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DOCTORAL THESIS

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Stochastic evolution equations with multiplicative fractional noise

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In Prague, 14th June 2012 Jana Šnupárková

Contents

Název práce: Stochastické evoluční rovnice s multiplikativním frakcionálním šumem
Autor: Jana Šnupárková
Katedra: Katedra pravděpodobnosti a matematické statistiky
Vedoucí doktorské práce: prof. RNDr. Bohdan Maslowski, DrSc.

Abstrakt: Frakcionální gaussovský šum je formální derivací frakcionálního Brownova pohybu s Hurstovým parametrem $H \in (0, 1)$. Je nalezen explicitní tvar pro řešení stochastických diferenciálních rovnic s multiplikativním frakcionálním gaussovským šumem v separabilním Hilbertově prostoru. Jest studováno asymptotické chování řešení na dlouhých časových intervalech. Dále jsou zkoumány rovnice s nelineární perturbací driftu v případě $H > 1/2$.

Klíčová slova: frakcionální Brownův pohyb, stochastické diferenciální rovnice v Hilbertově prostoru, explicitní tvar pro řešení

Title: Stochastic evolution equations with multiplicative fractional noise
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Abstract: The fractional Gaussian noise is a formal derivative of a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. An explicit formula for a solution to stochastic differential equations with a multiplicative fractional Gaussian noise in a separable Hilbert space is given. The large time behaviour of the solution is studied. In addition, equations of this type with a nonlinear perturbation of a drift part are investigated in the case $H > 1/2$.

Keywords: Fractional Brownian Motion, Stochastic Differential Equations in Hilbert Space, Explicit Formula for Solution

Introduction

Stochastic differential equations in separable Hilbert spaces where the driving process is a fractional Brownian motion, are intensively studied in recent years. Nowadays, stochastic calculus is developed enough to face obstacles which are caused by integrators which are not semimartingales.

Linear and semilinear stochastic equations with an additive fractional Brownian motion with Hurst parameter $H > 1/2$ were studied in [11], [14]. In [27] linear stochastic evolution equations with a multiplicative fractional Brownian motion are considered. In [17] it is shown that under usual dissipativity conditions linear and semilinear equations with an additive fractional noise form random dynamical systems. Weak and mild solutions to semilinear equations with a multiplicative fractional noise are studied in [12].

It is well known that the one-dimensional stochastic bilinear equation

$$\begin{aligned} \mathrm{d}X_t &= A(t)X_t\,\mathrm{d}t + BX_t\,\mathrm{d}W_t, \\ X_0 &= x, \end{aligned} \tag{1}$$

where $A \in \mathcal{C}([0,T])$, $B, x \in \mathbb{R}$ and $\{W_t, t \geq 0\}$ is a standard Wiener process, has a unique solution

$$X_t = \exp\left\{BW_t + \int_0^t A(r)\,\mathrm{d}r - \frac{1}{2}B^2 t\right\} x,\ \ t \in [0,T],$$

called geometric Brownian motion. An analogous formula can be obtained when one considers the equation (1) in a separable Hilbert space V. Linear operators $\{A(t), t \in [0,T]\}$ and B in V are typically densely defined and closed such that B generates a strongly continuous group $\{S_B(u), u \in \mathbb{R}\}$ on V, the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} y = \Big(A(t) - \frac{1}{2} B^2\Big) y, \; y(s) = x,$$

admits the classical solution $U_W = \{U_W(t,s)x, 0 \le s \le t \le T\}$ and the stochastic integral is understood in the Itô sense. The solution to the equation (1) can be found in the form

$$X_t = S_B(W_t) U_W(t,0)x, \; t \in [0,T],$$

(see Chapter 6 in [8]) under the essential assumption that S_B and $A(t)$ are commuting operators. Note that this requirement is already necessary in finite dimensional state space.

Using these ideas an explicit formula for a solution to more general equation

$$\begin{aligned} \mathrm{d}X_t &= A(t) X_t \,\mathrm{d}t + B X_t \,\mathrm{d}B_t^H, \\ X_0 &= x \in V, \end{aligned} \tag{2}$$

is given in [10] where the integrator $\{B_t^H, t \ge 0\}$ is a one-dimensional fractional Brownian motion with Hurst parameter $H > 1/2$. This process is a standard Wiener process for $H = 1/2$. The process defined as

$$X_t = S_B(B_t^H) U(t,0)x, \; t \in [0,T], \tag{3}$$

is the solution mentioned above, where $U = \{U(t,s)x, 0 \le s \le t \le T\}$ is a strongly continuous evolution system on V associated with operators

$$\{A(t) - H t^{2H-1} B^2, t \in [0,T]\} \tag{4}$$

and the stochastic integral is interpreted in the Skorokhod sense. Notice that U_W is a one-parametric system whenever A is independent of t contrary to the system U which is always two-parametric.

The extension of results from [10] to the case $H < 1/2$ is contained in paper [24]. Comparing to the regular case $H > 1/2$ two major obstacles have to be overcome. First, stochastic integration is much more difficult for $H < 1/2$. Secondly, the function $t \mapsto Ht^{2H-1}$ blows up as $t \to 0+$ if $H < 1/2$, so that it is not obvious, whether system (4) still generates an evolution system. Hence, it is not possible to apply the Itô formula to (3) directly and one has to resort to a suitable approximation procedure.

The results about the existence of a solution for the regular case contained in [10] and analogous results for $H < 1/2$ (previously published by the author in [24]) are described in Chapter 2, Theorem 2.4 and Theorem 2.6, respectively.

Chapter 3 is devoted to variants of examples which were originally discussed in [22], [23], [10], [24] and [25].

The second part of the work deals with a nonlinear equation

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + F(t, X_t)\,\mathrm{d}t + BX_t\,\mathrm{d}B_t^H,\ X_0 = x \in V, \tag{5}$$

where $F : [0,T] \times V \to V$ is a measurable function and $H > 1/2$. This problem was suggested by S. Bonaccorsi who studied a similar equation in the Wiener case (cf. [5]).

First define

$$U_Y(t,s)x = S_B(B_t^H - B_s^H)U(t-s,0)x,\ 0 \le s \le t \le T.$$

It is shown in Chapter 4 (Theorem 4.2) that $\{U_Y(t,s)x, s \le t \le T\}$ is a solution to the equation

$$\mathrm{d}Y_t = AY_t\,\mathrm{d}t + BY_t\,\mathrm{d}B_t^H,\ t > s,\ Y_s = x,$$

starting from any time $s \in [0,T]$.

Chapter 5 is devoted to the case when the drift part need not be linear. It may be shown that the equation

$$y(t) = U_Y(t,0)x + \int_0^t U_Y(t,r)F\big(r, y(r)\big)\,\mathrm{d}r,\ t \in [0,T],$$

has a solution $\{X_t, t \in [0,T]\}$ (Theorem 5.1) and try to find out whether this solution satisfies equation (5). This procedure gives a positive answer in the Wiener case as it was shown by S. Bonaccorsi [5]. But in a fractional case, apparently, this is possible only for F independent of a space variable (Theorem 5.4). Nevertheless, one can show that the "correct" equation which $\{X_t, t \in [0,T]\}$ solves, is

$$\begin{aligned} X_t = x &+ \int_0^t AX_r \,\mathrm{d}r + \int_0^t F(r, X_r)\,\mathrm{d}r + \int_0^t BX_r \,\mathrm{d}B_r^H \\ &+ \int_0^t \alpha_H \int_0^T \int_r^t |v-w|^{2H-2} BU_Y(v,r)F_x'(r,X_r)D_w^H X_r \,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}r \end{aligned}$$

(Theorem 5.8).

Chapter 1

Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A stochastic process $B^H = \{B_t^H, t \in [0,T]\}$ is said to be a **fractional Brownian motion** with Hurst parameter $H \in (0,1)$ if it is a real-valued centered Gaussian process with the covariance function given by

$$\mathbb{E}\,[B_t^H B_s^H] = \frac{1}{2}\big(s^{2H} + t^{2H} - |s-t|^{2H}\big),\ \ s,t \geq 0.$$

Note that

$$\mathbb{E}\left[(B_t^H - B_s^H)^2\right] = |t-s|^{2H},\ \ t,s \geq 0,$$

thus by the Kolmogorov continuity criterion there exists a version of B^H with Hölder continuous trajectories of order δ, $\delta < H$.
Define

$$K_H(t,s) = \frac{C_H}{H-\frac{1}{2}}\left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - \left(H-\frac{1}{2}\right)s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}\,du\right] I_{\{s<t\}},$$

where

$$C_H = \begin{cases} \sqrt{\frac{H(2H-1)}{\mathrm{B}\left(2-2H,\,H-\frac{1}{2}\right)}} \quad , & H > \frac{1}{2}, \\ \left(H-\frac{1}{2}\right)\sqrt{\frac{2H}{(1-2H)\mathrm{B}\left(1-2H,\,H+\frac{1}{2}\right)}} \quad , & H < \frac{1}{2}, \end{cases}$$

and $\mathrm{B}(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\,du,\ a>0, b>0$, denotes Beta function. The process B^H has an integral representation (see e.g. [9])

$$B_t^H = \int_0^t K_H(t,s)\,\mathrm{d}W_s,\ t \geq 0, \tag{1.1}$$

where $W = \{W_t, t \geq 0\}$ is a Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$.

Denote by $\mathcal{E}$ the set of real-valued step functions on the interval $[0,T]$, i.e. each $\varphi \in \mathcal{E}$ has a form

$$\varphi = \sum_{k=0}^{N-1} a_k\, I_{(t_k,t_{k+1}]}, \tag{1.2}$$

for some $N \in \mathbb{N}$, $0 = t_0 < t_1 < \ldots < t_N = T$, $a_k \in \mathbb{R}$, $k = 0,\ldots,N$. The integral of a function $\varphi \in \mathcal{E}$ of the form (1.2) with respect to a fractional Brownian motion is defined as

$$I(\varphi) \equiv \int_0^T \varphi(s)\,dB_s^H := \sum_{k=0}^{N-1} a_k \big(B^H(t_{k+1}) - B^H(t_k)\big).$$

Define a linear operator $\mathcal{K}_H^* : \mathcal{E} \to L^2\big([0,T]\big)$ by

$$\big(\mathcal{K}_H^*\varphi\big)(t) := K_H(T,t)\,\varphi(T) - \int_t^T \big(\varphi(s) - \varphi(t)\big)\frac{\partial K_H}{\partial s}(s,t)\,\mathrm{d}s,\ \varphi \in \mathcal{E}, t \in [0,T].$$

It follows (see [1])

$$\mathbb{E}\left\langle \int_0^T \varphi(s)\,\mathrm{d}B_s^H, \int_0^T \psi(s)\,\mathrm{d}B_s^H \right\rangle_{\mathbb{R}} = \big\langle \mathcal{K}_H^*(\varphi), \mathcal{K}_H^*(\psi)\big\rangle_{L^2([0,T])} =: \langle \varphi, \psi \rangle_{\mathcal{H}}$$

for all $\varphi, \psi \in \mathcal{E}$.

Let $(\mathcal{H}, \langle\,.\,,\,.\,\rangle_{\mathcal{H}})$ be the Hilbert space defined as the completion of $\mathcal{E}$ with respect to the scalar product $\langle\,.\,,\,.\,\rangle_{\mathcal{H}}$. Denote $\|\,.\,\|_{\mathcal{H}}$ the norm in $\mathcal{H}$ associated with $\langle\,.\,,\,.\,\rangle_{\mathcal{H}}$. The operator $\mathcal{K}_H^*$ provides an isometry between spaces $(\mathcal{H}, \|\,.\,\|_{\mathcal{H}})$ and $L^2\big(\Omega\big)$. Since $\mathcal{E}$ is dense in $\mathcal{H}$ the operator I can be uniquely extended on the whole $\mathcal{H}$ (the standard notation $I(\varphi) = B^H(\varphi) = \int_0^T \varphi(r)\,\mathrm{d}B_r^H$ is also used).

The process $W = \{W_t,\ 0 \leq t \leq T\}$ defined by

$$W_t = \int_0^T \big(\mathcal{K}_H^*\big)^{-1}(I_{(0,t]})(s)\,\mathrm{d}B_s^H,\ t \in [0,T],$$

is a Wiener proces and with this choice of W, the representation (1.1) holds (cf. [19]).

1.1 Skorokhod integral for $H > 1/2$

In this section the Skorokhod-type integral with respect to the fractional Brownian motion with Hurst parameter $H > 1/2$ is constructed.

Let $\mathcal{S}$ be a set of smooth cylindrical random variables of the form

$$F = f\big(B^H(\varphi_1), \ldots, B^H(\varphi_n)\big), \tag{1.3}$$

where $n \geq 1$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ (f and all its partial derivatives are bounded) and $\varphi_i \in \mathcal{H}, i = 1, \ldots, n$. The **derivative operator (Malliavin derivative)** of a smooth cylindrical random variable F of the form (1.3) is an $\mathcal{H}$-valued random variable

$$D^H F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(B^H(\varphi_1), \ldots, B^H(\varphi_n))\varphi_i.$$

The derivative operator D^H is closable from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$ for any $p \in [1, +\infty)$. Let $\mathbb{D}_H^{1,p}$ be the Sobolev space obtained as a closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} := \left(\mathbb{E}\left[|F|^p\right] + \mathbb{E}\left[\|D^H F\|_{\mathcal{H}}^p\right]\right)^{1/p}$$

for any $p \in [1, +\infty)$. Similarly, given a Hilbert space $\tilde{V} \subset \mathcal{H}$ denote by $\mathbb{D}_H^{1,p}(\tilde{V})$ the corresponding Sobolev space of $\tilde{V}$-valued random variables.

Definition 1.1 *The* ***divergence operator (Skorokhod integral)*** δ_H : $\operatorname{Dom}\delta_H \to L^2(\Omega)$ *is defined as the adjoint operator of the derivative operator* $D^H : L^2(\Omega) \to L^2(\Omega; \mathcal{H})$, *i.e. for any* $u \in \operatorname{Dom}\delta_H$ *the duality relationship*

$$\mathbb{E}\left[F\delta_H(u)\right] = \mathbb{E}\left[\langle D^H F, u\rangle_{\mathcal{H}}\right]$$

holds for any $F \in \mathbb{D}_H^{1,2}$.

A random variable $u \in L^2(\Omega; \mathcal{H})$ *belongs to the domain* $\operatorname{Dom}\delta_H$ *if there exists a constant* $c_u < +\infty$ *depending only on* u *such that*

$$\left|\mathbb{E}\left[\langle D^H F, u\rangle_{\mathcal{H}}\right]\right| \leq c_u \|F\|_{L^2(\Omega)}$$

for any $F \in \mathcal{S}$.

The useful facts listed below can be found e.g. in [19]. Let $|\mathcal{H}| \subset \mathcal{H}$ be a linear space of measurable functions φ on $[0,T]$ such that

$$\|\varphi\|_{|\mathcal{H}|}^2 = \alpha_H \int_0^T \int_0^T |\varphi(r)||\varphi(s)||r-s|^{2H-2}\,\mathrm{d}r\,\mathrm{d}s < +\infty,$$

where $\alpha_H = H(2H-1)$. Then $\mathcal{E}$ is dense in $|\mathcal{H}|$ and $(|\mathcal{H}|, \|\,.\,\|_{|\mathcal{H}|})$ is a Banach space. Moreover,

$$L^2([0,T]) \subset L^{1/H}([0,T]) \subset |\mathcal{H}| \subset \mathcal{H},$$

thus there exists a constant $K_e < +\infty$ such that

$$\|\mathcal{K}_H^*(\varphi)\|_{L^2([0,T])} = \|\varphi\|_{\mathcal{H}} \leq K_e \|\varphi\|_{L^2([0,T])} \tag{1.4}$$

for any $\varphi \in \mathcal{H}$. Note that

$$\mathbb{D}_H^{1,2}(|\mathcal{H}|) \subset \mathbb{D}_H^{1,2}(\mathcal{H}) \subset \mathrm{Dom}\,\delta_H \tag{1.5}$$

and for some constant $\tilde{C}_{H,2} < +\infty$

$$\mathbb{E}\left[\delta_H^2(u)\right] \leq \tilde{C}_{H,2}\left(\mathbb{E}\left[\|u\|_{|\mathcal{H}|}^2\right] + \mathbb{E}\left[\|D^H u\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^2\right]\right), \; u \in \mathbb{D}_H^{1,2}(|\mathcal{H}|),$$

where $\mathbb{D}_H^{1,p}(|\mathcal{H}|)$ $\big(p \in (1,+\infty)\big)$ contains processes $u \in \mathbb{D}_H^{1,p}(\mathcal{H})$ such that $u \in |\mathcal{H}|$, $D^H u \in |\mathcal{H}| \otimes |\mathcal{H}|$ $\quad \mathbb{P}$- a.s. and

$$\mathbb{E}\left[\|u\|_{|\mathcal{H}|}^p\right] + \mathbb{E}\left[\|D^H u\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^p\right] < +\infty.$$

The normed linear space $\big(|\mathcal{H}|\otimes|\mathcal{H}|, \|\,.\,\|_{|\mathcal{H}|\otimes|\mathcal{H}|}\big)$ is defined in a similar way as $(|\mathcal{H}|, \|\,.\,\|_{|\mathcal{H}|})$ (for a precise definition see e.g. [19]). Hence, for some constant $C_{H,2} < +\infty$

$$\mathbb{E}\big[\delta_H^2(u)\big] \leq C_{H,2}\Big(\mathbb{E}\left[\|u\|_{L^{1/H}([0,T])}^2\right] + \mathbb{E}\left[\|D^H u\|_{L^{1/H}([0,T]^2)}^2\right]\Big), \; u \in \mathbb{D}_H^{1,2}(|\mathcal{H}|). \tag{1.6}$$

Recall that there is a one to one correspondence between fractional Brownian motion B^H and Wiener process W via operator $\mathcal{K}_H^*$. Similar relation is valid for derivative and divergence operators, i.e.

(i) for any $F \in \mathbb{D}_W^{1,2}$

$$\mathcal{K}_H^*(D^H F) = D^W F,$$

where D^W denotes the derivative operator with respect to W and $\mathbb{D}_W^{1,2}$ the corresponding Sobolev space,

(ii) $\operatorname{Dom}\delta_W = \mathcal{K}_H^*(\operatorname{Dom}\delta_H)$ and

$$\delta_H(u) = \delta_W(\mathcal{K}_H^* u) \tag{1.7}$$

for any $u \in \operatorname{Dom}\delta_H$, where δ_W denotes the divergence operator with respect to W.

1.2 Skorokhod integral for $H < 1/2$

In the case $H < 1/2$ the operator δ_H may be constructed in the same way as in the case $H > 1/2$. However, as it is shown in [6], the space $\mathcal{H}$ is too small to include trajectories of the process $\{B_t^H, t \in [0,T]\}$ for $H \leq 1/4$. Hence, it is necessary to extend domain $\operatorname{Dom}\delta_H$ and divergence operator δ_H to processes whose trajectories need not to be in $\mathcal{H}$. The results originaly coming from [6] are described.

