Charles University in Prague
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DOCTORAL THESIS

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Biorthogonal Wavelets

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Supervisor: Doc. RNDr. Karel Najzar, CSc.
Field of study: m-6 Scientific and Technical Calculations

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I hereby declare that this thesis is my own work and contains no material previously published or written by another person except where specifically indicated in the text and bibliography. I agree with lending this thesis.

Liberec, 26$^{\text{th}}$ September 2008 Dana Černá

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Abstrakt: Tato práce se zabývá biortogonálními wavelety. Navrhneme zde novou metodu konstrukce některých tříd ortonormálních waveletů, konkrétně Daubechiesové waveletů, symmletů, coifletů a zobecněných coifletů. Tato metoda je založena na myšlence nahradit rovnice pro škálové koeficienty rovnicemi pro momenty. Ukážeme, že tímto postupem vyeliminujeme kvadratické podmínky v původní soustavě a zjednodušíme tak konstrukci. V některých případech dokonce získáme dané koeficienty v explicitním tvaru. Dále navrhneme metodu výpočtu momentů $M^i_{c,d} := \int_c^d x^i \phi(x)\, dx$, kde ϕ je zjemňující funkce s kompaktním nosičem a c, d jsou libovolné dyadické body. Konečně, modifikujeme konstrukci B-splinových waveletových bází na intervalu s cílem zlepšit podmíněnost těchto bází. Tyto báze pak adaptujeme na komplementární okrajové podmínky. Ukážeme, že zkonstruované báze řádu $N \leq 4$ jsou optimálně L^2-stabilní. Dále ukážeme, že adaptivní framové metody pro eliptické operátorové rovnice s bázemi zkonstruovanými v této práci dosahují optimální rychlosti konvergence, zejména v důležitém případě kubických splinových waveletů.

Klíčová slova: biortogonální wavelety, momenty, podmíněnost, splinové waveletové báze, adaptivní framové metody

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Abstract: The present work is concerned with biorthogonal wavelets. We propose novel methods for construction of some classes of orthonormal wavelets, namely Daubechies wavelets, symmlets, coiflets, and generalized coiflets. The idea of these methods is to replace equations for scaling coefficients representing necessary conditions by equations for scaling moments. We show that by this approach we are able to eliminate some quadratic conditions in the original system and thus simplify the construction and in some cases we are even able to find solutions in explicit form. Furthermore, we propose a method for the computation of moments $M^i_{c,d} := \int_c^d x^i \phi(x)\, dx$ for a refinable function ϕ with compact support and c, d any dyadic points. Finally, we modify the construction of B-spline wavelet bases on the interval in order to improve their condition. We adapt the constructed bases to the complementary boundary conditions. We show that the construction presented in this thesis yields optimal L^2-stability for spline-wavelet bases of the order $N \leq 4$. It will be shown that the adaptive frame methods for elliptic operator equations with the bases constructed in this thesis realizes the optimal convergence rate, in particular, in the important case of cubic spline wavelets.

Keywords: biorthogonal wavelets, moments, condition, spline-wavelet bases, adaptive frame methods

List of Symbols

Sets of Numbers

$\mathbb{N}$	set of all positive integers
$\mathbb{N}_0$	set of all nonnegative integers
$\mathbb{Z}$	set of all integers
$\mathbb{Q}$	set of all rational numbers
$\mathbb{R}$	set of all real numbers
$\mathbb{R}^d$	d-dimensional Euclidean vector space

General Notation

d	spatial dimension
Ω	an open bounded set in $\mathbb{R}^d$
$\partial\Omega$	the boundary of Ω
$\#$	cardinality of a set
$\lvert\cdot\rvert$	absolute value
$\lfloor\cdot\rfloor$, $\lceil\cdot\rceil$	the greatest (smallest) integer less (greater) than or equal to the argument
$\delta_{i,j}$, δ_n	the Kronecker delta, $\delta_{i,i} := 1$, $\delta_{i,j} := 0$ for $i \neq j$, $\delta_0 := 1$, $\delta_n := 0$ for $n \neq 0$
span	linear span
supp	support
diam	diameter
flops	floating point operations
$\mathbf{I}$, $\mathbf{I}_n$	the identity matrix, the identity matrix of the size $n \times n$

$[t_0, \ldots, t_n]f$	the divided difference
$\kappa(\cdot)$	spectral condition of an operator or a matrix
$f\|_A$	restriction of a function $f : \Omega \to \mathbb{R}$ to a subdomain $A \subset \Omega$
$a \lesssim b$	$a \leq Cb$, where C is a positive constant independent of any parameters of the arguments
$a \sim b$	$a \lesssim b$ and $b \lesssim a$

Function Spaces

H	separable Hilbert space
H'	dual space to H
$L^p(\Omega)$	Lebesgue space
$H^s(\Omega)$	Sobolev space of order $s \in \mathbb{R}$ on Ω
$C^m(\mathbb{R})$	$m \in \mathbb{N}$, the space of m-times continuously differentiable functions
$C_0^m(\mathbb{R})$	$m \in \mathbb{N}$, the space of m-times continuously differentiable functions with compact support
$B_q^t(L^p(\Omega))$	Besov space of smoothness s over $L^p(\Omega)$, with additional index q
$l^2(\mathcal{J})$	$l^2(\mathcal{J}) := \left\{\mathbf{v} : \mathcal{J} \to \mathbb{R}, \sum_{\lambda \in \mathcal{J}} \lvert\mathbf{v}_\lambda\rvert^2 < \infty\right\}$
$\Pi_m(\Omega)$	space of all algebraic polynomials on Ω of degree less or equal to $m \in \mathbb{N}$

Norms and Inner Products

$\lVert\cdot\rVert$	L^2-norm
$\lVert\cdot\rVert_H$	norm on some space H
$\lvert\cdot\rvert_{H^s(\Omega)}$	seminorm on $H^s(\Omega)$
$\langle\cdot,\cdot\rangle$	L^2-inner product or dual form
$\langle\cdot,\cdot\rangle_H$	inner product in H

Wavelets

j_0	the coarsest level in a multiresolution analysis in a given context
j	generic index for level of resolution
J	the finest level of a resolution in a given context

λ	wavelet index $\lambda := (j,k)$
N $(\tilde{N})$	primal (dual) order of the polynomial exactness
γ $(\tilde{\gamma})$	primal (dual) Sobolev exponent of smoothness,
ϕ $(\tilde{\phi})$	primal (dual) scaling function
ψ $(\tilde{\psi})$	primal (dual) wavelet
$\phi_{j,k},\ \phi_\lambda$	primal scaling functions
$\tilde{\phi}_{j,k},\ \tilde{\phi}_\lambda$	dual scaling functions
Φ_j $(\tilde{\Phi}_j)$	primal (dual) scaling basis
$\mathcal{I}_j$	index set for the scaling basis Φ_j
$\psi_{j,k},\ \psi_\lambda$	primal wavelets
$\tilde{\psi}_{j,k},\ \tilde{\psi}_\lambda$	dual wavelets
Ψ_j $(\tilde{\Psi}_j)$	primal (dual) single-scale wavelet basis
$\mathcal{J}_j$	index set for the single-scale wavelet basis Ψ_j
S_j $(\tilde{S}_j)$	closed subspace of the primal (dual) Hilbert space
W_j $(\tilde{W}_j)$	primal (dual) complement space
$\Psi_{(J)}$ $(\tilde{\Psi}_{(J)})$	primal (dual) wavelet basis up to level J
$\mathcal{S}$ $(\tilde{\mathcal{S}})$	primal (dual) multiresolution analysis
$\mathbf{M}_j$ $(\tilde{\mathbf{M}}_j)$	primal (dual) refinement matrix
$\mathbf{M}_{j,0}$ $(\tilde{\mathbf{M}}_{j,0})$	primal (dual) refinement matrix corresponding to scaling functions
$\mathbf{M}_{j,1}$ $(\tilde{\mathbf{M}}_{j,1})$	primal (dual) refinement matrix corresponding to wavelets
$\mathbf{G}_j$ $(\tilde{\mathbf{G}}_j)$	inverse of $\mathbf{M}_j$ $(\tilde{\mathbf{M}}_j)$
$\mathbf{G}_{j,0}$ $(\tilde{\mathbf{G}}_{j,0})$	upper part of $\mathbf{G}_j$ $(\tilde{\mathbf{G}}_j)$
$\mathbf{G}_{j,1}$ $(\tilde{\mathbf{G}}_{j,1})$	lower part of $\mathbf{G}_j$ $(\tilde{\mathbf{G}}_j)$

Introduction

This doctoral thesis is concerned with the theoretical and computational issues of the construction and the applications of biorthogonal wavelet bases.

Originally, wavelets were constructed on the whole real line. Since almost all constructions of wavelets on general domains start with wavelets as orthonormal or biorthogonal bases of the space $L^2(\mathbb{R})$, we describe this concept in Chapter 1.

In Chapter 2 we propose novel methods for construction of some classes of orthonormal wavelets, namely Daubechies wavelets, symmlets, coiflets, and generalized coiflets. The idea of these methods is to replace equations for scaling coefficients representing necessary conditions by equations for scaling moments. We show that by this approach we are able to eliminate some quadratic conditions in the original system and thus simplify the construction and in some cases we are even able to find solutions in explicit form. In the case of Daubechies wavelets our construction is just an alternative to very well-known spectral factorization. In the case of coiflets and generalized coiflets there is nothing similar to spectral factorization and the construction is more difficult. Coiflets are typically constructed by Newton's method [64, 86] or iterative numerical optimization [14]. These methods enable us to derive one particular solution for each system and the convergence and the obtained solution depends on the initial starting point, thus it is difficult to find all possible solutions. Moreover, the coefficients for length greater than 16 are given with less precision due to the roundoff error [63]. As an alternative one can use the Gröbner basis method [1, 13, 72]. The advantage of such an approach is that solutions can be computed to arbitrary precision and that in some cases it gives all possible solutions for a given system of polynomial equations. We aplly the Gröbner basis method to the simplified system of equations. By this approach we are able to find some exact values of filter coefficients and to find all possible solutions for filters up to length 20. The results of Chapter 2 were already published in [24, 25, 27].

In applications one often needs to compute the integrals involving scaling functions or wavelets. Since many types of scaling functions and wavelets have a low Sobolev as well as Hölder regularity, the classical quadratures such as the Simpsons rule are not useful in this case. In Chapter 3 we propose a novel method for the computation of moments $M^i_{c,d} := \int_c^d x^i \phi(x)\, dx$ for a refinable function ϕ with compact support and c, d any dyadic points.

Chapter 4 provides a short introduction to the theory of generalized wavelet bases, i.e. wavelet bases on bounded domains.

In Chapter 5 we briefly describe some methods of building wavelet bases on bounded domain. The first step is the adaptation from the real line to the interval. We identify the obstructions which arise when analyzing the function defined on the unit interval $(0, 1)$ from the viewpoint of numerical treatment of operator equations.

Chapter 6 is concerned with the construction of wavelet bases on the interval derived from B-splines. The resulting bases generate multiresolution analyses on the unit interval with the desired number of vanishing wavelet moments for primal and dual wavelets. Inner wavelets are translated and dilated versions of well-known wavelets designed by Cohen, Daubechies, Feauveau [41] while the construction of boundary wavelets is along the lines of [53, 80]. The disadvantage of popular bases from [53] is their bad condition which causes problems in practical applications. Some modifications which lead to better conditioned bases were proposed in [4, 54, 80, 97]. Our objective is now to modify the construction of B-spline wavelet bases on the interval from [53] in order to improve their condition. Then we adapt the constructed bases to the complementary boundary conditions. The results of Chapter 6 were published in [26, 28, 31].

In Chapter 7 we briefly review adaptive wavelet methods for the elliptic operator equations. Our intention is to show the dependence of the effectiveness of these methods on the condition of wavelet bases and to identify the routines which will be used in Chapter 8.

In the last Chapter 8 the condition of scaling bases, the single-scale wavelet bases and the multiscale wavelet bases is computed. It is improved by the L^2-normalization on the primal side. It will be shown that in the case of cubic spline wavelets bases the construction presented in this thesis yields optimal L^2-stability, which is not the case of constructions in [53, 80]. The condition of the wavelet transform is presented as well as the condition of scaling bases and wavelet bases satisfying the complementary boundary conditions of the first order. The other criteria for the effectiveness of wavelet bases is the condition number of the corresponding preconditioned stiffness matrix. To improve it further, we apply an orthogonal transformation to the scaling basis on the coarsest level as in [22] and then we use a diagonal matrix for preconditioning. Finally, it will be shown that the adaptive frame methods with the bases constructed in this thesis realizes the optimal convergence rate, in particular, in the important case of cubic spline wavelets. The results of Chapter 8 were published in [26, 28, 29, 30].

Chapter 1

Biorthogonal Wavelets on the Real Line

Originally, wavelets were constructed on the whole real line, then they were adapted to the interval and the n-dimensional cube and nowadays wavelet bases are available for fairly general domains and for a wide range of applications. Since almost all constructions of wavelets on general domains start with wavelets as orthonormal or biorthogonal bases of the space $L^2(\mathbb{R})$, we describe this concept here.
Let us start with the definition and properties of a Riesz basis.

Definition 1. A family $\{e_k\}_{k\in\mathbb{Z}}$ is called a *Riesz basis* of a Hilbert space H, if and only if it spans H, i.e. all finite linear combinations of the e_k are dense in H, and if there exist constants c, C such that $0 < c \le C$ and

$$c\left(\sum_{k\in\mathbb{Z}} |x_k|^2\right)^{1/2} \le \left\|\sum_{k\in\mathbb{Z}} x_k e_k\right\|_H \le C\left(\sum_{k\in\mathbb{Z}} |x_k|^2\right)^{1/2} \quad \text{for all} \quad \{x_k\} \in l^2(\mathbb{Z}). \tag{1.1}$$

Since any orthonormal system satisfies (1.1) with $c = C = 1$, a Riesz basis is a generalization of an orthonormal basis. The main properties of Riesz basis are summarized in the following theorem.

Theorem 2. *Let $\{e_k\}_{k\in\mathbb{Z}}$ be a Riesz basis in a separable Hilbert space H and let the operator $T : l^2(\mathbb{Z}) \to H$ be defined by*

$$T : \{c_k\}_{k\in\mathbb{Z}} \mapsto \sum_{k\in\mathbb{Z}} c_k e_k. \tag{1.2}$$

Then

a) *The series $\sum_{k\in\mathbb{Z}} c_k e_k$ converges unconditionally in H, i.e. its terms can be permuted without affecting the convergence, if and only if $\{c_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$.*

b) *Any $x \in H$ can be decomposed in a unique way according to*

$$x = \sum_{k\in\mathbb{Z}} c_k e_k \tag{1.3}$$

with $\{c_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$.

c) *T is an isomorphism from $l^2(\mathbb{Z})$ to H.*

d) *There exists a unique biorthogonal Riesz basis $\{\tilde{e}_k\}_{k\in\mathbb{Z}}$ in H, i.e. $\{\tilde{e}_k\}_{k\in\mathbb{Z}}$ is a Riesz basis and $\langle e_k, \tilde{e}_l\rangle = \delta_{k,l}$. This basis is defined by*

$$\tilde{e}_k = (TT^*)^{-1} e_k, \tag{1.4}$$

where T^* denotes the adjoint mapping to T.

e) *There exists constants $0 < c \leq C$ such that*

$$c\,\|x\|_H^2 \leq \sum_{k\in\mathbb{Z}} \left|\langle x, e_k\rangle_H^2\right| \leq C\,\|x\|_H^2\,. \tag{1.5}$$

The proof of Theorem 2 can be found in [70, 104]. From *b)* and *d)* it follows that

$$x = \sum_{k\in\mathbb{Z}} \langle x, \tilde{e}_k\rangle_H\, e_k = \sum_{k\in\mathbb{Z}} \langle x, e_k\rangle_H\, \tilde{e}_k, \quad x \in H. \tag{1.6}$$

A sequence $\{e_k\}_{k\in\mathbb{Z}}$ that satisfies (1.5) is called a *frame* and if in addition $c = C$ then the frame is called a *tight frame*. The important difference between frames and Riesz bases is that frames can be redundant. As a trivial example take vectors $(1,0)$, $(0,1)$, and $(-1,0)$ in $\mathbb{R}^2$ that satisfies (1.5) with $c = C = \frac{3}{2}$. In this chapter, we consider $H = L^2(\mathbb{R})$. A function ϕ such that $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a Riesz basis of its L^2-span is called L^2-*stable*.
Now we are able to introduce the definition of a wavelet and an orthonormal wavelet.

Definition 3. A function $\psi \in L^2(\mathbb{R})$ is called a *wavelet* if the family of function $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$, where $\psi_{j,k} = 2^{j/2}\psi(2^j \cdot - k)$, is a Riesz basis in $L^2(\mathbb{R})$. The wavelet is called orthonormal if $\langle \psi_{i,k}, \psi_{j,l}\rangle = \delta_{i,j}\delta_{k,l}$.

Wavelets are usually constructed starting from a multiresolution analysis.

Definition 4. A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ is called a *multiresolution analysis* if it satisfies the following conditions:

1) The sequence is nested, i.e.

$$V_j \subset V_{j+1} \quad \text{for all} \quad j \in \mathbb{Z}. \tag{1.7}$$

2) The spaces are related to each other by dyadic scaling, i.e.

$$f \in V_j \Leftrightarrow f(2\cdot) \in V_{j+1} \quad \text{for all} \quad j \in \mathbb{Z}. \tag{1.8}$$

3) The union of the spaces is dense, i.e.

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})\,. \tag{1.9}$$

4) The intersection of the spaces is reduced to the null function, i.e.

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}\,. \tag{1.10}$$

5) There exists a function $\phi \in V_0$ such that

$$\{\phi(\cdot - k)\,,\quad k\in\mathbb{Z}\} \tag{1.11}$$

is a Riesz basis of V_0.

A function ϕ from property 5) is called a *scaling function.* From property 2) it follows that the family

$$\phi_{j,k} = 2^{j/2}\phi\left(2^j\cdot - k\right),\quad k\in\mathbb{Z},$$

is a Riesz basis for the space V_j.
Since $V_0\subset V_1$ and (4.2), there exists a sequence $\{h_k\}_{k\in\mathbb{Z}}\in l^2(\mathbb{Z})$ such that

$$\phi(x) = \sum_{k\in\mathbb{Z}} h_k\phi(2x-k)\quad \text{for all}\quad x\in\mathbb{R}. \tag{1.12}$$

This equation is called *refinement* or *scaling equation* and the coefficients h_k are known as *scaling* or *refinement coefficients.*

Theorem 5. *If $\phi\in L^2(\mathbb{R})$, the series*

$$S_\phi(\omega) = \sum_{n\in\mathbb{Z}}\left|\hat{\phi}(\omega+2n\pi)\right|^2 \tag{1.13}$$

converges in $L^1(I)$ for any compact set I, to an L^1_{loc} 2π-periodic function. The function ϕ is L^2-stable if and only if there exist constants $c, C>0$ such that

$$c\le\sum_{n\in\mathbb{Z}}\left|\hat{\phi}(\omega+2n\pi)\right|^2\le C\quad a.e. \tag{1.14}$$

Moreover, ϕ is an orthonormal scaling function if and only if

$$\sum_{n\in\mathbb{Z}}\left|\hat{\phi}(\omega+2n\pi)\right|^2 = 1\quad a.e. \tag{1.15}$$

For the proof of Theorem 5 see [36, 78]. Note that the Fourier coefficients of S_ϕ are given by

$$S_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_\phi(\omega)\, \mathrm{e}^{-\mathrm{i}k\omega} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\hat{\phi}(\omega)\right|^2 \mathrm{e}^{-\mathrm{i}k\omega} d\omega = \langle \phi(\cdot - k), \phi \rangle . \qquad (1.16)$$

Thus ϕ is an orthonormal scaling function if and only if $S_\phi(\omega) = 1$ a.e. Now we are able to construct new scaling functions which generate orthonormal and biorthogonal bases.

Corollary 1. *Let $\phi \in L^2(\mathbb{R})$ satisfy (1.14). Then the function ϕ^o defined by*

$$\hat{\phi}^o(\omega) = [S_\phi(\omega)]^{-1/2}\, \hat{\phi}(\omega), \qquad (1.17)$$

generates an orthonormal basis of V_0. The function ϕ^d defined by

$$\hat{\phi}^d(\omega) = [S_\phi(\omega)]^{-1}\, \hat{\phi}(\omega), \qquad (1.18)$$

generates a biorthogonal Riesz basis of V_0. Moreover, if ϕ has a compact support and is bounded, then ϕ^0 and ϕ^d have exponential decay at infinity.

One can find the proof of Corollary 1 and other details in [36, 78]. For many scaling functions both ϕ^o and ϕ^d are globally supported on $\mathbb{R}$. For this reason we consider functions which are biorthogonal to ϕ and which are not enforced to be in V_0.

Definition 6. A refinable function $\tilde{\phi} = \sum_{k\in\mathbb{Z}} \tilde{h}_k \tilde{\phi}(2\cdot -k)$ in $L^2(\mathbb{R})$ is called *dual* to a refinable function ϕ if

$$\left\langle \phi(\cdot - k), \tilde{\phi}(\cdot - l) \right\rangle = \delta_{k,l} \quad \text{for all} \quad k, l \in \mathbb{Z}. \qquad (1.19)$$

It is known that if ϕ is a compactly supported L^2-stable refinable function then there always exists a dual scaling function to ϕ which is also compactly supported, see [73]. Now there are two scaling functions ϕ, $\tilde{\phi}$, which may generate different multiresolution analyses $\{V_j\}_{j\in\mathbb{Z}}$, $\left\{\tilde{V}_j\right\}_{j\in\mathbb{Z}}$, and accordingly two different wavelet functions ψ, $\tilde{\psi}$. Wavelet coefficients can be determined as

$$g_n = (-1)^n\, \tilde{h}_{1-n}, \quad \tilde{g}_n = (-1)^n\, h_{1-n}, \qquad (1.20)$$

where h_n and $\tilde{h}_n$ are scaling coefficients corresponding to ϕ and $\tilde{\phi}$, respectively. Wavelets are then given by

$$\psi(x) = \sum_{n\in\mathbb{Z}} g_n \phi(2x - n), \quad \tilde{\psi}(x) = \sum_{n\in\mathbb{Z}} \tilde{g}_n \tilde{\phi}(2x - n) \quad \text{for all} \quad x \in \mathbb{R}. \qquad (1.21)$$

The function ϕ is called a *primal* scaling function, the sequence $\{V_j\}_{j\in\mathbb{Z}}$ is called a *primal* multiresolution analysis and ψ is a *primal* wavelet, $\tilde{\phi}$, $\left\{\tilde{V}_j\right\}_{j\in\mathbb{Z}}$, and $\tilde{\psi}$ are called *dual*. The *symbols* of the scaling function ϕ and $\tilde{\phi}$ are defined by

$$m(\omega) = \frac{1}{2}\sum_{n\in\mathbb{Z}} h_n \mathrm{e}^{-\mathrm{i}n\omega}, \quad \tilde{m}(\omega) = \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{h}_n \mathrm{e}^{-\mathrm{i}n\omega} \quad \text{for all} \quad \omega \in \mathbb{R}. \qquad (1.22)$$

Necessary conditions on the symbols and scaling coefficients, which are useful for the construction of the dual scaling function, are given in the following lemma [36].

Lemma 7. *The scaling coefficients h_n, $\tilde{h}_n$ and the symbols $m(\omega)$, $\tilde{m}(\omega)$ satisfy*

$$\sum_{n\in\mathbb{Z}} h_n \tilde{h}_{n-2k} = 2\delta_{0,k} \quad \textit{for all} \quad k \in \mathbb{Z} \tag{1.23}$$

and

$$m(\omega)\overline{\tilde{m}(\omega)} + m(\omega+\pi)\overline{\tilde{m}(\omega+\pi)} = 1 \quad \textit{for all} \quad \omega \in \mathbb{R}. \tag{1.24}$$

Moreover, these two conditions are equivalent.

Let us define

$$W_j = \overline{\text{span}\{\psi_{j,k}, k \in \mathbb{Z}\}}, \quad \tilde{W}_j = \overline{\text{span}\left\{\tilde{\psi}_{j,k}, k \in \mathbb{Z}\right\}}. \tag{1.25}$$

It can be proved that W_j complements V_j in V_{j+1} and similarly $\tilde{W}_j$ complements $\tilde{V}_j$ in $\tilde{V}_{j+1}$ and that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ and $\left\{\tilde{\psi}_{j,k}\right\}_{j,k\in\mathbb{Z}}$ are Riesz bases of $L^2(\mathbb{R})$, for details see [104]. Moreover, $\tilde{\psi}$ is dual to ψ, i.e.

$$\left\langle \psi(\cdot - k), \tilde{\psi}(\cdot - l)\right\rangle = \delta_{k,l} \quad \text{for all} \quad k, l \in \mathbb{Z}, \tag{1.26}$$

and

$$\left\langle \psi(\cdot - k), \tilde{\phi}(\cdot - l)\right\rangle = \left\langle \tilde{\psi}(\cdot - k), \phi(\cdot - l)\right\rangle = \delta_{k,l} \quad \text{for all} \quad k, l \in \mathbb{Z}. \tag{1.27}$$

The consequence of (1.26) and (1.27) is that the spaces W_j and $\tilde{W}_l$ are orthogonal for all $j \neq l$, the space W_j is orthogonal to $\tilde{V}_l$ for all $l \leq j$ and the space $\tilde{W}_j$ is orthogonal to V_l for all $l \leq j$.

We define oblic projectors $P_j : L^2(\mathbb{R}) \to V_j$ and $\tilde{P}_j : L^2(\mathbb{R}) \to \tilde{V}_j$ by

$$P_j f = \sum_{k\in\mathbb{Z}} \left\langle f, \tilde{\phi}_{j,k}\right\rangle \phi_{j,k}, \quad \tilde{P}_j f = \sum_{k\in\mathbb{Z}} \langle f, \phi_{j,k}\rangle \tilde{\phi}_{j,k}, \tag{1.28}$$

and detail operators $Q_j : L^2(\mathbb{R}) \to W_j$, $\tilde{Q}_j : L^2(\mathbb{R}) \to \tilde{W}_j$ by

$$Q_j f = P_{j+1} f - P_j f, \quad \tilde{Q}_j f = \tilde{P}_{j+1} f - \tilde{P}_j f. \tag{1.29}$$

Theorem 8. *The oblic projector P_j is L^2-bounded independently of j. We have*

$$\lim_{j\to\infty} \|P_j f - f\| \to 0 \Leftrightarrow \int_{\mathbb{R}} \tilde{\phi}(x)\, dx \sum_{k\in\mathbb{Z}} \phi(x-k) = 1 \quad \textit{a.e.} \tag{1.30}$$

and

$$\lim_{j\to\infty} \left\|\tilde{P}_j f - f\right\| \to 0 \Leftrightarrow \int_{\mathbb{R}} \phi(x)\, dx \sum_{k\in\mathbb{Z}} \tilde{\phi}(x-k) = 1 \quad \textit{a.e.} \tag{1.31}$$

For the proof of Theorem 8 see [36]. Integrating (1.30) over $\langle 0,1\rangle$, we obtain

$$1=\int_{\mathbb{R}}\tilde{\phi}(x)\,dx\int_0^1\sum_{k\in\mathbb{Z}}\phi(x-k)\,dx=\int_{\mathbb{R}}\tilde{\phi}(x)\,dx\int_{\mathbb{R}}\phi(x)\,dx. \tag{1.32}$$

Thus we can renormalize the functions ϕ and $\tilde{\phi}$ and in the following we assume that $\int_{\mathbb{R}}\tilde{\phi}(x)\,dx=\int_{\mathbb{R}}\phi(x)\,dx=1$.
Since the spaces V_j are nested, we have $P_jP_l=P_j$ for all $j<l$ and due to the refinability of $\tilde{\phi}$ the spaces $\tilde{V}_j$ are also nested and $\tilde{P}_j\tilde{P}_l=\tilde{P}_j$ for all $j<l$. The detail operator Q_j is also a projector:

$$Q_j^2=P_{j+1}^2-P_{j+1}P_j-P_jP_{j+1}+P_j^2=Q_j, \tag{1.33}$$

and Q_j can be expanded into

$$Q_jf=\sum_{k\in\mathbb{Z}}\left\langle f,\tilde{\psi}_{j,k}\right\rangle\psi_{j,k}. \tag{1.34}$$

The space V_{j_1} can be decomposed:

$$V_{j_1}=V_{j_0}\oplus W_{j_0}\oplus W_{j_0+1}\ldots\oplus W_{j_1-1} \tag{1.35}$$

and any function $f\in V_{j_1}$ can be expanded into

$$f=\sum_{k\in\mathbb{Z}}c_{j_1,k}\phi_{j_1,k}=\sum_{k\in\mathbb{Z}}c_{j_0,k}\phi_{j_0,k}+\sum_{j=j_0}^{j_1-1}\sum_{k\in\mathbb{Z}}d_{j,k}\psi_{j,k}, \tag{1.36}$$

where due to duality property (1.26) we have

$$c_{j,k}=\langle f,\phi_{j,k}\rangle\quad\text{and}\quad d_{j,k}=\langle f,\psi_{j,k}\rangle. \tag{1.37}$$

The expansion (1.36) is called a *multiresolution* or *multiscale representation*. It can be proved [36] that $\{\phi_{j_0,k}\}_{k\in\mathbb{Z}}\cup\{\psi_{j,k}\}_{j_0\le j<j_1,k\in\mathbb{Z}}$ is a Riesz basis of V_{j_1}, but the Riesz constants may depend on j_1 and for this reason we cannot conclude that $\{\phi_{j_0,k}\}_{k\in\mathbb{Z}}\cup\{\psi_{j,k}\}_{j_0\le j,k\in\mathbb{Z}}$ is a Riesz basis of $L^2(\mathbb{R})$.
From Theorem 8, we can see that any $f\in L^2(\mathbb{R})$ satisfies

$$f=\lim_{j_1\to\infty}\left(\sum_{k\in\mathbb{Z}}c_{j_1,k}\phi_{j_1,k}\right)=\lim_{j_1\to\infty}\left(\sum_{k\in\mathbb{Z}}c_{j_0,k}\phi_{j_0,k}+\sum_{j=j_0}^{j_1-1}\sum_{k\in\mathbb{Z}}d_{j,k}\psi_{j,k}\right). \tag{1.38}$$

1.1 The Discrete Wavelet Transform

Computing the coefficients

$$c_{j_0,k}=\left\langle f,\tilde{\phi}_{j_0,k}\right\rangle=\int_{\mathbb{R}}f(x)\,2^{j_0/2}\tilde{\phi}\left(2^{j_0}x-k\right)dx\quad\text{for}\quad k\in\mathbb{Z} \tag{1.39}$$

and

$$d_{j,k} = \left\langle f, \tilde{\psi}_{j,k} \right\rangle = \int_{\mathbb{R}} f(x)\, 2^{j/2} \tilde{\psi}\left(2^j x - k\right) dx \quad \text{for} \quad j_0 \leq j < j_1, \quad k \in \mathbb{Z}, \tag{1.40}$$

in a multiscale decomposition (1.36) directly through quadrature formulas poses difficulties. Since the support of low level wavelets $\psi_{j_0,k}$ is large, a sufficiently accurate quadrature formula would involve too many samples of the function f. It might considerably slow down the computation. The computation of coefficients $c_{j_1,k}$, is much less expensive, due to their uniformly small support. Fortunately, there exists an algorithm which converts a scaling representation of the function f given by $c_{j_1,k}$ into the wavelet representation given by (1.39) and (1.40). It can be derived from the scaling equation (1.12) and the wavelet equation (1.21). We have

$$\begin{aligned} \tilde{\phi}_{j,k} &= 2^{j/2} \tilde{\phi}\left(2^j \cdot - k\right) = 2^{j/2} \sum_{n \in \mathbb{Z}} \tilde{h}_n \tilde{\phi}\left(2^{j+1} \cdot - 2k - n\right) \\ &= 2^{-1/2} \sum_{m \in \mathbb{Z}} \tilde{h}_{m-2k} 2^{j+1/2} \tilde{\phi}\left(2^{j+1} \cdot - m\right) = 2^{-1/2} \sum_{m \in \mathbb{Z}} \tilde{h}_{m-2k} \tilde{\phi}_{j+1,m}. \end{aligned} \tag{1.41}$$

This implies that

$$c_{j,k} = 2^{-1/2} \sum_{m \in \mathbb{Z}} \tilde{h}_{m-2k} c_{j+1,m}. \tag{1.42}$$

From (1.21) we obtain

$$\begin{aligned} \tilde{\psi}_{j,k} &= 2^{j/2} \tilde{\psi}\left(2^j \cdot - k\right) = 2^{j/2} \sum_{n \in \mathbb{Z}} \tilde{g}_n \tilde{\phi}\left(2^{j+1} \cdot - 2k - n\right) \\ &= 2^{-1/2} \sum_{m \in \mathbb{Z}} \tilde{g}_{m-2k} 2^{j+1/2} \tilde{\phi}\left(2^{j+1} \cdot - m\right) = 2^{-1/2} \sum_{m \in \mathbb{Z}} \tilde{g}_{m-2k} \tilde{\phi}_{j+1,m}. \end{aligned} \tag{1.43}$$

It follows that

$$d_{j,k} = 2^{-1/2} \sum_{m \in \mathbb{Z}} \tilde{g}_{m-2k} c_{j+1,m}. \tag{1.44}$$

The equations (1.42) and (1.44) represent the *decomposition algorithm*. The decomposition algorithm is the first half of the *discrete wavelet transform* (DWT). We can go also in the opposite direction and reconstruct coefficients $c_{j+1,k}$ from coefficients $c_{j,k}$ and $d_{j,k}$. By definition of the detail operator $\tilde{Q}_j$, we have

$$\tilde{P}_j = \tilde{P}_{j-1} + \tilde{Q}_{j-1}. \tag{1.45}$$

Therefore,

$$\begin{aligned} \sum_{k \in \mathbb{Z}} c_{j,k} \tilde{\phi}_{j,k} &= \sum_{k \in \mathbb{Z}} c_{j-1,k} \tilde{\phi}_{j-1,k} + \sum_{k \in \mathbb{Z}} d_{j-1,k} \tilde{\psi}_{j-1,k} \\ &= \sum_{k \in \mathbb{Z}} c_{j-1,k} \sum_{n \in \mathbb{Z}} \tilde{h}_{n-2k} \tilde{\phi}_{j,k} + \sum_{k \in \mathbb{Z}} d_{j-1,k} \sum_{n \in \mathbb{Z}} \tilde{g}_{n-2k} \tilde{\phi}_{j,k}. \end{aligned} \tag{1.46}$$

By matching coefficients, we obtain the *reconstruction algorithm*:

$$c_{j,k} = \sum_{n\in\mathbb{Z}} \tilde{h}_{k-2n} c_{j-1,n} + \sum_{n\in\mathbb{Z}} \tilde{g}_{k-2n} d_{j-1,n}. \tag{1.47}$$

The reconstruction algorithm is the second half of the discrete wavelet transform. In practice, we deal with functions with compact support. Then there exist $k_1 \in \mathbb{Z}$, $n \in \mathbb{N}$ such that $c_{j_1,k} = 0$ for $k < k_1$ and $k \geq k_1 + n$. In this case the discrete wavelet transform can be performed in $\mathcal{O}(n)$ operations.

