\right), i=0,1, \ldots \tag{7.48}
\end{equation*}
$$

Now we can define numbers

$$
\bar{n}_{1}=\max _{k=p+1, \ldots, n_{1}} r\left(x, x_{k q}\right)<\xi,
$$

$$
\begin{aligned}
& \underline{n}_{1}=\min _{k=1, \ldots, p-1} r\left(x, x_{k q}\right)<\xi, \\
& \bar{n}_{2}=\max _{l=q+1, \ldots, n_{2}} r\left(x, x_{p l}\right)<\xi,
\end{aligned}
$$

and

$$
\underline{n}_{2}=\min _{l=1, \ldots, q-1} r\left(x, x_{p l}\right)<\xi .
$$

Let us define

$$
D_{5}=\bigcup_{l=q+1}^{\bar{n}_{2}-2} D_{p l}
$$

From (7.33), (7.36) and (7.44) we get

$$
\begin{gather*}
\widetilde{C}_{5}\left(z_{x}\right) \leq \sum_{l=q+1}^{\bar{n}_{2}} \omega z_{x}\left(\bar{x}_{p l}\right) \leq \\
\leq(\gamma+1) \omega z(0)+\sum_{l=q+2+\gamma}^{\bar{n}_{2}} \omega z_{x}\left(\bar{x}_{p l}\right) \leq(\gamma+1) \omega z(0)+\sum_{l=q+\gamma}^{\bar{n}_{2}-2} \omega z_{x}\left(\underline{x}_{p l}\right) \leq \\
\leq(\gamma+1) \omega z(0)+\int_{\bigcup_{l=q+\gamma}^{\bar{x}_{2}-2} D_{p l}} z_{x}(t) d t \leq(\gamma+1) \omega z(0)+\int_{D_{5}} z_{x}(t) d t . \tag{7.49}
\end{gather*}
$$

Let us define

$$
D_{7}=\bigcup_{l=\underline{n}_{2}+2}^{q-1} D_{p l}
$$

From (7.33), (7.36) and (7.45) we get

$$
\begin{gather*}
\widetilde{C}_{7}\left(z_{x}\right) \leq \sum_{l=\underline{n}_{2}}^{q-1} \omega z_{x}\left(\bar{x}_{p l}\right) \leq \\
\leq(\gamma+1) \omega z(0)+\sum_{l=\underline{n}_{2}}^{q-2-\gamma} \omega z_{x}\left(\bar{x}_{p l}\right) \leq(\gamma+1) \omega z(0)+\sum_{l=\underline{n}_{2}+2}^{q-\gamma} \omega z_{x}\left(\underline{x}_{p l}\right) \leq \\
\leq(\gamma+1) \omega z(0)+\int_{\substack{\bigcup_{l=n_{2}+2}^{q-\gamma} D_{p l}}} z_{x}(t) d t \leq(\gamma+1) \omega z(0)+\int_{D_{7}} z_{x}(t) d t . \tag{7.50}
\end{gather*}
$$

Let us define

$$
D_{8}=\bigcup_{k=p+1}^{\bar{n}_{1}-2} D_{k q} .
$$

From (7.33), (7.36) and (7.47) we get

$$
\begin{gathered}
\widetilde{C}_{8}\left(z_{x}\right) \leq \sum_{k=p+1}^{\bar{n}_{1}} \omega z_{x}\left(\bar{x}_{k q}\right) \leq \\
\leq 2 \omega z(0)+\sum_{k=p+3}^{\bar{n}_{1}} \omega z_{x}\left(\bar{x}_{k q}\right) \leq 2 \omega z(0)+\sum_{k=p+1}^{\bar{n}_{1}-2} \omega z_{x}\left(\underline{x}_{k q}\right) \leq
\end{gathered}
$$

$$
\begin{equation*}
\leq 2 \omega z(0)+\int_{D_{8}} z_{x}(t) d t \tag{7.51}
\end{equation*}
$$

Let us define

$$
D_{6}=\bigcup_{k=\underline{n}_{1}+2}^{p-1} D_{k q} .
$$

From (7.33), (7.36) and (7.48) we get

$$
\begin{gather*}
\widetilde{C}_{6}\left(z_{x}\right) \leq \sum_{k=\underline{n}_{1}}^{p-1} \omega z_{x}\left(\bar{x}_{k q}\right) \leq \\
\leq 2 \omega z(0)+\sum_{k=\underline{n}_{1}}^{p-3} \omega z_{x}\left(\bar{x}_{k q}\right) \leq 2 \omega z(0)+\sum_{k=\underline{n}_{1}+2}^{p-1} \omega z_{x}\left(\underline{x}_{k q}\right) \leq \\
\leq 2 \omega z(0)+\int_{D_{6}} z_{x}(t) d t \tag{7.52}
\end{gather*}
$$

From (7.42), (7.49), (7.50), (7.51) and (7.52)

$$
\sum_{\substack{k=1, \ldots, n_{1} \\ l=1, \ldots, n_{2} \\ r\left(x, x_{k l}<\xi\right.}} \omega z_{x}\left(x_{k l}\right) \leq \omega z(0)+\sum_{i=1}^{4} \int_{D_{i}} z_{x}(t) d t+6(3+\gamma) \omega z(0)+3 \sum_{i=5}^{8} \int_{D_{i}} z_{x}(t) d t \leq
$$

$$
\begin{equation*}
\leq \omega z(0)(6 \gamma+19)+3 \sum_{i=1}^{8} \int_{D_{i}} z_{x}(t) d t \tag{7.53}
\end{equation*}
$$

From the definition od $D_{i}$ it holds

$$
D_{i} \subset B_{x}, i=1, \ldots, 8
$$

and (6.30) holds with $c=6 \gamma+19$.
The assumption (7.32) is only technical. A similar theorem can be formulated in case of

$$
h_{2}=\alpha h_{1} .
$$

Definition 7.4. Let the $D=\left\{\left(x_{1}, x_{2}, x_{3}\right), a \leq x_{1} \leq b, c \leq x_{2} \leq d, e \leq x_{3} \leq f\right\}$ be cuboid and let $v \in \mathcal{C}(D)$. The mid-cuboid integration rule is defined by

$$
\begin{equation*}
C_{m}(v)=(b-a)(d-c)(f-e) v\left(\frac{a+b}{2}, \frac{c+d}{2}, \frac{e+f}{2}\right) . \tag{7.54}
\end{equation*}
$$

Let the cuboid $D$ be partitioned by $n_{1} n_{2} n_{3}$ subcuboides ( $n_{1}$ in $x_{1}$ direction, $n_{2}$ in $x_{2}$ direction and $n_{3}$ in $x_{3}$ direction). Let

$$
\begin{align*}
h_{1} & =\frac{b-a}{n_{1}},  \tag{7.55}\\
h_{2} & =\frac{d-c}{n_{2}} \tag{7.56}
\end{align*}
$$

and

$$
\begin{equation*}
h_{3}=\frac{f-e}{n_{3}} . \tag{7.57}
\end{equation*}
$$

The compound mid-cuboid integration rule is defined as

$$
\begin{equation*}
C_{m}^{n_{1} n_{2} n_{3}}(v)=\sum_{k=1}^{n_{1}} \sum_{l=1}^{n_{2}} \sum_{m=1}^{n_{3}} \omega v\left(x_{k}, x_{l}, x_{m}\right) \tag{7.58}
\end{equation*}
$$

where

$$
\begin{gather*}
\omega=h_{1} h_{2} h_{3}, \\
x_{k}=a+k h_{1}-\frac{h_{1}}{2}, k=1, \ldots, n_{1},  \tag{7.59}\\
x_{l}=c+l h_{2}-\frac{h_{2}}{2}, l=1, \ldots, n_{2} \tag{7.60}
\end{gather*}
$$

and

$$
\begin{equation*}
x_{m}=e+m h_{3}-\frac{h_{3}}{2}, m=1, \ldots, n_{3} . \tag{7.61}
\end{equation*}
$$

Theorem 7.10. Let $z \in \mathcal{C}[0, \infty)$ be non-increasing function. Let the numerical integration rule be the compound mid-cuboid rule and assume there exists $1 \leq \beta \leq \alpha<\infty$ such that

$$
\begin{align*}
& h_{1}=\alpha h_{3},  \tag{7.62}\\
& h_{2}=\beta h_{3}, \tag{7.63}
\end{align*}
$$

where $h_{1}, h_{2}$ and $h_{3}$ are defined by (7.55) resp. (7.56), (7.57). Then there exists constant $c$ such that (6.30) is satisfied.

Proof. Let us also define a ball

$$
B_{x}=\left\{t \subset \mathbb{R}^{3}, r(x, t)<\xi\right\}
$$

and sub-cuboides $D_{i j}$ as

$$
\begin{gathered}
D_{k l m}=\left\{\left(x_{1}^{\prime}, x_{2}^{\prime}, x_{3}^{\prime}\right), x_{k}-\frac{h_{1}}{2} \leq x_{1}^{\prime} \leq x_{k}+\frac{h_{1}}{2}, x_{l}-\frac{h_{2}}{2} \leq x_{2}^{\prime} \leq x_{l}+\frac{h_{2}}{2},\right. \\
\left.x_{m}-\frac{h_{3}}{2} \leq x_{3}^{\prime} \leq x_{m}+\frac{h_{3}}{2}\right\} .
\end{gathered}
$$

Let us choose point $x=\left(\widehat{x}_{1}, \widehat{x}_{2}, \widehat{x}_{3}\right) \in D$. Then there exist $p, q, s$ such that $x \in D_{p q s}$. If the $p, q$ and $s$ are not unique we can take any of them. To simplify notation let us define function

$$
z_{x}(t)=z(r(x, t))
$$

and

$$
x_{k l m}=\left(x_{k}, x_{l}, x_{m}\right), k=1, \ldots, n_{1}, l=1, \ldots, n_{2}, m=1, \ldots, n_{3} .
$$

Now let us define points

The relation between points $x_{k l m}$ and $\bar{x}_{k l m}$ is similar as in the previous theorem. From their definition it holds

$$
r\left(x, x_{k l m}\right) \geq r\left(x, \bar{x}_{k l m}\right) \text { for all } k=1, \ldots, n_{1}, l=1, \ldots, n_{2}, m=1, \ldots, n_{3}
$$

and since $z$ is non-increasing we have

$$
\begin{equation*}
z\left(r\left(x, x_{k l m}\right)\right) \leq z\left(r\left(x, \bar{x}_{k l m}\right)\right) \text { for all } k=1, \ldots, n_{1}, l=1, \ldots, n_{2}, m=1, \ldots, s \tag{7.64}
\end{equation*}
$$

Now let us define points

The relation between points $x_{k l m}$ and $\underline{x}_{k l m}$ is the same as in previous theorem. From their definition it holds

$$
r\left(x, x_{k l m}\right) \leq r\left(x, \underline{x}_{k l m}\right) \text { for all } k=1, \ldots, n_{1}, l=1, \ldots, n_{2}, m=1, \ldots, n_{3}
$$

and since $z$ is non-increasing it holds:

$$
\begin{equation*}
z\left(r\left(x, x_{k l m}\right)\right) \geq z\left(r\left(x, \underline{x}_{k l m}\right)\right) \text { for all } k=1, \ldots, n_{1}, l=1, \ldots, n_{2}, m=1, \ldots, n_{3} \tag{7.65}
\end{equation*}
$$

As in the previous theorem we get

$$
\begin{equation*}
\left.r\left(x, \underline{x}_{k l m}\right)\right)=\sup _{t \in D_{k l m}} r(x, t) \text { for all } k=1, \ldots, n_{1}, l=1, \ldots, n_{2}, m=1, \ldots, n_{3} \tag{7.66}
\end{equation*}
$$

and hence

$$
\begin{equation*}
z\left(r\left(x, \underline{x}_{k l m}\right)\right)=\inf _{t \in D_{k l m}} z(r(x, t)) \text { for all } k=1, \ldots, n_{1}, l=1, \ldots, n_{2}, m=1, \ldots, n_{3} . \tag{7.67}
\end{equation*}
$$

As in previous theorem let us define

$$
\begin{aligned}
& \widetilde{C}_{1}\left(z_{x}\right)=\sum_{\substack{k=p+1, \ldots, n_{1} \\
=q+1, n_{2} \\
m=+1, \ldots, n_{3} \\
r\left(x, x_{k l m}\right)<\xi}} \omega z_{x}\left(x_{k l m}\right), \\
& \widetilde{C}_{2}\left(z_{x}\right)=\sum_{\substack{k=1, \ldots, p-1 \\
l=q+1, \ldots, n_{2} \\
m=s+\ldots, n_{3} \\
r\left(x, x_{k l m}\right)<\xi}} \omega z_{x}\left(x_{k l m}\right), \\
& \widetilde{C}_{3}\left(z_{x}\right)=\sum_{\substack{k=1, \ldots, p-1 \\
l=1, \ldots, q-1 \\
m=9+\ldots, n_{3} \\
r\left(x, x_{k l m}\right)<\xi}} \omega z_{x}\left(x_{k l m}\right), \\
& \widetilde{C}_{4}\left(z_{x}\right)=\sum_{\substack{k=p+1, \ldots, n_{1} \\
l=1, \ldots, q-1 \\
m=s+1, \ldots, n_{3} \\
r\left(x, x_{k l m}\right)<\xi}} \omega z_{x}\left(x_{k l m}\right), \\
& \widetilde{C}_{5}\left(z_{x}\right)=\sum_{\substack{k=p+1, \ldots, n_{1} \\
l=q+1, \ldots, n_{2} \\
m=1, \ldots, s^{-1} \\
r\left(x, x_{k l m}\right)<\xi}} \omega z_{x}\left(x_{k l m}\right), \\
& \widetilde{C}_{6}\left(z_{x}\right)=\sum_{\substack{k=1, \ldots, p-1 \\
l=q+\cdots, \ldots, n_{2} \\
m=1, \ldots, s-1 \\
r\left(x, x_{k l m}\right)<\xi}} \omega z_{x}\left(x_{k l m}\right), \\
& \widetilde{C}_{7}\left(z_{x}\right)=\sum_{\substack{k=1, \ldots, p-1 \\
l=1, \ldots-1 \\
m=1, \ldots, s-1 \\
r\left(x, x_{k l m}<\xi\right.}} \omega z_{x}\left(x_{k l m}\right),
\end{aligned}
$$

$$
\begin{aligned}
& \widetilde{C}_{8}\left(z_{x}\right)=\sum_{\begin{array}{c}
k=p+1, \ldots, n_{1} \\
l=1, \ldots, q_{1} \\
m=1, \ldots,-1 \\
r\left(x, x_{k l m}\right)<\xi
\end{array}} \omega z_{x}\left(x_{k l m}\right), \\
& \widetilde{C}_{9}\left(z_{x}\right)=\sum_{\substack{m=s+1, \ldots, n_{3} \\
r\left(x, x_{p q m}\right)<\xi}} \omega z_{x}\left(x_{p q m}\right), \\
& \widetilde{C}_{10}\left(z_{x}\right)=\sum_{\substack{m=1, \ldots, s-1 \\
r\left(x, x_{p q m}\right)<\xi}} \omega z_{x}\left(x_{p q m}\right), \\
& \widetilde{C}_{11}\left(z_{x}\right)=\sum_{\substack{l=q+1, \ldots, n_{2} \\
r\left(x, x_{p l s}\right)<\xi}} \omega z_{x}\left(x_{p l s}\right), \\
& \widetilde{C}_{12}\left(z_{x}\right)=\sum_{\substack{l=1, \ldots, q-1 \\
r\left(x, x_{p l s}\right)<\xi}} \omega z_{x}\left(x_{p l s}\right), \\
& \widetilde{C}_{13}\left(z_{x}\right)=\sum_{\substack{k=p+1, \ldots, n_{1} \\
r\left(x, x_{k q s}\right)<\xi}} \omega z_{x}\left(x_{k q s}\right), \\
& \widetilde{C}_{14}\left(z_{x}\right)=\sum_{\substack{k=1, \ldots, p-1 \\
r\left(x, k_{k q s}\right)<\xi}} \omega z_{x}\left(x_{k q s}\right), \\
& \widetilde{C}_{15}\left(z_{x}\right)=\sum_{\substack{k=p+1, \ldots, n_{1} \\
l=q+1, \ldots, n_{2} \\
r\left(x, x_{k l s}\right)<\xi}} \omega z_{x}\left(x_{k l s}\right), \\
& \widetilde{C}_{16}\left(z_{x}\right)=\sum_{\substack{k=1, \ldots, p-1 \\
l=q+\ldots, \ldots, n_{2} \\
r\left(x, x_{k l s}\right)<\xi}} \omega z_{x}\left(x_{k l s}\right), \\
& \widetilde{C}_{17}\left(z_{x}\right)=\sum_{\substack{k=1, \ldots, p-1 \\
l=1, \ldots, q-1 \\
r\left(x, x_{k l s}\right)<\xi}} \omega z_{x}\left(x_{k l s}\right), \\
& \widetilde{C}_{18}\left(z_{x}\right)=\sum_{\substack{\left.k=p+1, \ldots, n_{1} \\
l=1, \ldots q-1 \\
r\left(x, x_{k}\right) \\
r l_{s}\right)<\xi}} \omega z_{x}\left(x_{k l s}\right), \\
& \widetilde{C}_{19}\left(z_{x}\right)=\sum_{\substack{k=p+1, \ldots, n_{1} \\
m=s+1, n_{3} \\
r\left(x, x_{k q m}\right)<\xi}} \omega z_{x}\left(x_{k q m}\right), \\
& \widetilde{C}_{20}\left(z_{x}\right)=\sum_{\substack{k=1, \ldots, p-1 \\
m=s+1, \ldots, n_{3} \\
r\left(x, x_{k q m}\right)<\xi}} \omega z_{x}\left(x_{k q m}\right), \\
& \widetilde{C}_{21}\left(z_{x}\right)=\sum_{\substack{k=1, \ldots, p-1 \\
m=1, \ldots, s-1 \\
r\left(x, x_{k q m}\right)<\xi}} \omega z_{x}\left(x_{k q m}\right),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{C}_{22}\left(z_{x}\right) & =\sum_{\substack{k=p+1, \ldots, n_{1} \\
m=1, \ldots, s-1 \\
r\left(x, x_{k q m}\right)<\xi}} \omega z_{x}\left(x_{k q m}\right), \\
\widetilde{C}_{23}\left(z_{x}\right) & =\sum_{\substack{l=q+1, \ldots, n_{2} \\
m=s+1, \ldots, n_{3} \\
r\left(x, x_{p l m}\right)<\xi}} \omega z_{x}\left(x_{p l m}\right), \\
\widetilde{C}_{24}\left(z_{x}\right) & =\sum_{\substack{l=1, \ldots, q-1 \\
m=s+1, n_{3} \\
r\left(x, x_{p l m}\right)<\xi}} \omega z_{x}\left(x_{p l m}\right), \\
\widetilde{C}_{25}\left(z_{x}\right) & =\sum_{\substack{l=1, \ldots, c_{1}-1 \\
m=1, \ldots, s-1 \\
r\left(x, x_{p l m}\right)<\xi}} \omega z_{x}\left(x_{p l m}\right)
\end{aligned}
$$

and

$$
\widetilde{C}_{26}\left(z_{x}\right)=\sum_{\substack{l=q+1, \ldots, n_{2} \\ m=1, \ldots s-1 \\ r\left(x, x_{p l m}\right)<\xi}} \omega z_{x}\left(x_{p l m}\right) .
$$

Then

$$
\begin{equation*}
\sum_{\substack{k=1, \ldots, n_{1} \\ l=1, \ldots, n_{2} \\ m=1, \ldots, n_{3} \\ r\left(x, x_{k l m}\right)<\xi}} \omega z_{x}\left(x_{k l m}\right)=\sum_{i=1}^{26} \widetilde{C}_{i}\left(z_{x}\right)+\omega z\left(r\left(x, x_{p q s}\right)\right) \leq \sum_{i=1}^{26} \widetilde{C}_{i}\left(z_{x}\right)+\omega z(0) \tag{7.68}
\end{equation*}
$$

Let us define

$$
\begin{aligned}
& D_{1}=\quad U^{D_{k m},} \\
& k=p+1, \ldots, n_{1}-1 \\
& \begin{array}{c}
l=q+1, \ldots, n_{2}-1 \\
m=s+1, \ldots, n_{3}-1
\end{array} \\
& r\left(x, x_{k+1, l+1, m+1}\right)<\xi \\
& D_{2}=\bigcup_{\substack{k=2, \ldots, q-1 \\
m=q+1, \ldots, n_{2}-1 \\
m=s+1, \ldots, n_{3}-1 \\
r\left(x, x_{k-1, l+1, m+1}\right)<\xi}} D_{k l m}, \\
& D_{3}=\bigcup_{\substack{k=2, \ldots, p-1 \\
l=2, \ldots, q-1 \\
m=s+1, \ldots, n_{3}-1 \\
r\left(x, x_{k-1, l-1, m+1}\right)<\xi}} D_{k l m} \\
& D_{4}=\bigcup_{\substack{k=p+1, \ldots, n_{1}-1 \\
l=2, \ldots, q-1 \\
m=s+1, \ldots, n_{3}-1 \\
r\left(x, x_{k+1, l-1, m+1}\right)<\xi}} D_{k l m}, \\
& D_{5}=\quad D_{k l m} \\
& \begin{array}{c}
k=p+1, \ldots, n_{1}-1 \\
l=q+1, \ldots, n_{2}-1
\end{array} \\
& \begin{array}{c}
=q+1, \ldots, n_{2}-1 \\
m=2, \ldots, s-1
\end{array} \\
& r\left(x, x_{k+1, l+1, s-1}\right)<\xi
\end{aligned}
$$

$$
D_{5}=\bigcup_{\substack{k=2, \ldots, q-1 \\ l=q+1, \ldots, n_{2}-1 \\ m=2, \ldots, s-1 \\ r\left(x, x_{k-1, l+1, s-1}\right)<\xi}} D_{k l m},
$$

and

$$
D_{8}=\bigcup_{\substack{k=p+1, \ldots, n_{1}-1 \\ l=2, \ldots, q-1 \\ m=2, \ldots, s-1 \\ r\left(x, x_{k+1, l-1, s-1}\right)<\xi}} D_{k l m} .
$$

By similar way as proving (7.42) in previous theorem we get from (7.68)

$$
\sum_{\substack{k=1, \ldots, n_{1} \\ l=1, \ldots, n_{2} \\ m=1, \ldots, n_{3} \\ r\left(x, x_{k l}\right)<\xi}} \omega z_{x}\left(x_{k l m}\right) \leq \omega z(0)+\sum_{i=1}^{8} \sum_{\substack{k l m \\ x_{k l m} \in D_{i}}} \omega z_{x}\left(\underline{x}_{k l m}\right)+3 \sum_{i=15}^{26} \widetilde{C}_{i}\left(z_{x}\right)+\sum_{i=9}^{14} \widetilde{C}_{i}\left(z_{x}\right) \leq
$$

$$
\begin{equation*}
\leq \omega z(0)+\sum_{i=1}^{8} \int_{D_{i}} z_{x}(t) d t+3 \sum_{i=15}^{26} \widetilde{C}_{i}\left(z_{x}\right)+\sum_{i=9}^{14} \widetilde{C}_{i}\left(z_{x}\right) . \tag{7.69}
\end{equation*}
$$

To bound $\widetilde{C}_{9}-\widetilde{C}_{14}$ we need to derive several inequalites. Let us define

$$
\begin{align*}
& \gamma_{1}=\left[\frac{\alpha^{2}-1}{2}\right]+1  \tag{7.70}\\
& \gamma_{2}=\left[\frac{\beta^{2}-1}{2}\right]+1  \tag{7.71}\\
& \gamma_{3}=\left[\frac{\frac{\alpha^{2}}{\beta^{2}}-1}{2}\right]+1 \tag{7.72}
\end{align*}
$$

where [] denotes the whole part of number. Note that

$$
\gamma_{1}, \gamma_{2}, \gamma_{3} \geq 1
$$

Let $i=0,1, \ldots$ Then from (7.70) and (7.71)

$$
\frac{\alpha^{2}-1}{2}+\frac{\beta^{2}-1}{2} \leq \gamma_{1}+\gamma_{2} \leq \gamma_{1}+\gamma_{2}+i .
$$

Hence

$$
\begin{gathered}
\alpha^{2}-1+\beta^{2}-1 \leq 2\left(\gamma_{1}+\gamma_{2}+i\right) \\
\alpha^{2}+\beta^{2} \leq 2\left(\gamma_{1}+\gamma_{2}+i\right)+2 \leq 4\left(\gamma_{1}+\gamma_{2}+i\right)+4 \\
\alpha^{2}+\beta^{2}+\left(\gamma_{1}+\gamma_{2}+i\right)^{2} \leq\left(\gamma_{1}+\gamma_{2}+i\right)^{2}+4\left(\gamma_{1}+\gamma_{2}+i\right)+4 \\
\alpha^{2}+\beta^{2}+\left(\gamma_{1}+\gamma_{2}+i\right)^{2} \leq\left(\gamma_{1}+\gamma_{2}+i+2\right)^{2}
\end{gathered}
$$

$$
\alpha^{2} h_{3}^{2}+\beta^{2} h_{3}^{2}+\left(\gamma_{1}+\gamma_{2}+i\right)^{2} h_{3}^{2} \leq\left(\gamma_{1}+\gamma_{2}+i+2\right)^{2} h_{3}^{2} .
$$