Denote by $\mathcal{K}_H^{*,a}$ the adjoint operator of the operator $\mathcal{K}_H^*$ in $L^2([0,T])$ which has a form

$$\big(\mathcal{K}_H^{*,a} f\big)(s) = \frac{C_H}{H - \frac{1}{2}} \Gamma\Big(H + \frac{1}{2}\Big) s^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H} \big((\mathcal{K}_H^* f)_{\frac{1}{2}-H}\big)(s),$$

where $g_{\frac{1}{2}-H}(r) = r^{\frac{1}{2}-H} g(r)$ and

$$D_{0+}^{\frac{1}{2}-H} g(x) = \frac{1}{\Gamma\big(H + \frac{1}{2}\big)} \left(\frac{g(x)}{x^{\frac{1}{2}-H}} + \Big(\frac{1}{2} - H\Big) \int_0^x \frac{g(x) - g(y)}{(x-y)^{-\frac{1}{2}-H}} \,\mathrm{d}y \right).$$

Define the space

$$\mathcal{K} = \big((\mathcal{K}_H^*)^{-1} (\mathcal{K}_H^{*,a})^{-1}\big)(L^2([0,T]))$$

(compare with $\mathcal{H} = (\mathcal{K}_H^*)^{-1}(L^2([0,T]))$). Let $\mathcal{S}_{\mathcal{K}}$ be a set of smooth cylindrical random variables of the form (1.3) where $\varphi_i \in \mathcal{K}, i = 1, \ldots, n$.

Definition 1.2 *A random variable* $u \in L^2\big(\Omega; L^2([0,T])\big)$ *belongs to the extended domain of divergence operator* $\operatorname{Dom}^*\delta_H$ *if there exists a random variable* $\delta_H(u) \in L^2(\Omega)$ *such that*

$$\int_0^T \mathbb{E}\,\Big[u(t)\big(\mathcal{K}_H^{*,a}\mathcal{K}_H^*\big)(D^H F)(t)\Big]\,\mathrm{d}t = \mathbb{E}\,\big[\delta_H(u)F\big]$$

for any $F \in \mathcal{S}_{\mathcal{K}}$.

Note that $\delta_H : \operatorname{Dom}^*\delta_H \to \cup_{p>1} L^p(\Omega)$ is unique determined and linear. Moreover,

(i) $\operatorname{Dom}\delta_H \subset \operatorname{Dom}^*\delta_H$ and extended divergence operator δ_H restricted to $\operatorname{Dom}\delta_H$ coincides with the divergence operator,

(ii) $\operatorname{Dom}\delta_H = \operatorname{Dom}^*\delta_H \cap \big(\cup_{p>1} L^p(\Omega;\mathcal{H})\big)$,

(iii) extended divergence operator δ_H is closed in the following sense. Let $p \in (1,+\infty]$, $q \in \big(2/(1+2H),+\infty\big]$ and $\{u_k, k \in \mathbb{N}\}$ be a sequence in $\operatorname{Dom}^*\delta_H \cap L^p\big(\Omega; L^q([0,T])\big)$, $u \in L^p\big(\Omega; L^q([0,T])\big)$, such that

$$\lim_{k\to+\infty} u_k = u \quad \text{in} \quad L^p\big(\Omega; L^q([0,T])\big).$$

If there exists $\hat{p} \in (1,+\infty]$ and $X \in L^{\hat{p}}(\Omega)$ such that

$$\lim_{k\to+\infty} \delta_H(u_k) = X \quad \text{in} \quad L^{\hat{p}}(\Omega)$$

then $u \in \operatorname{Dom}^*\delta_H$ and $\delta_H(u) = X$.

In what follows the notation

$$\delta_H(u) = \int_0^T u(s)\,\mathrm{d}B_s^H$$

is used more often. The indefinite Skorokhod integral is defined as

$$\int_0^t u(s)\,\mathrm{d}B_s^H = \int_0^T u(s) I_{(0,t]}(s)\,\mathrm{d}B_s^H = \delta_H(uI_{(0,t]}).$$

Notice that the same construction of the integral remains valid for the integrands with values in a separable Hilbert space V.

Chapter 2

Stochastic bilinear equation

The aim of this chaper is to show that the stochastic differential equation

$$\begin{aligned} \mathrm{d}X_t &= A(t)X_t\,\mathrm{d}t + BX_t\,\mathrm{d}B_t^H, \\ X_0 &= x_0, \end{aligned} \tag{2.1}$$

in a separable Hilbert space V on the fixed finite interval $[0,T]$ has a solution in a sense which will be specified later.

The driving process $\{B_t^H, t \geq 0\}$ is a one-dimensional fractional Brownian motion with Hurst parameter $H \in (0,1)$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\{A(t), t \in [0,T]\}$ is the system of linear operators on V satisfying

(A1) for all $t \in [0,T]$ the operators $A(t)$ are closed and densely defined with the domain $D := \mathrm{Dom}(A(t))$ independent of t,

(A2) the resolvent set contains all $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) \geq \omega$ for some fixed $\omega \in \mathbb{R}$ and for some constant $M > 0$ independent of t the resolvent $R(\lambda, A(t))$ satisfies

$$\|R(\lambda, A(t))\|_{\mathcal{L}(V)} \leq \frac{M}{|\lambda - \omega| + 1}$$

for all $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) \geq \omega$, $t \in [0,T]$,

(A3) there exist constants $L > 0$ and $0 < \gamma \leq 1$ such that

$$\|A(t) - A(s)\|_{\mathcal{L}(D;V)} \leq L|t-s|^\gamma, \ \ s,t \in [0,T],$$

where the space D is equipped with the graph norm generated by the operator $A(0) - \omega I$, i.e. $\|x\|_V + \|(A(0) - \omega I)x\|_V$.

The linear operator B on V is

(B2) closed and densely defined and generates a strongly continuous group $\{S_B(u), u \in \mathbb{R}\}$ on V,

and $x_0 \in V$ is a deterministic initial value.
The stochastic integral is understood in the Skorokhod sense (see [6] or Section 1.2).

The conditions (A1), (A2), (A3) imply that the system of operators $\{A(t), t \in [0,T]\}$ generates a strongly continuous evolution system $\{U_A(t,s), 0 \le s \le t \le T\}$ satisfying (see e.g. [26], Theorem 5.2.1.)

$$\mathrm{Im}(U_A(t,s)) \subset D, \tag{2.2}$$

$$\|U_A(t,s)\|_{\mathcal{L}(V)} \le C, \tag{2.3}$$

$$\left\|\frac{\partial}{\partial t}U_A(t,s)\right\|_{\mathcal{L}(V)} = \|A(t)U_A(t,s)\|_{\mathcal{L}(V)} \le \frac{C}{t-s}, \tag{2.4}$$

$$\|A(t)U_A(t,s)(A(s) - \omega I)^{-1}\|_{\mathcal{L}(V)} \le C \tag{2.5}$$

for some constant $C > 0$ and any $0 \le s < t \le T$.
The condition (B2) ensures the existence of constants $M_B \ge 1, \omega_B \ge 0$ such that the inequality

$$\|S_B(u)\|_{\mathcal{L}(V)} \le M_B \exp\{\omega_B |u|\} \tag{2.6}$$

holds for each $u \in \mathbb{R}$.

The description of results contained in [10] ($H > 1/2$) and author's paper [24] ($H < 1/2$) is given now. It is shown that the process $\{X_t, t \in [0,T]\}$ defined as

$$X_t = S_B(B_t^H)U(t,0)x_0, \ 0 \le t \le T,$$

is a solution to the equation (2.1), where $\{U(t,0), 0 \le t \le T\}$ is a system of linear bounded operators on V associated with operators $\{A(t), t \in [0,T]\}$ and B.

2.1 Deterministic system

In this section the construction of the system of operators $\{U(t,0), 0 \leq t \leq T\}$ is outlined, that are contained in a formula for a solution.
In the **case $\boldsymbol{H > 1/2}$** define the operators $\bar{A}(t) : D \to V$ as

$$\bar{A}(t) = A(t) - Ht^{2H-1}B^2$$

for any $t \in [0,T]$. It is possible to show that the system $\{\bar{A}(t), t \in [0,T]\}$ satisfies (under some assumptions) (A1), (A2) and (A3). Therefore it generates a strongly continuous evolution system $\{U(t,s), 0 \leq s \leq t \leq T\}$ on V (e.g. [26]).

If one wants to follow the idea in the **case $\boldsymbol{H < 1/2}$** it is necessary to approximate the operators $\{\bar{A}(t), 0 < t \leq T\}$, because the function $t \mapsto Ht^{2H-1}$ blows up as $t \to 0+$ for $H < 1/2$. Let us define for any $n \in \mathbb{N}$ the system of operators $A_n(t) : D \to V$

$$A_n(t) = A(t) - Hu_n(t)B^2, \ t \in [0,T],$$

where $\{u_n, n \in \mathbb{N}\}$ is a sequence approximating the function

$$u(t) = t^{2H-1}, \ t > 0,$$

defined as

$$u_n(t) = \begin{cases} t^{2H-1} & , \quad t > \frac{1}{n}, \\ \left(\frac{1}{n}\right)^{2H-1} & , \quad 0 \leq t \leq \frac{1}{n}. \end{cases}$$

This sequence $\{u_n, n \in \mathbb{N}\}$ has the following properties

(U1) u_n is Lipschitz continuous on the interval $[0,T]$ for all $n \in \mathbb{N}$,

(U2) u_n converges to u in the space $L^1([0,T])$,

(U3) $0 \leq u_n(t) \leq u(t)$ for any $t > 0$ and $n \in \mathbb{N}$.

Proposition 2.1 *Assume that the conditions* (A1), (A2), (A3) *are satisfied for the system* $\{A(t), t \in [0,T]\}$. *Let* $B : \mathrm{Dom}(B) \to V$ *be a linear densely defined operator such that* B^2 *is closed and*

$$\mathrm{Dom}(B^2) \supset \mathrm{Dom}\big((-A(0))^{\alpha}\big) \tag{2.7}$$

for some $\alpha \in (0,1)$. *Then the conditions* (A1), (A2), (A3) *are satisfied for the system*

***H* > 1/2.** $\{\bar{A}(t), t \in [0,T]\}$ *which generates a strongly continous evolution system* $\{U(t,s), 0 \le s \le t \le T\}$ *on* V,

***H* < 1/2.** $\{A_n(t), t \in [0,T]\}$ *(for any fixed* $n \in \mathbb{N}$*) which generates a strongly continuous evolution system* $\{U_n(t,s), 0 \le s \le t \le T\}$ *on* V.

Proof See [10] for the case $H > 1/2$. The proof in the case $H < 1/2$ published by the author in [24] is given.

The assumption (A3) is equivalent to

$$\big\|\big(A(t) - A(s)\big)A^{-1}(0)\big\|_{\mathcal{L}(V)} \le L|t-s|^{\gamma} \tag{2.8}$$

which implies that there exists a constant $C_0 > 0$ independent of t such that

$$\|A(0)x\|_V \le C_0\|A(t)x\|_V \tag{2.9}$$

for all $t \in [0,T]$ and $x \in D$.

Indeed, (2.8) is equivalent to

$$\big\|A(0)\big(A^{-1}(t) - A^{-1}(s)\big)\big\|_{\mathcal{L}(V)} \le \tilde{L}|t-s|^{\gamma}$$

for some constants $\tilde{L} > 0$ and $0 < \gamma \le 1$ (see [7], p. 32). Hence for $s = 0$

$$\|A(0)A^{-1}(t) - I\|_{\mathcal{L}(V)} \le \tilde{L}T^{\gamma},\ 0 \le t \le T,$$

holds, so that

$$\|A(0)A^{-1}(t)\|_{\mathcal{L}(V)} \le 1 + \tilde{L}T^{\gamma},\ 0 \le t \le T,$$

which is equivalent to (2.9).

Now applying (2.9) and (A2) it follows

$$\|A(0)R(\lambda, A(t))x\|_V \le C_0\|A(t)R(\lambda, A(t))x\|_V \le C_0\big(M(1+\omega)+1\big)\|x\|_V \tag{2.10}$$

for any $x \in V$ and $\lambda \in \mathbb{C},\ \mathrm{Re}(\lambda) \ge \omega$. By the Corollary 2.6.11 from [20] there exists a constant $C_{A(0)} > 0$ depending on $A(0)$ such that for any $\rho > 0$ and $x \in V$

$$\|B^2R(\lambda, A(t))x\|_V \le C_{A(0)}\big[\rho^\alpha\|R(\lambda, A(t))x\|_V + \rho^{\alpha-1}\|A(0)R(\lambda, A(t))\|_V\big].$$

Using (A2) and (2.10)

$$\begin{aligned}\|B^2R(\lambda, A(t))x\|_V \le C_{A(0)}\Big[\rho^\alpha \frac{M}{1+|\lambda-\omega|}\|x\|_V \\ + \rho^{\alpha-1}C_0\big(M(1+\omega)+1\big)\|x\|_V\Big].\end{aligned}$$

Thus

$$\begin{aligned}\|Hu_n(t)B^2R(\lambda, A(t))\|_{\mathcal{L}(V)} \le H\|u_n\|_{\mathcal{C}([0,T])}C_{A(0)}\Big[\rho^\alpha \frac{M}{1+|\lambda-\omega|} \\ + \rho^{\alpha-1}C_0\big(M(1+\omega)+1\big)\Big].\end{aligned}$$

For $\rho > 0$ enough large

$$H\|u_n\|_{\mathcal{C}([0,T])}C_{A(0)}\rho^{\alpha-1}C_0\big(M(1+\omega)+1\big) < \frac{1}{2},$$

hence

$$\|Hu_n(t)B^2R(\lambda, A(t))\|_{\mathcal{L}(V)} \le H\|u_n\|_{\mathcal{C}([0,T])}C_{A(0)}\rho^\alpha \frac{M}{1+|\lambda-\omega|} + \frac{1}{2}.$$

Choosing some $\omega_1 \ge \omega$ such that for all $\lambda \in \mathbb{C},\ \mathrm{Re}(\lambda) \ge \omega_1$ and

$$2H\|u_n\|_{\mathcal{C}([0,T])}C_{A(0)}\rho^\alpha M - 1 + \omega < \mathrm{Re}(\lambda)$$

the inequality

$$\|Hu_n(t)B^2R(\lambda, A(t))\|_{\mathcal{L}(V)} \le K < 1$$

holds for all $t \in [0,T]$, where $K > 0$ is a constant strictly smaller than 1. Therefore

$$\begin{aligned}
\|R(\lambda, A_n(t))\|_{\mathcal{L}(V)} &= \|(\lambda I - A(t) + Hu_n(t)B^2)^{-1}\|_{\mathcal{L}(V)} \\
&= \left\| \left[I(\lambda I - A(t)) + Hu_n(t)B^2 R(\lambda, A(t))(\lambda I - A(t)) \right]^{-1} \right\|_{\mathcal{L}(V)} \\
&= \left\| \left\{ \left[I + Hu_n(t)B^2 R(\lambda, A(t)) \right] (\lambda I - A(t)) \right\}^{-1} \right\|_{\mathcal{L}(V)} \\
&= \left\| R(\lambda, A(t)) \left[I - \left(- Hu_n(t)B^2 R(\lambda, A(t)) \right) \right]^{-1} \right\|_{\mathcal{L}(V)} \\
&\leq \frac{M}{1 + |\lambda - \omega|} \times \frac{1}{1-K} \times \frac{1 + |\lambda - \omega_1|}{1 + |\lambda - \omega_1|} \\
&\leq \frac{M}{1-K} \times \frac{1}{1 + |\lambda - \omega_1|} \\
&\quad \times \left(\frac{1}{1 + |\lambda - \omega|} + \frac{|\lambda - \omega|}{1 + |\lambda - \omega|} + \frac{|\omega - \omega_1|}{1 + |\lambda - \omega|} \right) \\
&\leq \frac{M(2 + |\omega_1 - \omega|)}{1-K} \times \frac{1}{1 + |\lambda - \omega_1|}
\end{aligned}$$

which is (A2) for the system of operators $\{A_n(t), t \in [0,T]\}$.

From (A3) and (U1)

$$\begin{aligned}
\|A(t) - A(s)\|_{\mathcal{L}(D;V)} &\leq L|t-s|^\gamma, \\
|u_n(t) - u_n(s)| &\leq L_u|t-s|^\gamma
\end{aligned}$$

is obtained for some constants $L, L_u > 0$. Note that the norm $\|x\|_V + \|(A_n(t) - \omega_1 I)x\|_V$ is dominated by the norm $\|x\|_D$, so that

$$\begin{aligned}
\|A_n(t) - A_n(s)\|_{\mathcal{L}(D;V)} &\leq \|A(t) - A(s)\|_{\mathcal{L}(D;V)} \\
&\quad + H|u_n(t) - u_n(s)| \|B^2\|_{\mathcal{L}(D;V)} \\
&\leq L|t-s|^\gamma + HL_u|t-s|^\gamma \|B^2\|_{\mathcal{L}(D;V)} \leq L_{A_n}|t-s|^\gamma
\end{aligned}$$

holds for some constant $0 < L_{A_n} < +\infty$ because the operators $B^2 A^{-1}(0) \in \mathcal{L}(V)$ by the closed graph theorem. Hence (A3) is satisfied for the system of operators $\{A_n(t), t \in [0,T]\}$.

Q.E.D.

Remark In the case $H > 1/2$ one can directly assume that $\{\bar{A}(t), t \in [0,T]\}$ generates a strongly continuous evolution system $\{U(t,s), 0 \le s \le t \le T\}$ on V.

Since $\{U_n(t,s), 0 \le s \le t \le T\}$ is a strongly continuous evolution system for any $n \in \mathbb{N}$ it satisfies the equations

$$\frac{\partial}{\partial t} U_n(t,s)x = \big(A(t) - Hu_n(t)B^2\big)U_n(t,s)x, \; U_n(s,s)x = x,$$

and

$$U_n(t,s)x = U_A(t,s)x - \int_s^t Hu_n(r)U_A(t,r)B^2U_n(r,s)x \, \mathrm{d}r$$

for any $x \in V$ and $0 \le s \le t \le T$.

Corollary 2.2 *Consider the case $H < 1/2$. Suppose that the assumptions of Proposition 2.1 are satisfied and for some constants $C_A > 0$, $0 < \beta < 2H$,*

$$\|U_A(t,s)B^2\|_{\mathcal{L}(V)} \le \frac{C_A}{(t-s)^\beta}, \; 0 \le s < t \le T. \tag{2.11}$$

Then for any $x \in V$ there exists a constant $K_U > 0$ depending only on H, A, B and T such that

$$\sup\big\{\|U_n(t,0)x\|_V; \, n \in \mathbb{N}, \, 0 \le t \le T\big\} \le K_U\|x\|_V. \tag{2.12}$$

Moreover, the convergence

$$\|U_n(\,.\,,0)x - U(\,.\,,0)x\|_{\mathcal{C}([0,T];V)} \xrightarrow[n \to +\infty]{} 0 \tag{2.13}$$

holds for any $x \in V$ where $\{U(t,0)x, 0 \le t \le T\}$ is a unique continuous solution to the equation

$$y(t) = U_A(t,0)x - \int_0^t Hr^{2H-1}U_A(t,r)B^2y(r) \, \mathrm{d}r \tag{2.14}$$

on the interval $[0,T]$.

The set $\mathcal{C}([0,T];V)$ denotes the space of all continous functions from $[0,T]$ to V.