1.2 Approximation Properties

The rate of decay of the error of wavelet approximation is determined by the number of vanishing dual wavelet moments. A dual wavelet $\tilde{\psi}$ has m vanishing moments if and only if the scaling function ϕ can generate polynomials of degree smaller than or equal to m. In the following theorem further equivalent conditions are given.

Theorem 9. *Let ϕ be an L^1 function with compact support and with $\int_{\mathbb{R}} \phi(x)\, dx = 1$. The following properties are equivalent:*

i) ϕ satisfies the Strang-Fix conditions of order $L-1$, i.e.

$$\left(\frac{\partial}{\partial\omega}\right)^q \widehat{\phi}(2\pi n) = 0, \qquad n \in \mathbb{Z} - \{0\}, \quad 0 \leq q \leq L-1. \tag{1.48}$$

ii) For all $q = 0, \ldots, L-1$, we can expand the polynomial x^q according to

$$x^q = 2^{-jq} \sum_{k\in\mathbb{Z}} \left\langle (\cdot)^q, \tilde{\phi}(\cdot - k) \right\rangle \phi(\cdot - k), \quad x \in \mathbb{R} \text{ a.e.} \tag{1.49}$$

iii) The symbol of ϕ defined by (1.22) has the factorized form

$$m(\omega) = \left(\frac{1 + \mathrm{e}^{-\mathrm{i}\omega}}{2}\right)^L p(\omega), \tag{1.50}$$

where $p(\omega)$ is a trigonometric polynomial.

iv) The dual wavelet $\tilde{\psi}$ has L vanishing moments, i.e.

$$\int_{\mathbb{R}} x^r \tilde{\psi}(x)\, dx = 0 \quad \text{for all} \quad r = 0, \ldots, L-1. \tag{1.51}$$

v) There exists a constant $C > 0$ such that any $f \in H^{L+1}$ satisfies

$$\|f - P_j f\|_{H^s} \leq C 2^{-j(L-s)} |f|_{H^L}, \quad s = 0, \ldots, L-1. \tag{1.52}$$

The proof can be found in [36, 104].

1.3 B-Spline Wavelet Bases

In this section we review the construction of biorthogonal B-spline wavelet bases proposed in [41]. These bases will be adapted to the interval in Chapter 6 and then they will be used for the adaptive resolution of elliptic differential equations in Chapter 8.

Definition 10. The *(cardinal) B-spline* B_N of degree N, $N \in \mathbb{N}$, is defined by $B_1 = \chi_{\langle 0,1 \rangle}$ and

$$B_N = B_1 * B_{N-1} = \int_{\mathbb{R}} B_1(t)\, B_{N-1}(x-t)\, dt, \quad N \geq 2. \tag{1.53}$$

The following Lemma summarizes elementary properties of B-splines.

Lemma 11. *For $N \in \mathbb{N}$ the functions B_N have the following properties:*

1) B_N is supported in $\langle 0, N \rangle$.

2) $B_N(x) > 0 \quad$ for all $\quad x \in (0, N)$.

3) The function B_N is symmetric with respect to the point $\frac{N}{2}$, i.e.

$$B_N\left(\frac{N}{2} - x\right) = B_N\left(\frac{N}{2} + x\right) \quad \text{for all} \quad x \in \mathbb{R}. \tag{1.54}$$

4) $\int_{\mathbb{R}} B_N(x)\, dx = 1$.

5) For all $x \in \mathbb{R}$ we have

$$B_N(x) = \frac{1}{(N-1)!} \sum_{k=0}^{N} (-1)^k \binom{N}{k} (x-k)_+^{N-1}, \tag{1.55}$$

where $x_+^N = (max\{0, x\})^N$.

6) B_N generates the multiresolution spaces

$$V_j = \left\{ f \in L^2(\mathbb{R}) \cap C^{N-1}(\mathbb{R}) : f|_{\langle \frac{k}{2^j}, \frac{k+1}{2^j} \rangle} \in \Pi_N \quad \text{for all} \quad k \in \mathbb{Z} \right\}. \tag{1.56}$$

7) B_N is a refinable function, i.e. it satisfies (1.12), refinement coefficients are given by

$$h_n = 2^{-N} \binom{N}{n} \quad \text{for} \quad n = 0, \dots N, \quad h_n = 0 \quad \text{otherwise.} \tag{1.57}$$

The proof of Lemma 11 and other interesting properties of splines can be found in [18, 32, 106]. Due to Lemma 11 we can define primal scaling function as $_N\phi = B_N$, this function is exact of order N, i.e. it reproduces polynomials up to order $N-1$. It has been shown in [41] that for each N and any $\tilde{N} \in \mathbb{N}$, $\tilde{N} \geq N$, so that $N + \tilde{N}$ is even, there exists a

compactly supported dual scaling function ${}_{N,\tilde{N}}\tilde{\phi}$, which is exact of order $\tilde{N}$. We briefly review the construction of ${}_{N,\tilde{N}}\tilde{\phi}$ here.
Recall that the symbols of scaling functions have to satisfy:

$$\overline{m(\omega)}\tilde{m}(\omega) + \overline{m(\omega+\pi)}\tilde{m}(\omega+\pi) = 1. \tag{1.58}$$

Thus, for the given symbol $m(\omega)$ we try to find a trigonometric polynomial $\tilde{m}(\omega)$ so that the above identity is satisfied.

Lemma 12. *For $M \in \mathbb{N}$ let us define a polynomial*

$$p_M(x) = \sum_{n=0}^{M-1} \binom{M-1+n}{n} x^n. \tag{1.59}$$

Then

$$(1-x)^M p_M(x) + x^M p_M(1-x) = 1 \quad \textit{for all} \quad x \in \mathbb{R}. \tag{1.60}$$

This Lemma was proved by Daubechies in [62], where it was used for the construction of compactly supported orthogonal wavelets.
Replacing x by $\sin^2\frac{\omega}{2}$, we obtain

$$\left(\cos^2\frac{\omega}{2}\right)^M p_M\left(\sin^2\frac{\omega}{2}\right) + \left(\cos^2\frac{\omega+\pi}{2}\right)^M p_M\left(\sin^2\frac{\omega+\pi}{2}\right) = 1 \tag{1.61}$$

Therefore, it is sufficient to find trigonometric polynomials satisfying

$$\overline{m(\omega)}\tilde{m}(\omega) = \left(\cos^2\frac{\omega}{2}\right)^M p_M\left(\sin^2\frac{\omega}{2}\right). \tag{1.62}$$

The symbol of ${}_N\phi$ satisfies

$$m(\omega) = \frac{\left(\mathrm{e}^{-2\mathrm{i}\omega}-1\right)^N}{2^N\left(\mathrm{e}^{-\mathrm{i}\omega}-1\right)^N}. \tag{1.63}$$

We replace $e^{i\omega}$ by z. Let $2M > N$, then the scaling coefficients $\tilde{h}_n$ of the dual scaling function are given by:

$$\left(\frac{z+1}{2}\right)^N \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{h}_n z^{-n} = \frac{(z+1)^{2M}}{4^M z^M} p_M\left(1 - \frac{(z+1)^2}{4z}\right). \tag{1.64}$$

By Theorem 9 the polynomial exactness of the dual scaling function is $\tilde{N} = 2M - N$. The wavelets are then given by (1.20) and (1.21).

Example 13. Let $N = 3$ and $\tilde{N} = 5$. Then $M = 4$, $p_M(x) = 1 + 4x + 10x^2 + 20x^3$ and the scaling coefficients $\tilde{h}_n$ of the dual scaling function are given by:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \tilde{h}_n z^{-n} &= \frac{(z+1)^5}{2^4 z^4}\left(\frac{-5}{16z^3} + \frac{5}{2z^2} + \frac{-131}{16z} + 13 + \frac{-131z}{16} + \frac{5z^2}{2} + \frac{-5z^3}{16}\right) \\
&= -\frac{-5}{256}z^{-7} + \frac{15}{256}z^{-6} + \frac{19}{256}z^{-5} - \frac{97}{256}z^{-4} - \frac{13}{128}z^{-3} + \frac{175}{128}z^{-2} + \frac{175}{128}z^{-1} \\
&\quad -\frac{13}{128} - \frac{97}{256}z + \frac{19}{256}z^2 + \frac{15}{256}z^3 - \frac{5}{256}z^4.
\end{aligned}$$

The primal and dual scaling coefficients for some values of N and $\tilde{N}$ are listed in Table 1.1. Recall that the Sobolev regularity γ of a function f is defined by

$$\gamma := \sup \{s : f \in H^s\} . \tag{1.65}$$

It is known that the Sobolev regularity of the primal scaling function ${}_N\phi$ is $\gamma = N - \frac{1}{2}$. The Sobolev regularity of the dual scaling functions is shown in Table 1.2. It was computed by the well-known algorithm from [65]. Figure 1.1 shows the graphs of several B-spline scaling functions and wavelets.

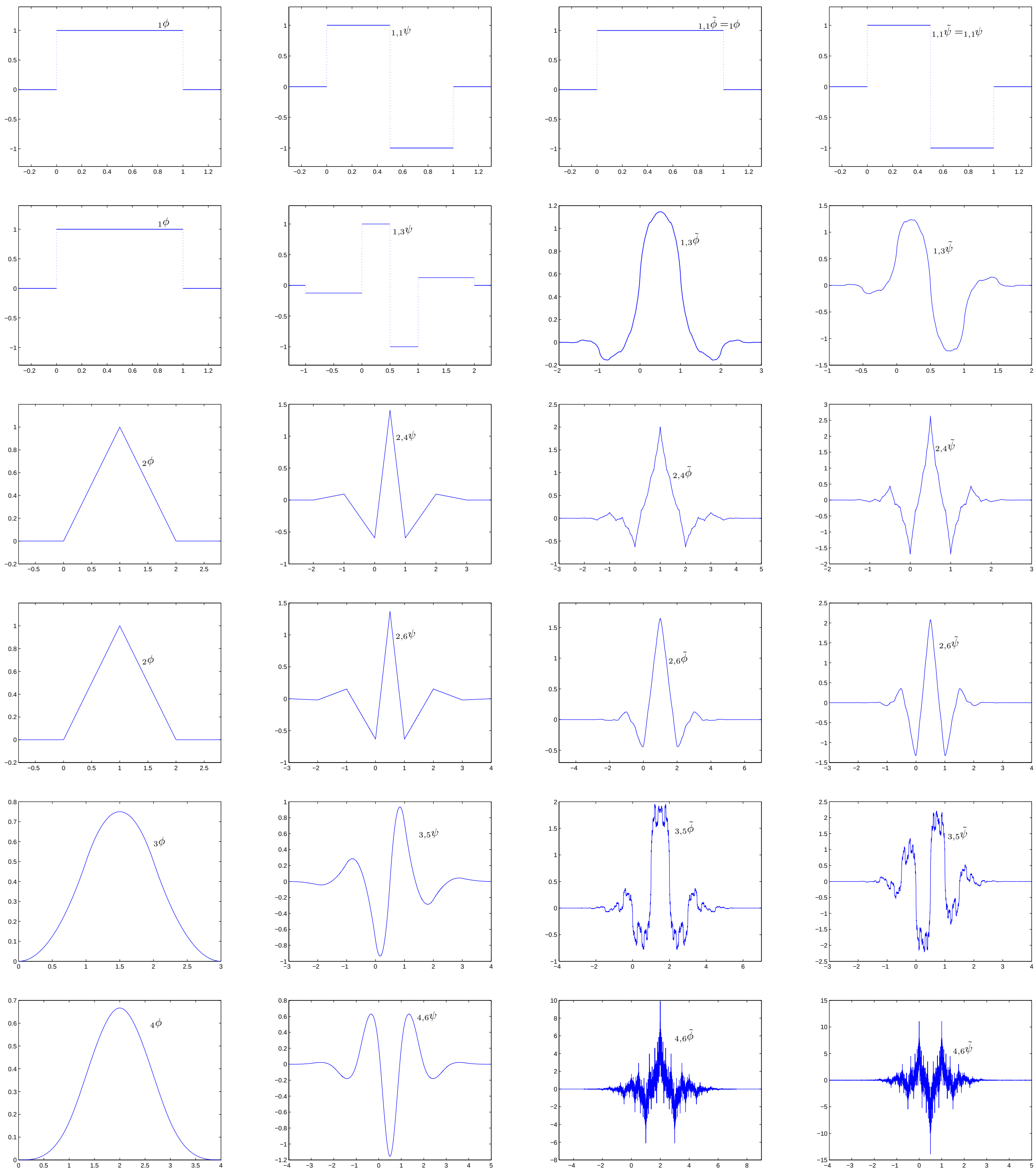
Table 1.1: Scaling coefficients of primal and dual scaling functions exact of order N and $\tilde{N}$, respectively.

N	$\{h_n\}$	$\tilde{N}$	$\{\tilde{h}_n\}$
1	$\{1,1\}$	1	$\{1,1\}$
		3	$\left\{\frac{-1}{8},\frac{1}{8},1,1,\frac{1}{8},\frac{-1}{8}\right\}$
		5	$\left\{\frac{3}{128},\frac{-3}{128},\frac{-11}{64},\frac{11}{64},1,1,\frac{11}{64},\frac{-11}{64},\frac{-3}{128},\frac{3}{128}\right\}$
2	$\left\{\frac{1}{2},1,\frac{1}{2}\right\}$	2	$\left\{\frac{-1}{4},\frac{1}{2},\frac{3}{2},\frac{1}{2},\frac{-1}{4}\right\}$
		4	$\left\{\frac{3}{64},\frac{-3}{64},\frac{-1}{4},\frac{19}{32},\frac{45}{32},\frac{19}{32},\frac{-1}{4},\frac{-3}{64},\frac{3}{64}\right\}$
3	$\left\{\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{1}{4}\right\}$	3	$\left\{\frac{3}{32},\frac{-9}{32},\frac{-7}{32},\frac{45}{32},\frac{45}{32},\frac{-7}{32},\frac{-9}{32},\frac{3}{32}\right\}$
		5	$\left\{\frac{-5}{256},\frac{15}{256},\frac{19}{256},\frac{-97}{256},\frac{-13}{128},\frac{175}{128},\frac{175}{128},\frac{-13}{128},\frac{-97}{256},\frac{19}{256},\frac{15}{256},\frac{-5}{256}\right\}$

Table 1.2: Sobolev exponent of smoothness $\tilde{\gamma}$ of the dual scaling function ${}_{N,\tilde{N}}\tilde{\phi}$

N	$\tilde{N}$	$\tilde{\gamma}$	N	$\tilde{N}$	$\tilde{\gamma}$	N	$\tilde{N}$	$\tilde{\gamma}$
2	2	0.441	3	3	0.175	4	6	0.344
2	4	1.175	3	5	0.793	4	8	0.862
2	6	1.793	3	7	1.344	4	10	1.363

Figure 1.1: Biorthogonal B-spline scaling functions and wavelets



Chapter 2

Construction of Orthonormal Wavelets

In this chapter we propose novel methods for construction of some classes of orthonormal wavelets, namely Daubechies wavelets, symmlets, coiflets, and generalized coiflets. The idea of these methods is to replace equations for scaling coefficients representing necessary conditions by equations for scaling moments. We show that by this approach we are able to eliminate some quadratic conditions in the original system and thus simplify the construction and in some cases we are even able to find solutions in explicit form. First of all we express some required properties of wavelets in terms of scaling coefficients.

Theorem 14. *Let $\phi(\cdot) = \sum h_k \phi(2 \cdot - k)$.*

i) If ϕ has a compact support, then only finite number of terms in h_k are nonzero.

ii) If a system $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is orthonormal in $L^2(\mathbb{R})$, then

$$\delta_m = \frac{1}{2} \sum_{j=N_1}^{N_2-2m} h_j h_{j+2m} \quad \text{for} \quad m \in \mathbb{Z}. \tag{2.1}$$

iii) Wavelet ψ corresponding to the scaling function ϕ has N vanishing moments, i.e.

$$\int_{-\infty}^{\infty} x^l \psi(x)\, dx \quad \text{for} \quad l = 0 \ldots N-1. \tag{2.2}$$

if and only if

$$\sum_{k\in\mathbb{Z}} (-1)^k k^l h_k = 0 \quad \text{for} \quad l = 0 \ldots N-1. \tag{2.3}$$

iv) The scaling function ϕ *has* $N-1$ *vanishing moments, i.e.*

$$\int_{-\infty}^{\infty} x^l \phi(x)\, dx \quad \textit{for} \quad l = 1 \dots N-1, \tag{2.4}$$

if and only if

$$\sum_{k\in\mathbb{Z}} k^l h_k = 0 \quad \textit{for} \quad l = 1 \dots N-1. \tag{2.5}$$

For the proof of Theorem 14 see for example [36, 78]. Note that condition (2.1) is only necessary for the scaling function to be orthogonal. Thus, after finding coefficients satisfying (2.1) and other required properties orthonormality should be verified, for example using Cohen [37] or Lawton [71] condition. Now we aim to replace the quadratic equations (2.1) by some linear ones. The essential part of this is the following theorem [27].

Theorem 15. *Let* $N_2 = N_1 + 2M - 1$ *then* (2.1) *is equivalent to*

$$\frac{1}{2}\,\delta_n = \sum_{i=0}^{2n} \binom{2n}{i} (-1)^i (a_i a_{2n-i} + b_i b_{2n-i}) \quad \textit{for} \quad 0 \le n \le M-1, \tag{2.6}$$

where

$$a_i = \sum_{k=0}^{M-1} (N_1 + 2k)^i h_{N_1+2k} \quad \textit{and} \quad b_i = \sum_{k=0}^{M-1} (N_1 + 2k + 1)^i h_{N_1+2k+1}. \tag{2.7}$$

Proof. The proof for the case $N_1 = 0$ is given in [67]. The proof for an arbitrary N_1 is similar. We briefly sketch it here. It is very well-known that the necessary condition of orthonormality (2.1) is equivalent to

$$|m(\omega)|^2 + |m(\omega+\pi)|^2 = 1, \tag{2.8}$$

where the symbol $m(\omega)$ is defined by (1.22). For the purpose of this proof we define operators G_0 and G_1

$$G_0(z) = \sum_{k=0}^{M-1} h_{N_1+2k} z^{N_1+2k} \qquad \text{and} \qquad G_1(z) = \sum_{k=0}^{M-1} h_{N_1+2k+1} z^{N_1+2k+1}. \tag{2.9}$$

Then (2.6) is equivalent to

$$G_0(z)G_0(z^{-1}) + G_1(z)G_1(z^{-1}) = \frac{1}{2}. \tag{2.10}$$

Now we define a function X by

$$X(z) := G_0(z)G_0(z^{-1}) + G_1(z)G_1(z^{-1}) - \frac{1}{2} = 0. \tag{2.11}$$

Then X includes only even polynomials in the form $c_k z^{2k}$, where k runs from $-M+1$ to $M-1$ and at the same time $c_k = c_{-k}$ for all k. Now, if we look closely how these c_k look like, we get

$$c_k = \sum_j h_j h_{2k+j}. \tag{2.12}$$

At the end, we prove the equivalence of (2.6) and (2.8). We apply the operator $(zD)^n$ on (2.11) and we obtain

$$(zD)^n X(1) = 0 \qquad \text{for} \quad n \geq 0, \tag{2.13}$$

where D denotes derivative. However, apparently only even derivatives up to degree M yield a basis for all functions in the form

$$c_{M-1}z^{2M-2} + \cdots + c_1 z^2 + c_0 z^0 + c_1 z^{-2} + \cdots + c_{M-1} z^{-2M+2}. \tag{2.14}$$

Because applying odd derivatives leads to the contradiction with the fact that the coefficients by z^k and by z^{-k} are the same. Furthermore, the system of functions in the form (2.14) contains only M free parameters and the even derivatives up to degree $2M-2$ are linearly independent and their number is also M. So, they generate a basis of this system of functions. Thus, (2.6) is equivalent to:

$$(zD)^{2n} X(1) = 0 \qquad \text{for} \quad 0 \leq n \leq M-1. \tag{2.15}$$

Applying the operator $(zD)^{2n}$ on X completes the proof:

$$\begin{aligned} \frac{1}{2}\,\delta_{0,n} &= (zD)^{2n}(G_0(z)G_0(z^{-1}) + G_1(z)G_1(z^{-1}))(1) \\ &= \sum_{i=0}^{2n} \binom{2n}{i} (-1)^i (a_i a_{2n-i} + b_i b_{2n-i}), \end{aligned} \tag{2.16}$$

where a_i, b_i are defined by (2.7).

□

Hence, we replace the original necessary ortonormality conditions by (2.7) and further we will work with a_i and b_i instead of h_k. Let us use the following notations:

$$M_n := \int_{-\infty}^{\infty} x^n \phi(x)\, dx \qquad \text{and} \qquad m_n := \sum_k h_k k^n \tag{2.17}$$

for the continuous scaling moments and for the discrete scaling moments, respectively. Note that $m_i = a_i + b_i$. We further investigate the relations among scaling moments and their parts a_i and b_i in the following theorem and lemma [27].

Theorem 16. *Let $\phi(\cdot) = 2\sum_0^{2N-1} h_k\, \phi(2\cdot - k)$ and let ϕ be an orthonormal scaling function with compact support and further let $M_0 = 1$. Then for $n < N$ the following properties are equivalent:*

1) $\sum_{k\in\mathbb{Z}} (-1)^k k^j h_k = 0 \quad$ *for* $j = 0, \dots, n$,

2) $\delta_{0,j} = \sum_{i=0}^{2j} \binom{2j}{i} (-1)^i\, m_i\, m_{2j-i} \quad$ *for* $\; j = 0, \dots, n$,

3) $\delta_{0,j} = \sum_{i=0}^{2j} \binom{2j}{i} (-1)^i\, M_i\, M_{2j-i} \quad$ *for* $\; j = 0, \dots, n$.

Proof. Implications '1) $\Rightarrow$ 2)' and '1) $\Rightarrow$ 3)' were already proved in [66, 67]. Here we propose simpler proof of these relations and prove '2) $\Rightarrow$ 1)' and '2) $\Leftrightarrow$ 3)'. Consider a_i, b_i as above.

First, we prove the implication '1) $\Rightarrow$ 2)'. The first condition implies

$$a_i = b_i \qquad \text{for} \quad i = 0, \dots, n \tag{2.18}$$

and in combination with (2.6), it results to

$$\delta_{0,j} = \sum_{i=0}^{2j} \binom{2j}{i} (-1)^i m_i m_{2j-i} \qquad \text{for} \quad j = 0, \dots, n. \tag{2.19}$$

The converse implication '2) $\Rightarrow$ 1)' will be proved by the mathematical induction. For $j = 0$, there is $a_0 + b_0 = 1$ and at the same time $a_0^2 + b_0^2 = \frac{1}{2}$ (which follows from (2.6)). Then

$$0 = a_0^2 + b_0^2 - \frac{1}{2} - \frac{1}{2}(a_0^2 + 2a_0 b_0 + b_0^2 - 1) = \frac{1}{2}(a_0 - b_0)^2. \tag{2.20}$$

This implies that $a_0 = b_0$. Further, let the implication holds for $j = 0, \dots, k-1$. Then from (2.6) it follows

$$\begin{aligned} 0 &= \sum_{i=0}^{k-1} \binom{2k}{i} (-1)^i\, m_i\, m_{2k-i} + \binom{2k}{k} (-1)^k (a_k^2 + b_k^2) \\ &= \binom{2k}{k} (-1)^k \left(a_k^2 + b_k^2 - \frac{m_k^2}{2} \right). \end{aligned} \tag{2.21}$$

Together with $a_k + b_k = m_k$, we have the same situation as before

$$0 = a_k^2 + b_k^2 - \frac{m_k^2}{2} - \frac{1}{2}\left(a_k^2 + 2a_k b_k + b_k^2 - m_k^2\right) = \frac{1}{2}\left(a_k - b_k\right)^2 \tag{2.22}$$

and thus $a_k = b_k$.

At the end, we prove the equivalence '2) $\Leftrightarrow$ 3)'. First, integrating the scaling equation immediately gives that $M_0 = 1 \Rightarrow m_0 = 1$. Then for $j = 0$ the equivalence holds. Further, for $j > 0$, by applying the scaling equation, interchanging summations and using some relations among binomial coefficients, it yields:

$$
\begin{aligned}
&\sum_{k=0}^{2j}\binom{2j}{k}(-1)^k M_{2j-k}M_k \qquad (2.23)\\
=\;& \sum_{k=0}^{2j}\binom{2j}{k}(-1)^k\;\frac{1}{2^{2j-k}}\sum_{i=0}^{2j-k}\binom{2j-k}{i}m_{2j-k-i}M_i\;\frac{1}{2^k}\sum_{l=0}^{k}\binom{k}{l}m_{k-l}M_l\\
=\;& \frac{1}{2^{2j}}\sum_{i=0}^{2j}\sum_{l=0}^{2j-i}\sum_{k=l}^{2j-i}(-1)^k\binom{2j}{k}\binom{2j-k}{i}\binom{k}{l}m_{2j-k-i}m_{k-l}M_iM_l\\
=\;& \frac{1}{2^{2j}}\sum_{i=0}^{2j}\sum_{l=0}^{2j-i}(-1)^l\binom{2j}{i}\binom{2j-i}{l}M_iM_l\sum_{k=l}^{2j-i}(-1)^{k-l}\binom{2j-i-l}{k-l}m_{2j-k-i}m_{k-l}
\end{aligned}
$$

Now if 2) is true, then

$$
\begin{aligned}
\sum_{k=0}^{2j}\binom{2j}{k}(-1)^k M_{2j-k}M_k &= \frac{1}{2^{2j}}\sum_{i=0}^{2j}\sum_{l=0}^{2j-i}(-1)^l\binom{2j}{i}\binom{2j-i}{l}M_iM_l\,\delta_{l,2j-i} \qquad (2.24)\\
&= \frac{1}{2^{2j}}\sum_{i=0}^{2j}(-1)^{2j-i}\binom{2j}{i}M_iM_{2j-i}
\end{aligned}
$$

and it implies

$$
\sum_{k=0}^{2j}\binom{2j}{k}(-1)^k M_{2j-k}M_k = 0 \qquad \text{for} \quad j = 1, ..., n. \qquad (2.25)
$$

Conversely if 3) holds true, then for $j = 0$ the property 2) is satisfied. Assume that the implication is true for $j = 0, \ldots, n-1$. By (2.23) we have

$$
\begin{aligned}
0 \;&=\; \sum_{k=0}^{2n}\binom{2n}{k}(-1)^k M_{2n-k}M_k \qquad (2.26)\\
&= \frac{1}{2^{2n}}\sum_{i=0}^{2n}\sum_{l=0}^{2n-i}(-1)^l\binom{2n}{i}\binom{2n-i}{l}M_iM_l\sum_{k=l}^{2n-i}(-1)^{k-l}\binom{2n-i-l}{k-l}m_{2n-k-i}m_{k-l}\\
&= \frac{1}{2^{2n}}\sum_{i=0}^{2n}(-1)^{2n-i}\binom{2n}{i}M_iM_{2n-i}+\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}m_{2n-k}m_k\\
&= \frac{1}{2^{2n}}\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}m_{2n-k}m_k
\end{aligned}
$$

□

Lemma 17. *Let a_i, b_i be defined by (2.7). Then a_i for $i \geq M$ is a linear combination of $a_0, \ldots a_{M-1}$ and b_i for $i \geq M$ is a linear combination of $b_0, \ldots b_{M-1}$*

Proof. Coefficients $h_{N_1}, h_{N_1+2}, \ldots h_{N_1+2M-2}$ are the solution of the system of linear algebraic equations (2.7). Since the matrix of this system is nonsingular, the solution exists and is unique. By definition, a_i is a linear combination of $h_{N_1}, h_{N_1+2}, \ldots h_{N_1+2M-2}$ and thus a_i for $i \geq M$ is a linear combination of a_i for $0 \leq i \leq M-1$:

$$a_i = c_{i0}a_0 + c_{i1}a_1 + \ldots c_{iM-1}a_{M-1}, \tag{2.27}$$

where the coefficients of this linear combination are given by

$$\begin{pmatrix} 1 & N_1 & N_1^2 & \ldots & N_1^{M-1} \\ 1 & N_1+2 & (N_1+2)^2 & \ldots & (N_1+2)^{M-1} \\ \vdots & & & & \vdots \\ 1 & N_1+2M-2 & (N_1+2M-2)^2 & \ldots & (N_1+2M-2)^{M-1} \end{pmatrix} \begin{pmatrix} c_{i0} \\ c_{i1} \\ \vdots \\ c_{iM-1} \end{pmatrix} = \begin{pmatrix} N_1^i \\ (N_1+2)^i \\ \vdots \\ (N_1+2M-2)^i \end{pmatrix}. \tag{2.28}$$

The situation for b_i is similar. □

2.1 Construction of Daubechies Wavelets and Symmlets

The symbol of Daubechies wavelets and symmlets of order N is defined as

$$|m(\omega)|^2 = \left(\frac{1+\mathrm{e}^{-\mathrm{i}\omega}}{2}\right)^{2N} P_N\left(\sin^2\frac{\omega}{2}\right), \tag{2.29}$$

where

$$P_N(x) = \sum_{k=0}^{N-1} \binom{N-1+k}{k} x^k. \tag{2.30}$$

Daubechies wavelets correspond to choosing the “minimum phase filter” m among all the possibilities once $|m(\omega)|^2$ is fixed, while symmlets correspond to the “least asymetric” wavelet with symbol satisfying (2.29), see [62, 64]. The roots of polynomial $m(\omega)$ can be found by *spectral factorization*, we refer to [62, 64] for details. Here, we propose a different method. Both of these types of wavelets are compactly supported orthonormal wavelets with a maximum number of vanishing wavelet moments for the given length of support. Daubechies wavelets and symlets of order N have N vanishing moments, the length of support is $2N-1$ and their scaling coefficients h_k satisfy conditions

i) $h_k = 0$ for $k < 0$ or $k > 2N-1$,

ii) $\delta_m = \frac{1}{2}\sum_{j=0}^{2N-1-2m} h_j h_{j+2m}$ for $0 \leq m \leq N-1$, *iii)* $\sum_{k\in\mathbb{Z}} (-1)^k k^l h_k = 0 \quad \text{for} \quad 0 \le l \le N-1.$

Clearly conditions *iii)* are equivalent to $a_i = b_i$ for $0 \le i \le N-1$. According to Lemma 17 the discrete scaling moments $m_i = a_i + b_i$ for $i > N-1$ can be expressed by linear combinations of those with index smaller than N and the same holds also for the continuous scaling moments. Thus the above system can be replaced by

i) $m_0 = 1$,

ii) $m_i = \sum_{j=0}^{N-1} c_{ij} m_j$ for $i \ge N$, where c_{ij} are given by (2.28),

iii) $\delta_{0,n} = \sum_{i=0}^{n} \binom{n}{i} (-1)^i m_i m_{n-i} \quad \text{for} \quad 1 \le m \le N-1.$

Thus, we obtain a system of $N-1$ equations for $N-1$ unknowns $m_1, \ldots m_{N-1}$. The previous approach can be further enhanced. As mentioned above, m_i for $i > N-1$ can be expressed by linear combinations of previous ones ($i < N$), but these linear combinations depend on the displacement of the scaling function or equivalently on the displacement of scaling coefficients. It means that shifted scaling moments

$$m'_i = \sum_{k=0}^{2N-1} h_k \left(k - \frac{2N-1}{2}\right)^i \tag{2.31}$$

can be treated instead of the originally defined moments $m_i = \sum_{k=0}^{2N-1} h_k k^i$. The main advantage of this approach consists in the symmetry of a'_i and b'_i (defined similar to the original ones) for even i and antisymmetry of odd ones, respectively. For example, the first term by a'_i is $\left(-\frac{2N-1}{2}\right)^i h_0$ and the last one by b'_i is $\left(\frac{2N-1}{2}\right)^i h_{2N-1}$ and so on. Thus the even moments for $i > N-1$ are expressed only by linear combinations of even moments (for $i < N$) and by analogy for the odd moments. This useful trick enables one to substantially reduce the complexity of arising system.