From here we get

$$
\sqrt{h_{1}^{2}+h_{2}^{2}+\left(\gamma_{1}+\gamma_{2}+i\right)^{2} h_{3}^{2}} \leq\left(\gamma_{1}+\gamma_{2}+i+2\right) h_{3}=r\left(\bar{x}_{p, q, s+1}, \bar{x}_{p, q, s+\gamma_{1}+\gamma_{2}+i+3}\right)
$$

and

$$
\begin{gathered}
\sqrt{h_{1}^{2}+h_{2}^{2}+\left(\gamma_{1}+\gamma_{2}+i\right)^{2} h_{3}^{2}}+r\left(x, \bar{x}_{p, q, s+1}\right) \leq \\
\leq r\left(\bar{x}_{p, q, s+1}, \bar{x}_{p, q, s+\gamma_{1}+\gamma_{2}+i+3}\right)+r\left(x, \bar{x}_{p, q, s+1}\right)=r\left(x, \bar{x}_{p, q, s+\gamma_{1}+\gamma_{2}+i+3}\right) .
\end{gathered}
$$

Since

$$
r\left(\underline{x}_{p, q, s+\gamma_{1}+\gamma_{2}+i}, \bar{x}_{p, q, s+1}\right) \leq \sqrt{h_{1}^{2}+h_{2}^{2}+\left(\gamma_{1}+\gamma_{2}+i\right)^{2} h_{3}^{2}}
$$

we get by triangular inequality

$$
\begin{equation*}
r\left(\underline{x}_{p, q, s+\gamma_{1}+\gamma_{2}+i}, x\right) \leq r\left(x, \bar{x}_{p, q, s+\gamma_{1}+\gamma_{2}+i+3}\right), i=0,1, \ldots . \tag{7.73}
\end{equation*}
$$

By similar way we get

$$
\begin{equation*}
r\left(\underline{x}_{p, q, s-\gamma_{1}-\gamma_{2}-i}, x\right) \leq r\left(x, \bar{x}_{p, q, s-\gamma_{1}-\gamma_{2}-i-3}\right), i=0,1, \ldots . \tag{7.74}
\end{equation*}
$$

From (7.32) is $h_{3} \leq h_{2} \leq h_{1}$. Now let us prove

$$
\begin{equation*}
h_{3}^{2}+h_{2}^{2}+(i+1)^{2} h_{1}^{2} \leq(i+2)^{2} h_{1}^{2}, i=1,2, \ldots . \tag{7.75}
\end{equation*}
$$

Since

$$
\begin{aligned}
& (i+2)^{2} h_{1}^{2}-(i+1)^{2} h_{1}^{2}-h_{2}^{2}-h_{3}^{2}=2(i+1) h_{1}^{2}+h_{1}^{2}-h_{2}^{3}-h_{3}^{2} \geq \\
& \geq 2 h_{1}^{2}-h_{2}^{2}-h_{3}^{2}=\left(h_{1}-h_{2}\right)\left(h_{1}+h_{2}\right)+\left(h_{1}+h_{3}\right)\left(h_{1}-h_{3}\right) \geq 0
\end{aligned}
$$

and (7.75) is proved. From

$$
r\left(\bar{x}_{p+1, q s}, \underline{x}_{p+i+1, q s}\right) \leq \sqrt{h_{3}^{2}+h_{2}^{2}+(i+1)^{2} h_{1}^{2}} \leq(i+2) h_{1}, i=0,1, \ldots
$$

we get

$$
r\left(\bar{x}_{p+1, q s}, \underline{x}_{p+i+1, q s}\right)+r\left(x, \bar{x}_{p+1, q s}\right) \leq(i+2) h_{1}+r\left(x, \bar{x}_{p+1, q s}\right)=r\left(x, \bar{x}_{p+3+i, q s}\right)
$$

and by triangular inequality we get

$$
\begin{equation*}
r\left(x, \underline{x}_{p+i+1, q s}\right) \leq r\left(x, \bar{x}_{p+3+i, q s}\right), i=0,1, \ldots . \tag{7.76}
\end{equation*}
$$

By similar way we get

$$
\begin{equation*}
r\left(x, \underline{x}_{p-i-1, q s}\right) \leq r\left(x, \bar{x}_{p-3-i, q s}\right), i=0,1, \ldots . \tag{7.77}
\end{equation*}
$$

Now let us prove

$$
\begin{equation*}
\left(\gamma_{3}+i+1\right)^{2}+1 \leq\left(\gamma_{3}+i+2\right)^{2}, i=0,1, \ldots . \tag{7.78}
\end{equation*}
$$

Since

$$
\left(\gamma_{3}+i+2\right)^{2}-\left(\gamma_{3}+i+1\right)^{2}-1=\left(\gamma_{3}+i+1\right)^{2}+2\left(\gamma_{3}+i+1\right)+1-\left(\gamma_{3}+i+1\right)^{2}-1=
$$

$$
=2\left(\gamma_{3}+i+1\right)>0
$$

and (7.78) is proved. Then from

$$
h_{1}=\frac{\alpha}{\beta} h_{2}
$$

and (7.72) we get

$$
\begin{gathered}
\frac{\frac{\alpha^{2}}{\beta^{2}}-1}{2} \leq \gamma_{3}+i, i=0,1, \ldots \\
\frac{\alpha^{2}}{\beta^{2}}-1 \leq 2\left(\gamma_{3}+i\right), i=0,1, \ldots \\
\frac{\alpha^{2}}{\beta^{2}}+\left(\gamma_{3}+i\right)^{2} \leq\left(\gamma_{3}+i\right)^{2}+2\left(\gamma_{3}+i\right)+1, i=0,1, \ldots \\
\frac{\alpha^{2}}{\beta^{2}}+\left(\gamma_{3}+i\right)^{2} \leq\left(\gamma_{3}+i+1\right)^{2}, i=0,1, \ldots \\
\frac{\alpha^{2}}{\beta^{2}} h_{2}^{2}+\left(\gamma_{3}+i\right)^{2} h_{2}^{2} \leq\left(\gamma_{3}+i+1\right)^{2} h_{2}^{2}, i=0,1, \ldots \\
h_{1}^{2}+\left(\gamma_{3}+i\right)^{2} h_{2}^{2} \leq\left(\gamma_{3}+i+1\right)^{2} h_{2}^{2}, i=0,1, \ldots
\end{gathered}
$$

By (7.78)

$$
h_{1}^{2}+\left(\gamma_{3}+i\right)^{2} h_{2}^{2}+h_{3}^{2} \leq\left(\gamma_{3}+i+1\right)^{2} h_{2}^{2}+h_{3}^{2} \leq\left(\gamma_{3}+i+2\right)^{2} h_{2}, i=0,1, \ldots
$$

and hence by the same way as in the previous theorem

$$
\begin{gathered}
r\left(x, \bar{x}_{p, q+1, s}\right)+\sqrt{h_{1}^{2}+\left(\gamma_{3}+i\right)^{2} h_{2}^{2}+h_{3}^{2}} \leq \\
\leq\left(\gamma_{3}+i+2\right) h_{2}+r\left(x, \bar{x}_{p, q+1, s}\right)=r\left(\bar{x}_{p, q+1, s}, \bar{x}_{p, q+3+\gamma_{3}+i, s}\right)+r\left(x, \bar{x}_{p, q+1, s}\right)= \\
=r\left(x, \bar{x}_{p, q+3+\gamma_{3}+i, s}\right), i=0,1, \ldots
\end{gathered}
$$

Since $r\left(\bar{x}_{p, q+1, s}, \underline{x}_{p, q+\gamma_{3}+i, s}\right) \leq \sqrt{h_{1}^{2}+\left(\gamma_{3}+i\right)^{2} h_{2}^{2}+h_{3}^{2}}$ we get by triangular inequality

$$
\begin{equation*}
r\left(x, \underline{x}_{p, q+\gamma_{3}+i, s}\right) \leq r\left(x, \bar{x}_{p, q+3+\gamma_{3}+i, s}\right), i=0,1, \ldots \tag{7.79}
\end{equation*}
$$

and by similar way we get

$$
\begin{equation*}
r\left(x, \underline{x}_{p, q-\gamma_{3}-i, s}\right) \leq r\left(x, \bar{x}_{p, q-3-\gamma_{3}-i, s}\right), i=0,1, \ldots . \tag{7.80}
\end{equation*}
$$

Let us define numbers

$$
\begin{aligned}
& \bar{n}_{1}=\max _{k=p+1, \ldots, n_{1}} r\left(x, x_{k q s}\right)<\xi, \\
& \underline{n}_{1}=\min _{k=1, \ldots, p-1} r\left(x, x_{k q s}\right)<\xi, \\
& \bar{n}_{2}=\max _{l=q+1, \ldots, n_{2}} r\left(x, x_{p l s}\right)<\xi, \\
& \underline{n}_{2}=\min _{l=1, \ldots, q-1} r\left(x, x_{p l s}\right)<\xi,
\end{aligned}
$$

$$
\bar{n}_{3}=\max _{m=s+1, \ldots, n_{3}} r\left(x, x_{p q m}\right)<\xi,
$$

and

$$
\underline{n}_{3}=\min _{m=1, \ldots, s-1} r\left(x, x_{p q m}\right)<\xi .
$$

Let us define

$$
D_{9}=\bigcup_{m=s+1}^{\bar{n}_{3}-3} D_{p q m} .
$$

From (7.64), (7.67) and (7.73) we get

$$
\begin{gather*}
\widetilde{C}_{9}\left(z_{x}\right) \leq \sum_{m=s+1}^{\bar{n}_{3}} \omega z_{x}\left(\bar{x}_{p q m}\right) \leq \\
\leq\left(\gamma_{1}+\gamma_{2}+2\right) \omega z(0)+\sum_{m=s+3+\gamma_{1}+\gamma_{2}}^{\bar{n}_{3}} \omega z_{x}\left(\bar{x}_{p q m}\right) \leq \\
\leq\left(\gamma_{1}+\gamma_{2}+2\right) \omega z(0)+\sum_{m=s+\gamma_{1}+\gamma_{2}}^{\bar{n}_{3}-3} \omega z_{x}\left(\underline{x}_{p q m}\right) \leq \\
\leq\left(\gamma_{1}+\gamma_{2}+2\right) \omega z(0)+\int_{\bigcup_{m=s+\gamma_{1}+\gamma_{2}}^{\bar{n}_{3}-3} D_{p q m}} z_{x}(t) d t \leq \\
\leq\left(\gamma_{1}+\gamma_{2}+2\right) \omega z(0)+\int_{D_{9}} z_{x}(t) d t . \tag{7.81}
\end{gather*}
$$

Let us define

$$
D_{10}=\bigcup_{m=\underline{n}_{3}+3}^{s-1} D_{p q m} .
$$

From (7.64), (7.67) and (7.74) we get

$$
\begin{gather*}
\widetilde{C}_{10}\left(z_{x}\right) \leq \sum_{m=\underline{n}_{3}}^{s-1} \omega z_{x}\left(\bar{x}_{p q m}\right) \leq \\
\leq\left(\gamma_{1}+\gamma_{2}+2\right) \omega z(0)+\sum_{m=\underline{n}_{3}}^{s-3-\gamma_{1}-\gamma_{2}} \omega z_{x}\left(\bar{x}_{p q m}\right) \leq \\
\leq\left(\gamma_{1}+\gamma_{2}+2\right) \omega z(0)+\sum_{m=\underline{n}_{3}+3}^{s-\gamma_{1}-\gamma_{2}} \omega z_{x}\left(\underline{x}_{p q m}\right) \leq \\
\leq\left(\gamma_{1}+\gamma_{2}+2\right) \omega z(0)+\int_{\bigcup_{m=n_{3}+3}^{s-\gamma_{1}-\gamma_{2}} D_{p q m}} z_{x}(t) d t \leq \\
\leq\left(\gamma_{1}+\gamma_{2}+2\right) \omega z(0)+\int_{D_{10}} z_{x}(t) d t . \tag{7.82}
\end{gather*}
$$

Let us define

$$
D_{11}=\bigcup_{l=q+1}^{\bar{n}_{2}-3} D_{p l s}
$$

From (7.64), (7.67) and (7.79) we get

$$
\begin{gather*}
\widetilde{C}_{11}\left(z_{x}\right) \leq \sum_{l=q+1}^{\bar{n}_{2}} \omega z_{x}\left(\bar{x}_{p l s}\right) \leq \\
\leq\left(\gamma_{3}+2\right) \omega z(0)+\sum_{l=q+3+\gamma_{3}}^{\bar{n}_{2}} \omega z_{x}\left(\bar{x}_{p l s}\right) \leq \\
\leq\left(\gamma_{3}+2\right) \omega z(0)+\sum_{l=q+\gamma_{3}}^{\bar{n}_{2}-3} \omega z_{x}\left(\underline{x}_{p l s}\right) \leq \\
\leq\left(\gamma_{3}+2\right) \omega z(0)+\int_{\bigcup_{l=q+\gamma_{3}}^{\bar{n}_{2}-3} D_{p l s}} z_{x}(t) d t \leq \\
\leq\left(\gamma_{3}+2\right) \omega z(0)+\int_{D_{11}} z_{x}(t) d t \tag{7.83}
\end{gather*}
$$

Let us define

$$
D_{12}=\bigcup_{l=\underline{n}_{2}+3}^{q-1} D_{p l s}
$$

From (7.64), (7.67) and (7.80) we get

$$
\begin{gather*}
\widetilde{C}_{12}\left(z_{x}\right) \leq \sum_{l=\underline{n}_{2}}^{q-1} \omega z_{x}\left(\bar{x}_{p l s}\right) \leq \\
\leq\left(\gamma_{3}+2\right) \omega z(0)+\sum_{l=\underline{n}_{2}}^{q-3-\gamma_{3}} \omega z_{x}\left(\bar{x}_{p l s}\right) \leq \\
\leq\left(\gamma_{3}+2\right) \omega z(0)+\sum_{l=\underline{n}_{2}+3}^{q-\gamma_{3}} \omega z_{x}\left(\underline{x}_{p l s}\right) \leq \\
\leq\left(\gamma_{3}+2\right) \omega z(0)+\int_{\bigcup_{l=\underline{n}_{2}+3}^{q-\gamma_{3}} D_{p l s}} z_{x}(t) d t \leq \\
\leq\left(\gamma_{3}+2\right) \omega z(0)+\int_{D_{12}} z_{x}(t) d t . \tag{7.84}
\end{gather*}
$$

Let us define

$$
D_{13}=\bigcup_{k=p+1}^{\bar{n}_{1}-2} D_{k q s} .
$$

From (7.64), (7.67) and (7.76)

$$
\begin{gathered}
\widetilde{C}_{13}\left(z_{x}\right) \leq \sum_{k=p+1}^{\bar{n}_{1}} \omega z_{x}\left(\bar{x}_{k q s}\right) \leq \\
\leq 2 \omega z(0)+\sum_{k=p+3}^{\bar{n}_{1}} \omega z_{x}\left(\bar{x}_{k q s}\right) \leq 2 \omega z(0)+\sum_{k=p+1}^{\bar{n}_{1}-2} \omega z_{x}\left(\underline{x}_{k q s}\right) \leq
\end{gathered}
$$

$$
\begin{equation*}
\leq 2 \omega z(0)+\int_{D_{8}} z_{x}(t) d t \tag{7.85}
\end{equation*}
$$

Let us define

$$
D_{14}=\bigcup_{k=\underline{n}_{1}+2}^{p-1} D_{k q s} .
$$

From (7.64), (7.67) and (7.77)

$$
\begin{gather*}
\widetilde{C}_{14}\left(z_{x}\right) \leq \sum_{k=\underline{n}_{1}}^{p-1} \omega z_{x}\left(\bar{x}_{k q s}\right) \leq \\
\leq 2 \omega z(0)+\sum_{k=\underline{n}_{1}}^{p-2} \omega z_{x}\left(\bar{x}_{k q s}\right) \leq 2 \omega z(0)+\sum_{k=\underline{n}_{1}+2}^{p-1} \omega z_{x}\left(\underline{x}_{k q s}\right) \leq \\
\leq 2 \omega z(0)+\int_{D_{14}} z_{x}(t) d t \tag{7.86}
\end{gather*}
$$

From (7.81), (7.82), (7.83), (7.84), (7.85) and (7.86) we get

$$
\begin{equation*}
\sum_{i=9}^{14} \widetilde{C}_{i}\left(z_{x}\right) \leq \bar{\gamma}_{1} \omega z(0)+\sum_{i=9}^{14} \int_{D_{i}} z_{x}(t) d t \tag{7.87}
\end{equation*}
$$

where

$$
\begin{equation*}
\bar{\gamma}_{1}=12+2 \gamma_{1}+2 \gamma_{2}+2 \gamma_{3} . \tag{7.88}
\end{equation*}
$$

Now we need to estimate $\widetilde{C}_{15}, \ldots, \widetilde{C}_{26}$. Here the situation is different that in previous theorem. From (7.72) we get for $i=0,1, \ldots$

$$
\begin{gathered}
\frac{\frac{\alpha^{2}}{\beta^{2}}-1}{2} \leq \gamma_{3} \leq \gamma_{3}+i \\
\frac{\alpha^{2}}{\beta^{2}}-1 \leq 2\left(\gamma_{3}+i\right) \\
\frac{\alpha^{2}}{\beta^{2}}+\left(\gamma_{3}+i\right)^{2} \leq\left(\gamma_{3}+i\right)^{2}+2\left(\gamma_{3}+i\right)+1 \\
\frac{\alpha^{2}}{\beta^{2}}+\left(\gamma_{3}+i\right)^{2} \leq\left(\gamma_{3}+i+1\right)^{2} \\
h_{1}^{2}+\left(\gamma_{3}+i\right)^{2} h_{2}^{2} \leq\left(\gamma_{3}+i+1\right)^{2} h_{2}^{2} .
\end{gathered}
$$

Now let us define

$$
\xi_{2}=x_{q}+\frac{h_{2}}{2}-\widehat{x}_{2}
$$

and

$$
\xi_{3}=x_{s}+\frac{h_{3}}{2}-\widehat{x}_{3} .
$$

Then for $j=0,1, \ldots$ we get

$$
h_{1}^{2}+\left(\gamma_{3}+i\right)^{2} h_{2}^{2}+\left[(j+1) h_{3}+\xi_{3}\right]^{2} \leq\left(\gamma_{3}+i+1\right)^{2} h_{2}^{2}+\left[(j+1) h_{3}+\xi_{3}\right]^{2}
$$

$$
\begin{aligned}
& h_{1}^{2}+\left(\gamma_{3}+i\right)^{2} h_{2}^{2}+2\left(\gamma_{3}+i\right) h_{2} \xi_{2}+\xi_{2}^{2}+\left[(j+1) h_{3}+\xi_{3}\right]^{2} \leq \\
& \leq\left(\gamma_{3}+i+1\right)^{2} h_{2}^{2}+2\left(\gamma_{3}+i\right) h_{2} \xi_{2}+\xi_{2}^{2}+\left[(j+1) h_{3}+\xi_{3}\right]^{2} \leq \\
& \leq\left(\gamma_{3}+i+1\right)^{2} h_{2}^{2}+2\left(\gamma_{3}+i+1\right) h_{2} \xi_{2}+\xi_{2}^{2}+\left[(j+1) h_{3}+\xi_{3}\right]^{2}
\end{aligned}
$$

and

$$
\begin{gathered}
r\left(x, \underline{x}_{p, q+\gamma_{3}+i, s+j+1}\right)^{2} \leq h_{1}^{2}+\left[\left(\gamma_{3}+i\right) h_{2}+\xi_{2}\right]^{2}+\left[(j+1) h_{3}+\xi_{3}\right]^{2} \leq \\
\leq\left[\left(\gamma_{3}+i+1\right) h_{2}+\xi_{2}\right]^{2}+\left[(j+1) h_{3}+\xi_{3}\right]^{2} .
\end{gathered}
$$

Since $r\left(x, \bar{x}_{p, q+2+\gamma_{3}+i, s+j+2}\right)=\sqrt{\left[\left(\gamma_{3}+i+1\right) h_{2}+\xi_{2}\right]^{2}+\left[(j+1) h_{3}+\xi_{3}\right]^{2}}$ we get

$$
\begin{equation*}
r\left(x, \underline{x}_{p, q+\gamma_{3}+i, s+j+1}\right) \leq r\left(x, \bar{x}_{p, q+2+\gamma_{3}+i, s+j+2}\right), i, j=0,1, \ldots \tag{7.89}
\end{equation*}
$$

By similar way (with different definition of $\xi_{2}$ and $\xi_{3}$ ) we get

$$
\begin{gather*}
r\left(x, \underline{x}_{p, q-\gamma_{3}-i, s-j-1}\right) \leq r\left(x, \bar{x}_{p, q-2-\gamma_{3}-i, s-j-2}\right), i, j=0,1, \ldots,  \tag{7.90}\\
r\left(x, \underline{x}_{p, q+\gamma_{3}+i, s-j-1}\right) \leq r\left(x, \bar{x}_{p, q+2+\gamma_{3}+i, s-j-2}\right), i, j=0,1, \ldots \tag{7.91}
\end{gather*}
$$

and

$$
\begin{equation*}
r\left(x, \underline{x}_{p, q-\gamma_{3}-i, s+j+1}\right) \leq r\left(x, \bar{x}_{p, q-2-\gamma_{3}-i, s+j+2}\right), i, j=0,1, \ldots \tag{7.92}
\end{equation*}
$$

Now let us define

$$
\xi_{1}=x_{p}+\frac{h_{1}}{2}-\widehat{x}_{1} .
$$

Then from (7.71) we get for $j=0,1, \ldots$

$$
\begin{gathered}
\frac{\beta^{2}-1}{2} \leq \gamma_{2} \leq \gamma_{2}+j \\
\beta^{2}-1 \leq 2\left(\gamma_{2}+j\right) \\
\beta^{2}+\left(\gamma_{2}+j\right)^{2} \leq\left(\gamma_{2}+j\right)^{2}+2\left(\gamma_{2}+j\right)+1=\left(\gamma_{2}+j+1\right)^{2} \\
h_{2}^{2}+\left(\gamma_{2}+j\right)^{2} h_{3}^{3} \leq\left(\gamma_{2}+j+1\right)^{2} h_{3}^{2} .
\end{gathered}
$$

For $i, j=0,1, \ldots$ we get

$$
\begin{gathered}
{\left[(i+1) h_{1}+\xi_{1}\right]^{2}+h_{2}^{2}+\left(\gamma_{2}+j\right)^{2} h_{3}^{3} \leq\left[(i+1) h_{1}+\xi_{1}\right]^{2}+\left(\gamma_{2}+j+1\right)^{2} h_{3}^{2}} \\
{\left[(i+1) h_{1}+\xi_{1}\right]^{2}+h_{2}^{2}+\left(\gamma_{2}+j\right)^{2} h_{3}^{3}+2 h_{3}\left(\gamma_{2}+j\right) \xi_{3}+\xi_{3}^{2} \leq} \\
\leq\left[(i+1) h_{1}+\xi_{1}\right]^{2}+\left(\gamma_{2}+j+1\right)^{2} h_{3}^{2}+2 h_{3}\left(\gamma_{2}+j\right) \xi_{3}+\xi_{3}^{2} \leq \\
\leq\left[(i+1) h_{1}+\xi_{1}\right]^{2}+\left(\gamma_{2}+j+1\right)^{2} h_{3}^{2}+2 h_{3}\left(\gamma_{2}+j+1\right) \xi_{3}+\xi_{3}^{2}
\end{gathered}
$$

and

$$
\begin{gathered}
r\left(x, \underline{x}_{p+i+1, q, s+\gamma_{2}+j}\right)^{2} \leq\left[(i+1) h_{1}+\xi_{1}\right]^{2}+h_{2}^{2}+\left[\left(\gamma_{2}+j\right) h_{3}+\xi_{3}\right]^{2} \leq \\
\leq\left[(i+1) h_{1}+\xi_{1}\right]^{2}+\left[\left(\gamma_{2}+j+1\right) h_{3}+\xi_{3}\right]^{2}
\end{gathered}
$$