Proof The assertion is published in [24] (Proposition 2.2 and 2.3). Fix $x \in V$. For any $n \in \mathbb{N}$ and $t \in [0,T]$ using (2.3), (U3) and (2.11)

$$\begin{aligned}\|U_n(t,0)x\|_V &\le \|U_A(t,0)x\|_V + \left\| \int_0^t Hu_n(r)U_A(t,r)B^2U_n(r,0)x\,\mathrm{d}r \right\|_V \\ &\le C\|x\|_V + HC_A \int_0^t \frac{r^{2H-1}}{(t-r)^\beta}\|U_n(r,0)x\|_V\,\mathrm{d}r\end{aligned}$$

holds. The generalized Gronwall inequality (see [15], Lemma 7.1.2) yields

$$\|U_n(t,0)x\|_V \le K_U\|x\|_V$$

for some finite constant $K_U > 0$ independent of n, t and (2.12) follows. The solution $\{U(t,0)x, 0 \le t \le T\}$ to the equation (2.14) can be obtained by the Banach fixed-point theorem. Define

$$\big(\Phi(y)\big)(t) = U_A(t,0)x - \int_0^t Hr^{2H-1}U_A(t,r)B^2y(r)\,\mathrm{d}r$$

and show that $\Phi : \mathcal{C}([0,T];V) \to \mathcal{C}([0,T];V)$ is a continuous and contraction mapping. Take $y \in \mathcal{C}([0,T];V)$ and $t_1, t_2 \in [0,T], t_1 < t_2$. Then

$$\begin{aligned}\big\|\big(\Phi(y)\big)(t_2) - \big(\Phi(y)\big)(t_1)\big\|_V &\le \|U_A(t_2,0)x - U_A(t_1,0)x\|_V \\ &\quad + \left\|\int_0^{t_2} Hr^{2H-1}U_A(t_2,r)B^2y(r)\,\mathrm{d}r - \int_0^{t_1} Hr^{2H-1}U_A(t_1,r)B^2y(r)\,\mathrm{d}r\right\|_V \\ &\le \|U_A(t_2,0)x - U_A(t_1,0)x\|_V \\ &\quad + \left\|\int_0^{t_1} Hr^{2H-1}\big(U_A(t_2,r) - U_A(t_1,r)\big)B^2y(r)\,\mathrm{d}r\right\|_V \\ &\quad + \left\|\int_{t_1}^{t_2} Hr^{2H-1}U_A(t_2,r)B^2y(r)\,\mathrm{d}r\right\|_V = T_1 + T_2 + T_3.\end{aligned}$$

Since for any fixed $s \in [0,T]$ the function $t \mapsto U_A(t,s)x$ is continuous on the interval $[s,T]$ for any $x \in V$

$$T_1 = \|U_A(t_2,0)x - U_A(t_1,0)x\|_V \longrightarrow 0$$

holds as $t_2 \to t_1+$ or $t_1 \to t_2-$ and for any fixed $0 < r < t_1$

$$\big\|Hr^{2H-1}\big(U_A(t_2,r) - U_A(t_1,r)\big)B^2y(r)\big\|_V \longrightarrow 0$$

as $t_2 \to t_1+$ or $t_1 \to t_2-$. By (2.11)

$$\begin{aligned}
&\Big\| \int_0^{t_1} Hr^{2H-1}\big(U_A(t_2,r) - U_A(t_1,r)\big)B^2 y(r)\,\mathrm{d}r\Big\|_V \\
&\quad \le H\|y\|_{\mathcal{C}([0,T];V)} \int_0^{t_1} r^{2H-1}\big[\|U_A(t_2,r)B^2\|_{\mathcal{L}(V)} + \|U_A(t_1,r)B^2\|_{\mathcal{L}(V)}\big]\,\mathrm{d}r \\
&\quad \le H\|y\|_{\mathcal{C}([0,T];V)} C_A \int_0^{t_1} \left[\frac{r^{2H-1}}{(t_2-r)^\beta} + \frac{r^{2H-1}}{(t_1-r)^\beta}\right]\,\mathrm{d}r \\
&\quad \le 2H\|y\|_{\mathcal{C}([0,T];V)} C_A \int_0^{t_1} \frac{r^{2H-1}}{(t_1-r)^\beta}\,\mathrm{d}r \\
&\quad = 2H\|y\|_{\mathcal{C}([0,T];V)} C_A t_1^{2H-\beta} \int_0^1 r^{2H-1}(1-r)^{-\beta}\,\mathrm{d}r \\
&\quad \le 2H\|y\|_{\mathcal{C}([0,T];V)} C_A T^{2H-\beta}\mathrm{B}(2H,1-\beta) < +\infty,
\end{aligned}$$

hence

$$T_2 = \Big\| \int_0^{t_1} Hr^{2H-1}\big(U_A(t_2,r) - U_A(t_1,r)\big)B^2 y(r)\ \mathrm{d}r\Big\|_V \longrightarrow 0$$

as $t_2 \to t_1+$ or $t_1 \to t_2-$ by the Lebesgue dominated convergence theorem. By (2.11) it follows

$$\begin{aligned}
T_3 &= \Big\| \int_{t_1}^{t_2} Hr^{2H-1} U_A(t_2,r) B^2 y(r)\,\mathrm{d}r\Big\|_V \\
&\le H\|y\|_{\mathcal{C}([0,T];V)} C_A \int_{t_1}^{t_2} \frac{r^{2H-1}}{(t_2-r)^\beta}\,\mathrm{d}r \\
&= H\|y\|_{\mathcal{C}([0,T];V)} C_A t_2^{2H-\beta} \int_{\frac{t_1}{t_2}}^1 r^{2H-1}(1-r)^{-\beta}\,\mathrm{d}r \longrightarrow 0
\end{aligned}$$

as $t_2 \to t_1+$ or $t_1 \to t_2-$. Therefore

$$\big\|\big(\Phi(y)\big)(t_2) - \big(\Phi(y)\big)(t_1)\big\|_V \longrightarrow 0$$

as $t_2 \to t_1+$ or $t_1 \to t_2-$ and the function $t \mapsto \big(\Phi(y)\big)(t)$ is continuous on $[0,T]$ for any $y \in \mathcal{C}([0,T];V)$.

For any $y_1, y_2 \in \mathcal{C}([0,T];V)$, $t \in [0,T]$ and $T > 0$ small enough there exists a constant $0 < L_T < 1$ depending only on A, B, T, H such that

$$\begin{aligned}
&\big\|\big(\Phi(y_2)\big)(t) - \big(\Phi(y_1)\big)(t)\big\|_V = \Big\| \int_0^t Hr^{2H-1} U_A(t,r) B^2\big(y_2(r) - y_1(r)\big)\,\mathrm{d}r\Big\|_V \\
&\quad \le H\|y_1 - y_2\|_{\mathcal{C}([0,T];V)} C_A \int_0^t \frac{r^{2H-1}}{(t-r)^\beta}\,\mathrm{d}r \\
&\quad \le H\|y_1 - y_2\|_{\mathcal{C}([0,T];V)} C_A T^{2H-\beta}\mathrm{B}(2H,1-\beta) \le L_T\|y_1 - y_2\|_{\mathcal{C}([0,T];V)}
\end{aligned}$$

holds and Φ is a contraction. Hence, by the Banach fixed-point theorem there exists a unique solution to the equation (2.14) for T enough small. Applying standard methods a unique continuous solution $\{U(t,0)x, t \in [0,T]\}$ to (2.14) for any $T > 0$ can be obtained.

The convergence (2.13) can be proved in a similar way as (2.12). Using (2.12) and (2.11)

$$\begin{aligned}
&\|U_n(t,0)x - U(t,0)x\|_V \\
&\quad = \Big\| \int_0^t H u_n(r) U_A(t,r) B^2 U_n(r,0)x \,\mathrm{d}r \\
&\qquad - \int_0^t H r^{2H-1} U_A(t,r) B^2 U(r,0)x \,\mathrm{d}r \Big\|_V \\
&\quad \leq \Big\| \int_0^t H \big(u_n(r) - r^{2H-1}\big) U_A(t,r) B^2 U_n(r,0)x \,\mathrm{d}r \Big\|_V \\
&\qquad + \Big\| \int_0^t H r^{2H-1} U_A(t,r) B^2 \big(U_n(r,0)x - U(r,0)x\big) \,\mathrm{d}r \Big\|_V \\
&\quad \leq H C_A K_U \|x\|_V \int_0^t \frac{r^{2H-1} - u_n(r)}{(t-r)^\beta} \,\mathrm{d}r \\
&\qquad + H C_A \int_0^t \frac{r^{2H-1}}{(t-r)^\beta} \|U_n(r,0)x - U(r,0)x\|_V \,\mathrm{d}r
\end{aligned}$$

follows for any $x \in V$ and $t \in [0,T]$. By the definition of $\{u_n, n \in \mathbb{N}\}$ the inequality

$$\int_0^t \frac{r^{2H-1} - u_n(r)}{(t-r)^\beta} \,\mathrm{d}r \leq \left(\frac{1}{n}\right)^{2H-\beta} \mathrm{B}(2H, 1-\beta)$$

is obtained and hence

$$\begin{aligned}
&\|U_n(t,0)x - U(t,0)x\|_V \\
&\quad \leq H C_A K_U \|x\|_V \left(\frac{1}{n}\right)^{2H-\beta} \mathrm{B}(2H, 1-\beta) \\
&\qquad + H C_A \int_0^t \frac{r^{2H-1}}{(t-r)^\beta} \|U_n(r,0)x - U(r,0)x\|_V \,\mathrm{d}r.
\end{aligned}$$

Using again the generalized Gronwall inequality ([15], Lemma 7.1.2)

$$\|U_n(t,0)x - U(t,0)x\|_V \leq H C_A K_U \|x\|_V \mathrm{B}(2H, 1-\beta) \left(\frac{1}{n}\right)^{2H-\beta} K_T,$$

where $0 < K_T < +\infty$ is a constant independent of n, t, therefore

$$\|U_n(\,.\,,0)x - U(\,.\,,0)x\|_{\mathcal{C}([0,T];V)} \xrightarrow[n\to+\infty]{} 0.$$

Q.E.D.

2.2 The concept of solution

In this section the concept of a solution to infinite-dimensional equations with unbounded operators is introduced. As it is usual in an infinite dimension three notions of the solution are given.

Let $A^*(t)$ be the adjoint operator to the operator $A(t)$ for each $t \in [0,T]$. Assume that the domain $\mathrm{Dom}(A^*(t)) = D^*$ of the operator $A^*(t)$ is independent of t. Moreover, suppose that

(B1) $D^* \subset \mathrm{Dom}((B^*)^2)$.

Definition 2.3 *A $\big(\mathcal{B}([0,T])\otimes\mathcal{F}\big)$-measurable stochastic process $\{X_t, t\in[0,T]\}$ is said to be*

(I) *a **strong solution** to the equation* (2.1) *if $X_t \in D$ $\mathbb{P}$-a.s. for all $t \in [0,T]$ and*

$$X_t = x_0 + \int_0^t A(r)X_r\,\mathrm{d}r + \int_0^t BX_r\,\mathrm{d}B^H_r \quad \mathbb{P}\text{-a.s.}$$

for all $t \in [0,T]$,

(II) *a **weak solution** to the equation* (2.1) *if for any $y \in D^*$*

$$\langle X_t, y\rangle_V = \langle x_0, y\rangle_V + \int_0^t \langle X_r, A^*(r)y\rangle_V\,\mathrm{d}r + \int_0^t \langle X_r, B^*y\rangle_V\,\mathrm{d}B^H_r$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$,

(III) *a **mild solution** to the equation* (2.1) *if*

$$X_t = U_A(t,0)x_0 + \int_0^t U_A(t,r)BX_r\,\mathrm{d}B^H_r \quad \mathbb{P}\text{-a.s.}$$

for all $t \in [0,T]$,

where the integrals in (I), (II) *and* (III) *have to be well-defined.*

While in a one-dimensional case the notion of strong and weak solution coincides it may not be true in an infinite-dimensional case.

2.3 Existence of solution

In this section the main results of the work about the existence of a solution to the equation (2.1) are contained.
The following commutativity condition (AB) is essential for the results given below.

(AB) The operators $A(t)$ and $\{S_B(u), u \in \mathbb{R}\}$ commute on the domain D for all $t \in [0, T]$.

2.3.1 Case $H > 1/2$

THEOREM 2.4 *Assume that* $\{A(t), t \in [0, T]\}$ *and* B *are linear operators on* V *satisfying* (A1), (A2), (A3) *and* (B1), (B2). *Moreover, assume* (2.7) *and* (AB). *If*

(i) $\boldsymbol{x_0} \in \boldsymbol{D}$ *then* $\{X_t, t \in [0, T]\}$ *given by*

$$X_t = S_B(B_t^H)U(t,0)x_0, \; 0 \le t \le T, \tag{2.15}$$

is a strong solution to the equation

$$\begin{aligned} \mathrm{d}X_t &= A(t)X_t \,\mathrm{d}t + BX_t \,\mathrm{d}B_t^H, \\ X_0 &= x_0, \end{aligned} \tag{2.16}$$

(ii) $\boldsymbol{x_0} \in \boldsymbol{V}$ *and for some constant* $C_0^* > 0$ *independent of* t

$$\|A^*(t)x\|_V \le C_0^* \|A^*(0)x\|_V, \; t \in [0, T], \tag{2.17}$$

holds for each $x \in D^*$ *then the process* $\{X_t, t \in [0, T]\}$ *is a weak solution to the equation* (2.16).

(iii) $\boldsymbol{x_0} \in \boldsymbol{V}$ *and* $\boldsymbol{B} \in \boldsymbol{\mathcal{L}(V)}$ *then* $\{X_t, t \in [0,T]\}$ *is a mild solution to* (2.16)*.*

Proof See [10] or slight modification of the proof in the case $H < 1/2$.

Q.E.D.

2.3.2 Case $H < 1/2$

This case is more complicated because it is necessary to aproximate the candidate for the solution $\{X_t, t \in [0,T]\}$. It turns out that the appropriate approximating processes $\{X_t^n, t \in [0,T]\}, n \in \mathbb{N}$, are defined as

$$X_t^n = S_B(B_t^H)U_n(t,0)x_0, \ t \in [0,T].$$

Proposition 2.5 *Assume that* $\{A(t), t \in [0,T]\}$ *and* B *are linear operators on* V *satisfying* (A1), (A2), (A3) *and* (B1), (B2)*. Moreover, assume* (2.7), (2.11) *and* (AB)*. If*

(i) $\boldsymbol{x_0} \in \boldsymbol{D}$ *then the process* $\{X_t^n, t \in [0,T]\}$ *is a strong solution to the equation*

$$\begin{aligned} \mathrm{d}X_t^n &= \Big(A(t) + H\big(t^{2H-1} - u_n(t)\big)B^2\Big)X_t^n \,\mathrm{d}t + BX_t^n \,\mathrm{d}B_t^H, \\ X_0^n &= x_0, \end{aligned} \tag{2.18}$$

(ii) $\boldsymbol{x_0} \in \boldsymbol{V}$ *and* (2.17) *then the process* $\{X_t, t \in [0,T]\}$ *is a weak solution to the equation* (2.18)*.*

Proof The proof published by the author in [24] is given for the convenience of the reader. Fix $y \in \mathrm{Dom}((B^*)^2)$. The idea is to apply the one-dimensional Itô formula for a fractional Brownian motion (see [6], Corollary 4.8) to the function

$$f(t,x) := \big\langle S_B(x)U_n(t,0)x_0, y\big\rangle_V = \big\langle U_n(t,0)x_0, S_B^*(x)y\big\rangle_V, \ t \geq 0, x \in \mathbb{R}.$$

Clearly, $f \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$,

$$\begin{aligned}\frac{\partial}{\partial t} f(t,x) &= \big\langle (A(t) - Hu_n(t)B^2)U_n(t,0)x_0, S_B^*(x)y\big\rangle_V,\\ \frac{\partial}{\partial x} f(t,x) &= \big\langle U_n(t,0)x_0, S_B^*(x)B^*y\big\rangle_V,\\ \frac{\partial^2}{\partial x^2} f(t,x) &= \big\langle U_n(t,0)x_0, S_B^*(x)(B^*)^2y\big\rangle_V.\end{aligned}$$

For applicability of the Itô formula it is necessary to check that for some constants $0 < C_f$, $0 < \lambda < 1/4T^{2H}$

$$\max\left\{\left|\frac{\partial}{\partial t} f(t,x)\right|, \left|\frac{\partial^2}{\partial x^2} f(t,x)\right|\right\} \le C_f \mathrm{e}^{\lambda x^2},\ t\in[0,T], x\in\mathbb{R}.$$

Note that for any fixed $b \in \mathbb{R}$ there exists a constant $C_b \ge 0$ such that

$$\exp\{bx\} \le \exp\{C_b + \lambda x^2\},\ x \in \mathbb{R}.$$

By (2.5) for $\{A_n(t), t \in [0,T]\}$ and (2.6)

$$\begin{aligned}\left|\frac{\partial}{\partial t} f(t,x)\right| &= \left|\big\langle (A(t) - Hu_n(t)B^2)U_n(t,0)x_0, S_B^*(x)y\big\rangle_V\right|\\ &\le \Big|\big\langle (A(t) - Hu_n(t)B^2)U_n(t,0)(A(0) - Hu_n(0)B^2)^{-1}(A(0)\\ &\qquad - Hu_n(0)B^2)x_0, S_B^*(x)y\big\rangle_V\Big|\\ &\le C\|(A(0) - Hu_n(0)B^2)x_0\|_V M_B \exp\{\omega_B|x|\}\|y\|_V \le C_f \mathrm{e}^{\lambda x^2}\end{aligned}$$

and by (2.12) and (2.6)

$$\begin{aligned}\left|\frac{\partial^2}{\partial x^2} f(t,x)\right| &= \big\langle U_n(t,0)x_0, S_B^*(x)(B^*)^2y\big\rangle_V\\ &\le \|U_n(t,0)x_0\|_V\|S_B^*(x)(B^*)^2y\|_V\\ &\le K_U\|x_0\|_V M_B\exp\{\omega_B|x|\}\|(B^*)^2y\|_V \le C_f\mathrm{e}^{\lambda x^2}.\end{aligned}$$

Now, Corollary 4.8 from [6] has a form

$$\begin{aligned}f(t,B_t^H) = f(0,B_0^H) &+ \int_0^t \frac{\partial}{\partial r} f(r,B_r^H)\,\mathrm{d}r + \int_0^t \frac{\partial}{\partial x} f(r,B_r^H)\,\mathrm{d}B_r^H\\ &+ \int_0^t Hr^{2H-1}\frac{\partial^2}{\partial x^2} f(r,B_r^H)\,\mathrm{d}r\end{aligned}$$

$\mathbb{P}$- a.s. for all $t \in [0,T]$, so that

$$\langle X_t^n, y\rangle_V = \langle x_0, y\rangle_V + \int_0^t \big\langle (A(r) - Hu_n(r)B^2)U_n(r,0)x_0, S_B^*(B_r^H)y\rangle_V \,\mathrm{d}r$$
$$+ \int_0^t \big\langle BS_B(B_r^H)U_n(r,0)x_0, y\big\rangle_V \,\mathrm{d}B_r^H$$
$$+ \int_0^t \big\langle Hr^{2H-1}B^2S_B(B_r^H)U_n(r,0)x_0, y\big\rangle_V \,\mathrm{d}r \quad \mathbb{P}\text{- a.s.}$$

for all $t \in [0,T]$. Using the commutativity assumption (AB)

$$\langle X_t^n, y\rangle_V = \langle x_0, y\rangle_V + \int_0^t \langle A(r)X_r^n, y\rangle_V \,\mathrm{d}r + \int_0^t \langle BX_r^n, y\rangle_V \,\mathrm{d}B_r^H$$
$$+ \int_0^t \big\langle H\big(r^{2H-1} - u_n(r)\big)B^2X_r^n, y\big\rangle_V \,\mathrm{d}r \quad \mathbb{P}\text{- a.s.}$$

holds for all $t \in [0,T]$ and $y \in \mathrm{Dom}((B^*)^2)$. Taking a countable subset of $\mathrm{Dom}((B^*)^2)$ dense in V it can be obtained that the process $\{X_t^n, t \in [0,T]\}$ is D-valued and it is a strong solution to the equation (2.18).

