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Alternative mathematical notation and its applications in calculus

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I would like to dedicate this thesis to my fiancée who supported me with her love during its creation.

I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Alternative mathematical notation and its applications in calculus

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Abstrakt: Práce zkoumá možnosti formalizace klasických pojmů matematické analýzy bez použití proměnných. Za tímto účelem vytváří nový matematický „jazyk", jenž je schopen popsat všechny klasické výpočty v matematické analýze (přesněji výpočty limit, konečných diferencí, jednorozměrných derivací a určitých a neurčitých integrálů) bez použití proměnných. Výpočty zapsané v tomto „jazyce" obsahují pouze symboly funkcí (a jsou tedy zcela rigorózní a nedávají prostor k vágnímu výkladu použitých symbolů). Obecně jsou také výrazně kratší a matematicky průhlednější než jejich tradiční verze (např. při výpočtech integrálů není potřeba zavádět žádné nové symboly a určitý integrál je formalizován tak, že všechna pravidla pro výpočet neurčitých integrálů (včetně „substitučních" pravidel) jsou přímo přenosná na případ určitých integrálů. Práce také formalizuje Landauovu o-notaci způsobem, díky němuž je možné provádět s ní výpočty limit zcela rigorózním způsobem.

Klíčová slova: alternativní matematická notace, kalkulus, konečné diference, integrace, derivování

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Abstract: We explore the possibility of formalizing classical notions in calculus without using the notion of variable. We provide a new mathematical 'language' capable of performing all classical computations (namely computing limits, finite differences, one-dimensional derivatives, and indefinite and definite integrals) without any need to introduce a variable. Equations written using our notation contain only function symbols (and as such are completely rigorous and don't leave any room for vague interpretations). They also tend to be much shorter and more mathematically transparent than their traditional counterparts (for example, there is no need for introduction of new symbols in integration, and definite integration is formalized in such a way that all rules (including 'substitution' rules) for indefinite integration translate directly to definite integration). We also fully formalize the Landau little-o notation in a way that makes computation of limits using it fully rigorous.

Keywords: alternative mathematical notation, calculus, finite differences, integration, differentiation

Contents

Motivation of a variableless approach

For centuries, mathematicians have been using notation that is "intuitively appealing" but not conforming to modern set theory and logic. Probably every mathematician believes that the traditional notation[1] could be translated into a completely rigorous symbolic language if necessary (and this is indeed true, as we will show in this text), and it is generally understood that elementary operations in calculus (i.e. computing limits, derivatives, integrals etc.) are just examples of application of an operator to a function (for example, if we denote the operator of one-dimensional derivative by ∂, it is completely natural to state the rule for the derivative of a sum of functions as $\partial(f+g) = \partial f + \partial g$, without any reference to a variable).

However, in practical computations, we always apply these operators "with respect to a variable". The reason why we cannot get rid of the variable lies basically in the fact that there is no function symbol for "x^n"; we cannot write, for example, $x^3(2) = 8$. In this text, we will use the symbol ι^n to denote this function, i.e. $\iota^n(x) = x^n$ (and, in particular, $\iota(x) = x$), so we can write, for example, $\sin \circ \iota^2$ to represent the function defined as $(\sin \circ \iota^2)(x) = \sin(x^2)$. However, because writing the circle symbol all the time would be impractical, we will write $f[g]$ instead of $f \circ g$. The chain rule $(f(g(x)))' = f'(g(x))g'(x)$ for differentiation translates as $\partial(f[g]) = \partial f[g]\, \partial g$ into a "variableless" notation, so we see that, for example, instead of $(\sin(x^2))' = \sin'(x^2)(x^2)' = \cos(x^2)2x$ we could write $\partial(\sin[\iota^2]) = \partial(\sin)[\iota^2]\, \partial(\iota^2) = \cos[\iota^2]\, 2\iota$ (and it is not hard to convince oneself that, thanks to "ι", any such computation could be written without using variables).

It may seem at first that the variableless notation only uses different symbols to denote the same thing, and so there is little reason to abandon the standard notation; however, we will also see that the variableless notation provides new methods of computation, not present in the traditional notation. For example, we can write $(\iota^2 + \iota^3 + \iota^4)[\sin + \cos]$ (or, more traditionally, $(\iota^2 + \iota^3 + \iota^4) \circ (\sin + \cos)$), a perfectly well defined function, whose derivative could be computed using the chain rule:

$$\partial\big((\iota^2 + \iota^3 + \iota^4)[\sin + \cos]\big) = (2\iota + 3\iota^2 + 4\iota^3)[\sin + \cos]\,(\cos - \sin)\ .$$

This process doesn't have any direct traditional notational counterpart; we just have to write $(\sin x + \cos x)^2 + (\sin x + \cos x)^3 + (\sin x + \cos x)^4$, which also means that, to differentiate it, we have to perform many operations repeatedly. The reader can surely imagine that by nesting the previous construction (e.g. by writing $(\iota^2 + \iota^3 + \iota^4)[\sin + \cos][\tan + \cot + \iota^5]$) we can denote by relatively simple expressions functions whose expression in the traditional notation would become ridiculously long, and whose differentiation would be a very time-consuming process.

In integration, the inability of the standard notation to record the real mathematical structure becomes even more pronounced. For simplicity, let's think about

[1] When we refer to *the* traditional (or standard) notation, we mean any of the conventions that are standardly in use. For example, both meaning of the symbol $\subset$ (a subset or a proper subset) are considered to be a part of the standard notation, although we always use only one meaning in this text.

the symbol $\int f$ as about one particular primitive function of f, i.e. a function such that $\partial(\int f) = f$ (later on, the symbol $\int f$ will denote the *set* of all primitive functions of f, and this fact will be crucial for the whole process to be completely correct). By differentiating the function $(\int f)[g]$ (for some function g), we get (using the chain rule) $\partial\big((\int f)[g]\big) = f[g]\,\partial g$, i.e. $(\int f)[g]$ is a primitive function of $f[g]\,\partial g$. If we postpone the question of integration constants for a moment, we can write

$$\int f[g]\,\partial g = \Big(\int f\Big)[g] \ .$$

This is intuitively appealing (provided that we know that constants do not cause a problem), because we get an equality upon differentiating both sides of this equation. In the traditional notation, this would be written as

$$\int f(g(x))g'(x)\,dx = \int f(t)\,dt\,\bigg|_{t=g(x)}$$

which one readily recognizes as the traditional integration by substitution. The reason we have to use the "substitution" is that we are not able to work with the function f independently, without a parameter it is applied to (and this makes the whole process of integration much less transparent). For example, in the variableless notation we would write

$$\int \cos[\sin]\cos = \Big(\int \cos\Big)[\sin] = \sin[\sin] \ ,$$

and using variables (forgetting about the constants again):

$$\int \cos(\sin(x))\cos(x)\,dx = \int \cos(t)\,dt\,\bigg|_{t=\sin(x)} = \sin(t) = \sin(\sin(x)) \ .$$

Definite integration in the traditional notation is performed in two independent steps; first, the indefinite integral of a function is computed, and then the limits are applied to the result. Although these steps are independent, they are inseparable in the traditional notation; we may, however, separate them by defining a new symbol (resp. operator):

$$\mathop{\mathrm{I}}\limits_a^b f = f(b) - f(a) \ .$$

This operator has some obvious properties, such as $\mathrm{I}_a^b\, f[g] = f[g](b) - f[g](a) = f(g(b)) - f(g(a)) = \mathrm{I}_{g(a)}^{g(b)}\, f$. The definite integral of f from a to b can be written as $\mathrm{I}_a^b \int f$. Using the mentioned property, we get the substitution rule for definite integration for free:

$$\mathop{\mathrm{I}}\limits_a^b \int f[g]\,\partial g = \mathop{\mathrm{I}}\limits_a^b \Big(\int f\Big)[g] = \mathop{\mathrm{I}}\limits_{g(a)}^{g(b)} \int f \ ,$$

whose mathematical meaning is much clearer at first sight than in the traditional notation:

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(t)\,dt \ .$$

1. Basic notational conventions

The purpose of this section is to both enrich and regularize the language of mathematics. The standard mathematical notation lacks symbols for some mathematical constructions used in sections 2 and 3; however, most symbols defined in section 1 completely or partially agree with the traditional notation. The following table contains a list of all used symbols that agree with the standard notation. Descriptions typeset in *italics* signify that, in practical terms, the symbol means the same as in the traditional notation, but its mathematical interpretation slightly changes.

e, π	common mathematical constants
$\{x : \phi(x)\}$	the set of all x such that $\phi(x)$ holds
$A \subseteq B$	A is a subset of B
$A \subset B$	A is a proper subset of B
$<, \leq, >, \geq$	common order relations on $\mathbb{R}$
(a, b)	the open interval $\{x \in \mathbb{R} : a < x < b\}$
$[a, b]$	the closed interval $\{x \in \mathbb{R} : a \leq x \leq b\}$
$[a, b), [b, a)$	half-open/half-closed intervals
$f : A \to B$	f is a mapping from A to B
f^{-1}	the inverse function of f
$(x_1, x_2, \ldots, x_n)$	an ordered n-tuple
a_n	*the n^{th} member of a sequence a*
$a + b, a \cdot b, \frac{a}{b}, \ldots$	*common binary operations*
$A \cap B, A \cup B, A \setminus B, \ldots$	*common set operations*
a^b	*a to the power b*
$\sqrt[n]{x}$	*the n^{th} root of x*
$\sum_{i=m}^{n}, \prod_{i=m}^{n}$	*finite sums and products*
Δ	*the forward difference operator*
$\lim_{x \to a} (\ldots)$	*the limit of "$(\ldots)$" for $x \to a$*

Furthermore, there are many symbols that don't fully agree with the traditional notation or don't have a traditional counterpart. Descriptions are listed together with the number of section these notions are defined in.

$f[x]$	application of a mapping f to x	1.1
$R[\![X]\!]$	the image of a set X under a relation R	1.1
$R[\![x]$	set of all y such that yRx	1.1
$R[X]\!]$	y such that $(\forall x \in X)(yRx)$	1.1
Ω	the "undefined"	1.1
ι	the identity function on $\mathbb{R}$	1.2
f_A	restriction of a function f to a set A	1.2
$f[g]$	composition of f and g	1.2
$\bullet$	a placeholder	1.3
ι^n, a^ι	power and exponential functions	1.4
$\underset{x}{D}$ or $D_{:x}$	application of D with respect to x	1.5
$\sum_m^n, \prod_m^n$	sum and product operators	1.6

Δ_h	growth operators	2.2
$\lim_a$	the limit operator at a	2.1
Lim	the sequence limit operator	2.1
∂	the derivative operator	2.3
$o_a^n[f]$	remainder of the Taylor polynomial of f of order n at a	2.4
$\sum_h, \sum$	indefinite sum operators	3.1
C_h	the set of all h-periodic functions on $\mathbb{R}$	3.1
$\sum_{a,h}^b$	definite sum operators	3.2
I_a^b	the (continuous) operator of limits from a to b	3.2
$\int$	the indefinite integral operator	3.3
C	the set of all constant functions on $\mathbb{R}$	3.3
$\int_a^b$	definite integral operators	3.4

Note (*on typographical conventions*). Typographical conventions used in this text are mostly standard. We number only referenced equations, and, to avoid possible confusion, we always insert one blank space before punctuation in separate equations. Definitions, theorems, lemmas and larger examples share common numbering (for example, after Theorem 2.2 follows Example 2.3 and Theorem 2.4).

Definitions, theorems etc. are thought of as blocks of text separated from the rest of the text. To help the reader to quickly see the boundaries of these blocks, they always end with a small black square placed to the right bottom corner (this note is an example of such a block).

In definitions, the ***term being defined*** is typeset in bold italic type, as are titles of theorems and lemmas (and this note). Theorems and lemmas generally consist of two parts, the statement of the theorem and the proof. The black square symbol is inserted at the end of the proof (and so we don't have to mark it further by adding a white square symbol). If the statement itself ends with a black square (and no proof part is present), it means that the proof has already been explained in foregoing text. ■

1.1 Operations with relations

A binary relation is commonly defined as a set of pairs. For us, the set-theoretical representation of relations is irrelevant; we will only use the notation yRx for "y and x are in relation through R".

Let $f : X \to Y$ be a function. The application of f to $x \in X$ will be denoted by $f[x]$, i.e.

$$f[x] = y \iff (\forall z)(zfx \Leftrightarrow z = y) \ .$$

This would be traditionally denoted $f(x)$, so a natural question of why to abandon this convention arises[1]. The problem is in the ambiguity of expressions like $f(x+y)$; it is not clear whether we mean application of f to $x+y$, or the product $f \cdot (x+y)$.

[1] One could also note that we write functions as relations in the "wrong way", i.e. we write $f[x] = y$ as yfx instead of the more traditional xfy. This is to assure consistency with notation used for general relations (in fact, even in the notation $f[x] = y$, f doesn't have to be a function, the symbol merely means that there is only one value in relation with x).

While this is usually obvious in traditional calculus, it would reduce intelligibility in variableless calculus presented in this text; for us, round parentheses will always denote boundaries of expressions.

Let R be a relation and X a set. $R[\![X]\!]$ denotes the image of X under R, that is, the set consisting of all y for which yRx for some $x \in X$. Written symbolically:

$$R[\![X]\!] = \{y : (\exists x \in X)(yRx)\} \ .$$

For example, $\sin[\![\mathbb{R}]\!] = [-1, 1]$. When X doesn't contain any element in the domain of R, $R[\![X]\!]$ is the empty set.