Since $r\left(x, \bar{x}_{p+i+2, q, s+\gamma_{2}+2+j}\right)=\sqrt{\left[(i+1) h_{1}+\xi_{1}\right]^{2}+\left[\left(\gamma_{2}+j+1\right) h_{3}+\xi_{3}\right]^{2}}$ we get

$$
\begin{equation*}
\left.r\left(x, \underline{x}_{p+i+1, q, s+\gamma_{2}+j}\right)\right) \leq r\left(x, \bar{x}_{p+i+2, q, s+\gamma_{2}+2+j}\right), i, j=0,1, \ldots . \tag{7.93}
\end{equation*}
$$

By similar way (with different definition of $\xi_{1}$ and $\xi_{3}$ ) we get

$$
\begin{align*}
& \left.r\left(x, \underline{x}_{p-i-1, q, s-\gamma_{2}-j}\right)\right) \leq r\left(x, \bar{x}_{p-i-2, q, s-\gamma_{2}-2-j}\right), i, j=0,1, \ldots,  \tag{7.94}\\
& \left.r\left(x, \underline{x}_{p-i-1, q, s+\gamma_{2}+j}\right)\right) \leq r\left(x, \bar{x}_{p-i-2, q, s+\gamma_{2}+2+j}\right), i, j=0,1, \ldots \tag{7.95}
\end{align*}
$$

and

$$
\begin{equation*}
\left.r\left(x, \underline{x}_{p+i+1, q, s-\gamma_{2}-j}\right)\right) \leq r\left(x, \bar{x}_{p+i+2, q, s-\gamma_{2}-2-j}\right) i, j=0,1, \ldots . \tag{7.96}
\end{equation*}
$$

Let $i, j=0,1, \ldots$ From

$$
\begin{gathered}
r\left(x, \underline{x}_{p+i+1, q+j+1, s}\right)^{2} \leq\left[(i+1) h_{1}+\xi_{1}\right]^{2}+\left[(j+1) h_{2}+\xi_{2}\right]^{2}+h_{3}^{3} \leq \\
\leq\left[(i+1) h_{1}+\xi_{1}\right]^{2}+\left[(j+1) h_{2}+\xi_{2}\right]^{2}+h_{2}^{2} \leq \\
\leq\left[(i+1) h_{1}+\xi_{1}\right]^{2}+\left[(j+1) h_{2}+\xi_{2}\right]^{2}+2\left[(j+1) h_{2}+\xi_{2}\right] h_{2}+h_{2}^{2}= \\
=\left[(i+1) h_{1}+\xi_{1}\right]^{2}+\left[(j+2) h_{2}+\xi_{2}\right]^{2} .
\end{gathered}
$$

Since $r\left(x, \bar{x}_{p+i+2, q+j+3, s}\right)=\sqrt{\left[(i+1) h_{1}+\xi_{1}\right]^{2}+\left[(j+2) h_{2}+\xi_{2}\right]^{2}}$ we get

$$
\begin{equation*}
r\left(x, \underline{x}_{p+i+1, q+j+1, s}\right) \leq r\left(x, \bar{x}_{p+i+2, q+j+3, s}\right), i, j=0,1, \ldots . \tag{7.97}
\end{equation*}
$$

By similar way (with different definition of $\xi_{1}$ and $\xi_{2}$ ) we get

$$
\begin{align*}
& r\left(x, \underline{x}_{p-i-1, q-j-1, s}\right) \leq r\left(x, \bar{x}_{p-i-2, q-j-3, s}\right), i, j=0,1, \ldots,  \tag{7.98}\\
& r\left(x, \underline{x}_{p-i-1, q+j+1, s}\right) \leq r\left(x, \bar{x}_{p-i-2, q+j+3, s}\right), i, j=0,1, \ldots \tag{7.99}
\end{align*}
$$

and

$$
\begin{equation*}
r\left(x, \underline{x}_{p+i+1, q+j+1, s}\right) \leq r\left(x, \bar{x}_{p+i+2, q-j-3, s}\right) i, j=0,1, \ldots \tag{7.100}
\end{equation*}
$$

Let us define

$$
\begin{aligned}
& D_{15}=\bigcup_{\substack{k=p+1, \ldots, n_{1}-1 \\
l=q+1, \ldots, n_{2}-2 \\
r\left(x, x_{k+1}, l+2, s\right)<\xi}} D_{k l s}, \\
& D_{16}=\bigcup_{\substack{k=2, \ldots, p-1 \\
l=q+1, \ldots, n_{2}-2 \\
r\left(x, x_{k-1, l+s, s}\right)<\xi}} D_{k l s}, \\
& D_{17}=\bigcup_{\substack{k=2, \ldots, p-1 \\
l=3, \ldots, q^{2} \\
r\left(x, x_{k-1, l-2, l s}\right)<\xi}} D_{k l s}
\end{aligned}
$$

and

$$
D_{18}=\bigcup_{\substack{k=p+1, \ldots, n_{1}-1 \\ l=3, \ldots,-1 \\ r\left(x, x_{k+1, l-2, s)}\right)<\xi}} D_{k l s} .
$$

Since $z$ is non-increasing we get from the definition of $\bar{x}_{k l m}, \underline{x}_{k l m}$ and (7.97)

$$
\begin{align*}
& \widetilde{C}_{15}\left(z_{x}\right)=\sum_{\substack{k=p+1, \ldots, n_{1} \\
r\left(x, x_{k, q+1, s}\right)<\xi}} \omega z_{x}\left(x_{k, q+1, s}\right)+\sum_{\substack{k=p+, \ldots, n_{1} \\
r\left(x, x_{k, q+2, s}<\xi\right.}} \omega z_{x}\left(x_{k, q+2, s}\right)+ \\
& +\sum_{\substack{l=q+1, \ldots, n_{2} \\
r\left(x, x_{p+1, l, s}\right)<\xi}} \omega z_{x}\left(x_{p+1, l, s}\right)+\sum_{\substack{k=p+2, \ldots, n_{1} \\
l q+3, \ldots, n_{2} \\
r\left(x, x_{k l s}<\xi\right.}} \omega z_{x}\left(x_{k l s}\right) \leq \\
& \leq 2 \sum_{\substack{k=p+1, \ldots, n_{1} \\
r\left(x, x_{k q s}\right)<\xi}} \omega z_{x}\left(x_{k q s}\right)+\sum_{\substack{l=q+1, \ldots, n_{2} \\
r\left(x, x_{p l s}\right)<\xi}} \omega z_{x}\left(x_{p l s}\right)+ \\
& +\sum_{\substack{k=p+2, \ldots, n_{1} \\
l q+3, \ldots, 2 \\
r\left(x, x_{k l s}\right)<\xi}} \omega z_{x}\left(x_{k l s}\right) \leq 2 \widetilde{C}_{13}\left(z_{x}\right)+\widetilde{C}_{11}\left(z_{x}\right)+\sum_{\substack{k=p+2, \ldots, n_{1} \\
l=q+3, \ldots, n_{2} \\
r\left(x, x_{k l s}\right)<\xi}} \omega z_{x}\left(\bar{x}_{k l s}\right)= \\
& =2 \widetilde{C}_{13}\left(z_{x}\right)+\widetilde{C}_{11}\left(z_{x}\right)+\sum_{\begin{array}{c}
k=p+1, \ldots, n_{1}-1 \\
l=++1, \ldots n_{2}-2 \\
r\left(x, x_{k+1,1,2, s}<\xi\right.
\end{array}} \omega z_{x}\left(\bar{x}_{k+1, q+2, s}\right) \leq \\
& \leq 2 \widetilde{C}_{13}\left(z_{x}\right)+\widetilde{C}_{11}\left(z_{x}\right)+\sum_{\begin{array}{c}
k=p+1, \ldots, n_{1}-1 \\
l=q+1, \ldots n_{2}-2 \\
r\left(x, x_{k+1, l+2, s}<\xi\right.
\end{array}} \omega z_{x}\left(\underline{x}_{k l s}\right) \leq \\
& =2 \widetilde{C}_{13}\left(z_{x}\right)+\widetilde{C}_{11}\left(z_{x}\right)+\int_{D_{15}} z(t) d t . \tag{7.101}
\end{align*}
$$

Analogously using (7.98), (7.99) and (7.100) we get

$$
\begin{align*}
& \widetilde{C}_{16}\left(z_{x}\right) \leq 2 \widetilde{C}_{14}\left(z_{x}\right)+\widetilde{C}_{11}\left(z_{x}\right)+\int_{D_{16}} z(t) d t,  \tag{7.102}\\
& \widetilde{C}_{17}\left(z_{x}\right) \leq 2 \widetilde{C}_{14}\left(z_{x}\right)+\widetilde{C}_{12}\left(z_{x}\right)+\int_{D_{17}} z(t) d t \tag{7.103}
\end{align*}
$$

and

$$
\begin{equation*}
\widetilde{C}_{18}\left(z_{x}\right) \leq 2 \widetilde{C}_{13}\left(z_{x}\right)+\widetilde{C}_{12}\left(z_{x}\right)+\int_{D_{18}} z(t) d t . \tag{7.104}
\end{equation*}
$$

From (7.101), (7.102), (7.103) and (7.104) we get

$$
\begin{equation*}
\sum_{j=15}^{18} \widetilde{C}_{j}\left(z_{x}\right) \leq 2 \widetilde{C}_{11}\left(z_{x}\right)+2 \widetilde{C}_{12}\left(z_{x}\right)+4 \widetilde{C}_{13}\left(z_{x}\right)+4 \widetilde{C}_{14}\left(z_{x}\right)+\sum_{j=15}^{18} \int_{D_{j}} z(t) d t \tag{7.105}
\end{equation*}
$$

Let us define

$$
D_{19}=\bigcup_{\substack{k=p+1, \ldots, n_{1}-1 \\ m=+1, \ldots, n_{3}-2 \\ r\left(x, x_{k+1, q, m+2}\right)<\xi}} D_{k q m},
$$

$$
D_{21}=\bigcup_{\substack{k=2, \ldots, p-1 \\ m=3, \ldots, s-1 \\ r\left(x, x_{k-1, q, m-2}\right)<\xi}} D_{k q m}
$$

and

$$
D_{22}=\bigcup_{\substack{k=p+1, \ldots, n_{1}-1 \\ m=3, \ldots, s-1 \\ r\left(x, x_{k+1, q, m-2}\right)<\xi}} D_{k q m} .
$$

From (7.93) we get

$$
\begin{align*}
& \widetilde{C}_{19}\left(z_{x}\right)= \sum_{\substack{k=p+1, \ldots, n_{1} \\
r\left(x, x_{k, q, s+1}\right)<\xi}} \omega z_{x}\left(x_{k, q, s+1}\right)+\ldots+\sum_{\substack{k=p+1, \ldots, n_{1} \\
r\left(x, x_{k, q, s+\gamma_{2}+1}\right)<\xi}} \omega z_{x}\left(x_{k, q, s+\gamma_{2}+1}\right) \\
&+\sum_{\substack{m=s+1, \ldots, n_{3} \\
r\left(x, x_{p+1, q, m}\right)<\xi}} \omega z_{x}\left(x_{p+1, q, m}\right)+\sum_{\substack{k=p+2, \ldots, n_{1} \\
m=s+2+\gamma_{2}, \ldots, n_{3} \\
r\left(x, x_{k q m}\right)<\xi}} \omega z_{x}\left(x_{k q m}\right) \leq \\
& \leq\left(1+\gamma_{2}\right) \sum_{\substack{k=p+1, \ldots, n_{1} \\
r\left(x, x_{k q s}\right)<\xi}} \omega z_{x}\left(x_{k q s}\right)+\sum_{\substack{m=s+1, \ldots, n_{3} \\
r\left(x, x_{p q m}\right)<\xi}} \omega z_{x}\left(x_{p q m}\right)+ \\
&+\sum_{\substack{k=p+2, \ldots, n_{1} \\
m=s+2+\gamma_{2}, \ldots, n_{3} \\
r\left(x, x_{k q m}\right)<\xi}} \omega z_{x}\left(x_{k q m}\right) \leq\left(1+\gamma_{2}\right) \widetilde{C}_{13}\left(z_{x}\right)+\widetilde{C}_{9}\left(z_{x}\right)+\sum_{\substack{k=p+2, \ldots, n_{1} \\
m=s+2+n_{2}, \ldots, n_{3} \\
r\left(x, x_{k q m}\right)<\xi}} \omega z_{x}\left(\bar{x}_{k q m}\right)= \\
&=\left(1+\gamma_{2}\right) \widetilde{C}_{13}\left(z_{x}\right)+\widetilde{C}_{9}\left(z_{x}\right)+\sum_{\substack{k=p+1, \ldots, n_{1}-1 \\
m=s+\gamma_{2}, \ldots, n_{3}-2 \\
r\left(x, x_{k+1, q, m+2)<\xi}\right.}} \omega z_{x}\left(\bar{x}_{k+1, q, m+2}\right) \leq \\
& \leq\left(1+\gamma_{3}\right) \widetilde{C}_{13}\left(z_{x}\right)+\widetilde{C}_{9}\left(z_{x}\right)+\sum_{\substack{k=p+1, \ldots, n_{1}-1 \\
m=s+1, \ldots, n_{3}-2 \\
r\left(x, x_{k+1, q, s+2)<\xi}\right.}} \omega z_{x}\left(\underline{x}_{k q m}\right) \leq \\
&=\left(1+\gamma_{2}\right) \widetilde{C}_{13}\left(z_{x}\right)+\widetilde{C}_{9}\left(z_{x}\right)+\int_{D_{19}} z(t) d t . \tag{7.106}
\end{align*}
$$

Analogously using (7.94), (7.95) and (7.96) we get

$$
\begin{align*}
& \widetilde{C}_{16}\left(z_{x}\right) \leq\left(1+\gamma_{2}\right) \widetilde{C}_{14}\left(z_{x}\right)+\widetilde{C}_{9}\left(z_{x}\right)+\int_{D_{16}} z(t) d t  \tag{7.107}\\
& \widetilde{C}_{17}\left(z_{x}\right) \leq\left(1+\gamma_{2}\right) \widetilde{C}_{14}\left(z_{x}\right)+\widetilde{C}_{10}\left(z_{x}\right)+\int_{D_{17}} z(t) d t \tag{7.108}
\end{align*}
$$

and

$$
\begin{equation*}
\widetilde{C}_{18}\left(z_{x}\right) \leq\left(1+\gamma_{2}\right) \widetilde{C}_{13}\left(z_{x}\right)+\widetilde{C}_{10}\left(z_{x}\right)+\int_{D_{18}} z(t) d t . \tag{7.109}
\end{equation*}
$$

From (7.106), (7.107), (7.108) and (7.109) we get

$$
\begin{equation*}
\sum_{j=19}^{22} \widetilde{C}_{j}\left(z_{x}\right) \leq 2 \widetilde{C}_{9}\left(z_{x}\right)+2 \widetilde{C}_{10}\left(z_{x}\right)+2\left(\gamma_{2}+1\right) \widetilde{C}_{13}\left(z_{x}\right)+2\left(\gamma_{2}+1\right) \widetilde{C}_{14}\left(z_{x}\right)+\sum_{j=19}^{22} \int_{D_{j}} z(t) d t . \tag{7.110}
\end{equation*}
$$

Finally let us define

$$
D_{23}=\bigcup_{\substack{l=q+1, \ldots, n_{2}-2 \\ m=s+1, \ldots, n_{3} \\ r\left(x, x_{p, l+2, m+1}\right)<\xi}} D_{p l m},
$$

and

$$
D_{26}=\bigcup_{\substack{l=q+1, \ldots, n_{2}-2 \\ m=2, \ldots, s-1 \\ r\left(x, x_{p, l+2, m-1}\right)<\xi}} D_{p l m}
$$

From (7.89) we get

$$
\begin{align*}
& \widetilde{C}_{23}\left(z_{x}\right)=\sum_{\substack{m=s+1, \ldots, n_{3} \\
r\left(x, x_{p, q+1, m}\right)<\xi}} \omega z_{x}\left(x_{p, q+1, m}\right)+\ldots+\sum_{\substack{m=s+1, \ldots, n_{3} \\
r\left(x, x_{p, q+\gamma_{3}+1, m}\right)<\xi}} \omega z_{x}\left(x_{p, q+\gamma_{3}+1, m}\right) \\
& +\sum_{\substack{l=q+1, \ldots, n_{2} \\
r\left(x, x_{p, l}, s+1\right)<\xi}} \omega z_{x}\left(x_{p, l, s+1}\right)+\sum_{\substack{l=q+\gamma_{3}+2, \ldots, n_{2} \\
m=s+2, \ldots, n_{3} \\
r\left(x, x_{p l m}\right)<\xi}} \omega z_{x}\left(x_{p l m}\right) \leq \\
& \leq\left(1+\gamma_{3}\right) \sum_{\substack{m=s+1, \ldots, n_{3} \\
r\left(x, x_{p q m}\right)<\xi}} \omega z_{x}\left(x_{p q m}\right)+\sum_{\substack{l=\left(+1, \ldots, n_{2} \\
r\left(x, x_{p l s}\right)<\xi\right.}} \omega z_{x}\left(x_{p l s}\right)+ \\
& +\sum_{\substack{l=q+2+\gamma_{3}, \ldots, n_{2} \\
m=s+2, \ldots, n_{3} \\
r\left(x, x_{p l m}\right)<\xi}} \omega z_{x}\left(x_{p l m}\right) \leq\left(1+\gamma_{3}\right) \widetilde{C}_{9}\left(z_{x}\right)+\widetilde{C}_{11}\left(z_{x}\right)+\sum_{\substack{l=q+2+\gamma_{3}, \ldots, n_{2} \\
m=s+2, \ldots, n_{3} \\
r\left(x, x_{p l m}\right)<\xi}} \omega z_{x}\left(\bar{x}_{p l m}\right)= \\
& =\left(1+\gamma_{3}\right) \widetilde{C}_{9}\left(z_{x}\right)+\widetilde{C}_{11}\left(z_{x}\right)+\sum_{\substack{l=q+\gamma_{3}, \ldots, n_{2}-2 \\
m=q+1, \ldots, n_{3}-1 \\
r\left(x, x_{p, l+2, m+1}\right)<\xi}} \omega z_{x}\left(\bar{x}_{p, l+2, m+1}\right) \leq \\
& \leq\left(1+\gamma_{3}\right) \widetilde{C}_{9}\left(z_{x}\right)+\widetilde{C}_{11}\left(z_{x}\right)+\sum_{\substack{l=q+1, \ldots, n_{2}-2 \\
m=s+1, \ldots, n_{3}-1 \\
r\left(x, x_{p, l+2, m+1}\right)<\xi}} \omega z_{x}\left(\underline{x}_{p l m}\right) \leq \\
& =\left(1+\gamma_{3}\right) \widetilde{C}_{9}\left(z_{x}\right)+\widetilde{C}_{11}\left(z_{x}\right)+\int_{D_{23}} z(t) d t . \tag{7.111}
\end{align*}
$$

Analogously from (7.90), (7.91) and (7.92) we get

$$
\begin{equation*}
\widetilde{C}_{24}\left(z_{x}\right) \leq\left(1+\gamma_{3}\right) \widetilde{C}_{10}\left(z_{x}\right)+\widetilde{C}_{11}\left(z_{x}\right)+\int_{D_{24}} z(t) d t, \tag{7.112}
\end{equation*}
$$

$$
\begin{equation*}
\widetilde{C}_{25}\left(z_{x}\right) \leq\left(1+\gamma_{3}\right) \widetilde{C}_{10}\left(z_{x}\right)+\widetilde{C}_{12}\left(z_{x}\right)+\int_{D_{25}} z(t) d t \tag{7.113}
\end{equation*}
$$

and

$$
\begin{equation*}
\widetilde{C}_{26}\left(z_{x}\right) \leq\left(1+\gamma_{3}\right) \widetilde{C}_{9}\left(z_{x}\right)+\widetilde{C}_{12}\left(z_{x}\right)+\int_{D_{26}} z(t) d t \tag{7.114}
\end{equation*}
$$

From (7.111), (7.112), (7.113) and (7.114) we get

$$
\begin{equation*}
\sum_{j=23}^{26} \widetilde{C}_{j}\left(z_{x}\right) \leq 2\left(1+\gamma_{3}\right) \widetilde{C}_{9}\left(z_{x}\right)+2\left(1+\gamma_{3}\right) \widetilde{C}_{10}\left(z_{x}\right)+2 \widetilde{C}_{11}\left(z_{x}\right)+2 \widetilde{C}_{12}\left(z_{x}\right)+\sum_{j=23}^{26} \int_{D_{j}} z(t) d t . \tag{7.115}
\end{equation*}
$$

From (7.105), (7.110) and (7.115) we get

$$
\begin{gather*}
\sum_{j=15}^{26} \widetilde{C}_{j}\left(z_{x}\right) \leq 2\left(\gamma_{3}+2\right) \widetilde{C}_{9}\left(z_{x}\right)+2\left(\gamma_{3}+2\right) \widetilde{C}_{10}\left(z_{x}\right)+ \\
+4 \widetilde{C}_{11}\left(z_{x}\right)+4 \widetilde{C}_{12}\left(z_{x}\right)+2\left(\gamma_{2}+3\right) \widetilde{C}_{13}\left(z_{x}\right)+2\left(\gamma_{2}+3\right) \widetilde{C}_{14}\left(z_{x}\right)+\sum_{j=15}^{26} \int_{D_{j}} z(t) d t \leq \\
\leq \bar{\gamma}_{2} \sum_{j=9}^{13} \widetilde{C}_{j}\left(z_{x}\right)+\sum_{j=15}^{26} \int_{D_{j}} z(t) d t \tag{7.116}
\end{gather*}
$$

where

$$
\bar{\gamma}_{2}=\max \left\{2\left(\gamma_{3}+2\right), 2\left(\gamma_{2}+3\right), 4\right\}
$$

From (7.69), (7.87) and (7.116) is (6.30) with $c=\max \left\{\bar{\gamma}_{1}\left(\overline{\gamma_{2}}+4\right), \bar{\gamma}_{2}+3\right\}$.
Also here assumptions (7.62) and (7.63) are only technical. A similar theorem can be proved for every compound mid-cuboid integration rule. So we found integration rules for $D \subset \mathbb{R}^{2}$ resp. $\mathbb{R}^{3}$ that satisfy (6.30). As we will see in the next chapter the integration rule does not need to have high degree of precision because the one of the error factors is the singularity.