Example 18. For $N = 3$, the following system will be obtained

$$m'_0 = 1, \qquad m'_2 = {m'_1}^2, \qquad m'_4 - 4m'_1 m'_3 + 3{m'_2}^2 = 0, \tag{2.32}$$

and further,

$$m'_3 = \frac{13}{4} m'_1, \qquad m'_4 = \frac{11}{2} m'_2 - \frac{45}{16}, \tag{2.33}$$

after substitution and elimination

$$16{m'_1}^4 - 40{m'_1}^2 - 15 = 0. \tag{2.34}$$

Now, we will study further properties of m'_1. Are solutions in variable m'_1 symmetric with respect to the point 0? Is there any estimate for the localization of m'_1? We start with the first problem. First of all, we notice some symmetry in scaling coefficients.

Lemma 19. [24] *If*

$$\{h_0, \dots, h_{2N-1}\} \tag{2.35}$$

satisfies the necessary conditions for the scaling coefficients, then

$$\{h_{2N-1}, \dots, h_0\} \tag{2.36}$$

also satisfies these conditions.

Proof. First we check the condition ensuring approximation properties. The sum

$$\sum_{k=0}^{2N-1} (-1)^k h_k k^n = 0 \quad \text{for} \quad 0 \le n \le N-1 \tag{2.37}$$

can be replaced by

$$\sum_{k=0}^{2N-1} (-1)^k h_k (k-l)^n = 0 \quad \text{for} \quad 0 \le n \le N-1 \tag{2.38}$$

for any fixed integer l. The choice $l = 2N-1$ brings

$$(-1)^n \sum_{k=0}^{2N-1} (-1)^k h_k (2N-1-k)^n = 0 \quad \text{for} \quad 0 \le n \le N-1. \tag{2.39}$$

At last the symmetry of the orthonormality conditions $\delta_{m,0} = 2^{-1} \sum_{j=0}^{2N-1} h_j h_{2m+j}$ finishes the proof.

□

Note that Lemma 19 means that if ψ is a scaling function corresponding to MRA, then its mirror image

$$\phi^s(x) := \phi(2N-1-x) \tag{2.40}$$

is a scaling function corresponding to the same MRA and the computation of the first continuous scaling moments gives

$$M_1^s = \int_{\mathbb{R}} x\, \phi^s(x)\, dx = \int_{\mathbb{R}} x\, \phi(2N-1-x)\, dx = \int_{\mathbb{R}} (2N-1-y)\, \phi(y)\, dy = 2N-1-M_1. \tag{2.41}$$

Furthermore, $m_0 = 1, M_0 = 1$, and

$$\begin{aligned} M_1 &= \int_{\mathbb{R}} x\phi(x)\, dx = 2 \int_{\mathbb{R}} \sum_k x h_k \phi(2x-k)\, dx = \frac{1}{2} \int_{\mathbb{R}} \sum_k (y+k)\, h_k\, \phi(y)\, dy \\ &= \frac{1}{2} \int_{\mathbb{R}} (m_0 y + m_1)\, \phi(y)\, dy = \frac{1}{2}(m_0 M_1 + m_1 M_0), \end{aligned}$$

imply $M_1 = m_1$ and consequently $m_1^s = 2N-1-m_1$, where m_1^s is the symmetric discrete moment defined analogously to the continuous one. It means that all solutions in variable m_1 are symmetric with respect to the point $\frac{2N-1}{2}$. We formulate it in the next lemma [24].

Lemma 20. *The solutions in variable m_1 coming from* i), ii), iii) *possess pairs of roots symmetric with respect to the point $\frac{2N-1}{2}$ or equivalently the solutions in variable m_1' possess pairs of roots symmetric with respect to the point* 0.

Now, we will study the second question concerning the localization of m_1. We are mainly concerned with the real solution.

Lemma 21. [24] *Let $H_N(z) = \sum_{k=0}^{2N-1} h_k z^k$ be any nonconstant polynomial with real coefficients and $h_0 h_{2N-1} \neq 0$. Further, let $H_N(1) = 1$ and $\sup_{|z|=1} |H_N(z)| \leq 1$,. Then $H_N'(1)$ belongs to the interval $(0, 2N-1)$.*

Proof. We use the following version of the Bernstein inequality for trigonometric polynomials: *Let p be any polynomial with complex coefficients and degree at most $2N-1$. Then $\max_{|z|=1} |p'(z)| \leq (2N-1) \max_{|z|=1} |p(z)|$. The equality holds if and only if there exists a constant c such that $p(z) = cz^{2N-1}$.*
We apply this inequality to the polynomials $H_N(z)$ an $z^{2N-1}H_N(z)$ and we obtain $|H_N'(1)| < 2N-1$ and $|2N-1-H_N'(1)| < 2N-1$, respectively. Since H_N has real coefficients, $H_N'(1)$ is a real number and the proof is finished. □

To conclude: The original system of necessary conditions on scaling coefficients is partly linear and partly quadratic. We eliminated some quadratic conditions - even variables can be namely immediately expressed by combinations of odd ones. Furthermore, solving these systems for shifted moments defined by (2.31) can again reduce the complexity. For filter lengths up to 16 this system can be explicitly solved via algebraic methods like the Gröbner bases. Its particularly simple structure allows one to find all possible solutions. At the end, the two theorems mentioned above imply that m_1' belongs to the interval $(\frac{1-2N}{2}, \frac{2N-1}{2})$ and due to the symmetry, we can restrict it only to the interval $[0, \frac{2N-1}{2})$. The obtained results are presented in tables in Appendix A. We should also note that exact expressions in closed forms of Daubechies orthogonal filters for $N = 4$ and 5 are given in [90]. For each filter the first collection of scaling coefficients corresponds to the Daubechies wavelet and the last one corresponds to symmlet, both originally constructed by Daubechies. Hence, our construction is an alternative to very well-known spectral factorization [62, 64].

2.2 Construction of Coiflets

As mentioned in Chapter 1, approximation properties of multiresolution analysis and the smoothness of wavelet and the scaling function depend on the number of vanishing wavelet moments. In [62] Daubechies constructed orthonormal wavelets with an arbitrary number N of vanishing wavelet moments and the minimal length of support $2N-1$. The filter

coefficients were computed there by an analytical method and exact values could be found only for filters up to length 6. In [90] Shann and Yen calculated the exact values of filter coefficients of Daubechies wavelets of length 8 and 10. Other approaches for constructing Daubechies wavelets which enables us to find exact values of some coefficients can be found in [24, 35, 84, 85].
In addition to the orthogonality, compact supports and vanishing wavelet moments, Coifman has suggested that also requiring vanishing scaling moments has some advantages. In practical applications these wavelets are useful due to their "nearly linear phase" and "almost interpolating property", see [77]. Daubechies created coiflets by setting an equal number N of vanishing wavelet moments and vanishing scaling moments for even N and the length of support $3N$, see [64, 63]. It was noticed in [8] that these coiflets have one additional vanishing scaling moment than is imposed. Tian constructed coiflets with N vanishing moments for odd N and the length of support $3N-1$ in [98, 100]. Burrus and Odegard constructed coiflets with N vanishing moments for odd N and the length of support $3N+1$ which has two additional vanishing scaling moments, see [14]. In this thesis the computation of exact values of filter coefficients of coiflets up to filter length 14 is presented.
There exist a number of coiflet filter design methods, such as Newton's method [64, 86] or iterative numerical optimization [14]. However, in contrast to construction of Daubechies wavelets there is nothing similar to spectral factorization and the construction of coiflets is more difficult. These methods enable to derive one particular solution for each system and the convergence and the obtained solution depends on the initial starting point, thus it is difficult to find all possible solutions. Moreover, the coefficients for length greater than 16 are given with less precision due to the roundoff error [63]. As an alternative one can use the Gröbner basis method [1, 13, 72]. This method is geared toward solving a polynomial system of equations with finite solutions. The idea consists of finding a new set of equations equivalent to the original set, which can be solved more easily. The advantage of such an approach is that solutions can be computed to arbitrary precision and that in some cases it gives all possible solutions for a given system of polynomial equations. In this section we derive a redundant free and simplified system of equations and then aplly the Gröbner basis method. By this approach we are able to find some exact values of filter coefficients and to find all possible solutions for filters up to length 20.

2.2.1 Preliminaries

Let us start with the definition of coiflet.

Definition 22. An orthonormal wavelet ψ with compact support is called a *coiflet* of order N, if the following conditions are satisfied:

i) $\int_{-\infty}^{\infty} x^n \psi(x)\, dx = 0$ for $n = 0, \dots N-1$,

ii) $\int_{-\infty}^{\infty} x^n \phi(x)\, dx = \delta_n$ for $n = 0, \dots N-1$, where ϕ is scaling function corresponding to ψ and δ_n is the Kronecker delta, i.e. $\delta_0 = 1$ and $\delta_n = 0$ for $n \neq 0$.

Since also a length of support plays a role, it is common to consider a wavelet satisfying *i)* and *ii)* which has the minimal length of support. The existence of coiflet for an arbitrary order N is still an open question. We rewrite this definition in terms of filter coefficients $\{h_k\}$ in the following lemma which is a direct consequence of Theorem 14. It is known that for orthonormal wavelet with compact support a number of filter coefficients is even, we denote it by $2M$.

Lemma 23. *Let $\{h_k\}_{k=N_1}^{N_2}$ be the real coefficients and $N_2 = N_1+2M-1$. If the orthonormal wavelet corresponding to the scaling function $\phi(\cdot) = 2\sum_{k=N_1}^{N_2} h_k\,\phi(2\cdot -k)$ is a coiflet of order N, then the following three conditions are satisfied:*

i) $\delta_m = \frac{1}{2}\sum_{j=0}^{N_2-N_1-2m} h_{N_1+j}h_{N_1+2m+j} \quad \textit{for} \quad 0 \leq m \leq M-1,$

ii) $\sum_{k=N_1}^{N_2} h_k k^n = \delta_n \quad \textit{for} \quad 0 \leq n \leq N-1,$

iii) $\sum_{k=N_1}^{N_2} (-1)^k h_k k^n = 0 \quad \textit{for} \quad 0 \leq n \leq N-1.$

As mentioned above, condition *i)* is necessary but not sufficient for wavelet to be orthonormal. Conditions *ii)* and *iii)* are equivalent to vanishing wavelet and vanishing scaling function moments, respectively. In summary, conditions in Lemma 23 are only necessary. It is known that they are not sufficient to generate a coiflet system. Hence, after finding coefficients satisfying *i)* $-$ *iii)* orthonormality should be verified, for example using Cohen [37] or Lawton [71] condition. There are typically more than one wavelet satisfying these conditions and some of them, despite zero wavelet moments, are very rough. Likewise, despite zero scaling function moments, some are not at all symmetric. In practical applications the most regular wavelet or the wavelet with the most symmetric scaling function is typically chosen.

2.2.2 Further Properties

It is well known that coiflets have more vanishing scaling moments than required in the above definition. This was first noted by G. Beylkin at al. in [8]. In this thesis, we derive redundant free and simpler definition of coiflets. Due to Theorem 15 we are now able to set necessary conditions to filter coefficients to generate a coiflet which are equivalent to conditions from Lemma 23 and the system is without redundant conditions.

Lemma 24. *Let $\{h_k\}_{k=N_1}^{N_2}$ be the real coefficients, $N_2 = N_1 + 2M - 1$, and let a_i and b_i be defined by* (2.7)*. Then conditions i)* $-$ *iii) from Lemma* 23 *are equivalent to the following conditions:*

$i')$ $\frac{1}{2}\,\delta_n = \sum_{i=0}^{2n} \binom{2n}{i} (-1)^i (a_i a_{2n-i} + b_i b_{2n-i})$ *for* $N \le 2n \le 2M-2$,

$ii')$ $a_0 = b_0 = \frac{1}{2}$,

$iii')$ $a_n = b_n = 0$ *for* $1 \le n \le N-1$.

Proof. It is clear that $ii), iii)$ are equivalent to $ii'), iii')$. The condition

$$\frac{1}{2}\,\delta_n = \sum_{i=0}^{2n} \binom{2n}{i} (-1)^i (a_i a_{2n-i} + b_i b_{2n-i}) \quad \text{for} \quad 0 \le n \le M-1, \tag{2.42}$$

is equivalent to $i)$ according to Theorem 15. It remains to prove that equations from (2.42) are redundant for $0 \le 2n \le N-1$. But this is a simple consequence of $ii')$ and $iii')$. □

The consequence of this lemma is that the minimal length of support of coiflet of order N is $3N$ for even N and $3N-1$ for odd N and that some coiflets have more vanishing moments than is imposed. Thus, we have three classes of coiflets, see Table 2.1.

Table 2.1: The length of filter $2M$, the number of vanishing scaling and wavelet moments for coiflet of order N

N	$2M$	number of vanishing scaling moments		number of vanishing wavelet moments	
		set	actual	set	actual
even	$3N$	N	$N+1$	N	N
odd	$3N-1$	N	N	N	N
odd	$3N+1$	$N+1$	$N+2$	N	N

Now we further simplify the system by replacing some quadratic conditions by linear ones.

Lemma 25. [27] *Let* $\{h_k\}_{k=N_1}^{N_2}$ *be the real coefficients,* $N_2 = N_1 + 2M - 1$ *and let* a_i *and* b_i *be defined by* (2.7). *Then conditions* $i) - iii)$ *from Lemma* 23 *are equivalent to the following four conditions:*

$i*)$ $\frac{1}{2}\,\delta_n = \sum_{i=0}^{2n} \binom{2n}{i} (-1)^i (a_i a_{2n-i} + b_i b_{2n-i})$ *for* $N \le n \le M-1$,

$ii*)$ $a_0 = b_0 = \frac{1}{2}$,

$iii*)$ $a_n = b_n = 0$ *for* $1 \le n \le N-1$,

$iv*)$ $a_{2n} + b_{2n} = 0$ *for* $N \le 2n \le 2N-2$.

Proof. Putting $ii')$ and $iii')$ into $i')$ for $N \leq 2n \leq 2N$, we obtain that $iv*)$ holds and conversely putting $ii'), ii')$, and $iv*)$ into the right-hand side of equation in $i')$, we find that $i')$ holds for $N \leq 2n \leq 2N$. □

Now we summarize the simplified construction of coiflets which enables us to find exact values of coiflets up to length 14 of support:

1. For a given N take the system of polynomial equations given by Lemma 25.

2. Replace $a_M, \dots a_{2M-2}$ by linear combinations of $a_0, \dots a_{M-1}$ and $b_M, \dots b_{2M-2}$ by linear combinations of $b_0, \dots b_{M-1}$.

3. Solve the arising system for $a_0, \dots a_{M-1}, b_0, \dots b_{M-1}$. For greater N use the Gröbner basis method to simplify the system.

4. Compute the filter coefficients $h_{N_1}, \dots, h_{N_2}$ by solving the system of linear algebraic equations (2.7).

2.2.3 Examples

At last we provide two examples to illustrate our approach based on Lemma 25.

Example 26. For $N = 4$ and $N_1 = -5$, the following system will be obtained

$$a_0 = b_0 = \frac{1}{2} \quad \text{and} \quad a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0, \tag{2.43}$$

$$a_4 + b_4 = 0 \quad \text{and} \quad a_6 + b_6 = 0, \tag{2.44}$$

$$a_8 + b_8 + 140\, b_4^2 = 0 \quad \text{and} \quad a_{10} + b_{10} + 840\, b_4\, b_6 - 252(a_5^2 + b_5^2) = 0. \tag{2.45}$$

Now $a_6,\ a_8,\ a_{10},\ b_6,\ b_8,\ b_{10}$ are linear combinations of $a_0, \dots, a_5,\ b_0, \dots, b_5$. We find these linear combinations and substitute them to (2.44) and (2.45). Then after simplification we obtain the system

$$-135 + 12b_4 + 8b_4^2 = 0, \quad a_4 + b_4 = 0, \tag{2.46}$$

$$75 - 10b_4 + 4b_5 = 0, \quad 32a_5^2 + 12300b_4 - 28575 = 0. \tag{2.47}$$

In this case we can easily find both real solutions in closed form. See Table A.9 and Table A.10.

Example 27. For $N = 5$ and $N_1 = -5$, the following system will be obtained

$$a_0 = b_0 = \frac{1}{2} \quad \text{and} \quad a_1 = a_2 = a_3 = a_4 = b_1 = b_2 = b_3 = b_4 = 0, \tag{2.48}$$

$$a_6 + b_6 = 0 \quad \text{and} \quad a_8 + b_8 = 0, \tag{2.49}$$

$$a_{10}+b_{10}-252(a_5^2+b_5^2) = 0 \quad \text{and} \quad a_{12}+b_{12}-1584\, b_5\, b_7-1584\, a_5\, a_7+924(a_5^2+b_5^2) = 0. \tag{2.50}$$

Now $a_7,\ a_8,\ a_{10},\ a_{12},\ b_6,\ b_8,\ b_{10},\ b_{12}$ are linear combinations of $a_0, \ldots, a_6,\ b_0, \ldots, b_6$. We find these linear combinations and substitute them to (2.49) and (2.50). Consequently we simplify arising system and finally we compute its Gröbner bases:

$$11419648\, b_5^4 + 246374400\, b_5^3 - 13765248000\, b_5^2 - 497539800000\, b_5 - 4303042734375 = 0, \tag{2.51}$$

$$298890000\, a_5 - 5709824\, b_5^3 + 3945600\, b_5^2 + 6931764000\, b_5 + 94943559375 = 0, \tag{2.52}$$

$$8\, a_6 + 64\, b_5 + 525 = 0, \qquad -525 - 64\, b_5 + 8\, b_6 = 0. \tag{2.53}$$

Then by using an algebraic formula for the solution of polynomials of degree 4, we obtain two different real roots:

$$b_5 = \frac{15\left(\sqrt{15}u^{3/4} - 4010u^{1/6}v^{1/4} \pm \sqrt{15}\sqrt{w}\right)}{11152u^{1/6}v^{1/4}}, \tag{2.54}$$

where

$$u = 4854802096 + 369\sqrt{15}\sqrt{66685436848043}, \quad v = 8475076\, u^{1/3} + 697\, u^{2/3} - 3366028373, \tag{2.55}$$

$$w = 16950152\, u^{1/3}\sqrt{v} - 697\sqrt{v}\, u^{2/3} + 3366028373\sqrt{v} + 13383342756\sqrt{15}\sqrt{u}. \tag{2.56}$$

Once we have the values of b_5, we simply find a_5, a_6, and b_6. And finally we transform coefficients a_i and b_i to scaling coefficients h_i.

2.2.4 Properties of Coiflets

Let us now mention the properties of constructed wavelets. It is well-known that the approximation properties depend on the number of vanishing wavelet moments. More precisely, let $P_j f$ be an approximation of $f \in L^2(\mathbb{R})$ on level j, i.e.

$$P_j f = \sum_{k\in\mathbb{Z}} \langle f, \phi_{j,k}\rangle\, \phi_{j,k} \tag{2.57}$$

and for $J < j$, we have

$$P_j f = \sum_{k\in\mathbb{Z}} \langle f, \phi_{J,k}\rangle\, \phi_{J,k} + \sum_{l=J}^{j-1} \sum_{k\in\mathbb{Z}} \langle f, \psi_{l,k}\rangle\, \psi_{l,k}, \tag{2.58}$$

where $\phi_{l,k} = 2^{l/2}\phi\left(2^l \cdot -k\right)$ and $\psi_{l,k} = 2^{l/2}\psi\left(2^l \cdot -k\right)$ for $l, k \in \mathbb{Z}$. Let us further set $I_{l,k} = \text{supp}\ \phi_{l,k}$ and $J_{l,k} = \text{supp}\ \psi_{l,k}$. Wavelet coefficients satisfy

$$\langle f, \psi_{l,k} \rangle = \int_{-\infty}^{\infty} f(x)\, 2^{l/2}\psi\left(2^l x - k\right) dx. \tag{2.59}$$

and if $f \in C^N(J_{l,k})$, then expanding f about $\frac{k}{2^l}$ by Taylor's formula, it follows that for all $x \in J_{l,k}$,

$$f(x) = f\left(\frac{k}{2^l}\right) + f'\left(\frac{k}{2^l}\right)\left(x - \frac{k}{2^l}\right) + \ldots + \frac{f^{(N-1)}\left(\frac{k}{2^l}\right)}{(N-1)!}\left(x - \frac{k}{2^l}\right)^{N-1} + \frac{f^{(N)}(\xi)}{N!}\left(x - \frac{k}{2^l}\right)^N,$$

where ξ depends on x and belongs to the interval $J_{l,k}$. If ψ has N vanishing moments, i.e. condition $i)$ in Definition 22 is satisfied, then the first N terms do not contribute and

$$|\langle f, \psi_{l,k} \rangle| = \left| \int_{-\infty}^{\infty} \frac{f^{(N)}(\xi(x))}{N!}\left(x - \frac{k}{2^l}\right)^N 2^{l/2}\psi\left(2^l x - k\right) dx \right| \leq C 2^{-l(N+1/2)}, \tag{2.60}$$

where

$$C = \frac{\max_{\xi \in J_{l,k}} \left| f^{(N)}(\xi) \right|}{N!} \int_{J_{l,k}} |y|^N \psi(y)\, dy. \tag{2.61}$$

Thus, for l large, the wavelet coefficients are small except for those which are near singularities of the function f or its derivatives. Small coefficients can be set up to zero and the function f can be represented by a small number of coefficients. This compression property of wavelets has many applications. The most important are data compression, signal analysis, and efficient adaptive schemes for PDE's. Note that more vanishing wavelet moments implies a faster decay of wavelet coefficients and that only a local smoothness of the function f is involved in the above estimate. It was observed in [2] that also the regularity of the scaling function plays a role. We confirmed in our experiments that this is true for coiflets as well. As an example, let us consider

$$\begin{aligned} f(x) &= x^5 && \text{if} \quad 0 \leq x \leq 0.5, \\ &= (1-x)^5 && \text{if} \quad 0.5 < x \leq 1, \\ &= 0 && \text{otherwise}, \end{aligned} \tag{2.62}$$

and its n-term approximation

$$f_n(x) = \sum_{\lambda=(l,k)\in\Lambda_\phi^n} \langle f, \phi_\lambda \rangle \phi_\lambda + \sum_{\lambda=(l,k)\in\Lambda_\psi^n} \langle f, \psi_\lambda \rangle \psi_\lambda, \tag{2.63}$$

where $\Lambda_\phi^n \subset \{\lambda = (J,k), k \in \mathbb{Z}\}$, $\Lambda_\psi^n \subset \{\lambda = (l,k), J \leq l < j, k \in \mathbb{Z}\}$, and $\Lambda_\phi^n \cup \Lambda_\psi^n$ is the set of indexes of the n largest coefficients. In our case, the coarsest level is $J = 3$, the finest level is $j = 9$ and the number of preserved coefficients is $n = 50$. The function f has sharp

derivative near the point $x = 0.5$ and the approximation is automatically refined near this point. Errors of approximation for some of the constructed coiflets are shown in Table 2.2. We can see that the most regular coiflet of prescribed order gives the best result.
The significance of vanishing scaling moments highly depends on the type of application. In [64], it is proved that all real orthonormal wavelets with compact support are asymmetric. However, vanishing scaling moments result in "almost symmetry" of the scaling function and filter. In image coding, more symmetry would result in greater compressibility for the same perceptual error and it makes easier to deal with the boundaries of the image. Vanishing scaling moments also causes "nearly linear phase", which is a desired quality in many applications, e.g. transmission of audio and video signals, because it does not cause phase distortion. In numerical analysis, vanishing scaling moments are important due to their "almost interpolating property". It means that any $f \in C_0^N(\mathbb{R})$ can be approximated by

$$f_j = 2^{-j/2} \sum_{k \in \mathbb{Z}} f\left(\frac{k}{2^j}\right) \phi_{j,k} \tag{2.64}$$

and if the number of vanishing scaling and wavelet moments is N then this approximation satisfies the following estimate

$$\|f - f_j\| \leq C 2^{-jN}, \tag{2.65}$$

Table 2.2: Error of approximation of the function f by 50 coefficients for coiflets of order N, length of support $2M$ and Sobolev exponent of smoothness γ

N	$2M$	γ	L^∞ of error $\times 10^{-6}$	L^2 norm of error $\times 10^{-7}$	H^1 seminorm of error $\times 10^{-4}$
1	4	0.604	743	1986	1358
1	4	0.050	2800	7332	5642
2	6	0.041	402	978	706
2	6	1.232	44	116	46
2	6	0.590	184	469	234
2	6	1.022	83	200	87
3	8	0.147	103	225	137
3	8	1.775	2	6	1
3	8	1.422	20	31	13
3	8	0.936	44	97	33
3	8	1.464	15	33	10
3	8	1.773	3	5	1

where C depends only on f and the scaling function ϕ, see [99]. Due to this property, some types of operators can be treated efficiently. Thus coiflets have some interesting properties and for some applications are more suitable than orthonormal wavelets with vanishing wavelet moments only. The price to pay is of course the length of support, which can make the computation more expensive. We should also mention that we can obtain symmetric wavelets by giving up orthonormality. Symmetric biorthogonal wavelets were constructed in [41], and construction of biorthogonal coiflets can be found in [99, 100]. However, there are applications where the orthogonality plays an important role and the disadvantage of many biorthogonal wavelets is their bad stability when adapted to the interval, see [12, 53]. In literature, one can find coiflets which are the most symmetric among all coiflets of given order and given length of support, see [14, 63, 64, 98, 100]. As we could see above, these coiflets need not be the best and other solutions of equations given in Lemma 25 may be better suited for some type of applications. Typically the most regular coiflet for a given order N has the best compression property and due to almost interpolating property and ability to generate a stable wavelet basis on bounded domain it seems to be very well suited for some applications.

2.2.5 Conclusion

The arising system from the Corollary 25 is redundant-free, more simple than the original one, and it enables us to find directly the exact values of the scaling coefficients of coiflets up to length 8 and two further with length 12 in closed form. The results are given in tables in Appendix A. We verified orthonormality by the Lawton criterion, all the results correspond to orthonormal scaling function. As mentioned earlier, the solutions are not of the same quality, because also smoothness and symmetry plays a role. For this reason the most symmetric scaling function among all scaling functions of order N is denoted in tables and the Sobolev exponents of smoothness are computed by the method from [65, 103]. Furthermore, for remaining coiflets up to length 14 we obtain two quadratic equations of two variables, which can be transformed to polynomial of degree 4 and there is an algebraic formula to find solutions in closed form. These solutions we do not provide because of their length and complicated structure. Moreover, one can use our approach to find all possible solutions to given system up to the length of filter 20. For longer filters the computation failed, because the coefficients of polynomials in the Gröbner basis were too large numbers.

2.3 Construction of Generalized Coiflets

In the previous section we dealt with coiflets – ortonormal wavelets with vanishing both wavelet and scaling moments. Strictly speaking, the vanishing scaling moments means that shifted scaling moments are vanishing and in [63] these shifts were considered to be an integer. Several other examples of coiflets, still for integer shifts, can be found in the literature [14, 15]. In [77], there was used the fact that the shifts does not have to be necessarily an integer. As a matter of the fact, the noninteger shifts may be used to optimize the construction of coiflets. A similar example of this approach can be found in [16].

Here, we will use the notation “generalized coiflets” for orthonormal wavelets which possess maximal number of vanishing shifted scaling moments for given number of scaling coefficients. Generalized coiflets have the same interesting properties as coiflets that make them useful both in numerical analysis and in signal processing. Namely we “almost interpolating property” is preserved in the following sense. Consider the scaling function ϕ corresponding to a generalized coiflet of order N with shift α. Then for any polynomial p of order less or equal N,

$$\int_{\mathbb{R}} p(x)\,\phi(2^j x - k)\,dx = p\left(\frac{\alpha + k}{2^j}\right), \tag{2.66}$$

and if the degree of p is less or equal $\frac{N}{2}$ then

$$p(x) = \sum_k p\left(\frac{\alpha + k}{2^j}\right)\phi(2^j x - k). \tag{2.67}$$

Since at some scale every smooth function resembles polynomial, we have the almost interpolation property.

2.3.1 Generalized Coiflets

We start with the definition of a generalized coiflet.

Definition 28. An orthonormal wavelet ψ with compact support is called *generalized coiflet* of shift α and with order N, if the following conditions are satisfied:

i) $\int_{\mathbb{R}} x^n \psi(x)\,dx = 0$ for $n = 0, \ldots, N-1$,

ii) $\int_{\mathbb{R}} x^n \phi(x)\,dx = \alpha^n$ for $n = 0, \ldots, N-1$, where ϕ is the scaling function corresponding to ψ.

Then the necessary conditions for scaling coefficients of a generalized coiflet of order N are:

i) $h_k = 0$ for $k \notin \{0, 1, \ldots, 2N-1\}$,

ii) $\delta_{m,0} = 2^{-1} \sum_{j=0}^{2N-1} h_j h_{2m+j}$ for $0 \leq m \leq N-1$,

iii) $\sum_{k=0}^{2N-1} h_k k^n = \alpha^n$ for $0 \leq n \leq N$.

Here, we mention that it was not proved yet the existence of generalized coiflets for an arbitrary number of moments. It is still an interesting open problem.

First we give notice of the fact, that the condition *iii*) already implies that generalized coiflets possess certain number of vanishing wavelet moments. This is equivalent with polynomial exactness of the corresponding scaling function. For more details on this subject see for example [44, 64, 94, 106].
Using the notation $\lfloor x \rfloor$ for the integer part of $x \in \mathbb{R}$, we have the following lemma. With assistance of Theorem 16, the proof is trivial.

Lemma 29. *Let ϕ be a generalized coiflet of order N, then*

$$\sum_{k \in \mathbb{Z}} (-1)^k k^j h_k = 0 \quad \text{for } j = 0, \ldots, \left\lfloor \frac{N}{2} \right\rfloor. \tag{2.68}$$

And this is equivalent to the fact that any polynomial up to degree $\lfloor \frac{N}{2} \rfloor$ can be exactly expressed by linear combinations of integer translations of ϕ.

Furthermore, we can use Theorem 16 to derive equivalent definition for generalized coiflets. The reason for restating this definition consists in the fact that the approximation properties can be treated more easily than originally imposed conditions in Definition 28.

Lemma 30. [25] *Necessary conditions $i), ii)$ and $iii)$ for scaling coefficients h_k of a generalized coiflet of order N and shift α are equivalent to*

i') $h_k = 0 \quad \text{for} \quad k \notin \{0, 1, \ldots, 2N-1\}$,

ii') $\delta_{m,0} = 2^{-1} \sum_{j=0}^{2N-1} h_j h_{2m+j} \quad \text{for} \quad 0 \le m \le N-1$,

iii') $\sum_{k=0}^{2N-1} h_k k^{2n+1} = \alpha^{2n+1} \quad \text{for} \quad 0 \le n \le \lfloor \frac{N-1}{2} \rfloor$.

iv') $\sum_{k=0}^{2N-1} (-1)^k h_k k^n = 0 \quad \text{for} \quad 0 \le n \le \lfloor \frac{N}{2} \rfloor$.

If we compare these conditions with conditions for Daubechies' scaling coefficients, then we have the condition $\sum_{k=0}^{2N-1} (-1)^k h_k k^n = 0 \quad \text{for} \quad 0 \le n \le N-1$ instead of the conditions $iii')$ and $iv')$, respectively. Thus we can immediately see that for $N = 1, 2$ we have for generalized coiflets the same conditions as for Daubechies' wavelets.