There are also two notions lying somewhere between the previous two. Let R be a relation. Define

$$R[\![x] = \{y : yRx\} \ ,$$

that is, $R[\![x]$ is the set of all elements that are in relation with x. A few examples will quickly clarify the meaning. If R is an equivalence relation, then $R[\![x]$ is simply the equivalence class of x. Indeed, by definition, $R[\![x]$ is the set of all y such that y is equivalent to x (i.e. yRx holds). Let $\leq$ denote the standard order on the set of real numbers. What is $\leq[\![x]$? The set of all y such that $y \leq x$, i.e. $\leq[\![x] = (-\infty, x]$.

It might seem strange to use a closing bracket that differs from the opening one; however, it is logical: single brackets symbolize an "element", whereas double brackets symbolize a "set". The symbol $R[\![X]\!]$ should be thought of as "taking a set, producing a set" and the symbol $R[\![x]$ as "taking an element, producing a set".

What should $R[X]\!]$ mean? It's "taking a set, producing an element", so the following definition shouldn't be surprising:

$$R[X]\!] = y \Leftrightarrow R[\![X]\!] = \{y\} \ .$$

If all elements of X are mapped to the same y by R, then $R[X]\!]$ is this y (for example, $\sin[\![\{0, \pi, 2\pi\}]\!] = \{0\}$, whereas $\sin[\{0, \pi, 2\pi\}]\!] = 0$).

Example 1.1. Why should be the $R[X]\!]$ notation an important concept in calculus? Let ∂ denote the operator of derivative in one dimension (so $\partial[f] = f'$ in the traditional notation, for example $\partial[\sin] = \cos$). Let $F = \{\sin + c : c \in \mathbb{R}\}$ (meaning the set of all functions f of the form $f(x) = \sin(x) + c$ where c is a real constant, i.e. the indefinite integral (in the sense to be made precise later) of cos). What is $\partial[F]\!]$? Obviously, $\partial[\![F]\!] = \{\partial[\sin + c] : c \in R\} = \{\cos\}$, hence $\partial[F]\!] = \cos$. ∎

It is customary to drop parentheses when denoting application of an operator to a function. We shall follow this convention and often use expressions like ∂f instead of $\partial[f]$. Similarly, if F is a set of functions, we will often write ∂F instead of $\partial[F]\!]$. The same rule applies to $R[\![\ldots]$; for example, if $\sim$ is an equivalence relation, the equivalence class of x could be denoted $\sim x$.

However, we will always write brackets in application of a function that is not an operator (e.g. $\sin[x]$), so when such a function symbol is not followed by brackets, it is immediately clear that we mean multiplication (thus, strictly speaking, $\sin x$ would denote $\sin \cdot x$ in our notation, but we will avoid such confusing expressions).

The undefined

The 'element-producing' constructions can, from the traditional viewpoint, be "undefined". If x is not in the domain of f, then $f(x)$ "doesn't make sense". However, it is somewhat unfortunate that the very meaningfulness of a formula depends on the particular value of symbols it contains, and, strictly speaking, such approach is incorrect from the viewpoint of logic and set theory. We will adopt a different approach. Let Ω (symbolizing the letter U written upside down) be a constant (that is, from the viewpoint of set theory just a set) whose set structure is irrelevant; it is just a symbol in a manner analogous to ∞. Whenever a set-theoretical construction should be "undefined", we define it to be equal to Ω and call Ω the ***undefined***.

In other words, if x doesn't lie in the domain of f, then $f[x] = \Omega$ (which would be read as "f of x is undefined"). Similarly, if $f[\![X]\!]$ is not a singleton, then $f[X] = \Omega$. This will simplify statements like "a limit doesn't exist", "a sum isn't convergent" etc. We will also follow the convention that Ω doesn't lie in the domain of any function we work with. This means in particular that whenever any part of an expression is undefined, the whole expression is undefined, for example $\sin[5 + \frac{1}{0}] = \sin[5 + \Omega] = \sin[\Omega] = \Omega$.

1.2 Basic constructions with functions

The identity function (on some set understood from the context) is commonly denoted by 1 or id, neither of which would be convenient for our purposes. The real identity function will be denoted by ι (the Greek letter iota, slightly vertically extended), that is[2]

$$\iota[x] = \begin{cases} x & \text{for } x \in \mathbb{R} \\ \Omega & \text{otherwise} \end{cases} .$$

Restriction of a function f to a set A (which is usually a subset of its original domain) is denoted f_A and defined by

$$f_A[x] = \begin{cases} f[x] & \text{for } x \in A \\ \Omega & \text{otherwise} \end{cases} .$$

For example, $\log_{(0,2]}$ is exactly the Taylor series of log around 1. Using this notation, the identity function on a set $A \subset \mathbb{R}$ can be naturally written as ι_A.

For a constant c, the symbol c will represent both the constant c and the constant function $x \mapsto c$ whose domain is understood from the context (which is a common convention). Combining this notion with the notation defined in the previous paragraph, we can denote naturally constant functions on a particular domain. For example, $1_{\mathbb{R}}$ denotes the constant function 1 restricted to $\mathbb{R}$, that is, the function $1_{\mathbb{R}}[x] = 1$ for $x \in \mathbb{R}$, Ω otherwise.

[2]One might be wondering why we use a variable to define elementary notions in *variableless* calculus. The reason is that there are two different ways in which variables are used (which is explained in greater detail in section 1.5). To use variables to *define* functions may be quite useful; however, what we will be trying to avoid is a reference to variables when we apply operators to functions that are already defined.

Composition of functions f and g is commonly denoted $f \circ g$ and defined by $(f \circ g)[x] = f[g[x]]$. Since composition of functions will be perhaps the most commonly used concept in this text, a more convenient notation is in order here. We shall use the notation[3]

$$f[g] = f \circ g \ ,$$

i.e. $f[g][x] = f[g[x]]$. This notation creates ambiguity with the notation for function application, but this ambiguity is actually desirable, because composition and application are virtually the same concept (for example, if π denotes the constant function $\pi_{\mathbb{R}}$, the composition $\sin[\pi]$ is the constant function 0); in other words, composition of functions is just an "application" of a function to a "parameter" that varies as well.

Since application of a function to a parameter is sometimes denoted without parentheses (e.g. $\sin\cos x$), one might be tempted to write also composition without brackets (e.g. $\sin\cos$ instead of $\sin[\cos]$). However, we will not use this notation for functions that are not operators (such as sin or cos) as we do not denote application of such functions without brackets (that is, $\sin\cos$ will always denote the function defined as $(\sin\cos)[x] = \sin[x]\cos[x]$). We may, however, use this notation for operators.

Functions of multiple variables

An n-tuple will be denoted by round parentheses and commas. Its set-theoretical definition is immaterial, the only important property is that $(a, b, \dots) = (a', b', \dots)$ is equivalent to $a = a', b = b', \dots$ The Cartesian product of $X_1, \dots, X_n$ is the set of n-tuples $(x_1, x_2, \dots, x_n)$ such that $x_i \in X_i$. That is:

$$X_1 \times \cdots \times X_n = \big\{(x_1, \dots, x_n) : x_i \in X_i\big\} \ .$$

Functions of more than one variable are simply functions operating on some Cartesian product, i.e. their parameters are n-tuples. Instead of $f[(a, b, \dots)]$ we shall use also a shorter version $f[a, b, \dots]$.

1.3 Arbitrarily placed parameters

It is customary to denote application of a function to parameters by placing these parameters into larger symbols, for example, $\sqrt[n]{x}$ denotes application of a function of two variables to (n, x). This process is often not understood as function application, but merely as a symbol (that can be used to define a function). For us, this will be just a typographical convention to denote parameters of functions.

Whenever a function symbol contains the symbol $\bullet$ (called a ***placeholder***), the parameter of this function can be written instead of it (and resized in accordance with its size). For example, let $a_\bullet : \mathbb{N} \to \mathbb{R}$ be a function. Instead of $a_\bullet[n]$ we can write a_n. We will usually write the parameter instead of $\bullet$ when it is "small", e.g.

[3] While this might seem unnecessary at first sight, it is a very useful convention, as it reduces greatly the number of parentheses required in commonly used expressions. For example, instead of $\big(f \circ (g + h)\big)\big((p \circ (q + r)\big)$ we would write simply $f[g + h]p[q + r]$.

a_n, and ordinarily after the function symbol when it is "large", e.g. $a_{\bullet}[\frac{a+b}{c+\frac{d}{e}}]$ instead of $a_{\frac{a+b}{c+\frac{d}{e}}}$ (which is typographically inappropriate).

Composition of a function whose symbol contains a placeholder with another function will be denoted simply by putting it in place of the placeholder. For example, if $\sqrt{\bullet}$ denotes the square root function, $\sqrt{\bullet} \circ f$ would be denoted $\sqrt{f}$. We will also avoid placeholders whenever possible and write expressions like $\sqrt{l}$ instead of $\sqrt{\bullet}$.

This concept generalizes naturally to any number of placeholders[4]. The most important instance of this construction is the notation $\bullet\ \bullet\ \bullet$ for binary operations. We define

$$(\bullet\ \bullet\ \bullet)[a, f, b] = f[a, b]\ .$$

For example, for $+ : \mathbb{R}^2 \to \mathbb{R}$ being the ordinary addition of real numbers, $a + b = (\bullet\ \bullet\ \bullet)[a, +, b] = +[a, b]$. This construction is defined for any binary function but will be used only for those thought of as binary operations.

Every placeholder can be replaced by a parameter with any type of brackets, because virtually for any combination of brackets there is only one reasonable interpretation of the resulting expression. For example, it is obvious that $\frac{1}{2} + [\![\mathbb{Z}]\!]$ denotes the set $\{\frac{1}{2} + k : k \in \mathbb{Z}\}$. In this case, the meaning is so obvious that we would drop the brackets altogether and write simply $\frac{1}{2} + \mathbb{Z}$. Nevertheless, there are also non-trivial cases.

Example 1.2. Let T be a topology on a set X. The set

$$X \setminus [\![T]\!] = \{X \setminus t : t \in T\}$$

is the set of all closed sets in the topology T. Let $A \subseteq X$, the set

$$A \cap [\![T]\!] = \{A \cap t : t \in T\}$$

is the subspace topology on A (topology is a large source of such examples for its broad use of operations with sets). ∎

Note. It is often typographically convenient to expand horizontal components of a function symbol when its parameter is too wide. For example, $\frac{\bullet}{\bullet}$ is defined by $\frac{\bullet}{\bullet}[a, b] = a/b$. When larger parameters are written in place of $\bullet$, the horizontal line extends to fit their width (e.g. $\frac{a+b}{c+d}$).

Function symbols often consist of several horizontally separated symbols, for example $[\bullet, \bullet)$ is the function which returns the half open interval $[a, b)$ when applied to the pair (a, b). In this case, when the parameter is too wide, the distance of individual symbols increases, e.g. $[a + b, c + d) = [\bullet, \bullet)[a + b, c + d]$. ∎

[4] One might argue that we should specify the order in which the parameters of a function symbol containing multiple placeholders should be written; however, there will be usually only one reasonable choice.

1.4 The exponential and the power function

Let $n \in \mathbb{N}$ and let $*$ be an associative binary operation (or a binary operation with a natural n-ary version). The construction $\bullet^{\bullet\bullet}$ is defined by

$$x^{*n} = \underbrace{x * x * \cdots * x}_{n \text{ times}} . \tag{1.1}$$

This can often be further generalized to $n \in \mathbb{Z}$ (for example when $*$ is a group operation). With this notation, we can distinguish various "power" functions, such as $X^{\cdot n}$ (the matrix power) for a matrix X, $X^{\times n}$ (the Cartesian power) for a set X, $X^{\times n}$ or $X^{\oplus n}$ (the tensor power and the direct sum power) for a vector space X etc., without any need for further verbal explanation. This is, in fact, a standard notation; the symbols $V^{\oplus n}$ and $V^{\otimes n}$ are common in algebraic texts. We just generalize this notion, so we could write, for example, $x^{+n} = x + x + \cdots + x = nx$. Whenever $*$ is obvious from the context, we will drop it and write simply X^n.

Usually, this construction denotes the power function "x^n" defined on real numbers. That is,

$$x^n = \underbrace{x \cdot x \cdot \cdots \cdot x}_{n \text{ times}} .$$

The symbol $\bullet^{\bullet}$ is defined only for integer exponents by (1.1); however, a^b is defined much more generally in $\mathbb{R}$ in the traditional notation and **we shall use it in this generalized sense**, i.e. the symbol a^b denotes what it would denote in any common real calculus textbook.

The function ι^n (i.e. $\bullet^{\bullet}$ composed with ι in the first parameter and applied to n in the second) shall be called the n^{th} ***power*** function. Similarly, a^ι denotes the ***exponential*** function with base a.

More (and also less) generally, $\bullet^{\bullet}_{\bullet}$ will denote the construction defined by

$$x^n_h = x(x-h)\dots(x-(n-1)h) .$$

This is a slight generalization of the so called Pochhammer symbol (or ***falling factorial***, which is the name we will use), which would be traditionally denoted $(x)_n$ or $x^{\underline{n}}$ (which is equal to x^n_1 in our notation). Also, notice that $x^n_0 = x^n$.

In accordance with the traditional notation, we define the symbol $\sqrt[\bullet]{\bullet}$ to denote the "root" function; in particular, $\sqrt[n]{\iota}$ is called the n^{th} ***root function***.