## 8. Example in one dimensional <br> case

To verify theory of collocation methods and two Nyström methods described in previous chapters let us make numerical tests on a simple integral equation

$$
\begin{equation*}
y(x)-\int_{0}^{1} \frac{y(t)}{|x-t|^{\gamma}} d t=f(x) \tag{8.1}
\end{equation*}
$$

where $\gamma \in(0,1)$. To show that (8.1) has a unique solution for all $f(x)$ we need to show that the operator

$$
\begin{equation*}
\mathcal{K} y(x)=\int_{0}^{1} \frac{y(t)}{|x-t|^{\gamma}} d t \tag{8.2}
\end{equation*}
$$

is compact on $\mathcal{C}[0,1]$ and $\lambda=1$ is not eigenvalue of operator $\mathcal{K}$.
Theorem 8.1. Operator $\mathcal{K}$ defined in (8.2) is compact operator on $\mathcal{C}[0,1]$.
Proof. The kernel function of operator $\mathcal{K}$ is defined by

$$
\begin{equation*}
k(x, t)=|x-t|^{-\gamma} . \tag{8.3}
\end{equation*}
$$

We will use lemma 2.14. We need to construct approximation to kernel function $k_{n}(x, t)$. Let

$$
k_{n}(x, t)=\left\{\begin{array}{l}
|x-t|^{-\gamma},|x-t| \geq \frac{1}{n}  \tag{8.4}\\
n^{\gamma},|x-t|<\frac{1}{n}
\end{array}\right.
$$

Note that

$$
\begin{equation*}
k_{n}(x, t) \leq k(x, t), \text { for all } x, t \in[0,1] \tag{8.5}
\end{equation*}
$$

and

$$
\begin{equation*}
k_{n}(x, t)=k(x, t), \text { if }|x-t| \geq \frac{1}{n} . \tag{8.6}
\end{equation*}
$$

We need to fulfill (2.13). By (8.4), (8.5) and (8.6) we get

$$
\begin{gathered}
\int_{0}^{1}\left|k(x, t)-k_{n}(x, t)\right| d t=\int_{0}^{1}\left(k(x, t)-k_{n}(x, t)\right) d t= \\
=\int_{\left\{t, t \in[0,1],|x-t|<\frac{1}{n}\right\}}\left(|x-t|^{-\gamma}-n^{\gamma}\right) d t \leq \int_{\left\{t,|x-t|<\frac{1}{n}\right\}}\left(|x-t|^{-\gamma}-n^{\gamma}\right) d t= \\
=\int_{x-\frac{1}{n}}^{x}\left[(x-t)^{-\gamma}-n^{\gamma}\right] d t+\int_{x}^{x+\frac{1}{n}}\left[(-x+t)^{-\gamma}-n^{\gamma}\right] d t .
\end{gathered}
$$

Let's make following transformation

$$
r=t-x .
$$

Then for the first integral we have

$$
\int_{x-\frac{1}{n}}^{x}\left[(x-t)^{-\gamma}-n^{\gamma}\right] d t=\int_{-\frac{1}{n}}^{0}\left[(-r)^{-\gamma}-n^{\gamma}\right] d r=
$$

$$
\begin{gathered}
=\int_{-\frac{1}{n}}^{0}(-r)^{-\gamma} d r-\int_{-\frac{1}{n}}^{0} n^{\gamma} d r= \\
=\left[\frac{(-r)^{(-\gamma+1)}}{\gamma-1}\right]_{r=-\frac{1}{n}}^{0}-n^{\gamma}[r]_{r=-\frac{1}{n}}^{0}=0-\frac{\left(\frac{1}{n}\right)^{1-\gamma}}{\gamma-1}-n^{\gamma}\left[0-\left(-\frac{1}{n}\right)\right]= \\
=\frac{\left(\frac{1}{n}\right)^{1-\gamma}}{1-\gamma}-n^{\gamma} \frac{1}{n}=\frac{1}{(1-\gamma) n^{1-\gamma}}-\frac{1}{n^{1-\gamma}}
\end{gathered}
$$

and for the second integral we have

$$
\begin{gathered}
\int_{x}^{x+\frac{1}{n}}\left[(-x+t)^{-\gamma}-n^{\gamma}\right] d t=\int_{0}^{\frac{1}{n}}\left(r^{-\gamma}-n^{\gamma}\right) d r= \\
=\left[\frac{r^{(-\gamma+1)}}{-\gamma+1}\right]_{0}^{r=\frac{1}{n}}-n^{\gamma}[r]_{0}^{r=\frac{1}{n}}=\frac{\left(\frac{1}{n}\right)^{1-\gamma}}{1-\gamma}-n^{\gamma} \frac{1}{n}=\frac{1}{(1-\gamma) n^{1-\gamma}}-\frac{1}{n^{1-\gamma}} .
\end{gathered}
$$

Hence

$$
\begin{gather*}
\int_{0}^{1}\left|k(x, t)-k_{n}(x, t)\right| d t=2\left[\frac{1}{(1-\gamma) n^{1-\gamma}}-\frac{1}{n^{1-\gamma}}\right]= \\
=\frac{2}{1-\gamma} \frac{\gamma}{n^{1-\gamma}} \rightarrow 0 \text { as } n \rightarrow \infty \tag{8.7}
\end{gather*}
$$

and by theorem 2.14 is $\mathcal{K}$ compact integral operator.
We will solve equation (4.1) with parameter $\gamma=1 / 2$. We will show error behavior of collocation and Nyström methods. They all lead to solving a system of linear equations. The system will be solved by Gauss elimination. The columns labeled "Ratio" give the ratio of successive errors. All results are calculated with Maple [13] with library Int1D (attachment no. 1). Maple is also used for calculation of integrals and Gauss elimination.

### 8.1 Collocation method

### 8.1.1 Piecewise constant collocation

Firstly let us solve equation (4.1) with parameter $\gamma=1 / 2$ by piecewise constant collocation method. If we define approximation $k_{n}$ by the same way as in the proof of previous theorem - (8.4) we have by (8.7) and proposition 5.8 item (c) that operator $\mathcal{K}$ defined by (8.2) is compact operator from $L^{\infty}[0,1]$ into $\mathcal{C}[0,1]$. The sets $D_{i}$ are defined by

$$
D_{i}=\left\{\begin{array}{l}
{[(i-1) h, i h), \text { when } i=1, \ldots, n-1}  \tag{8.8}\\
{[(i-1) h, i h], \text { when } i=n}
\end{array}\right.
$$

where

$$
h=\frac{1}{n}
$$

and the approximation points are

$$
\begin{equation*}
x_{i}=\frac{h}{2}+(i-1) h, i=1, \ldots, n . \tag{8.9}
\end{equation*}
$$

Since the approximation points $x_{i}$ are at the interior of $D_{i}$ we have by (5.43) in the corollary 5.7 for $y \in \mathcal{C}[0,1]$ (note that $\tau_{n}=h$ )

$$
\begin{equation*}
\left\|P_{n} y-y\right\|_{\infty} \leq \sup _{\substack{x, t \in[0,1] \\|x-t|<h}}|y(t)-y(x)| \tag{8.10}
\end{equation*}
$$

where projection $\mathcal{P}_{n}$ is defined by (5.31). The error estimate is given by theorem 5.1. The speed of convergence of the approximate solution is the same as the speed of convergence $\left\|\mathcal{P}_{n} y-y\right\|_{\infty}$. In the first example let's choose

$$
f(x)=x^{2}-\left(\frac{16}{15} x^{2} \sqrt{x}+\frac{16}{15} x^{2} \sqrt{1-x}+\frac{8}{15} x \sqrt{1-x}+\frac{2}{5} \sqrt{1-x}\right)
$$

The exact solution is $y(x)=x^{2}$. Let $y$ be the exact solution and let $y_{n}$ be approximate solution obtained by piecewise constant collocation. Tables 8.1 and 8.2 show numerical solution $y_{n}(x)$ and error at the approximation points.

| 5 approximation points |  |  |  |
| :---: | :---: | :---: | :---: |
| $x$ | $y(x)$ | $y_{n}(x)$ | $\left\|y(x)-y_{n}(x)\right\|$ |
| 0,1 | 0,0100000 | $-0,0486860$ | 0,0586860 |
| 0,3 | 0,0900000 | 0,0772023 | 0,0127977 |
| 0,5 | 0,2500000 | 0,2999383 | 0,0499383 |
| 0,7 | 0,4900000 | 0,5355457 | 0,0455457 |
| 0,9 | 0,8100000 | 0,7870752 | 0,0229248 |

Table 8.1: Piecewise constant collocation, exact solution $x^{2}, 5$ approximation points

| 10 approximation points |  |  |  |
| :---: | :---: | :---: | :---: |
| $x$ | $y(x)$ | $y_{n}(x)$ | $\left\|y(x)-y_{n}(x)\right\|$ |
| 0,05000000 | 0,0025000 | $-0,0130852$ | 0,0155852 |
| 0,15000000 | 0,0225000 | 0,0066933 | 0,0158068 |
| 0,25000000 | 0,0625000 | 0,0530598 | 0,0094402 |
| 0,35000000 | 0,1225000 | 0,1231183 | 0,0006183 |
| 0,45000000 | 0,2025000 | 0,2129520 | 0,0104520 |
| 0,55000000 | 0,3025000 | 0,3188175 | 0,0163175 |
| 0,65000000 | 0,4225000 | 0,4384674 | 0,0159674 |
| 0,75000000 | 0,5625000 | 0,5719405 | 0,0094405 |
| 0,85000000 | 0,7225000 | 0,7214450 | 0,0010550 |
| 0,95000000 | 0,9025000 | 0,8898013 | 0,0126987 |

Table 8.2: Piecewise constant collocation, exact solution $x^{2}, 10$ approximation points

Since the solution $y(x)$ is Lipschitz-continuous function on $[0,1]$ from here and (8.10) we have that there exists constant $L<\infty$ such that

$$
\left\|\mathcal{P}_{n} y-y\right\|_{\infty} \leq \sup _{x, t \in[0,1],|x-t|<h} L|x-t| \leq L h .
$$

So the error should decrease by a factor approximately 2 whenever $n$ is doubled. Table 8.3 shows error development. From the table 8.3 we can see that the error

| nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio | nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 0,0586860 | - | 40 | 0,0015759 | 3,01 |
| 10 | 0,0163175 | 3,60 | 80 | 0,0005093 | 3,09 |
| 20 | 0,0047499 | 3,43 | 160 | 0,0001650 | 3,09 |

Table 8.3: Piecewise constant collocation, exact solution $x^{2}$, error development
behavior is even better. In other examples of piecewise constant collocation we will only show error development table.

Now let us choose $f(x)$ such that the exact solution is $y(x)=e^{x}$. Table 8.4 shows error development.

| nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio | nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 0,0664882 | - | 40 | 0,0017259 | 3,26 |
| 10 | 0,0206985 | 3,21 | 80 | 0,0005929 | 2,91 |
| 20 | 0,0056190 | 3,63 | 160 | 0,0002002 | 2,96 |

Table 8.4: Piecewise constant collocation, exact solution $e^{x}$, error development
Also here the function $y(x)$ is Lipschitz-continuous on $[0,1]$ so the error should decrease by a factor approximately 2 whenever $n$ is doubled. Table 8.4 shows that theory is consistent to the example. Also here the error behavior is better. In the last example let us choose $f(x)$ such that the exact solution is $y(x)=\sqrt{x}$. Table 8.5 shows error development.

| nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio | nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 0,0665780 | - | 40 | 0,0046562 | 2,18 |
| 10 | 0,0259178 | 2,57 | 80 | 0,0021031 | 2,21 |
| 20 | 0,0101622 | 2,55 | 160 | 0,0009614 | 2,19 |

Table 8.5: Piecewise constant collocation, exact solution $\sqrt{x}$, error development
The function $\sqrt{x}$ is Hölder continuous function (see definition A-2) with constant $\alpha=1 / 2$ and $A=1$. Hence by (A-2) and (5.43) in corollary 5.7 we have that

$$
\left\|\mathcal{P}_{n} y-y\right\|_{\infty} \leq \sup _{x, t \in[0,1],|x-t|<h} \sqrt{|x-t|} \leq \sqrt{h} .
$$

So the error should decrease by factor approximately $\sqrt{2} \approx 1.41$ when $n$ is doubled. From the table 8.5 we can see that also here the theory is consistent to the example.

### 8.1.2 Piecewise linear collocation

Now let's try piecewise linear collocation method for (8.1). First let us remember some important properties of approximation theory.

Theorem 8.2 (Lagrange interpolation). Let $f$ be continuous function defined on closed interval $[a, b]$. Let

$$
a \leq x_{0}<x_{1}<\ldots<x_{n} \leq b
$$

be interpolation points. Let's define polynomial

$$
\begin{equation*}
p_{n}(x)=\sum_{i=0}^{n} f\left(x_{i}\right) \phi_{i}(x) \text { where } \phi_{i}(x)=\prod_{j=0, j \neq i}^{n} \frac{x-x_{j}}{x_{i}-x_{j}} . \tag{8.11}
\end{equation*}
$$

$p_{n}$ is called Lagrange interpolation polynomial and satisfies

$$
\begin{equation*}
f\left(x_{i}\right)=p_{n}\left(x_{i}\right), \text { for all } i=0, \ldots, n \tag{8.12}
\end{equation*}
$$

The functions $\phi_{i}$ satisfy

$$
\begin{equation*}
\phi_{i}\left(x_{j}\right)=\delta_{i j} \tag{8.13}
\end{equation*}
$$

where $\delta_{i j}$ is Kronecker delta. If $f \in \mathcal{C}^{n+1}[a, b]$ then there exists $\xi \in[a, b]$ such that

$$
\begin{equation*}
f(x)-p_{n}(x)=\frac{\omega_{n}(x)}{(n+1)!} f^{(n+1)}(\xi), \text { where } \omega_{n}=\prod_{i=0}^{n}\left(x-x_{i}\right) \tag{8.14}
\end{equation*}
$$

Now let's make piecewise linear approximation of $y$ on interval $[0,1]$. Lets define

$$
\begin{equation*}
h=\frac{1}{n} \tag{8.15}
\end{equation*}
$$

and the approximation points $x_{j}$ as

$$
\begin{equation*}
x_{j}=a+j h \text { where } j=0,1, \ldots, n . \tag{8.16}
\end{equation*}
$$

Let us define functions $l_{i}$ as

$$
\begin{align*}
& l_{0}(x)=\left\{\begin{array}{l}
\frac{x_{1}-x}{h} \text { when } x \in\left[x_{0}, x_{1}\right] \\
0 \text { when } x \notin\left[x_{0}, x_{1}\right]
\end{array}\right. \\
& l_{n}(x)=\left\{\begin{array}{l}
\frac{x-x_{n-1}}{h} \text { when } x \in\left[x_{n-1}, x_{n}\right] \\
0 \text { when } x \notin\left[x_{n-1}, x_{n}\right]
\end{array}\right. \\
& l_{i}(x)=\left\{\begin{array}{l}
0, \text { when } x \notin\left[x_{i-1}, x_{i+1}\right] \\
1-\frac{\left|x-x_{i}\right|}{h}, \text { when } x \in\left[x_{i-1}, x_{i+1}\right]
\end{array} \text { for } i=1, \ldots, n-1 .\right. \tag{8.17}
\end{align*}
$$

Functions defined above satisfy

$$
\begin{equation*}
l_{i}\left(x_{j}\right)=\delta_{i j} \tag{8.18}
\end{equation*}
$$

where $\delta$ is Kronecker delta function. The piecewise linear interpolation is then given by projection operator

$$
\begin{equation*}
\mathcal{P}_{n} y(x)=y_{n}(x)=\sum_{i=0}^{n} y\left(x_{i}\right) l_{i}(x) . \tag{8.19}
\end{equation*}
$$

Properties of piecewise linear interpolation is given by following lemma:

Lemma 8.3. Approximation (8.19) satisfies for all $x \in[0,1]$

$$
\begin{equation*}
\left\|y(x)-y_{n}(x)\right\|_{\infty} \leq \omega(y, h) \tag{8.20}
\end{equation*}
$$

where $\omega(y, h)$ is modulus of continuity defined as (A-1) in appendix. If $f \in \mathcal{C}^{1}([0,1])$ then it holds

$$
\begin{equation*}
\left|y(x)-y_{n}(x)\right| \leq \frac{h}{2}\left\|y^{\prime}\right\|_{\infty} . \tag{8.21}
\end{equation*}
$$

If $f \in \mathcal{C}^{2}([0,1])$ then it holds

$$
\begin{equation*}
\left|y(x)-y_{n}(x)\right| \leq \frac{h^{2}}{8}\left\|y^{\prime \prime}\right\|_{\infty} . \tag{8.22}
\end{equation*}
$$

Proof. If $x$ is an approximation point the proof is trivial because at the collocation function coincide with the approximation. Let us find error for fixed $x \in(0,1)$ which is not approximation point. Then there exists $i$ such that $x \in\left(x_{i-1}, x_{i}\right)$. For such $x$ is (8.19) equivalent to

$$
\begin{gather*}
y_{n}(x)=y\left(x_{i-1}\right) l_{i-1}(x)+y\left(x_{i}\right) l_{i}(x)= \\
=\frac{1}{h}\left[\left(x_{i}-x\right) y\left(x_{i-1}\right)+\left(x-x_{i-1}\right) y\left(x_{i}\right)\right] \text { for } i=0, \ldots, n . \tag{8.23}
\end{gather*}
$$

Hence

$$
\begin{gathered}
y_{n}(x)-y(x)= \\
\frac{1}{h}\left[\left(x_{i}-x\right) y\left(x_{i-1}\right)+\left(x-x_{i-1}\right) y\left(x_{i}\right)\right]-y(x) \frac{x_{i}-x+x-x_{i-1}}{h}= \\
=\frac{1}{h}\left[\left(x_{i}-x\right)\left(y\left(x_{i-1}\right)-y(x)\right)+\left(x-x_{i-1}\right)\left(y\left(x_{i}\right)-y(x)\right)\right]
\end{gathered}
$$

and

$$
\begin{gathered}
\left|y_{n}(x)-y(x)\right| \leq \frac{1}{h}\left|x_{i}-x\right| \cdot\left|y\left(x_{i-1}\right)-y(x)\right|+\frac{1}{h}\left|x-x_{i-1}\right| \cdot\left|y\left(x_{i}\right)-y(x)\right| \leq \\
\leq \frac{1}{h}\left|x_{i}-x\right| \max _{x, t \in\left[x_{i-1}, x_{i}\right]}|y(x)-y(t)|+\frac{1}{h}\left|x-x_{i-1}\right| \max _{x, t \in\left[x_{i-1}, x_{i}\right]}|y(x)-y(t)| \leq \\
\leq \frac{1}{h}\left|x_{i}-x\right| \omega(y, h)+\frac{1}{h}\left|x-x_{i-1}\right| \omega(y, h)= \\
=\frac{1}{h}\left(x_{i}-x\right) \omega(y, h)+\frac{1}{h}\left(x-x_{i-1}\right) \omega(y, h)=\omega(y, h) .
\end{gathered}
$$

To prove (8.21) from mean value theorem we get for $\xi_{1} \in\left(x_{i-1}, x_{i}\right)$

$$
y^{\prime}\left(\xi_{1}\right)=\frac{y(x)-y\left(x_{i-1}\right)}{x-x_{i-1}}
$$

and hence

$$
\begin{equation*}
y\left(x_{i-1}\right)=y(x)-y^{\prime}\left(\xi_{1}\right)\left(x-x_{i-1}\right) . \tag{8.24}
\end{equation*}
$$

From mean value theorem for $\xi_{2} \in\left(x, x_{i}\right)$ we have

$$
y^{\prime}\left(\xi_{2}\right)=\frac{y\left(x_{i}\right)-y(x)}{x_{i}-x}
$$

and hence

$$
\begin{equation*}
y\left(x_{i}\right)=y(x)+y^{\prime}\left(\xi_{2}\right)\left(x_{i}-x\right) . \tag{8.25}
\end{equation*}
$$

From (8.24), (8.25) and (8.23) we get

$$
\begin{gathered}
y(x)-y_{n}(x)= \\
y(x)-\frac{x_{i}-x}{h} y(x)-\frac{\left(x-x_{i-1}\right)\left(x_{i}-x\right)}{h}\left(y^{\prime}\left(\xi_{2}\right)-y^{\prime}\left(\xi_{1}\right)\right)-\frac{x-x_{i-1}}{h} y(x)
\end{gathered}
$$

and hence

$$
\left|y_{n}(x)-y(x)\right| \leq\left\|y^{\prime}\right\|_{\infty} \frac{2\left(x-x_{i-1}\right)\left(x_{i}-x\right)}{h} .
$$

If we put $u=x-x_{i-1}$ we get

$$
\left|y_{n}(x)-y(x)\right| \leq\left\|y^{\prime}\right\|_{\infty} \frac{2|(u)(h-u)|}{h} \leq \frac{h}{2}\left\|y^{\prime}\right\|_{\infty}
$$

where we have used

$$
\begin{equation*}
2|u(h-u)| \leq \frac{h^{2}}{2} \text { for } u \in[0, h], h>0 \tag{8.26}
\end{equation*}
$$

and the proof of (8.21) is complete. $y_{n}$ defined by (8.23) is Lagrange interpolant of $y$ on $\left[x_{i-1}, x_{i}\right]$ and from (8.14) in theorem 8.2 it holds for error

$$
y(x)-y_{n}(x)=\frac{\left(x-x_{i}\right)\left(x-x_{i-1}\right)}{2} y^{\prime \prime}(\xi), \xi \in\left[x_{i-1}, x_{i}\right] .
$$

If we put $u=x-x_{i-i}$ we get

$$
y_{n}(x)-y(x)=\frac{(h-u) u}{2} y^{\prime \prime}(\xi), \xi \in\left[x_{i-1}, x_{i}\right] .
$$

From here and (8.26) it holds

$$
\left|y_{n}(x)-y(x)\right| \leq \frac{h^{2}}{8}\left\|y^{\prime \prime}\right\|_{\infty}
$$

and the proof is complete.
Now let us use approximation $y_{n}$ defined in (8.19). The system of equations defined by (5.4) for (8.1) becomes:

$$
y_{n}\left(x_{i}\right)-\sum_{j=0}^{n} y_{n}\left(x_{j}\right) \int_{0}^{1} k\left(x_{i}, t\right) l_{j}(t) d t=f\left(x_{i}\right), i=0, \ldots, n
$$

and after simplifications we finally get

$$
\begin{equation*}
y_{n}\left(x_{i}\right)-\sum_{j=0}^{n} y\left(x_{j}\right) \phi_{j}\left(x_{i}\right)=f\left(x_{i}\right) \tag{8.27}
\end{equation*}
$$

where
$\phi_{0}\left(x_{i}\right)=\frac{1}{h} \int_{x_{0}}^{x_{1}} k\left(x_{i}, t\right)\left(x_{1}-t\right) d t$
$\left.\phi_{j}\left(x_{i}\right)=\frac{1}{h} \int_{x_{j-1}}^{x_{j}} k\left(x_{i}, t\right)\left(t-x_{j-1}\right)\right) d t+\frac{1}{h} \int_{x_{j}}^{x_{j+1}} k\left(x_{i}, t\right)\left(x_{j+1}-t\right) d t, j=1, \ldots, n-1$
$\phi_{n}\left(x_{i}\right)=\frac{1}{h} \int_{x_{n-1}}^{x_{n}} k\left(x_{i}, t\right)\left(t-x_{n-1}\right) d t$.

In order to apply theorem 5.1 we need to satisfy (5.11). We will use lemma 5.2. The projection $\mathcal{P}_{n}$ is defined by $\mathcal{P}_{n} y=y_{n}$. From lemma 8.3 we have

$$
\left\|P_{n} y-y\right\| \leq \omega(y, h)
$$

Since $y \in \mathcal{C}[0,1]$ it holds from the definition of the modulus of continuity that

$$
\omega(y, h) \rightarrow 0 \text { as } h \rightarrow 0
$$

and since $n \rightarrow \infty$ we have that $h \rightarrow 0$. Hence the (5.24) is satisfied for all $x \in[0,1]$ and by lemma 5.2 we have satisfied (5.11).

In the first example let's choose

$$
f(x)=x^{2}-\left(\frac{16}{15} x^{2} \sqrt{x}+\frac{16}{15} x^{2} \sqrt{1-x}+\frac{8}{15} x \sqrt{1-x}+\frac{2}{5} \sqrt{1-x}\right) .
$$

The exact solution is $y(x)=x^{2}$. Tables 8.6 and 8.7 show numerical solution $y_{n}(x)$ and error at the approximation points, table 8.8 error development.

| 5 approximation points |  |  |  |
| :---: | :---: | :---: | :---: |
| $x$ | $y(x)$ | $y_{n}(x)$ | $\left\|y(x)-y_{n}(x)\right\|$ |
| 0,0 | 0,0000000 | $-0,0054518$ | 0,0054518 |
| 0,25 | 0,0625000 | 0,0478201 | 0,0146799 |
| 0,50 | 0,2500000 | 0,2292963 | 0,0207037 |
| 0,75 | 0,5625000 | 0,5478200 | 0,0146800 |
| 1,00 | 1.0000000 | 0,9945482 | 0,0054518 |

Table 8.6: Piecewise linear collocation, exact solution $x^{2}, 5$ approximation points

| 10 approximation points |  |  |  |
| :---: | :---: | :---: | :---: |
| $x$ | $y(x)$ | $y_{n}(x)$ | $\left\|y(x)-y_{n}(x)\right\|$ |
| 0 | 0,0000000 | $-0,0009781$ | 0,0009789 |
| 0,11111111 | 0,2345679 | 0,0108997 | 0,0014460 |
| 0,22222222 | 0,4938272 | 0,0468316 | 0,0025511 |
| 0,33333333 | 0,1111111 | 0,1073395 | 0,0037717 |
| 0,44444444 | 0,1975309 | 0,1929765 | 0,0045543 |
| 0,55555555 | 0,3086420 | 0,3040876 | 0,0045543 |
| 0,66666666 | 0,4444444 | 0,4406728 | 0,0037717 |
| 0,77777777 | 0,6049383 | 0,6023872 | 0,0025511 |
| 0,88888888 | 0,7901235 | 0,7886775 | 0,0014460 |
| 1,00000000 | 1,0000000 | 0,9990210 | 0,0009790 |

Table 8.7: Piecewise linear collocation, exact solution $x^{2}, 10$ approximation points
Since the solution $y(x)=x^{2} \in \mathcal{C}^{2}[0,1]$ we have by (8.22) that the error should decrease by factor approximately 4 when $n$ is doubled. From the table 8.8 we can see that the theory is consistent to the example. In another examples only error development tables will be shown.

| nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio | nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 0,0207037 | - | 40 | 0,0002984 | 3,92 |
| 10 | 0,0045543 | 4,54 | 80 | 0,0000766 | 3,90 |
| 20 | 0,0011694 | 3,89 | 160 | 0,0000179 | 4,20 |

Table 8.8: Piecewise linear collocation, exact solution $x^{2}$, error development
In the second example let us choose $f(x)$ such that the exact solution is $y(x)=e^{x}$. Table 8.9 shows error development.

| nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio | nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 0,0490883 | - | 40 | 0,0003671 | 3,98 |
| 10 | 0,0061967 | 7,92 | 80 | 0,0000951 | 3,86 |
| 20 | 0,0014616 | 4,23 | 160 | 0,0000221 | 4,30 |

Table 8.9: Piecewise linear collocation, exact solution $e^{x}$, error development
Since the solution $y(x)=e^{x} \in \mathcal{C}^{2}[0,1]$ we have by (8.22) that the error should decrease by factor approximately 4 when $n$ is doubled. From the table 8.9 we can see that also here the theory is consistent to the example.