2.3.2 The Construction of Generalized Coiflets

In [77], a system for generalized coiflets was not described in terms of $\{h_k\}_{k=0}^{2N-1}$ but in terms of the new variables parametrized by α. (The parameter α is, as before, the first discrete moment of ϕ.) Unlike [77] we present here a more general approach based on the

Theorem 16 and on the following theorem characterizing the orthonormality conditions for the scaling function with a compact support.
Now, we can use Theorems 16 and 15, respectively to derive further equivalent conditions for generalized coiflets. This third set of conditions will be most appropriate and it will be later used to compute the corresponding scaling coefficients.

Lemma 31. [25] *Conditions $i'), ii'), iii')$, and $iv')$ from Lemma 30 are equivalent to*

i)* $h_k = 0$ *for* $k \notin \{0, 1, \ldots, 2N-1\}$,

ii)* $\frac{1}{2}\,\delta_{0,n} = \sum_{i=0}^{2n} \binom{2n}{i}(-1)^i(a_i a_{2n-i} + b_i b_{2n-i})$ *for* $\left\lfloor \frac{N}{2} \right\rfloor < n \leq N-1$,

iii)* $\sum_{k=0}^{2N-1} h_k k^n = \alpha^n$ *for* $0 \leq n \leq N$,

iv)* $\sum_{k=0}^{2N-1}(-1)^k h_k k^n = 0$ *for* $0 \leq n \leq \left\lfloor \frac{N}{2} \right\rfloor$,

where a_i and b_i, respectively, are defined above in Theorem 15.

Proof. Combining Theorems 16 and 15, and considering that due to the conditions $iii)$ and $iv)$, respectively, in the Definition 31 both a_i and b_i can be replaced by $\frac{m_1^i}{2}$ for $i \leq \left\lfloor \frac{N}{2} \right\rfloor$. □

Furthermore, the coefficients a_i for $i > N-1$ are a linear combination of a_i for $i < N$ and the same holds for the coefficients b_i as well.

2.3.3 Free Parameter

It is clear that Theorem 16 holds for the shifted moments as well. Then this shift can be treated as a free parameter and consequently can be used to optimize the arising system (or to reduce its complexity). Now, we will study how to choose this parameter.

First of all, we notice some symmetry in the definition of generalized coiflets.

Lemma 32. [25] *If*

$$\{h_0, \ldots, h_{2N-1}\} \tag{2.69}$$

satisfies the conditions placed to the scaling coefficients of generalized coiflets then

$$\{h_{2N-1}, \ldots, h_0\} \tag{2.70}$$

satisfies these conditions as well.

Proof. To prove that we employ conditions $i)-iii)$. First we check the moments' conditions:

$$\sum_{k=0}^{2N-1} h_k k^n = \alpha^n \quad \text{for} \quad 0 \leq n \leq N-1 \tag{2.71}$$

can be replaced by

$$\sum_{k=0}^{2N-1} h_k (l-k)^n = (l-\alpha)^n \quad \text{for} \quad 0 \le n \le N-1 \tag{2.72}$$

for any fixed integer l. And the choice $l = 2N-1$ brings

$$\sum_{k=0}^{2N-1} h_k (2N-1-k)^n = (2N-1-\alpha)^n \quad \text{for} \quad 0 \le n \le N-1. \tag{2.73}$$

At last the symmetry of the orthonormality conditions $\delta_{m,0} = 2^{-1} \sum_{j=0}^{2N-1} h_j h_{2m+j}$ finishes the proof.

□

Thus, the corresponding symmetric scaling function can be defined as follows

$$\phi^s(.) := \phi(2N-1-.) \tag{2.74}$$

and the computation of the first continuous scaling moments gives

$$M_1^s = \int_{\mathbb{R}} x\,\phi^s(x)\,dx = \int_{\mathbb{R}} x\,\phi(2N-1-x)\,dx = \int_{\mathbb{R}} (2N-1-y)\,\phi(y)\,dy = 2N-1-M_1. \tag{2.75}$$

Furthermore, $m_0 = 1, M_0 = 1$, and

$$\begin{aligned} M_1 &= \int_{\mathbb{R}} x^1 \phi(x)\,dx = 2\int_{\mathbb{R}} \sum_k x h_k \phi(2x-k)\,dx = \frac{1}{2}\int_{\mathbb{R}} \sum_k (y+k)\,h_k\,\phi(y)\,dy \\ &= \frac{1}{2}\int_{\mathbb{R}} (m_0 y + m_1)\,\phi(y)\,dy = \frac{1}{2}(m_0 M_1 + m_1 M_0) \end{aligned} \tag{2.76}$$

imply $M_1 = m_1$ and consequently, $m_1^s = 2N-1-m_1$, where m_1^s is the symmetric discrete moment defined analogously to the continuous one. It means that all solutions in variable m_1 are symmetric with respect to the point $\frac{2N-1}{2}$. We formulate it in the next lemma. This fact was also noticed in [77].

Lemma 33. *The solutions in variable m_1 coming from the above definitions of generalized coiflets' filters possess pairs of roots symmetric with respect to the point $\frac{2N-1}{2}$ or equivalently the solutions in variable m_1' possess pairs of roots symmetric with respect to the point* 0.

Then more convenient is the following shifted scaling moments

$$m_i' = \sum_{k=0}^{2N-1} \left(k - \frac{2N-1}{2}\right)^i h_k. \tag{2.77}$$

We explain the reason for this choice below. These shifted moments can be treated instead of originally defined moments $m_i = \sum_{k=0}^{2N-1} h_k k^i$. Then we define similarly as in the Theorem 15

$$a'_i = \sum_{k=0}^{N-1} \left(2k - \frac{2N-1}{2}\right)^i h_{2k} \qquad \text{and} \qquad b'_i = \sum_{k=0}^{N-1} \left(2k + 1 - \frac{2N-1}{2}\right)^i h_{2k+1}. \tag{2.78}$$

We can see some symmetry in a'_i and b'_i for even i and some anti-symmetry in odd ones, respectively. For example the first term by a'_i is $\left(-\frac{2N-1}{2}\right)^i h_0$ and the last one by b'_i is $\left(\frac{2N-1}{2}\right)^i h_{2N-1}$, respectively and so on. As mentioned above the coefficients a'_i for $i > N-1$ are a linear combination of a'_i for $i < N$ and the same holds for the coefficients b'_i as well. First we compute these linear combinations and then this anti-symmetry comes into play by calculating $m'_i = a'_i + b'_i$. Due to this anti-symmetry many coefficients are vanished and this useful trick enables us substantially reduce the complexity of arising system. At last we provide an example to illustrate our approach based on Lemma 31.

Example 34. For $N = 3$, the following system will be obtained

$$m'_0 = 1, \qquad m'_2 = {m'_1}^2, \qquad m'_3 = {m'_1}^3, \tag{2.79}$$

further

$$m'_3 = \frac{13}{4} m'_1 + \frac{3}{2} {m'_1}^2 - 3a'_2, \qquad m'_4 = \frac{11}{2} {m'_1}^2 - \frac{45}{16}, \tag{2.80}$$

and from orthonormality condition

$$\frac{11}{2} {m'_1}^2 - \frac{45}{16} + 2{m'_1}^4 - 12a'_2(a'_2 - {m'_1}^2) = 0 \tag{2.81}$$

after substitution and elimination we get

$$\frac{4}{3} {m'_1}^6 - \frac{29}{3} {m'_1}^4 + \frac{235}{12} {m'_1}^2 - \frac{45}{16} = 0. \tag{2.82}$$

At the end, we can use Lemma 21 for localization of m_1.

To conclude: The arising system from the Lemma 31 is partly linear and partly quadratic. This definition enables us simply eliminate some quadratic equations. Furthermore, solving these systems for shifted moments can again reduce the complexity. For small filter lengths this system can be explicitly solved via algebraic methods like the Gröbner bases. The result of these methods is a polynomial in one variable. As mentioned above m'_1 belongs to the interval $(\frac{1-2N}{2}, \frac{2N-1}{2})$ and due to the symmetry, we can restrict it only to the interval $[0, \frac{2N-1}{2})$. Some of the computed scaling coefficients are given in tables in Appendix A.

Chapter 3

Numerical Integration

In applications one often needs to compute the integrals involving scaling functions or wavelets. Since many types of scaling functions and wavelets have a low Sobolev as well as Hölder regularity, see Table 1.2, tables in the Appendix or [64], the classical quadratures such as the Simpson rule are not useful in this case. Therefore, other quadratures were designed in [9, 56, 96]. They require the knowledge of the precise values of scaling moments, that means the integrals of the form

$$M_{c,d}^i := \int_c^d x^i \phi(x)\, dx, \tag{3.1}$$

where $i = 0, \ldots, M$ for some given M and ϕ is a refinable function. If supp $\phi \subset \langle c, d \rangle$, then $M_i := M_{c,d}^i$ can be simply computed by the reccurence formula

$$M_i := \int_{-\infty}^{\infty} x^i \phi(x)\, dx = \frac{1}{2^{i+1} - 2} \sum_{l=1}^{i} \sum_{n \in \mathbb{Z}} \binom{i}{l} h_n n^{i-l} M_l. \tag{3.2}$$

If supp ϕ is not contained in $\langle c, d \rangle$, then the computation of $M_{c,d}^i$ is more difficult. The method of computation of precise values of $M_{c,d}^i$ in the case that supp ϕ is not a subset of $\langle c, d \rangle$, $c, d \in \mathbb{N}$, has been proposed in [68, 75] for the Daubechies wavelets and in [56] for more general wavelets. In this chapter we propose a novel method for the computation of the moments $M_{c,d}^i$ for a refinable function ϕ with compact support and c, d any dyadic points. In Chapter 6 we use our method for the computation of the entries of the biorthogonalization matrix.

3.1 The Computation of Antiderivatives of Scaling Function and Wavelet

It is easy to verify that the antiderivative of scaling function $\Phi^{[1]}(x) = \int_{-\infty}^{x} \phi(s)\, ds$ satisfies the refinement equation

$$\Phi^{[1]}(x) = \sum_{k \in \mathbb{Z}} \frac{h_k}{2} \Phi^{[1]}(2x - k) \tag{3.3}$$

and similarly the antiderivative of wavelet $\Psi^{[1]}(x) = \int_{-\infty}^{x} \psi(s)\, ds$ satisfies

$$\Psi^{[1]}(x) = \sum_{k\in\mathbb{Z}} \frac{g_k}{2} \Phi^{[1]}(2x-k). \tag{3.4}$$

Note that the function $\Phi^{[1]}$ is generally not compactly supported. Assuming that supp $\phi = \langle a, b\rangle$, the function $\Phi^{[1]}$ is vanishing on $(-\infty, a)$ and $\Phi^{[1]}$ is constant on (b, ∞). Due to the non-compactness of the support we could not find exact values of $\Phi^{[1]}$ at integer points by solving some eigenvalue problem as in [36]. However, we show that we are able to find these values as a solution of a system of linear algebraic equations.

Lemma 35. *Let ϕ be a refinable function with compact support $\langle a, b\rangle$, where $a, b \in \mathbb{Z}$, and let ϕ be normalized as $\int_{\mathbb{R}} \phi(x)\, dx = 1$. Then the values of $\Phi^{[1]}$ at integers are given by*

$$\begin{aligned}
\Phi^{[1]}(k) &= 0, && k \le a, \\
\Phi^{[1]}(k) &= \sum_{l=a}^{b-1} \frac{h_{2k-l}}{2} \Phi(l) + \sum_{l=b}^{\infty} \frac{h_{2k-l}}{2}, && a < k < b, \quad k \in \mathbb{Z}, \\
\Phi^{[1]}(k) &= 1, && b \le k.
\end{aligned} \tag{3.5}$$

Proof. It is clear that $\Phi^{[1]}(k) = 0$ for $k \le a$ and $\Phi^{[1]}(k) = \int_a^b \phi(x)\, dx = 1$ for $k \ge b$. From the refinement equation (3.3) it follows that

$$\begin{aligned}
\Phi^{[1]}(k) &= \sum_{l\in\mathbb{Z}} \frac{h_l}{2} \Phi^{[1]}(2k-l) = \sum_{l\in\mathbb{Z}} \frac{h_{2k-l}}{2} \Phi^{[1]}(l) \\
&= \sum_{l=a}^{b-1} \frac{h_{2k-l}}{2} \Phi^{[1]}(l) + \sum_{l=b}^{\infty} \frac{h_{2k-l}}{2}, \quad a < k < b, \quad k \in \mathbb{Z}.
\end{aligned} \tag{3.6}$$

□

Hence, by solving system (3.6) we can find precise values of $\Phi^{[1]}$ at integers and then applying refinement equation J times, we obtain precise value of $\Phi^{[1]}$ at any dyadic point $\frac{k}{2^J}$, $k \in \mathbb{Z}$.
For evaluating integrals of refinable functions we will also need $\Phi^{[n+1]}(x) := \int_{-\infty}^{x} \Phi^{[n]}(s)\, ds$, $\Psi^{[n+1]}(x) := \int_{-\infty}^{x} \Psi^{[n]}(s)\, ds$. By integrating (1.12) and (1.21), we obtain refinement equations:

$$\Phi^{[n]}(x) = \sum_{k\in\mathbb{Z}} \frac{h_k}{2^n} \Phi^{[n]}(2x-k) \tag{3.7}$$

and

$$\Psi^{[n]}(x) = \sum_{k\in\mathbb{Z}} \frac{g_k}{2^n} \Phi^{[n]}(2x-k). \tag{3.8}$$

In the next theorem we show that we are able to find the values of $\Phi^{[n]}$ and $\Psi^{[n]}$ at integers and so at any dyadic point.

Theorem 36. *Let ϕ be a refinable function with compact support $\langle a,b\rangle$, where $a,b\in\mathbb{Z}$ and ϕ is normalized by $\int_{\mathbb{R}}\phi(x)\,dx=1$. Then the following relations hold:*

$$\begin{aligned}
\Phi^{[n]}(k) &= 0, && k\le a, \\
\Phi^{[n]}(k) &= \sum_{l=a}^{b-1}\frac{h_{2k-l}}{2^n}\Phi^{[n]}(l)+\sum_{l=b}^{\infty}\frac{h_{2k-l}}{2^n}\sum_{m=1}^{n}\Phi^{[m]}(b)\frac{(l-b)^{n-m}}{(n-m)!}, && a<k<b,\ k\in\mathbb{Z}, \\
\Phi^{[n]}(b) &= \frac{1}{2^n-2}\sum_{m=1}^{n-1}\frac{\Phi^{[m]}(b)}{(n-m)!}\sum_{l=a}^{b}h_l(b-l)^{n-m}, \\
\Phi^{[n]}(k) &= \sum_{m=1}^{n}\Phi^{[m]}(b)\frac{(l-b)^{n-m}}{(n-m)!}, && b\le k.
\end{aligned} \tag{3.9}$$

Proof. We prove this theorem by induction. For $n=1$ relations (3.9) hold by Lemma 35. Now let us suppose that the above system of equations hold. Then

$$\begin{aligned}
\Phi^{[n]}(b) &= \sum_{m\in\mathbb{Z}}\frac{h_{2b-m}}{2^n}\Phi^{[n]}(m)=\sum_{m\in\mathbb{Z}}\frac{h_{2b-m}}{2^n}\left(\Phi^{[n]}(b)+\int_b^m\Phi^{[n-1]}(x)\,dx\right) \\
&= \sum_{m\in\mathbb{Z}}\frac{h_{2b-m}}{2^n}\left(\Phi^{[n]}(b)+\int_b^m\sum_{l=1}^{n-1}\Phi^{[l]}(b)\frac{(x-b)^{n-1-m}}{(n-1-m)!}dx\right) \\
&= \sum_{m\in\mathbb{Z}}\frac{h_{2b-m}}{2^n}\Phi^{[n]}(b)+\sum_{m\in\mathbb{Z}}\frac{h_{2b-m}}{2^n}\sum_{l=1}^{n-1}\Phi^{[l]}(b)\frac{(m-b)^{n-m}}{(n-m)!}.
\end{aligned} \tag{3.10}$$

Integrating (1.12), we obtain $\sum_{m\in\mathbb{Z}}h_m=2$. Thus

$$\Phi^{[n]}(b)=\frac{1}{2^n-2}\sum_{m=1}^{n-1}\frac{\Phi^{[m]}(b)}{(n-m)!}\sum_{l=a}^{b}h_l(b-l)^{n-m}. \tag{3.11}$$

For $k\ge b$ we have

$$\begin{aligned}
\Phi^{[n]}(k) &= \int_a^b\Phi^{[n-1]}(x)\,dx+\int_b^k\Phi^{[n-1]}(x)\,dx \\
&= \Phi^{[n]}(b)+\int_b^k\sum_{m=1}^{n-1}\Phi^{[m]}(b)\frac{(x-b)^{n-1-m}}{(n-1-m)!}dx \\
&= \Phi^{[n]}(b)+\sum_{m=1}^{n-1}\Phi^{[m]}(b)\frac{(k-b)^{n-m}}{(n-m)!} \\
&= \sum_{m=1}^{n}\Phi^{[m]}(b)\frac{(l-b)^{n-m}}{(n-m)!}.
\end{aligned} \tag{3.12}$$

Now using refinement equation (3.7), we obtain for $k \in \mathbb{Z}$, $a < k < b$,

$$\begin{aligned}\Phi^{[n]}(k) &= \sum_{l=a}^{b} \frac{h_l}{2^n} \Phi^{[n]}(2k-l) = \sum_{m\in\mathbb{Z}} \frac{h_{2k-l}}{2^n} \Phi^{[n]}(l) \\ &= \sum_{l=a}^{b-1} \frac{h_{2k-l}}{2^n} + \sum_{l=b}^{\infty} \frac{h_{2k-l}}{2^n} \sum_{m=1}^{n} \Phi^{[m]}(b) \frac{(l-b)^{n-m}}{(n-m)!}.\end{aligned} \tag{3.13}$$

□

3.2 Evaluation of Moments

Now we can evaluate moments $M^i_{c,d}$ by the formula in the following theorem.

Theorem 37. *Let ϕ be a refinable function and let $\Phi^{[n]}$ be defined as above. Then*

$$\int_c^d x^i \phi(x)\, dx = \sum_{l=0}^{i} (-1)^l \frac{i!}{(i-l)!} \left(d^{i-l}\Phi^{[l+1]}(d) - c^{i-l}\Phi^{[l+1]}(c)\right). \tag{3.14}$$

Proof. We prove Theorem 37 by induction. Clearly for $i = 0$ relation (3.14) is valid. Let us now suppose that (3.14) holds for $0, \ldots, i$. Using integration by parts, we obtain

$$\begin{aligned}\int_c^d x^{i+1}\phi(x)\, dx &= \left[x^{i+1}\Phi^{[1]}(x)\right]_c^d - (i+1)\int_c^d x^i \Phi^{[1]}(x)\, dx \\ &= \left[x^{i+1}\Phi^{[1]}(x)\right]_c^d - (i+1)\sum_{l=0}^{i} \frac{(-1)^l i!}{(i-l)!} \left(d^{i-l}\Phi^{[l+2]}(d) - c^{i-l}\Phi^{[l+2]}(c)\right) \\ &= \left[x^{i+1}\Phi^{[1]}(x)\right]_c^d - (i+1)\sum_{l=1}^{i+1} \frac{(-1)^{l-1} i!}{(i-l+1)!} \left(d^{i+1-l}\Phi^{[l+1]}(d) - c^{i+1-l}\Phi^{[l+1]}(c)\right) \\ &= \sum_{l=0}^{i+1} (-1)^l \frac{(i+1)!}{(i+1-l)!} \left(d^{i+1-l}\Phi^{[l+1]}(d) - c^{i+1-l}\Phi^{[l+1]}(c)\right),\end{aligned} \tag{3.15}$$

which proves the assertion. □

Chapter 4

Generalization of Wavelet Bases

The above setting is clearly not suitable yet for applications such as the treatment of differential equations, since they are usually defined on bounded domains. This chapter provides a short introduction to the theory of generalized wavelet bases. The results in this chapter are known and can be found in [36, 40, 52].

4.1 Wavelet Basis and Multiresolution Analysis

Let us consider a separable Hilbert space H with inner product $\langle \cdot, \cdot \rangle_H$ and induced norm $\|\cdot\|_H$. Here $\mathcal{J}$ is an index set. Each index $\lambda \in \mathcal{J}$ takes the form $\lambda = (j, k)$, where $|\lambda| = j$ denotes the *scale* or *level* and k represents the *spatial location*.

Definition 38. A family $\Psi := \{\psi_\lambda \in \mathcal{J}\} \subset H$ is called a *wavelet basis* of H, if the following conditions are satisfied:

i) Ψ is a Riesz bases for H, that means Ψ generates H, i.e.

$$H = \overline{\operatorname{span}\Psi}^H, \tag{4.1}$$

and there exist constants $c, C \in (0, \infty)$ such that for all $\mathbf{b} := \{b_\lambda\}_{\lambda \in \mathcal{J}} \in l^2(\mathcal{J})$ we have

$$c \|\mathbf{b}\|_{l_2(\mathcal{J})} \leq \left\| \sum_{\lambda \in \mathcal{J}} b_\lambda \psi_\lambda \right\|_H \leq C \|\mathbf{b}\|_{l_2(\mathcal{J})}. \tag{4.2}$$

Constants $c_\psi := \sup\{c : c \text{ satisfies } (4.2)\}$, $C_\psi := \inf\{C : C \text{ satisfies } (4.2)\}$ are called the *Riesz bounds* and C_ψ / c_ψ is called the *condition* of Ψ.

ii) The functions ψ_λ are *local* in the sense that

$$\operatorname{diam}(\Omega_\lambda) \leq \mathrm{C}2^{-|\lambda|}, \quad \lambda \in \mathcal{J}, \tag{4.3}$$

where Ω_λ is support of ψ_λ.

From the property *ii*) it follows that the length of support of the wavelet basis function depends on the scale j. For large j this support is very small. In this aspect, wavelet bases are close to hierarchical bases, but the classical hierarchical bases are not stable in the sense of *i*). On the other hand, spectral methods use Fourier decomposition, which satisfy *i*) but do not fullfil *ii*). Due to properties *i*) and *ii*) wavelets are often said to be well-localized both in space and frequency.
Since Ψ generates H any function $f \in H$ has an expansion

$$f = \sum_{\lambda \in J} c_\lambda \psi_\lambda. \tag{4.4}$$

Due to the estimate (4.2), the coefficient functionals $c_\lambda = c_\lambda(f)$ are bounded on H. By the Riesz representation theorem, there exists a unique family of dual functions $\tilde{\Psi} = \left\{ \tilde{\psi}_\lambda, \lambda \in \tilde{\mathcal{J}} \right\} \subset H$ such that $c_\lambda = c_\lambda(f) = \left\langle f, \tilde{\psi}_\lambda \right\rangle$. Moreover, it is $\|c_\lambda\|_{H'} = \|\psi_\lambda\|_H$. As a consequence of duality, the sets Ψ and $\tilde{\Psi}$ are biorthogonal, i.e. we have

$$\left\langle \psi_{i,k}, \tilde{\psi}_{j,l} \right\rangle_H = \delta_{i,j}\delta_{k,l} \quad \text{for all} \quad (i,k) \in \mathcal{J}, \quad (j,l) \in \tilde{\mathcal{J}}. \tag{4.5}$$

This dual family is also a Riesz basis for H with Riesz bounds C^{-1}, c^{-1}. By the above argument, biorthogonality is a necessary for the Riesz basis property (4.2) to hold. But unfortunately it is not sufficient, see [49].
In the sequel, we use a convenient shorthand notation

$$\langle \Theta, \Phi \rangle = \left(\langle \theta, \phi \rangle \right)_{\theta \in \Theta, \phi \in \Phi}, \tag{4.6}$$

for collections of functions Θ, $\Phi \in H$ and bilinear form $\langle \cdot, \cdot \rangle$ on $H \times H$. Thus, (4.5) can be written as

$$\left\langle \Psi, \tilde{\Psi} \right\rangle = \mathbf{I}. \tag{4.7}$$

In many cases, the wavelet system Ψ is constructed with the aid of a multiresolution analysis.

Definition 39. A sequence $\mathcal{S} = \{S_j\}_{j \in \mathbb{N}_{j_0}}$ of closed linear subspaces $S_j \subset H$ is called a *multiresolution* or *multiscale analysis*, if the subspaces are nested, i.e.,

$$S_{j_0} \subset S_{j_0+1} \subset \ldots \subset S_j \subset S_{j+1} \subset \ldots \subset H \tag{4.8}$$

and is dense in H, i.e.

$$\overline{\cup_{j \in \mathbb{N}_{j_0}} S_j}^H = H. \tag{4.9}$$

We now assume that S_j is spanned by set of basis functions

$$\Phi_j := \{\phi_{j,k}, k \in \mathcal{I}_j\}, \tag{4.10}$$

where $\mathcal{I}_j$ is a finite index set. We refer to $\phi_{j,k}$ as *(generalized) scaling functions*. Moreover, the collections Φ_j will always be assumed to be *uniformly stable*, it means that there exists two constants $c, C > 0$ independent of j such that the Riesz property

$$c\left(\sum_{k\in\mathcal{I}_j}|c_{j,k}|^2\right)^{1/2} \leq \left\|\sum_{k\in\mathcal{I}_j}c_{j,k}\phi_{j,k}\right\|_H \leq C\left(\sum_{k\in\mathcal{I}_j}|c_{j,k}|^2\right)^{1/2} \tag{4.11}$$

holds for all $\{c_{j,k}\}_{k\in\mathcal{I}_j} \in l^2(\mathcal{I}_j)$. Furthermore, it is important to require that

$$\text{diam supp}(\phi_{j,k}) \leq C2^{-j}, \quad j \geq j_0. \tag{4.12}$$

From the nestedness of S and the uniform stability of the Riesz bases, we conclude the existence of a bounded linear operator $\mathbf{M}_{j,0} = \left(m_{l,k}^{j,0}\right)_{l\in\mathcal{I}_{j+1},k\in\mathcal{I}_j}$ such that

$$\phi_{j,k} = \sum_{l\in\mathcal{I}_{j+1}} m_{l,k}^{j,0}\phi_{j+1,l}. \tag{4.13}$$

Viewing Φ_j as a column vector, the *two-scale* or *refinement relation* (4.13) can be written as

$$\Phi_j = \mathbf{M}_{j,0}^T\Phi_{j+1}. \tag{4.14}$$

As a consequence of locality (4.12), the matrices $\mathbf{M}_{j,0}$ are *uniformly sparse* which means that the number of entries per each row and column is uniformly bounded.
The nestedness of the multiresolution analysis implies the existence of the *complement* or *wavelet spaces* W_j such that

$$S_{j+1} = S_j \oplus W_j. \tag{4.15}$$

Let

$$\Psi_j := \{\psi_{j,k}, k \in \mathcal{J}_j\}, \quad \mathcal{J}_j := \mathcal{I}_{j+1} - \mathcal{I}_j, \quad j \geq j_0, \tag{4.16}$$

be a Riesz basis of W_j. Functions in Ψ_j are called *wavelets*. Since every $\psi_{j,k} \in \Psi_j$ is also in the space S_{j+1} generated by Φ_{j+1} it has a unique representation

$$\psi_{j,k} = \sum_{l\in\mathcal{I}_{j+1}} m_{l,k}^{j,1}\phi_{j,l}, \tag{4.17}$$

which can be expressed as

$$\Psi_j = \mathbf{M}_{j,1}^T\Phi_{j+1}, \tag{4.18}$$

where $\mathbf{M}_{j,1}$ is a bounded linear operator given by $\mathbf{M}_{j,1} = \left(m_{l,k}^{j,1}\right)_{l\in\mathcal{I}_{j+1},k\in\mathcal{J}_j}$. We assume that Ψ_j satisfy *locality conditions*

$$\text{diam supp}(\psi_{j,k}) \leq C2^{-j}, \quad j \geq j_0. \tag{4.19}$$

Then $\mathbf{M}_{j,1}$ is also uniformly sparse. The refinement relations (4.14) and (4.18) lead to

$$\begin{pmatrix} \Phi_j \\ \Psi_j \end{pmatrix} = \mathbf{M}_j^T \Phi_{j+1}, \tag{4.20}$$

with $\mathbf{M}_j := (\mathbf{M}_{j,0}, \mathbf{M}_{j,1})$. Note that $\mathbf{M}_j$ as a basis transformation is invertible and let the inverse transformation matrix be defined by

$$\mathbf{G}_j = \begin{pmatrix} \mathbf{G}_{j,0} \\ \mathbf{G}_{j,1} \end{pmatrix} := \mathbf{M}_j^{-1}. \tag{4.21}$$

The inverses of sparse matrices are usually densely populated. However, for numerical purposes, we consider those matrices which are uniformly sparse.

Definition 40. Any $\mathbf{M}_{j,1} \in [l_2(\mathcal{J}_j), l_2(\mathcal{I}_{j+1})]$ is called a *stable completion* of $\mathbf{M}_{j,0}$, if

$$\|\mathbf{M}_j\|, \left\|\mathbf{M}_j^{-1}\right\| = \mathcal{O}(1), \quad j \to \infty, \tag{4.22}$$

where $\mathbf{M}_j := (\mathbf{M}_{j,0}, \mathbf{M}_{j,1})$.

It is known that $\Phi_j \cup \Psi_j$ is uniformly stable if and only if $\mathbf{M}_{j,1}$ is a stable completion of $\mathbf{M}_{j,0}$, see [50]. Iterating (4.15) up to some level J yields a multiscale decomposition of the space S_J

$$S_J = S_{j_0} \oplus \bigoplus_{j=j_0}^{J-1} W_j. \tag{4.23}$$

Thus, the multiscale basis of S_J contains scaling functions on the coarsest level j_0 and wavelets on the level j for $j_0 \leq j \leq J-1$

$$\Psi_{(J)} = \Phi_{j_0} \cup \bigcup_{j=j_0}^{J-1} \Psi_j. \tag{4.24}$$

Since the union of subspaces S_j is dense in H, a basis of H is given by

$$\Psi = \Phi_{j_0} \cup \bigcup_{j=j_0}^{\infty} \Psi_j. \tag{4.25}$$

For convenience we define $\Psi_{j_0-1} := \Phi_{j_0}$. Then

$$\Psi = \bigcup_{j=j_0-1}^{\infty} \Psi_j, \tag{4.26}$$

and we split $\mathcal{J}$ into two index sets

$$\mathcal{J}_\phi := \{(j_0-1, k), k \in \mathcal{I}_j\}, \quad \mathcal{J}_\psi := \{(j,k), j \geq j_0, k \in \mathcal{J}_j\}. \tag{4.27}$$

In order to ensure that Ψ forms a Riesz basis for H, we need to verify sufficient conditions from the following theorem given in [50]. Note that these conditions are independent of the basis and refer to properties of the multiresolution analysis.

Theorem 41. *Assume that* $\mathcal{P} = (P_j)_{j \geq j_0}$ *is a sequence of uniformly bounded projectors* $P_j : H \to S_j$ *such that*

$$P_l P_j = P_l, \quad l \leq j. \tag{4.28}$$

Let $\tilde{\mathcal{S}} = \left\{ \tilde{S}_j \right\}$ *be the ranges of the sequence of adjoints* $\mathcal{P}^* = ({P^*}_j)_{j \geq j_0}$. *Moreover, suppose that there exists a family of uniformly bounded subadditive functionals* $\omega(\cdot, t) : H \to \mathbb{R}^0_+$, $t > 0$, *such that* $\lim_{t \to 0+} \omega(f, t) = 0$ *for each* $f \in H$ *and that a pair of estimates*

$$\inf_{v \in V_j} \|f - v\|_H \lesssim \omega\left(f, 2^{-j}\right), \quad f \in H, \tag{4.29}$$

and

$$\omega(v_j, t) \lesssim \left(min\left\{1, t2^j\right\}\right)^\gamma \|v_j\|_H, \quad v_j \in V_j, \tag{4.30}$$

holds for $\mathcal{V} := \{V_j\} = \mathcal{S}$ *and* $V := \{V_j\} = \tilde{\mathcal{S}}$ *with some* γ, $\tilde{\gamma} > 0$, *respectively. Then we have a norm equivalence*

$$\|\cdot\|_H \sim N_P(\cdot) \sim N_{P*}(\cdot), \quad v \in H, \tag{4.31}$$

where

$$N_P(v) := \left(\sum_{j \geq j_0} \|(P_j - P_{j-1}) v\|_H^2 \right)^{1/2}, \quad v \in H, \tag{4.32}$$

and

$$N_{P^*}(v) := \left(\sum_{j \geq j_0} \left\|\left(P_j^* - P_{j-1}^*\right) v\right\|_H^2 \right)^{1/2}, \quad v \in H, \tag{4.33}$$

with $P_{j_0 - 1} := P^*_{j_0 - 1} := 0$.