Example 1.3 (*Usage of* ι^n). Sometimes the fact that the standard notation lacks a function symbol for ι^n disallows us to denote functions that are simple in their nature, but require nesting of rational functions or polynomials. Consider the following function:

$$f = (\iota^2 + \iota^3 + \iota^4)[2\iota^2 + 3\iota^3 + 4\iota^4][\iota + \iota^2]$$

(i.e. $f = (\iota^2 + \iota^3 + \iota^4) \circ (2\iota^2 + 3\iota^3 + 4\iota^4) \circ (\iota + \iota^2)$). The structure of this function is clear; it wouldn't be hard to compute its value at some point using only a pen and paper, for example

$$f[1] = (\iota^2 + \iota^3 + \iota^4)[2\iota^2 + 3\iota^3 + 4\iota^4][2] = (\iota^2 + \iota^3 + \iota^4)[96] \approx 86 \cdot 10^6 ,$$

nor would it be hard to differentiate it (differentiation will be considered in greater detail in section 2.3):

$$\partial f = (2\iota + 3\iota^2 + 4\iota^3)[2\iota^2 + 3\iota^3 + 4\iota^4][\iota + \iota^2]\,(4\iota + 9\iota^2 + 16\iota^3)[\iota + \iota^2]\,(1 + 2\iota)\ .$$

In the standard notation, there's no way to write f using a simple formula. Subsequent polynomials are simply put "inside" the preceding ones:

$$\begin{aligned} f(x) = (2(x+x^2)^2 + 3(x+x^2)^3 + 4(x+x^2)^4)^2 + (2(x+x^2)^2 + 3(x+x^2)^3 + \\ + 4(x+x^2)^4)^3 + (2(x+x^2)^2 + 3(x+x^2)^3 + 4(x+x^2)^4)^4\ . \end{aligned}$$

Such a function is too complicated to be of concern, and calculating its derivative by hand would be a nightmare. However, in fact, this complicatedness is caused only by lack of appropriate notation. ■

1.5 Variables

In practice, we often work in an abstract manner with letters (which may represent some physical properties of an object under consideration, mathematical constants etc.) as with "dummy variables" that have the same value throughout the whole computation. For example, x in the expression $f(x) = x^2$ is used as a bond between the sides of the equality, meaning "if you replace me by a particular value, you still get a correct equality", e.g. $f(3) = 3^2$. From the set-theoretical viewpoint, these dummy variables are just variables bound to an implicit universal quantifier, e.g. $f(x) = x^2$ means $(\forall x)(f(x) = x^2)$.

However, expressions like $\frac{\partial}{\partial x}(\sin(xy))$ are of different kind; they denote a vague process of application of an operator *with respect to a variable.* This process will be formalized in this section.

Let F be a functional (i.e. an operator that returns a number when applied to a function). Define:

$$\underset{x}{F}(\text{“expression with } x\text{”}) \equiv F[x \mapsto \text{“expression with } x\text{”}]\ .$$

In text style in LaTeX (i.e. inside text, in the numerator or the denominator of a fraction etc.) we would write $F_{:x}$ instead (with a colon to avoid confusion with a bottom index; the colon can be read as "with respect to").

Example 1.4. Let $\mathrm{Eval}_a[f] = f[a]$ be the *evaluation operator* at a. We can either write, for example $\mathrm{Eval}_2[\iota^2 + \iota + 1] = (\iota^2 + \iota + 1)[2] = 7$, or, using the notion just defined,

$$\underset{x}{\mathrm{Eval}_2}\,(x^2 + x + 1) = 2^2 + 2 + 1 = 7\ .$$

However, it would be more natural to write $\mathrm{Eval}_{x=a} f[x]$ instead of $\mathrm{Eval}_{x:a} f[x]$, resp.

$$\underset{x=a}{\mathrm{Eval}} f[x] \quad \text{instead of} \quad \underset{x}{\mathrm{Eval}_a} f[x]\ .$$

This will often be the case; when there exists a more natural notation for variables than the two possibilities just defined (usually because there is already a standard way to denote variables), we will use this "irregular" natural version. ■

There is also another important class of operators—those that return a function (such as ∂). Let D be such an operator. Define:

$$\underset{x}{D}(\text{“expression with } x\text{”}) \equiv D[x \mapsto \text{“expression with } x\text{”}][x] \ .$$

The letter x is used in two meanings in this expression; firstly, it is a variable with respect to which the operator D is applied, and secondly, it is a "dummy variable". For example, we can write

$$\underset{x}{\partial} \sin[x] = \partial[x \mapsto \sin[x]][x] = \partial[\sin][x] = \cos[x] \ .$$

Example 1.5. Expressions can contain more than one dummy variable; it is completely correct to write

$$f[x,y] = \underset{x}{\partial}(x^2y^3) = 2xy^3 \ . \tag{*}$$

However, notice that this concept is independent of the concept of partial derivatives; no partial derivative operator was used here. Formula (*) can be rewritten as

$$f[x,y] = \underset{x}{\partial}(x^2y^3) = \partial[x \mapsto x^2y^3][x] = \partial[\imath^2y^3][x] = (2\imath y^3)[x] = 2xy^3 \ .$$

■

1.6 Classical operators

When we have an associative binary operation, or an operation having a natural n-ary version for every $n \in \mathbb{N}$ (for example $\oplus$, the direct sum of vector spaces), and a sequence of its parameters (e.g. $V_\bullet$ being a sequence of vector spaces), it is customary to write

$$V_m \oplus V_{m+1} \oplus \cdots \oplus V_{m+n} = \bigoplus_{i=m}^{n} V_i \ .$$

Similarly, for example, $V_m \otimes \cdots \otimes V_n = \bigotimes_{i=m}^n V_i$, or $V_m \times \cdots \times V_n = \bigtimes_{i=m}^n V_i$ etc. This suggests the following construction:

$$\bullet\!\!:[*, m, n][f] = \mathop{\ast}_m^n[f] = f[m] * f[m+1] * \cdots * f[n] \ .$$

The operator $\mathop{\ast}_m^n$ will be usually called by the name of $*$ followed by the word "operator", for example $\bigoplus_m^n$ is a direct sum operator, $\bigotimes_m^n$ is a tensor product operator etc. Its synonymous form can be used in display style (i.e. in a separate formula):

$$\underset{\cdot}{\overset{\cdot}{\bullet}} = \bullet\!\!: \ .$$

There's usually a conventional way to denote variables for these "classical" operators, and we shall follow this way instead of the two defined possibilities. For example, instead of $\bigoplus_{a:k}^b V_k$, we would write

$$\bigoplus_{k=m}^{n} V_k \quad \text{or} \quad \textstyle\bigoplus_{k=m}^n V_k \ .$$

In accordance with the traditional notation, we let $\sum_m^n = \mathop{+}_m^n$ and $\prod_m^n = \bullet_m^n$ (i.e. $\sum_m^n f = f[m] + \cdots + f[n]$ and analogously for $\prod_m^n$) and call $\sum_m^n$ a sum operator and $\prod_m^n$ a product operator.

Another common version of $\sum_m^n$ is traditionally denoted as $\sum_{x \in X} f(x)$, meaning the sum of $f[x]$ over all elements of X. This expression can be understood as the traditional way to write variables for the operator $\sum_X$, that is, $\sum_X f = \sum_{x \in X} f[x]$. In display style, we can use expressions like

$$\sum_{\{0,\pi,2\pi\}} \cos = \sum_{x \in \{0,\pi,2\pi\}} \cos[x] = 1 + (-1) + 1 = 1 \ .$$

It is common to use the same operator symbol for functions as for elements. For example, $\sum_{k=1}^n f_k$ could traditionally denote a sum of functions, that is, a function defined by $(\sum_{k=1}^n f_k)(x) = \sum_{k=1}^n f_k(x)$. We shall follow this convention; $\sum_1^n f_{\bullet} = \sum_{k=1}^n f_k$ will denote the function $f_1 + \cdots + f_n$.

2. Differential calculus

In this section, we present basics of variableless differential calculus. We will learn how to compute limits, finite differences, and derivatives without using variables, and how to translate these computations into computations with variables, when this is desirable. Since we will develop only real one-dimensional calculus in this text, the following convention will save us a lot of repetition.

Convention. In what follows, f and g will always denote real functions of one "real variable", i.e. functions $M \to \mathbb{R}$ where $M \subseteq \mathbb{R}$, and h denotes a real number not equal to zero, unless stated otherwise. ∎

This convention applies here, as well as in the section about integral calculus.

2.1 Limits

Every mathematician has learned how to compute limits using symbolical manipulation. The symbol

$$\lim_{x \to a} f(x)$$

is traditionally defined to "mean" the value L such that $(\forall \varepsilon > 0)(\exists \delta > 0)(f[\![U_\delta[a]]\!] \subseteq B_\varepsilon[L])$, where $B_\alpha[x]$ denotes the open ball of (positive) radius α around x (in some metric space) and $U_\alpha[x] = B_\alpha[x] \setminus \{x\}$.

In $\mathbb{R}$ with the Euclidean metric this translates as $B_\alpha[x] = (x - \alpha, x + \alpha)$ and is extended to $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ by $B_\alpha[\infty] = (1/\alpha, \infty)$, and similarly for $-\infty$. This definition of limit can be rewritten using a more mathematical way of thinking:

Definition 2.1. Let $a \in \bar{\mathbb{R}}$. A function f lies in the domain D of ***the limit operator*** $\lim_a : D \to \bar{\mathbb{R}}$ whenever there is a number $L \in \bar{\mathbb{R}}$ such that $(\forall \varepsilon > 0)(\exists \delta > 0)(f[\![U_\delta[a]]\!] \subseteq B_\varepsilon[L])$. For such f we define[1] $\lim_a f = L$. ∎

The parameter a of $\lim_a$ will be placed below it when written in a separate formula. This definition says nothing else than

$$\lim_{a} f = \lim_{x \to a} f(x) \text{ (in the traditional notation).}$$

If the limit "doesn't exist" (i.e. f doesn't lie in the domain of $\lim_a$), we write simply $\lim_a f = \Omega$ (in agreement with the definiton of Ω).

The arithmetics on $\bar{\mathbb{R}}$ is defined in the standard way, that is $x + \infty = \infty + x = \infty$ for $x \neq -\infty$, $x \cdot \infty = \infty \cdot x = \infty$ for $x \neq 0$, $x/\infty = 0$ for all $x \in \mathbb{R}$, and similarly for $-\infty$. Other expressions are left undefined, that is, $-\infty + \infty = 0 \cdot \infty = 0 \cdot -\infty = \Omega$. Also remember that $a + b = a \cdot b = \Omega$ whenever $a = \Omega$ or $b = \Omega$.

The following elementary theorems about **E**valuation, **N**eighbourhood, **C**omposition, and **A**rithmetics are standard and we will not prove them in this text. Above every equality, one or more of the letters E, N, C or A can be written to express which theorem was used.

[1] Of course, for this definition to be correct, one has to check that if such L exists, it is unique (as in the traditional approach) which is an easy exercise.

Theorem 2.2 (*Evaluation*). If f is continuous at a, then $\lim_a f = f[a]$.

Proof. This is, basically, one way to state the definition of continuity. ∎

Example 2.3. Since $\iota + 1$ is a continuous function (and therefore in particular continuous at 1), we can write

$$\lim_1 (\iota + 1) \overset{\mathrm{E}}{=} (\iota + 1)[1] = 1 + 1 = 2 \ .$$

∎

Theorem 2.4 (*Neighbourhood*). If f and g agree on some neighbourhood of a (excluding a), i.e. $f_{U_\alpha(a)} = g_{U_\alpha(a)}$ for some α (recall that f_A denotes the restriction of f to a set A), then $\lim_a f = \lim_a g$.

Proof. This is a trivial consequence of the definition of the limit operator. ∎

Example 2.5. Consider the function $\frac{\iota-1}{\iota-1}$. It is equal to $1_{\mathbb{R}\setminus\{1\}}$, that is, for any neighbourhood of 1 (excluding 1 itself) it is equal to $1_{\mathbb{R}}$, so using the two previous theorems we can write

$$\lim_1 \frac{\iota - 1}{\iota - 1} \overset{\mathrm{N}}{=} \lim_1 1_{\mathbb{R}} \overset{\mathrm{E}}{=} 1_{\mathbb{R}}[1] = 1 \ .$$

∎

Theorem 2.6 (*Composition*). If f is continuous at $\lim_a g$, then $\lim_a f[g] = f[\lim_a g]$. If g doesn't equal $\lim_a g$ on some neighborhood of a, then $\lim_a f[g] = \lim_{\lim_a g} f$.

Proof. See, for example, [Lang96, p. 49]. ∎

Example 2.7. It is well known that $\lim \frac{\sin}{\iota} = 1$. Since e^ι is continuous, we can write

$$\lim_0 e^\iota\left[\tfrac{\sin}{\iota}\right] = e^\iota\left[\lim_0 \tfrac{\sin}{\iota}\right] = e^\iota[1] = e \ ,$$

which would be usually written more concisely as

$$\lim_0 e^{\frac{\sin}{\iota}} = e^{\lim_0 \frac{\sin}{\iota}} = e^1 = e \ .$$

∎

Theorem 2.8 (*Arithmetics*).