For the last example let us choose $f(x)$ such that the exact solution is $y(x)=$ $\sqrt{x}$. Table 8.10 shows error development.

| nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio | nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 0,4046966 | - | 40 | 0,0126216 | 2,55 |
| 10 | 0,0802910 | 5,04 | 80 | 0,0060760 | 2,08 |
| 20 | 0,0322054 | 2,49 | 160 | 0,0029034 | 2,10 |

Table 8.10: Piecewise linear collocation, exact solution $\sqrt{x}$, error development
The solution $y(x)=\sqrt{x} \in \mathcal{C}[0,1]$ but not in $\mathcal{C}^{1}[0,1]$. The function $\sqrt{x}$ is Hölder continuous function (see definition A-2) with constant $\alpha=1 / 2$ and $A=1$. Hence by (A-2), (8.20) and (A-1) the error should decrease by factor approximately $\sqrt{2} \approx 1.41$ when $n$ is doubled. From the table 8.10 we can see that also here the theory is consistent to the example.

### 8.2 Nyström method

In this section we will use Nyström method to find numerical solution of (8.1). The kernel function

$$
\begin{equation*}
k(x, t)=|x-t|^{-\gamma}, \text { for } \gamma \in(0,1) \tag{8.29}
\end{equation*}
$$

is of the form as in (6.14) with $h(x, t) \equiv 1, r(x, t)=|x-t|$. The singular factor - function $g$ is of the following form:

$$
\begin{equation*}
g(u)=u^{-\gamma} . \tag{8.30}
\end{equation*}
$$

Function $g$ is positive non-increasing and continuous function on $(0, \infty)$. We need to define sequence $\mu_{n}$ and choose numerical integration rule. Let us define a sequence

$$
\begin{equation*}
\mu_{n}=\frac{1}{n} \tag{8.31}
\end{equation*}
$$

and let us choose the compound midpoint integration rule. We need to show that the behavior of both Nyström methods is consistent to the theory. For the error estimation we have following lemma.

Lemma 8.4. Let operator $\mathcal{K}$ be defined as in (4.3), where the kernel function $k$ is of the form (6.14) with with $h(x, t)=1$ and $g(u)=u^{-\gamma}$. Let $\mu_{n}$ be defined by (8.31), $k_{n}$ as (6.16), operator $\mathcal{K}_{n}$ as (6.33). Let $y$ be the solution of (8.1). Let the numerical integration rule be compound midpoint rule. Then there exists $N_{1}$ and $c_{N_{1}}<\infty$ such that for the solution of Nyström method $1 y_{n}$ it holds

$$
\begin{equation*}
\left\|y-y_{n}\right\|_{\infty} \leq c_{N_{1}}\left\|\mathcal{K} y-\mathcal{K}_{n} y\right\|_{\infty} \text { when } n \geq N_{1} \tag{8.32}
\end{equation*}
$$

and there exist $N_{2}$ and $c_{N_{2}}<\infty$ such that for the solution of Nyström method 2 $\widetilde{y}_{n}$ it holds

$$
\begin{equation*}
\left\|y-\widetilde{y}_{n}\right\|_{\infty} \leq c_{N_{2}}\left\|\mathcal{K} y-\widetilde{\mathcal{K}}_{n} y\right\|_{\infty} \text { when } n \geq N_{2} . \tag{8.33}
\end{equation*}
$$

Proof. The compound midpoint rule converges for all continuous functions - see [10] chapter 2.4 (Compound rules) and hence the rule satisfies (6.23). From lemma 7.5 the rule satisfies ( 7.2 ) with $c_{1}=1$. Hence by lemma 7.3 is (6.30) satisfied. From definition 7.1 is $\bar{\omega}_{n}=1 / n$. Hence (6.26) is satisfied with $\rho=1$. For $\gamma$ from (8.1) was assumed that $\gamma \in(0,1)$. Hence

$$
g\left(\mu_{n}\right) \bar{\omega}_{n} \leq \frac{n^{\gamma}}{n}=\frac{1}{n^{1-\gamma}} \leq 1
$$

and (6.27) is also satisfied. For (6.28) and (6.29) with transformation $v=t-x$ we get

$$
\begin{align*}
& \int_{\{t, r(x, t)<\tau\}} g(r(x, t)) d t=\int_{\{t, r(x, t)<\tau\}}(|x-t|)^{-\gamma} d t= \\
= & \int_{\{t, t \in(x, x+\tau)\}}(t-x)^{-\gamma} d t+\int_{\{t, t \in(x-\tau, x]\}}(x-t)^{-\gamma} d t= \\
= & \int_{0}^{\tau} v^{-\gamma} d v+\int_{-\tau}^{0}(-v)^{-\gamma} d v=2 \int_{0}^{\tau} v^{-\gamma} d v=\frac{2}{1-\gamma} \tau^{1-\gamma} . \tag{8.34}
\end{align*}
$$

Since right hand side of (8.34) is independent to $x$ (6.28) and (6.29) immediately follows. Inequalities (8.32), (8.33) and existence of $y_{n}$ and $\widetilde{y}_{n}$ for all sufficiently large $n$ follows from theorems 6.12 and 6.15.

The last lemma give us that the speed of convergence of $y_{n}$ (resp. $\widetilde{y}_{n}$ ) to $y$ is the same as the speed of convergence of $\left\|\mathcal{K}_{n} y-\mathcal{K} y\right\|_{\infty}\left(\right.$ resp. $\left.\left\|\widetilde{\mathcal{K}}_{n} y-\mathcal{K} y\right\|_{\infty}\right)$. This means that error depends on the solution $y$ and the numerical integration rule. In following two subsections we well give numerical examples of both Nyström methods with compound midpoint rule.

### 8.2.1 Nyström method 1

The speed of convergence of Nyström method 1 with compound midpoint is described by following theorem.

Theorem 8.5. Let $y$ be the solution of (8.1). Let $y_{n}$ be the solution of Nyström method 1 where the numerical integration rule is compound midpoint rule and $k_{n}$ is defined by (8.4). Then for sufficiently large $n$ there exists $c_{M}$ such that it holds

$$
\begin{equation*}
\left\|y-y_{n}\right\|_{\infty} \leq \frac{2}{1-\gamma} \omega\left(y, \frac{1}{n}\right)+\frac{c_{M}}{n^{1-\gamma}} \tag{8.35}
\end{equation*}
$$

where $\omega$ is the modulus of continuity.
Proof. For the compound midpoint rule on $[0,1]$ we have that

$$
\omega_{j}=\frac{1}{n}, j=1, \ldots, n
$$

and

$$
x_{j}=\frac{j}{n}-\frac{1}{2 n}, j=1, \ldots, n
$$

Let us take $x \in[0,1]$ and $n$. Then there exist $l$ such that

$$
x \in\left[x_{l}-\frac{1}{2 n}, x_{l}+\frac{1}{2 n}\right] .
$$

Let us take such $l$. Note that there is one special case when $x$ is between two node points. Then there exist two such $l$. We can take anyone of them. Let us define

$$
a=\max \left\{0, x_{l}-\frac{3}{2 n}\right\}
$$

and

$$
b=\min \left\{1, x_{l}+\frac{3}{2 n}\right\}
$$

Then

$$
k_{n}(x, t)=k(x, t) \text { when } t \in[0, a] \text { or } t \in[b, 1]
$$

and

$$
\begin{gather*}
\left|\mathcal{K}_{n} y(x)-\mathcal{K} y(x)\right|= \\
\left.=\left|\sum_{j=1}^{n} \frac{1}{n} y\left(x_{j}\right) k_{n}\left(x, x_{j}\right)-\int_{0}^{1}\right| x-\left.t\right|^{-\gamma} y(t) d t \right\rvert\, \leq E_{1}(x)+E_{2}(x)+E_{3}(x), \tag{8.36}
\end{gather*}
$$

where $k_{n}$ is defined by (8.4),

$$
\begin{aligned}
& E_{1}(x)=\left|\sum_{j, x_{j} \in[0, a]} \frac{1}{n}\left(x-x_{j}\right)^{-\gamma} y\left(x_{j}\right)-\int_{0}^{a}(x-t)^{-\gamma} y(t) d t\right|, \\
& E_{2}(x)=\left|\sum_{j, x_{j} \in[b, 1]} \frac{1}{n}\left(x_{j}-x\right)^{-\gamma} y\left(x_{j}\right)-\int_{b}^{1}(t-x)^{-\gamma} y(t) d t\right|,
\end{aligned}
$$

and

$$
\left.E_{3}(x)=\left|\sum_{j, x_{j} \in[a, b]} \frac{1}{n} k_{n}\left(x, x_{j}\right) y\left(x_{j}\right)-\int_{a}^{b}\right| x-\left.t\right|^{-\gamma} y(t) d t \right\rvert\, .
$$

From (A-3) in theorem A-1 we have that for each $j$ there exist $\xi_{j} \in\left[x_{j}-\frac{1}{2 n}, x_{j}+\frac{1}{2 n}\right]$ such that

$$
\begin{align*}
E_{1}(x)= & \left|\sum_{j, x_{j} \in[0, a]} \frac{1}{n}\left(x-x_{j}\right)^{-\gamma} y\left(x_{j}\right)-\sum_{j, x_{j} \in[0, a]} \int_{x_{j}-\frac{1}{2 n}}^{x_{j}+\frac{1}{2 n}}(x-t)^{-\gamma} y(t) d t\right| \leq \\
\leq & \sum_{j, x_{j} \in[0, a]}\left|\frac{1}{n}\left(x-x_{j}\right)^{-\gamma} y\left(x_{j}\right)-\int_{x_{j}-\frac{1}{2 n}}^{x_{j}+\frac{1}{2 n}}(x-t)^{-\gamma} y(t) d t\right|= \\
= & \sum_{j, x_{j} \in[0, a]}\left|\frac{1}{n}\left(x-x_{j}\right)^{-\gamma} y\left(x_{j}\right)-y\left(\xi_{j}\right) \int_{x_{j}-\frac{1}{2 n}}^{x_{j}+\frac{1}{2 n}}(x-t)^{-\gamma} d t\right| \leq \\
& \leq \sum_{j, x_{j} \in[0, a]}\left|\frac{1}{n}\left(x-x_{j}\right)^{-\gamma} y\left(x_{j}\right)-\frac{1}{n}\left(x-x_{j}\right)^{-\gamma} y\left(\xi_{j}\right)\right|+ \\
& \sum_{j, x_{j} \in[0, a]}\left|\frac{1}{n}\left(x-x_{j}\right)^{-\gamma} y\left(\xi_{j}\right)-y\left(\xi_{j}\right) \int_{x_{j}-\frac{1}{2 n}}^{x_{j}+\frac{1}{2 n}}(x-t)^{-\gamma} d t\right| \leq \\
\leq \omega\left(y, \frac{1}{n}\right) & \sum_{j, x_{j} \in[0, a]} \frac{\left(x-x_{j}\right)^{-\gamma}}{n}+\|y\|_{\infty} \sum_{j, x_{j} \in[0, a]}\left|\frac{\left(x-x_{j}\right)^{-\gamma}}{n}-\int_{x_{j}-\frac{1}{2 n}}^{x_{j}+\frac{1}{2 n}}(x-t)^{-\gamma} d t\right| \tag{8.37}
\end{align*}
$$

Since $(x-t)^{-\gamma} \in \mathcal{C}^{2}[0, a]$ we have from (7.24) in 7.7 for each $j$ such that $x_{j} \in[0, a]$

$$
\left|\frac{\left(x-x_{j}\right)^{-\gamma}}{n}-\int_{x_{j}-\frac{1}{2 n}}^{x_{j}+\frac{1}{2 n}}(x-t)^{-\gamma} d t\right| \leq \frac{\gamma(1+\gamma)}{24 n^{3}}\left[\frac{(l-j-1)}{n}\right]^{-2-\gamma} .
$$

Hence

$$
\begin{gather*}
\sum_{j, x_{j} \in[0, a]}\left|\frac{1}{n}\left(x-x_{j}\right)^{-\gamma}-\int_{x_{j}-\frac{1}{2 n}}^{x_{j}+\frac{1}{2 n}}(x-t)^{-\gamma} d t\right| \leq \frac{\gamma(1+\gamma)}{24 n^{1-\gamma}} \sum_{j, x_{j} \in[0, a]}\left[\frac{1}{l-j-1}\right]^{2+\gamma}= \\
=\frac{\gamma(1+\gamma)}{24 n^{1-\gamma}} \sum_{j=1}^{l-2}\left[\frac{1}{l-j-1}\right]^{2+\gamma} \leq \frac{\gamma(1+\gamma)}{24 n^{1-\gamma}} \sum_{j=1}^{l-2}\left[\frac{1}{l-j-1}\right]^{2}= \\
=\frac{\gamma(1+\gamma)}{24 n^{1-\gamma}}\left[\frac{1}{(l-2)^{2}}+\frac{1}{(l-3)^{2}}+\ldots+\frac{1}{1^{2}}\right] \leq \\
\leq \frac{\gamma(1+\gamma)}{24 n^{1-\gamma}} \sum_{j=1}^{\infty} \frac{1}{j^{2}} \leq \frac{\gamma(1+\gamma)}{24 n^{1-\gamma}} \frac{\pi^{2}}{6} . \tag{8.38}
\end{gather*}
$$

By (A-4) in theorem A-2 we have

$$
\sum_{j, x_{j} \in[0, a]} \frac{\left(x-x_{j}\right)^{-\gamma}}{n}=\sum_{j=1}^{l-2} \frac{\left(x-x_{j}\right)^{-\gamma}}{n} \leq \sum_{j=1}^{l-2} \frac{\left(x_{l-1}-x_{j}\right)^{-\gamma}}{n}=
$$

$$
\begin{gather*}
=\sum_{j=1}^{l-2} \frac{1}{n} \frac{n^{\gamma}}{(l-j-1)^{\gamma}}=\frac{1}{n^{1-\gamma}}\left[\frac{1}{(l-2)^{\gamma}}+\frac{1}{(l-1)^{\gamma}}+\ldots+\frac{1}{1^{\gamma}}\right] \leq \\
\leq \frac{1}{n^{1-\gamma}} \sum_{j=1}^{n} \frac{1}{j^{\gamma}} \leq \frac{1}{n^{\gamma-1}} \int_{0}^{n} \frac{d t}{t^{\gamma}}=\frac{1}{n^{1-\gamma}} \frac{n^{1-\gamma}}{1-\gamma}=\frac{1}{1-\gamma} . \tag{8.39}
\end{gather*}
$$

From (8.37), (8.38) and (8.39) we get

$$
\begin{equation*}
E_{1}(x) \leq \omega\left(y, \frac{1}{n}\right) \frac{1}{1-\gamma}+\|y\|_{\infty} \frac{\pi^{2} \gamma(\gamma+1)}{144 n^{1-\gamma}} . \tag{8.40}
\end{equation*}
$$

From (A-3) in theorem A-1 we have that for each $j$ there exist $\xi_{j} \in\left[x_{j}-\right.$ $\left.\frac{1}{2 n}, x_{j}+\frac{1}{2 n}\right]$ such that

$$
\begin{align*}
& E_{2}(x)=\left|\sum_{j, x_{j} \in[b, 1]} \frac{1}{n}\left(x_{j}-x\right)^{-\gamma} y\left(x_{j}\right)-\sum_{j, x_{j} \in[b, 1]} \int_{x_{j}-\frac{1}{2 n}}^{x_{j}+\frac{1}{2 n}}(t-x)^{-\gamma} y(t) d t\right| \leq \\
& \leq \sum_{j, x_{j} \in[b, 1]}\left|\frac{1}{n}\left(x_{j}-x\right)^{-\gamma} y\left(x_{j}\right)-\int_{x_{j}-\frac{1}{2 n}}^{x_{j}+\frac{1}{2 n}}(t-x)^{-\gamma} y(t) d t\right|= \\
&= \sum_{j, x_{j} \in[b, 1]}\left|\frac{1}{n}\left(x_{j}-x\right)^{-\gamma} y\left(x_{j}\right)-y\left(\xi_{j}\right) \int_{x_{j}-\frac{1}{2 n}}^{x_{j}+\frac{1}{2 n}}(t-x)^{-\gamma} d t\right| \leq \\
& \leq \sum_{j, x_{j} \in[b, 1]}\left|\frac{1}{n}\left(x_{j}-x\right)^{-\gamma} y\left(x_{j}\right)-\frac{1}{n}\left(x_{j}-x\right)^{-\gamma} y\left(\xi_{j}\right)\right|+ \\
&+\sum_{j, x_{j} \in[b, 1]}\left|\frac{1}{n}\left(x_{j}-x\right)^{-\gamma} y\left(\xi_{j}\right)-y\left(\xi_{j}\right) \int_{x_{j}-\frac{1}{2 n}}^{x_{j}+\frac{1}{2 n}}(t-x)^{-\gamma} d t\right| \leq \\
& \leq \omega\left(y, \frac{1}{n}\right) \sum_{j, x_{j} \in[b, 1]} \frac{\left(x_{j}-x\right)^{-\gamma}}{n}+\|y\|_{\infty} \sum_{j, x_{j} \in[0, a]}\left|\frac{\left(x_{j}-x\right)^{-\gamma}}{n}-\int_{x_{j}-\frac{1}{2 n}}^{x_{j}+\frac{1}{2 n}}(t-x)^{-\gamma} d t\right| . \tag{8.41}
\end{align*}
$$

Since $(t-x)^{-\gamma} \in \mathcal{C}^{2}[b, 1]$ we have from (7.24) in 7.7 for each $j$ such that $x_{j} \in[b, 1]$

$$
\left|\frac{\left(x_{j}-x\right)^{-\gamma}}{n}-\int_{x_{j}-\frac{1}{2 n}}^{x_{j}+\frac{1}{2 n}}(t-x)^{-\gamma} d t\right| \leq \frac{\gamma(1+\gamma)}{24 n^{3}}\left[\frac{(j-l-1)}{n}\right]^{-2-\gamma} .
$$

Hence

$$
\begin{gathered}
\sum_{j, x_{j} \in[b, 1]}\left|\frac{1}{n}\left(x_{j}-x\right)^{-\gamma}-\int_{x_{j}-\frac{1}{2 n}}^{x_{j}+\frac{1}{2 n}}(t-x)^{-\gamma} d t\right| \leq \frac{\gamma(1+\gamma)}{24 n^{1-\gamma}} \sum_{j, x_{j} \in[b, 1]}\left[\frac{1}{j-l-1}\right]^{2+\gamma}= \\
=\frac{\gamma(1+\gamma)}{24 n^{1-\gamma}} \sum_{j=l+2}^{n}\left[\frac{1}{j-l-1}\right]^{2+\gamma} \leq \frac{\gamma(1+\gamma)}{24 n^{1-\gamma}} \sum_{j=l+2}^{n}\left[\frac{1}{j-l-1}\right]^{2}= \\
=\frac{\gamma(1+\gamma)}{24 n^{1-\gamma}}\left[\frac{1}{1^{2}}+\frac{1}{2^{2}}+\ldots+\frac{1}{(n-l-1)^{2}}\right] \leq
\end{gathered}
$$

$$
\begin{equation*}
\leq \frac{\gamma(1+\gamma)}{24 n^{1-\gamma}} \sum_{j=1}^{\infty} \frac{1}{j^{2}} \leq \frac{\gamma(1+\gamma)}{24 n^{1-\gamma}} \frac{\pi^{2}}{6} \tag{8.42}
\end{equation*}
$$

By (A-4) in theorem A-2 we have

$$
\begin{gather*}
\sum_{j, x_{j} \in[b, 1]} \frac{\left(x_{j}-x\right)^{-\gamma}}{n}=\sum_{j=l+1}^{n} \frac{\left(x_{j}-x\right)^{-\gamma}}{n} \leq \sum_{j=l+2}^{n} \frac{\left(x_{j}-x_{l+1}\right)^{-\gamma}}{n}= \\
=\sum_{j=l+2}^{n} \frac{1}{n} \frac{n^{\gamma}}{(j-l-1)^{\gamma}}=\frac{1}{n^{1-\gamma}}\left[\frac{1}{(1)^{\gamma}}+\frac{1}{(2)^{\gamma}}+\ldots+\frac{1}{(n-l-1)^{\gamma}}\right] \leq \\
\leq \frac{1}{n^{1-\gamma}} \sum_{j=1}^{n} \frac{1}{j^{\gamma}} \leq \frac{1}{n^{1-\gamma}} \int_{0}^{n} \frac{d t}{t^{\gamma}}=\frac{1}{n^{1-\gamma}} \frac{n^{1-\gamma}}{1-\gamma}=\frac{1}{1-\gamma} . \tag{8.43}
\end{gather*}
$$

From (8.41), (8.42) and (8.43) we get

$$
\begin{equation*}
E_{2}(x) \leq \omega\left(y, \frac{1}{n}\right) \frac{1}{1-\gamma}+\|y\|_{\infty} \frac{\pi^{2} \gamma(\gamma+1)}{144 n^{1-\gamma}} . \tag{8.44}
\end{equation*}
$$

Let

$$
z(r)=\left\{\begin{array}{l}
r^{-\gamma} \text { if } r \geq \frac{1}{n} \\
n^{\gamma} \text { if } r<\frac{1}{n} .
\end{array}\right.
$$

Since $z$ is continuous and non-increasing function on $[0, \infty)$, by (7.11) in lemma 7.3, lemma 7.5 and (8.34) we get

$$
\begin{gather*}
E_{3}(x) \leq \sum_{j,\left|x-x_{j}\right| \leq \frac{2}{n}}\left|\omega_{j} z\left(\left|x-x_{j}\right|\right) y\left(x_{j}\right)\right|+\int_{\left\{t,|x-t|<\frac{2}{n}\right\}}|x-t|^{-\gamma}|y(t)| d t \leq \\
\leq 2\|y\|_{\infty}\left(\frac{1}{n^{1-\gamma}}+\int_{\left\{t,|x-t|<\frac{2}{n}\right\}}|x-t|^{-\gamma} d t\right) \leq \\
\leq 2\|y\|_{\infty}\left(\frac{1}{n^{1-\gamma}}+\frac{2}{1-\gamma} \frac{2^{1-\gamma}}{n^{1-\gamma}}\right) . \tag{8.45}
\end{gather*}
$$

From (6.93) in theorem 6.12, (8.40), (8.44) and (8.45) inequality (8.35) follows with

$$
\begin{equation*}
C_{M}=2\left\|\left(\lambda \mathcal{I}-\mathcal{K}_{n}\right)^{-1}\right\|\|y\|_{\infty}\left[\frac{\pi^{2} \gamma(\gamma+1)}{144}+1+\frac{2^{2-\gamma}}{1-\gamma}\right] . \tag{8.46}
\end{equation*}
$$

Corollary 8.6. Under assumption of theorem 8.5 for sufficiently large $n$ there exists $C_{M}$ such that for the solution of the Nyström method $2 \widetilde{y}_{n}$ we have

$$
\begin{equation*}
\left\|y-\widetilde{y}_{n}\right\|_{\infty} \leq \frac{2}{1-\gamma} \omega\left(y, \frac{1}{n}\right)+\frac{2 c_{M}}{n^{1-\gamma}} . \tag{8.47}
\end{equation*}
$$

Proof. Follows from the proof of previous theorem with function $y(x)-y(t)$ instead of $y(t)$.