Let $x_0 \in V$. To prove the second part take a sequence $\{x_k, k \in \mathbb{N}\}$ in D converging to x_0 in V and consider the approximating processes $\{Y_t^k, t \in [0,T]\}, k \in \mathbb{N}$, of the process $\{X_t^n, t \in [0,T]\}$ defined as

$$Y_t^k = S_B(B_t^H)U_n(t,0)x_k.$$

By the previous part of the proof it is known that $\{Y_t^k, t \in [0,T]\}$ is a strong solution to the equation (2.18) with the initial value $Y_0^k = x_k$ and for each $y \in D^*$

$$\langle Y_t^k, y\rangle_V = \langle x_k, y\rangle_V + \int_0^t \langle Y_r^k, A^*(r)y\rangle_V \,\mathrm{d}r + \int_0^t \langle Y_r^k, B^*y\rangle_V \,\mathrm{d}B_r^H \tag{2.19}$$
$$+ \int_0^t \big\langle H\big(r^{2H-1} - u_n(r)\big)Y_r^k, (B^*)^2y\big\rangle_V \,\mathrm{d}r \quad \mathbb{P}\text{- a.s.}$$

for all $t \in [0,T]$.

The aim is to pass to the limit in the equation (2.19) in the space $L^2(\Omega)$ for any fixed $t \in [0,T]$, $y \in D^*$ and to use the closedness of the Skorokhod integral.

By the Fernique theorem (see [13]) it is well-known that

$$\mathbb{E}\Big[\exp\big\{\zeta \sup\{|B_t^H|; t \in [0,T]\}\big\}\Big] < +\infty \tag{2.20}$$

for any constant $\zeta > 0$.

Using (2.6), (2.20) and (2.12)

$$\begin{aligned}
&\mathbb{E}\left|\langle Y_t^k, y\rangle_V - \langle X_t^n, y\rangle_V\right|^2 = \mathbb{E}\left|\langle Y_t^k - X_t^n, y\rangle_V\right|^2 \\
&= \mathbb{E}\left|\left\langle S_B(B_t^H)U_n(t,0)(x_k - x_0), y\right\rangle_V\right|^2 \\
&\leq M_B^2 \mathbb{E}\left[\exp\left\{2\omega_B \sup\{|B_r^H|; r \in [0,T]\}\right\}\right] K_U^2 \|y\|_V^2 \|x_k - x_0\|_V^2 \xrightarrow[k\to+\infty]{} 0,
\end{aligned} \tag{2.21}$$

$$\mathbb{E}\left|\langle x_k, y\rangle_V - \langle x_0, y\rangle_V\right|^2 = \langle x_k - x_0, y\rangle_V^2 \leq \|y\|_V^2 \|x_k - x_0\|_V^2 \xrightarrow[k\to+\infty]{} 0,$$

by (2.17)

$$\begin{aligned}
&\mathbb{E}\left|\int_0^t \left\langle \left(Y_r^k - X_r^n\right), A^*(r)y\right\rangle_V \mathrm{d}r\right|^2 \\
&\quad = \mathbb{E}\left|\int_0^t \left\langle \left(S_B(B_t^H)U_n(t,0)(x_k - x_0)\right), A^*(r)y\right\rangle_V \mathrm{d}r\right|^2 \\
&\quad \leq M_B^2 \mathbb{E}\left[\exp\left\{2\omega_B \sup\{|B_r^H|; r \in [0,T]\}\right\}\right] \\
&\qquad \times K_U^2 T^2 \|x_k - x_0\|_V^2 (C_0^*)^2 \|A^*(0)y\|_V^2 \xrightarrow[k\to+\infty]{} 0,
\end{aligned}$$

and by (U3)

$$\begin{aligned}
&\mathbb{E}\left|\int_0^t \left\langle H(r^{2H-1} - u_n(r))(Y_r^k - X_r^n), (B^*)^2 y\right\rangle_V \mathrm{d}r\right|^2 \\
&\quad = \mathbb{E}\left|\int_0^t \left\langle H(r^{2H-1} - u_n(r)) S_B(B_t^H)U_n(t,0)(x_k - x_0), (B^*)^2 y\right\rangle_V \mathrm{d}r\right|^2 \\
&\quad \leq M_B^2 \mathbb{E}\left[\exp\left\{2\omega_B \sup\{|B_r^H|; r \in [0,T]\}\right\}\right] \\
&\qquad \times K_U^2 T^{4H} \|x_k - x_0\|_V^2 \|(B^*)^2 y\|_V^2 \xrightarrow[k\to+\infty]{} 0.
\end{aligned}$$

Therefore it is possible to pass to the limit in the equation (2.19) in the space $L^2(\Omega)$ and there exists a random variable $Y_t^{(n,y)}$ such that

$$\int_0^t \langle Y_r^k, B^* y\rangle_V \,\mathrm{d}B_r^H \xrightarrow[n\to+\infty]{} Y_t^{(n,y)} \quad \text{in} \quad L^2(\Omega).$$

Analogously to (2.21) it follows

$$\int_0^t \mathbb{E}\left|\langle Y_r^k, B^* y\rangle_V - \langle X_r^n, B^* y\rangle_V\right|^2 \mathrm{d}r \xrightarrow[k\to+\infty]{} 0$$

and

$$\{\langle Y_r^k, B^*y\rangle_V, r\in[0,t]\},\ \{\langle X_r^n, B^*y\rangle_V, r\in[0,t]\}\in L^2\big(\Omega; L^2([0,t])\big)$$

for any $k\in\mathbb{N}$. The Itô formula yields that $\{\langle Y_r^k, B^*y\rangle_V, r\in[0,t]\}$ is Skorokhod integrable with respect to the fractional Brownian motion. Hence by the closedness of the Skorokhod integral the process $\{\langle X_r^n, B^*y\rangle_V, r\in[0,t]\}$ is Skorokhod integrable with respect to the fractional Brownian motion and

$$Y_t^{(n,y)} = \int_0^t \langle X_r^n, B^*y\rangle_V \,\mathrm{d}B_r^H \quad \mathbb{P}\text{-a.s.}$$

(see [6], Remark 3.4.2, or Section 1.2) for any $t\in[0,T]$. Therefore the process $\{X_t^n, t\in[0,T]\}$ is a weak solution to the equation (2.18).

Q.E.D.

In the case $H<1/2$ the singularity of the system of operators $\{A(t)-Ht^{2H-1}B^2, t\in(0,T]\}$ at zero (and hence the lack of information about the system $\{U(t,0), t\in[0,T]\}$) admits only the weak solution.

THEOREM 2.6 *Suppose that the assumptions of Proposition 2.5 hold. Then for each $x_0\in V$ the process $\{X_t, t\in[0,T]\}$ defined by* (2.15) *is a weak solution to the equation* (2.16).

Proof The proof is similar to the last part of the proof of Proposition 2.5. The first step is to pass to the limit in the equation

$$\langle X_t^n, y\rangle_V = \langle x_0, y\rangle_V + \int_0^t \langle X_r^n, A^*(r)y\rangle_V \,\mathrm{d}r + \int_0^t \langle X_r^n, B^*y\rangle_V \,\mathrm{d}B_r^H + \int_0^t \big\langle H\big(r^{2H-1}-u_n(r)\big)X_r^n, (B^*)^2 y\big\rangle_V \,\mathrm{d}r$$

in the space $L^2\big(\Omega\big)$ for any fixed $t\in[0,T]$ and any fixed $y\in D^*$. Let $Y_y(t)$ be the limit

$$\int_0^t \langle X_r^n, B^*y\rangle_V \,\mathrm{d}B_r^H \xrightarrow[n\to+\infty]{} Y_y(t) \quad \text{in} \quad L^2(\Omega).$$

The second step is to identify the limit $Y_y(t)$ as the Skorokhod integral $\int_0^t \langle X_r, B^* y \rangle_V \, \mathrm{d}B_r^H$ applying the closedness of Skorokhod integral. For detailed proof see [24].

Q.E.D.

Chapter 3

Examples

This chapter is focused on examples which can be covered by Theorem 2.4 and Theorem 2.6.

EXAMPLE 3.1 Consider the one-dimensional equation

$$\begin{aligned} \mathrm{d}X_t &= a(t)X_t\mathrm{d}t + bX_t\mathrm{d}B_t^H, \\ X_0 &= x_0, \end{aligned} \tag{3.1}$$

where $a \in \mathcal{C}([0,T])$ is a function, $0 \neq b \in \mathbb{R}$ is a constant and $x_0 \in \mathbb{R}$. Then the process $\{X_t, t \in [0,T]\}$ defined as

$$\begin{aligned} X_t &= \exp\left\{bB_t^H + \int_0^t \left(a(r) - Hb^2r^{2H-1}\right)\mathrm{d}r\right\}x_0 \\ &= \exp\left\{bB_t^H + \int_0^t a(r)\,\mathrm{d}r - \frac{1}{2}b^2t^{2H}\right\}x_0 \end{aligned}$$

is a strong solution to the equation (3.1). In this case the three notions of solution coincide. The results contained in this example are well-known for the Wiener case $H = 1/2$, can be easily obtained from [10] for $H > 1/2$ and are described in the previous author's paper [22] for $H < 1/2$.

Using Law of Iterated Logarithm for a fractional Brownian motion (e.g. [3])

$$\limsup_{t\to+\infty} \frac{B_t^H}{t^H\sqrt{\log\log t}} = C_H \quad \mathbb{P}\text{- a.s.}$$

and

$$\liminf_{t\to+\infty} \frac{B_t^H}{t^H\sqrt{\log\log t}} = -C_H \quad \mathbb{P}\text{- a.s.},$$

where $C_H > 0$ is a constant independent of $\omega \in \Omega$, the limit behaviour of the solution $\{X_t, t \in [0,T]\}$ can be studied.

For the simplicity let $a \equiv a(t), t \in [0,T]$, be independent of t. Then the solution is given by a formula

$$X_t = \exp\left\{bB_t^H + \left(at - \frac{1}{2}b^2t^{2H}\right)\right\}x_0, \ \ t \in [0,T].$$

Without loss of generality assume that $b > 0$ and $x_0 > 0$. Then

$H > \frac{1}{2}$. $\quad X_t \xrightarrow[t\to+\infty]{} 0 \quad \mathbb{P}\text{- a.s.},$

$H < \frac{1}{2}$. $\quad$ if $a > 0$ then $X_t \xrightarrow[t\to+\infty]{} +\infty \quad \mathbb{P}\text{- a.s.},$

if $a \leq 0$ then $X_t \xrightarrow[t\to+\infty]{} 0 \quad \mathbb{P}\text{- a.s.},$

$H = \frac{1}{2}$. $\quad$ if $a < \frac{1}{2}b^2$ then $X_t \xrightarrow[t\to+\infty]{} 0 \quad \mathbb{P}\text{- a.s.},$

if $a > \frac{1}{2}b^2$ then $X_t \xrightarrow[t\to+\infty]{} +\infty \quad \mathbb{P}\text{- a.s.},$

if $a = \frac{1}{2}b^2$ then

$$\limsup_{t\to+\infty} X_t = +\infty \quad \mathbb{P}\text{- a.s.} \quad \text{and} \quad \liminf_{t\to+\infty} X_t = 0 \quad \mathbb{P}\text{- a.s.}$$

△

Example 3.2 Let

$$\begin{aligned} \mathrm{d}X_t &= AX_t\mathrm{d}t + BX_t\mathrm{d}B_t^H, \\ X_0 &= x_0, \end{aligned}$$

be the equation in a separable Hilbert space V with A and B which are supposed to be linear bounded operators on V. This case was studied in the previous author's paper [23] for $H < 1/2$. Nevertheless, this equation can be covered by more general Theorems 2.4 and 2.6. The solution $\{X_t, t \in [0,T]\}$ defined as

$$X_t = S_B(B_t^H)U(t,0)x_0, \ \ t \in [0,T],$$

is strong and weak. Moreover, for $H \geq 1/2$ the solution is also mild.

△

EXAMPLE 3.3 Consider the stochastic equation

$$\begin{aligned} \mathrm{d}X_t &= AX_t\,\mathrm{d}t + bX_t\,\mathrm{d}B_t^H, \\ X_0 &= x_0, \end{aligned}$$

in a separable Hilbert space V, where the linear closed and densely defined operator A generates a strongly continuous semigroup $\{S_A(t), t \in [0,T]\}$ on V and $b \in \mathbb{R}$ is a nonzero deterministic constant. Then there exists a weak solution $\{X_t, t \in [0,T]\}$ (the case $H = 1/2$ is omitted) for any $0 < T < +\infty$ which has a form

$$X_t = \exp\left\{ bB_t^H - \frac{1}{2}b^2 t^{2H} \right\} S_A(t)x_0, \ t \in [0,T].$$

Using Law of Iterated Logarithm similar to the Example 3.1 yields

$$\lim_{t\to+\infty} \|X_t\|_V = 0 \quad \mathbb{P}\text{ - a.s.}$$

in the case $H > 1/2$ with the rate of convergence which is faster than exponential, namely there exist constants $C_1(\omega)$ depending on $\omega \in \Omega$ and $c_2 > 0$ such that

$$\|X(t,\omega)\|_V \leq C_1(\omega)\exp\{-c_2 t^{2H}\}\|x_0\|_V, \ t \to \infty,$$

because $2H > 1$.

The case $H < 1/2$ is more complicated. Under the above assumptions nothing can be said about limit behaviour of the solution. Some more restrictive conditions on $\{S_A(t), t \in [0,T]\}$ have to be given. For instance, suppose that $\{S_A(t), t \in [0,T]\}$ is an exponentially stable semigroup, i.e. for some constants $M \geq 1$, $\omega > 0$, the inequality

$$\|S_A(t)\|_{\mathcal{L}(V)} \leq M\mathrm{e}^{-\omega t}$$

holds for each $t \in (0, +\infty)$. Then $\|X_t\|_V \to 0$ as $t \to +\infty$ for $\mathbb{P}$- almost all $\omega \in \Omega$ and the solution is pathwise exponentially stable.

An example of this situation is a Dirichlet problem for the stochastic fractional heat equation, i.e. the equation described formally as

$$\frac{\partial y}{\partial t}(t,x) = \Delta y(t,x) + by(t,x)\frac{\mathrm{d}B_t^H}{\mathrm{d}t},\ (t,x) \in \mathbb{R}^+ \times \mathcal{O},$$
$$y(t,x) = 0,\ (t,x) \in \mathbb{R}^+ \times \partial\mathcal{O},$$

where Δ denotes the Laplace operator and $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain with smooth boundary. Set $A := \Delta$ with the domain $\mathrm{Dom}(A) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$. Then the strongly continuous semigroup $\{S_A(t), t \in [0,T]\}$ associated with A is exponentially stable.

The Neumann problem for the heat equation, i.e.

$$\frac{\partial y}{\partial t}(t,x) = \Delta y(t,x) + by(t,x)\frac{\mathrm{d}B_t^H}{\mathrm{d}t},\ (t,x) \in \mathbb{R}^+ \times \mathcal{O},$$
$$\frac{\partial y}{\partial \nu}(t,x) = 0,\ (t,x) \in \mathbb{R}^+ \times \partial\mathcal{O},$$

is more interesting. Consider the Laplace operator $\Delta = A$ with the domain $\mathrm{Dom}(A) = \{\varphi \in H^2(\mathcal{O}); \frac{\partial \varphi}{\partial \nu} = 0\}$, where $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain with smooth boundary and ν is a normal vector to the boundary $\partial\mathcal{O}$. Then the strongly continuous semigroup $\{S_A(t), t \in [0,T]\}$ associated with operator A is a contraction, i.e.

$$\|S_A(t)\|_{\mathcal{L}(V)} \leq 1,\ t \in [0,T].$$

Therefore the stability of the solution with the rate of convergence which is slower than exponential is obtained, i.e. there exist constants $C_1 \equiv C_1(\omega)$ depending on $\omega \in \Omega$ and $c_2 > 0$ such that

$$\|X(t)\|_V \leq C_1 \exp\{-c_2 t^{2H}\}\|x_0\|_V,\ t \to \infty,$$

because $2H < 1$.

Moreover, for initial values x_0 which are constant functions on $\mathcal{O}$ more can be said. Since constant functions are eigenfunctions of the operator

$S_A(t)$ for any t it is obtained using Law of Iterated Logarithm that there exists a constant $c > 0$ depending on $\omega \in \Omega$ such that

$$\begin{aligned}\|X_t\|_V &= \exp\left\{bB_t^H - \frac{1}{2}b^2t^{2H}\right\}\|x_0\|_V \\ &\geq \exp\{-bct^H\sqrt{\log\log t} - \frac{1}{2}b^2t^{2H}\}\|x_0\|_V\end{aligned}$$

as $t \to +\infty$, hence there exist constants $C_3 \equiv C_3(\omega)$ depending on $\omega \in \Omega$ and $c_4 > 0$ such that

$$\|X_t\|_V \geq C_3 \exp\{-c_4t^{2H}\}\|x_0\|_V,\ t \to +\infty,$$

so that the convergence rate is sharp.

This example was originally described in [25].

△

Example 3.4 Consider the stochastic parabolic equation of the second order

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) &= L(t,x)u + bu(t,x)\frac{\mathrm{d}B^H}{\mathrm{d}t}, \qquad (3.2)\\ u(0,x) &= x_0(x),\ x \in \mathcal{O}\\ u(t,x) &= 0,\ (t,x) \in [0,T] \times \partial\mathcal{O},\end{aligned}$$

where $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain with the boundary of class $\mathcal{C}^2$, $b \in \mathbb{R} \setminus \{0\}$ and

$$L(t,x)u = a_0(t,x)u(t,x) + \sum_{i=1}^d a_i(t,x)\frac{\partial u}{\partial x_i}(t,x) + \sum_{i,j=1}^d a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x)$$

is a uniformly strongly elliptic operator on $\mathcal{O}$, i.e. there exists a constant $\vartheta > 0$ such that

$$\sum_{i,j=1}^d a_{ij}(t,x)\zeta_i\zeta_j > \vartheta\|\zeta\|_{\mathbb{R}^d}^2$$

for all $(t,x) \in [0,T] \times \bar{\mathcal{O}}$ and $0 \neq \zeta = (\zeta_1,\dots,\zeta_d) \in \mathbb{R}^d$.

Suppose that the functions $a_0(t,\,.\,), a_i(t,\,.\,), a_{ij}(t,\,.\,) \in \mathcal{C}^\infty(\bar{\mathcal{O}})$ and for some constants $M > 0$, $0 < \gamma < 1$

$$\sup_{x\in\mathcal{O}}\{|a_0(t,x)-a_0(s,x)|, |a_i(t,x)-a_i(s,x)|, |a_{ij}(t,x)-a_{ij}(s,x)|\} \leq M|t-s|^\gamma$$

holds for any $s,t \in [0,T]$, $i,j = 1,\dots,d$.

Equation (3.2) can be rewritten in the form (2.1), where $V = L^2(\mathcal{O})$,

$$\big(A(t)u\big)(x) = L(t,x)u,$$

with $\mathrm{Dom}(A(t)) = D = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$ and $B = bI \in \mathcal{L}(V)$.