1. If $\lim_a f + \lim_a g \neq \Omega$, then $\lim_a (f + g) = \lim_a f + \lim_a g$.

2. If $\lim_a f \cdot \lim_a g \neq \Omega$, then $\lim_a (fg) = \lim_a f \cdot \lim_a g$.

3. If $\lim_a f / \lim_a g \neq \Omega$, then $\lim_a (f/g) = \lim_a f / \lim_a g$.

Proof. See, for example, [Lang96, p. 45]. ∎

Example 2.9. Combining Examples 2.3, 2.5 and Theorem 2.8, we can write

$$\begin{aligned}\lim_1 \frac{\iota^2-1}{\iota-1} &= \lim_1 \frac{(\iota-1)(\iota+1)}{\iota-1} = \lim_1 \left(1_{\mathbb{R}\setminus\{1\}}\,(\iota+1)\right) \overset{\text{A}}{=} \lim_1 \left(1_{\mathbb{R}\setminus\{1\}}\right) \lim_1 \left(\iota+1\right) \\ &\overset{\text{N}}{=} \lim_1 \left(1_{\mathbb{R}}\right) \lim_1 \left(\iota+1\right) \overset{\text{E}}{=} 1_{\mathbb{R}}[1]\,(\iota+1)[1] = 1\cdot 2 = 2\ . \end{aligned} \tag{2.1}$$

This derivation is as formal as it could possibly be; in practice, however, most of the steps are obvious, so a typical computation will look more like

$$\lim_1 \frac{\iota^2-1}{\iota-1} = \lim_1 \frac{(\iota-1)(\iota+1)}{\iota-1} = \lim_1 \left(\iota+1\right) = 2 \tag{2.2}$$

which resembles traditional

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1}(x+1) = 2\ . \tag{2.3}$$

Nevertheless, it is almost impossible to write a purely formal derivation in the standard notation, whereas in our notation, it's always possible to write all mathematical steps (containing only functions and set-theoretical operations) if necessary. ▪

It might be sometimes convenient to work with variables. This is captured by the notion of "applying a functional with respect to a variable" defined in section 1.5. The notation for variables of $\lim_a$ is borrowed from the traditional notation, that is, we use expressions of the following kind:

$$\lim_{x\to a} f[x] = \lim_a f\ .$$

We see that, in fact, equation (2.3) is just the equation (2.2) rewriten using variables.

The limit operator extends naturally to families of functions and operators:

Definition 2.10. Let $\{f_h : h \in M \subseteq \mathbb{R}\}$ be a family of functions. Define $\lim_a f_{\bullet}$ by $(\lim_a f_{\bullet})[x] = \lim_{h\to a} f_h[x]$. ▪

This is the "pointwise" limit, which might be clearer when written using a variable: $(\lim_{h\to a} f_h)[x] = \lim_{h\to a} f_h[x]$. For example $\lim_{h\to 0}(\iota_h^n) = \iota^n$. This can be further generalized to operators:

Definition 2.11. Let $\{F_h : h \in M \subseteq \mathbb{R}\}$ be a family of operators. Define $\lim_a F_{\bullet}$ by $(\lim_a F_{\bullet})[f] = \lim_{h\to a} F_h[f]$. ▪

This captures the idea of having a family of operators which converges to a single operator whose action on functions is defined pointwise. The definition can be rewritten as $(\lim_{h\to a} F_h)[f][x] = \lim_{h\to a} F_h[f][x]$. The most important example of such operator will be the operator of derivative.

Limits of sequences

There is another important notion of limit in calculus—the limit of a sequence. In the traditional notation, the distinction between the limit of a function and that of a sequence is usually provided by the symbol for the variable, thus, for example, $\lim_{n\to\infty} f(n)$ would most probably denote the limit of a sequence, whereas $\lim_{x\to\infty} f(x)$ would be the limit of a function.

However, the reader may have noticed that the definition of the limit operator is broad enough to include also sequences, for if $a_\bullet : \mathbb{N} \to \mathbb{R}$ is a sequence, $\lim_\infty a_\bullet = \lim_{n\to\infty} a_n$ is already defined to be the limit of this sequence by Definition 2.1 (because it is defined using images of sets, and it is not necessary for a function to be defined for all elements of those sets). So, in fact, the information saying what kind of limit is to be computed is encoded in the domain of the function we apply the limit operator to. For example,

$$\lim_\infty \sin[\pi \iota] = \Omega \ ,$$

whereas

$$\lim_\infty \sin[\pi \iota_\mathbb{N}] = \lim_\infty 0_\mathbb{N} = 0 \ .$$

The last example indicates how to denote the traditional notion of limit over a set:

$$\lim_a f_A = \lim_{\substack{x\to a\\ x\in A}} f(x) \ .$$

Notice that it subsumes also the notion of limit from above and from below. For example the limit traditionally written as $\lim_{x\to a-} f(x)$ can be denoted $\lim_a f_{<a}$; however, we define also $\lim_{a-} f = \lim_a f_{<a}$ and $\lim_{a+} f = \lim_a f_{>a}$ in agreement with the traditional notation.

It is convenient to define a simpler notation for sequences, so we don't have to write ∞ and ${}_\mathbb{N}$ every time. Define:

$$\operatorname{Lim} f = \lim_\infty f_\mathbb{N} \ .$$

Example 2.12. As Lim is just a special case of lim, all the rules of arithmetics hold for it, for example

$$\operatorname{Lim} \frac{2\iota^2 + \iota}{3\iota^2 + (-1)^\iota} = \operatorname{Lim} \frac{2 + 3/\iota}{3 + (-1)^\iota/\iota^2} = \frac{\operatorname{Lim} 2 + \operatorname{Lim}(3/\iota)}{\operatorname{Lim} 3 + \operatorname{Lim}((-1)^\iota/\iota^2)} = \frac{2+0}{3+0} = \tfrac{2}{3} \ .$$

Again, if convenient, we can introduce a variable:

$$\operatorname*{Lim}_n \frac{an^2 + (-1)^n}{n^2 + a} = \operatorname*{Lim}_n \frac{a + (-1)^n/n^2}{1 + a/n^2} = \frac{a+0}{1+0} = a$$
$$\operatorname*{Lim}_a \frac{an^2 + (-1)^n}{n^2 + a} = \operatorname*{Lim}_a \frac{n^2 + (-1)^n/a}{n^2/a + 1} = \frac{n^2+0}{0+1} = n^2 \ .$$

However, such computations can always be translated into computations without variables if desired:

$$\operatorname*{Lim}_n \frac{an^2 + (-1)^n}{n^2 + a} = \operatorname{Lim} \frac{a\iota^2 + (-1)^\iota}{\iota^2 + a} = \operatorname{Lim} \frac{a + (-1)^\iota/\iota^2}{1 + a/\iota^2} = \frac{a+0}{1+0} = a \ .$$

■

As the example of $\sin[\pi\iota]$ suggests, the operators $\lim_\infty$ and Lim are not generally interchangeable. If $\lim_\infty f \neq \Omega$, then obviously $\operatorname{Lim} f = \lim_\infty f$, but the converse doesn't generally hold.

2.2 Finite differences

Differential calculus can be thought of as a "continuous" version of difference calculus. It is therefore pedagogical to define first the difference quotient operators and to show some of their properties (which are similar to those of the derivative operator), and then define derivatives in terms of these simple operators.

Definition 2.13. The ***difference quotient operator*** with step h will be denoted Δ_h and defined by $\Delta_h f = \frac{f[\iota+h]-f}{h}$. ■

The definition can be more traditionally written as $(\Delta_h f)[x] = \frac{f[x+h]-f[x]}{h}$. When no h is given, we suppose that it is equal to 1, i.e. $\Delta f = f[\iota+1] - f$.

Example 2.14. Lets show a few simple computations:

$$\begin{aligned}
\Delta\iota^2 &= (\iota+1)^2 - \iota^2 = \iota^2 + 2\iota + 1 - \iota^2 = 2\iota + 1 \\
\Delta\iota^3 &= (\iota+1)^3 - \iota^3 = \iota^3 + 3\iota^2 + 3\iota + 1 - \iota^3 = 3\iota^2 + 3\iota + 1 \\
\Delta_\pi \sin &= \tfrac{\sin[\iota+\pi]-\sin}{\pi} = \tfrac{-\sin-\sin}{\pi} = -\tfrac{2}{\pi}\sin \ .
\end{aligned}$$

■

The following theorem about basic arithmetics of Δ_h can be proven by a simple computation (however, there is no general formula for $\Delta_h(f[g])$).

Theorem 2.15 (*Arithmetics of* Δ_h).

1. $\Delta_h(cf) = c\,\Delta_h f$, for $c \in \mathbb{R}$,
2. $\Delta_h(f+g) = \Delta_h f + \Delta_h g$,
3. $\Delta_h(fg) = f\Delta_h g + g\Delta_h f + h(\Delta_h f)(\Delta_h g)$,
4. $\Delta_h(f/g) = \dfrac{g\Delta_h f - f\Delta_h g}{g^2 + hg\Delta_h g}$.

■

In fact, the condition $c \in \mathbb{R}$ in point 1. of the previous theorem could be replaced by a weaker condition that c be an h-periodic function.

In the traditional approach, operators of this kind are usually thought of as a symbolical process with "variables"; one would write, for example

$$\Delta(n^2) = (n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1$$

where it is implicitly understood that the difference is "with respect" to n. This approach is captured by the notion of application of an operator with respect to a variable defined in section 1.5:

$$\underset{n}{\Delta}(n^2) = \Delta[\iota^2][n] = (2\iota+1)[n] = 2n + 1 \ .$$

Example 2.16. We have seen that there is no need to introduce a variable in a case we have a function and want to compute its difference. However, in practice we often work with expressions containing several dummy variables and want to apply an operator "with respect" to one of them, for example

$$\underset{n}{\Delta}(n^3m^2) = \Delta[\iota^3m^2][n] = (\Delta[\iota^3]m^2)[n] = ((3\iota^2+3\iota+1)m^2)[n] = (3n^2+3n+1)m^2$$
$$\underset{m}{\Delta}(n^3m^2) = \Delta[n^3\iota^2][m] = (n^3\Delta[\iota^2])[m] = (n^3(2\iota+1))[m] = n^3(2m+1)\ .$$

Of course, one wouldn't write the middle steps in practice, because the variableless definition translates in the obvious way into expressions with variables. ■

Example 2.17. We still miss some functions that behave nicely in connection with Δ_h and could therefore serve as building blocks for more complicated functions, whose difference would be easy to compute. Those are exactly falling factorials:

$$\begin{aligned}\Delta_h(\iota_h^n) &= \frac{(\iota+h)\iota\cdots(\iota-(n-2)h) - \iota\cdots(\iota-(n-1)h)}{h}\\ &= \frac{\iota\cdots(\iota-(n-2)h)(\iota+h-(\iota-(n-1)h))}{h} = \frac{\iota\cdots(\iota-(n-2)h)nh}{h}\\ &= n\iota_h^{n-1}\ .\end{aligned}$$

One way to compute the difference of a power function is to express it through falling factorials (for example $\iota^3 = \iota_1^3 + 3\iota_1^2 + \iota_1^1$, and indeed, $\Delta(\iota_1^3 + 3\iota_1^2 + \iota_1^1) = 3\iota_1^2 + 6\iota_1^1 + 1 = 3\iota^2 - 3\iota + 6\iota + 1 = 3\iota^2 + 3\iota + 1$, in agreement with what we computed previously). Although this is superfluous for computing differences, it will be a useful tool for computing sums. ■

Example 2.18. We would like to find a "finite exponential" that is, a function such that $\Delta_h f = f$. Lets suppose it is of the form a^ι for some a. The equation $\Delta_h(a^\iota) = a^\iota$ can be rewritten as

$$\frac{a^{\iota+h} - a^\iota}{h} = a^\iota \ \Leftrightarrow\ a^\iota(a^h-1) = a^\iota h \ \Leftrightarrow\ a^h - 1 = h\ .$$

The last equation can be rewritten as $a = (1+h)^{1/h}$. Since for $h = 1$, $a = 2$,

$$\Delta(2^\iota) = 2^\iota\ .$$

Furthermore, as $h \to 0$, a tends to e, that is, a^ι are "finite approximations" of e^ι. ■

2.3 Differentiation

The concept of finite differences is much more accessible than the notion of derivative, because all of its properties can be easily derived using only elementary mathematical knowledge. Once the concept of limits is understood, a natural question of what Δ_h as h tends to 0 should mean arises (since $(\Delta_h f)[x]$ expresses the slope of the secant line of f going through points $(x, f[x])$ and $(x+h, f[x+h])$, the natural geometric interpretation is that for $h \to 0$, $(\Delta_h f)[x]$ is the slope of the tangent line of f at x).

Definition 2.19. The ***operator of derivative*** ∂ is defined by $\partial f = \lim_{h\to 0}(\Delta_h f)$, which can be written more concisely as $\partial = \lim_0 \Delta_\bullet$. ▪

In fact, we already know elementary rules of arithmetics for ∂; applying $\lim_{h\to 0}$ to both sides of the equalities in Theorem 2.15 simply changes all occurrences of Δ_h to ∂ and h to 0.

Theorem 2.20 (*Arithmetics of* ∂).

1. $\Delta_h(cf) = c(\Delta_h f) \implies \partial(cf) = c(\partial f)$,
2. $\Delta_h(f+g) = \Delta_h f + \Delta_h g \implies \partial(f+g) = \partial f + \partial g$,
3. $\Delta_h(fg) = f\Delta_h g + g\Delta_h f + h(\Delta_h f)(\Delta_h g) \implies \partial(fg) = f\partial g + g\partial f$,
4. $\Delta_h(f/g) = \dfrac{g\Delta_h f - f\Delta_h g}{g^2 + hg(\Delta_h g)} \implies \partial(f/g) = \dfrac{g\partial f - f\partial g}{g^2}$.