Let us choose $\gamma=1 / 2$ as in case of collocation method. In first example let's choose

$$
f(x)=x-\left(\frac{4}{3} x \sqrt{x}+\frac{4}{3} x \sqrt{1-x}+\frac{2}{3} \sqrt{1-x}\right) .
$$

The exact solution is $y(x)=x$. Since

$$
\omega\left(x, \frac{1}{n}\right)=\frac{1}{n}
$$

we have from the bound (8.35) in theorem 8.5 that the ration should be $\sqrt{2} \approx 1,41$. Tables 8.11 and 8.12 show numerical solution $y_{n}(x)$, table 8.13 error behavior.

| $x$ | $y(x)$ | $y_{n}(x)$ | $\left\|y(x)-y_{n}(x)\right\|$ |
| :---: | :---: | :---: | :---: |
| 0,1 | 0,1 | 1,0290752 | 0,9290752 |
| 0,3 | 0,3 | 1,1409041 | 0,8409041 |
| 0,5 | 0,5 | 1,0718740 | 0,5718740 |
| 0,7 | 0,7 | 0,9261074 | 0,2261074 |
| 0,9 | 0,9 | 0,8124308 | 0,0875692 |

Table 8.11: Nyström method 1 , exact solution $x, 5$ node points

| $x$ | $y(x)$ | $y_{n}(x)$ | $\left\|y(x)-y_{n}(x)\right\|$ |
| :---: | :---: | :---: | :---: |
| 0,05 | 0,01 | 0,7898022 | 0,7398022 |
| 0,15 | 0,15 | 0,9289965 | 0,7789965 |
| 0,25 | 0,25 | 0,9718612 | 0,7218612 |
| 0,35 | 0,35 | 0,9497689 | 0,5997689 |
| 0,45 | 0,45 | 0,8828211 | 0,4328211 |
| 0,55 | 0,55 | 0,7897835 | 0,2397835 |
| 0,65 | 0,65 | 0,6899107 | 0,0399107 |
| 0,75 | 0,75 | 0,6035352 | 0,1464648 |
| 0,85 | 0,85 | 0,5536158 | 0,2963842 |
| 0,95 | 0,95 | 0,5772388 | 0,3727612 |

Table 8.12: Nyström method 1, exact solution $x, 10$ node points

| nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | $r_{n}$ | nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 0,9290752 | - | 160 | 1,2387265 | 3,50 |
| 10 | 0,7789965 | 1,19 | 320 | 0,4556743 | 2,72 |
| 20 | 0,8032121 | 1,03 | 640 | 0,2457405 | 1,85 |
| 40 | 1,0995873 | 0,71 | 1280 | 0,1508253 | 1,63 |
| 80 | 4,3409017 | 0,25 | 2560 | 0,0984279 | 1,53 |

Table 8.13: Nyström method 1, exact solution $x$, error development

From the first point of view the scheme does'n seem to converge anymore. However we can see that the method correspondents with theory if the number of node points is 640 and more. The same situation is if we take $f(x)$ such that the exact solution is

$$
y(x)=\sqrt{x} .
$$

Since $\sqrt{x}$ is Hölder continuous with $\alpha=1 / 2$ we have

$$
\omega\left(\sqrt{x}, \frac{1}{n}\right)=\frac{1}{\sqrt{n}}
$$

and from the bound (8.35) in theorem 8.5 that the ration should be $\sqrt{2} \approx 1,41$. Tables 8.14 and 8.15 show numerical solution $y_{n}(x)$, table 8.16 error behavior.

| $x$ | $y(x)$ | $y_{n}(x)$ | $\left\|y(x)-y_{n}(x)\right\|$ |
| :---: | :---: | :---: | :---: |
| 0,1 | 0,3162278 | 1,3209227 | 1,0046948 |
| 0,3 | 0,5477226 | 1.4715382 | 0,9238157 |
| 0,5 | 0,7071068 | 1.4318693 | 0,7247625 |
| 0,7 | 0,8366600 | 1.2971632 | 0,4605032 |
| 0,9 | 0,9486833 | 1.1437625 | 0,1950792 |

Table 8.14: Nyström method 1, exact solution $\sqrt{x}, 5$ node points

| $x$ | $y(x)$ | $y_{n}(x)$ | $\left\|y(x)-y_{n}(x)\right\|$ |
| :---: | :---: | :---: | :---: |
| 0,05 | 0,2236068 | 0,9810204 | 0,7574136 |
| 0,15 | 0,3872983 | 1.1496600 | 0,7623617 |
| 0,25 | 0,5000000 | 1.2093524 | 0,7093524 |
| 0,35 | 0,5916080 | 1.2051308 | 0,6135228 |
| 0,45 | 0,6708204 | 1.1578175 | 0,4869971 |
| 0,55 | 0,7416198 | 1.0833584 | 0,3417386 |
| 0,65 | 0,8062258 | 0.9965144 | 0,1902886 |
| 0,75 | 0,8660254 | 0.9120142 | 0,0459888 |
| 0,85 | 0,9219544 | 0.8458348 | 0,0761197 |
| 0,95 | 0,9746794 | 0.8236537 | 0,1510257 |

Table 8.15: Nyström method 1, exact solution $\sqrt{x}, 10$ node points

| nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | $r_{n}$ | nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 1,0046949 | - | 160 | 1,000474 | 0,93 |
| 10 | 0,7623617 | 1,31 | 320 | 0,379390 | 2,64 |
| 20 | 0,7285222 | 1,04 | 640 | 0,208483 | 1,82 |
| 40 | 0,9254966 | 0,79 | 1280 | 0,129346 | 1,61 |
| 80 | 0,9254966 | 1,00 | 2560 | 0,084869 | 1,52 |

Table 8.16: Nyström method 1, exact solution $\sqrt{x}$, error development

We can see that this method is not useable because large number of nodes is needed.

### 8.2.2 Nyström method 2

Now let's test Nyström method 2. For the speed of convergence we have corollary 8.6. But its result is very pessimistic. To get better result we need to make more assumption on the solution $y$.

Theorem 8.7. Let $y \in \mathcal{C}^{2}[0,1]$ be the solution of (8.1). Let $\widetilde{y}_{n}$ be the solution of Nyström method 2, where the numerical integration rule is compound midpoint rule and $k_{n}$ is defined by (8.4). Then for sufficiently large $n$ there exists $c_{\widetilde{M}}$ such that it holds

$$
\begin{equation*}
\left\|y-\widetilde{y}_{n}\right\|_{\infty} \leq \frac{C_{\widetilde{M}}}{n^{2-\gamma}} . \tag{8.48}
\end{equation*}
$$

Proof. Let us define numbers $a, b, l$ by the same way as in the beginning of the proof of the theorem 8.5. Then

$$
k_{n}(x, t)=k(x, t) \text { when } t \in[0, a] \text { or } t \in[b, 1]
$$

and

$$
\begin{gather*}
\left|\widetilde{\mathcal{K}}_{n} y(x)-\mathcal{K} y(x)\right|= \\
\left.=\left|\sum_{j=1}^{n} \frac{1}{n}\left[y\left(x_{j}\right)-y(x)\right] k_{n}\left(x, x_{j}\right)-\int_{0}^{1}\right| x-\left.t\right|^{-\gamma}[y(t)-y(x)] d t \right\rvert\, \leq \\
\leq \widetilde{E}_{1}(x)+\widetilde{E}_{2}(x)+\widetilde{E}_{3}(x) \tag{8.49}
\end{gather*}
$$

where $k_{n}$ is defined by (8.4),

$$
\begin{aligned}
& \widetilde{E}_{1}(x)=\left|\sum_{j, x_{j} \in[0, a]} \frac{1}{n}\left(x-x_{j}\right)^{-\gamma}\left[y\left(x_{j}\right)-y(x)\right]-\int_{0}^{a}(x-t)^{-\gamma}[y(t)-y(x)] d t\right|, \\
& \widetilde{E}_{2}(x)=\left|\sum_{j, x_{j} \in[b, 1]} \frac{1}{n}\left(x_{j}-x\right)^{-\gamma}\left[y\left(x_{j}\right)-y(x)\right]-\int_{b}^{1}(t-x)^{-\gamma}[y(t)-y(x)] d t\right|,
\end{aligned}
$$

and

$$
\widetilde{E}_{3}(x)=\left|\sum_{j, x_{j} \in[a, b]} \frac{1}{n} k_{n}\left(x, x_{j}\right)\left[y\left(x_{j}\right)-y(x)\right]-\int_{a}^{b} k(x, t)[y(t)-y(x)] d t\right| .
$$

From $y(t) \in \mathcal{C}^{2}[0,1]$ we have that $(x-t)^{-\gamma}[y(x)-y(t)] \in \mathcal{C}^{2}[0, a]$ and from the mean value theorem we get

$$
\begin{equation*}
|y(x)-y(t)| \leq|x-t|\left\|y^{\prime}\right\|_{\infty} . \tag{8.50}
\end{equation*}
$$

For future use let us define for each $j$ intervals

$$
I_{j}=\left[x_{j}-\frac{1}{2 n}, x_{j}+\frac{1}{2 n}\right]
$$

and a constant

$$
\widetilde{c}=\left\|y^{\prime}\right\|_{\infty}[\gamma(\gamma+1)+2 \gamma] .
$$

From (8.50), (7.24) in theorem 7.7 and (A-4) in theorem A-2 we have

$$
\begin{aligned}
& \widetilde{E}_{1}(x)=\left|\sum_{j, x_{j} \in[0, a]} \frac{1}{n}\left(x-x_{j}\right)^{-\gamma}\left[y\left(x_{j}\right)-y(x)\right]-\sum_{j, x_{j} \in[0, a]} \int_{I_{j}}(x-t)^{-\gamma}[y(t)-y(x)] d t\right| \leq \\
& \leq \sum_{j, x_{j} \in[0, a]}\left|\frac{1}{n}\left(x-x_{j}\right)^{-\gamma}\left[y\left(x_{j}\right)-y(x)\right]-\int_{I_{j}}(x-t)^{-\gamma}[y(t)-y(x)] d t\right| \leq \\
& \left.\leq \sum_{j, x_{j} \in[0, a]} \frac{1}{24 n^{3}} \max _{\xi_{j} \in I_{j}} \right\rvert\, \gamma(\gamma+1)\left(x-\xi_{j}\right)^{-2-\gamma}\left[y\left(\xi_{j}\right)-y(x)\right]+ \\
& +2 y^{\prime}\left(\xi_{j}\right) \gamma\left(x-\xi_{j}\right)^{-1-\gamma}+y^{\prime \prime}\left(\xi_{j}\right)\left(x-\xi_{j}\right)^{-\gamma} \mid \leq \\
& \leq \sum_{j, x_{j} \in[0, a]} \frac{1}{24 n^{3}} \max _{\xi_{j} \in I_{j}}\left[\gamma(\gamma+1)\left(x-\xi_{j}\right)^{-2-\gamma}\left\|y^{\prime}\right\|_{\infty}\left(x-\xi_{j}\right)+2\left\|y^{\prime}\right\|_{\infty} \gamma\left(x-\xi_{j}\right)^{-1-\gamma}\right]+ \\
& +\sum_{j, x_{j} \in[0, a]} \frac{1}{24 n^{3}} \max _{\xi_{j} \in I_{j}}\left[\left|y^{\prime \prime}\left(\xi_{j}\right)\right|\left(x-\xi_{j}\right)^{-\gamma}\right] \leq \\
& \leq \frac{\widetilde{c}}{24 n^{3}} \sum_{j, x_{j} \in[0, a]} \max _{\xi_{j} \in I_{j}}\left(x-\xi_{j}\right)^{-1-\gamma}+\frac{\left\|y^{\prime \prime}\right\|_{\infty}}{24 n^{3}} \sum_{j, x_{j} \in[0, a]} \max _{\xi_{j} \in I_{j}}\left(x-\xi_{j}\right)^{-\gamma} \leq \\
& \leq \frac{\widetilde{c}}{24 n^{3}} \sum_{j, x_{j} \in[0, a]}\left[\frac{l-j-1}{n}\right]^{-1-\gamma}+\frac{\left\|y^{\prime \prime}\right\|_{\infty}}{24 n^{3}} \sum_{j, x_{j} \in[0, a]}\left[\frac{l-j-1}{n}\right]^{-\gamma}= \\
& =\frac{\widetilde{c}}{24 n^{2-\gamma}} \sum_{j=1}^{l-2}\left[\frac{1}{l-j-1}\right]^{1+\gamma}+\frac{\left\|y^{\prime \prime}\right\|_{\infty}}{24 n^{3-\gamma}} \sum_{j=1}^{l-2}\left[\frac{1}{l-j-1}\right]^{\gamma}= \\
& =\frac{\widetilde{c}}{24 n^{2-\gamma}}\left[\frac{1}{(l-2)^{1+\gamma}}+\frac{1}{(l-3)^{1+\gamma}}+\ldots+\frac{1}{1^{1+\gamma}}\right]+ \\
& +\frac{\left\|y^{\prime \prime}\right\|_{\infty}}{24 n^{3-\gamma}}\left[\frac{1}{(l-2)^{\gamma}}+\frac{1}{(l-3)^{\gamma}}+\ldots+\frac{1}{1^{\gamma}}\right] \leq \\
& \leq \frac{\widetilde{c}}{24 n^{2-\gamma}}\left(1+\sum_{j=2}^{\infty} \frac{1}{j^{1+\gamma}}\right)+\frac{\left\|y^{\prime \prime}\right\|_{\infty}}{24 n^{3-\gamma}} \sum_{j=1}^{n} \frac{1}{j^{\gamma}} \leq \\
& \leq \frac{\widetilde{c}}{24 n^{2-\gamma}}\left(1+\int_{1}^{\infty} \frac{d t}{t^{1+\gamma}}\right)+\frac{\left\|y^{\prime \prime}\right\|_{\infty}}{24 n^{3-\gamma}} \int_{0}^{n} \frac{d t}{t^{\gamma}}= \\
& =\frac{\widetilde{c}}{24 n^{2-\gamma}}\left(1+\frac{1}{\gamma}\right)+\frac{\left\|y^{\prime \prime}\right\|_{\infty}}{24 n^{3-\gamma}}(1-\gamma) n^{1-\gamma}=
\end{aligned}
$$

$$
\begin{equation*}
=\frac{\widetilde{c}}{24 n^{2-\gamma}}\left(1+\frac{1}{\gamma}\right)+\frac{\left\|y^{\prime \prime}\right\|_{\infty}}{24 n^{2}}(1-\gamma) . \tag{8.51}
\end{equation*}
$$

By the same way for $\widetilde{E}_{2}$ we have from (8.50), (7.24) in theorem 7.7 and (A-4) in theorem A-2

$$
\begin{aligned}
& \widetilde{E}_{2}(x)=\left|\sum_{j, x_{j} \in[b, 1]} \frac{1}{n}\left(x_{j}-x\right)^{-\gamma}\left[y\left(x_{j}\right)-y(x)\right]-\sum_{j, x_{j} \in[b, 1]} \int_{I_{j}}(t-x)^{-\gamma}[y(t)-y(x)] d t\right| \leq \\
& \leq \sum_{j, x_{j} \in[b, 1]}\left|\frac{1}{n}\left(x_{j}-x\right)^{-\gamma}\left[y\left(x_{j}\right)-y(x)\right]-\int_{I_{j}}(t-x)^{-\gamma}[y(t)-y(x)] d t\right| \leq \\
& \left.\leq \sum_{j, x_{j} \in[b, 1]} \frac{1}{24 n^{3}} \max _{\xi_{j} \in I_{j}} \right\rvert\, \gamma(\gamma+1)\left(\xi_{j}-x\right)^{-2-\gamma}\left[y\left(\xi_{j}\right)-y(x)\right]- \\
& -2 y^{\prime}\left(\xi_{j}\right) \gamma\left(\xi_{j}-x\right)^{-1-\gamma}+y^{\prime \prime}\left(\xi_{j}\right)\left(\xi_{j}-x\right)^{-\gamma} \mid \leq \\
& \leq \sum_{j, x_{j} \in[b, 1]} \frac{1}{24 n^{3}} \max _{\xi_{j} \in I_{j}}\left[\gamma(\gamma+1)\left(\xi_{j}-x\right)^{-2-\gamma}\left\|y^{\prime}\right\|_{\infty}\left(\xi_{j}-x\right)+2\left\|y^{\prime}\right\|_{\infty} \gamma\left(\xi_{j}-x\right)^{-1-\gamma}\right]+ \\
& +\sum_{j, x_{j} \in[b, 1]} \frac{1}{24 n^{3}} \max _{\xi_{j} \in I_{j}}\left[\left|y^{\prime \prime}\left(\xi_{j}\right)\right|\left(\xi_{j}-x\right)^{-\gamma}\right] \leq \\
& \leq \frac{\widetilde{c}}{24 n^{3}} \sum_{j, x_{j} \in[b, 1]} \max _{\xi_{j} \in I_{j}}\left(\xi_{j}-x\right)^{-1-\gamma}+\frac{\left\|y^{\prime \prime}\right\|_{\infty}}{24 n^{3}} \sum_{j, x_{j} \in[b, 1]} \max _{\xi_{j} \in I_{j}}\left(\xi_{j}-x\right)^{-\gamma} \leq \\
& \leq \frac{\widetilde{c}}{24 n^{3}} \sum_{j, x_{j} \in[b, 1]}\left[\frac{j-l-1}{n}\right]^{-1-\gamma}+\frac{\left\|y^{\prime \prime}\right\|_{\infty}}{24 n^{3}} \sum_{j, x_{j} \in[b, 1]}\left[\frac{j-l-1}{n}\right]^{-\gamma}= \\
& =\frac{\widetilde{c}}{24 n^{2-\gamma}} \sum_{j=l+2}^{n}\left[\frac{1}{j-l-1}\right]^{1+\gamma}+\frac{\left\|y^{\prime \prime}\right\|_{\infty}}{24 n^{3-\gamma}} \sum_{j=l+2}^{n}\left[\frac{1}{j-l-1}\right]^{\gamma}= \\
& =\frac{\widetilde{c}}{24 n^{2-\gamma}}\left[\frac{1}{1^{1+\gamma}}+\frac{1}{2^{1+\gamma}}+\ldots+\frac{1}{(n-l-1)^{1+\gamma}}\right]+ \\
& +\frac{\left\|y^{\prime \prime}\right\|_{\infty}}{24 n^{3-\gamma}}\left[\frac{1}{1^{\gamma}}+\frac{1}{2^{\gamma}}+\ldots+\frac{1}{(n-l-1)^{\gamma}}\right] \leq \\
& \leq \frac{\widetilde{c}}{24 n^{2-\gamma}}\left(1+\sum_{j=2}^{\infty} \frac{1}{j^{1+\gamma}}\right)+\frac{\left\|y^{\prime \prime}\right\|_{\infty}}{24 n^{3-\gamma}} \sum_{j=1}^{n} \frac{1}{j^{\gamma}} \leq \\
& \leq \frac{\widetilde{c}}{24 n^{2-\gamma}}\left(1+\int_{1}^{\infty} \frac{d t}{t^{1+\gamma}}\right)+\frac{\left\|y^{\prime \prime}\right\|_{\infty}}{24 n^{3-\gamma}} \int_{0}^{n} \frac{d t}{t^{\gamma}}= \\
& =\frac{\widetilde{c}}{24 n^{2-\gamma}}\left(1+\frac{1}{\gamma}\right)+\frac{\left\|y^{\prime \prime}\right\|_{\infty}}{24 n^{3-\gamma}}(1-\gamma) n^{1-\gamma}=
\end{aligned}
$$

$$
\begin{equation*}
=\frac{\widetilde{c}}{24 n^{2-\gamma}}\left(1+\frac{1}{\gamma}\right)+\frac{\left\|y^{\prime \prime}\right\|_{\infty}}{24 n^{2}}(1-\gamma) . \tag{8.52}
\end{equation*}
$$

We need to bound $\widetilde{E}_{3}$. From (8.4) is

$$
\begin{equation*}
k_{n}(x, t) \leq n^{\gamma} \text { for all } x, t \in[0,1] . \tag{8.53}
\end{equation*}
$$

From the definition of $a$ and $b$ it holds

$$
\begin{equation*}
\max _{x_{i} \in[a, b]}\left|x_{i}-x\right| \leq \frac{3}{2 n} \tag{8.54}
\end{equation*}
$$

Since in $[a, b]$ are maximal three nodes we have from (8.53), (8.54) and (8.50) we get

$$
\begin{align*}
& \sum_{j, x_{j} \in[a, b]}\left|\frac{1}{n} k_{n}\left(x, x_{j}\right)\left[y\left(x_{j}\right)-y(x)\right]\right| \leq \frac{1}{n^{1-\gamma}} \sum_{j, x_{j} \in[a, b]}\left|y\left(x_{j}\right)-y(x)\right| \leq \\
& \leq \frac{\|y\|_{\infty}}{n^{1-\gamma}} \sum_{x_{j} \in[a, b]}\left|x_{j}-x\right| \leq \frac{3\left\|y^{\prime}\right\|_{\infty}}{n^{1-\gamma}} \max _{x_{j} \in[a, b]}\left|x_{j}-x\right| \leq \frac{9\left\|y^{\prime}\right\|_{\infty}}{2 n^{2-\gamma}} . \tag{8.55}
\end{align*}
$$

(8.50) also implies

$$
\begin{gather*}
\int_{a}^{b}|k(x, t)[y(t)-y(x)]| d t=\int_{a}^{b}|x-t|^{-\gamma} \mid y(t)-y(x) \| d t \leq \\
\leq\left\|y^{\prime}\right\|_{\infty} \int_{a}^{b}|x-t|^{1-\gamma} d t=\left\|y^{\prime}\right\|_{\infty}\left(\int_{a}^{x}(x-t)^{1-\gamma} d t+\int_{x}^{b}(t-x)^{1-\gamma} d t\right) \leq \\
\leq\left\|y^{\prime}\right\|_{\infty}\left(\int_{x-\frac{2}{n}}^{x}(x-t)^{1-\gamma} d t+\int_{x}^{x+\frac{2}{n}}(t-x)^{1-\gamma} d t\right)= \\
=2\left\|y^{\prime}\right\|_{\infty} \int_{0}^{\frac{2}{n}} r^{1-\gamma} d r=\frac{2\left\|y^{\prime}\right\|_{\infty} 2^{2-\gamma}}{\left.n^{2-\gamma}(2-\gamma)\right)} . \tag{8.56}
\end{gather*}
$$

From (8.55) and (8.56) we get

$$
\begin{gather*}
\widetilde{E}_{3}(x) \leq \sum_{j, x_{j} \in[a, b]}\left|\frac{1}{n} k_{n}\left(x, x_{j}\right)\left[y\left(x_{j}\right)-y(x)\right]\right|+\int_{a}^{b}|k(x, t)[y(t)-y(x)]| d t \leq \\
\leq \frac{\left\|y^{\prime}\right\|_{\infty}}{n^{2-\gamma}} \widetilde{c}_{3} \tag{8.57}
\end{gather*}
$$

where

$$
\widetilde{c}_{3}=\frac{9}{2}+\frac{2^{3-\gamma}}{2-\gamma} .
$$

From (8.49), (8.51), (8.52) and (8.57) theorem follows with

$$
C_{\widetilde{M}}=\frac{1}{12}\left[\widetilde{c}+\frac{\widetilde{c}}{\gamma}+\left\|y^{\prime \prime}\right\|_{\infty}(1-\gamma)\right]+\widetilde{c}_{3}\left\|y^{\prime}\right\|_{\infty}
$$

Let's choose five different functions $f(x)$. From (8.48) in theorem 8.7 the ratio should be $\sqrt{2^{3}} \approx 2,83$ for all solutions $y \in \mathcal{C}^{2}[0,1]$. In first example let's choose

$$
f(x)=x-\left(\frac{4}{3} x \sqrt{x}+\frac{4}{3} x \sqrt{1-x}+\frac{2}{3} \sqrt{1-x}\right) .
$$

The exact solution is $y(x)=x$. Tables 8.17 and 8.18 show numerical solution $y_{n}(x)$, table 8.19 error behavior.

| $x$ | $y(x)$ | $y_{n}(x)$ | $\left\|y(x)-y_{n}(x)\right\|$ |
| :---: | :---: | :---: | :---: |
| 0,1 | 0,1 | 0,09664074 | 0,00335926 |
| 0,3 | 0,3 | 0,29256321 | 0,00743679 |
| 0,5 | 0,5 | 0,50000001 | 0,00000001 |
| 0,7 | 0,7 | 0,70743679 | 0,00743679 |
| 0,9 | 0,9 | 0,90335924 | 0,00335924 |

Table 8.17: Nyström method 2, exact solution $x, 5$ node points

| $x$ | $y(x)$ | $y_{n}(x)$ | $\left\|y(x)-y_{n}(x)\right\|$ |
| :---: | :---: | :---: | :---: |
| 0,05 | 0,01 | 0,04976397 | 0,00023603 |
| 0,15 | 0,15 | 0,14825262 | 0,00174739 |
| 0,25 | 0,25 | 0,24768600 | 0,00231400 |
| 0,35 | 0,35 | 0,34809194 | 0,00190806 |
| 0,45 | 0,45 | 0,44926656 | 0,00073344 |
| 0,55 | 0,55 | 0,55073344 | 0,00073344 |
| 0,65 | 0,65 | 0,65190806 | 0,00190806 |
| 0,75 | 0,75 | 0,75231400 | 0,00231400 |
| 0,85 | 0,85 | 0,85174739 | 0,00174739 |
| 0,95 | 0,95 | 0,95023603 | 0,00023603 |

Table 8.18: Nyström method 2, exact solution $x, 10$ node points

| nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio | nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 0,00743679 | - | 80 | 0,00004444 | 3,84 |
| 10 | 0,00231400 | 3,21 | 160 | 0,00001384 | 3,21 |
| 20 | 0,00638130 | 3,63 | 320 | 0,00000507 | 2,73 |
| 40 | 0,00017080 | 3,74 | 640 | 0,00000181 | 2,80 |

Table 8.19: Nyström method 2, exact solution $x$, error development
Let's now choose

$$
f(x)=x^{2}-\left(\frac{16}{15} x^{2} \sqrt{x}+\frac{16}{15} x^{2} \sqrt{1-x}+\frac{8}{15} x \sqrt{1-x}+\frac{2}{5} \sqrt{1-x}\right)
$$

The exact solution is $y(x)=x^{2}$. Tables 8.20 and 8.21 show numerical solution $y_{n}(x)$, table 8.22 error behavior.

| $x$ | $y(x)$ | $y_{n}(x)$ | $\left\|y(x)-y_{n}(x)\right\|$ |
| :---: | :---: | :---: | :---: |
| 0,10 | 0,01000000 | $-0,010771$ | 0,020770912 |
| 0,30 | 0,09000000 | 0,091105 | 0,001104519 |
| 0,50 | 0,25000000 | 0,275946 | 0,025946124 |
| 0,70 | 0,49000000 | 0,505978 | 0,015978092 |
| 0,90 | 0,81000000 | 0,795948 | 0,014052409 |

Table 8.20: Nyström method 2, exact solution $x^{2}, 5$ node points

| $x$ | $y(x)$ | $y_{n}(x)$ | $\left\|y(x)-y_{n}(x)\right\|$ |
| :---: | :---: | :---: | :---: |
| 0,05 | 0,0025 | $-0,001615225$ | 0,004115225 |
| 0,15 | 0,0225 | 0,018477734 | 0,004022266 |
| 0,25 | 0,0625 | 0,060584529 | 0,001915471 |
| 0,35 | 0,1225 | 0,123700923 | 0,001200923 |
| 0,45 | 0,2025 | 0,206560107 | 0,004060107 |
| 0,55 | 0,3025 | 0,308026989 | 0,005526989 |
| 0,65 | 0,4225 | 0,427517047 | 0,005017047 |
| 0,75 | 0,5625 | 0,565212522 | 0,002712522 |
| 0,85 | 0,7225 | 0,721972503 | 0,000527497 |
| 0,95 | 0,9025 | 0,898856838 | 0,003643162 |

Table 8.21: Nyström method 2, exact solution $x^{2}, 10$ node points

| nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio | nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 0,02595461 | - | 80 | 0,00012399 | 3,11 |
| 10 | 0,00552699 | 4,69 | 160 | 0,00040150 | 3,09 |
| 20 | 0,00142514 | 3,89 | 320 | 0,00001316 | 3,05 |
| 40 | 0,00038570 | 3,69 | 640 | 0,00000435 | 3,03 |

Table 8.22: Nyström method 2, exact solution $x^{2}$, error development
In other examples only error development tables will be showed. Let's now choose $f(x)$ such that exact solution is $y(x)=e^{x}$. Table 8.23 show error behavior.

| nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio | nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 0,02306451 | - | 80 | 0,00014216 | 3,08 |
| 10 | 0,00644676 | 3,58 | 160 | 0,00004821 | 2,95 |
| 20 | 0,00168100 | 3,83 | 320 | 0,00001636 | 2,95 |
| 40 | 0,00043755 | 3,84 | 640 | 0,00000549 | 2,98 |

Table 8.23: Nyström method 2, exact solution $e^{x}$, error development
From tables 8.19, 8.22 and 8.23 we can see that theory is consistent to the examples. In last examples let's choose $f(x)$ such that $y \in \mathcal{C}[0,1]$ but $y \notin \mathcal{C}^{1}[0,1]$.