In this case the assumptions (A1), (A2), (A3) are satisfied (cf. Theorem 3.8.3, [26]). The adjoint operator $A^*(t)$ has the same form as the operator $A(t)$ only with other coefficients. Hence $\mathrm{Dom}(A^*(t)) = D^* = D = \mathrm{Dom}(A(t))$ is independent of t. Also conditions (B1), (B2), (2.11), (AB) and (2.7) are trivially satisfied. In the case $H > 1/2$ using Theorem 2.4 it can be concluded that there exists a mild solution to the equation (3.2) for any $x_0 \in V$ and if $x_0 \in D$ then the solution is also strong. Moreover, if it is assumed (2.17) the solution is weak for any $x_0 \in V$ in both cases $H > 1/2$ and $H < 1/2$ applying Theorem 2.4 and Theorem 2.6, respectively.

△

EXAMPLE 3.5 Consider the equation

$$\frac{\partial u}{\partial t}(t,x) = -\frac{\partial^4 u}{\partial x^4}(t,x) - \alpha u(t,x) + \frac{\partial u}{\partial x}(t,x)\frac{\mathrm{d}B^H}{\mathrm{d}t}, \qquad (3.3)$$
$$u(0,x) = x_0(x),$$

in the weighted space $V = L^2_\rho(\mathbb{R})$ with the weight $\mathrm{e}^{-\rho|x|}$, $x \in \mathbb{R}$, and some fixed positive constant ρ, where $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$. The operator

$$A = -\frac{\partial^4}{\partial x^4} - \alpha I$$

defined on the domain $D = \mathrm{Dom}(A) = W^{4,2}(\mathbb{R})$ generates a strongly continuous semigroup $\{S_A(t), t \in [0,T]\}$ on V which is exponentially stable for any fixed $\alpha > 0$ (see e.g. [21]). The operator $B = \frac{\partial}{\partial x}$ with the domain $\mathrm{Dom}(B) = W^{1,2}(\mathbb{R})$ generates a strongly continuous group $\{S_B(t), t \in \mathbb{R}\}$ on V which is a shift operator

$$\big(S_B(t)u\big)(x) = u(t+x),\ \ t,x \in \mathbb{R}.$$

Moreover, $D = D^* = \mathrm{Dom}(A^*)$, $\mathrm{Dom}(B^2) = \mathrm{Dom}((B^*)^2) = W^{2,2}(\mathbb{R})$ and $S_B(t)$ commute with A on D for each $t \in [0,T]$.

$H > 1/2$. The operators

$$\left\{\bar{A}(t) = -\frac{\partial^4}{\partial x^4} - \alpha I - Ht^{2H-1}\frac{\partial^2}{\partial x^2},\ t \in [0,T]\right\}$$

are strongly elliptic and generate a strongly continuous evolution system $\{U(t,s), 0 \leq s \leq t \leq T\}$. Thus there exists a strong (if $x_0 \in D$) and weak (if $x_0 \in V$) solution to the equation (3.3).

$H < 1/2$. The operators

$$\left\{A_n(t) = -\frac{\partial^4}{\partial x^4} - \alpha I - Hu_n(t)\frac{\partial^2}{\partial x^2},\ t \in [0,T]\right\}$$

are strongly elliptic and generate a strongly continuous evolution system $\{U_n(t,s), 0 \leq s \leq t \leq T\}$.

It remains to show (2.11), i.e. for some constants $C_A > 0$, $0 < \beta < 2H$

$$\|S_A(t)B^2\|_{\mathcal{L}(V)} \leq \frac{C_A}{t^\beta},\ t \in (0,T].$$

In fact

$$\|S_A(t)B^2\|_{\mathcal{L}(V)} \leq \frac{C_A}{t^{1/2}},\ t \in (0,T],$$

holds (for technical detailes see [24]). Hence the condition $1/2 < 2H$ can be satisfied only for $H > 1/4$. Therefore, under this hypothesis $H > 1/4$ the equation (3.3) has a weak solution for any initial value $x_0 \in V$.

$\triangle$

Chapter 4

Random evolution system

In this chapter the result from Chapter 2 is extended to obtain a random two-parameter evolution system representing the solution with general initial time. Only the case $H > 1/2$ is studied.

Consider the stochastic equation

$$\begin{aligned} \mathrm{d}Y_t &= AY_t\,\mathrm{d}t + BY_t\,\mathrm{d}B_t^H,\ t > s, \\ Y_s &= x, \end{aligned} \tag{4.1}$$

where the solution can start from any fixed time $s \in [0, T]$. The aim of this chapter is to show that the equation (4.1) has a weak solution $\{U_Y(t, s)x, s \leq t \leq T\}$ given by a formula

$$U_Y(t, s)x = S_B(B_t^H - B_s^H)U(t - s, 0)x,\ s \leq t \leq T, \tag{4.2}$$

for any initial value $x \in V$. Note that $\{U(t, s), 0 \leq s \leq t \leq T\}$ is a strongly continuous evolution system associated with operators $\{A - Ht^{2H-1}B^2, t \in [0, T]\}$ and $\{S_B(u), u \in \mathbb{R}\}$ is a strongly continuous group associated with operator B satisfying conditions from Chapter 2.

The suitable version of Itô formula will be used.

Lemma 4.1 *Let $s \in [0, T]$, $f \in \mathcal{C}^{1,2}([s, T] \times \mathbb{R})$ and $u = \{u_t, t \in [0, T]\}$ be a process in $\mathbb{D}^{2,2}(|\mathcal{H}|)$ such that $\{X_t = \int_0^t u_r\,\mathrm{d}B_r^H, t \in [0, T]\}$ is $\mathbb{P}$ - a.s.*

continuous. Assume that $\|u\|_{L^2(\Omega)} \in \mathcal{H}$. *Then the following formula*

$$
\begin{aligned}
f(t, X_t) = f(s, X_s) &+ \int_s^t \frac{\partial}{\partial r} f(r, X_r)\,\mathrm{d}r + \int_s^t \frac{\partial}{\partial x} f(r, X_r)\,\mathrm{d}B_r^H \\
&+ \alpha_H \int_s^t \frac{\partial^2}{\partial x^2} f(r, X_r) u_r \\
&\quad \times \Big(\int_0^T |r-w|^{2H-2} \Big(\int_s^r D_w^H u_\sigma \,\mathrm{d}B_\sigma^H \Big) \mathrm{d}w \Big)\,\mathrm{d}r \\
&+ \alpha_H \int_s^t \frac{\partial^2}{\partial x^2} f(r, X_r) u_r \Big(\int_s^r u_w (r-w)^{2H-2}\,\mathrm{d}w \Big)\,\mathrm{d}r
\end{aligned}
$$

holds $\mathbb{P}$ *- a.s. for any* $t \in [s, T]$.

Proof Basically, the lemma is proved in [2], Theorem 8. Nevertheless, in [2] the function f does not depend on time and the starting point is zero. The version with f dependent on time can be obtained by standard methods. The possibility to start from a nonzero point $s \in [0, T]$ can be checked by passing through the proof of Theorem 8.

Q.E.D.

THEOREM 4.2 *The process* $\{U_Y(t,s)x, s \le t \le T\}$ *is a weak solution to the equation* (4.1) *for any fixed* $x \in V$ *and* $s \in [0, T]$ *under the assumptions of Theorem 2.4 on the existence of a weak solution.*

Proof First assume that $x \in D$, fix $\zeta \in \mathrm{Dom}((B^*)^2)$ and set

$$f(t, y) = \big\langle U(t-s, 0)x, S_B^*(y)\zeta \big\rangle_V, \; s \le t \le T, y \in \mathbb{R}.$$

Clearly, $f \in \mathcal{C}^{1,2}([s, T] \times \mathbb{R})$ and

$$
\begin{aligned}
\frac{\partial}{\partial t} f(t, y) &= \big\langle (A - H(t-s)^{2H-1} B^2) U(t-s, 0)x, S_B^*(y)\zeta \big\rangle_V, \\
\frac{\partial}{\partial y} f(t, y) &= \big\langle U(t-s, 0)x, S_B^*(y) B^* \zeta \big\rangle_V, \\
\frac{\partial^2}{\partial y^2} f(t, y) &= \big\langle U(t-s, 0)x, S_B^*(y)(B^*)^2 \zeta \big\rangle_V.
\end{aligned}
$$

Since

$$B_t^H - B_s^H = \int_s^t 1\,\mathrm{d}B_r^H,$$

the integrand satisfies the conditions of Lemma 4.1. Therefore

$$\begin{aligned}
\langle U_Y(t,s)x,\zeta\rangle_V &= f(t,B_t^H-B_s^H) = f(s,B_s^H-B_s^H) \\
&\quad + \int_s^t \frac{\partial}{\partial r} f(r,B_r^H-B_s^H)\,\mathrm{d}r + \int_s^t \frac{\partial}{\partial y} f(r,B_r^H-B_s^H)\,\mathrm{d}B_r^H \\
&\quad + \alpha_H \int_s^t \int_s^r (r-u)^{2H-2}\,\mathrm{d}u \frac{\partial^2}{\partial y^2} f(r,B_r^H-B_s^H)\,\mathrm{d}r \\
&= \langle x,\zeta\rangle_V \\
&\quad + \int_s^t \langle (A-H(r-s)^{2H-1}B^2)U(r-s,0)x, S_B^*(B_r^H-B_s^H)\zeta\rangle_V\,\mathrm{d}r \\
&\quad + \int_s^t \left\langle S_B(B_r^H-B_s^H)U(r-s,0)x, B^*\zeta\right\rangle_V \mathrm{d}B_r^H \\
&\quad + \int_s^t \left\langle H(r-s)^{2H-1} S_B(B_r^H-B_s^H)U(r-s,0)x, (B^*)^2\zeta\right\rangle_V \mathrm{d}r
\end{aligned}$$

holds $\mathbb{P}$ - a.s. for any $s \le t \le T$ by Lemma 4.1. The commutativity assumption (AB) yields

$$\begin{aligned}
\langle U_Y(t,s)x,\zeta\rangle_V = \langle x,\zeta\rangle_V &+ \int_s^t \left\langle AU_Y(r,s)x,\zeta\right\rangle_V \mathrm{d}r \\
&+ \int_s^t \left\langle BU_Y(r,s)x,\zeta\right\rangle_V \mathrm{d}B_r^H \quad \mathbb{P}\text{ - a.s.}
\end{aligned}$$

for any $s \le t \le T$, hence $\{U_Y(t,s)x, s \le t \le T\}$ is a strong solution to the equation (4.1).

Finally, assume that $x \in V$ and let $\{x_k, k \in \mathbb{N}\} \subset D$ be a sequence converging to x in V. To get a weak solution it sufficies to pass to the limit in the equation

$$\begin{aligned}
\langle U_Y(t,s)x_k,\zeta\rangle_V = \langle x_k,\zeta\rangle_V &+ \int_s^t \langle U_Y(r,s)x_k, A^*\zeta\rangle_V\,\mathrm{d}r \\
&+ \int_s^t \langle U_Y(r,s)x_k, B^*\zeta\rangle_V\,\mathrm{d}B_r^H
\end{aligned}$$

in $L^2(\Omega)$ and to use the closedness of the Skorokhod integral as in the proof of Proposition 2.5.

Q.E.D.

Remark The system $\{U_Y(t,s), 0 \le s \le t \le T\}$ is not a random continuous evolution system because it does not possess the standard composition property.

Chapter 5

Nonlinear equation

This chapter is devoted to equations with a nonlinear perturbation of a drift part.

Let $\{U_Y(t,s), 0 \le s \le t \le T\}$ be the system of operators defined as

$$U_Y(t,s)x := S_B(B_t^H - B_s^H)U(t-s,0)x, \ x \in V,$$

where $\{U(t,s), 0 \le s \le t \le T\}$ is a strongly continuous evolution system associated with operators $\{A - Ht^{2H-1}B^2, t \in [0,T]\}$ and $\{S_B(u), u \in \mathbb{R}\}$ is a strongly continuous group associated with operator B satisfying conditions from Chapter 2. Note that in Chapter 4 it has been shown that for any fixed $s \in [0,T]$ the process $\{U_Y(t,s)x, s \le t \le T\}$ is a weak solution to the equation

$$\begin{aligned} \mathrm{d}Y_t &= AY_t\,\mathrm{d}t + BY_t\,\mathrm{d}B_t^H, \ t > s, \\ Y_s &= x \in V. \end{aligned} \tag{5.1}$$

Denote by $C_U > 0$ a constant such that

$$\|U(t,s)\|_{\mathcal{L}(V)} \le C_U, \ 0 \le s \le t \le T. \tag{5.2}$$

Theorem 5.1 *Let $F : [0,T] \times V \to V$ be a measurable function satisfying*

$(\mathrm{i})_{\mathrm{F}}$ *there exists a function $\bar{L} \in L^1([0,T])$ such that*

$$\|F(t,x) - F(t,y)\|_V \le \bar{L}(t)\|x-y\|_V, \ x,y \in V, \ t \in [0,T].$$

(ii)$_{\mathrm{F}}$ *for some function* $\bar{K} \in L^1([0,T])$

$$\|F(t,0)\|_V \le \bar{K}(t), \;\; t \in [0,T].$$

Then the equation

$$y(t) = U_Y(t,0)x + \int_0^t U_Y(t,r)F\big(r, y(r)\big)\,\mathrm{d}r \tag{5.3}$$

has a unique solution in the space $\mathcal{C}([0,T];V)$ *for a.e.* $\omega \in \Omega$ *and any initial value* $x \in V$.

Remark (i) In the Wiener case $H = 1/2$ the solution to the equation (5.3) is the so-called mild solution to the equation

$$\begin{aligned} \mathrm{d}X_t &= AX_t\,\mathrm{d}t + F(t,X_t)\,\mathrm{d}t + BX_t\,\mathrm{d}W_t, \\ X_0 &= x \in V, \end{aligned}$$

(cf. [5]).

(ii) The assumptions (i)$_{\mathrm{F}}$ and (ii)$_{\mathrm{F}}$ imply that there exists a function $\bar{C} \in L^1([0,T])$

$$\|F(t,x)\|_V \le \bar{C}(t)(1 + \|x\|_V), \;\; x \in V, \, t \in [0,T]. \tag{5.4}$$

Proof Fix $x \in V$ and show that the mapping

$$\big(\mathcal{K}(y)\big)(t) = U_Y(t,0)x + \int_0^t U_Y(t,r)F\big(r,y(r)\big)\,\mathrm{d}r$$

is continuous from $\mathcal{C}([0,T];V)$ into $\mathcal{C}([0,T];V)$ and that $\mathcal{K}$ is a contraction mapping.

Take $y \in \mathcal{C}([0,T];V)$ and $t, s \in [0,T]$. Then

$$\begin{aligned} &\big\|\big(\mathcal{K}(y)\big)(t) - \big(\mathcal{K}(y)\big)(s)\big\|_V \le \|U_Y(t,0)x - U_Y(s,0)x\|_V \\ &\quad + \Big\| \int_0^t U_Y(t,r)F\big(r,y(r)\big)\,\mathrm{d}r - \int_0^s U_Y(s,r)F\big(r,y(r)\big)\,\mathrm{d}r \Big\|_V = I_1 + I_2. \end{aligned}$$

Note that applying (2.6) and continuity of trajectories of $\{B_t^H, t \in [0,T]\}$

$$\sup_{t\in[0,T]} \|S_B(B_t^H(\omega))\|_{\mathcal{L}(V)} \le M_B \exp\{\omega_B \|B^H(\omega)\|_{\mathcal{C}([0,T])}\} \le C_B(\omega)$$
$$\sup_{s,t\in[0,T]} \big\|S_B\big(B_t^H(\omega) - B_s^H(\omega)\big)\big\|_{\mathcal{L}(V)} \le M_B \exp\{2\omega_B \|B^H(\omega)\|_{\mathcal{C}([0,T])}\} \le C_B(\omega) \tag{5.5}$$

hold for some constant $0 < C_B(\omega) < +\infty$ depending on $\omega \in \Omega$.

Since strongly continuous groups and evolution systems are continuous for any fixed element in V it follows

$$\begin{aligned} I_1 = \|U_Y(t,0)x - U_Y(s,0)x\|_V &\le \big\|\big(S_B(B_t^H) - S_B(B_s^H)\big)U(t,0)x\big\|_V \\ &\quad + \big\|S_B(B_s^H)\big(U(t,0) - U(s,0)\big)x\big\|_V \\ &\le \big\|\big(S_B(B_t^H) - S_B(B_s^H)\big)U(t,0)x\big\|_V \\ &\quad + C_B(\omega)\big\|\big(U(t,0) - U(s,0)\big)x\big\|_V \xrightarrow[s\to t]{} 0. \end{aligned}$$

Now, let $t > s$. It follows

$$\begin{aligned} I_2 &= \Big\| \int_0^t U_Y(t,r)F\big(r,y(r)\big)\,\mathrm{d}r - \int_0^s U_Y(s,r)F\big(r,y(r)\big)\,\mathrm{d}r \Big\|_V \\ &\le \Big\| \int_0^s \big(U_Y(t,r) - U_Y(s,r)\big)F\big(r,y(r)\big)\,\mathrm{d}r \Big\|_V \\ &\quad + \Big\| \int_s^t U_Y(t,r)F\big(r,y(r)\big)\,\mathrm{d}r \Big\|_V = J_1 + J_2. \end{aligned}$$

Using (5.5), (5.2) and (5.4)

$$\begin{aligned} J_2 &= \Big\| \int_s^t U_Y(t,r)F\big(r,y(r)\big)\,\mathrm{d}r \Big\|_V \\ &\le \int_s^t C_U \|S_B(B_t^H - B_r^H)\|_{\mathcal{L}(V)} \big\|F\big(r,y(r)\big)\big\|_V \,\mathrm{d}r \\ &\le C_U C_B(\omega)(1 + \|y\|_{\mathcal{C}([0,T];V)}) \int_s^t \bar{C}(r)\,\mathrm{d}r \longrightarrow 0 \end{aligned}$$

as $s \to t-$ or $t \to s+$.