▪

The only result that crucially distinguishes ∂ and Δ is the chain rule for derivatives which doesn't have any direct analogue in the finite case. A simple version of it can be stated as

Theorem 2.21. Let f and g be functions differentiable on their respective domains, then $\partial(f[g]) = \partial f[g]\,\partial g$.

This can be easily generalized to more functions; the case of three functions should make the idea clear (here h is also supposed to be a differentiable function):

$$\partial(f[g][h]) = \partial f[g][h]\,\partial g[h]\,\partial h\ .$$

Proof. See, for example, [Lang96, p. 67]. ▪

The standard rules of differentiation are probably well known to the reader, so it shouldn't be hard to translate statements such as $\partial[\arctan] = \frac{1}{1+\iota^2}$ into the standard notation and vice versa.

Example 2.22. Lets differentiate a more complicated function that would be written as

$$f(x) = \frac{(\sin x + \cos x)^3 + e^{\sin x + \cos x}}{1 + (\sin x + \cos x)^4}$$

in the traditional notation. In our notation, we can notice that the function actually has a simpler structure:

$$f = \frac{(\sin + \cos)^3 + e^{\sin + \cos}}{1 + (\sin + \cos)^4} = \frac{\iota^3 + e^\iota}{1 + \iota^4}[\sin + \cos]\ ,$$

and so we would differentiate it in this form:

$$\partial\left(\frac{\iota^3 + e^\iota}{1 + \iota^4}[\sin + \cos]\right) = \frac{(3\iota^2 + e^\iota)(1 + \iota^4) - (\iota^3 + e^\iota)4\iota^3}{(1 + \iota^4)^2}[\sin + \cos]\,(\cos - \sin)\,.$$

This could, of course, be further simplified, but it is a complete result. There is no way to "notice" the simpler structure in the traditional notation (because there is no way to denote it), thus we are forced to repeat steps many times:

$$\left(\frac{(\sin x+\cos x)^3+e^{\sin x+\cos x}}{1+(\sin x+\cos x)^4}\right)' = ((3(\sin x+\cos x)^2(\cos x-\sin x)+e^{\sin x+\cos x}\cdot$$
$$\cdot(\cos x-\sin x))(1+(\sin x+\cos x)^4)-((\sin x+\cos x)^3+e^{\sin x+\cos x})\cdot$$
$$\cdot 4(\sin x+\cos x)^3(\cos x-\sin x))/(1+(\sin x+\cos x)^4)^2 \ .$$

Further algebraical manipulations with such an expression would be very complicated and its structure is unclear. Also, in terms of computational complexity, the first expression is very friendly, because the value at a particular point (e.g. 0) can be computed quite simply:

$$\begin{aligned}\partial f[0] &= \frac{(3\iota^2+e^\iota)(1+\iota^4)-(\iota^3+e^\iota)4\iota^3}{(1+\iota^4)^2}[\sin 0+\cos 0]\,(\cos 0-\sin 0) = \\ &= \frac{(3\iota^2+e^\iota)(1+\iota^4)-(\iota^3+e^\iota)4\iota^3}{(1+\iota^4)^2}[1] = \frac{(3+e)(1+1)-(1+e)4}{2^2} = \frac{1-e}{2} \ .\end{aligned}$$

An algorithm computing the value using the traditionally written expression would require much more time (because it is necessary to repeat the most time-consuming operations many times). ∎

Of course, everything we have done so far can be rewritten using variables, that is, we can write expressions like

$$\underset{x}{\partial}\big(\sin[x]\cos[x]\big) = \cos^2[x]-\sin^2[x]$$

that resemble the traditional notation. However, in such a situation we don't have to use the variable at all and could write

$$\partial\big(\sin\cos\big) = \cos^2-\sin^2 \ ,$$

which is briefer and more convenient. The real reason to use variables arises when there is more of them.

Example 2.23. We can write

$$\underset{y}{\partial}(x^2y^3) = 3x^2y^2 \ .$$

However, this construction can also be nested. For example, we can write

$$\underset{x}{\partial}\underset{y}{\partial}(x^2y^3) = \underset{x}{\partial}(3x^2y^2) = 6xy^2 \ .$$

Nevertheless, this has nothing to do with more-dimensional differentiation. The whole process would be correctly rewritten as

$$\begin{aligned}\underset{x}{\partial}\underset{y}{\partial}(x^2y^3) &= \underset{x}{\partial}\,\partial[x^2\iota^3][y] = \underset{x}{\partial}(x^2\,3\iota^2)[y] = \underset{x}{\partial}(3x^2y^2) \\ &= \partial[3\iota^2y^2][x] = (6\iota y^2)[x] = 6xy^2 \ .\end{aligned}$$

∎

Traditional counterparts of $\partial_{:x}$ are $\frac{d}{dx}$ and $\frac{\partial}{\partial x}$. The symbol $\frac{\partial}{\partial x}$ is used in two different meanings in the traditional notation. In expressions like

$$\frac{\partial}{\partial x}(x^3y^2) = 3x^2y^2$$

it is synonymous to $\partial_{:x}$, whereas in expressions like

$$\frac{\partial f}{\partial x}(1,1) = 3$$

it denotes the directional derivative of f in direction $(1,0)$. Both $\frac{d}{dx}$ and $\frac{\partial}{\partial x}$ are quite inconvenient and will be avoided in our notation.

2.4 L'Hôpital's rule and the Landau notation

Most limits in practice are computed using l'Hôpital's rule.

Theorem 2.24 (*l'Hôpital's rule*). If

1. $\lim_a f = \lim_a g = 0$ or $\lim_a f = \pm \lim_a g = \pm\infty$, and
2. $\lim_a(\partial f/\partial g) \neq \Omega$,

then $\lim_a(f/g) = \lim_a(\partial f/\partial g)$.

Proof. For a very nice unified proof of all cases, see [Tayl52]. ∎

Its usage can be expressed by putting H above =, for example

Example 2.25.

$$\lim_0 \frac{\sin - \iota}{\iota^3} \overset{\text{H}}{=} \lim_0 \frac{\cos - 1}{3\iota^2} \overset{\text{H}}{=} \lim_0 \frac{-\sin}{6\iota} \overset{\text{H}}{=} \lim_0 \frac{-\cos}{6} \overset{\text{E}}{=} -\tfrac{1}{6}\ .$$

∎

For sequences, we can use the fact that if $\lim_\infty f \neq \Omega$, then $\lim_\infty f = \operatorname{Lim} f$. In combination with l'Hôpital's rule, this leads to a series of equalities $\operatorname{Lim}(f/g) = \lim_\infty(f/g) = \lim_\infty(\partial f/\partial g)$.

Example 2.26.

$$\operatorname{Lim} \frac{e^{1/\iota} - 1}{1/\iota} = \lim_\infty \frac{e^\iota - 1}{\iota}\left[\tfrac{1}{\iota}\right] \overset{\text{C}}{=} \lim_{\lim_\infty \frac{1}{\iota}} \frac{e^\iota - 1}{\iota} = \lim_0 \frac{e^\iota - 1}{\iota} \overset{\text{H}}{=} \lim_0 \frac{e^\iota}{1} \overset{\text{E}}{=} 1 \neq \Omega\ .$$

This would be traditionally denoted by something like

$$\lim_{n\to\infty} \frac{e^{1/n} - 1}{1/n} = \lim_{x\to\infty} \frac{e^{1/x} - 1}{1/x} = \lim_{y \to \lim_{x\to\infty} \frac{1}{x}} \frac{e^y - 1}{y} = \lim_{y\to 0} \frac{e^y - 1}{y} = \lim_{y\to 0} \frac{e^y}{1} = 1\ .$$

∎

The fact that Lim changes to $\lim_\infty$ is important. One could be tempted to compose the equalities into one as $\operatorname{Lim}(f/g) = \operatorname{Lim}(\partial f/\partial g)$ whenever f and g are not only sequences, but are defined on some neighbourhood of ∞, and $\operatorname{Lim}(\partial f/\partial g) \neq \Omega$. However, this is not correct, for example

$$0 = \operatorname{Lim} \frac{\sin[2\pi\iota]}{1/\iota} \neq \operatorname{Lim} \frac{\partial(\sin[2\pi\iota])}{\partial(1/\iota)} = \operatorname{Lim} \frac{\cos[2\pi\iota]\,2\pi}{-1/\iota^2} = -\infty\ .$$

The reason is that $\lim_\infty(\frac{\cos[2\pi\iota]\,2\pi}{-1/\iota^2}) = \Omega$, so the assumptions of Theorem 2.24 aren't satisfied.

Landau notation

Lets define first the notion of Taylor polynomials.

Definition 2.27. The operator T_a^n of ***Taylor polynomial*** of order n at a is defined as

$$\mathrm{T}_a^n[f] = \sum_{k=0}^{n} \frac{\partial^k f[a]}{k!} (\iota - a)^k \ .$$

For $a = 0$, this will be denoted also $\mathrm{T}^n[f]$. ∎

Functions of the form $\mathrm{T}^n[f]$ are functions fully in their own right; for example, we can write $\mathrm{T}^3[\sin][\pi] = (\iota - \iota^3/6)[\pi] = \pi(1-\pi^2/6)$. If f is not n times differentiable at a, then $\mathrm{T}_a^n[f] = \Omega$, i.e. the condition "$f$ is n times differentiable at a" can be symbolically written as $\mathrm{T}_a^n[f] \neq \Omega$.

With the notions we have developed so far, we can fully formalize the Landau little-o notation.

Definition 2.28. The ***remainder*** of the Taylor polynomial of function f around a is defined as $o_a^n[f] = f - \mathrm{T}_a^n[f]$. In particular, define $o^n[f] = o_0^n[f]$. ∎

The following well-known theorem gives us an important information about o^n.

Theorem 2.29 (*Taylor's theorem*). Let $f : O \to \mathbb{R}$ be an n times differentiable function on some open neighborhood $O \subseteq \mathbb{R}$ of a. Then

$$\lim_a \frac{o_a^n[f]}{(\iota - a)^n} = 0 \ .$$

In particular, $\lim_0 \frac{o^n[f]}{\iota^n} = 0$.

Proof. See, for example [Lang96, p. 109]. ∎

Example 2.30. Lets compute Example 2.25 again using this notation:

$$\lim_0 \frac{\sin - \iota}{\iota^3} = \lim_0 \frac{\iota - \frac{1}{6}\iota^3 + o^3[\sin] - \iota}{\iota^3} = \lim_0 \left(-\tfrac{1}{6} + \frac{o^3[\sin]}{\iota^3} \right) = -\tfrac{1}{6} \ .$$

∎

The difference between this notation and the traditional notation is that $o^3[\sin]$ is a well defined function, that is, we can boldly use expressions like $o^3[\sin][\pi] = \sin[\pi] - \mathrm{T}^3[\sin][\pi] = \pi(\pi^2/6 - 1)$.

Some elementary rules follow directly from the definition and well known theorems about the Taylor series:

Theorem 2.31 (*Arithmetics of o_a^n*). Let f and g be (for simplicity) infinitely differentiable functions on some neighbourhood of a, then

1. $o_a^n[f] + o_a^m[g] = o_a^{\min(m,n)}[h]$ for some h,
2. $c o_a^n[f] = o_a^n[cf]$ for $c \in \mathbb{R}$,

3. $o_a^n[f]o_a^m[g] = o_a^{n+m}[h]$ for some h,

4. $(\iota - a)^m o_a^n[f] = o_a^{n+m}[(\iota - a)^m f]$,

5. $o_a^n[f] = o_a^m[h]$ for any $m < n$ for some h.

The argument of o_a^n is often immaterial for the computation, and in such a case we will simply omit it. With this vague notation, we can rewrite the rules from this theorem as

1. $o_a^n + o_a^m = o_a^{\min(m,n)}$,

2. $co_a^n = o_a^n$ *for* $c \in \mathbb{R}$,

3. $o_a^n o_a^m = o_a^{n+m}$,

4. $(\iota - a)^m o_a^n = o_a^{n+m}$,

5. $o_a^n = o_a^m$ for any $m < n$. ∎

However, it is important to keep in mind that these equalities hold only when the correct arguments are filled in (in particular, they are usable only in the left-to-right direction, because the full version states that there *exists* an argument for the right hand side such that the equality holds). Nevertheless, in computing limits, the argument is usually immaterial.

Example 2.32. Let's compute a more complicated example:

$$\lim_0 \left(\frac{1}{\log[1+\iota]} - \frac{1}{\tan} \right) = \lim_0 \frac{\tan - \log[1+\iota]}{\log[1+\iota]\tan} = \lim_0 \frac{\iota + o^2 - \iota + \frac{1}{2}\iota^2 - o^2}{(\iota + o^2)(\iota - \frac{1}{2}\iota^2 + o^2)}$$
$$= \lim_0 \frac{\frac{1}{2}\iota^2 + o^2}{\iota^2 - \frac{1}{2}\iota^3 - \iota o^2 + \iota o^2 - \frac{1}{2}\iota^2 o^2 + o^2 o^2} = \lim_0 \frac{\frac{1}{2}\iota^2 + o^2}{\iota^2 + o^2} = \lim_0 \frac{\frac{1}{2} + o^2/\iota^2}{1 + o^2/\iota^2} = \frac{1}{2}\ .$$

Every time we "merge" more o^n or a function with some o^n its argument changes. It would be possible to write the correct argument to all o^n in the previous computation, but we know it wouldn't change the result, because the fact that $\lim_0(o^2[f]/\iota^2) = 0$ doesn't depend on f. ∎

This vague notation can be also adopted to what would be traditionally written as $f(x) = g(x) + o(x^n)$. For example, instead of $\sin(x) = x + o(x^2)$ we can write simply $\sin = \iota + o^2$ (meaning $\sin = \iota + o^2[\sin]$). When using variables, we would write vaguely $\sin[x] = x + o^2[x]$.