Here we have only bound by corollary 8.6 , which is very pessimistic and the ratio is expected to be $\sqrt{2} \approx 1,41$. Table 8.24 shows error development when exact solution is $y(x)=\sqrt{x}$.

| nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio | nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 0,06566880 | - | 80 | 0,00150604 | 2,18 |
| 10 | 0,02042666 | 3,21 | 160 | 0,00069379 | 2,17 |
| 20 | 0,00752862 | 2,71 | 320 | 0,00032326 | 2,15 |
| 40 | 0,00329685 | 2,28 | 640 | 0,00015234 | 2,12 |

Table 8.24: Nyström method 2, exact solution $\sqrt{x}$, error development
Table 8.25 shows error development when exact solution is $y(x)=\sqrt[4]{x}$. Since $\sqrt[4]{x}$ is Hölder continuous with constant $\alpha=1 / 4$ the ratio should be approximately $\sqrt[4]{2} \approx 1.19$.

| nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio | nodes | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 0,09604551 | - | 80 | 0,00418252 | 1,87 |
| 10 | 0,03427357 | 2,80 | 160 | 0,00227937 | 1,83 |
| 20 | 0,01472064 | 2,32 | 320 | 0,00125873 | 1,81 |
| 40 | 0,00776413 | 1,90 | 640 | 0,00070393 | 1,79 |

Table 8.25: Nyström method 2, exact solution $\sqrt[4]{x}$, error development
We can see that corollary has very pessimistic bound. But in contrast to the Nyström method 1 the number of node points does not need to be so high.

Last question is whether the error development can be improved by taking more precise integration rule. From the proof of theorem 8.7 the answer should be no. The reason is the error $\widetilde{E}_{3}$. It does not depend on the numerical integration rule and it is $\mathcal{O}\left(1 / n^{2-\gamma}\right)$. So there is no need to take more good integration rule than given speed of $\mathcal{O}\left(1 / n^{2}\right)$. To demonstrate this let us use compound Simpson rule which is of order $\mathcal{O}\left(1 / n^{4}\right)$. We will use same example functions. Instead of number of nodes we will use the number of cells that are dividing original interval. Note that in the case of compound midpoint rule the number of nodes equals the number of cells. Also here we will show only error development tables.

For the first example let's choose

$$
f(x)=x-\left(\frac{4}{3} x \sqrt{x}+\frac{4}{3} x \sqrt{1-x}+\frac{2}{3} \sqrt{1-x}\right) .
$$

The exact solution is $y(x)=x$. Table 8.26 shows error evolution.

| cells | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio | cells | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 0,00345657 | - | 80 | 0,00004433 | 2,94 |
| 10 | 0,00115105 | 3,00 | 160 | 0,00001524 | 2,90 |
| 20 | 0,00038555 | 2,99 | 320 | 0,00000528 | 2,88 |
| 40 | 0,00013011 | 2,96 | 640 | 0,00000185 | 2,86 |

Table 8.26: Nyström method 2, Simpson rule, exact solution $x$, error development

For the second example let's now choose

$$
f(x)=x^{2}-\left(\frac{16}{15} x^{2} \sqrt{x}+\frac{16}{15} x^{2} \sqrt{1-x}+\frac{8}{15} x \sqrt{1-x}+\frac{2}{5} \sqrt{1-x}\right)
$$

The exact solution is $y(x)=x^{2}$. Table 8.27 shows error evolution.

| cells | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio | cells | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 0,00814766 | - | 80 | 0,0000901 | 2,97 |
| 10 | 0,00254656 | 3,20 | 160 | 0,0000307 | 2,93 |
| 20 | 0,00081629 | 3,12 | 320 | 0,0000106 | 2,89 |
| 40 | 0,00026832 | 3,04 | 640 | 0,0000037 | 2,87 |

Table 8.27: Nyström method 2, Simpson rule, exact solution $x^{2}$, error development

For the third example let's now choose $f(x)$ such that exact solution is $y(x)=$ $e^{x}$. Table 8.28 shows error evolution.

| cells | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio | cells | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 0,01065630 | - | 80 | 0,00012186 | 2,96 |
| 10 | 0,00336930 | 3,16 | 160 | 0,00004168 | 2,92 |
| 20 | 0,00109000 | 3,09 | 320 | 0,00001439 | 2,90 |
| 40 | 0,00036147 | 3,02 | 640 | 0,00000503 | 2,86 |

Table 8.28: Nyström method 2, Simpson rule, exact solution $e^{x}$, error development

From the tables 8.26, 8.27 and 8.28 we can see that the behavior of ratio didn't change. For the fourth example Let's choose $f(x)$ so that the exact solution is $y(x)=\sqrt{x}$. Table 8.29 shows error evolution.

| cells | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio | cells | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 0,04636450 | - | 80 | 0,00242788 | 2,08 |
| 10 | 0,02197312 | 2,11 | 160 | 0,00117522 | 2,07 |
| 20 | 0,01051308 | 2,09 | 320 | 0,00057204 | 2,06 |
| 40 | 0,00504212 | 2,08 | 640 | 0,00027988 | 2,04 |

Table 8.29: Nyström method 2, Simpson rule, exact solution $\sqrt{x}$, error development

For last example let's choose $f(x)$ so that the exact solution is $y(x)=\sqrt[4]{x}$. Table 8.30 shows error evolution.

| cells | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio | cells | $\left\\|y(x)-y_{n}(x)\right\\|_{\infty}$ | Ratio |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 0,15521530 | - | 80 | 0,01650029 | 1,75 |
| 10 | 0,08863859 | 1,75 | 160 | 0,00947577 | 1,74 |
| 20 | 0,05064411 | 1,75 | 320 | 0,00547279 | 1,73 |
| 40 | 0,02887358 | 1,75 | 640 | 0,00317827 | 1,72 |

Table 8.30: Nyström method 2, Simpson rule, exact solution $\sqrt[4]{x}$, error development

Tables 8.29 and 8.30 show that the error behavior didn't change by changing the numerical integration rule.

### 8.3 Method comparison

As we saw, the Nyström method 1 is not usable. However let us make comparison of Nyström method 2 with both integration rules and collocation method. Following three tables compare errors for various functions. $n$ is the number of cells in case of Nyström method with compound Simpson integration rule, number of nodes in other cases.

| n | constant col. | linear coll. | Nyström - midpoint | Nyström - Simpson |
| :---: | :---: | :---: | :---: | :---: |
| 10 | 0,0163175 | 0,0045543 | 0,0055269 | 0,0025466 |
| 20 | 0,0047499 | 0,0011694 | 0,0014251 | 0,0008163 |
| 40 | 0,0015759 | 0,0002984 | 0,0003857 | 0,0002683 |
| 80 | 0,0005093 | 0,0000766 | 0,0001230 | 0,0000901 |
| 160 | 0,0001650 | 0,0000179 | 0,0000402 | 0,0000307 |

Table 8.31: Exact solution $x^{2}$, error comparison

| n | constant col. | linear coll. | Nyström - midpoint | Nyström - Simpson |
| :---: | :---: | :---: | :---: | :---: |
| 10 | 0,0206985 | 0,0061967 | 0,0064467 | 0,0033693 |
| 20 | 0,0056190 | 0,0014616 | 0,0016810 | 0,0010900 |
| 40 | 0,0017259 | 0,0003671 | 0,0004375 | 0,0003615 |
| 80 | 0,0005929 | 0,0000951 | 0,0001421 | 0,0001219 |
| 160 | 0,0002002 | 0,0000221 | 0,0000482 | 0,0000417 |

Table 8.32: Exact solution $e^{x}$, error comparison

| n | constant col. | linear coll. | Nyström - midpoint | Nyström - Simpson |
| :---: | :---: | :---: | :---: | :---: |
| 10 | 0,0259178 | 0,0802910 | 0,0204266 | 0,0219731 |
| 20 | 0,0101621 | 0,0322054 | 0,0075286 | 0,0105131 |
| 40 | 0,0046562 | 0,0126216 | 0,0032969 | 0,0050421 |
| 80 | 0,0021031 | 0,0060760 | 0,0015060 | 0,0024279 |
| 160 | 0,0009614 | 0,0029034 | 0,0006938 | 0,0011752 |

Table 8.33: Exact solution $\sqrt{x}$, error comparison
From tables above we can see that when $y \in \mathcal{C}^{2}[0,1]$ the best is piecewise linear collocation. But the Nyström method have also good results. Last table shows case that function is only continuous. Here the best is the Nyström method with midpoint rule and the second best is piecewise linear collocation. When using Simpson rule or piecewise linear collocation the effect of singularity is bigger and the results are not so good.

Now let us compare methods by computing time for exact solution $e^{x}$. The time is measured by Maple [13] on operating system Windows 7 with Inter Core i5, 4GB ram.

| n | constant col. | linear coll. | Nyström - midpoint | Nyström - Simpson |
| :---: | :---: | :---: | :---: | :---: |
| 10 | 3,7 | 6,6 | 1,6 | 1,6 |
| 20 | 11,9 | 26,6 | 4,5 | 3,7 |
| 40 | 41,7 | 120,1 | 8,9 | 8,9 |
| 80 | 189,4 | 688,5 | 19,8 | 19,2 |

Table 8.34: Exact solution $e^{x}$, computing time (in seconds)
From last table we can see that Nyström method has very low computing time. The slowest is piecewise linear collocation. The reason is simple. Both methods generate fully populated matrixes. In case of collocation method all members of the matrix are integrals which need to be calculated. In case of piecewise linear collocation the integrals are more complicated than in case of piecewise constant collocation. However in the Nyström method only diagonal entries of the matrix are integrals. Since the Nyström method has two error factors - singularity and the numerical integration rule error there is no need to use too good integration rule.

## 9. Numerical Solution of <br> Induction Heating Model

In this chapter we will derive numerical model of for solving (1.15). Equation (1.15) is of the form (4.1) with $\lambda=1$, the kernel function

$$
\begin{equation*}
k(x, t)=\frac{-\iota \kappa(x)}{r(x, t)} \tag{9.1}
\end{equation*}
$$

where $r(x, t)$ is the Euclidian distance of $x$ and $t$ and

$$
f(x)=-\iota I_{e x t} F(x) .
$$

Assume that $\Omega_{1}=\left\{\left(x_{1}, x_{2}, x_{3}\right), a \leq x_{1} \leq b, c \leq x_{2} \leq d, e \leq x_{3} \leq f\right\}$ is cuboid.
We will use piecewise constant collocation and Nyström method 2. The reason against piecewise linear collocation (or more precise collocation) is large computing time. The reason against Nyström method 1 is large number of node points as so much memory usage. All was showed in previous chapter.

### 9.1 Collocation method

Let us cover $\Omega_{1}$ by $n_{1}$ sub-cuboides in $x_{1}$ direction, $n_{2}$ sub-cuboides in $x_{2}$ direction and $n_{3}$ sub-cuboides in $x_{3}$ direction. Let

$$
\begin{align*}
h_{1} & =\frac{b-a}{n_{1}}  \tag{9.2}\\
h_{2} & =\frac{d-c}{n_{2}} \tag{9.3}
\end{align*}
$$

and

$$
\begin{equation*}
h_{3}=\frac{f-e}{n_{3}} \tag{9.4}
\end{equation*}
$$

Let us define collocation points

$$
\begin{gather*}
x_{k}=a+k h_{1}-\frac{h_{1}}{2}, k=1, \ldots, n_{1},  \tag{9.5}\\
x_{l}=c+l h_{2}-\frac{h_{2}}{2}, l=1, \ldots, n_{2} \tag{9.6}
\end{gather*}
$$

and

$$
\begin{equation*}
x_{m}=e+m h_{3}-\frac{h_{3}}{2}, m=1, \ldots, n_{3} \tag{9.7}
\end{equation*}
$$

The sub-cuboides are for $k=1, \ldots, n_{1}-1, l=1, \ldots, n_{2}-1, m=1, \ldots, n_{3}-1$ defined by

$$
\begin{gathered}
\Omega_{1, k l m}=\left\{\left(x_{1}^{\prime}, x_{2}^{\prime}, x_{3}^{\prime}\right), x_{k}-\frac{h_{1}}{2} \leq x_{1}^{\prime}<x_{k}+\frac{h_{1}}{2}, x_{l}-\frac{h_{2}}{2} \leq x_{2}^{\prime}<x_{l}+\frac{h_{2}}{2}\right. \\
\left.x_{m}-\frac{h_{3}}{2} \leq x_{3}^{\prime}<x_{m}+\frac{h_{3}}{2}\right\} .
\end{gathered}
$$

For $k=n_{1}, l=n_{2}$ and $m=n_{3}$ the definition is the same as above with " $\leq "$ instead of " $<"$ in corresponding variable. Let $n=n_{1} n_{2} n_{3}$. For $j=1, \ldots, n$ let us define bijection $(k, l, m) \longleftrightarrow j$

$$
\begin{equation*}
j=k+n_{1}(l-1)+n_{1} n_{2}(m-1) \tag{9.8}
\end{equation*}
$$

for simplification of the text. From (5.37) and (5.38) with $y=J_{\text {eddy }, x_{1}}, y_{n}=\widetilde{J}_{n}$, $\widetilde{y}_{i}=J_{i}, \lambda=1, D=\Omega_{1}, D_{i}=\Omega_{1, i}, f(x)=-\iota F(x) I_{\text {ext }}$ and the kernel function $k$ defined as (9.1) we get the piecewise constant collocation $\widetilde{J}_{n}$ of $J_{\text {eddy }, x_{1}}$

$$
\begin{equation*}
\widetilde{J}_{n}(x)=\sum_{i=1}^{n} \chi_{i}(x) J_{i} \tag{9.9}
\end{equation*}
$$

where

$$
\chi_{i}=\chi_{\Omega_{1, i}}
$$

is characteristic function of $\Omega_{1, i}$ and $J_{i}$ is the solution of system of linear equations

$$
\begin{equation*}
J_{i}+\iota \sum_{j=1}^{n} \kappa\left(x_{i}\right) J_{j} \int_{\Omega_{1, j}} \frac{1}{r\left(x_{i}, t\right)} d t_{1} d t_{2} d t_{3}=-\iota I_{e x t} F\left(x_{i}\right), i=1, \ldots, n \tag{9.10}
\end{equation*}
$$

The operator form of (9.10) is

$$
\begin{equation*}
\left(\mathcal{I}-\mathcal{P}_{n} \mathcal{K}\right) J_{n}=\mathcal{P}_{n} \widetilde{F} \tag{9.11}
\end{equation*}
$$

where $\mathcal{K}$ is defined in (3.3), $\widetilde{F}$ is defined by (3.1) and $\mathcal{P}_{n}$ is a projection operator defined as

$$
\begin{equation*}
\mathcal{P}_{n} y(x)=y\left(x_{i}\right), x \in \Omega_{1, i} . \tag{9.12}
\end{equation*}
$$

$J_{i}$ is complex number. Let us define for each $i=1, \ldots, n$

$$
\begin{equation*}
J_{i}^{(R)}=\operatorname{Re} J_{i} \text { and } J_{i}^{(I)}=\operatorname{Im} J_{i} . \tag{9.13}
\end{equation*}
$$

Then (9.10) is equivalent to

$$
\begin{equation*}
J_{i}^{(R)}+\iota J_{i}^{(I)}+\sum_{j=1}^{n} \kappa\left(x_{i}\right)\left[\iota J_{j}^{(R)}-J_{j}^{(I)}\right] \int_{\Omega_{1, j}} \frac{d t_{1} d t_{2} d t_{3}}{r\left(x_{i}, t\right)}=-\iota I_{e x t} F\left(x_{i}\right), i=1, \ldots, n . \tag{9.14}
\end{equation*}
$$

When rewriting (9.14) into two real equations we get system of linear equations for $J_{i}^{(R)}$ and $J_{i}^{(I)}$ :

$$
\begin{align*}
J_{i}^{(R)}-\sum_{j=1}^{n} \kappa\left(x_{i}\right) J_{j}^{(I)} \int_{\Omega_{1, j}} \frac{d t_{1} d t_{2} d t_{3}}{r\left(x_{i}, t\right)}=\operatorname{Im} I_{e x t} F\left(x_{i}\right), i=1, \ldots, n \\
-J_{i}^{(I)}-\sum_{j=1}^{n} \kappa\left(x_{i}\right) J_{j}^{(R)} \int_{\Omega_{1, j}} \frac{d t_{1} d t_{2} d t_{3}}{r\left(x_{i}, t\right)}=\operatorname{Re} I_{e x t} F\left(x_{i}\right), i=1, \ldots, n . \tag{9.15}
\end{align*}
$$

The numerical solution is (9.9) with $J_{i}=J_{i}^{(R)}+\iota J_{i}^{(I)}$. The convergence of the numerical solution is described by following theorem.

Theorem 9.1. Let $\widetilde{J}_{n}$ be the solution of (9.10) and $J_{\text {eddy, } x_{1}}$ be solution of (1.15). Then it holds for all sufficiently large $n \geq N$

$$
\begin{equation*}
\left\|J_{e d d y, x_{1}}-\widetilde{J}_{n}\right\|_{\infty} \leq C_{N} \sup _{r(x, t)<\tau_{n}}\left|J_{e d d y, x_{1}}(x)-J_{e d d y, x_{1}}(t)\right| \tag{9.16}
\end{equation*}
$$

where $C_{N}$ is defined in (5.12) in theorem 5.1 and

$$
\tau_{n}=\max _{i=1, \ldots, n} \max _{x, t \in \Omega_{1, i}} r(x, t) .
$$

Proof. Let us use theorem 5.1. We need to verify (5.11). First let us prove that operator $\mathcal{K}$ defined by (3.3) is compact operator from $L^{\infty}\left(\Omega_{1}\right)$ to $\mathcal{C}\left(\Omega_{1}\right)$. The proof is analog to the proof of theorem 3.3. Let us define operators $\mathcal{K}_{1}$ as (3.10), $\mathcal{M}_{1}$ as (3.11) and $\mathcal{N}$ as (3.12). The operator $\mathcal{K}_{1}$ is of the form (4.3) with

$$
k(x, t)=\frac{1}{r(x, t)} .
$$

If we define approximation

$$
k_{n}(x, t)=\frac{1}{r_{n}(x, t)}
$$

where $r_{n}$ is defined by (3.13) we have by (3.14) that (2.13) holds and by proposition 5.8 that $\mathcal{K}_{1}$ is compact operator from $L^{\infty}\left(\Omega_{1}\right)$ to $\mathcal{C}\left(\Omega_{1}\right)$. It was proved in theorem 3.3 that operators $\mathcal{N}$ and $\mathcal{M}_{1}$ are continuous linear operators from $\mathcal{C}\left(\Omega_{1}\right)$ to $\mathcal{C}\left(\Omega_{1}\right)$. Since $\mathcal{K}=\mathcal{N} \mathcal{M}_{1} \mathcal{K}_{1}$ is $\mathcal{K}$ compact operator from $L^{\infty}\left(\Omega_{1}\right)$ to $\mathcal{C}\left(\Omega_{1}\right)$. By lemma 5.6

$$
\left\|\mathcal{K}-\mathcal{P}_{n} \mathcal{K}\right\| \rightarrow 0 \text { as } n \rightarrow \infty
$$

and (9.16) follows from (5.43) in corollary 5.7, (5.12) and (5.14) in theorem 5.1 with $y=J_{\text {eddy }, x_{1}}, y_{n}=\widetilde{J}_{n}$ and $\lambda=1$.