Estimates of type (4.29) indicating the approximation properties of $\mathcal{S}$ are called the *direct* or *Jackson estimates*, while estimates like (4.30) describe smoothness properties of $\mathcal{S}$ and are often referred to as the *inverse* or *Bernstein estimate.*
For a given basis Ψ and basis $\tilde{\Psi}$ which is biorthogonal to Ψ the operator $Q_j : H \to S_j$ is defined by

$$P_j(v) := \left\langle v, \tilde{\Psi}_{(j)} \right\rangle \Psi_{(j)}, \quad j \geq j_0. \tag{4.34}$$

and the adjoint of Q_j is given by

$$P_j^*(v) := \left\langle v, \Psi_{(j)} \right\rangle \tilde{\Psi}_{(j)}, \quad j \geq j_0. \tag{4.35}$$

The biorthogonality of Ψ, $\tilde{\Psi}$ means that the operators P_j are indeed projectors and that they satisfy *the commutator property* (4.28). To ensure that Ψ is a Riesz basis it remains to verify (4.29) and (4.30).
Wavelet basis Ψ is called a *primal* wavelet basis, $\mathcal{S}$ is called *primal* multiresolution analysis, $\phi_{j,k} \in \Phi_j$ are called *primal* scaling functions, and $\psi_{j,k} \in \Psi_j$ are called *primal* wavelets. Analogically, a basis $\tilde{\Psi}$ which is biorthogonal to Ψ is called a *dual* wavelet basis. It is a

part of a *dual* multiresolution analysis $\tilde{\mathcal{S}} = \{\mathcal{S}_j\}_{j \geq j_0}$. A *dual* wavelet basis consists of *dual* scaling functions and *dual* wavelets. The complement spaces $\tilde{W}_j$ are defined by

$$\tilde{S}_{j+1} = \tilde{S}_j \oplus \tilde{W}_j, \quad j \geq j_0. \tag{4.36}$$

Morever, we have

$$S_j \perp \tilde{S}_j, \quad W_j \perp \tilde{S}_j, \quad \tilde{W}_j \perp S_j, \quad j \geq j_0. \tag{4.37}$$

4.2 Multiscale Transformation

From (4.24) it follows that any $v \in S_J$ has a *single-scale representation*

$$v = \mathbf{c}_J^T \Phi = \sum_{k \in \mathcal{I}_j} c_{j,k} \phi_{j,k}, \tag{4.38}$$

as well as a *multiscale representation*

$$v = \mathbf{d}_{(J)}^T \Psi_{(J)} = \mathbf{c}_{j_0}^T \Phi_{j_0} + \mathbf{d}_{j_0}^T \Psi_{j_0} + \ldots + \mathbf{d}_{J-1}^T \Psi_{J-1} = \sum_{k \in I_{j_0}} c_{j_0,k} \phi_{j_0,k} + \sum_{j=j_0}^{J-1} \sum_{k \in J_j} d_{j,k} \psi_{j,k}. \tag{4.39}$$

The corresponding vectors of single-scale coefficients $\mathbf{c}_J$ and multiscale coefficients

$$\mathbf{d}_{(J)} := \left(\mathbf{c}_{j_0}^T, \mathbf{d}_{j_0}^T, \ldots \mathbf{d}_{J-1}^T\right)^T \tag{4.40}$$

are interrelated by the *multiscale transformation* $\mathbf{T}_J : l^2(\mathcal{I}_J) \to l^2(\mathcal{I}_J)$,

$$\mathbf{c}_J = \mathbf{T}_J \mathbf{d}_{(J)}. \tag{4.41}$$

From refinement relations (4.14) and (4.18), it follows that

$$\mathbf{c}_j^T \Phi_j + \mathbf{d}_j^T \Psi_j = \left(\mathbf{M}_{j,0} \mathbf{c}_j + \mathbf{M}_{j,1} \mathbf{d}_j\right)^T \Phi_{j+1} = \mathbf{c}_{j+1}^T \Phi_j. \tag{4.42}$$

Consequently, the transform $\mathbf{T}_J$ consists of the successive applications of two-scale operators,

$$\mathbf{T}_J = \mathbf{T}_{J,J-1} \ldots \mathbf{T}_{J,j_0}, \quad \text{where} \quad \mathbf{T}_{J,j} = \begin{pmatrix} \mathbf{M}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}. \tag{4.43}$$

Schematically $\mathbf{T}_J$ can be visualized as a *pyramid scheme*,

$$\begin{array}{ccccccccccc}
\mathbf{c}_{j_0} & \overset{\mathbf{M}_{j_0,0}}{\longrightarrow} & \mathbf{c}_{j_0+1} & \overset{\mathbf{M}_{j_0+1,0}}{\longrightarrow} & \mathbf{c}_{j_0+2} & \longrightarrow & \ldots & \mathbf{c}_{J-1} & \overset{\mathbf{M}_{J-1,0}}{\longrightarrow} & \mathbf{c}_J & . \\
& \mathbf{M}_{j_0,1} \nearrow & & \mathbf{M}_{j_0+1,1} \nearrow & & \mathbf{M}_{j_0+2,1} \nearrow & & & \mathbf{M}_{J,1} \nearrow & & \\
& \mathbf{d}_{j_0} & & \mathbf{d}_{j_0+1} & & \mathbf{d}_{j_0+2} & \ldots & & \mathbf{d}_{J-1} & &
\end{array}$$

In order to determine the inverse transform $\mathbf{T}_J^{-1}$ note that

$$\mathbf{c}_{j+1}^T \Phi_{j+1} = \mathbf{G}_{j,0}^T \mathbf{c}_{j+1}^T \Phi_j + \mathbf{G}_{j,1}^T \mathbf{c}_{j+1}^T \Psi_j = \mathbf{c}_j^T \Phi_j + \mathbf{d}_j^T \Psi_j. \tag{4.44}$$

Thus, the inverse transform $\mathbf{T}_J^{-1}$ can be written also in product structure by applying the inverses of the matrices $\mathbf{T}_{J,j}$ in reverse order as follows:

$$\mathbf{T}_J^{-1} = \mathbf{T}_{J,j_0}^{-1} \dots \mathbf{T}_{J,J-1}^{-1}, \quad \text{where} \quad \mathbf{T}_{J,j}^{-1} = \begin{pmatrix} \mathbf{G}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}. \tag{4.45}$$

The corresponding pyramid scheme is then

$$\begin{array}{ccccccccccc}
\mathbf{c}_J & \overset{\mathbf{G}_{J-1,0}}{\longrightarrow} & \mathbf{c}_{J-1} & \overset{\mathbf{G}_{J-2,0}}{\longrightarrow} & \mathbf{c}_{J-2} & \overset{\mathbf{G}_{J-3,0}}{\longrightarrow} & \dots & \mathbf{c}_{j_0+1} & \overset{\mathbf{G}_{j_0,0}}{\longrightarrow} & \mathbf{c}_{j_0} & . \\
& \searrow^{\mathbf{G}_{J-1,1}} & & \searrow^{\mathbf{G}_{J-2,1}} & & \searrow^{\mathbf{G}_{J-3,1}} & & & \searrow^{\mathbf{G}_{j_0,1}} & & \\
& \mathbf{d}_{J-1} & & \mathbf{d}_{J-2} & & \mathbf{d}_{J-3} & \dots & & \mathbf{d}_{j_0} & &
\end{array}$$

From the uniform sparseness of matrices $\mathbf{M}_j$ and $\mathbf{G}_j$ we conclude that they can be applied in $\mathcal{O}(N_j)$ operations, where $N_j := \dim S_j$. It follows that $\mathbf{T}_J$ and $\mathbf{T}_J^{-1}$ can be applied in $\mathcal{O}(N_J)$ operations when using a pyramid scheme. Note that these matrices are not explicitly computed and stored in computer memory.
The next theorem illustrates the interdependence between Ψ and $\mathbf{T}_J$.

Theorem 42. *Assume that Φ_j are uniformly stable. Then $\mathbf{T}_J$ are well conditioned or stable in the sense of $\|\mathbf{T}_J\|$, $\|\mathbf{T}_J^{-1}\| = \mathcal{O}(1)$ if and only if Ψ is a Riesz basis in H.*

For the proof see [19, 50].

4.3 Cancellation Properties

In the case $H \subset L^2$, another main requirement on the primal wavelet basis is that it has a *cancellation property* of order $\tilde{m}$ for some given $\tilde{m} \in \mathbb{N}$. This means that integration of a function against a wavelet annihilates smooth parts, in the sense that

$$|\langle v, \psi_\lambda \rangle| \lesssim 2^{-|\lambda|\left(\frac{d}{2}+\tilde{m}-\frac{d}{p}\right)} |v|_{W_p^{\tilde{m}}(\Omega_\lambda)}, \quad v \in W_p^{\tilde{m}}(\Omega_\lambda), \quad \lambda \in \mathcal{J}_\psi. \tag{4.46}$$

Similarly, we require cancellation property of order $m \in \mathbb{N}$ for dual wavelet basis.
When wavelets live on a bounded domain $\Omega \subset \mathbb{R}^d$, then cancellation property of order $\tilde{m}$ for primal wavelet basis is equivalent to the *polynomial exactness* of multiresolution analysis of order $\tilde{m}$, i.e. the ability of spaces S_j to reproduce polynomials up to order $\tilde{m}$. More precisely, let Π_r denote the space of all polynomials of degree less or equal to $r-1$. It is said that the multiresolution analysis $\mathcal{S}$ is *exact* of order $\tilde{m} \in \mathbb{N}$, if $\Pi_{\tilde{m}} \subset S_{j_0}$. Due to

the biorthogonality this holds if and only if the corresponding wavelets have $\tilde{m}$ *vanishing moments*, i.e. wavelets are orthogonal to polynomials up to order $\tilde{m}$,

$$\langle P, \psi_\lambda \rangle = 0, \quad P \in \Pi_{\tilde{m}}, \quad \lambda \in \mathcal{J}_\psi. \tag{4.47}$$

By the Taylor expansion, this in turn yields the *cancellation properties* of order $\tilde{m}$. Conversely, by substituting $v = P$, $P \in \Pi_{\tilde{m}}$, (4.46) immediately implies that the wavelets have $\tilde{m}$ vanishing moments.
In some cases we refer to the cancellation properties rather than to vanishing moments, because it makes sense also for domains, where ordinary polynomials are not defined.

4.4 Norm Equivalences and Function Spaces

One of the most important property of wavelet bases is that they can be used to characterize function spaces in terms of expansion coefficients. The Riesz basis property (4.2) is a special case of similar relations called *norm equivalences* (4.31). In this section, we formulate a fundamental theorem similar to Theorem 41 for Sobolev spaces and we discuss how to verify the validity of direct and inverse estimates in the form of (4.49) and (4.50). In Section 7 we will see that norm equivalences for Sobolev spaces play a vital role for preconditioning of systems arising from discretization of elliptic problems, matrix compression, and adaptive techniques.
In the following, we assume that Ω is a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary and $H = L^2(\Omega)$.

Theorem 43. *Assume that $\mathcal{Q} = (Q_j)_{j \geq j_0}$ is a sequence of uniformly bounded projectors $Q_j : L^2(\Omega) \to S_j$ such that*

$$Q_l \, Q_j = Q_l, \quad l \leq j. \tag{4.48}$$

Let $\tilde{\mathcal{S}} = \left\{ \tilde{S}_j \right\}$ be the ranges of the sequence of adjoints $\mathcal{Q}^* = \left(Q^*{}_j \right)_{j \geq j_0}$. Assume that the Jackson estimate holds in the form

$$\inf_{v_j \in V_j} \|v - v_j\|_{L^2(\Omega)} \lesssim 2^{-sj} \|v\|_{H^s(\Omega)}, \quad v \in H^s(\Omega), \quad 0 \leq s \leq m_{\mathcal{V}} \tag{4.49}$$

and the Bernstein estimate holds in the form

$$\|v_j\|_{H^s(\Omega)} \lesssim 2^{sj} \|v_j\|_{L^2(\Omega)}, \quad v_j \in V_j, \quad s < \gamma_{\mathcal{V}}, \tag{4.50}$$

for $\mathcal{V} = \{V_j\} := \mathcal{S}$ and $\mathcal{V} = \{V_j\} := \tilde{\mathcal{S}}$, where

$$0 < \gamma := min\{\gamma_{\mathcal{S}}, m_{\mathcal{S}}\} \quad and \quad 0 < \tilde{\gamma} := min\{\gamma_{\tilde{\mathcal{S}}}, m_{\tilde{\mathcal{S}}}\}. \tag{4.51}$$

Then the norm equivalence

$$\|v\|_{H^s(\Omega)} \sim \left(\sum_{j=0}^{\infty} 2^{2sj} \|(Q_j - Q_{j-1}) v\|^2_{L^2(\Omega)} \right)^{1/2}, \quad v \in H^s(\Omega), \tag{4.52}$$

holds for $s \in (-\tilde{\gamma}, \gamma)$.

Thus, if Ψ is a wavelet basis of $L^2(\Omega)$ and Q_j are defined by (4.34), then Theorem 43 implies the equivalence of the H^s–norm and the l^2–norm of coefficients in the wavelet expansion

$$\|v\|_{H^s(\Omega)} \sim \left(\sum_{j=0}^{\infty} 2^{2sj} \sum_{|\lambda|=j} \left| \left\langle v, \tilde{\psi}_\lambda \right\rangle \right|^2 \right)^{1/2}, \quad v \in H^s(\Omega), \quad s \in (-\tilde{\gamma}, \gamma). \tag{4.53}$$

Let $\mathbf{D}^{+s}$, $\mathbf{D}^{-s}$ be a diagonal matrices given by

$$\mathbf{D}^{+s}_{\lambda,\lambda'} := 2^{s|\lambda|}\delta_{\lambda,\lambda'}, \quad \mathbf{D}^{-s}_{\lambda,\lambda'} := 2^{-s|\lambda|}\delta_{\lambda,\lambda'}. \tag{4.54}$$

Then we can rewrite (4.53) in the form

$$\|v\|_{H^s(\Omega)} \sim \left\|\mathbf{D}^{+s}\mathbf{v}\right\|_{l^2(J)}, \tag{4.55}$$

where $\mathbf{v}$ is a vector of coefficients of v in the wavelet expansion, i.e.

$$v = \mathbf{v}^T\Psi = \left\langle v, \tilde{\Psi} \right\rangle \Psi. \tag{4.56}$$

Hence, the *diagonally rescaled* basis $\mathbf{D}^{-s}\Psi$ constitutes a Riesz basis for $H^s(\Omega)$. Similarly, $\mathbf{D}^{+s}\tilde{\Psi}$ is a Riesz basis for $H^{-s}(\Omega)$ and we have

$$\left\langle \mathbf{D}^{-s}\Psi, \mathbf{D}^{+s}\tilde{\Psi} \right\rangle = \mathbf{I}. \tag{4.57}$$

In many cases, the validity of Jackson and Bernstein estimates follows from the polynomial reproduction properties and smoothness properties of multiresolution analysis.
In some applications the above norm equivalences are needed only for a certain $s > 0$. In this case, it suffices to require the validity of the Jackson and Bernstein estimates for $\mathcal{S} = \{S_j\}_{j \geq j_0}$.
The norm equivalence of the type (4.53) can be extended to other function spaces such as the Sobolev spaces in $L^p(\Omega)$. Moreover, interpolation between such spaces provides the norm equivalences for a whole range of the Besov spaces, for details we refer to [36]. Let $0 < p, q \leq \infty$. Whenever we have an embedding $B^t_q(L^p(\Omega)) \subset L^r(\Omega)$ for some $r \geq 1$, then Q_j are bounded on $L^r(\Omega)$ and we have

$$\|v\|_{B^s_q(L_p(\Omega))} \sim \left(\sum_{j=j_0}^{\infty} 2^{qj\left(s+\frac{d}{2}-\frac{d}{p}\right)} \left(\sum_{|\lambda|=j} \left| \left\langle v, \tilde{\psi}_\lambda \right\rangle \right|^p \right)^{q/p} \right)^{1/q} \tag{4.58}$$

for all $t > 0$ such that

$$\frac{1}{p} < \frac{t}{d} + \frac{1}{r} \quad \text{and} \quad t < \min\{s, n\}, \tag{4.59}$$

where $n-1$ is the order of polynomial exactness of S_j and s is such that $\psi_\lambda \in B^s_{q_0}(L^p(\Omega))$ for all $\lambda \in \mathcal{J}$ and some $q_0 > 0$.

Chapter 5

Construction of Wavelets on Bounded Domains

At present many constructions of wavelet bases on fairly general domains with prescribed boundary conditions are available. However, their efficiency seems to be dependent on the type of application. In this chapter, we are mainly concerned with wavelet systems which are suitable for solving operator equations. The important properties are smoothness and local support of basis functions, sparseness of refinement matrices and validity of the Jackson and Bernstein inequalities. The main difficulty is the large condition of wavelet bases resulting in a bad numerical stability and bad spectral properties of the corresponding stiffness matrices. The construction of wavelets on a fairly general domain Ω is typically achieved by the following steps

$$\mathbb{R} \to (0,1) \to (0,1)^d \to \Omega.$$

Wavelets on the real line are adapted to the interval by restriction and adaptation at the edges. This adaptation and the prescription of boundary conditions is a delicate task. By a tensor product technique wavelet bases are designed on n-dimensional cube. Finally, there are several strategies for adapting them to a bounded domain $\Omega \subset \mathbb{R}^d$, namely fictious domain method or domain decomposition methods. These issues will be discussed below.

5.1 Wavelets on the Interval

Starting from a pair of biorthogonal wavelets on the line, there are many constructions of biorthogonal wavelet bases on the interval $(0,1)$. Our goal is to preserve the important properties of wavelets on the line such as the biorthogonality, local support of basis functions, sparseness of refinement matrices as well as smoothness and vanishing moments ensuring the validity of the Jackson and Bernstein inequalities. First of all, we describe some methods and obstructions which arise when analyzing a function defined on the unit interval $(0,1)$ from the viewpoint of numerical treatment of operator equations.

Extension by zero
A square integrable function f supported on $\langle 0,1\rangle$ can be extended by zero to the whole line. This extension then belongs to $L^2(\mathbb{R})$ and can be analyzed using wavelet basis on the line. This naive approach has a serious drawback. There is typically a discontinuity in f at 0 and 1, which is reflected by many large wavelet coefficients near the boundary.

Restrictions
A wavelet basis could be obtained by restricting all those functions to $(0,1)$ whose support intersect the interval. This basis inherits many useful properties such as polynomial exactness, smoothness and locality. However, the biorthogonality on the real line does not imply biorthogonality of the restrictions. Moreover, there are in general some small overlapping pieces at the edges which causes serious numerical instabilities in the process of biorthogonalization.

Periodization
For $f \in L^2(\mathbb{R})$ we define its *periodized version* by

$$f^{per}(x) = \sum_{k\in\mathbb{Z}} f(x-k). \tag{5.1}$$

The periodization $\phi_{j,k}$ and $\psi_{j,k}$ are then denoted by $\phi_{j,k}^{per}$ and $\psi_{j,k}^{per}$, respectively. It is clear that $\phi_{j,k}^{per} = \phi_{j,k+2^j}^{per}$, so for given scale $j \geq 0$ we have 2^j different periodized functions $\phi_{j,k}^{per}$. The system $\left\{V_j^{per}\right\}_{j\geq 0}$, where

$$V_j^{per} := span\left\{\phi_{j,k}^{per} | k = 0 \dots 2^j - 1\right\} \tag{5.2}$$

is called *periodized MRA* and

$$\bigcup_{j\geq 0} V_j^{per} = L^2(0,1). \tag{5.3}$$

Since

$$\int_0^1 f(x)\phi_{j,k}^{per}(x)dx = \int_{-\infty}^{\infty} \tilde{f}(x)\phi_{j,k}(x)dx, \tag{5.4}$$

where

$$\tilde{f}(x+l) := f(x) \quad \text{for } x \in \langle 0,1\rangle, l \in \mathbb{Z}, \tag{5.5}$$

expanding a function on $\langle 0,1\rangle$ into periodized wavelets is equivalent to periodic extension of function and analyzing this extension with the standard wavelets. It follows that this approach has the same drawback as the extension by zero unless the extension $\tilde{f}$ was smooth enough. Thus, bases obtained via periodization are well-suited only for periodic problems.

Biorthogonal bases reproducing polynomials

For above reasons other constructions were proposed. The main idea is to retain most of the inner functions, i.e. the scaling functions and wavelets whose supports belongs to $(0, 1)$, and to treat the boundary scaling functions and wavelets separately. In [42, 53] the following construction of stable wavelet bases which preserves the polynomial exactness was proposed. Let $\mathrm{supp}(\phi) = \langle 0, N\rangle$ and $\tilde{m}$ be the order of polynomial exactness of ϕ. We define V_j as the space generated by the interior functions $\phi_{j,k}, \quad M_1 \le k \le 2^j - M_2$, for some fixed $M_1 \ge 0$ and $M_2 \ge N$, and the left and right edge functions $\phi^L_{j,q}$, $\phi^R_{j,q}$, $q = 0, \dots, \tilde{m}$ defined by

$$\begin{aligned} \phi^L_{j,q} &= 2^{j(1/2-q)} \sum_{k<M_1} \left\langle (.)^q , \tilde{\phi}_{j,k} \right\rangle \phi_{j,k}, \\ \phi^R_{j,q} &= 2^{j(1/2-q)} \sum_{k>2^j-M_2} \left\langle (1-.)^q , \tilde{\phi}_{j,k} \right\rangle \phi_{j,k}. \end{aligned}$$

Due to the particular structure of coefficients of these linear combinations, these functions are independent and reproduce polynomials up to order $\tilde{m} - 1$. It can be shown that these functions are better adapted for orthonormalization or biorthogonalization and that the spaces V_j are nested. Boundary wavelets are constructed by orthonormalization of the functions $\phi^L_{j+1,q} - P_j \phi^L_{j+1,q}$ and $\phi^R_{j+1,q} - P_j \phi^R_{j+1,q}$ or in the case of biorthogonal wavelets by a method called *stable completion*. We refer to [36, 42, 53] for more details. In the case of orthonormal wavelets this approach leads to wavelet bases well-suited for some applications. However, in the case of bases which are biorthogonal but not orthonormal such as B-spline wavelet bases, their condition is typically too large.

5.2 Tensor Product Wavelets

A wavelet basis on the unit cube $\Box = (0,1)^d$ is built from the univariate wavelet basis by a tensor product. We define the multivariate scaling functions by

$$\phi^{\Box}_{j,k}(x) := \Pi^d_{l=1} \phi_{j,k_l}(x_l), \quad x = (x_1, \dots, x_d) \in \Box, \tag{5.6}$$

with $k = (k_1, \dots, k_d)$ now being a multiindex, $k \in \mathcal{I}^{\Box}_j := \mathcal{I}_j \times \dots \times \mathcal{I}_j$. We introduce the abbreviation

$$\mathcal{J}_{j,e} := \begin{cases} \mathcal{J}_j, & e = 1, \\ \mathcal{I}_j & e = 0, \end{cases}, \tag{5.7}$$

i.e. the parameter e allows to distinguish between scaling functions and wavelets. Furthermore, we denote

$$\mathcal{J}^{\Box}_{j,e} := \mathcal{J}_{j,e_1} \times \dots \times \mathcal{J}_{j,e_d}, \quad \mathcal{J}^{\Box}_j := \bigcup_{e \in E} \mathcal{J}_{j,e}, \quad E := \{0,1\}^d - \{0,0\}. \tag{5.8}$$

For any $e = (e_1, \dots, e_d) \in E$, $j \ge j_0$ and $k = (k_1, \dots, k_d) \in \mathcal{J}^{\Box}_j$, we define the multivariate wavelet

$$\psi_\lambda(x) := \Pi^d_{l=1} \psi_{j,e_l,k_l}(x_l), \quad x = (x_1, \dots, x_d) \in \Box, \quad \lambda = (j, e, k), \tag{5.9}$$

where

$$\psi_{j,e_l,k_l} := \begin{cases} \phi_{j,k_l}, & e_l = 1, \\ \psi_{j,k_l}, & e_l = 0. \end{cases} \tag{5.10}$$

The wavelet basis on the unit cube $\square$ is then given by

$$\Psi^{\square} = \left\{\psi_{\lambda}^{\square},\ \lambda \in \mathcal{J}_j^{\square}, j \geq j_0\right\} \cup \left\{\phi_{j_0,k},\ k \in \mathcal{I}_j^{\square}\right\}. \tag{5.11}$$

The dual wavelet basis $\tilde{\Psi}^{\square}$ is defined similarly. Obviously, the regularity γ and $\tilde{\gamma}$ and the full degree of polynomial exactness N and $\tilde{N}$ is preserved on the primal and dual side, respectively.

5.3 Wavelet Bases and Wavelet Frames on General Domain

Most of the constructions of a wavelet basis on a general domain $\Omega \subset \mathbb{R}^d$ consist in splitting Ω into not-overlapping subdomains Ω_i, which are images of the reference element $\square = (0,1)^d$ under appropriate parametric mappings κ_i. More precisely,

$$\overline{\Omega} = \bigcup_{i=1}^{M} \overline{\Omega}_i, \quad \Omega_i = \kappa_i\left(\square\right),\ i = 1, \ldots, M, \tag{5.12}$$

where κ_i are considered to be sufficiently smooth. The multiresolution analysis on Ω is obtained by transformations of systems on $\square$ and glueing together the boundary functions across the inter-faces between adjacent subdomains. Such wavelet bases constructed for a variety of operator equations can be found in [21, 22, 23, 43, 58, 59, 60, 69, 93]. In some cases it can be very difficult to find the parametric mappings κ_i. For this reason in some applications a frame instead of a wavelet basis is used, see e.g. [47, 91]. A frame on a bounded domain can be obtained by a union of wavelet bases on the overlapping subdomains, which are lifted tensor products of a basis on the unit interval. Such constructed frame is called an aggregated Gelfand frame. Hence, the construction of wavelet frames is much simpler than the construction of wavelet bases. For details, we refer to [47, 83, 91].

Chapter 6

Construction of Stable B-spline Wavelet Bases on the Interval

In this chapter, we are concerned with the construction of wavelet bases on the interval derived from B-splines. The resulting bases generate multiresolution analyses on the unit interval with the desired number of vanishing wavelet moments for primal and dual wavelets. Inner wavelets are translated and dilated versions of well-known wavelets designed by Cohen, Daubechies, Feauveau [41] whereas the construction of boundary wavelets is along the lines of [53, 80]. The disadvantage of popular bases from [53] is their bad condition which causes problems in practical applications. Some modifications which lead to better conditioned bases were proposed in [4, 54, 80, 97]. Our objective is now to modify the construction of B-spline wavelet bases on the interval from [53] in order to improve their condition. Then we adapt the constructed bases to the complementary boundary conditions. Quantitative properties of these bases and numerical experiments are presented in Chapter 8.
First of all, we summarize the desired properties:

- *Riesz basis property.* The functions form a Riesz basis of the space $L^2(\langle 0, 1\rangle)$.
- *Locality.* The basis functions are local in the sense of (4.3).
- *Biorthogonality.* The primal and dual wavelet bases form a biorthogonal pair.
- *Polymial exactness.* The primal MRA has polynomial exactness of order N and the dual MRA has polynomial exactness of order $\tilde{N}$. As in [41], $N + \tilde{N}$ has to be even and $\tilde{N} \geq N$.
- *Smoothness.* The smoothness of primal and dual wavelet bases is another desired property. It ensures the validity of the Bernstein inequality (4.30).
- *Closed form.* The primal scaling functions and wavelets are known in the closed form. It is desirable property for the fast computation of integrals involving primal scaling functions and wavelets.

- *Well-conditioned bases.* Our objective is to construct wavelet bases with improved condition number, especially for larger values of N and $\tilde{N}$.

From the viewpoint of numerical stability, ideal wavelet bases are orthogonal wavelet bases. However, they are usually avoided in numerical treatment of partial differential equations, because they are not accessible analytically, the complementary boundary conditions can not be satisfied and it is not possible to increase the number of vanishing wavelet moments independent from the order of accuracy.
Biorthogonal B-spline wavelet bases on the unit interval were constructed in [53]. Their advantage is that the primal basis is known explicitly. Bases constructed in [33] are well-conditioned, but have globally supported dual basis functions. B-spline bases from [53] have large condition number that cause problems in practical applications. Some modifications which lead to better conditioned bases were proposed in [4, 54, 97]. We should also mention the construction in [11], though the corresponding dual bases are unknown so far. The recent construction in [80] seems to outperform the previous constructions for the spline order $N \leq 3$, for some numerical experiments with these bases see [46, 79, 83]. In this chapter, we combine approaches from [33, 53, 80] to further improve stability properties of B-spline wavelet bases on the interval, especially for larger values of N and $\tilde{N}$.

6.1 Primal Scaling Basis

The primal scaling bases will be the same as bases designed by Chui and Quak in [33], because they are known to be well-conditioned. A big advantage of this approach is that it readily adapts to the bounded interval by introducing multiple knots at the endpoints. Let N be the desired order of polynomial exactness of primal scaling basis and let $\mathbf{t}^j = \left(t_k^j\right)_{k=-N+1}^{2^j+N-1}$ be a *Schoenberg sequence of knots* defined by

$$\begin{aligned} t_k^j &:= 0, \qquad k = -N+1, \ldots, 0, \\ t_k^j &:= \frac{k}{2^j}, \qquad k = 1, \ldots, 2^j - 1, \\ t_k^j &:= 1, \qquad k = 2^j, \ldots, 2^j + N - 1. \end{aligned} \tag{6.1}$$

The corresponding B-splines of order N are defined by

$$B_{k,N}^j(x) := \left(t_{k+N}^j - t_k^j\right) \left[t_k^j, \ldots, t_{k+N}^j\right]_t (t-x)_+^{N-1}, \quad x \in \langle 0, 1 \rangle, \tag{6.2}$$

where $(x)_+ := \max\{0, x\}$. The symbol $[t_k, \ldots t_{k+N}]_t f$ is the N-th divided difference of f which is recursively defined as

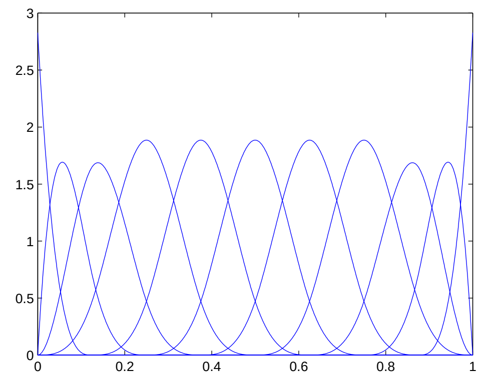
$$\begin{aligned} [t_k, \ldots, t_{k+N}] f &= \frac{[t_{k+1}, \ldots, t_{k+N}] f - [t_k, \ldots, t_{k+N-1}] f}{t_{k+N} - t_k} \quad \text{if} \quad t_k \neq t_{k+N}, \\ &= \frac{f^{(N)}(t_k)}{N!} \quad \text{if} \quad t_k = t_{k+N}, \end{aligned}$$

with $[t_k] f = f(t_k)$.
The set $\Phi_j = \{\phi_{j,k}, k = -N+1, \ldots, 2^j - 1\}$ of primal scaling functions is then simply defined by

$$\phi_{j,k} = 2^{j/2} B_{k,N}^j, \quad k = -N+1, \ldots, 2^j - 1, \quad j \geq 0. \tag{6.3}$$

Thus there are $2^j - N + 1$ inner scaling functions and $N - 1$ functions at each boundary. Figure 6.1 shows the primal scaling functions for $N = 4$ and $j = 3$. Inner scaling functions are translations and dilations of one function ϕ which corresponds to primal scaling function constructed by Cohen, Daubechies, Feauveau in [41], see also Section 1.3. In the following, we consider ϕ from [41] which is shifted so that its support is $[0, N]$.

Figure 6.1: Primal scaling functions for $N = 4$ and $j = 3$ without boundary conditions.



We define the primal multiresolution spaces by

$$S_j = \text{span } \Phi_j. \tag{6.4}$$

Lemma 44. *Under the above assumptions, the following holds:*
i) For any $j_0 \in \mathbb{N}$ the sequence $\mathcal{S} = \{S_j\}_{j \geq j_0}$ forms a multiresolution analysis of $L^2(\langle 0, 1 \rangle)$.
ii) The spaces S_j are exact of order N, i.e.

$$\Pi_N([0,1]) \subset S_j, \quad j \geq 1. \tag{6.5}$$

The proof can be found in [33, 80, 88].