3. Integral calculus

3.1 Indefinite summation

We have already defined sum operators by $\sum_m^n f = f[m] + \cdots + f[n]$. There is a certain connection between these operators and the difference operator Δ. We shall present some basics of "sum calculus" that will illustrate elementary ideas of integral calculus.

Definition 3.1. The ***indefinite sum operator*** with step h is defined as $\sum_h = (\Delta_h)^{-1}$, i.e. the inverse relation to Δ_h. In particular, define $\sum = \Delta^{-1}$. ▪

$\sum_h$ is certainly not a function (since Δ_h is not an injective function); nevertheless, we shall call $\sum_h$ an operator. Obviously, $\Delta_h F = \Delta_h G$ iff $\Delta_h(F - G) = 0$ iff $F - G$ is an h-periodic function, that is, $(F-G)[x+h] = (F-G)[x]$ for all x. This motivates the following definition.

Definition 3.2. The symbol C_h will denote the set of all h-periodic functions on $\mathbb{R}$, i.e. $C_h = \{c : \mathbb{R} \to \mathbb{R} : c[\iota + h] = c\}$. ▪

With this notation, we can rewrite the previous paragraph.

Lemma 3.3. $\Delta_h F = \Delta_h G$ if and only if $F \in G + C_h$. In other words, $\sum_h [\![f]\!] = F + C_h$ where F is any function such that $\Delta_h F = f$. ▪

For example, we already know that $\Delta(\frac{1}{3}\iota_1^3) = \iota_1^2$. This means that $\sum [\![\iota_1^2]\!] = \frac{1}{3}\iota_1^3 + C_1$. The symbol C_h has some "unusual" properties, such as $C_h + C_h = C_h$, meaning $[\![C_h]\!] + [\![C_h]\!] = C_h$ (because the sum of two h-periodic functions is again h periodic). Some elementary arithmetical rules of $\sum_h$ follow from its definition.

Theorem 3.4 (*Arithmetics of* $\sum_h$).

1. $\sum_h [\![f + g]\!] = \sum_h [\![f]\!] + \sum_h [\![g]\!]$,
2. $\sum_h [\![cf]\!] = c \sum_h [\![f]\!]$ for $c \in C_h$,
3. $\sum_h [\![\Delta_h f]\!] = f + C_h$,
4. $\Delta_h \big[\sum_h [\![f]\!] \big]\!\!] = f$. ▪

To make expressions more readable, we will usually omit brackets; that is, we will denote $\sum_h [\![f]\!]$ as $\sum_h f$.

The difference quotient of a product of functions can be rewritten as $f \Delta_h g = \Delta_h(fg) - g\Delta_h f - h(\Delta_h f)(\Delta_h g)$. By applying $\sum_h$ to both sides of this equality, we get a form of "summation by parts".

Theorem 3.5 (*Summation by parts*).

$$\sum_h f \Delta_h g = fg - \sum_h \big(g\Delta_h f + h(\Delta_h f)(\Delta_h g) \big) \ .$$

▪

The reader might be wondering whether we shouldn't write $fg + C_h$ instead of fg (according to the third rule in theorem 3.4), but this would be superfluous, since the "constant" $+C_h$ is already included in the subsequent sum.

Example 3.6. Since $\Delta 2^\iota = 2^\iota$, it follows that $\sum 2^\iota = 2^\iota + C_1$, so, for example

$$\sum \iota 2^\iota = \iota 2^\iota - \sum (2^\iota + 2^\iota) = \iota 2^\iota - 2 \cdot 2^\iota + C_1 = (\iota - 2)2^\iota + C_1 \ .$$

■

3.2 Definite summation

The result of $\sum_h f$ itself is not very interesting; it is not even clear why $\sum_h$ is called the indefinite *sum* operator. Lets define another operator for which the "sum" in its name is obviously appropriate:

Definition 3.7. Let $a, h \in \mathbb{R}$ and $b \in a + h\mathbb{N}$ (i.e. $b = a + nh$ for some $n \in \mathbb{N}$). The ***definite sum operator*** from a to b with step h is defined by

$$\sum_{a}^{b}{}_h f = \sum_{\{a, a+h, \ldots, b-h\}} hf = h\big(f[a] + f[a+h] + \cdots + f[b-h]\big)$$

(in text we will write $\sum_{a,h}^{b}$). ■

Geometrically, $\sum_{a,h}^{b}$ is an approximation of the area under f on the interval $[a, b]$ by rectangles of base size h. Exclusion of the term $hf[b]$ tells us that $\sum_{a,h}^{b}$ is the sum over $(b - a)/h$ numbers starting with a (in particular $\sum_{m,1}^{n}$ is the sum over $n - m$ integers starting with m, whereas $\sum_{m}^{n} = \sum_{m,1}^{n+1}$ is the sum over $n - m + 1$ integers). Thanks to this property, this operator is "additive" in limits, that is $\sum_{a,h}^{b} + \sum_{b,h}^{c} = \sum_{a,h}^{c}$.

The following notion will provide a way to compute definite sums through indefinite sums.

Definition 3.8. The ***operator of limits*** from a to b (denoted $\big[_a^b$) is defined as $\big[_a^b f = f[b] - f[a]$ for $a, b \in \mathbb{R}$. ■

This operator has some obvious properties, such as $\big[_a^b cf = c \big[_a^b f$ for $c \in \mathbb{R}$ and $\big[_a^b (f + g) = \big[_a^b f + \big[_a^b g$. Lets show how it cooperates with $\sum_h$.

Theorem 3.9 (*Fundamental theorem of finite calculus*). Let $b \in a + h\mathbb{N}$ (that is, $b = a + hn$ for some $n \in \mathbb{N}$), then

$$\Big[_a^b \Big[\sum_h f \Big] = \sum_a^b{}_h f \ .$$

Proof. If $c \in C_h$, then $\mathrm{I}_a^b\, c = 0$ (because $\mathrm{I}_a^{a+hn}\, c = c[a+hn] - c[a] = c[a] - c[a] = 0$). Let $F \in \sum_h f$. $\sum_h f = F + C_h$ by lemma 3.3, so we can write

$$\mathop{\mathrm{I}}\limits_a^b \left[\!\!\left[\sum\nolimits_h f \right]\!\!\right] = \mathop{\mathrm{I}}\limits_a^b [\![F + C_h]\!] = \left\{ \mathop{\mathrm{I}}\limits_a^b \big(F + c\big) : c \in C_h \right\} = \left\{ \Big(\mathop{\mathrm{I}}\limits_a^b F \Big) + 0 : c \in C_h \right\} = \left\{ \mathop{\mathrm{I}}\limits_a^b F \right\}$$

It follows that $\mathrm{I}_a^b [\![\sum_h f]\!] = \mathrm{I}_a^b F$. This can be even further rewritten as

$$\begin{aligned} &\mathop{\mathrm{I}}\limits_a^b F = F[b] - F[a] = (F[b] - F[b-h]) + (F[b-h] - F[b-2h]) + \dots + \\ &+ (F[a+h] - F[a]) = h(\Delta_h F)[b-h] + h(\Delta_h F)[b-2h] + \dots + h(\Delta_h F)[a] \\ &= h\big(f[b-h] + f[b-2h] + \dots + f[a]\big) = \sum\nolimits_{h\,a}^{\;b} f \ , \end{aligned}$$

from which the assertion follows. ∎

We will usually write this relation without brackets:

$$\mathop{\mathrm{I}}\limits_a^b \sum\nolimits_h f = \mathop{\sum\nolimits_h}\limits_a^b f \ . \tag{3.1}$$

The left-hand side of this equation can be understood as the composition $\mathrm{I}_m^n \sum_h$ applied to f. In particular, $\sum_m^n$ can be rewritten as $\mathrm{I}_m^{n+1} \sum$. Theorem 3.9 provides a way to compute ordinary sums using indefinite summation.

Example 3.10. Say, we wanted to compute $\sum_{i=1}^n i2^i$. This can be rewritten as $\sum_1^n \iota 2^\iota$. Using (3.1) and Example 3.6 we can write:

$$\begin{aligned} \sum_1^n \iota 2^\iota &= \mathop{\mathrm{I}}\limits_1^{n+1} \sum \iota 2^\iota = \mathop{\mathrm{I}}\limits_1^{n+1} (\iota - 2)2^\iota = (n+1-2)2^{n+1} - (1-2)2^1 \\ &= (n-1)2^{n+1} + 2 \ . \end{aligned}$$

∎

It is possible to compute $\sum_{i=m}^n P(i)a^i$ for any polynomial P and $a \in (0, \infty)$ in this way using summation by parts; however, we need a way to compute the indefinite sum of ι^n. This can be done by rewriting it using falling factorials.

Example 3.11. It is easy to compute that $\iota^3 = \iota_1^3 + 3\iota_1^2 + \iota_1^1$, so:

$$\begin{aligned} \sum_{i=1}^n i^3 &= \sum_1^n \iota^3 = \mathop{\mathrm{I}}\limits_1^{n+1} \sum \iota^3 = \mathop{\mathrm{I}}\limits_1^{n+1} \sum \big(\iota_1^3 + 3\iota_1^2 + \iota_1^1\big) = \mathop{\mathrm{I}}\limits_1^{n+1} \big(\tfrac{1}{4}\iota_1^4 + \iota_1^3 + \tfrac{1}{2}\iota_1^2\big) \\ &= \big(\tfrac{1}{4}\iota_1^4 + \iota_1^3 + \tfrac{1}{2}\iota_1^2\big)[n+1] - \big(\tfrac{1}{4}\iota_1^4 + \iota_1^3 + \tfrac{1}{2}\iota_1^2\big)[1] = \\ &= (n+1)n(\tfrac{1}{4}(n-1)(n-2) + n - 1 + \tfrac{1}{2}) = \frac{(n+1)n(n^2+n)}{4} \\ &= \frac{(n+1)^2 n^2}{4} \ . \end{aligned}$$

∎

Variables

We cannot use variables in indefinite summation in the same way as in limits or derivatives, because $\sum f$ is not a function; it is a set of functions. However, it is natural to define $\sum_{h:i} f[i] = \sum_h [\![f]\!]$. Then, in expressions of the form $\big[_m^n \sum f$, a variable can "travel" from $\sum$ to $\big[_m^n$ in the following way:

Lemma 3.12. If $F \in \sum f$, then $\big[_m^n \sum_{:i} f[i] = \big[_{i=m}^n F[i]$. ▪

Example 3.13. We can rewrite Example 3.10 using variables as

$$\sum_{i=1}^{n} i2^i = \Big[_1^{n+1} \sum_i i2^i = \Big[_{i=1}^{n+1} (i-2)2^i = (n-1)2^{n+1} + 2 \ .$$

Nevertheless, there is no natural way to work with variables in the case of indefinite summation; we have no other choice than to write

$$\sum_i i2^i = (\iota - 2)2^\iota + C_1$$

if we want to be mathematically rigorous. ▪

3.3 Indefinite integration

In chapter 2, we presented difference calculus first and then differential calculus as its limit version, whose arithmetics was analogous. As summation calculus is just "the opposite" of difference calculus, while integral calculus is just "the opposite" of differential calculus, one would expect certain analogies between summation and integration calculus to be present as well, and this is indeed the case.

To simplify statements of theorems, we shall impose one assumption on functions we work with.

Convention. In what follows, f and g will denote functions continuous on some open interval. F and G will denote functions continuously differentiable on some open interval. ▪

This assumption is in fact not very restrictive, as virtually all functions appearing in practical computations satisfy it. It is probably expectable what shall follow.

Definition 3.14. The ***indefinite integral operator***[1] $\int$ is defined as $\int = \partial^{-1}$. ▪

We are in the situation analogous to $\sum$. Since $\partial f = \partial g$ iff $f = g + c$ for some $c \in \mathbb{R}$, it is natural to restate Definition 3.1 as

Definition 3.15. The letter C will denote the set of all constant functions on $\mathbb{R}$, that is $C = \{c_{\mathbb{R}} : c \in \mathbb{R}\}$. ▪

[1]We use the symbol $\int$ instead of the traditional slanted $\int$, because it is typographically more convenient when used in a manner analogous to $\sum$, and because it avoids ambiguity with the traditional notation.

Using this notation, we can rewrite lemma 3.3 as

Lemma 3.16. $\partial F = \partial G$ if and only if $F \in G + C$, that is, $\int [\![f]\!] = F + C$ if and only if $\partial F = f$. ■

For example, $\int [\![\cos]\!] = \sin + C$, $\int [\![\iota^2]\!] = \frac{1}{3}\iota^3 + C$ etc. The basic arithmetics of $\int$ is the same as that of $\sum$ and follows immediately from its definition.

Theorem 3.17 (*Arithmetics of* $\int$).

1. $\int [\![f + g]\!] = \int [\![f]\!] + \int [\![g]\!]$,
2. $\int [\![cf]\!] = c \int [\![f]\!]$ for $c \in \mathbb{R}$,
3. $\int [\![\partial F]\!] = F + C$,
4. $\partial \big[\int [\![f]\!] \big] = f$. ■

In practice, we will omit brackets when appropriate and use expressions like $\int \cos = \sin + C$.