By the last theorem approximate solution $\widetilde{J}_{n}$ converges to the exact solution $J_{\text {eddy }, x_{1}}$ if the diameter of sub-cuboides goes to zero. Problem to be solved in the system of linear equations (9.15) is singular integral

$$
\begin{equation*}
I=\int_{-\frac{h_{1}}{2}}^{\frac{h_{1}}{2}} \int_{-\frac{h_{2}}{2}}^{\frac{h_{2}}{2}} \int_{-\frac{h_{3}}{2}}^{\frac{h_{3}}{2}} \frac{d x_{3} d x_{2} d x_{1}}{\sqrt{x_{1}^{2}+x_{2}^{2}+x_{3}^{2}}} \tag{9.17}
\end{equation*}
$$

From the figure 9.1 we can see that the integral can be calculated analytically with


Figure 9.1: Mathematica integral
software Mathematica [12] and we don't need to use special numerical integration rules. For the integral we have

$$
\begin{gather*}
I=2 h_{2} h_{3} \ln \left(h_{1}+\sqrt{h_{1}^{2}+h_{2}^{2}+h_{3}^{2}}\right)+2 h_{1} h_{3} \ln \left(h_{2}+\sqrt{h_{1}^{2}+h_{2}^{2}+h_{3}^{2}}\right)+ \\
+2 h_{1} h_{2} \ln \left(h_{3}+\sqrt{h_{1}^{2}+h_{2}^{2}+h_{3}^{2}}\right)-h_{2} h_{3} \ln \left(h_{2}^{2}+h_{3}^{2}\right)- \\
-h_{1} h_{3} \ln \left(h_{1}^{2}+h_{3}^{2}\right)-h_{1} h_{2} \ln \left(h_{1}^{2}+h_{2}^{2}\right)- \\
-h_{2}^{2} \arctan \frac{h_{1} h_{3}}{h_{2} \sqrt{h_{1}^{2}+h_{2}^{2}+h_{3}^{2}}}-h_{1}^{2} \arctan \frac{h_{2} h_{3}}{h_{1} \sqrt{h_{1}^{2}+h_{2}^{2}+h_{3}^{2}}}- \\
-h_{3}^{2} \arctan \frac{h_{1} h_{2}}{h_{3} \sqrt{h_{1}^{2}+h_{2}^{2}+h_{3}^{2}}} . \tag{9.18}
\end{gather*}
$$

### 9.2 Nyström method

Let the numerical integration rule be compound mid-cuboid rule - definition 7.4. First we need to define kernel approximation $k_{n}(x, t)$. Let

$$
\begin{equation*}
\mu_{n}=\sqrt[3]{\frac{\widetilde{\omega}}{n}} \tag{9.19}
\end{equation*}
$$

where

$$
\widetilde{\omega}=(b-a)(d-c)(f-e) .
$$

The kernel function $k(x, t)$ is of the form (6.14) with with $h(x, t)=-\iota \kappa(x)$ and $g(u)=u^{-1}$. By (6.16) with $\mu_{n}$ defined as (9.19) we get

$$
\begin{equation*}
k_{n}(x, t)=\frac{-\iota \kappa(x)}{r_{n}(x, t)} \tag{9.20}
\end{equation*}
$$

where

$$
r_{n}(x, t)=\left\{\begin{array}{l}
r(x, t) \text { if } r(x, t) \geq \mu_{n}  \tag{9.21}\\
\mu_{n} \text { if } r(x, t)<\mu_{n}
\end{array} .\right.
$$

Let $\widetilde{J}_{n}$ be Nyström approximation of $J_{\text {eddy, } x_{1}}$. From (6.7) with $\lambda=1$, $f(x)=-\iota I_{\text {ext }} F(x), \omega_{j}=\omega$, the kernel function $k(x, t)$ defined as (9.1) and the approximation $k_{n}(x, t)$ defined as (9.20) we get

$$
\begin{align*}
& {\left[1-\iota \kappa\left(x_{i}\right) \sum_{j=1, j \neq i}^{n} \frac{\omega}{r_{n}\left(x_{i}, x_{j}\right)}+\iota \kappa\left(x_{i}\right) \int_{\Omega_{1, j}} \frac{1}{r\left(x_{i}, t\right)} d t_{1} d t_{2} d t_{3}\right] \widetilde{J}_{n}\left(x_{i}\right)+} \\
& \quad+\iota \kappa\left(x_{i}\right) \sum_{j=1, j \neq i}^{n} \omega \frac{\widetilde{J}_{n}\left(x_{j}\right)}{r_{n}\left(x_{i}, x_{j}\right)}=-\iota I_{e x t} F\left(x_{i}\right), i=1, \ldots, n . \tag{9.22}
\end{align*}
$$

$\widetilde{J}_{n}\left(x_{i}\right)$ is complex number. Let us define for each $i=1, \ldots, n$

$$
\begin{equation*}
\widetilde{J}_{i}^{(R)}=\operatorname{Re} \widetilde{J}_{n}\left(x_{i}\right) \text { and } \widetilde{J}_{i}^{(I)}=\operatorname{Im} \widetilde{J}_{n}\left(x_{i}\right) . \tag{9.23}
\end{equation*}
$$

Then (9.22) is equivalent to

$$
\begin{aligned}
& {\left[1-\iota \kappa\left(x_{i}\right) \sum_{j=1, j \neq i}^{n} \frac{\omega}{r_{n}\left(x_{i}, x_{j}\right)}+\iota \kappa\left(x_{i}\right) \int_{\Omega_{1, j}} \frac{1}{r\left(x_{i}, t\right)} d t_{1} d t_{2} d t_{3}\right]\left(\widetilde{J}_{i}^{(R)}+\iota \widetilde{J}_{i}^{(I)}\right)+} \\
& \quad+\iota \kappa\left(x_{i}\right) \sum_{j=1, j \neq i}^{n} \omega \frac{\widetilde{J}_{j}^{(R)}+\iota \widetilde{J}_{j}^{(I)}}{r_{n}\left(x_{i}, x_{j}\right)}=-\iota I_{e x t} F\left(x_{i}\right), i=1, \ldots, n .
\end{aligned}
$$

When rewriting last equation into two real equations we get system of linear equations for $\widetilde{J}_{i}^{(R)}$ and $\widetilde{J}_{i}^{(I)}$

$$
\begin{align*}
& \widetilde{J}_{i}^{(R)}+\kappa\left(x_{i}\right)\left[\sum_{j=1, j \neq i}^{n} \frac{\omega}{r_{n}\left(x_{i}, x_{j}\right)}-\int_{\Omega_{1}} \frac{1}{r\left(x_{i}, t\right)} d t_{1} d t_{2} d t_{3}\right] \widetilde{J}_{i}^{(I)}- \\
& -\kappa\left(x_{i}\right) \sum_{j=1, j \neq i}^{n} \omega \frac{\widetilde{J}_{j}^{(I)}}{r_{n}\left(x_{i}, x_{j}\right)}=\operatorname{Im} I_{e x t} F\left(x_{i}\right), i=1, \ldots, n \\
& -\widetilde{J}_{i}^{(I)}+\kappa\left(x_{i}\right)\left[\sum_{j=1, j \neq i}^{n} \frac{\omega}{r_{n}\left(x_{i}, x_{j}\right)}-\int_{\Omega_{1}} \frac{1}{r\left(x_{i}, t\right)} d t_{1} d t_{2} d t_{3}\right] \widetilde{J}_{i}^{(R)}- \\
& -\kappa\left(x_{i}\right) \sum_{j=1, j \neq i}^{n} \omega \frac{\widetilde{J}_{j}^{(R)}}{r_{n}\left(x_{i}, x_{j}\right)}=\operatorname{Re} I_{e x t} F\left(x_{i}\right), i=1, \ldots, n . \tag{9.24}
\end{align*}
$$

From the interpolation formula (6.8) we get the numerical solution $\widetilde{J}_{n}$

$$
\begin{equation*}
\widetilde{J}_{n}(x)=\frac{-\iota I_{e x t} F(x)-\iota \kappa(x) \sum_{j=1}^{n} \frac{\omega}{r_{n}\left(x, x_{j}\right)} \widetilde{J}_{n}\left(x_{j}\right)}{1-\iota \kappa(x) \sum_{j=1}^{n} \frac{\omega}{r_{n}\left(x, x_{j}\right)}+\iota \kappa(x) \int_{\Omega_{1}} \frac{1}{r(x, t)} d t}, \tag{9.25}
\end{equation*}
$$

where

$$
\widetilde{J}_{n}\left(x_{i}\right)=\widetilde{J}_{i}^{(R)}+\iota \widetilde{J}_{i}^{(I)} .
$$

The convergence is described by following theorem
Theorem 9.2. Let operator $\mathcal{K}$ be defined as in (4.3), where the kernel function $k$ is of the form (6.14) with with $h(x, t)=-\iota \kappa(x)$ and $g(u)=u^{-1}, k_{n}$ as (9.20), operator $\mathcal{K}_{n}$ as (6.35). Let $J_{\text {eddy }, x_{1}}$ be the solution of (1.15). Let the numerical integration rule be compound mid-cuboid rule. Then there exist $N$ and $c_{N}<\infty$ such that for the solution of Nyström method $2 \widetilde{J}_{n}$ it holds

$$
\begin{equation*}
\left\|J_{e d d y, x_{1}}-\widetilde{J}_{n}\right\|_{\infty} \leq c_{N}\left\|\mathcal{K} J_{e d d y, x_{1}}-\widetilde{\mathcal{K}}_{n} J_{e d d y, x_{1}}\right\|_{\infty} \text { when } n \geq N . \tag{9.26}
\end{equation*}
$$

Proof. From the definition of (9.19) is satisfied (6.15). Function $\kappa$ was assumed to be bounded and hence $|-\iota \kappa(x)|$ is bounded and (6.22) is satisfied. The compound mid-cuboid integration rule converges for all continuous function (see [10]) and (6.23) is satisfied. Since $\bar{\omega}_{n}=\omega_{j}=\widetilde{\omega} / n>0$ for all $j=1, \ldots, n$ is (6.24) also satisfied. Condition (6.26) is satisfied with $\rho=1$. Since

$$
g\left(\mu_{n}\right) \bar{\omega}_{n}=\frac{g\left(\mu_{n}\right) \widetilde{\omega}}{n}=\frac{\widetilde{\omega}^{\frac{2}{3}}}{n^{\frac{2}{3}}} \leq \widetilde{\omega}^{\frac{2}{3}} \text { for all } n
$$

is (6.27) satisfied. From (3.5) in lemma 3.1 follow (6.28) and (6.29). From theorem 7.10 is (6.30) satisfied and by theorem 6.15 (9.26) follows.

The convergence of $\left\|\mathcal{K} J_{e d d y, x_{1}}-\widetilde{\mathcal{K}}_{n} J_{\text {eddy, } x_{1}}\right\|_{\infty}$ is done by corollary 6.14. Hence we have that $\widetilde{J}_{n}$ converges to the exact solution $J_{\text {eddy, }} x_{1}$. Also in (9.24) is problem with computing singular integral. It can be done analytically or with software Mathematica [12].

### 9.3 Example of induction heating

A brass cuboid body with the measures $0,15 \times 0,01 \times 0,01 \mathrm{~m}$ (see figure 9.2 ) is heated with a stationary inductor starting at the room temperature $20^{\circ} \mathrm{C}$. The inductor has the form of a coil which turns around the heated body in the $x_{1}$-direction in 6 loops. Radius of the coil is $0,015 \mathrm{~m}$, exciting current 500 A , frequency 150 kHz . The length of the coil is $0,15 \mathrm{~m}$. The cuboid is partitioned


Figure 9.2: Heating of a brass body, 6 loops, visualization by Maple [13]
by 75 elements in $x_{1}$ direction, 5 elements in $x_{2}$ and $x_{3}$ direction. Figures 9.3 and 9.4 show the specific Joule losses distribution calculated by piecewise constant collocation and Nyström method with compound mid-cuboid integration rule on the $x_{1}$ axes with $x_{2}=-0.004$ where the blue color matches $x_{3}=-0.004$, the red color matches $x_{3}=0$ and the black color matches $x_{3}=0.004$.

Computation was made by script heatCol (attachment no. 2) and heatNyst (attachment no. 3) in Matlab [14]. The reason for choosing Matlab is that Maple has problems with solution of large matrixes. The integration for computing $F\left(x_{i}\right)$ is done by Matlab method quad. It uses adaptive Simpson quadrature. The computation of non-singular integrals of the matrix of collocation method is used simp3D [15]. It uses compound product Simpson integration rule. Matlab has tool triplequad for computing triple integrals, but it is too slow.

The same situation with $x_{2}=0$ show figures 9.5 resp. 9.6 and the same situation with $x_{2}=0.004$ show figures 9.7 resp. 9.8.


Figure 9.3: Brass body, piecewise constant collocation, 6 loops, $x_{2}=-0.004$


Figure 9.4: Brass body, Nyström method, 6 loops, $x_{2}=-0.004$


Figure 9.5: Brass body, piecewise constant collocation, 6 loops, $x_{2}=0$


Figure 9.6: Brass body, Nyström method, 6 loops, $x_{2}=0$


Figure 9.7: Brass body, piecewise constant collocation, 6 loops, $x_{2}=0.004$


Figure 9.8: Brass body, Nyström method, 6 loops, $x_{2}=0.004$

From the pictures we can see that both methods have similar results. Also the dependence of distance of the point in the body from the coil is well described by the pictures above. If we compare computing time on operating system Windows 7 with Inter Core i5, 4GB ram we get approximately 2 minutes for the Nyström method and approximately 12 minutes for piecewise constant collocation. To see the dependence of Joule looses and coil structure let us make another example with one change. The coil turns around the heated body in the $x_{1}$-direction in 3 loops.


Figure 9.9: Heating of a brass body, 3 loops, visualization by Maple [13]

Figures 9.10 and 9.11 show the specific Joule losses distribution calculated by piecewise constant collocation and Nyström method with compound midcuboid integration rule on the $x_{1}$ axes with $x_{2}=-0.004$ where the blue color matches $x_{3}=-0.004$, the red color matches $x_{3}=0$ and the black color matches $x_{3}=0.004$. The same situation with $x_{2}=0$ show figures 9.12 resp. 9.13 and the same situation with $x_{2}=0.004$ show figures 9.14 resp. 9.15. Also here both methods have similar results. All important show following figures.


Figure 9.10: Brass body, piecewise constant collocation, 3 loops, $x_{2}=-0.004$


Figure 9.11: Brass body, Nyström method, 3 loops, $x_{2}=-0.004$


Figure 9.12: Brass body, piecewise constant collocation, 3 loops, $x_{2}=0$


Figure 9.13: Brass body, Nyström method, 3 loops, $x_{2}=0$


Figure 9.14: Brass body, piecewise constant collocation, 3 loops, $x_{2}=0.004$


Figure 9.15: Brass body, Nyström method, 3 loops, $x_{2}=0.004$

## Conclusion

Now let us make comparison of results in the previous chapters. Both methods generate fully populated matrixes. This is disadvantage in case of 2D or 3D problem because of the large memory usage.

The theory of collocation methods have almost no limitations to the domain of integration and kernel function. Their applying to integral equation of the second kind with singular kernel generates two problems. First one is the calculation of singular integral. In one dimensional case it can be done with mathematical software (in chapter 8 it was Maple). In the situation of more dimensions the calculation can be a problem. The second problem is computing time. The more precise collocation generates more time. The reason is simple. Every element of collocation matrix is an integral. In chapter 8 table 8.34 was showed the time difference between piecewise constant and piecewise linear collocation. However from table 8.33 we can see that if the solution has not continuous derivatives the advantage of precise collocation loses.

The Nyström method 1 was showed to converge but it is not usable. From tables 8.11-8.16 we can see that large number of node points is needed even in one dimensional case. The Nyström method 2 has big advantage. As we can see in the table 8.34 it has very short computing time. The reason for it is simple. Only diagonal elements of the matrix are integrals. Other elements are very easy to be calculated. First disadvantage of this method is that the proof of convergence is connected with domain of integration, kernel function and numerical integration rule. The main problem is to satisfy (6.30). In the chapter 7 was showed some integration rules that satisfy it. In the chapter 8 was showed that the integration rule does not need to be enough precise due to singularity of kernel function. Second problem is that the error estimate is very complicated even in one dimensional case as we saw in chapter 8.

In the chapter 9 were piecewise constant collocation and Nyström method 2 applied to three dimensional problem of induction heating. Both methods have similar results and the convergence of them is proved. In case of Nyström method only for cuboid domain with compound mid-cuboid rule but it is not against using it to more general case.

## Appendix

In this chapter several well known theorems and definitions used in the thesis are remembered.
Definition A-1 (Modulus of Continuity). Let $f$ be continuous function in closed and bounded set $D$. Then the modulus of continuity of $f$ on $D$ is defined by

$$
\begin{equation*}
\omega(f, h)=\max _{\substack{x, t \in D \\ r(x, t) \leq h}}|f(x)-f(t)| \tag{A-1}
\end{equation*}
$$

where $r(x, t)$ is Euclidean distance od $x$ and $t$.
Definition A-2 (Hölder continuous function). A real or complex-valued function $y$ defined on $D$ is Hölder continuous function if there are nonnegative real constants $A<\infty$ and $\alpha \in(0,1)$ such that for all $x, t \in D$ it holds

$$
\begin{equation*}
|y(t)-y(x)| \leq A . r(x, t)^{\alpha} . \tag{A-2}
\end{equation*}
$$

If $\alpha=1$ then the function $y(x)$ is called Lipschitz continuous function.
Theorem A-1. Let $f \in \mathcal{C}[a, b]$ and let $g$ be integrable nonnegative function on $[a, b]$. Then there exists $\xi \in[a, b]$ such that

$$
\begin{equation*}
\int_{a}^{b} f(t) g(t) d t=f(\xi) \int_{a}^{b} g(t) d t \tag{A-3}
\end{equation*}
$$

Theorem A-2. Let $f$ be positive non-increasing function on $[0, \infty)$, let $x_{0}<x_{1}<\ldots<x_{n}$. Then for the sequence $\left\{f_{k}=f\left(x_{k}\right)\right\}_{k=n_{0}}^{n}$ it holds

$$
\begin{equation*}
\sum_{i=n_{0}+1}^{n} f_{i} \leq \int_{n_{0}}^{n} f(t) d t \tag{A-4}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{i=n_{0}}^{n} f_{i} \leq f_{n_{0}}+\int_{n_{0}}^{n} f(t) d t \tag{A-5}
\end{equation*}
$$

Note that if the right hand sides of (A-4) and (A-5) are finite the inequalities hold also for $n=\infty$.
Definition A-3 (Partition of interval). A partition of an interval $[a, b]$ is finite set $P=\left\{a=x_{0}<x_{1}<\ldots<x_{n-1}<x_{n}=b\right\}$. Each $\left[x_{i}, x_{i+1}\right]$ is called a subinterval of the partition. The norm of a partition is defined by

$$
\nu(P)=\max _{i=0, \ldots, n-1}\left(x_{i+1}-x_{i}\right)
$$

Definition A-4 (Riemann sums). Let $f$ be bounded function on $[a, b]$ and $P$ is partition of interval $[a, b]$. The upper Riemann sum is defined by

$$
S(f, P)=\sum_{j=1}^{n} \sup _{x \in\left[x_{j-1}, x_{j}\right]} f(x)\left(x_{j}-x_{j-1}\right) .
$$

The lower Riemann sum is defined by

$$
s(f, P)=\sum_{j=1}^{n} \inf _{x \in\left[x_{j-1}, x_{j}\right]} f(x)\left(x_{j}-x_{j-1}\right) .
$$

Definition A-5 (Riemann integral). Let $f$ be bounded a function on $[a, b]$. The upper Riemann integral is defined by

$$
\overline{\int_{a}^{b}} f=\inf \{S(f, P), P \text { is partition of }[a, b]\} .
$$

The lower Riemann integral is defined by

$$
\underline{\int_{a}^{b}} f=\sup \{s(f, P), P \text { is partition of }[a, b]\}
$$

where $P$ is partition of interval $[a, b]$ Function $f$ is Riemann-integrable if

$$
\underline{\int_{a}^{b}} f=\overline{\int_{a}^{b}} f .
$$

Riemann integral of function $f$ is then defined by

$$
(R) \int_{a}^{b} f=\overline{\int_{a}^{b}} f
$$

Theorem A-3 (Fubini's theorem). Suppose that $M \subset \mathbb{R}^{n+k}$ is measurable set. Let $M^{x}=\left\{y \in \mathbb{R}^{k},[x, y] \in M\right\}$ and $M^{y}=\left\{x \in \mathbb{R}^{n},[x, y] \in M\right\}$. Let

$$
\int_{M}|f(x, y)| d x d y<\infty
$$

Then for almost all $x \in \mathbb{R}^{n}$ the integral

$$
\int_{M^{x}} f(x, y) d y
$$

have sense, for almost all $y \in \mathbb{R}^{k}$ integral

$$
\int_{M^{y}} f(x, y) d x
$$

have sense and it holds

$$
\int_{M} f(x, y) d x d y=\int_{\mathbb{R}^{n}}\left(\int_{M^{x}} f(x, y) d y\right) d x=\int_{\mathbb{R}^{k}}\left(\int_{M^{y}} f(x, y) d x\right) d y
$$

Corollary A-4. Suppose that

$$
\int_{A}|g(x)| d x<\infty \text { and } \int_{B}|h(y)| d y<\infty .
$$

If $f(x, y)=g(x) h(y)$ then it holds

$$
\int_{A} g(x) d x \int_{B} h(y) d y=\int_{A \times B} f(x, y) d x d y .
$$

Theorem A-5. Let $G \subset \mathbb{R}^{n}$ be open set. Let $\phi: G \rightarrow \mathbb{R}^{n}$ be one-to-one and continuously differentiable. Let the Jacobian matrix of $\phi J_{\phi}(x) \neq 0$ for all $x \in G$. Let $f$ be a function defined on $\phi(G)$. Then it holds

$$
\int_{\phi(G)} f(x) d x=\int_{G} f(\phi(t))\left|J_{\phi}(t)\right| d t
$$

if any of the integrals above exist.

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## List of Figures

1.1 Heated body and coil. ..... 3
7.1 Point $x$, points $x_{k l}$, points $\bar{x}_{k l}$, and sums $\widetilde{C}_{i}$ ..... 62
7.2 Point $x$, points $x_{k l}$, points $\underline{\mathrm{x}}_{k l}$, and sums $\widetilde{C}_{i}$ ..... 63
9.1 Mathematica integral ..... 116
9.2 Heating of a brass body, 6 loops, visualization by Maple [13] ..... 119
9.3 Brass body, piecewise constant collocation, 6 loops, $x_{2}=-0.004$ ..... 120
9.4 Brass body, Nyström method, 6 loops, $x_{2}=-0.004$ ..... 120
9.5 Brass body, piecewise constant collocation, 6 loops, $x_{2}=0$ ..... 121
9.6 Brass body, Nyström method, 6 loops, $x_{2}=0$ ..... 121
9.7 Brass body, piecewise constant collocation, 6 loops, $x_{2}=0.004$ ..... 122
9.8 Brass body, Nyström method, 6 loops, $x_{2}=0.004$ ..... 122
9.9 Heating of a brass body, 3 loops, visualization by Maple [13] ..... 123
9.10 Brass body, piecewise constant collocation, 3 loops, $x_{2}=-0.004$ ..... 124
9.11 Brass body, Nyström method, 3 loops, $x_{2}=-0.004$ ..... 124
9.12 Brass body, piecewise constant collocation, 3 loops, $x_{2}=0$ ..... 125
9.13 Brass body, Nyström method, 3 loops, $x_{2}=0$ ..... 125
9.14 Brass body, piecewise constant collocation, 3 loops, $x_{2}=0.004$ ..... 126
9.15 Brass body, Nyström method, 3 loops, $x_{2}=0.004$ ..... 126

## List of Tables

8.1 Piecewise constant collocation, exact solution $x^{2}, 5$ approximation points ..... 90
8.2 Piecewise constant collocation, exact solution $x^{2}, 10$ approximation points ..... 90
8.3 Piecewise constant collocation, exact solution $x^{2}$, error development ..... 91
8.4 Piecewise constant collocation, exact solution $e^{x}$, error development ..... 91
8.5 Piecewise constant collocation, exact solution $\sqrt{x}$, error development ..... 91
8.6 Piecewise linear collocation, exact solution $x^{2}, 5$ approximation points ..... 95
8.7 Piecewise linear collocation, exact solution $x^{2}, 10$ approximation points ..... 95
8.8 Piecewise linear collocation, exact solution $x^{2}$, error development ..... 96
8.9 Piecewise linear collocation, exact solution $e^{x}$, error development ..... 96
8.10 Piecewise linear collocation, exact solution $\sqrt{x}$, error development ..... 96
8.11 Nyström method 1, exact solution $x, 5$ node points ..... 102
8.12 Nyström method 1, exact solution $x, 10$ node points ..... 102
8.13 Nyström method 1, exact solution $x$, error development ..... 102
8.14 Nyström method 1, exact solution $\sqrt{x}, 5$ node points ..... 103
8.15 Nyström method 1, exact solution $\sqrt{x}, 10$ node points ..... 103
8.16 Nyström method 1 , exact solution $\sqrt{x}$, error development ..... 103
8.17 Nyström method 2, exact solution $x, 5$ node points ..... 108
8.18 Nyström method 2, exact solution $x, 10$ node points ..... 108
8.19 Nyström method 2, exact solution $x$, error development ..... 108
8.20 Nyström method 2, exact solution $x^{2}, 5$ node points ..... 109
8.21 Nyström method 2, exact solution $x^{2}, 10$ node points ..... 109
8.22 Nyström method 2, exact solution $x^{2}$, error development ..... 109
8.23 Nyström method 2, exact solution $e^{x}$, error development ..... 109
8.24 Nyström method 2, exact solution $\sqrt{x}$, error development ..... 110
8.25 Nyström method 2, exact solution $\sqrt[4]{x}$, error development ..... 110
8.26 Nyström method 2, Simpson rule, exact solution $x$, error development ..... 110
8.27 Nyström method 2, Simpson rule, exact solution $x^{2}$, error development ..... 111
8.28 Nyström method 2, Simpson rule, exact solution $e^{x}$, error development ..... 111
8.29 Nyström method 2, Simpson rule, exact solution $\sqrt{x}$, error development ..... 111
8.30 Nyström method 2, Simpson rule, exact solution $\sqrt[4]{x}$, error development ..... 112
8.31 Exact solution $x^{2}$, error comparison ..... 112
8.32 Exact solution $e^{x}$, error comparison ..... 112
8.33 Exact solution $\sqrt{x}$, error comparison ..... 113
8.34 Exact solution $e^{x}$, computing time (in seconds) ..... 113

## Attachments

Following applications are attached in the CD contained in this thesis.

1. int1D - Maple library for testing the error behavior of collocation and Nyström methods described in chapter 4
2. heatCol - Matlab m-file for computation of the specific Joule losses by piecewise constant collocation
3. heatNyst - Matlab m-file for computation of the specific Joule losses by Nyström method with compound mid-cuboid numerical integration rule