Also

$$\begin{aligned} J_1 &= \Big\| \int_0^s \big(U_Y(t,r) - U_Y(s,r)\big) F\big(r, y(r)\big)\, \mathrm{d}r \Big\|_V \\ &\le \Big\| \int_0^s \big(S_B(B_t^H - B_r^H) - S_B(B_s^H - B_r^H)\big) U(t-r,0) F\big(r, y(r)\big)\, \mathrm{d}r \Big\|_V \\ &\quad + \Big\| \int_0^s S_B(B_s^H - B_r^H)\big(U(t-r,0) - U(s-r,0)\big) F\big(r, y(r)\big)\, \mathrm{d}r \Big\|_V \\ &= K_1 + K_2. \end{aligned}$$

Since for any fixed $0 \le r \le s$

$$\big\|\big(U(t-r,0) - U(s-r,0)\big) F\big(r, y(r)\big)\big\|_V \longrightarrow 0$$

as $s \to t-$ or $t \to s+$ and by (5.2)

$$\begin{aligned} &\int_0^s \big\|\big(U(t-r,0) - U(s-r,0)\big) F\big(r, y(r)\big)\big\|_V \, \mathrm{d}r \\ &\qquad \le 2C_U \int_0^s \big\|F\big(r, y(r)\big)\big\|_V \, \mathrm{d}r \\ &\qquad \le 2C_U (1 + \|y\|_{\mathcal{C}([0,T];V)}) \int_0^s \bar{C}(r)\, \mathrm{d}r < +\infty, \end{aligned}$$

the convergence

$$\begin{aligned} K_2 &= \Big\| \int_0^s S_B(B_s^H - B_r^H)\big(U(t-r,0) - U(s-r,0)\big) F\big(r, y(r)\big)\, \mathrm{d}r \Big\|_V \\ &\le C_B(\omega) \int_0^s \big\|\big(U(t-r,0) - U(s-r,0)\big) F\big(r, y(r)\big)\big\|_V \, \mathrm{d}r \longrightarrow 0 \end{aligned}$$

is obtained as $s \to t-$ or $t \to s+$ by the Lebesgue dominated convergence theorem. Note that the set

$$\begin{aligned} K := \Big\{ &\bar{y} \in V; \exists\, 0 \le s_1 \le t_1 \le T \\ &\bar{y} = \int_0^{s_1} S_B(-B_r^H) U(t_1 - r, 0) F\big(r, y(r)\big)\, \mathrm{d}r \Big\} \end{aligned}$$

is compact as a continuous image of a compact set and

$$\lim_{t \to s} \sup_{z \in K} \big\|\big(S_B(B_t^H) - S_B(B_s^H)\big) z\big\|_V = 0.$$

Therefore

$$
\begin{aligned}
K_1 &= \Big\| \int_0^s \big(S_B(B_t^H - B_r^H) - S_B(B_s^H - B_r^H)\big) U(t-r,0) F\big(r, y(r)\big)\,\mathrm{d}r \Big\|_V \\
&= \Big\| \big(S_B(B_t^H) - S_B(B_s^H)\big) \int_0^s S_B(-B_r^H) U(t-r,0) F\big(r, y(r)\big)\,\mathrm{d}r \Big\|_V \\
&\le \sup_{z\in K} \big\| \big(S_B(B_t^H) - S_B(B_s^H)\big) z \big\|_V \longrightarrow 0
\end{aligned}
$$

as $s \to t-$ or $t \to s+$. Thus

$$
\big\| \big(\mathcal{K}(y)\big)(t) - \big(\mathcal{K}(y)\big)(s) \big\|_V \longrightarrow 0
$$

as $s \to t-$ or $t \to s+$ and the function $t \mapsto \big(\mathcal{K}(y)\big)(t)$ is continuous on the interval $[0,T]$ for any $y \in \mathcal{C}([0,T];V)$.
For any $y_1, y_2 \in \mathcal{C}([0,T];V)$, $t \in [0,T]$ and $T > 0$ small enough there exists a constant $0 < L_T < 1$ such that

$$
\begin{aligned}
&\big\| \big(\mathcal{K}(y_1)\big)(t) - \big(\mathcal{K}(y_2)\big)(t) \big\|_V = \\
&\Big\| \int_0^t U_Y(t,r)\big(F\big(r, y_1(r)\big) - F\big(r, y_2(r)\big)\big)\,\mathrm{d}r \Big\|_V = \\
&\le C_B(\omega) C_U \int_0^t \big\| \big(F\big(r, y_1(r)\big) - F\big(r, y_2(r)\big)\big) \big\|_V \,\mathrm{d}r \\
&\le C_B(\omega) C_U \| y_1 - y_2 \|_{\mathcal{C}([0,T];V)} \int_0^T \bar{L}(r)\,\mathrm{d}r \le L_T \| y_1 - y_2 \|_{\mathcal{C}([0,T];V)}
\end{aligned}
$$

holds so that $\mathcal{K}$ is a contraction mapping. Hence, by the Banach fixed-point theorem there exists a unique solution to the equation (5.3) for T small enough. Applying standard methods a unique continuous solution to (5.3) for any $T > 0$ can be obtained.

Q.E.D.

A slight extension of the definition of a weak solution to the linear case is necessary. Consider an equation with a nonlinear perturbation of a drift part

$$
\begin{aligned}
\mathrm{d}X_t &= AX_t\,\mathrm{d}t + F(t, X_t)\,\mathrm{d}t + BX_t\,\mathrm{d}B_t^H, \\
X_0 &= x \in V.
\end{aligned}
\tag{5.6}
$$

Definition 5.2 *A* $(\mathcal{B}([0,T])\otimes\mathcal{F})$*-measurable process* $\{X_t, t\in[0,T]\}$ *is said to be a* ***weak solution*** *to the equation* (5.6) *if for any* $y\in D^*$

$$\langle X_t, y\rangle_V = \langle x, y\rangle_V + \int_0^t \langle X_r, A^* y\rangle_V \,\mathrm{d}r + \int_0^t \langle F(r, X_r), y\rangle_V \,\mathrm{d}r + \int_0^t \langle X_r, B^* y\rangle_V \,\mathrm{d}B_r^H$$

$\mathbb{P}$ *- a.s. for all* $t\in[0,T]$*, where the integrals have to be well-defined.*

The aim of the next theorem is to check that the solution $\{X_t, t\in[0,T]\}$ to the equation

$$y(t) = U_Y(t,0)x + \int_0^t U_Y(t,r)F(r)\,\mathrm{d}r$$

stated in Theorem 5.1 is a weak solution under the additional assumption that the coefficient F does not depend on the space variable. The main idea is to use standard and stochastic Fubini theorem for the Skorokhod integral stated in [16], Lemma 2.10, or [18], Exercise 3.2.8.

Lemma 5.3 *Consider a random field* $\{u(t,x), t\in[0,T], x\in G\}$*, where* $G\subset\mathbb{R}$ *is a bounded set, such that*

$(\mathrm{i})_{\mathrm{W}}$ $u\in L^2(\Omega\times[0,T]\times G)$,

$(\mathrm{ii})_{\mathrm{W}}$ $u(\,.\,,x)\in \mathrm{Dom}\,\delta_W$ *for a.e.* $x\in G$,

$(\mathrm{iii})_{\mathrm{W}}$ $\mathbb{E}\left[\int_G \left(\int_0^T u(t,x)\,\mathrm{d}W_t\right)^2 \mathrm{d}x\right] < +\infty$.

Then the process $\left\{\int_G u(t,x)\,\mathrm{d}x, t\in[0,T]\right\}\in \mathrm{Dom}\,\delta_W$ *and*

$$\int_0^T \left(\int_G u(t,x)\,\mathrm{d}x\right)\mathrm{d}W_t = \int_G \left(\int_0^T u(t,x)\,\mathrm{d}W_t\right)\mathrm{d}x.$$

Due to the relationship between Skorokhod integral with respect to Wiener process and fractional Brownian motion (see (1.7) or [19] for more detailes) $(\mathrm{ii})_{\mathrm{W}}$, $(\mathrm{iii})_{\mathrm{W}}$ are equivalent to

$(\mathrm{ii})_{\mathrm{H}}$ $u_H(\,.\,,x)\in \mathrm{Dom}\,\delta_H$ for a.e. $x\in G$, (iii)$_{\mathrm{H}}$ $\mathbb{E}\left[\int_G \left(\int_0^T u_H(t,x)\,\mathrm{d}B_t^H\right)^2 \mathrm{d}x\right] < +\infty,$

respectively, where $u_H(t,x) = \left(\mathcal{K}_H^*\right)^{-1}\big(u(\,.\,,x)\big)(t), t \in [0,T]$. The conclusion of Lemma 5.3 can be reformulated in the following way. The process $\left\{\int_G u_H(t,x)\,\mathrm{d}x, t \in [0,T]\right\} \in \mathrm{Dom}\,\delta_H$ and

$$\int_0^T \left(\int_G u_H(t,x)\,\mathrm{d}x\right)\mathrm{d}B_t^H = \int_G \left(\int_0^T u_H(t,x)\,\mathrm{d}B_t^H\right)\mathrm{d}x.$$

THEOREM 5.4 *Assume that the measurable function $F : [0,T] \to V$ is independent of a space variable and $\|F\|_V \in L^2([0,T])$. Then the unique continuous solution $\{X_t, t \in [0,T]\}$ to the equation*

$$y(t) = U_Y(t,0)x + \int_0^t U_Y(t,r)F(r)\,\mathrm{d}r \tag{5.7}$$

stated in Theorem 5.1 is a weak solution to the equation

$$\begin{aligned} \mathrm{d}X_t &= AX_t\,\mathrm{d}t + F(t)\,\mathrm{d}t + BX_t\,\mathrm{d}B_t^H, \\ X_0 &= x \in V. \end{aligned} \tag{5.8}$$

The proof of the theorem is based on the following lemma.

LEMMA 5.5 *The equalities*

$$\int_0^t \int_0^r \left\langle U_Y(r,v)F(v), A^*\zeta\right\rangle_V \mathrm{d}v\,\mathrm{d}r = \int_0^t \int_v^t \left\langle U_Y(r,v)F(v), A^*\zeta\right\rangle_V \mathrm{d}r\,\mathrm{d}v \tag{5.9}$$

and

$$\int_0^t \int_0^r \left\langle U_Y(r,v)F(v), B^*\zeta\right\rangle_V \mathrm{d}v\,\mathrm{d}B_r^H = \int_0^t \int_v^t \left\langle U_Y(r,v)F(v), B^*\zeta\right\rangle_V \mathrm{d}B_r^H\mathrm{d}v$$

hold $\mathbb{P}$ - a.s. for any $t \in [0,T]$ and fixed $\zeta \in D^*$.

Proof It is necessary to verify the assumptions of standard and stochastic Fubini theorem.

Notice that the Fernique theorem (see [13]) yields that there exists a random variable $C_{B^H}(\omega)$ such that $C_{B^H} \in L^q(\Omega)$ for any $q \in [1, +\infty)$ and

$$M_B \exp\{l\omega_B \|B^H(\omega)\|_{\mathcal{C}([0,T])}\} \le C_{B^H}(\omega), \ \omega \in \Omega, l = 1, 2. \tag{5.10}$$

Since by (5.10) and (5.2)

$$\begin{aligned}
&\int_0^t \int_0^r \left|\left\langle U_Y(r,v)F(v), A^*\zeta\right\rangle_V\right| \mathrm{d}v\, \mathrm{d}r \\
&\quad \le \int_0^T \int_0^T C_{B^H}(\omega) C_U \|F(v)\|_V \|A^*\zeta\|_V \,\mathrm{d}v\, \mathrm{d}r \\
&\quad \le K(\omega) \int_0^T \|F(v)\|_V \,\mathrm{d}v < +\infty
\end{aligned}$$

for a.e. $\omega \in \Omega$, (5.9) follows by the standard Fubini theorem.

Denote

$$\begin{aligned}
u_H(r,s) &= \left\langle U_Y(r,s)F(s), B^*\zeta\right\rangle_V, \ 0 \le s \le r \le t, \\
u(r,s) &= \left(\mathcal{K}_H^* u_H(\,.\,,s)\right)(r), \ 0 \le s \le r \le t,
\end{aligned}$$

and verify that $(\mathrm{i})_{\mathrm{W}}, (\mathrm{ii})_{\mathrm{H}}$ and $(\mathrm{iii})_{\mathrm{H}}$ hold for the corresponding processes. First show that $u \in L^2([0,t]^2 \times \Omega)$. Using (1.4)

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \int_0^t u^2(r,s)\,\mathrm{d}r\,\mathrm{d}s\right] \le K_e \mathbb{E}\left[\int_0^t \int_0^t u_H^2(r,s)\,\mathrm{d}r\,\mathrm{d}s\right] \\
&\quad \le K_e \mathbb{E}\left[\int_0^t \int_0^t \left(C_{B^H}(\omega) C_U \|F(s)\|_V \|B^*\zeta\|_V\right)^2 \mathrm{d}r\,\mathrm{d}s\right] < +\infty,
\end{aligned}$$

and $(\mathrm{i})_{\mathrm{W}}$ follows. To prove $(\mathrm{ii})_{\mathrm{H}}$ it sufficies to show (in the view of (1.5)) that $u_H(\,.\,,s) \in \mathbb{D}_H^{1,2}(|\mathcal{H}|)$ for a.e. $s \in [0,t]$ which is true whenever

$$\max\left\{\sup_{r\in[0,t]} \mathbb{E}\left[u_H^2(r,s)\right], \sup_{r\in[0,t]} \sup_{v\in[0,t]} \mathbb{E}\left[(D_v^H u_H(r,s))^2\right]\right\} < +\infty \tag{5.11}$$

for a.e. $s \in [0,t]$. Since

$$D_v^H u_H(r,s) = \left\langle U_Y(r,s)F(s), (B^*)^2\zeta\right\rangle_V I_{(s,r]}(v)$$

the inequalities

$$\begin{aligned}
&\sup_{r\in[0,t]} \sup_{v\in[0,t]} \mathbb{E}\left[(D_v^H u_H(r,s))^2\right] \\
&\quad \le \sup_{r\in[0,t]} \mathbb{E}\left[\left(C_{B^H}(\omega) C_U \|F(s)\|_V \|(B^*)^2\zeta\|_V\right)^2\right] = K\|F(s)\|_V^2 < +\infty
\end{aligned}$$

and

$$\sup_{r\in[0,t]} \mathbb{E}\left[u_H^2(r,s)\right] \le \mathbb{E}\left[(C_{B^H}(\omega)C_U\|F(s)\|_V\|B^*\zeta\|_V)^2\right]$$
$$\le K\|F(s)\|_V^2 < +\infty$$

hold for a.e. $s \in [0,t]$ which completes the proof of (5.11).

Finally, applying the estimate on the Skorokhod integral (1.6) and the previous part of the proof of (5.11)

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \Big(\int_0^t u_H(r,s)\,\mathrm{d}B_r^H\Big)^2 \mathrm{d}s\right] = \int_0^t \mathbb{E}\left[\Big(\int_0^t u_H(r,s)\,\mathrm{d}B_r^H\Big)^2\right] \mathrm{d}s \\
&\le C_{H,2}\int_0^t \Big(\mathbb{E}\left[\|u_H(\,.\,,s)\|^2_{L^2([0,t])}\right] + \mathbb{E}\left[\|D^H u_H(\,.\,,s)\|^2_{L^2([0,t]^2)}\right]\Big)\,\mathrm{d}s \\
&\le C_{H,2}\int_0^t (t+t^2)K\|F(s)\|_V^2\,\mathrm{d}s < +\infty
\end{aligned}$$

and $(\mathrm{iii})_{\mathrm{H}}$ follows.

Q.E.D.

Proof of Theorem 5.4 Fix $\zeta \in D^*$. It sufficies to show

$$\begin{aligned}
\langle X_t,\zeta\rangle_V = \langle x,\zeta\rangle_V &+ \int_0^t \langle X_r, A^*\zeta\rangle_V\,\mathrm{d}r + \int_0^t \langle F(r),\zeta\rangle_V\,\mathrm{d}r \\
&+ \int_0^t \langle X_r, B^*\zeta\rangle_V\,\mathrm{d}B_r^H \quad \mathbb{P}\text{- a.s.}
\end{aligned}$$

for all $t \in [0,T]$.

Since $\{X_t, t\in[0,T]\}$ satisfies (5.7) and $\{U_Y(t,s)x, s\le t\le T\}$ is a weak solution to the equation (5.1)

$$\begin{aligned}
&\int_0^t \langle X_r, A^*\zeta\rangle_V\,\mathrm{d}r + \int_0^t \langle X_r, B^*\zeta\rangle_V\,\mathrm{d}B_r^H = \int_0^t \big\langle U_Y(r,0)x, A^*\zeta\big\rangle_V\,\mathrm{d}r \\
&\quad + \int_0^t\int_0^r \big\langle U_Y(r,v)F(v), A^*\zeta\big\rangle_V\,\mathrm{d}v\,\mathrm{d}r + \int_0^t \big\langle U_Y(r,0)x, B^*\zeta\big\rangle_V\,\mathrm{d}B_r^H \\
&\quad + \int_0^t\int_0^r \big\langle U_Y(r,v)F(v), B^*\zeta\big\rangle_V\,\mathrm{d}v\,\mathrm{d}B_r^H \\
&= \langle U_Y(t,0)x,\zeta\rangle_V - \langle x,\zeta\rangle_V + \int_0^t\int_v^t \big\langle U_Y(r,v)F(v), A^*\zeta\big\rangle_V\,\mathrm{d}r\,\mathrm{d}v \\
&\quad + \int_0^t\int_v^t \big\langle U_Y(r,v)F(v), B^*\zeta\big\rangle_V\,\mathrm{d}B_r^H\,\mathrm{d}v \quad \mathbb{P}\text{- a.s.}
\end{aligned}$$

holds for any $t \in [0,T]$, where in the last equality Lemma 5.5 is used. Applying again that $\{U_Y(t,s)x, s \le t \le T\}$ is a weak solution to the equation (5.1)

$$\begin{aligned}\int_0^t \langle X_r, A^*\zeta\rangle_V \,\mathrm{d}r + \int_0^t \langle X_r, B^*\zeta\rangle_V \,\mathrm{d}B_r^H &= \langle U_Y(t,0)x,\zeta\rangle_V - \langle x,\zeta\rangle_V \\ &+ \int_0^t \left\langle U_Y(t,v)F(v),\zeta\right\rangle_V \mathrm{d}v - \int_0^t \langle F(v),\zeta\rangle_V \,\mathrm{d}v \\ &= \langle X_t,\zeta\rangle_V - \langle x,\zeta\rangle_V - \int_0^t \langle F(v),\zeta\rangle_V \,\mathrm{d}v \quad \mathbb{P}\text{- a.s.}\end{aligned}$$

is obtained for any $t \in [0,T]$ and the conclusion follows.

Q.E.D.

The next example is a counterexample that the solution to the equation (5.3) need not be a weak solution to the equation (5.6) if the function F depends on the solution.

Example 5.6 Consider a one-dimensional equation

$$\mathrm{d}X_t = aX_t\,\mathrm{d}t + bX_t\,\mathrm{d}B_t^H,\ X_0 = 1, \tag{5.12}$$

where $a, b \in \mathbb{R}$ are nonzero constants. In the previous notation, the equation (5.12) takes a form

$$\mathrm{d}X_t = F(t,X_t)\,\mathrm{d}t + BX_t\,\mathrm{d}B_t^H,\ X_0 = 1,$$

where $F(t,x) = ax$, $A = 0$ and $B = bI$.

Define a system $\{\bar{U}_Y(t,s), 0 \le s \le t \le T\}$ as

$$\begin{aligned}\bar{U}_Y(t,s) &= S_B(B_t^H - B_s^H)U(t,s) \\ &= \exp\left\{b(B_t^H - B_s^H) - \frac{1}{2}b^2\left(t^{2H} - s^{2H}\right)\right\},\ 0 \le s \le t \le T.\end{aligned}$$

The aim is to show that the unique continuous solution to the equation

$$y(t) = \bar{U}_Y(t,0) + \int_0^t \bar{U}_Y(t,r)F(r,y(r))\,\mathrm{d}r \tag{5.13}$$

is also strong solution to the equation (5.12). Note that in a one-dimensional case the notion of weak and strong solution coincides.

In fact Theorem 2.4 yields the formula

$$X_t = \exp\left\{bB_t^H - \frac{1}{2}b^2t^{2H} + at\right\},\ t \in [0,T],$$

for a solution to the equation (5.12), therefore

$$\begin{aligned}
&\bar{U}_Y(t,0) + \int_0^t \bar{U}_Y(t,r)F(r,X_r)\,\mathrm{d}r \\
&\quad= \bar{U}_Y(t,0) + \int_0^t \exp\left\{b(B_t^H - B_r^H) - \frac{1}{2}b^2\big(t^{2H} - r^{2H}\big)\right\} aX_r\,\mathrm{d}r \\
&\quad= \exp\left\{bB_t^H - \frac{1}{2}b^2t^{2H}\right\} + \int_0^t a\exp\left\{bB_t^H - \frac{1}{2}b^2t^{2H} + ar\right\}\mathrm{d}r \\
&\quad= \exp\left\{bB_t^H - \frac{1}{2}b^2t^{2H}\right\} + \exp\left\{bB_t^H - \frac{1}{2}b^2t^{2H}\right\}(\mathrm{e}^{at} - 1) \\
&\quad= \exp\left\{bB_t^H - \frac{1}{2}b^2t^{2H} + at\right\} = X_t,
\end{aligned}$$

i.e. $\{X_t, t \in [0,T]\}$ is a solution to the equation (5.13). By the uniqueness the solution to (5.13) is strong.