If we rewrite the derivative of a product of functions as $f \partial g = \partial(fg) - g \partial f$, we can derive integration by parts by applying $\int$ to both sides of this equality.

Theorem 3.18 (*Integration by parts*). $\int f \partial g = fg - \int g \partial f$. ■

Example 3.19. Lets recompute Example 3.6 with $\int$ instead of $\sum$ and e^ι instead of 2^ι:

$$\int \iota e^\iota = \iota e^\iota - \int e^\iota = \iota e^\iota - e^\iota + C = (\iota - 1)e^\iota + C \ .$$

As you can see, the finite and the continuous cases are completely analogous. ■

Another trivial fact is how $\int f$ composes with other functions.

Lemma 3.20. Let $F \in \int f$, then

$$\Big(\int f \Big)[g] = F[g] + C \ ,$$

where $(\int f)[g]$ denotes the the set $\{F[g] : F \in \int f\}$.

Proof. $(\int f)[g] = \{F + c : c \in C\}[g] = \{F[g] + c : c \in C\} = F[g] + C$, since $c[g] = c$. ■

Theorem 3.21 (*First substitution rule*). Let g be a differentiable function, then

$$\int f[g] \partial g = \Big(\int f \Big)[g] \ .$$

Proof. By lemma 3.20, $(\int f)[g] = F[g] + C$ for any $F \in \int f$, and so by lemma 3.16 it is the integral of the function $\partial(F[g]) = f[g]\partial g$. ■

This theorem is traditionally written as

$$\int f(g(x))g'(x)\, dx = \int f(t)\, dt \Big|_{t=g(x)} \ .$$

One may think that we use this formula, because it is practical. We will show that this is not really true.

Example 3.22. Lets compute the integral of $\sin^3 \cos$ (that is, the product of $\sin^3$ and $\cos$):

$$\int \sin^3 \cos = \Big(\int \iota^3\Big)[\sin] = \tfrac{1}{4}\iota^4[\sin] + C = \tfrac{1}{4}\sin^4 + C\ .$$

(If the way we used the first substitution rule is not immediately obvious, one may rewrite $\sin^3 \cos$ as $\iota^3[\sin]\cos$). In the traditional notation:

$$\int \sin^3(x)\cos(x)\,dx \ = \int s^3\,ds\Big|_{s=\sin(x)} = \tfrac{1}{4}s^4 + c = \tfrac{1}{4}\sin^4(x) + c\ .$$

■

It is quite hard to see what the traditional notation means from the set-theoretical viewpoint. Nevertheless, denoting a larger function by a new letter may help to find an appropriate substitution.

Example 3.23. The integral $\int \frac{1}{\sqrt{x^2+1}}\,dx$ can be computed using the first Euler's substitution $\sqrt{x^2+1} = -x + s$ in the traditional notation. From this equation, one would derive $x = \frac{s^2-1}{2s}$, so "$dx = \frac{s^2+1}{2s^2}ds$" and put together:

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \int \frac{\frac{s^2+c}{2s^2}\,ds}{\sqrt{\left(\frac{s^2-1}{2s}\right)^2+1}} = \int \frac{1}{s}\,ds = \log s + c = \log(x+\sqrt{x^2+1}) + c\ .$$

It seems that such a process cannot be rewritten without variables. However, in fact, it can be rewritten in almost the same way: we define a new function s by $\sqrt{\iota^2+1} = -\iota + s$ (that is, $s = \sqrt{\iota^2+1}+\iota$). From this equality we can derive in the same way as with variables that $\iota = \frac{s^2-1}{2s}$ and that $1 = \partial\iota = \frac{s^2+1}{2s^2}\partial s$. Put together:

$$\int \frac{1}{\sqrt{\iota^2+1}} = \int \frac{\frac{s^2+1}{2s^2}\partial s}{\sqrt{\left(\frac{s^2-1}{2s}\right)^2+1}} = \int \tfrac{1}{s}\partial s = \log[s] + C = \log[\iota+\sqrt{\iota^2+1}] + C\ .$$

Again, it is completely clear what mathematically happens. In the first step we just express ι^2 using s, 1 using s and ∂s, and then simplify the expression. The result is exactly of the form $\int f[s]\partial s$ for $f = \frac{1}{\iota}$, so we can apply the first substitution rule. ■

Example 3.24. Using the first substitution rule, we can also perform separation of variables in differential equations. Suppose we have a differential equation

$$\partial y = g[y]f\ .$$

This is a variableless way to write what would be traditionally written as $y'(x) = f(x)g(y(x))$, or slightly incorrectly as $y' = f(x)g(y)$. The latter (incorrect) expression can be rewritten as $\frac{dy}{dx} = f(x)g(y)$ which may be further nonsensically rewritten as $\frac{dy}{g(y)} = f(x)dx$ from which "follows" that $\int \frac{dy}{g(y)} = \int f(x)\,dx$ (and most physicists and engineers do solve differential equations using this notation). However, this nonsensical process gives the correct result. This is not a coincidence; it mimics the correct mathematical derivation, which is the following:

Suppose that $g \neq 0$, points t for which $g[t] = 0$ provide stationary solutions and can be treated separately. Rewrite $\partial y = g[y]f$ as

$$\frac{1}{g[y]}\partial y = f$$

and further using $\frac{1}{g[y]} = \frac{1}{g}[y]$ as

$$\tfrac{1}{g}[y]\partial y = f \ .$$

Apply $\int$ to both sides (which can be easily shown to be equivalent to the previous equality):

$$\int \tfrac{1}{g}[y]\partial y = \int f \ .$$

Apply the first substitution rule:

$$\left(\int \tfrac{1}{g}\right)[y] = \int f \ .$$

This process resembles the traditional nonsensical computation, but all steps are completely rigorous. ∎

There is another also another substitution rule:

Theorem 3.25 (*Second substitution rule*). Let $g : I \to J$ be a differentiable and invertible function from an open interval I onto an open interval J (i.e. there is a function g^{-1} such that $g[g^{-1}] = \iota_J$). Then

$$\int f_J = \left(\int f[g]\partial g\right)[g^{-1}] \ ,$$

where f_J denotes the restriction of f to J.

Proof. $(\int f[g]\partial g)[g^{-1}] = (\int f)[g][g^{-1}] = (\int f)[g[g^{-1}]] = (\int f)[\iota_J] = \int f_J$. ∎

Traditionally, this substitution rule is written as

$$\int f(x)\, dx = \int f(g(t))g'(t)\, dt \bigg|_{t=g^{-1}(x)} \ .$$

Example 3.26. Suppose that we know that $\int \cos^2 = \frac{1}{2}(\iota + \sin\cos)$. We use the preceding theorem for $g = \sin_I : I \to J$ where $I = (-\frac{\pi}{2}, \frac{\pi}{2})$, $J = (-1, 1)$. We also write arcsin instead of $\sin_I^{-1}$:

$$\begin{aligned}\int \sqrt{1-\iota^2} &= \int \left(\sqrt{1-\iota^2}\right)_J = \left(\int \sqrt{1-\sin_I^2}\,\cos_I\right)[\arcsin] = \left(\int \cos_I^2\right)[\arcsin] \\ &= \tfrac{1}{2}(\iota_I + \sin_I\cos_I)[\arcsin] + C = \tfrac{1}{2}\left(\iota_I + \sin_I\sqrt{1-\sin_I^2}\right)[\arcsin] + C \\ &= \tfrac{1}{2}\left(\arcsin + \iota_J\sqrt{1-\iota_J^2}\right) + C\end{aligned}$$

and in the traditional notation:

$$\begin{aligned}\int \sqrt{1-x^2}\,dx &= \int \sqrt{1-(\sin t)^2}\cos(t)\,dt\Big|_{t=\arcsin x} = \int \cos^2(t)\,dt \\ &= \tfrac{1}{2}\big(t+\sin(t)\cos(t)\big)+c = \tfrac{1}{2}\Big(t+\sin(t)\sqrt{1-(\sin t)^2}\Big)+c \\ &= \tfrac{1}{2}\Big(\arcsin(x)+x\sqrt{1-x^2}\Big)+c\ .\end{aligned}$$

Notice how our notation already subsumes further discussion that would be necessary in the traditional notation. For example, it is true that $\sqrt{1-\sin_I^2}=\cos_I^2$, but it is not possible to write $\sqrt{1-\sin^2(t)}=\cos(t)$ without further explanation in the traditional notation. ■

Note. This substitution rule is sometimes used in cases that require solving an integral on several intervals separately and than it is necessary to 'glue' these parts together. This can be also addressed by our approach. For example, if $C=A\cup B$ where $A=(a,c)$ and $B=(c,b)$ (meaning open intervals), then $\int f_C = \int(f_A\cup f_B) = [\![\int f_A]\!] \cup'_c [\![\int f_B]\!]$ where $\cup'_c$ denotes the ***differentiable union*** of functions defined as follows: if the domains of f and g are disjoint and $\lim_c f=\lim_c g = L\in\mathbb{R}$, and if $f\cup L_{\{c\}}\cup g$ is differentiable at c, then $f\cup'_c g = f\cup L_{\{c\}}\cup g$, otherwise it is left undefined. ■

3.4 Definite integration

The following notion is a direct generalization of $\sum_{a,h}^b$.

Definition 3.27. Let $a,b\in\mathbb{R}$, $a<b$, and let f be continuous on $[a,b]$. The ***area operator*** $\int_a^b$ is defined as

$$\int\limits_a^b f = \lim_{h\to 0+} \sum\limits_a^b{}_h\, f\ . \tag{3.2}$$

and

$$\int\limits_b^a f = -\int\limits_a^b f$$

■

The right-hand side of (3.2) is essentially the Riemann integral in which only equidistant divisions with one "pivot point" are taken into account. Notice that $\sum_{a,h}^b f$ is defined only for h of the form $(b-a)/n$, $n\in\mathbb{N}$. So, in fact, the definition can be rewritten as

$$\int\limits_a^b f = \operatorname*{Lim}_n \sum\limits_a^b{}_{(b-a)/n}\, f\ .$$

In teaching, this could be the first definition of integration; it has an obvious geometric interpretation—we approximate the area under a curve by smaller and smaller rectangles (all of the same base size). This definition is fully sufficient as long as we deal with continuous or piecewise continuous functions (and we usually do in practice).

Theorem 3.28 (*Area of a continuous function*). $\int_a^b f$ is a real number for every function f continuous on $[a, b]$ (that is, the limit in the definition exists and is finite).

Proof. As $\int_a^b$ is just a special case of the Riemann integral, and for the Riemann integral this is true, we know that this must hold for our area operator as well. The reader interested in the general proof can find it in [Lang96, p. 106]. The proof in our situation can be, of course, much simplified, because we need to take only equidistant divisions into account. ∎

Lemma 3.29. Let $f : [a, b] \to \mathbb{R}$ be continuous, then $(b - a)(\min_{[a,b]} f) \le \int_a^b f \le (b - a)(\max_{[a,b]} f)$.

Proof. Since $f \le \max_{[a,b]} f$, it follows that $\sum_{a,h}^b f \le \sum_{a,h}^b (\max_{[a,b]} f) = (b - a)(\max_{[a,b]} f)$, and so the same must hold for the limit for $h \to 0$. The other inequality is analogous. ∎

Lemma 3.30. Let $f : [a, \beta] \to \mathbb{R}$ be continuous, then

$$\int_a^\beta f = \lim_{b\to\beta-} \int_a^b f\ . \tag{3.3}$$

Proof. Let $b \in (a, \beta)$. Since $\sum_{a,h}^b = \sum_{a,h}^c + \sum_{c,h}^b$, it follows from arithmetics of limits that $\int_a^b f = \int_a^c f + \int_c^b f$ for $c \in (a, b)$. We can therefore write $\lim_{b\to\beta-} \int_a^b f = \lim_{b\to\beta-}(\int_a^\beta f - \int_b^\beta f)$. However, $\lim_{b\to\beta-} \int_b^\beta f = 0$, since $\int_b^\beta f$ is bounded by $(\beta - b)(\max_{[b,\beta]} f)$ above and similarly from below by Lemma 3.29, from which the assertion follows. ∎

In the light of this lemma, we can extend the definition of $\int_a^b$ to functions continuous on $[a, \beta)$ by formula (3.3), and similarly for functions continuous on $(\alpha, b]$.

Lemma 3.31. Let f be continuous on $[a, \beta)$ and let $c \in (a, \beta)$. Then $\int_a^\beta f = \int_a^c f + \int_c^\beta f$.

Proof.

$$\int_a^c f + \int_c^\beta f = \int_a^c f + \lim_{b\to\beta-} \int_c^b f = \lim_{b\to\beta-} \left(\int_a^c f + \int_c^b f \right) = \lim_{b\to\beta-} \int_a^b f = \int_a^\beta f$$

∎

Of course, an analogous result holds for f continuous on $(\alpha, b]$. We can use this fact to further extend Definition 3.27 for functions continuous on (α, β) as

$$\int_\alpha^\beta f = \int_\alpha^c f + \int_c^\beta f \tag{3.4}$$

for some $c \in (\alpha, \beta)$. Let $c' \in (\alpha, \beta)$. Using the previous theorem we can write

$$\int_{\alpha}^{c} f + \int_{c}^{\beta} f = \int_{\alpha}^{c} f + \left(\int_{c}^{c'} f + \int_{c'}^{\beta} f \right) = \left(\int_{\alpha}^{c} f + \int_{c}^{c'} f \right) + \int_{c'}^{\beta} f = \int_{\alpha}^{c'} f + \int_{c'}^{b} f \ ,$$

so the right-hand side of (3.4) is independent of the choice of c. Similarly as in the case of $[a, \beta)$, we can derive that, for f continuous on $[\alpha, \beta]$, this extension agrees with Definition 3.27. Now, we can derive some elementary facts about $\int_{\alpha}^{\beta}$.