6.2 Dual Scaling Basis

The desired property of dual scaling basis $\tilde{\Phi}$ is the biorthogonality to Φ and the polynomial exactness of order $\tilde{N}$. Let $\tilde{\phi}$ be dual scaling function which was designed by Cohen, Daubechies, and Feauveau in [41] and which is shifted so that $\left\langle \phi, \tilde{\phi} \right\rangle = 0$, i.e. its support is $\left[-\tilde{N}+1, N+\tilde{N}-1\right]$, see also Section 1.3. In this case $\tilde{N} \geq N$ and $\tilde{N}+N$ has to be an even number. In the sequel, we assume that

$$j \geq j_0 := \left\lceil \log_2 \left(N + 2\tilde{N} - 3 \right) \right\rceil \tag{6.6}$$

so that the supports of the boundary functions are contained in $[0,1]$. We define inner scaling functions as translations and dilations of $\tilde{\phi}$:

$$\theta_{j,k} = 2^{j/2} \tilde{\phi} \left(2^j \cdot -k \right), \quad k = \tilde{N}-1, \ldots, 2^j - N - \tilde{N} + 1. \tag{6.7}$$

There will be two types of basis functions at each boundary. In the following, it will be convenient to abbreviate the boundary and inner index sets by

$$\begin{aligned}
\mathcal{I}_j^{L,1} &= \left\{ -N+1, \ldots, -N+\tilde{N} \right\}, &(6.8)\\
\mathcal{I}_j^{L,2} &= \left\{ -N+\tilde{N}+1, \ldots, \tilde{N}-2 \right\}, &(6.9)\\
\mathcal{I}_j^{0} &= \left\{ \tilde{N}-1, \ldots, 2^j - N - \tilde{N} + 1 \right\}, &(6.10)\\
\mathcal{I}_j^{R,2} &= \left\{ 2^j - N - \tilde{N} + 2, \ldots 2^j - \tilde{N} - 1 \right\}, &(6.11)\\
\mathcal{I}_j^{R,1} &= \left\{ 2^j - \tilde{N}, \ldots, 2^j - 1 \right\}, &(6.12)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}_j^{L} &= \mathcal{I}_j^{L,1} \cup \mathcal{I}_j^{L,2} = \left\{ -N+1, \ldots, \tilde{N}-2 \right\}, &(6.13)\\
\mathcal{I}_j^{R} &= \mathcal{I}_j^{R,2} \cup \mathcal{I}_j^{R,1} = \left\{ 2^j - N - \tilde{N} + 2, \ldots, 2^j - 1 \right\}, &(6.14)\\
\mathcal{I}_j &= \mathcal{I}_j^{L,1} \cup \mathcal{I}_j^{L,2} \cup \mathcal{I}_j^{0} \cup \mathcal{I}_j^{R,2} \cup \mathcal{I}_j^{R,1} = \left\{ -N+1, \ldots, 2^j - 1 \right\}. &(6.15)
\end{aligned}$$

Basis functions of the first type are defined to preserve polynomial exactness by the same way as in [42, 53] :

$$\theta_{j,k} = 2^{j/2} \sum_{l=-N-\tilde{N}+2}^{\tilde{N}-2} \langle p_{k+N-1}, \phi(\cdot - l) \rangle \, \tilde{\phi} \left(2^j \cdot -l \right) |_{[0,1]}, \quad k \in \mathcal{I}_j^{L,1}. \tag{6.16}$$

Here $\left\{ p_0, \ldots, p_{\tilde{N}-1} \right\}$ is a basis of $\Pi_{\tilde{N}}([0,1])$. In our case, p_k are the Bernstein polynomials defined by

$$p_k(x) := b^{-\tilde{N}+1} \binom{\tilde{N}-1}{k} x^k (b-x)^{\tilde{N}-1-k}, \quad k = 0, \ldots, \tilde{N}-1, \tag{6.17}$$

because they are known to be well-conditioned on $[0,b]$ relative to the supremum norm. The Bernstein polynomials were used also in [53]. On the contrary to [53], in our case the

choice of polynomials does not affect the resulting dual scaling basis $\tilde{\Psi}$, but it has only the effect of stabilization of the computation, for details see Lemma 47 and the discussion below.
The basis functions of the second type are defined as

$$\theta_{j,k} = 2^{\frac{j}{2}} \sum_{l=\tilde{N}-1-2k}^{N+\tilde{N}-1} \tilde{h}_l \tilde{\phi}\left(2^{j+1} \cdot -2k - l\right)|_{[0,1]}, \quad k \in \mathcal{I}_j^{L,2}, \tag{6.18}$$

where $\tilde{h}_i$ are the scaling coefficients corresponding to the scaling function $\tilde{\phi}$ given by (1.64). The boundary functions at the right boundary are defined to be symmetric with the left boundary functions:

$$\theta_{j,k} = \theta_{j,2^j-k}\left(1-\cdot\right), \quad k \in \mathcal{I}_j^R. \tag{6.19}$$

It is easy to see that

$$\theta_{j+1,k} = 2^{1/2}\theta_{j,k}\left(2\cdot\right), \quad k \in \mathcal{I}_j^L \tag{6.20}$$

for left boundary functions and

$$\theta_{j+1,k}\left(1-\cdot\right) = 2^{1/2}\theta_{j,k}\left(1-2\cdot\right), \quad k \in \mathcal{I}_j^R \tag{6.21}$$

for right boundary functions.
Since the set $\Theta_j := \{\theta_{j,k}, k \in \mathcal{I}_j\}$ is not biorthogonal to Φ_j, we derive a new set

$$\tilde{\Phi}_j := \left\{\tilde{\phi}_{j,k}, k \in \mathcal{I}_j\right\} \tag{6.22}$$

from Θ_j by biorthogonalization. Let

$$\mathbf{\Gamma}_j = \left(\langle \phi_{j,k}, \theta_{j,l} \rangle\right)_{k,l \in \mathcal{I}_j}. \tag{6.23}$$

Then viewing $\tilde{\Phi}_j$ and Θ_j as column vectors we define

$$\tilde{\Phi}_j := \mathbf{\Gamma}_j^{-T}\Theta_j, \tag{6.24}$$

assuming that $\mathbf{\Gamma}_j$ is invertible, which is the case of all choices of N and $\tilde{N}$ considered in Chapter 8.
Then $\tilde{\Phi}_j$ is biorthogonal to Φ_j, because

$$\left\langle \Phi_j, \tilde{\Phi}_j \right\rangle = \left\langle \Phi_j, \mathbf{\Gamma}_j^{-T}\Theta_j \right\rangle = \mathbf{\Gamma}_j \mathbf{\Gamma}_j^{-1} = \mathbf{I}. \tag{6.25}$$

Lemma 45. [31] *i) Let Φ_j, Θ_j be defined as above. Then the matrices*

$$\mathbf{\Gamma}_{j,L} = \left(\langle \phi_{j,k}, \theta_{j,l} \rangle\right)_{k,l \in \mathcal{I}_j^L} \quad \text{and} \quad \mathbf{\Gamma}_{j,R} = \left(\langle \phi_{j,k}, \theta_{j,l} \rangle\right)_{k,l \in \mathcal{I}_j^R} \tag{6.26}$$

are independent of j, i.e. there are matrices $\mathbf{\Gamma}_L$, $\mathbf{\Gamma}_R$ such that

$$\mathbf{\Gamma}_{j,L} = \mathbf{\Gamma}_L, \quad \mathbf{\Gamma}_{j,R} = \mathbf{\Gamma}_R. \tag{6.27}$$

Moreover, the matrix $\mathbf{\Gamma}_R$ *results from the matrix* $\mathbf{\Gamma}_L$ *by reversing the ordering of rows and columns, which means that*

$$(\mathbf{\Gamma}_R)_{k,l} = (\mathbf{\Gamma}_L)_{2^j-N-k,2^j-N-l}\,, \quad k,l \in \mathcal{I}_j^R. \tag{6.28}$$

ii) The following holds:

$$(\mathbf{\Gamma}_j)_{k,l} = \delta_{k,l}, \quad k \in \mathcal{I}_j,\ l \in \mathcal{I}_j^0. \tag{6.29}$$

iii) The following holds:

$$(\mathbf{\Gamma}_j)_{k,l} = 0, \quad k \in \mathcal{I}_j^0,\ l \in \mathcal{I}_j^L \cup \mathcal{I}_j^R. \tag{6.30}$$

Proof. Due to (6.20) and by substitution we have for $k,l \in \mathcal{I}_j^L$

$$\langle \phi_{j,k}, \theta_{j,l}\rangle = \left\langle 2^{\frac{j-j_0}{2}}\phi_{j_0,k}\left(2^{j-j_0}\cdot\right), 2^{\frac{j-j_0}{2}}\theta_{j_0,l}\left(2^{j-j_0}\cdot\right)\right\rangle = \langle \phi_{j_0,k}, \theta_{j_0,l}\rangle\,. \tag{6.31}$$

Therefore, $\mathbf{\Gamma}_{j,L} = \mathbf{\Gamma}_{j_0,L} = \mathbf{\Gamma}_L$, i.e. the matrix $\mathbf{\Gamma}_{j,L}$ is independent of j. Due to (6.21) $\mathbf{\Gamma}_{j,R}$ is independent of j too. The property (6.28) is a direct consequence of the reflection invariance (6.19).

The property *ii*) follows from the biorthogonality of $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ and $\left\{\tilde{\phi}(\cdot - l)\right\}_{l\in\mathbb{Z}}$. It also implies (6.30) for $k \in \mathcal{I}_j^0$, $l \in \mathcal{I}_j^{L,1} \cup \mathcal{I}_j^{R,1}$. It remains to prove (6.30) for $k \in \mathcal{I}_j^0$, $l \in \mathcal{I}_j^{L,2} \cup \mathcal{I}_j^{R,2}$. By the definition of dual scaling functions of the second type (6.18), refinement relation (1.12) for the dual scaling function $\tilde{\phi}$ and Lemma 7, we have for $k \in \mathcal{I}_j^0$, $l \in \mathcal{I}_j^{L,2}$,

$$\begin{aligned}
\langle \phi_{j,k}, \theta_{j,l}\rangle &= \left\langle \phi(\cdot - k), \sqrt{2}\sum_{m=\tilde{N}-1-2k}^{N+\tilde{N}-1} \tilde{h}_l\tilde{\phi}(2\cdot -2l - m)\,|_{[0,1]}\right\rangle && (6.32)\\
&= 2\left\langle \sum_{n=0}^{N} h_n\phi(2\cdot -2k-n), \sum_{m=\tilde{N}-1-2k}^{N+\tilde{N}-1} \tilde{h}_m\tilde{\phi}(2\cdot -2l-m)\,|_{[0,1]}\right\rangle && (6.33)\\
&= 2\sum_{n=0}^{N}\sum_{m=\tilde{N}-1-2k}^{N+\tilde{N}-1} h_n\tilde{h}_m\delta_{2k+n,2l+m} = 2\sum_{m=\tilde{N}-1-2k}^{N+\tilde{N}-1} h_{2l-2k+m}\tilde{h}_m && (6.34)\\
&= 2\sum_{m\in\mathbb{Z}} h_{2l-2k+m}\tilde{h}_m = 0. && (6.35)
\end{aligned}$$

By (6.19), the relation (6.30) holds for $k \in \mathcal{I}_j^0$, $l \in \mathcal{I}_j^{R,2}$ is similar.

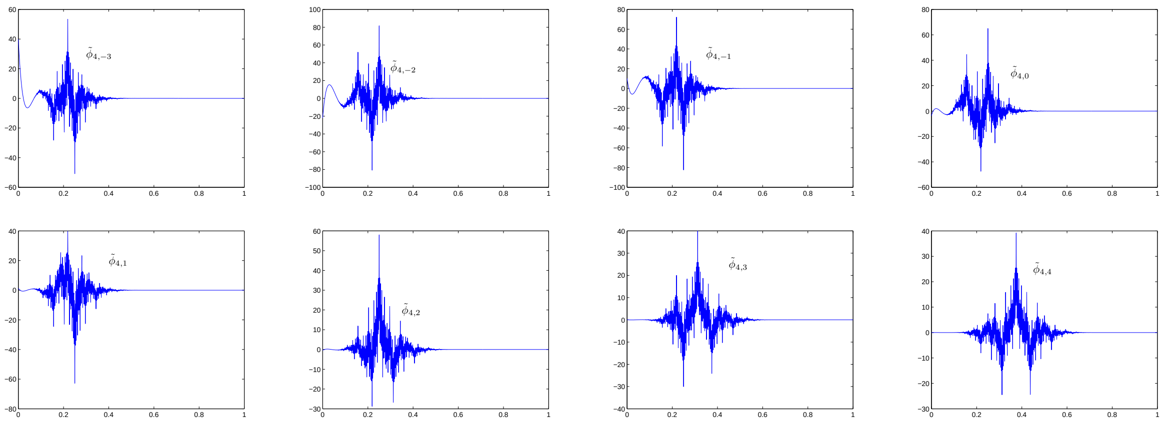
□

Thus, we can write

$$\tilde{\Phi}_j := \mathbf{\Gamma}_j^{-T}\Theta_j = \begin{pmatrix} \mathbf{\Gamma}_L & & \\ & \mathbf{I}_{\#\mathcal{I}_j^0} & \\ & & \mathbf{\Gamma}_R \end{pmatrix}^{-T}\Theta_j = \begin{pmatrix} \mathbf{\Gamma}_L^{-T} & & \\ & \mathbf{I}_{\#\mathcal{I}_j^0} & \\ & & \mathbf{\Gamma}_R^{-T} \end{pmatrix}\Theta_j. \tag{6.36}$$

Since the matrix $\mathbf{\Gamma}_j$ is symmetric in the sense of (6.28), the properties (6.19), (6.20), and (6.21) hold for $\tilde{\phi}_{j,k}$ as well.

Figure 6.2: Boundary dual scaling functions for $N = 4$ without boundary conditions.



Remark 46. *As mentioned in Chapter 1, the scaling function $\tilde{\phi}$ has typically a low Sobolev regularity for smaller values of $\tilde{N}$. Thus the functions $\theta_{j,k}$ have a low Sobolev regularity for smaller values of $\tilde{N}$, too. Therefore, we do not obtain the sufficiently accurate entries of the matrix $\mathbf{\Gamma}_j$ directly by classical quadratures. Fortunately, we are able to compute the matrix $\mathbf{\Gamma}_j$ precisely up to the round off errors. For $k \in \mathcal{I}_j^{L,1} \cup \mathcal{I}_j^{L,2}$, $l \in \mathcal{I}_j^{L,1}$ we have*

$$\langle \phi_{j,k}, \theta_{j,l} \rangle = \sum_{m=-N-\tilde{N}+2}^{\tilde{N}-2} \sum_{n=0}^{\tilde{N}-1} c_{l,n} \left\langle (\cdot)^n, \phi(\cdot - m) \right\rangle \left\langle \phi(\cdot - k), \tilde{\phi}(\cdot - m) \right\rangle_{L^2(\langle 0,1 \rangle)}, \tag{6.37}$$

with $c_{l,n}$ given by (6.49)*. Since ϕ is a piecewise polynomial function and $\tilde{\phi}$ is refinable, for $k \in \mathcal{I}_j^{L,1} \cup \mathcal{I}_j^{L,2}$, $l \in \mathcal{I}_j^{L,1}$ we can compute the entries of $\mathbf{\Gamma}_j$ by the method from the Chapter 3. By refinement relation we easily obtain the following relations for the computation of the remaining entries of $\mathbf{\Gamma}^L$:*

$$\begin{aligned} \langle \phi_{j,k}, \theta_{j,l} \rangle &= \sum_{m=\tilde{N}-1-2l}^{N+\tilde{N}-1} \tilde{h}_m \left\langle \phi_{0,k}, \tilde{\phi}(\cdot - 2k - m) \right\rangle, \quad k = -N+1, \ldots, -1,\ l \in \mathcal{I}_j^{L,2}, \\ &= 2^{-1} \sum_{m=\tilde{N}-1-2l}^{N+\tilde{N}-1} h_{2k-2l+m} \tilde{h}_m, \qquad k = 0, \ldots, \tilde{N}-2,\ l \in \mathcal{I}_j^{L,2}. \end{aligned}$$

Since the submatrix $\mathbf{\Gamma}_R$ is obtained from a matrix $\mathbf{\Gamma}_L$ by reversing the ordering of rows and columns, the matrix $\mathbf{\Gamma}_j$ can be indeed computed precisely up to the round off errors.

Now we show that the resulting dual scaling basis $\tilde{\Phi}$ does not depend on the choice of bases of the space $\Pi_{\tilde{N}}([0,1])$ in the formula (6.16).

Lemma 47. [31] *We suppose that* $P^1 = \left\{p_0^1, \ldots, p_{\tilde{N}-1}^1\right\}$, $P^2 = \left\{p_0^2, \ldots, p_{\tilde{N}-1}^2\right\}$ *are two different bases of the space* $\Pi_{\tilde{N}}([0,1])$ *and for* $i = 1, 2$ *we define the sets* $\Theta_j^i = \left\{\theta_{j,k}^i\right\}_{k=-N+1}^{2^j-1}$ *by*

$$\begin{aligned}
\theta_{j,k}^i &= 2^{j/2} \sum_{l=-N-\tilde{N}+2}^{\tilde{N}-2} \left\langle p_{k+N-1}^i, \phi(\cdot - l)\right\rangle \tilde{\phi}\left(2^j \cdot -l\right)|_{[0,1]}, && k \in \mathcal{I}_j^{L,1}, \\
\theta_{j,k}^i &= \theta_{j,2^j-N-k}^i, && k \in \mathcal{I}_j^{R,1}, \\
\theta_{j,k}^i &= \theta_{j,k}, && k \in \mathcal{I}_j^{L,2} \cup \mathcal{I}_j^0 \cup \mathcal{I}_j^{R,2}.
\end{aligned}$$

Furthermore, we define

$$\boldsymbol{\Gamma}_j^i = \left\langle \Phi_j, \Theta_j^i \right\rangle, \quad \tilde{\Phi}_j^i = \left(\boldsymbol{\Gamma}_j^i\right)^{-T} \Theta_j^i, \quad i = 1, 2. \tag{6.38}$$

Then $\tilde{\Phi}_j^1 = \tilde{\Phi}_j^2$.

Proof. Since P^1 and P^2 are both bases of the space $\Pi_{\tilde{N}}([0,1])$, there exists a regular matrix $\mathbf{B}_L$ such that $P^2 = \mathbf{B}_L P^1$. The consequence is that

$$\Theta^2 = \mathbf{B}_j \Theta^1, \tag{6.39}$$

with

$$\mathbf{B}_j = \begin{pmatrix} \mathbf{B}_L & & \\ & \mathbf{I}_{\#\mathcal{I}_j^0} & \\ & & \mathbf{B}_R \end{pmatrix}, \tag{6.40}$$

where $\mathbf{B}_R$ results from a matrix $\mathbf{B}_L$ by reversing the ordering of rows and columns, which means that

$$(\mathbf{B}_R)_{k,l} = (\mathbf{B}_L)_{2^j-N-k, 2^j-N-l}, \quad k, l \in \mathcal{I}_j^{L,1}. \tag{6.41}$$

Therefore, we have

$$\tilde{\Phi}_j^2 = \left(\boldsymbol{\Gamma}_j^2\right)^{-T} \Theta_j^2 = \left(\boldsymbol{\Gamma}_j^1\right)^{-T} \mathbf{B}_j^{-1} \mathbf{B}_j \Theta_j^1 = \tilde{\Phi}_j^1. \tag{6.42}$$

□

Although the choice of polynomial basis does not influence the resulting dual scaling basis, it has an influence on the stability of the computation and the preciseness of the results, because some choices of the polynomial bases leads to the critical values of the condition number of the biorthogonalization matrix. We present the condition numbers of the matrix $\boldsymbol{\Gamma}_L$ for the monomial basis $\left\{1, x, x^2, \ldots x^{\tilde{N}-1}\right\}$ and Bernstein polynomials (6.17) with the parameters $b = 4$ and $b = 10$ in Table 6.2. In our numerical experiments in Chapter 8 we choose $b = 10$.

Remark 48. *In the case of linear primal basis, i.e. $N = 2$, there are no boundary dual functions of the second type. In [80] the primal scaling functions and the inner dual scaling functions are the same as ours. The boundary dual functions before biorthogonalizations are defined by (6.16) with the same choice of polynomials $p_0, \ldots, p_{\tilde{N}-1}$ as in [43]. Due to the Lemma 47, for $N = 2$ the wavelet basis in [80] is identical to the wavelet basis constructed in this chapter.*

For the proof of Theorem 50 below and also for deriving of refinement matrices we will need the following lemma.

Lemma 49. [31] *For the left boundary functions of the first type there exist refinement coefficients $m_{n,k}$, $k \in \mathcal{I}_j^{L,1}$, $n \in \mathcal{I}_j^{L,1} \cup \mathcal{I}_j^3$, $\mathcal{I}_j^3 := \left\{\tilde{N} - 1, \ldots, 3\tilde{N} + N - 5\right\}$ such that*

$$\theta_{j,k} = \sum_{n=-N+1}^{-N+\tilde{N}} m_{n,k}\theta_{j+1,n} + \sum_{n=\tilde{N}-1}^{3\tilde{N}+N-5} m_{n,k}\theta_{j+1,n}, \quad k \in \mathcal{I}_j^{L,1}. \tag{6.43}$$

Proof. Let $\Theta_j^0 = \left\{\theta_{j,k}, k \in \mathcal{I}_j^3\right\}$ and $\Theta_j^{L,1,mon} = \left\{\theta_{j,k}^{mon}, k \in \mathcal{I}_j^{L,1}\right\}$ be defined by

$$\theta_{j,k}^{mon} = 2^{j/2} \sum_{l=-N-\tilde{N}+2}^{\tilde{N}-2} \left\langle (\cdot)^i, \phi(\cdot - l)\right\rangle \tilde{\phi}\left(2^j \cdot -l\right)|_{[0,1]}, \quad k \in \mathcal{I}_j^{L,1}. \tag{6.44}$$

Then

$$\Theta_j^{L,1,mon} = \left(\mathbf{M}^{mon}\right)^T \begin{pmatrix} \Theta_{j+1}^{L,1,mon} \\ \Theta_{j+1}^0 \end{pmatrix}, \tag{6.45}$$

Table 6.1: Condition numbers of the matrices $\mathbf{\Gamma}_L$

N	$\tilde{N}$	mon.	$b=4$	$b=10$	N	$\tilde{N}$	mon.	$b=4$	$b=10$
2	2	6.68e+00	9.94e+00	3.16e+01	4	4	2.46e+04	6.75e+02	1.33e+04
2	4	4.66e+02	1.94e+01	9.48e+02	4	6	1.30e+07	2.94e+04	7.34e+04
2	6	1.40e+05	1.00e+02	4.47e+03	4	8	1.24e+10	6.24e+06	9.42e+04
2	8	1.03e+08	8.52e+03	5.81e+03	4	10	1.92e+13	2.26e+09	5.24e+04
2	10	1.48e+11	1.67e+06	1.58e+03	5	5	5.34e+06	3.29e+04	1.26e+05
3	3	2.18e+02	1.07e+02	1.00e+03	5	7	5.62e+09	6.91e+06	3.73e+05
3	5	3.73e+04	1.88e+02	1.05e+04	5	9	9.39e+12	2.57e+09	3.47e+05
3	7	1.64e+07	1.20e+04	2.26e+04	6	6	1.20e+09	3.68e+06	6.81e+05
3	9	1.54e+10	2.90e+06	1.33e+04	6	8	2.97e+12	1.92e+09	1.81e+06

where the refinement matrix $\mathbf{M}^{mon} = \left\{m_{n,k}^{mon}\right\}_{n\in\mathcal{I}_j^{L,1}\cup\mathcal{I}_j^3, k\in\mathcal{I}_j^{L,1}}$ is given by

$$\begin{aligned} m_{n,k}^{mon} &= \frac{1}{\sqrt{2}}2^{-k}, \quad k=n,\ n\in\mathcal{I}_j^{L,1}, && (6.46)\\ &= \frac{1}{\sqrt{2}}\sum_{q=\left\lceil\frac{n-N-\tilde{N}+1}{2}\right\rceil}^{\tilde{N}-2}\left\langle(\cdot)^{k+N-1},\phi(\cdot-q)\right\rangle\tilde{h}_{n-2q}, \quad k\in\mathcal{I}_j^{L,1},\ n\in\mathcal{I}_j^3, && (6.47)\\ &= 0, \quad \text{otherwise}. && (6.48)\end{aligned}$$

For deriving of $\mathbf{M}^{mon}$ see [53]. It is known that the coefficients of Bernstein polynomials in a monomial basis are given by

$$\begin{aligned} c_{l,n} &= (-1)^{l-n}\binom{\tilde{N}-1}{n}\binom{n}{l}b^{-n}, \quad \text{if} \quad n\geq l, && (6.49)\\ &= 0, \quad \text{otherwise}. && (6.50)\end{aligned}$$

Hence, the matrix $\mathbf{C} = \{C_{l,n}\}_{l,n=-N+1}^{-N+\tilde{N}}$ is an upper triangular matrix with nonzero entries on the diagonal which implies that $\mathbf{C}$ is invertible. We denote $\Theta_j^{L,1} = \left\{\theta_{j,k}, k\in\mathcal{I}_j^{L,1}\right\}$ and we obtain

$$\Theta_j^{L,1} = \mathbf{C}\Theta_j^{L,1,mon} = \mathbf{C}\left(\mathbf{M}^{mon}\right)^T\begin{pmatrix}\Theta_{j+1}^{L,1,mon}\\ \Theta_{j+1}^0\end{pmatrix} = \mathbf{C}\left(\mathbf{M}^{mon}\right)^T\begin{pmatrix}\mathbf{C}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\Theta_{j+1}^{L,1}\\ \Theta_{j+1}^0\end{pmatrix}. \quad (6.51)$$

Therefore, the refinement matrix $\mathbf{M} = \{m_{n,k}\}_{n\in\mathcal{I}_j^{L,1}\cup\mathcal{I}_j^3, k\in\mathcal{I}_j^{L,1}}$ is given by

$$\mathbf{M} = \begin{pmatrix}\mathbf{C}^{-T} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}\mathbf{M}^{mon}\mathbf{C}^T. \quad (6.52)$$

□

We define the dual multiresolution spaces by

$$\tilde{S}_j = \text{span}\ \tilde{\Phi}_j. \quad (6.53)$$

Theorem 50. [31] *Under the above assumptions, the following holds*
i) The sequence $\tilde{\mathcal{S}} = \left\{\tilde{S}_j\right\}_{j\geq j_0}$ *forms a multiresolution analysis of* $L^2(\langle 0,1\rangle)$.
ii) The spaces $\tilde{S}_j$ *are exact of order* $\tilde{N}$, *i.e.*

$$\Pi_{\tilde{N}}([0,1]) \subset \tilde{S}_j, \quad j > j_0. \quad (6.54)$$

Proof. To prove i) we have to show the nestedness of the spaces $\tilde{S}_j$, i.e. $\tilde{S}_j \subset \tilde{S}_{j+1}$. Note that

$$\tilde{S}_j = \text{span } \tilde{\Phi}_j = \text{span } \Theta_j. \qquad (6.55)$$

Therefore, it is sufficient to prove that any function from Θ_j can be written as a linear combination of the functions from Θ_{j+1}. For the left boundary functions of the first type it is a consequence of Lemma 49. By definition (6.18) it holds also for the left boundary functions of the second type. Since the inner basis functions are just translated and dilated scaling function $\tilde{\phi}$, they obviously satisfy the refinement relation. Finally, right boundary scaling functions are derived by reflection from the left boundary scaling functions and therefore, they satisfy the refinement relation, too. It remains to prove that

$$\overline{\bigcup_{j\geq j_0} \tilde{S}_j} = L^2\left(\langle 0,1\rangle\right). \qquad (6.56)$$

It is known [82] that for the spaces generated by inner functions

$$\tilde{S}_j^0 = \left\{\theta_{j,k}, k \in \mathcal{I}_j^0\right\} \qquad (6.57)$$

we have

$$\overline{\bigcup_{j\geq j_0} \tilde{S}_j^0} = L^2\left(\langle 0,1\rangle\right). \qquad (6.58)$$

Hence, (6.56) holds independently of the choice of boundary functions.
To prove ii) we recall that the scaling function $\tilde{\phi}$ is exact of order $\tilde{N}$, i.e.

$$2^{j(r+1/2)}x^r = \sum_{k\in\mathbb{Z}} \alpha_{k,r} 2^{j/2}\tilde{\phi}\left(2^j x - k\right), \quad x \in \mathbb{R} \text{ a.e.}, \; r = 0,\ldots,\tilde{N}-1, \qquad (6.59)$$

where

$$\alpha_{k,r} = \left\langle (\cdot)^k, \phi\left(\cdot - r\right)\right\rangle. \qquad (6.60)$$

It implies that for $r = 0,\ldots,\tilde{N}-1$, $x \in \langle 0,1\rangle$, the following holds

$$\begin{aligned} 2^{j(r+1/2)}x^r|_{\langle 0,1\rangle} &= \sum_{k=-N-\tilde{N}+2}^{\tilde{N}-2} \alpha_{k,r} 2^{j/2}\tilde{\phi}\left(2^j x - k\right)|_{\langle 0,1\rangle} + \sum_{k=\tilde{N}-1}^{2^j-N-\tilde{N}+1} \alpha_{k,r} 2^{j/2}\tilde{\phi}\left(2^j x - k\right)|_{\langle 0,1\rangle} \\ &+ \sum_{k=2^j-N-\tilde{N}+2}^{2^j+\tilde{N}-2} \alpha_{k,r} 2^{j/2}\tilde{\phi}\left(2^j x - k\right)|_{\langle 0,1\rangle}. \end{aligned}$$

By definitions (6.16) and (6.19), we immediately have

$$\Pi_{\tilde{N}} \subset \text{span}\left\{\tilde{\phi}_{j,k}, k \in \mathcal{I}_j^{L,1} \cup \mathcal{I}_j^0 \cup \mathcal{I}_j^{R,1}\right\} \subset \tilde{S}_j. \qquad (6.61)$$

□

6.3 Refinement Matrices

Due to the length of support of primal scaling functions, the refinement matrix $M_{j,0}$ corresponding to Φ has the following structure:

$$\mathbf{M}_{j,0} = \begin{pmatrix} \mathbf{M}_L & & \\ & \mathbf{A}_j & \\ & & \mathbf{M}_R \end{pmatrix}. \tag{6.62}$$

where $\mathbf{M}_L$, $\mathbf{M}_R$ are blocks $(2N-2) \times (N-1)$ and $\mathbf{A}_j$ is a $(2^{j+1} - N + 2) \times (2^j - N + 2)$ matrix given by

$$(\mathbf{A}_j)_{m,n} = \frac{1}{\sqrt{2}} h_{m-2n}, \quad 0 \leq m - 2n \leq N. \tag{6.63}$$

Since the matrix $\mathbf{M}_L$ is given by

$$\begin{pmatrix} \phi_{j,-N+1} \\ \phi_{j,-N+2} \\ \vdots \\ \phi_{j,-1} \end{pmatrix} = \mathbf{M}_L^T \begin{pmatrix} \phi_{j+1,-N+1} \\ \phi_{j+1,-N+2} \\ \vdots \\ \phi_{j+1,N-1} \end{pmatrix}, \tag{6.64}$$

it could be computed by solving the system

$$\mathbf{P}_1 = \mathbf{M}_L^T \mathbf{P}_2, \tag{6.65}$$

where

$$\mathbf{P}_1 = \begin{pmatrix} \phi_{0,-N+1}(0) & \phi_{0,-N+1}(1) & \dots & \phi_{0,-N+1}(2N-3) \\ \phi_{0,-N+2}(0) & \phi_{0,-N+2}(1) & \dots & \phi_{0,-N+2}(2N-3) \\ \vdots & & & \vdots \\ \phi_{0,-1}(0) & \phi_{0,-1}(1) & \dots & \phi_{0,-1}(2N-3) \end{pmatrix} \tag{6.66}$$

and

$$\mathbf{P}_2 = \begin{pmatrix} \phi_{1,-N+1}(0) & \phi_{1,-N+1}(1) & \dots & \phi_{1,-N+1}(2N-3) \\ \phi_{1,-N+2}(0) & \phi_{1,-N+2}(1) & \dots & \phi_{1,-N+2}(2N-3) \\ \vdots & & & \vdots \\ \phi_{1,N-1}(0) & \phi_{1,N-1}(1) & \dots & \phi_{1,N-1}(2N-3) \end{pmatrix}. \tag{6.67}$$

The solution of system (6.65) exists and is unique if and only if the matrix $\mathbf{P}_2$ is nonsingular. The proof of regularity of $\mathbf{P}_2$ can be found [105].