The same computations with $\{U_Y(t,s), 0 \le s \le t \le T\}$ give that the strong solution $\{X_t, t \in [0,T]\}$ is NOT a solution to the equation (5.3). Nevertheless, we are not able to prove the uniqueness of the solution to the equation (5.12) so that we do not know whether another strong solution satisfies (5.3).

△

Remark Let the assumptions of Theorem 2.4 be satisfied. Then the system $\{\bar{U}_Y(t,s), 0 \le s \le t \le T\}$ defined as

$$\bar{U}_Y(t,s)x = S_B(B_t^H - B_s^H)U(t,s)x,\ x \in V,\ 0 \le s \le t \le T,$$

is a weak solution to the equation

$$\begin{aligned}
\mathrm{d}Y_t &= A(t)Y_t\,\mathrm{d}t + H\big((t-s)^{2H-1} - t^{2H-1}\big)B^2Y_t\,\mathrm{d}t + BY_t\,\mathrm{d}B_t^H,\ t > s, \\
Y_s &= x.
\end{aligned}$$

This result can be obtained in the same way as Theorem 4.2. Moreover, this system has a composition property unlike $\{U_Y(t,s), 0 \le s \le t \le T\}$.

Let $\{X_t^n, t \in [0,T]\}, n \in \mathbb{N}$, be the Picard approximations of the solution $\{X_t, t \in [0,T]\}$ to the equation (5.1) obtained from the proof of the Banach fixed-point theorem, i.e.

$$\begin{aligned} &X_t^0 \equiv x,\ t \in [0,T], \\ &X_t^n = U_Y(t,0)x + \int_0^t U_Y(t,r)F\big(r, X_r^{n-1}\big)\,\mathrm{d}r,\ t \in [0,T]. \end{aligned} \tag{5.14}$$

Proposition 5.7 *Let the assumptions of Theorem 5.1 be satisfied. Moreover, assume that the function F is Fréchet differentiable with respect to the space variable for any time $t \in [0,T]$ and that there exists a function $C \in L^2([0,T])$ such that*

$$\max\{\|F(t,x)\|_V, \|F'_x(t,x)\|\} \le C(t),\ t \in [0,T], \tag{5.15}$$

and $B \in \mathcal{L}(V)$. Then $X^n \in \mathbb{D}_H^{1,p}(|\mathcal{H}|)$ for any $p \in [1,+\infty)$ and fixed $n \in \mathbb{N}$, i.e. X^n is Skorokhod integrable for any $n \in \mathbb{N}$.

Proof It sufficies to check as in the proof of Lemma 5.5

$$\max\left\{\sup_{t\in[0,T]} \mathbb{E}\,\|X_t^n\|_V^p,\ \sup_{t\in[0,T]}\sup_{v\in[0,T]} \mathbb{E}\,\|D_v^H X_t^n\|_V^p\right\} < +\infty,\ p \in [1,+\infty),$$

for any fixed $n \in \mathbb{N}$.

Fix $p \in [1,+\infty)$. By (5.15), (5.2) and (5.10)

$$\begin{aligned} &\sup_{t\in[0,T]} \mathbb{E}\,\|X_t^n\|_V^p \\ &\le K \sup_{t\in[0,T]} \left\{\mathbb{E}\left[\|U_Y(t,0)x\|_V^p + \left(\int_0^t \|U_Y(t,s)F(s,X_s^{n-1})\|_V\,\mathrm{d}s\right)^p\right]\right\} \\ &\le K \sup_{t\in[0,T]} \left\{\mathbb{E}\left[(C_{B^H}(\omega)C_U\|x\|_V)^p + \left(\int_0^t C_{B^H}(\omega)C_U C(s)\,\mathrm{d}s\right)^p\right\} < +\infty. \end{aligned}$$

The proof of the finiteness of the second term is done by induction. First note that

$$D_v^H X_t^1 = BU_Y(t,0)xI_{(0,t]}(v) + \int_0^t BU_Y(t,s)F(s,x)I_{(s,t]}(v)\,\mathrm{d}s$$

and

$$\sup_{t\in[0,T]}\sup_{v\in[0,T]}\mathbb{E}\,\|D_v^H X_t^1\|_V^p \le K\mathbb{E}\left[(\|B\|_{\mathcal{L}(V)}C_{B^H}(\omega)C_U\|x\|_V)^p\right]$$
$$+K\mathbb{E}\left[\left(\|B\|_{\mathcal{L}(V)}C_{B^H}(\omega)C_U\int_0^T C(s)\,\mathrm{d}s\right)^p\right] < +\infty.$$

Assume that there exists a constant $0 < C_{n-1}^{(p)} < +\infty$ such that

$$\sup_{t\in[0,T]}\sup_{v\in[0,T]}\mathbb{E}\,\|D_v^H X_t^{n-1}\|_V^p \le C_{n-1}^{(p)}$$

holds for any $p\in[1,+\infty)$ and fixed $n-1$. Then

$$D_v^H X_t^n = BU_Y(t,0)xI_{(0,t]}(v) + \int_0^t BU_Y(t,s)F(s,X_s^{n-1})I_{(s,t]}(v)\,\mathrm{d}s$$
$$+\int_0^t U_Y(t,s)F_x'(s,X_s^{n-1})D_v^H X_s^{n-1}\,\mathrm{d}s$$

and

$$\sup_{t\in[0,T]}\sup_{v\in[0,T]}\mathbb{E}\,\|D_v^H X_t^n\|_V^p \le K\mathbb{E}\left[(\|B\|_{\mathcal{L}(V)}C_{B^H}(\omega)C_U\|x\|_V)^p\right]$$
$$+K\mathbb{E}\left[\left(\|B\|_{\mathcal{L}(V)}C_{B^H}(\omega)C_U\int_0^T C(s)\,\mathrm{d}s\right)^p\right]$$
$$+K\sup_{v\in[0,T]}\mathbb{E}\left[\left(C_{B^H}(\omega)C_U\int_0^T C(s)\|D_v^H X_s^{n-1}\|_V\,\mathrm{d}s\right)^p\right] < +\infty,$$

because the third term

$$K\sup_{v\in[0,T]}\mathbb{E}\left[\left(C_{B^H}(\omega)C_U\int_0^T C(s)\|D_v^H X_s^{n-1}\|_V\,\mathrm{d}s\right)^p\right]$$
$$\le \tilde{K}\left(\mathbb{E}\left[C_{B^H}^{2p}(\omega)\right]\right)^{1/2}\left(\int_0^T C^2(s)\,\mathrm{d}s\right)^{p/2}\left(C_{n-1}^{(2p)}\right)^{1/2} < +\infty$$

applying the induction assumption.

Q.E.D.

There is a natural question whether the solution to the equation (5.3) is a weak one to some equation. By Proposition 5.7 the Piccard approximations $\{X_t^n, t\in[0,T]\}$ are in $\mathbb{D}_H^{1,p}(|\mathcal{H}|)$ for any $p\in[1,+\infty)$ and any fixed $n\in\mathbb{N}$. But we are not able to prove that $\{D^H X^n, n\in\mathbb{N}\}$ are uniformly bounded (in n) in $L^p(\Omega \times [0,T]^2; V)$ thus it is not possible to show that $X \in \mathbb{D}_H^{1,p}(|\mathcal{H}|)$ by the closedness of Malliavin derivative which is essential in solving the stated question.

Another way how to try to solve the problem is to find an equation to which $\{X_t^n, t \in [0,T]\}$ is a weak solution for any fixed $n \in \mathbb{N}$ and pass to the limit in a suitable sense. From the proof of the next theorem it is clear that one would have to pass in the weak formulation for approximations to the limit in the Skorokhod integral and in the integral with integrand containing the Malliavin derivative of the Piccard approximation. This seems to be very difficult. Therefore the additional assumptions are required on $\{X_t, t \in [0,T]\}$.

THEOREM 5.8 *Let the assumptions of Theorem 5.1 hold and $\{X_t, t \in [0,T]\}$ be the solution to the equation* (5.3) *such that there exists a constant $C_X < +\infty$*

$$\max\left\{\sup_{t\in[0,T]} \mathbb{E}\,\|X_t\|_V^4,\ \sup_{t\in[0,T]}\sup_{v\in[0,T]} \mathbb{E}\,\|D_v^H X_t\|_V^4\right\} \le C_X. \tag{5.16}$$

In addition, let F be Fréchet differentiable with respect to the space variable for any time $t \in [0,T]$. Suppose that there exists a function $C \in L^4([0,T])$ such that (5.15) *holds. Then $\{X_t, t \in [0,T]\}$ is a solution to the integral equation*

$$\begin{aligned} X_t = x &+ \int_0^t AX_r\,\mathrm{d}r + \int_0^t F(r,X_r)\,\mathrm{d}r + \int_0^t BX_r\,\mathrm{d}B_r^H \\ &+ \int_0^t \alpha_H \int_0^T \int_r^t |v-w|^{2H-2} BU_Y(v,r)F_x'(r,X_r)D_w^H X_r\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}r \end{aligned}$$

in a weak sense, i.e. for any $y \in D^*$

$$\begin{aligned} \langle X_t, y\rangle_V = \langle x, y\rangle_V &+ \int_0^t \langle X_r, A^*y\rangle_V\,\mathrm{d}r + \int_0^t \langle F(r,X_r), y\rangle_V\,\mathrm{d}r \\ &+ \int_0^t \langle X_r, B^*y\rangle_V\,\mathrm{d}B_r^H \\ &+ \int_0^t \alpha_H \int_0^T \int_r^t |v-w|^{2H-2} \big\langle U_Y(v,r)F_x'(r,X_r)D_w^H X_r, B^*y\big\rangle_V\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}r \end{aligned}$$

$\mathbb{P}$ - a.s. for all $t \in [0,T]$.

Remark The condition (5.16) implies that $X \in \mathbb{D}_H^{1,4}(|\mathcal{H}|)$.

The proof of the theorem requires some technical lemmas.

LEMMA 5.9 *The equalities*

$$\int_0^t \int_0^r \left\langle U_Y(r,v)F(v,X_v), A^*\zeta\right\rangle_V \mathrm{d}v\,\mathrm{d}r = \int_0^t \int_v^t \left\langle U_Y(r,v)F(v,X_v), A^*\zeta\right\rangle_V \mathrm{d}r\,\mathrm{d}v \quad \mathbb{P}\text{-a.s.} \tag{5.17}$$

and

$$\int_0^t \int_0^r \left\langle U_Y(r,v)F(v,X_v), B^*\zeta\right\rangle_V \mathrm{d}v\,\mathrm{d}B_r^H = \int_0^t \int_v^t \left\langle U_Y(r,v)F(v,X_v), B^*\zeta\right\rangle_V \mathrm{d}B_r^H\,\mathrm{d}v \quad \mathbb{P}\text{-a.s.}$$

hold for any $t \in [0,T]$ and fixed $\zeta \in D^*$.

Proof The assumptions of Fubini theorem's need to be verified. Since

$$\begin{aligned}\int_0^t \int_0^r &\left|\left\langle U_Y(r,v)F(v,X_v), A^*\zeta\right\rangle_V\right| \mathrm{d}v\,\mathrm{d}r \\ &\le \int_0^T \int_0^T C_{B^H}(\omega)C_U\|F(v,X_v)\|_V\|A^*\zeta\|_V\,\mathrm{d}v\,\mathrm{d}r \\ &\le K(\omega)\int_0^T C(v)\,\mathrm{d}v < +\infty \quad \mathbb{P}\text{-a.s.},\end{aligned}$$

the standard Fubini theorem may be applied to obtain (5.17).

As in the proof of Lemma 5.5 let

$$\begin{aligned}u_H(r,v) &= \left\langle U_Y(r,v)F(v,X_v), B^*\zeta\right\rangle_V,\ 0 \le v \le r \le t, \\ u(r,v) &= \left(\mathcal{K}_H^* u_H(\,.\,,v)\right)(r),\ 0 \le v \le r \le t,\end{aligned}$$

and verify that $(\mathrm{i})_{\mathrm{W}}, (\mathrm{ii})_{\mathrm{H}}$ and $(\mathrm{iii})_{\mathrm{H}}$ hold for the corresponding processes. First show that $u \in L^2([0,t]^2 \times \Omega)$, i.e.

$$\begin{aligned}\mathbb{E}\left[\int_0^t\int_0^t u^2(r,v)\,\mathrm{d}r\,\mathrm{d}v\right] &\le K_e\mathbb{E}\left[\int_0^t\int_0^t u_H^2(r,v)\,\mathrm{d}r\,\mathrm{d}v\right] \\ &\le K_e\mathbb{E}\left[\int_0^t\int_0^t \left(C_{B^H}(\omega)C_U C(v)\|B^*\zeta\|_V\right)^2\mathrm{d}r\,\mathrm{d}v\right] < +\infty,\end{aligned}$$

and $(\mathrm{i})_{\mathrm{W}}$ follows. To prove $(\mathrm{ii})_{\mathrm{H}}$ it sufficies to show

$$\max\left\{\sup_{r\in[0,t]}\mathbb{E}\left[u_H^2(r,v)\right],\ \sup_{r\in[0,t]}\sup_{w\in[0,t]}\mathbb{E}\left[(D_w^H u_H(r,v))^2\right]\right\}<+\infty \quad (5.18)$$

for a.e. $v\in[0,t]$. Applying

$$\begin{aligned}D_w^H u_H(r,v)=&\left\langle U_Y(r,v)F(v,X_v),(B^*)^2\zeta\right\rangle_V I_{(v,r]}(w)\\&+\left\langle U_Y(r,v)F_x'(v,X_v)D_w^H X_v,B^*\zeta\right\rangle_V,\end{aligned}$$

using (5.16)

$$\begin{aligned}&\sup_{r\in[0,t]}\sup_{w\in[0,t]}\mathbb{E}\left[(D_w^H u_H(r,v))^2\right]\\&\quad\le K\sup_{r\in[0,t]}\sup_{w\in[0,t]}\Big\{\mathbb{E}\left[\left(C_{B^H}(\omega)C_U C(v)\|(B^*)^2\zeta\|_V\right)^2\right]\\&\qquad+\mathbb{E}\left[\left(C_{B^H}(\omega)C_U C(v)\|D_w^H X_v\|_V\|B^*\zeta\|_V\right)^2\right]\Big\}\\&\quad\le\tilde{K}C^2(v)\left(1+\left(\mathbb{E}\left[C_{B^H}^4(\omega)\right]\right)^{1/2}\sup_{w\in[0,t]}\sup_{z\in[0,t]}\left(\mathbb{E}\left[\|D_w^H X_z\|_V^4\right]\right)^{1/2}\right)\\&\quad\le\bar{K}C^2(v)<+\infty\end{aligned}$$

and

$$\sup_{r\in[0,t]}\mathbb{E}\left[u_H^2(r,v)\right]\le\mathbb{E}\left[\left(C_{B^H}(\omega)C_U C(v)\|B^*\zeta\|_V\right)^2\right]\le\bar{K}C^2(v)<+\infty$$

is obtained for a.e. $v\in[0,t]$ which completes the proof of (5.18).
Finally, the previous part of the proof of (5.18) yields

$$\begin{aligned}&\mathbb{E}\left[\int_0^t\left(\int_0^t u_H(r,v)\,\mathrm{d}B_r^H\right)^2\mathrm{d}v\right]=\int_0^t\mathbb{E}\left[\left(\int_0^t u_H(r,v)\,\mathrm{d}B_r^H\right)^2\right]\mathrm{d}v\\&\quad\le C_{H,2}\int_0^t\left(\mathbb{E}\left[\|u_H(\,.\,,v)\|_{L^2([0,t])}^2\right]+\mathbb{E}\left[\|D^H u_H(\,.\,,v)\|_{L^2([0,t]^2)}^2\right]\right)\mathrm{d}v\\&\quad\le C_{H,2}\int_0^t(t+t^2)\bar{K}C^2(v)\,\mathrm{d}v<+\infty\end{aligned}$$

and $(\mathrm{iii})_{\mathrm{H}}$ follows.

Q.E.D.

The second lemma is based on the basic property of the Skorokhod integral which can be obtained directly from the definition. Let $\tilde{F}\in\mathbb{D}_H^{1,2}$ and

$u \in \operatorname{Dom} \delta_H$ such that $\tilde{F} u$ and $\tilde{F} \delta_H(u) + \langle D^H \tilde{F}, u \rangle_{\mathcal{H}}$ are square integrable. Then $\tilde{F} u \in \operatorname{Dom} \delta_H$ and

$$\delta_H(\tilde{F} u) = \tilde{F} \delta_H(u) - \langle D^H \tilde{F}, u \rangle_{\mathcal{H}}. \tag{5.19}$$

Let $\{e_i\}_{i=1}^{+\infty}$ be an orthonormal basis in V. Fix $i \in \mathbb{N}$ and set

$$F_i = \langle F(v, X_v), e_i \rangle_V, \; u_i(r) = \langle U_Y^*(r, v) B^* \zeta, e_i \rangle_V I_{(v,t]}(r), \; 0 \le v \le r \le T.$$

Lemma 5.10 *The equality*

$$\int_v^t F_i u_i(r) \, \mathrm{d}B_r^H = F_i \int_v^t u_i(r) \, \mathrm{d}B_r^H - \alpha_H \int_0^T \int_v^t \langle F'_x(v, X_v) D_w^H X_v, e_i \rangle_V u_i(r) \, \mathrm{d}r \, \mathrm{d}w$$

holds $\mathbb{P}$ - *a.s. for any* $0 \le v \le t \le T$, $i \in \mathbb{N}$.