Theorem 3.32 (*Arithmetics of* $\int_{\alpha}^{\beta}$). Let $f, g : (\alpha, \beta) \to \mathbb{R}$ be continuous and $c \in \mathbb{R}$, then

1. If $\int_{\alpha}^{\beta} f + \int_{\alpha}^{\beta} g \neq \Omega$, then $\int_{\alpha}^{\beta} \big(f + g\big) = \int_{\alpha}^{\beta} f + \int_{\alpha}^{\beta} g$,
2. $\int_{a}^{b} \big(cf\big) = c \int_{a}^{b} f$.

Proof. First, we prove the first assertion for f, g continuous on $[a, b] \subseteq (\alpha, \beta)$. For such f, g we can write

$$\int_{a}^{b} \big(f + g\big) = \lim_{h \to 0+} \sum\nolimits_{a,h}^{b} \big(f + g\big) = \lim_{h \to 0+} \left(\sum\nolimits_{a,h}^{b} f + \sum\nolimits_{a,h}^{b} g \right) = \int_{a}^{b} f + \int_{a}^{b} g \ ,$$

where we used arithmetics of limits and the fact that the integrals of f and g exist and are finite. Let $c \in (\alpha, \beta)$, then

$$\int_{\alpha}^{\beta} \big(f + g\big) = \int_{\alpha}^{c} \big(f + g\big) + \int_{c}^{\beta} \big(f + g\big) = \lim_{a \to \alpha+} \int_{a}^{c} \big(f + g\big) + \lim_{b \to \beta-} \int_{c}^{b} \big(f + g\big)$$

$$= \lim_{a \to \alpha+} \left(\int_{a}^{c} f + \int_{a}^{c} g \right) + \lim_{b \to \beta-} \left(\int_{c}^{b} f + \int_{c}^{b} g \right) = \int_{\alpha}^{c} f + \int_{\alpha}^{c} g + \int_{c}^{\beta} f + \int_{c}^{\beta} g = \int_{\alpha}^{\beta} f + \int_{\alpha}^{\beta} g$$

provided that the rightmost expression is defined, which was the assumption. The second assertion would be proven similarly. ∎

In the traditional notation, Riemann and indefinite integration are usually taught as two related but different mathematical concepts. However, our notions of the area operator and the indefinite integration can be shown to be in some sense equivalent using the following notion:

Definition 3.33. The ***continuous operator of limits*** $\big[_{a}^{b}$ is defined as $\big[_{a}^{b} = \lim_{b-} - \lim_{a+}$ (that is, $\big[_{a}^{b} f = \lim_{b-} f - \lim_{a+} f$) when $b \geq a$, and by $\big[_{a}^{b} = -\big[_{b}^{a}$ when $b < a$. ∎

One could expect that $\sum_{a,h}^{b} = \big[_{a}^{b} \sum_{h}$ translates as $\int_{a}^{b} = \big[_{a}^{b} \int$. This is true and we shall prove it.

Theorem 3.34 (*Fundamental theorem of calculus*). Let f be a continuous function on (α, β). Then

$$\int_{\alpha}^{\beta} f = \Big[_{\alpha}^{\beta} \int f \ .$$

Proof. Let $\alpha < c < \beta$. Define

$$F[x] = \int\limits_c^x f$$

for $x \in (\alpha, \beta)$. Suppose that $x \geq c$, then

$$\partial F[x] = \lim_{\delta \to 0} \tfrac{1}{\delta}\Big(\int\limits_c^{x+\delta} f - \int\limits_c^{x} f \Big) = \lim_{\delta \to 0} \tfrac{1}{\delta} \int\limits_x^{x+\delta} f \tag{3.5}$$

Let $\varepsilon > 0$. Since f is continuous on $[x, \beta)$, we can always find δ sufficiently close to 0 such that $f[\![[x - |\delta|, x + |\delta|]]\!] \subseteq (f[x] - \varepsilon, f[x] + \varepsilon)$. That is, $(\int_x^{x+\delta} f) \in ((f[x] - \varepsilon)|\delta|, (f[x] + \varepsilon)|\delta|)$ and $\frac{1}{\delta}(\int_x^{x+\delta} f) \in (f[x] - \varepsilon, f[x] + \varepsilon)$. Since this holds for any ε, it follows that the limit on the right-hand side of (3.5) is equal to $f[x]$. That is, $F \in \int f$ and it follows that

$$\begin{aligned} \int\limits_\alpha^\beta f &= \int\limits_c^\beta f + \int\limits_\alpha^c f = \lim_{x \to \beta-} \int\limits_c^x f + \lim_{x \to \alpha+} \int\limits_x^c f = \lim_{x \to \beta-} \int\limits_c^x f - \lim_{x \to \alpha+} \int\limits_c^x f \\ &= \lim_{x \to \beta-} F[x] - \lim_{x \to \alpha+} F[x] = \mathop{\mathrm{I}}\limits_a^b F = \mathop{\mathrm{I}}\limits_a^b \int f \ . \end{aligned}$$

∎

Lets show a table of correspondence between discrete and continuous calculus:

Discrete	Defined as	Computed using	Cont.	Defined as	Computed using
Δ_h	$\frac{f[\iota+h]-f}{h}$	Arithmetics Formulas	∂	$\lim\limits_{h \to 0} \Delta_h$	Arithmetics Formulas
$\sum_h$	$(\Delta_h)^{-1}$	Arithmetics Formulas	$\int$	∂^{-1}	Arithmetics Formulas
$\sum\limits_a^b{}_h$	Evaluation	$\mathop{\mathrm{I}}\limits_a^b \sum_h$	$\int\limits_a^b$	$\lim\limits_{h \to 0} \sum\limits_a^b{}_h$	$\mathop{\mathrm{I}}\limits_a^b \int$

As you can see, the correspondence is very precise. The expression $\mathrm{I}_a^b \int f$ agrees with the standard notion of definite integration which is usually defined as $\int_a^b f(x)\,dx = \lim_{x \to b-} F(x) - \lim_{x \to a+} F(x)$ where F is a primitive function to f. However, a huge difference between the two notions is that I_a^b is a separate operator, and as such has some interesting properties. If f and g are continuous and $a, b \in \mathbb{R}$, then the limits in Definition 3.33 reduce to evaluation, and we can write

$$\mathop{\mathrm{I}}\limits_a^b f[g] = f[g[a]] - f[g[b]] = \mathop{\mathrm{I}}\limits_{g[a]}^{g[b]} f \ .$$

This can be further generalized if f and g are such that composition rule for limits holds for them:

$$\mathop{\mathrm{I}}\limits_a^b f[g] = \mathop{\mathrm{I}}\limits_{\lim_{a+} g}^{\lim_{b-} g} f \ .$$

However, a primitive function of a function is continuous (it is even differentiable). This means that there are actually **no separate substitution rules for definite integration**, since they are already subsumed by rules for indefinite integration, because

$$\int\limits_a^b f[g]\partial g = \mathop{\Large\mathrm{I}}\limits_a^b \int f[g]\partial g = \mathop{\Large\mathrm{I}}\limits_a^b \Big(\int f\Big)[g] = \mathop{\Large\mathrm{I}}\limits_{\lim_{a+} g}^{\lim_{b-} g} \int f = \int\limits_{\lim_{a+} g}^{\lim_{b-} g} f \ .$$

Example 3.35. Say, we want to compute the area under the graph of $\sin^2 \cos$ over the interval $[0, \pi/2]$, that is

$$\int\limits_0^{\pi/2} \sin^2 \cos \ .$$

We know that we can rewrite this problem using indefinite integration:

$$\mathop{\Large\mathrm{I}}\limits_0^{\pi/2} \int \sin^2 \cos = \mathop{\Large\mathrm{I}}\limits_0^{\pi/2} \Big(\int \iota^2\Big)[\sin] = \mathop{\Large\mathrm{I}}\limits_{\sin 0}^{\sin(\pi/2)} \int \iota^2 = \mathop{\Large\mathrm{I}}\limits_0^1 \tfrac{1}{3}\iota^3 = \tfrac{1}{3}1^3 - \tfrac{1}{3}0^3 = \tfrac{1}{3} \ .$$

This is completely clear; we just use the first substitution rule and change the limits by putting sin into them. In the traditional notation:

$$\int_0^{\pi/2} \sin^2(x)\cos(x)\,dx\bigg|_{t=\sin x} = \int_{\sin 0}^{\sin(\pi/2)} t^2\,dt = [\tfrac{1}{3}t^3]_{t=0}^1 = \tfrac{1}{3}1^3 - \tfrac{1}{3}0^3 = \tfrac{1}{3} \ .$$

It is intuitively plausible to say that "if x ranges from 0 to $\pi/2$, then $t = \sin x$ ranges from 0 to 1, and since the substitution rule works for indefinite integration, it should work for definite as well", but it is not completely clear why this is correct. ■

In the very same manner, we can rewrite the second substitution rule. Suppose that g is injective and differentiable. Then

$$\int\limits_a^b f = \mathop{\Large\mathrm{I}}\limits_a^b \int f = \mathop{\Large\mathrm{I}}\limits_a^b \Big(\int f[g]\partial g\Big)[g^{-1}] = \mathop{\Large\mathrm{I}}\limits_{\lim_{a+} g^{-1}}^{\lim_{b-} g^{-1}} \int f[g]\partial g = \int\limits_{\lim_{a+} g^{-1}}^{\lim_{b-} g^{-1}} f[g]\partial g \ .$$

In the traditional notation, this theorem is usually written as

$$\int_a^b f(x)\,dx = \int_{\lim_{x\to a+} g^{-1}(x)}^{\lim_{x\to b-} g^{-1}(x)} f(g(t))g'(t)\,dt \ ,$$

which is not only less mathematically clear, but also much longer than

$$\int\limits_a^b f = \int\limits_{\lim_{a+} g^{-1}}^{\lim_{b-} g^{-1}} f[g]\partial g \ .$$

Variables

In definite integration, variables can travel from $\int$ to I in the very same manner as in definite summation, that is, if $F \in \int f$, we can write

$$\mathop{\mathrm{I}}\limits_{a}^{b} \int\limits_{x} f[x] = \mathop{\mathrm{I}}\limits_{x=a}^{b} F[x]\ .$$

In practice, rewriting $\int_a^b$ as $\mathrm{I}_a^b \int$ is usually superfluous (one can always imagine this process), and with variables it would be also unnatural, so we would just write something like

$$\int\limits_{x=a}^{b} f[x] = \mathop{\mathrm{I}}\limits_{x=a}^{b} F[x]\ .$$

When using a substitution rule, we would put the function in substitution directly into limits:

$$\int\limits_{x=a}^{b} f[g[x]]\partial g[x] = \int\limits_{t=g[a]}^{g[b]} f[t]\ .$$

This resembles the traditional notation, but it is generally better to use variables only when this is really necessary, since expressions like

$$\int\limits_{a}^{b} f[g]\partial g = \int\limits_{g[a]}^{g[b]} f$$

Conclusion

The purpose of this thesis was to change the way one understands classical notions in calculus from just a 'vague notation whose only meaning is to express theorems' to a more algebraic way of thinking in which computations are performed by algebraic manipulations (knowing, of course, which manipulations are correct for particular objects under consideration).

In computation of limits and derivatives, the difference between the traditional notation and our notation is mostly formal. Our notation provides a way to write mathematical concepts more fundamentally with all symbols representing clearly defined objects, not just 'something to be justified by further explanation'. However, by exploiting the composition of functions, our notation also provides new ways to rewrite functions (such as writing $\sqrt{\iota + \sin}[1 + \iota + \iota^2]$ instead of the traditional $\sqrt{1 + x + x^2 + \sin(1 + x + x^2)}$), which, especially in differentiation, can lead to substantive reduction in length (and computational complexity) of expressions. In addition to that, our definition of the little-o symbol makes computations of limits with it completely rigorous.

In integration, the difference between the traditional notation and our notation is more than just formal. In indefinite integration, there is no need for a 'substitution' and all common computations can be performed in an almost purely algebraical manner. The results do not contain any vague symbols such as "$+\,c$"; instead, they contain only clearly defined function and set symbols. This leads to 'algebraic' computations (such as solving a differential equation by separation of 'variables') without having to worry about correctness; if one writes everything according to the rules, then it is correct, and it is not necessary to define auxiliary functions and provide further verbal explanation to make all the steps correct. Furthermore, by rewriting functions in the way as in the previous paragraph, it may be possible to notice 'substitutions' that would be very hard to see in the traditional notation. In definite integration, there are no separate rules, because thanks to the notion of the operator of limits, they are already included in the rules for indefinite integration.

Another purpose of this thesis was to show possible ways to teach the variableless approach to students. We have shown that finite differences and differentiation are almost completely analogous, the only substantial difference being the chain rule. Similarly, summation and integral calculus are also almost completely analogous. It would be therefore possible to teach difference and summation calculus first, let students get used to the algebraical manipulations involved, and than introduce differentiation and integration as its limit cases. The concept of the area operator was also defined for pedagogical reasons; it is the limit case of the definite sum operator and it turns out that it completes the analogy by translating the fundamental theorem of finite calculus to the fundamental theorem of (integral) calculus. This definition of integral would be, of course, insufficient for most parts of mathematics, and Riemann and Lebesgue integration could be taught as a generalization of this most intuitive concept.

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