To compute the refinement matrix corresponding to the dual scaling functions, we need to identify first the structure of refinement matrices $\mathbf{M}_{j,0}^{\Theta}$ corresponding to Θ.

$$\mathbf{M}_{j,0}^{\Theta} = \begin{pmatrix} \mathbf{M}_L^{\Theta} & & \\ & \tilde{\mathbf{A}}_j & \\ & & \mathbf{M}_R^{\Theta} \end{pmatrix}, \tag{6.68}$$

where $\mathbf{M}_L^{\Theta}$ and $\mathbf{M}_R^{\Theta}$ are blocks $\left(2N + 3\tilde{N} - 5\right) \times \left(N + \tilde{N} - 2\right)$ and $\tilde{\mathbf{A}}_j$ is a matrix of the size $\left(2^{j+1} - N - 2\tilde{N} + 3\right) \times \left(2^j - N - 2\tilde{N} + 3\right)$ given by

$$\left(\tilde{\mathbf{A}}_j\right)_{m,n} = \frac{1}{\sqrt{2}} \tilde{h}_{m-2n}, \quad 0 \le m - 2n \le N + \tilde{N} - 2. \tag{6.69}$$

The receipt for the computation of the refinement coefficients for the left boundary functions of the first type is the proof of Lemma 49. The refinement coefficients for the left boundary functions of the second type are given by definition (6.18). The matrix $\mathbf{M}_R^{\Theta}$ can be computed by the similar way.
Since we have

$$\tilde{\Phi}_j = \mathbf{\Gamma}_j^{-T} \Theta_j = \mathbf{\Gamma}_j^{-T} \left(\mathbf{M}_{j,0}^{\Theta}\right)^T \Theta_{j+1} = \mathbf{\Gamma}_j^{-T} \left(\mathbf{M}_{j,0}^{\Theta}\right)^T \mathbf{\Gamma}_{j+1}^T \tilde{\Phi}_{j+1}, \tag{6.70}$$

the refinement matrix $\tilde{\mathbf{M}}_{j,0}$ corresponding to the dual scaling basis $\tilde{\Phi}_j$ is given by

$$\tilde{\mathbf{M}}_{j,0} = \mathbf{\Gamma}_{j+1} \mathbf{M}_{j,0}^{\Theta} \mathbf{\Gamma}_j^{-1}. \tag{6.71}$$

6.4 Wavelets

Our next goal is to determine the corresponding wavelet bases. This is directly connected to the task of determining an appropriate matrices $\mathbf{M}_{j,1}$ and $\tilde{\mathbf{M}}_{j,1}$. Thus, the problem has been transferred from functional analysis to linear algebra. We follow a general principle called *stable completion* which was proposed in [19], see Definition 40. The idea is to determine first an initial stable completion and then to project it to the desired complement space W_j determined by $\left\{\tilde{V}_j\right\}_{j \ge j_0}$. This is summarized in the following theorem [19].

Theorem 51. *Let Φ_j and $\tilde{\Phi}_j$ be primal and dual scaling basis, respectively. Let $\mathbf{M}_{j,0}$ and $\tilde{\mathbf{M}}_{j,0}$ be refinement matrices corresponding to these bases. Suppose that $\check{\mathbf{M}}_{j,1}$ is some stable completion of $\mathbf{M}_{j,0}$ and $\check{\mathbf{G}}_j = \check{\mathbf{M}}_j^{-1}$. Then*

$$\mathbf{M}_{j,1} := \left(\mathbf{I} - \mathbf{M}_{j,0} \tilde{\mathbf{M}}_{j,0}^T\right) \check{\mathbf{M}}_{j,1} \tag{6.72}$$

is also a stable completion and $\mathbf{G}_j = \mathbf{M}_j^{-1}$ *has the form*

$$\mathbf{G}_j = \begin{pmatrix} \check{\mathbf{M}}_{j,0}^T \\ \check{\mathbf{G}}_{j,1} \end{pmatrix}. \tag{6.73}$$

Moreover, the collections

$$\Psi_j := \mathbf{M}_{j,1}^T \Phi_{j+1}, \quad \tilde{\Psi}_j := \check{\mathbf{G}}_{j,1}^T \tilde{\Phi}_{j+1} \tag{6.74}$$

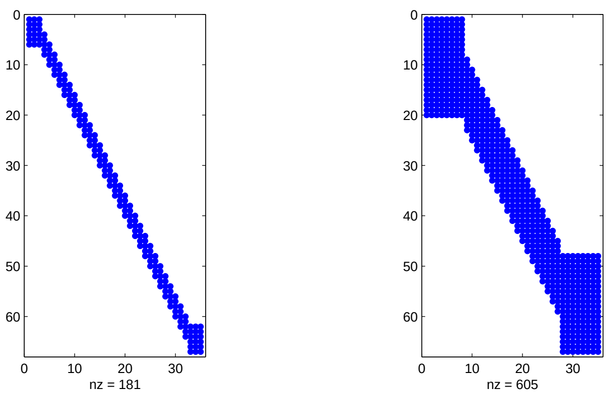
form biorthogonal systems

$$\left\langle \Psi_j, \tilde{\Psi}_j \right\rangle = \mathbf{I}, \quad \left\langle \Phi_j, \tilde{\Psi}_j \right\rangle = \left\langle \Psi_j, \tilde{\Phi}_j \right\rangle = \mathbf{0}. \tag{6.75}$$

We found the initial stable completion by the method from [53, 55] with some small changes. The difference is only in the dimensions of the involved matrices. Recall that $\mathbf{A}_j$ is the interior block in the matrix $\mathbf{M}_{j,0}$ of the form

$$\mathbf{A}_j = \frac{1}{\sqrt{2}} \begin{pmatrix} h_0 & 0 & \dots & 0 \\ h_1 & 0 & & \vdots \\ h_3 & h_0 & & \\ \vdots & \vdots & & \vdots \\ h_N & h_{N-2} & & \vdots \\ 0 & h_{N-1} & & 0 \\ 0 & h_N & & h_0 \\ \vdots & & & \vdots \\ 0 & & & h_N \end{pmatrix}, \tag{6.76}$$

Figure 6.3: Nonzero pattern of the matrices $\mathbf{M}_{4,0}$ and $\tilde{\mathbf{M}}_{4,0}$ for $N = 4$ and $\tilde{N} = 6$, nz is the number of nonzero entries.



where $h_0, \dots, h_N$ are scaling coefficients corresponding to ϕ. By a suitable elimination we will successively reduce the upper and lower bands from $\mathbf{A}_j$ such that after i steps we obtain

$$
\mathbf{A}_j^{(i)} = \left(\begin{array}{cccc}
0 & 0 & & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & & \\
h^{(i)}_{\lceil \frac{i}{2}\rceil} & 0 & & \\
h^{(i)}_{\lceil \frac{i}{2}\rceil+1} & 0 & & \\
\vdots & h^{(i)}_{\lceil \frac{i}{2}\rceil} & & \\
\vdots & \vdots & & \\
h^{(i)}_{N-\lfloor \frac{i}{2}\rfloor} & & & \\
0 & & & \\
\vdots & & & \\
 & & & h^{(i)}_{N-\lfloor \frac{i}{2}\rfloor} \\
 & & & 0 \\
 & & & \vdots \\
0 & & & 0
\end{array}\right)
\begin{array}{l}
\left.\vphantom{\begin{array}{c}0\\ \vdots\\ 0\end{array}}\right\}\lceil \frac{i}{2}\rceil \\
\vphantom{\begin{array}{c}0\\0\\0\\ \vdots\\ \vdots\\ 0\\ 0\\ \vdots\\ 0\end{array}} \\
\left.\vphantom{\begin{array}{c}0\\ \vdots\\ 0\end{array}}\right\}\lfloor \frac{i}{2}\rfloor
\end{array}
, \quad \mathbf{A}_j^{(0)} := \mathbf{A}_j. \tag{6.77}
$$

In [53], it was proved for B-spline scaling functions that

$$
h^{(i)}_{\lceil i/2\rceil}, \dots, h_{N-\lfloor i/2\rfloor} \neq 0, \quad i = 1, \dots, N. \tag{6.78}
$$

Therefore, the ellimination is always possible. The elimination matrices are of the form

$$
H_j^{(2i-1)} := \operatorname{diag}\left(\mathbf{I}_{i-1}, \mathbf{U}_{2i-1}, \dots, \mathbf{U}_{2i-1}, \mathbf{I}_{N-1}\right), \tag{6.79}
$$

$$
H_j^{(2i)} := \operatorname{diag}\left(\mathbf{I}_{N-i}, \mathbf{L}_{2i}, \dots, \mathbf{L}_{2i}, \mathbf{I}_{i-1}\right), \tag{6.80}
$$

where

$$
\mathbf{U}_{i+1} := \begin{pmatrix} 1 & -\frac{h^{(i)}_{\lceil i/2\rceil}}{h^{(i)}_{\lceil i/2\rceil+1}} \\ 0 & 1 \end{pmatrix}, \quad \mathbf{L}_{i+1} := \begin{pmatrix} 1 & 0 \\ -\frac{h^{(i)}_{N-\lfloor i/2\rfloor}}{h^{(i)}_{N-\lfloor i/2\rfloor-1}} & 1 \end{pmatrix}. \tag{6.81}
$$

It is easy to see that indeed

$$
\mathbf{A}_j^{(i)} = \mathbf{H}_j^{(i)} \mathbf{A}_j^{(i-1)}. \tag{6.82}
$$

After N elimination steps we obtain the matrix $\mathbf{A}_j^{(N)}$ which looks as follows

$$\mathbf{A}_j^{(N)} = \mathbf{H}_j \mathbf{A}_j = \begin{pmatrix} 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \\ b & 0 & & \\ 0 & 0 & & \\ 0 & b & & \\ \vdots & 0 & \ddots & \\ & & & b \\ & & & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} \lceil \frac{N}{2} \rceil \\ \\ \\ \\ \\ \\ \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} \lfloor \frac{N}{2} \rfloor \end{matrix}, \quad \text{where} \quad \mathbf{H}_j := \mathbf{H}_j^{(N)} \dots \mathbf{H}_j^{(1)}, \tag{6.83}$$

with $b \neq 0$. We define

$$\mathbf{B}_j := \left(\mathbf{A}_j^{(N)}\right)^{-1} = \begin{pmatrix} \underbrace{0 \dots 0}_{\lceil \frac{N}{2} \rceil} & b^{-1} & 0 & 0 & 0 & \dots & & 0 \\ 0 \dots 0 & 0 & 0 & b^{-1} & 0 & \dots & & 0 \\ & & & & & \ddots & & \\ & & & & & & b^{-1} & \underbrace{0 \ \dots \ 0}_{\lfloor \frac{N}{2} \rfloor} \end{pmatrix} \tag{6.84}$$

and

$$\mathbf{F}_j := \begin{pmatrix} 0 & 0 & & \\ \vdots & \vdots & & \\ 0 & 0 & & \\ 1 & 0 & & \\ 0 & 0 & & \\ 0 & 1 & & \\ \vdots & 0 & \ddots & \\ & & & 1 \\ & & & 0 \\ & & & \vdots \\ & & & 0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} \lceil \frac{N}{2} \rceil - 1 \\ \\ \\ \\ \\ \\ \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} \lfloor \frac{N}{2} \rfloor + 1 \end{matrix}. \tag{6.85}$$

Then, we have

$$\mathbf{B}_j \mathbf{F}_j = \mathbf{0}. \tag{6.86}$$

After these preparations we define extended versions of the matrices $\mathbf{H}_j$, $\mathbf{A}_j$, $\mathbf{A}_j^{(N)}$, and $\mathbf{B}_j$ by

$$\hat{\mathbf{H}}_j := \begin{pmatrix} \mathbf{I}_{N-1} & & \\ & \mathbf{H}_j & \\ & & \mathbf{I}_{N-1} \end{pmatrix}, \quad \hat{\mathbf{A}}_j^{(N)} := \begin{pmatrix} \mathbf{I}_{N-1} & & \\ & \mathbf{A}_j^{(N)} & \\ & & \mathbf{I}_{N-1} \end{pmatrix}, \tag{6.87}$$

$$\hat{\mathbf{A}}_j := \begin{pmatrix} \mathbf{I}_{N-1} & & \\ & \mathbf{A}_j & \\ & & \mathbf{I}_{N-1} \end{pmatrix}, \quad \hat{\mathbf{B}}_j^T := \begin{pmatrix} \mathbf{I}_{N-1} & & \\ & \mathbf{B}_j^T & \\ & & \mathbf{I}_{N-1} \end{pmatrix}. \tag{6.88}$$

Note that $\mathbf{H}_j$, $\mathbf{A}_j$, $\mathbf{A}_j^{(N)}$, and $\mathbf{B}_j$ are all matrices of the size $(\#\mathcal{I}_{j+1}) \times (\#\mathcal{I}_j)$. Hence, the completion of $\mathbf{A}_j^{(N)}$ has to be a $(\#\mathcal{I}_{j+1}) \times 2^j$. On the contrary to the construction in [53], we define an expanded version of $\mathbf{F}_j$ as in [17], because it leads to a more natural formulation, when then the entries of both the refinement matrices belong to $\sqrt{2}\mathbb{Q}$. The difference is in multiplication by $\sqrt{2}$.

$$\hat{\mathbf{F}}_j := \sqrt{2} \begin{pmatrix} & \mathbf{O} & \\ \mathbf{I}_{\lceil \frac{N}{2} \rceil - 1} & & \\ & \mathbf{F}_j & \\ & & \mathbf{I}_{\lfloor \frac{N}{2} \rfloor} \\ & \mathbf{O} & \end{pmatrix} \begin{matrix} \} N-1 \\ \\ \\ \\ \} N-1 \end{matrix}. \tag{6.89}$$

The above findings can be summarized as follows.

Lemma 52. *The following relations hold:*

$$\hat{\mathbf{B}}_j \hat{\mathbf{A}}_j^{(N)} = \mathbf{I}_{\#\mathcal{I}_j}, \quad \frac{1}{2} \hat{\mathbf{F}}_j^T \hat{\mathbf{F}}_j = \mathbf{I}_{2^j} \tag{6.90}$$

and

$$\hat{\mathbf{B}}_j \hat{\mathbf{F}}_j = \mathbf{0}, \quad \hat{\mathbf{F}}_j^T \hat{\mathbf{A}}_j^{(N)} = \mathbf{0}. \tag{6.91}$$

The proof of this lemma is similar to the proof in [53]. Note the refinement matrix $\mathbf{M}_{j,0}$ can be factorized as

$$\mathbf{M}_{j,0} = \mathbf{P}_j \hat{\mathbf{A}}_j = \mathbf{P}_j \hat{\mathbf{H}}_j^{-1} \hat{\mathbf{A}}_j^{(N)} \tag{6.92}$$

with

$$\mathbf{P}_j := \begin{pmatrix} \mathbf{M}_L & & \\ & \mathbf{I}_{\#\mathcal{I}_{j+1}-2N} & \\ & & \mathbf{M}_R \end{pmatrix}. \tag{6.93}$$

Now we are able to define the initial stable completions of the refinement matrices $\mathbf{M}_{j,0}$.

Lemma 53. *Under the above assumptions, the matrices*

$$\check{\mathbf{M}}_{j,1} := \mathbf{P}_j \hat{\mathbf{H}}_j^{-1} \hat{\mathbf{F}}_j, \quad j \geq j_0, \tag{6.94}$$

are uniformly stable completions of the matrices $\mathbf{M}_{j,0}$. *Moreover, the inverse*

$$\check{\mathbf{G}}_j = \begin{pmatrix} \check{\mathbf{G}}_{j,0} \\ \check{\mathbf{G}}_{j,1} \end{pmatrix} \tag{6.95}$$

of $\check{\mathbf{M}}_j = \left(\mathbf{M}_{j,0}, \check{\mathbf{M}}_{j,1}\right)$ *is given by*

$$\check{\mathbf{G}}_{j,0} = \hat{\mathbf{B}}_j \hat{\mathbf{H}}_j \mathbf{P}_j^{-1}, \quad \check{\mathbf{G}}_{j,1} = \frac{1}{2} \hat{\mathbf{F}}_j^T \hat{\mathbf{H}}_j \mathbf{P}_j^{-1}. \tag{6.96}$$

The proof of this lemma is straightforward and similar to the proof in [53]. Then using the initial stable completion $\check{\mathbf{M}}_{j,1}$ we are already able to contruct wavelets according to the Theorem 51.

6.5 Adaptation to Complementary Boundary Conditions

In this section, we introduce a construction of stable spline-wavelet bases on the interval satisfying complementary boundary conditions of the first order. This means that the primal wavelet basis is adapted to homogeneous Dirichlet boundary conditions of the first order, whereas the dual wavelet basis preserves the full degree of polynomial exactness. This construction is based on the spline-wavelet bases constructed above. In the linear case, i.e. $N = 2$, our bases are the same as bases constructed in [80]. The adaptation of these bases to complementary boundary conditions can be found in [80]. Thus, we consider only the case $N \geq 3$.
Let $\Phi_j = \{\phi_{j,k}, k = -N+1, \ldots, 2^j - 1\}$ be defined as above. Note that the functions $\phi_{j,-N+1}$, $\phi_{j,2^j-1}$ are the only two functions which do not vanish at zero. Therefore, defining

$$\Phi_j^{comp} = \left\{\phi_{j,k}, k = -N+2, \ldots, 2^j - 2\right\} \tag{6.97}$$

we obtain primal scaling bases satisfying complementary boundary conditions of the first order.

On the dual side, we also need to omit one scaling function at each boundary, because the number of primal scaling functions must be the same as the number of dual scaling functions. Let $\Theta_j = \{\theta_{j,k}, k \in \mathcal{I}_j\}$ be the dual scaling basis on level j before biorthogonalization from Section 6.2. There are boundary functions of two types. Recall that functions $\theta_{j,-N+1}, \ldots, \theta_{j,-N+\tilde{N}}$ are left boundary functions of the first type which are defined to preserve polynomial exactness of the order $\tilde{N}$. Functions $\theta_{j,-N+\tilde{N}+1}, \ldots, \theta_{j,\tilde{N}-2}$ are left boundary functions of the second type. The right boundary scaling functions are then

Figure 6.4: Some primal wavelets for $N = 4$ without boundary conditions.

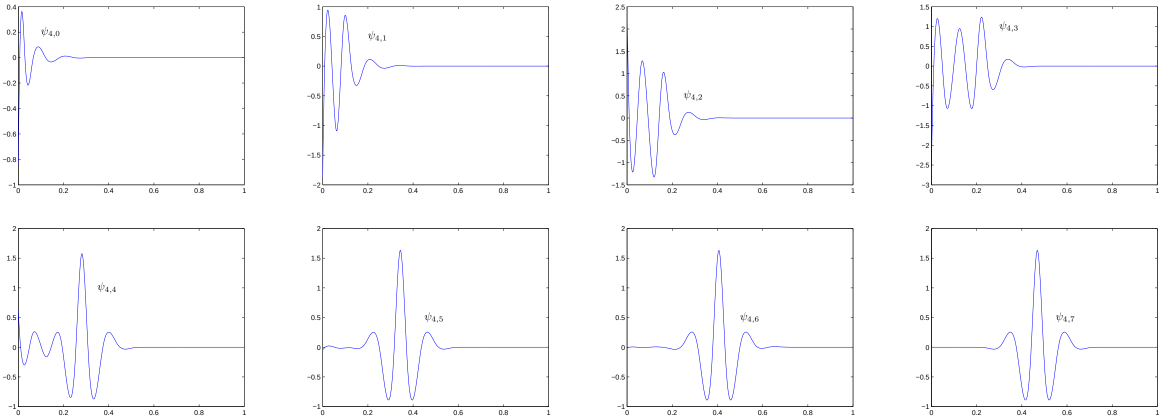
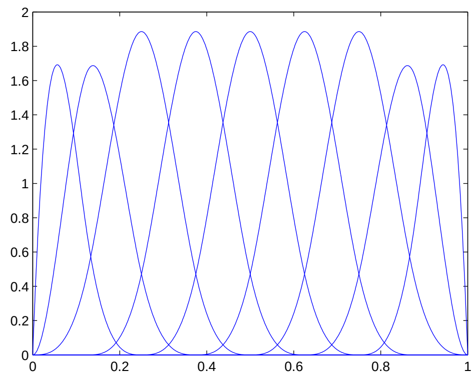


Figure 6.5: Primal scaling basis for $N = 4$, $j = 3$ satisfying complementary boundary conditions of the first order.



derived by reflection of the left boundary functions. Since we want to preserve the full degree of polynomial exactness, we omit one function of the second type at each boundary. Thus, we define

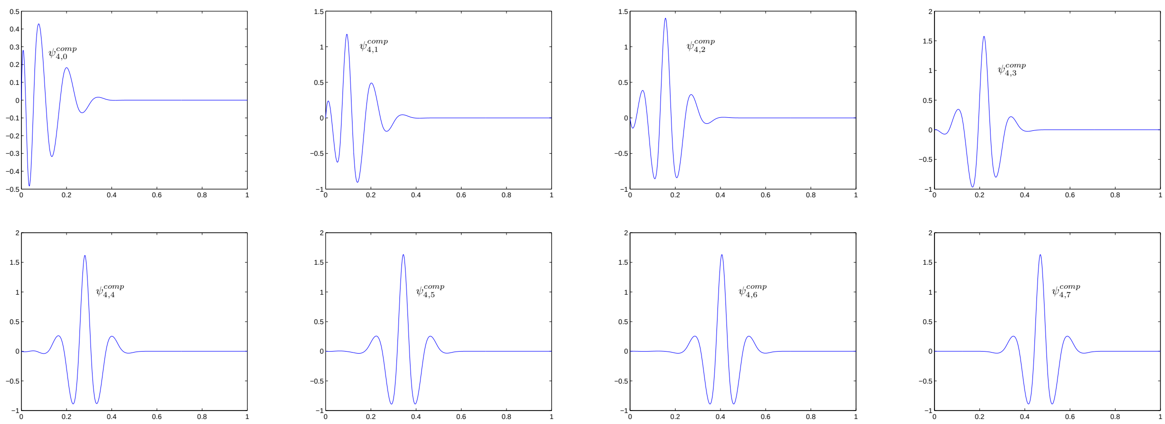
$$\begin{aligned}
\theta_{j,k}^{comp} &= \theta_{j,k-1}, & k &= -N+2, \dots, -N+\tilde{N}+1, \\
\theta_{j,k}^{comp} &= \theta_{j,k}, & k &= -N+\tilde{N}+2, \dots, 2^j-\tilde{N}-2, \\
\theta_{j,k}^{comp} &= \theta_{j,k+1}, & k &= 2^j-\tilde{N}-1, \dots, 2^j-2.
\end{aligned}$$

Since the set $\Theta_j^{comp} := \left\{\theta_{j,k}^{comp} : k = -N+2, \dots, 2^j-2\right\}$ is not biorthogonal to Φ_j, we derive a new set $\tilde{\Phi}_j^{comp}$ from Θ_j^{comp} by biorthogonalization. Let $\mathbf{\Gamma}_j^{comp} = \left(\langle \phi_{j,k}, \theta_{j,l}^{comp} \rangle\right)_{k,l=-N+2}^{2^j-2}$, then viewing $\tilde{\Phi}_j^{comp}$ and Θ_j^{comp} as column vectors we define

$$\tilde{\Phi}_j^{comp} := \left(\mathbf{\Gamma}_j^{comp}\right)^{-T} \Theta_j^{comp}. \tag{6.98}$$

Our next goal is to determine the corresponding wavelets $\Psi_j^{comp} := \left\{\psi_{j,k}^{comp}, k = 1, \dots, 2^j\right\}$, $\tilde{\Psi}_j^{comp} := \left\{\tilde{\psi}_{j,k}^{comp}, k = 1, \dots, 2^j\right\}$. It can be done by the method of stable completion as in Section 6.4.

Figure 6.6: Some primal wavelets for $N = 4$ satisfying the complementary boundary conditions of the first order.



Chapter 7

Adaptive Wavelet and Frame Methods for Elliptic Operator Equations

In recent years adaptive wavelet methods were successfully used for solving partial differential as well as integral equations, both linear and nonlinear [3, 4, 38, 39, 43, 45]. It has been shown that these methods converge and that they are asymptotically optimal in the sense that storage and number of floating point operations, needed to resolve the problem with desired accuracy, remain proportional to the problem size when the resolution of the discretization is refined. Therefoe, the computational complexity for all steps of the algorithm is controlled.
The effectiveness of these methods is strongly influenced by the choice of a wavelet basis. Our intention is to show the theoretical dependence of the convergence speed on the condition number of the underlying wavelet basis. In Chapter 8 we show that the adaptive wavelet frame method from [47] with bases constructed in this thesis realizes the optimal convergence rate. Although adaptive wavelet methods are available for a large class of equations, both linear and nonlinear, for simplicity, we focus on the linear elliptic case.
The results of this chapter are well-known and can be found in [39, 40, 45, 46, 47, 52, 83, 91].

7.1 Scope of Problems

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle_H$ and the induced norm $\|\cdot\|_H$. Let $A : H \to H'$ be the selfadjoint and H-elliptic operator, i.e.

$$a(v, w) := \langle Av, w \rangle \lesssim \|v\|_H \|w\|_H \quad \text{and} \quad a(v, v) \sim \|v\|_H^2 . \tag{7.1}$$

By the Lax-Milgram theorem, A is an isomorphism from H to H', i.e. there exist positive constants c_A and C_A such that

$$c_A \|v\|_H \leq \|Av\|_{H'} \leq C_A \|v\|_H , \quad v \in H. \tag{7.2}$$

Therefore, the equation

$$Au = f \tag{7.3}$$

has for any $f \in H'$ a unique solution. If (7.2) holds, then (7.3) is called *well-posed* (on H). Typical examples are second order elliptic boundary value problems with homogeneous Dirichlet boundary conditions on some open domain $\Omega \subset \mathbb{R}^d$. In this case $H = H_0^1(\Omega)$ and $H' = H^{-1}(\Omega)$. Other examples are singular integral equations on the boundary $\partial\Omega$ with $H = H^{-1/2}(\partial\Omega)$, $H' = H^{1/2}(\partial\Omega)$.
Thus H is typically a Sobolev space. In the following, we assume that

$$H \subset L^2 \subset H' \quad \text{or} \quad H' \subset L^2 \subset H. \tag{7.4}$$

7.2 An Equivalent l^2-Problem

We assume that $\mathbf{D}^{-t}\Psi$ is a wavelet basis in the energy space H. Thus, we have

$$c_\psi \left\|\mathbf{v}\right\|_{l^2} \leq \left\|\mathbf{v}^T\mathbf{D}^{-t}\Psi\right\|_H \leq C_\psi \left\|\mathbf{v}\right\|_{l^2}, \quad \mathbf{v} \in l^2(\mathcal{J}), \tag{7.5}$$

where $c_\psi > 0$. Then the original equation (7.3) can be reformulated as an equivalent biinfinite matrix equation

$$\mathbf{Au} = \mathbf{f}, \tag{7.6}$$

where $\mathbf{A} = \mathbf{D}^{-t}\langle A\Psi, \Psi\rangle \mathbf{D}^{-t}$ is a diagonally preconditioned stiffness matrix, $u = \mathbf{u}^T D^{-t}\Psi$ and $\mathbf{f} = \mathbf{D}^{-t}\langle f, \Psi\rangle$. The following theorem from [52] will be crucial in what follows.

Theorem 54. *Under the above assumptions, u solves (7.3) if and only if $\mathbf{u}$ solves the matrix equation (7.6). Moreover, the matrix $\mathbf{A}$ satisfies*

$$\left\|\mathbf{A}\right\|_{l^2}, \left\|\mathbf{A}^{-1}\right\|_{l^2} \leq \frac{C_\psi^2 C_A}{c_\psi^2 c_A} < +\infty. \tag{7.7}$$

As an immediate consequence there exists a finite number κ such that all finite sections

$$\mathbf{A}_\Lambda := \mathbf{D}^{-t}\langle A\Psi_\Lambda, \Psi_\Lambda\rangle \mathbf{D}^{-t}, \quad \Psi_\Lambda := \{\psi_\lambda, \lambda \in \Lambda\}, \quad \Lambda \subset \mathcal{J}, \tag{7.8}$$

have uniformly bounded condition numbers

$$\kappa(\mathbf{A}_\Lambda) \leq \frac{C_\psi^2 C_A}{c_\psi^2 c_A}, \quad \Lambda \subset \mathcal{J}. \tag{7.9}$$

Thus the original problem is equivalent to the well-posed problem in l^2.
While the classical adaptive methods uses refining and derefining step by step a given mesh according to a-posteriori local error indicators, the wavelet approach is somewhat different and follows a paradigm which comprises the following steps:

1. One starts with a variational formulation but instead of turning to a finite dimensional approximation, using the suitable wavelet basis the continuous problem is transformed into an infinite-dimensional l^2-problem, which is well-conditioned.

2. One then tries to devise a *convergent iteration* for the l^2-problem.

3. Finally, one derives a practicle version of this idealized iteration. All infinite-dimensional quantities have to be replaced by finitely supported ones and the routine for the application of the biinfinite-dimensional matrix $\mathbf{A}$ approximately have to be designed.

The simplest convergent iteration for the l^2-problem is a *Richardson iteration* which has the following form:

$$\mathbf{u}_0 := 0, \quad \mathbf{u}_{n+1} := \mathbf{u}_n + \omega\left(\mathbf{f} - \mathbf{A}\mathbf{u}_n\right), \quad n = 0, 1, \ldots. \tag{7.10}$$

For the convergence, the relaxation parameter ω has to satisfy

$$\rho := \|\mathbf{I} - \omega\mathbf{A}\|_{\mathcal{L}(l^2)} < 1. \tag{7.11}$$

Then the iteration (7.10) convergence with an error reduction per step

$$\|\mathbf{u}_{n+1} - \mathbf{u}\|_{l^2} \leq \rho\,\|\mathbf{u}_n - \mathbf{u}\|_{l^2}. \tag{7.12}$$

In the case that $\mathbf{A}$ is symmetric and positive definite, then (7.11) is satisfied if

$$0 < \omega < \frac{2}{\lambda_{max}}, \tag{7.13}$$

where λ_{max} is the largest eigenvalue of $\mathbf{A}$. It is known that the optimal relaxation parameter is given by

$$\hat{\omega} = \frac{2}{\lambda_{min} + \lambda_{max}}, \tag{7.14}$$

where λ_{min} is the smallest eigenvalue of $\mathbf{A}$. For $\hat{\omega}$ the estimate of the error reduction can be computed as

$$\rho\left(\hat{\omega}\right) = \frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} = \frac{\kappa\left(\mathbf{A}\right) - 1}{\kappa\left(\mathbf{A}\right) + 1} = 1 - \frac{1}{\kappa\left(\mathbf{A}\right) + 1} \leq 1 - \frac{1}{\frac{C_\psi^2 C_A}{c_\psi^2 c_A} + 1}. \tag{7.15}$$

Hence, if the wavelet basis $\mathbf{D}^{-t}\Psi$ is badly conditioned, i.e. cond $\mathbf{D}^{-t}\Psi = \frac{C_\psi}{c_\psi}$ is large, then the right-hand side of (7.15) is close to 1, which is clearly useless.

7.3 Quasi-Sparse Matrices

Due to the cancellation property (4.46) the matrix $\mathbf{A}$ exhibits fast decay away from the diagonal for a large class of elliptic operators. Therefore, by setting to zero the matrix entries that are small in modulus, the matrix $\mathbf{A}$ may be approximated well by a sparse matrix with a finite number of entries per row and column.
When $H \subset H^t(\Omega)$, the following estimate holds:

$$2^{-(|\lambda|+|\lambda'|)t} |\langle A\psi_\lambda, \psi_{\lambda'}\rangle| \lesssim 2^{-||\lambda|-|\lambda'||\sigma} (1 + d(\lambda, \lambda'))^{-\beta}, \quad \lambda, \lambda' \in \mathcal{J}, \tag{7.16}$$

where $\sigma > \frac{d}{2}$, $\beta > d$, and d is defined by

$$d(\lambda, \lambda') := 2^{min\{|\lambda|,|\lambda'|\}} \operatorname{dist}(\operatorname{supp} \psi_\lambda, \operatorname{supp} \psi_{\lambda'}). \tag{7.17}$$

The constant σ depends on the regularity of the wavelets ψ_λ. More precisely,

$$\sigma = \min\left\{\tau, \gamma - t, t + \tilde{N}\right\}, \tag{7.18}$$

if A is a bounded operator from H^{t+s} to H^{t-s} for $|s| \leq \tau$, Ψ satisfies (4.31) for $s \in (-\tilde{\gamma}, \gamma)$, and $\tilde{N}$ is the polynomial exactness of the dual scaling basis. The constant β satisfies

$$\beta \geq d + 2\tilde{N} + 2t \tag{7.19}$$

for a wide range of cases, including classical pseudo-differential operators and the Calderon-Zygmund operators, see e.g. [57].

Definition 55. The matrix $\mathbf{A}$ is said to be s^*-*compressible*, $\mathbf{A} \in \mathcal{A}_{s^*}$, if for any $0 < s < s^*$ and for every $j \geq 0$, there exists a positive summable sequence $(\alpha_j)_{j\geq 0}$ and a matrix $\mathbf{A}_j$ with at most $2^j\alpha_j$ nonzero entries per row and column such that

$$\|\mathbf{A}_j - \mathbf{A}\| \lesssim \alpha_j 2^{-js}. \tag{7.20}$$

In the following lemma from [38], it is stated for which values of s^* the matrix $\mathbf{A}$ is s^*-compressible.