Proof Since

$$D_w^H F_i = \langle F'_x(v, X_v) D_w^H X_v, e_i \rangle_V, \; D_w^H u_i(r) = \langle U_Y^*(r, v) (B^*)^2 \zeta, e_i \rangle_V I_{(v,r]}(w),$$

the following estimates

$$\mathbb{E}\,[F_i^4] \le C^4(v) < +\infty,$$
$$\sup_{w \in [0,T]} \mathbb{E}\left[|D_w^H F_i|^2\right] \le C^2(v) \sup_{w \in [0,T]} \sup_{v \in [0,T]} \mathbb{E}\, \|D_w^H X_v\|_V^2 \le C_X C^2(v) < +\infty,$$

and

$$\mathbb{E}\left[\|u_i\|_{L^2([0,T])}^4\right] \le K (C_U \|B^* \zeta\|_V)^4 \mathbb{E}\,[C_{B^H}^4(\omega)] < +\infty,$$
$$\mathbb{E}\left[\|D^H u_i\|_{L^2([0,T]^2)}^2\right] \le K \big(C_U \|(B^*)^2 \zeta\|_V\big)^2 \mathbb{E}\,[C_{B^H}^2(\omega)] < +\infty,$$

hold. Therefore $F_i \in \mathbb{D}_H^{1,2}$, $u_i \in \mathbb{D}_H^{1,2}(|\mathcal{H}|) \subset \operatorname{Dom} \delta_H$ and

$$\mathbb{E}\,[\|F_i u_i\|_{L^2([0,T])}^2] \le \big(\mathbb{E}\,[F_i^4]\big)^{1/2} \Big(\mathbb{E}\,[\|u_i\|_{L^2([0,T])}^4]\Big)^{1/2} < +\infty,$$

which implies $F_i u_i \in L^2(\Omega; |\mathcal{H}|)$. Furthermore,

$$\mathbb{E}\left[\big(F_i \delta_H(u_i)\big)^2\right] \le C^2(v) \mathbb{E}\,[\delta_H^2(u_i)]$$
$$\le C^2(v) C_{H,2} \Big\{\mathbb{E}\left[\|u_i\|_{L^2([0,T])}^2\right] + \mathbb{E}\left[\|D^H u_i\|_{L^2([0,T]^2)}^2\right]\Big\} < +\infty$$

and

$$\mathbb{E}\left[\langle D^H F_i, u_i\rangle_{\mathcal{H}}^2\right] = \alpha_H \mathbb{E}\left[\Big(\int_0^T \int_0^T |r-w|^{2H-2} (D_w^H F_i) u_i(r)\,\mathrm{d}r\,\mathrm{d}w\Big)^2\right]$$
$$\leq \alpha_H C^2(v) \big(C_U \|B^*\zeta\|_V\big)^2 \Big(\int_0^T \int_0^T |r-w|^{2H-2}\,\mathrm{d}r\,\mathrm{d}w\Big)^2$$
$$\times \mathbb{E}\left[C_{B^H}^2(\omega) \sup_{w\in[0,T]} \sup_{v\in[0,T]} \|D_w^H X_v\|_V^2\right]$$
$$\leq K C^2(v) \big(\mathbb{E}\left[C_{B^H}^4(\omega)\right]\big)^{1/2} \sqrt{C_X} < +\infty,$$

hence $F_i \delta_H(u_i) + \langle D^H F_i, u_i\rangle_{\mathcal{H}} \in L^2(\Omega)$ and the assumptions of (5.19) for F_i and u_i are verified. It follows that

$$\int_v^t F_i u_i(r)\,\mathrm{d}B_r^H = F_i \int_v^t u_i(r)\,\mathrm{d}B_r^H - \langle D^H F_i, u_i\rangle_{\mathcal{H}}$$
$$= F_i \int_v^t u_i(r)\,\mathrm{d}B_r^H - \alpha_H \int_0^T \int_v^t \big\langle F_x'(v, X_v) D_w^H X_v, e_i\big\rangle_V u_i(r)\,\mathrm{d}r\,\mathrm{d}w$$

$\mathbb{P}$ - a.s. for any $0 \leq v \leq t \leq T$.

Q.E.D.

The next lemma verifies a limit passage in the Skorokhod integral.

Lemma 5.11 *The identity*

$$\int_v^t \sum_{i=1}^{+\infty} \big\langle F(v, X_v), e_i\big\rangle_V \big\langle U_Y^*(r,v) B^*\zeta, e_i\big\rangle_V\,\mathrm{d}B_r^H$$
$$= \sum_{i=1}^{+\infty} \int_v^t \big\langle F(v, X_v), e_i\big\rangle_V \big\langle U_Y^*(r,v) B^*\zeta, e_i\big\rangle_V\,\mathrm{d}B_r^H \quad \mathbb{P}\text{ - a.s.}$$

holds for any $0 \leq v \leq t \leq T$ *and* $\zeta \in D^*$.

Proof Set

$$\varphi_n(r) = \sum_{i=1}^{n} \big\langle F(v, X_v), e_i\big\rangle_V \big\langle U_Y^*(r,v) B^*\zeta, e_i\big\rangle_V I_{(v,t]}(r),\ n \in \mathbb{N},$$
$$\varphi(r) = \sum_{i=1}^{+\infty} \big\langle F(v, X_v), e_i\big\rangle_V \big\langle U_Y^*(r,v) B^*\zeta, e_i\big\rangle_V I_{(v,t]}(r)$$
$$= \big\langle F(v, X_v), U_Y^*(r,v) B^*\zeta\big\rangle_V I_{(v,t]}(r).$$

Then

$$\varphi_n \xrightarrow[n\to+\infty]{} \varphi \qquad \text{for a.e. } (\omega, r) \in \Omega \times [0,T].$$

Since $\varphi \in L^2\big(\Omega \times L^2([0,T])\big)$ and

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \big(\varphi_n(r) - \varphi(r)\big)^2 \,\mathrm{d}r\right] \\
&= \mathbb{E}\left[\int_0^T \Big(\sum_{i=n+1}^{+\infty} \big\langle F(v, X_v), e_i\big\rangle_V \big\langle U_Y^*(r,v)B^*\zeta, e_i\big\rangle_V I_{(v,t]}(r)\Big)^2 \,\mathrm{d}r\right] \\
&\le \mathbb{E}\left[\int_0^T \Big(\sum_{i=1}^{+\infty} \big\langle F(v, X_v), e_i\big\rangle_V^2\Big)\Big(\sum_{i=1}^{+\infty} \big\langle U_Y^*(r,v)B^*\zeta, e_i\big\rangle_V^2\Big) \,\mathrm{d}r\right] \\
&= \mathbb{E}\left[\int_0^T \|F(v,X_v)\|_V^2 \|U_Y^*(r,v)B^*\zeta\|_V^2 \,\mathrm{d}r\right] \\
&\le C^2(v)(C_U\|B^*\zeta\|_V)^2 T\mathbb{E}\left[C_{B^H}^2(\omega)\right] < +\infty,
\end{aligned}$$

it follows

$$\varphi_n \xrightarrow[n\to+\infty]{} \varphi \qquad \text{in } L^2(\Omega \times [0,T])$$

by the Lebesgue dominated convergence theorem.

The proof of Lemma 5.10 yields $F_i u_i \in \operatorname{Dom}\delta_H$ for any $i \in \mathbb{N}$ so that $\varphi_n \in \operatorname{Dom}\delta_H$. Therefore it sufficies to show that $\left\{\int_0^T \varphi_n(r)\,\mathrm{d}B_r^H, n \in \mathbb{N}\right\}$ converges in $L^2(\Omega)$. This will be proved when one checks that $\{D^H\varphi_n, n \in \mathbb{N}\}$ is convergent in $L^2\big(\Omega \times [0,T]^2\big)$ because

$$\begin{aligned}
&\mathbb{E}\left[\Big(\int_0^T \big(\varphi_n(r) - \varphi_m(r)\big)\,\mathrm{d}B_r^H\Big)^2\right] \\
&\qquad \le C_{H,2}\big\{\mathbb{E}\left[\|\varphi_n - \varphi_m\|^2_{L^2([0,T])}\right] + \mathbb{E}\left[\|D^H\big(\varphi_n - \varphi_m\big)\|^2_{L^2([0,T]^2)}\right]\big\}
\end{aligned}$$

and $\{\varphi_n, n \in \mathbb{N}\}$ is a Cauchy sequence in $L^2(\Omega \times [0,T])$.

Note that

$$\begin{aligned}
D_w^H\varphi_n(r) &= D_w^H\varphi_n^1(r) + D_w^H\varphi_n^2(r) \\
&= \sum_{i=1}^n \big\langle F_x'(v,X_v)D_w^H X_v, e_i\big\rangle_V \big\langle U_Y^*(r,v)B^*\zeta, e_i\big\rangle_V I_{(v,t]}(r) \\
&\quad + \sum_{i=1}^n \big\langle F(v,X_v), e_i\big\rangle_V \big\langle U_Y^*(r,v)(B^*)^2\zeta, e_i\big\rangle_V I_{(v,r]}(w)
\end{aligned}$$

and

$$D_w^H \varphi_n^1(r) \xrightarrow[n\to+\infty]{} \big\langle F_x'(v,X_v) D_w^H X_v, U_Y^*(r,v) B^* \zeta \big\rangle_V I_{(v,t]}(r)$$
$$D_w^H \varphi_n^2(r) \xrightarrow[n\to+\infty]{} \big\langle F(v,X_v), U_Y^*(r,v) (B^*)^2 \zeta \big\rangle_V I_{(v,r]}(w)$$

almost everywhere on $\Omega \times [0,T]^2$. Hence

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \int_0^T \big(D_w^H \varphi_n^1(r)\big)^2 \,\mathrm{d}w\,\mathrm{d}r\right] \\
&\quad\leq \mathbb{E}\left[\int_0^T\!\!\int_0^T \Big(\sum_{i=1}^{+\infty} \big\langle F_x'(v,X_v) D_w^H X_v, e_i\big\rangle_V^2\Big)\Big(\sum_{i=1}^{+\infty} \big\langle U_Y^*(r,v)B^*\zeta, e_i\big\rangle_V^2\Big) \mathrm{d}w\,\mathrm{d}r\right] \\
&\quad= \mathbb{E}\left[\int_0^T \int_0^T \big\|F_x'(v,X_v) D_w^H X_v\big\|_V^2 \big\|U_Y^*(r,v)B^*\zeta\big\|_V^2 \,\mathrm{d}w\,\mathrm{d}r\right] \\
&\quad\leq C^2(v)\big(C_U \|B^*\zeta\|_V\big)^2 T^2 \sqrt{C_X}\,\big(\mathbb{E}\left[C_{B^H}^4(\omega)\right]\big)^{1/2} < +\infty
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \int_0^T \big(D_w^H \varphi_n^2(r)\big)^2 \,\mathrm{d}w\,\mathrm{d}r\right] \\
&\quad\leq \mathbb{E}\left[\int_0^T \int_0^T \big\|F(v,X_v)\big\|_V^2 \big\|U_Y^*(r,v)(B^*)^2\zeta\big\|_V^2 \,\mathrm{d}w\,\mathrm{d}r\right] \\
&\quad\leq C^2(v)\big(C_U \|(B^*)^2\zeta\|_V\big)^2 T^2 \mathbb{E}\left[C_{B^H}^2(\omega)\right] < +\infty.
\end{aligned}$$

Therefore $\{D^H \varphi_n, n \in \mathbb{N}\}$ is convergent in $L^2\big(\Omega \times [0,T]^2\big)$ by the Lebesgue dominated convergence theorem and the assertion follows by the closedness of the Skorokhod integral.

Q.E.D.

In the last lemma Lemma 5.10 and Lemma 5.11 are merged.

LEMMA 5.12 *Under the assumptions of Theorem 5.8 the equality*

$$\begin{aligned}
\int_v^t \big\langle F(v,X_v), U_Y^*(r,v)B^*\zeta\big\rangle_V \,\mathrm{d}B_r^H &= \left\langle F(v,X_v), \int_v^t U_Y^*(r,v)B^*\zeta \,\mathrm{d}B_r^H \right\rangle_V \\
&- \alpha_H \int_0^T \int_v^t |r-w|^{2H-2} \big\langle U_Y(r,v)F'(v,X_v) D_w^H X_v, B^*\zeta\big\rangle_V \,\mathrm{d}r\,\mathrm{d}w
\end{aligned}$$

holds $\mathbb{P}$ *- a.s. for any* $0 \leq v \leq t \leq T$ *and* $\zeta \in D^*$.

Proof Let $\{e_i\}_{i=1}^{+\infty}$ be an orthonormal basis in V. Using Lemma 5.11 and Lemma 5.10 it follows

$$\begin{aligned}
&\int_v^t \big\langle F(v, X_v), U_Y^*(r,v)B^*\zeta\big\rangle_V \,\mathrm{d}B_r^H \\
&\quad = \int_v^t \sum_{i=1}^{+\infty} \big\langle F(v, X_v), e_i\big\rangle_V \big\langle U_Y^*(r,v)B^*\zeta, e_i\big\rangle_V \,\mathrm{d}B_r^H \\
&\quad = \sum_{i=1}^{+\infty} \int_v^t \big\langle F(v, X_v), e_i\big\rangle_V \big\langle U_Y^*(r,v)B^*\zeta, e_i\big\rangle_V \,\mathrm{d}B_r^H \\
&\quad = \sum_{i=1}^{+\infty} \big\langle F(v, X_v), e_i\big\rangle \int_v^t \big\langle U_Y^*(r,v)B^*\zeta, e_i\big\rangle_V \,\mathrm{d}B_r^H \\
&\qquad\quad - \alpha_H \sum_{i=1}^{+\infty} \int_0^T \int_v^t |r-w|^{2H-2} \\
&\qquad\qquad \times \big\langle F_x'(v, X_v) D_w^H X_v, e_i\big\rangle_V \big\langle U_Y^*(r,v)B^*\zeta, e_i\big\rangle_V \,\mathrm{d}r \,\mathrm{d}w \\
&\quad = \left\langle F(v, X_v), \int_v^t U_Y^*(r,v)B^*\zeta \,\mathrm{d}B_r^H \right\rangle_V \\
&\qquad\quad - \alpha_H \int_0^T \int_v^t |r-w|^{2H-2} \big\langle U_Y(r,v) F_x'(v, X_v) D_w^H X_v, B^*\zeta\big\rangle_V \,\mathrm{d}r \,\mathrm{d}w
\end{aligned}$$

$\mathbb{P}$- a.s. for any $0 \le v \le t \le T$.

Q.E.D.

Proof of Theorem 5.8 Fix $\zeta \in D^*$. Similarly as in the proof of Theorem 5.4 applying Lemma 5.9 the equality

$$\begin{aligned}
&\int_0^t \langle X_r, A^*\zeta\rangle_V \,\mathrm{d}r + \int_0^t \langle X_r, B^*\zeta\rangle_V \,\mathrm{d}B_r^H \\
&\quad = \langle U_Y(t,0)x, \zeta\rangle_V - \langle x, \zeta\rangle_V + \int_0^t \int_v^t \big\langle U_Y(r,v)F(v, X_v), A^*\zeta\big\rangle_V \,\mathrm{d}r \,\mathrm{d}v \\
&\qquad + \int_0^t \int_v^t \big\langle U_Y(r,v)F(v, X_v), B^*\zeta\big\rangle_V \,\mathrm{d}B_r^H \,\mathrm{d}v \quad \mathbb{P}\text{- a.s.}
\end{aligned}$$

is obtained for any $t \in [0,T]$.

Note that $\{U_Y(t,s)x, 0 \le s \le t \le T\}$ as a weak solution to the equation (4.1) satisfies

$$\begin{aligned}
\langle U_Y(t,v)x, \zeta\rangle_V = \langle x, \zeta\rangle_V &+ \left\langle x, \int_v^t U_Y^*(r,v)A^*\zeta \,\mathrm{d}r \right\rangle_V \\
&+ \left\langle x, \int_v^t U_Y^*(r,v)B^*\zeta \,\mathrm{d}B_r^H \right\rangle_V \quad \mathbb{P}\text{- a.s.}
\end{aligned}$$

for any $0 \leq v \leq t \leq T$. By Lemma 5.12 it follows

$$
\begin{aligned}
&\int_0^t \langle X_r, A^*\zeta\rangle_V \,\mathrm{d}r + \int_0^t \langle X_r, B^*\zeta\rangle_V \,\mathrm{d}B_r^H \\
&= \langle U_Y(t,0)x, \zeta\rangle_V - \langle x, \zeta\rangle_V + \int_0^t \left\langle F(v, X_v), \int_v^t U_Y^*(r,v)A^*\zeta \,\mathrm{d}r \right\rangle_V \mathrm{d}v \\
&\quad + \int_0^t \left\langle F(v, X_v), \int_v^t U_Y^*(r,v)B^*\zeta \,\mathrm{d}B_r^H \right\rangle_V \mathrm{d}v \\
&\quad - \int_0^t \alpha_H \int_0^T \int_v^t |r-w|^{2H-2} \big\langle U_Y(r,v)F_x'(v, X_v)D_w^H X_v, B^*\zeta\big\rangle_V \mathrm{d}r \,\mathrm{d}w \,\mathrm{d}v \\
&= \langle U_Y(t,0)x, \zeta\rangle_V - \langle x, \zeta\rangle_V \\
&\quad + \int_0^t \left(\big\langle U_Y(t,v)F(v, X_v), \zeta\big\rangle_V - \langle F(v, X_v), \zeta\rangle_V \right) \mathrm{d}v \\
&\quad - \int_0^t \alpha_H \int_0^T \int_v^t |r-w|^{2H-2} \big\langle U_Y(r,v)F_x'(v, X_v)D_w^H X_v, B^*\zeta\big\rangle_V \mathrm{d}r \,\mathrm{d}w \,\mathrm{d}v \\
&= \langle X_t, \zeta\rangle_V - \langle x, \zeta\rangle_V - \int_0^t \langle F(v, X_v), \zeta\rangle_V \,\mathrm{d}v \\
&\quad - \int_0^t \alpha_H \int_0^T \int_v^t |r-w|^{2H-2} \big\langle U_Y(r,v)F_x'(v, X_v)D_w^H X_v, B^*\zeta\big\rangle_V \mathrm{d}r \,\mathrm{d}w \,\mathrm{d}v
\end{aligned}
$$

$\mathbb{P}$ - a.s. for any $t \in [0, T]$ and the assertion holds.

Q.E.D.

Conclusion

Stochastic differential equations with a multiplicative fractional noise in a separable Hilbert space were studied using the semigroup approach. An explicit formula for a solution to the linear equation was given. The regular case $H > 1/2$ provides all types of solution depending on the choice of initial value (Theorem 2.4) contrary to the singular case $H < 1/2$ which admits only a weak solution (Theorem 2.6). An application of these theorems was illustrated on several examples in Chapter 3. Moreover, the large time behaviour of the solution was studied.

While the linear problem is an expected generalization of results obtained in [8] for a Wiener case, the nonlinear perturbation of a drift part seems to be more complicated and studying of this problem does not give an expected result inspired by the corresponding equation in [5] with a Wiener process.

The regular case $H > 1/2$ and a solution U_Y to the linear equation starting from any time (not only from zero), Theorem 4.2, were considered. Then a "mild" formulation of the nonlinear problem using this solution (Theorem 5.1) implies a weak one only for a space independent perturbation (Theorem 5.4). When a space dependent perturbation is assumed, the "mild" formulation implies an integral weak formulation with added term (Theorem 5.8).

As it was shown in Example 5.6, if U_Y is replaced by other family of operators, the "mild" formulation implies a weak one for a perturbation depending on a solution. Therefore there is a natural question whether it is possible to find some "mild" formulation (using some unknown U_Y) which would imply the weak one.

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List of Abbreviations

$\mathcal{L}(V)$	the space of all linear bounded operators from V to V
$\mathcal{L}(D;V)$	the space of all linear bounded operators from D to V
$\mathcal{C}([0,T];V)$	the space of all continuous functions from the interval $[0,T]$ to V
$\mathcal{C}_b^\infty(\mathbb{R}^n)$	the space of all $\mathbb{R}^n$-valued functions which are bounded with all their partial derivatives
$(-A(0))^\alpha$	the fractional power of the operator $-A(0)$ defined for any $\alpha \in (0,1]$
D^H	derivative operator (Malliavin derivative) with respect to fractional Brownian motion
$\mathbb{D}_H^{1,p}$	the Sobolev space associated with Malliavin derivative D^H(domain of D^H), $p \geq 1$
D^W	derivative operator (Malliavin derivative) with respect to Wiener process
$\mathbb{D}_W^{1,2}$	the Sobolev space associated with D^W(domain of D^W)
$\operatorname{Dom}\delta_H$	the domain of the Skorokhod integral with respect to fractional Brownian motion
$\operatorname{Dom}\delta_W$	the domain of the Skorokhod integral with respect to Wiener process
$\mathrm{B}(a,b)$	the Beta function, $a>0, b>0$