Lemma 56. *Let*

$$s^* := min\left\{\frac{\sigma}{d} - \frac{1}{2}, \frac{2t + 2\tilde{N}}{d}\right\}. \tag{7.21}$$

Then $\mathbf{A}$ *is* s^**-compressible.*

For other results we refer to [3, 45, 51, 92]. In [92], it was shown that in the case of spline-wavelet basis the matrix $\mathbf{A}$ is s^*-compressible for

$$s^* > \frac{N - t}{d}, \tag{7.22}$$

where N is the order of the spline basis, t is the order of differential or integral operator and d is the spatial dimension.
In [38] the truncantion rules were given. The first step is a *truncation in scale*: Given j, set

$$\tilde{\mathbf{A}}_{\lambda,\nu} := \begin{cases} \mathbf{A}_{\lambda,\nu}, & ||\lambda| - |\nu|| \leq \frac{j}{d}, \\ 0, & otherwise. \end{cases} \tag{7.23}$$

Then a *spatial truncation* is done:

$$\mathbf{A}'_{\lambda,\nu} := \begin{cases} \tilde{\mathbf{A}}_{\lambda,\nu}, & d(\lambda,\nu) \leq 2^{j/d - ||\lambda| - |\nu||}\gamma(||\lambda| - |\nu||), \\ 0, & otherwise. \end{cases} \tag{7.24}$$

Here $\gamma(n)$ is any summable sequence, e.g. $\gamma(n) := (1+n)^{-2/d}$. In the case that the matrix $\mathbf{A}$ is the preconditioned matrix representation of an elliptic operator which is local, then the truncation in space is not needed.
In the sequel, we assume that the entries of the matrix $\mathbf{A}$ can be computed (up to round off errors) at unit cost. This assumption is realistic for a constant coefficient differential operator and piecewise polynomial wavelets such as the spline-wavelets constructed in the previous chapter. For the computation of the entries of the matrix $\mathbf{A}$ for a more general situation we refer to [5, 6].

Algorithm 57. APPLY [A, v, ϵ] $\rightarrow \mathbf{w}_\epsilon$
For $j \in \mathbb{N}_0$ let C_j be such that $\|\mathbf{A} - \mathbf{A}_j\| \leq C_j$.
1. Set $q := \left\lceil \log\left((\#\text{supp } \mathbf{v})^{1/2} \|\mathbf{v}\|_{l^2} \|\mathbf{A}\|_{\mathcal{L}(l^2)} 2/\epsilon \right) \right\rceil$.
2. Regroup the elements of $\mathbf{v}$ into the sets $B_0, \ldots, B_q$, where $\mathbf{v}_\lambda \in B_i$ if and only if

$$2^{-(i+1)} \|\mathbf{v}\|_{l^2} < |\mathbf{v}_\lambda| \leq 2^{-i} \|\mathbf{v}\|_{l^2}, \quad 0 \leq i < q. \tag{7.25}$$

Possible remaining elements are put into the set B_q.
3. For $k = 0, 1, \ldots,$ generate vectors $\mathbf{z}_k$ by subsequently extracting $2^k - \lfloor 2^{k-1} \rfloor$ elements from $\cup_i B_i$, starting from B_0 and when it is empty continuing with B_1 and so forth until for some $k = l$ either $\cup_i B_i$ becomes empty or

$$\|\mathbf{A}\|_{\mathcal{L}(l^2)} \left\| \mathbf{v} - \sum_{k=0}^{l} \mathbf{z}_k \right\|_{l^2} \leq \epsilon/2. \tag{7.26}$$

4. Compute the smallest $j \geq l$ such that

$$\sum_{k=0}^{l} C_{j-k} \|\mathbf{z}_k\|_{l^2} \leq \epsilon/2. \tag{7.27}$$

5. Compute

$$\mathbf{w}_\epsilon := \sum_{k=0}^{l} \mathbf{A}_{j-k}\mathbf{z}_k. \tag{7.28}$$

7.4 Best N-term Approximation

Let N be the number of degrees of freedom produced by the algorithm. The optimal outcome of an adaptive scheme is then given by a function u_N which satisfies

$$\|u - u_N\|_H = \min_{v \in \Sigma_N} \|u - v\|_H , \tag{7.29}$$

where

$$\Sigma_N = \left\{ v = \sum_{\lambda \in \Lambda} c_\lambda \psi_\lambda, \#\Lambda \leq N \right\}. \tag{7.30}$$

Such a function u_n is called a *best N-term approximation.* By Theorem 54

$$u_N = \mathbf{u}_N^T \mathbf{D}^{-t} \Psi, \tag{7.31}$$

where $\mathbf{u}_N$ is a best N-term approximation of $\mathbf{u}$ in l^2, i.e. $\mathbf{u}_N$ is obtained by retaining the N largest components of $\mathbf{u}$. Of course, since $\mathbf{u}$ is not known, $\mathbf{u}_N$ is not directly available. In summary, the best that could be achieved by an adaptive scheme is to produce approximate solution $\mathbf{u}_\Lambda$ such that

$$\|\mathbf{u} - \mathbf{u}_\Lambda\|_{l^2} \sim \sigma_N(\mathbf{u}) := \inf \{ \|\mathbf{u} - \mathbf{v}\|_{l^2} : \#\text{supp } \mathbf{v} \leq N \}. \tag{7.32}$$

Thus, we will have to compute the best or near best N-term approximation of a given finitely supported vector $\mathbf{v}$. Our aim is to obtain algorithm of linear complexity. The sorting of all nonzero elements of $\mathbf{v}$ requires $M \log M$ arithmetic operations, where $M := \#\text{supp } \mathbf{v}$. Fortunately, it has been shown in [4, 91] that the complete sorting is not necessary and may be substitute by so called binning. Then the algorithm for the computation of a near best N-term approximation looks as follows.

Algorithm 58. COARSE[$\mathbf{v}$, ϵ] $\rightarrow \mathbf{v}_\epsilon$
1. Set $q := \left\lceil \log \left((\#\text{supp } \mathbf{v})^{1/2} \|\mathbf{v}\|_{l^2} / \epsilon \right) \right\rceil$.
2. Regroup the elements of $\mathbf{v}$ *into the sets* $B_0, \ldots, B_q$*, where* $\mathbf{v}_\lambda \in B_i$ *if and only if*

$$2^{-(i+1)} \|\mathbf{v}\|_{l^2} < |\mathbf{v}_\lambda| \leq 2^{-i} \|\mathbf{v}\|_{l^2}, \quad 0 \leq i < q. \tag{7.33}$$

Possible remaining elements are put into the set B_q.
3. Create $\mathbf{v}_\epsilon$ *by collecting nonzero entries from* B_0 *and when it is exhausted from* B_1 *and so forth until*

$$\|\mathbf{v} - \mathbf{v}_\epsilon\|_{l^2} \leq \epsilon. \tag{7.34}$$

7.5 Approximation of the Right-Hand Side

We assume that it is possible to compute the wavelet coefficients $\mathbf{f} = \mathbf{D}^{-t} \langle f, \Psi \rangle$ of the right-hand side $f \in H'$ up to a desired accuracy. More precisely, we require that for any $\epsilon > 0$, there exists a computable, finitely supported vector $\mathbf{f}_\epsilon \in l^2(\mathcal{J})$ such that

$$\|\mathbf{f} - \mathbf{f}_\epsilon\|_{l^2} \leq \epsilon. \tag{7.35}$$

In the following, the computation of $\mathbf{f}_\epsilon$ will be reffered to as the subroutine

$$\mathbf{RHS}\,[\mathbf{f},\epsilon] \to \mathbf{f}_\epsilon. \tag{7.36}$$

It can be realized by computing in a preprocessing step a highly accurate approximation to f in the dual basis along with the corresponding coefficients and then applying of **COARSE** to this finitely supported array of coefficients.

7.6 Adaptive Wavelet and Frame Schemes

In [91] the following implementable version of the ideal iteration (7.10) was proposed. This algorithm can be applied in the case that Ψ is a wavelet basis or an aggregated Gelfand frame. It was proved that such an algorithm converge and is asymptotically optimal. Under the assumption that $\mathbf{A}$ is the biinfinite matrix corresponding to Ψ, this algorithm looks as follows.

Algorithm 59. SOLVE2 [A, f, ϵ] $\to$ $\mathbf{u}_\epsilon$
Let $\theta < 1/3$ and $K \in \mathbb{N}$ be fixed such that $3\rho^K < \theta$.
1. Set $j := 0$, $\mathbf{u}_0 := 0$, $\epsilon_0 := \left\| \left(\mathbf{A}|_{Ran(\mathbf{A})}\right)^{-1} \right\|_{\mathcal{L}(l^2)} \|\mathbf{f}\|_{l^2}$.
2. While $\epsilon_j > \epsilon$ do
$j := j+1$,
$\epsilon_j := 3\rho^K \epsilon_{j-1}/\theta$,
$\mathbf{f}_j := \mathbf{RHS}[\mathbf{f}, \frac{\theta\epsilon_j}{6\omega K}]$,
$\mathbf{z}_0 := \mathbf{u}_{i-1}$,
For $l = 1, \ldots, K$ do
$\mathbf{z}_l := \mathbf{z}_{l-1} + \omega\left(\mathbf{f}_j - \mathbf{APPLY}[\mathbf{A}, \mathbf{z}_{l-1}, \frac{\theta\epsilon_j}{6\omega K}]\right)$,
end for,
$\mathbf{u}_j := \mathbf{COARSE}[\mathbf{z}_K, (1-\theta)\,\epsilon_j]$,
end while,
$\mathbf{u}_\epsilon := \mathbf{u}_j$.

The main result can be formulated as follows.

Theorem 60. *Let $\mathbf{A} \in \mathcal{A}_s$ for some $s^* > 0$. Then if the exact solution $u = \mathbf{u}^T D^{-t}\Psi$ of (7.3) satisfies for some $s < s^*$*

$$\inf_{\#\mathrm{supp}\mathbf{v} \le N} \left\| u - \mathbf{v}^T \mathbf{D}^{-1}\Psi \right\| \lesssim N^{-s}, \tag{7.37}$$

then, for any $\epsilon > 0$, the approximations $\mathbf{u}_\epsilon$ produced by **SOLVE** *satisfy*

$$\left\| u - \mathbf{u}_\epsilon^T \mathbf{D}^{-1}\Psi \right\| \lesssim \epsilon \tag{7.38}$$

and

$$\#\mathbf{u}_\epsilon, \#flops \lesssim \epsilon^{-1/s}. \tag{7.39}$$

For the proof of Theorem 60 under the assumtion that Ψ is a wavelet basis see [39] and the proof of Theorem 60 in the case of frames can be found in [91].
As an alternative to the Richardson iterations the steepest descent approach was studied both for wavelet bases [61, 20] and frames [48]. It converges with the same error reduction factor ρ as in (7.15). The advantage of this approach is that there is no relaxation parameter. However, one iteration step is slightly more expensive.

Chapter 8

Quantitative Properties of Constructed Bases and Numerical Examples

In this chapter the condition of scaling bases, the single-scale wavelet bases and the multiscale wavelet bases is computed. As in [80] it can be improved by the L^2-normalization on the primal side. It will be shown that in the case of cubic spline wavelets bases the construction presented in this thesis yields optimal L^2-stability, which is not the case of constructions in [53, 80]. The condition of the wavelet transform is presented as well as the condition of scaling bases and wavelet bases satisfying the complementary boundary conditions of the first order. The other criteria for the effectiveness of wavelet bases is the condition number of the corresponding preconditioned stiffness matrix. To improve it further we apply orthogonal transformation to the scaling basis on the coarsest level as in [22] and then we use a diagonal matrix for preconditioning. Finally, it will be shown that the adaptive frame methods with the bases constructed in this thesis realizes the optimal convergence rate, in particular, in the important case of cubic spline wavelets.

8.1 Quantitative Properties of Constructed Bases

In this section, quantitative properties of bases constructed in Chapter 6 are presented. It is known that Riesz bounds (4.2) of basis Φ_j can be computed by

$$c = \sqrt{\lambda_{min}\left(\mathbf{G}_j\right)}, \quad C = \sqrt{\lambda_{max}\left(\mathbf{G}_j\right)}, \tag{8.1}$$

where $\mathbf{G}_j$ is the Gram matrix, i.e. $\mathbf{G}_j = \left(\langle \phi_{j,k}, \phi_{j,l} \rangle\right)_{k,l \in \mathcal{I}_j}$, and $\lambda_{min}\left(\mathbf{G}_j\right)$, $\lambda_{max}\left(\mathbf{G}_j\right)$ denote the smallest and the largest eigenvalue of $\mathbf{G}_j$, respectively. The Riesz bounds of $\tilde{\Phi}_j$, Ψ_j and $\tilde{\Psi}_j$ are computed in a similar way.
The condition of constructed bases is presented in Table 8.1. To improve it further we

provide a diagonal rescaling in the following way:

$$\phi_{j,k}^N = \frac{\phi_{j,k}}{\sqrt{\langle \phi_{j,k}, \phi_{j,k} \rangle}}, \quad \tilde{\phi}_{j,k}^N = \tilde{\phi}_{j,k} * \sqrt{\langle \phi_{j,k}, \phi_{j,k} \rangle}, \quad k \in \mathcal{I}_j, \quad j \geq j_0, \tag{8.2}$$

$$\psi_{j,k}^N = \frac{\psi_{j,k}}{\sqrt{\langle \psi_{j,k}, \psi_{j,k} \rangle}}, \quad \tilde{\psi}_{j,k}^N = \tilde{\psi}_{j,k} * \sqrt{\langle \psi_{j,k}, \psi_{j,k} \rangle}, \quad k \in \mathcal{J}_j, \quad j \geq j_0. \tag{8.3}$$

Then the new primal scaling and wavelet bases are normalized with respect to the L^2-norm. As already mentioned in Remark 48, the resulting bases for $N = 2$ are the same as those designed in [80, 81]. For quadratic spline-wavelet bases, i.e. $N = 3$, the condition of our bases is comparable to the condition of the bases from [80, 81]. In [10], it was shown that for any spline wavelet basis of order N on the real line, the condition is bounded below by 2^{N-1}. This result readily carries over to the case of spline wavelet bases on the interval. Now, the constructions from [80, 81] yields wavelet bases whose Riesz bounds are nearly optimal, i.e. cond $\Psi_j^N \approx 2^{N-1}$ for $N = 2$ and $N = 3$. Unfortunately, the L^2-stability gets considerably worse for $N \geq 4$. As can be seen in Table 8.1, the column "Ψ_j^N", the presented construction seems to yield the optimal L^2-stability also for $N = 4$. Note that the case $N = 4$, $\tilde{N} = 4$ is not included in Table 8.1. It was shown in [41] that the corresponding scaling functions and wavelets do not belong to the space L^2.

Table 8.1: The condition of single-scale scaling and wavelet bases

N	$\tilde{N}$	j	Φ_j	Φ_j^N	$\tilde{\Phi}_j$	$\tilde{\Phi}_j^N$	Ψ_j	Ψ_j^N	$\tilde{\Psi}_j$	$\tilde{\Psi}_j^N$
2	2	5	2.00	1.73	2.30	1.97	1.99	1.99	2.01	1.99
2	4	5	2.00	1.73	2.09	1.80	1.99	1.99	2.04	1.99
2	6	5	2.00	1.73	2.26	2.03	1.99	1.99	2.29	2.25
2	8	5	2.00	1.73	2.89	2.78	2.33	2.22	3.13	3.80
3	3	5	3.25	2.76	7.56	6.36	4.48	3.98	7.00	4.24
3	5	5	3.25	2.76	3.93	3.49	4.61	3.97	5.50	4.02
3	7	5	3.25	2.76	3.52	3.11	4.52	3.96	5.09	3.98
3	9	5	3.25	2.76	3.75	3.32	4.41	3.96	5.47	4.20
4	6	6	5.18	4.42	10.87	9.07	14.00	7.98	24.22	9.18
4	8	6	5.18	4.42	6.69	5.88	13.94	7.98	16.91	8.16
4	10	6	5.18	4.42	5.82	5.16	13.78	7.97	15.21	7.97
5	9	6	8.32	7.13	29.86	25.23	67.74	27.44	168.35	68.30

In Table 8.2 the condition of the multiscale wavelet bases $\Psi_{j_0,s} = \Phi_{j_0} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j$ is presented.

Table 8.2: The condition of the multiscale wavelet bases

N	$\tilde{N}$	j_0	$\Psi^N_{j_0,1}$	$\Psi^N_{j_0,2}$	$\Psi^N_{j_0,3}$	$\Psi^N_{j_0,4}$	$\Psi^N_{j_0,5}$	$\tilde{\Psi}^N_{j_0,1}$	$\tilde{\Psi}^N_{j_0,2}$	$\tilde{\Psi}^N_{j_0,3}$	$\tilde{\Psi}^N_{j_0,4}$	$\tilde{\Psi}^N_{j_0,5}$
2	2	2	1.98	2.27	2.52	2.65	2.76	2.20	2.42	2.65	2.78	2.87
2	4	3	2.13	2.25	2.30	2.33	2.34	2.15	2.26	2.31	2.33	2.35
2	6	4	2.47	2.71	2.84	2.92	2.99	2.60	2.78	2.88	2.94	3.00
2	8	4	3.71	4.77	5.35	5.68	5.89	4.44	5.17	5.57	5.82	5.98
3	3	3	4.92	6.01	7.15	7.87	8.50	7.25	8.54	9.50	10.08	10.48
3	5	4	4.51	4.82	5.01	5.10	5.14	4.63	4.98	5.11	5.15	5.16
3	7	4	4.19	4.38	4.44	4.46	4.48	4.24	4.39	4.45	4.48	4.49
3	9	5	4.44	4.55	4.61	4.64	4.65	4.48	4.58	4.62	4.64	4.66
4	6	4	9.55	10.90	11.88	12.50	12.90	10.88	12.90	13.35	13.48	13.58
4	8	5	8.01	8.31	8.54	8.68	8.76	8.23	8.60	8.73	8.79	8.81
4	10	5	7.89	8.02	8.09	8.12	8.13	7.93	8.05	8.11	8.13	8.14
5	9	5	30.22	64.60	75.17	81.03	84.81	72.34	83.19	87.93	90.11	91.27

The stability of the wavelet transform $\mathbf{T}_{j,s}$ is quantified by the condition of $\mathbf{T}_{j,s}$ defined by

$$\text{cond}\left(\mathbf{T}_{j,s}\right) := \left\|\mathbf{T}_{j,s}\right\| \left\|\mathbf{T}^{-1}_{j,s}\right\|. \tag{8.4}$$

It is well known that the Riesz stability of wavelet bases Ψ, $\tilde{\Psi}$ is equivalent to

$$\text{cond}\left(\mathbf{T}_{j,s}\right) = \mathcal{O}\left(1\right), \tag{8.5}$$

for details see [50]. The condition of the wavelet transform $\mathbf{T}_{j,s}$ for various constructions of wavelet bases derived from B-splines is presented in Table 8.3. Here, j is the coarsest possible level. For the case of linear splines, i.e. $N = 2$, our wavelet bases are the same as those constructed by M. Primbs [80]. For $N = 3$ the results for basis from the Chapter 6 and [80] are comparable. The condition of wavelet transform for cubic spline-wavelets, $N = 4$, is significantly better for our basis than for those from [53, 80]. In [80, 101] the condition is further improved by scaling of the wavelet basis. However, in our case this approach does not lead to the significant improvement.

8.2 Quantitative Properties of Bases with Boundary Conditions

In this section, quantitative properties of the wavelet bases adapted to complementary boundary condition of the first order are presented. As in the previous section, we improve

the condition of constructed bases by L^2-normalization. The condition of single-scale bases are listed in Table 8.4. For $N = 4$ the condition of bases constructed in this contribution is again significantly better than the condition of bases from [53, 80].
The other criteria for the effectiveness of wavelet bases is the condition number of the corresponding stiffness matrix. Here, let us consider the stiffness matrix for the Poisson equation:

$$\mathbf{A}_{j_0,s} = \left(\langle \psi'_{j,k}, \psi'_{l,m} \rangle\right)_{\psi_{j,k},\psi_{l,m} \in \Psi_{j_0,s}}, \tag{8.6}$$

where $\Psi_{j_0,s} = \Phi_{j_0} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j$ denotes the multiscale basis. It is well-known that the condition number of $\mathbf{A}_{j_0,s}$ increases quadratically with the matrix size. To remedy this, we

Table 8.3: Condition numbers of wavelet transforms for our bases denoted by CF and bases from [53, 80]

N	$\tilde{N}$	s	DKU	Primbs	CF	N	$\tilde{N}$	s	DKU	Primbs	CF
2	2	1	5.2e+00	2.4e+00	2.4e+00	3	7	1	1.2e+01	5.5e+00	3.0e+00
		2	5.7e+00	3.1e+00	3.1e+00			2	3.0e+01	8.0e+00	4.5e+00
		3	6.3e+00	3.5e+00	3.5e+00			3	2.2e+01	9.3e+00	5.9e+00
		4	7.0e+00	3.7e+00	3.7e+00			4	2.6e+01	1.0e+01	6.6e+00
		5	7.6e+00	3.9e+00	3.9e+00			5	2.8e+01	1.1e+01	7.1e+00
2	4	1	5.1e+00	2.1e+00	2.1e+00	4	6	1	2.6e+01	3.1e+01	8.8e+00
		2	5.3e+00	2.6e+00	2.6e+00			2	7.8e+01	7.6e+01	2.4e+01
		3	6.0e+00	2.9e+00	2.9e+00			3	1.1e+02	1.2e+02	4.0e+01
		4	6.9e+00	3.1e+00	3.1e+00			4	1.2e+02	1.5e+02	4.9e+01
		5	7.5e+00	3.2e+00	3.2e+00			5	1.3e+02	1.7e+02	5.5e+01
3	3	1	8.4e+00	4.1e+00	4.5e+00	4	8	1	1.6e+01	7.8e+01	6.1e+00
		2	2.1e+01	6.1e+00	7.2e+00			2	7.5e+01	1.8e+02	1.6e+01
		3	2.6e+01	7.8e+00	1.0e+01			3	1.2e+02	3.0e+02	2.4e+01
		4	2.9e+01	9.1e+00	1.2e+01			4	1.4e+02	3.6e+02	2.7e+01
		5	3.1e+01	1.0e+01	1.4e+01			5	1.6e+02	4.0e+02	2.9e+01
3	5	1	7.3e+00	4.3e+00	3.5e+00	5	5	1	1.4e+02	6.8e+02	3.7e+01
		2	1.8e+01	6.1e+00	5.4e+00			2	9.1e+02	3.0e+03	3.7e+02
		3	2.2e+01	7.3e+00	7.2e+00			3	1.4e+03	7.9e+03	1.1e+03
		4	2.6e+01	7.9e+00	8.2e+00			4	2.2e+03	1.4e+04	1.9e+03
		5	2.8e+01	8.2e+00	8.6e+00			5	3.5e+03	1.7e+04	3.1e+03

use the diagonal matrix for preconditioning

$$\mathbf{A}_{j_0,s}^{prec} = \mathbf{D}_{j_0,s}^{-1}\mathbf{A}_{j_0,s}\mathbf{D}_{j_0,s}^{-1}, \quad \mathbf{D}_{j_0,s} = \mathrm{diag}\left(\left\langle \psi'_{j,k}, \psi'_{j,k}\right\rangle^{1/2}\right)_{\psi_{j,k}\in\Psi_{j_0,s}}. \tag{8.7}$$

To improve further the condition number of $\mathbf{A}_{j_0,s}^{prec}$ we apply the orthogonal transformation to the scaling basis on the coarsest level as in [22] and then we use the diagonal matrix for preconditioning. We denote the obtained matrix by $\mathbf{A}_{j_0,s}^{ort}$. Condition numbers of resulting matrices are listed in Table 8.5.

8.3 Adaptive Frame Method with Constructed Bases

As stated above, adaptive frame methods are designed in particular for solving operator equations on complicated domains. However, even in some one-dimensional numerical examples the optimal convergence rate was not realized [87], probably due to stability problems of the used bases. Our intention is to show that the optimal convergence rates of adaptive wavelet frame methods can be achieved also for the case of higher order spline wavelets. We should emphasize that we consider the one-dimensional example as a milestone on the way to treat higher-dimensional problems.
We consider the same test example as in [46], i.e. the Poisson equation with Dirichlet boundary conditions

$$-u'' = f \quad \text{in} \quad \Omega = (0,1), \quad u(0) = u(1) = 0 \tag{8.8}$$

whose solution u is given by

$$u(x) = -\sin(3\pi x) + 2x^2, \qquad x \in [0, 0.5), \tag{8.9}$$

$$= -\sin(3\pi x) + 2(1-x)^2, \quad x \in [0.5, 1]. \tag{8.10}$$

Table 8.4: The condition of scaling bases and single-scale wavelet bases satisfying complementary boundary conditions of the first order

N	$\tilde{N}$	j	Φ_j	Φ_j^N	$\tilde{\Phi}_j$	$\tilde{\Phi}_j^N$	Ψ_j	Ψ_j^N	$\tilde{\Psi}_j$	$\tilde{\Psi}_j^N$
3	3	5	2.74	2.72	4.49	4.34	3.98	3.98	4.08	3.96
3	5	5	2.74	2.72	4.94	4.58	3.98	3.98	6.62	6.22
3	7	5	2.74	2.72	8.60	8.32	4.82	4.28	12.02	15.92
3	9	5	2.74	2.72	17.92	17.76	8.14	6.25	24.98	45.69
4	4	6	4.53	4.30	12.29	11.89	9.13	7.99	11.53	7.90
4	6	6	4.53	4.30	7.89	6.83	9.46	8.00	16.37	7.96
4	8	6	4.53	4.30	11.15	10.05	8.45	8.02	25.30	15.26
4	10	6	4.53	4.30	17.89	16.97	8.39	8.42	37.65	35.80

To test our bases, we construct a wavelet frame on Ω just as the union of interval wavelet bases on $\Omega_1 = (0, 0.7)$ and $\Omega_2 = (0.3, 1)$. Note that the singularity is contained in the overlapping part and thus the boundary scaling functions and wavelets, which may potentially cause instabilities, are more involved in the frame than in the wavelet approach. This is the reason why we use a wavelet frame instead of a wavelet basis directly.
Let us define the diagonal matrix

$$\mathbf{D} = \operatorname{diag}\left(\left\langle \psi'_{j,k}, \psi'_{j,k} \right\rangle^{1/2}\right)_{\psi_{j,k} \in \Psi} \tag{8.11}$$

and operators

$$\mathbf{A} = \mathbf{D}^{-1} \left\langle \Psi', \Psi' \right\rangle \mathbf{D}^{-1}, \quad \mathbf{f} = \mathbf{D}^{-1} \left\langle f, \Psi \right\rangle. \tag{8.12}$$

Then the variational formulation of (8.8) is equivalent to

$$\mathbf{A}\mathbf{u} = \mathbf{f} \tag{8.13}$$

and the solution u is given by $u = \mathbf{u}\mathbf{D}^{-1}\Psi$. We solve the infinite dimensional problem (8.13) by the inexact damped Richardson iterations, i.e. we use the algorithm **SOLVE** from Chapter 7.
The solution u has a limited Sobolev regularity, $u \in H^s(\Omega) \cap H_0^1(\Omega)$ only for $s < 1.5$. Thus the linear methods can only converge with limited order. On the other hand, it can

Table 8.5: The condition number of the stiffness matrices $\mathbf{A}_{j,s}^{prec}$, $\mathbf{A}_{j,s}^{ort}$ of the size $M \times M$

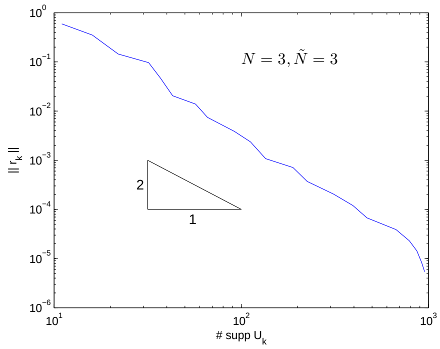
N	$\tilde{N}$	j	s	M	$\mathbf{A}_{j,s}^{prec}$	$\mathbf{A}_{j,s}^{ort}$	N	$\tilde{N}$	j	s	M	$\mathbf{A}_{j,s}^{prec}$	$\mathbf{A}_{j,s}^{ort}$
3	3	5	1	16	12.24	3.78	4	4	4	1	33	47.02	15.38
			3	64	12.72	4.79				3	125	49.56	17.40
			5	256	12.85	5.20				5	513	50.17	18.52
			7	1024	12.86	5.37				7	2049	50.28	18.91
3	5	5	1	32	52.97	4.20	4	6	4	1	33	48.98	15.25
			3	128	54.88	7.65				3	125	49.56	15.94
			5	512	55.19	8.91				5	513	50.17	16.24
			7	2048	55.24	9.47				7	2049	50.28	16.31
3	7	5	1	32	71.07	10.74	4	8	5	1	65	205.56	15.92
			3	128	71.86	29.56				3	257	208.37	25.04
			5	512	71.91	35.97				5	1025	209.12	27.47
			7	2048	71.91	38.66				7	4097	209.31	27.69

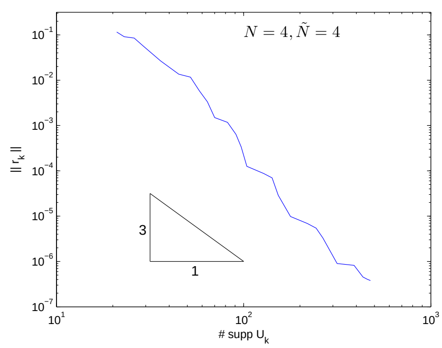
be shown that $u \in B_\tau^{s+1}(L^\tau(\Omega))$ for any positive s and $\tau = (s+0.5)^{-1}$. Therefore, we have

$$\|\mathbf{u} - \mathbf{u}_k\|_{l^2} \le C\,(\#\ \mathrm{supp}\ \mathbf{u}_k)^{-n}\,, \tag{8.14}$$

where $\mathbf{u}_k$ is the k-th approximate iteration. The rate of convergence n is limited only by the polynomial exactness of underlying wavelet bases. It can be shown that in our case relation (8.14) holds for any $n < N-1$. Figure 8.3 shows a logarithmic plot of the realized convergence rate for the spline-wavelet bases designed in this thesis with $N = 3$, $\tilde{N} = 3$ and $N = 4$, $\tilde{N} = 4$.

Figure 8.1: The l^2–norm of the residual $\mathbf{r}_k = \mathbf{f} - \mathbf{A}\mathbf{u}_k$ versus the number of degrees of freedom





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Appendix A

The Computed Scaling Coefficients

A.1 Scaling Coefficients of Daubechies Wavelets and Symmlets

Table A.1: Scaling coefficients of Daubechies wavelets and symmlets of the order N

	n	$h_n/2$		n	$h_n/2$
$N=1$	0	0.5	$N=4$	0	0.162901714025
Haar	1	0.5	Daubechies	1	0.505472857545
$N=2$	0	$(1-\sqrt{3})/8$		2	0.446100069123
Daubechies	1	$(3-\sqrt{3})/8$		3	-0.019787513117
	2	$(3+\sqrt{3})/8$		4	-0.132253583684
	3	$(1+\sqrt{3})/8$		5	0.021808150237
$N=3$	0	$(1+\sqrt{10}+\sqrt{5+2\sqrt{10}})/32$		6	0.023251800535
Daubechies	1	$(5+\sqrt{10}+3\sqrt{5+2\sqrt{10}})/32$		7	-0.007493494665
	2	$(5-\sqrt{10}+\sqrt{5+2\sqrt{10}})/16$			
	3	$(5-\sqrt{10}-\sqrt{5+2\sqrt{10}})/16$			
	4	$(5+\sqrt{10}-3\sqrt{5+2\sqrt{10}})/32$			
	5	$(1+\sqrt{10}-\sqrt{5+2\sqrt{10}})/32$			

Table A.2: Scaling coefficients of Daubechies wavelets and symmlets of the order N

	n	$h_n/2$		n	$h_n/2$
$N = 4$	0	−0.053574450709	$N = 5$	0	0.113209491291
symmlet	1	−0.020955482562	Daubechies	1	0.426971771352
	2	0.351869534328		2	0.512163472129
	3	0.568329121703		3	0.097883480673
	4	0.210617267101		4	−0.171328357691
	5	−0.070158812089		5	−0.022800565941
	6	−0.008912350720		6	0.054851329321
	7	0.022785172947		7	−0.004413400054
$N = 5$	0	0.019327397977		8	−0.008895935050
symmlet	1	0.020873432210		9	0.002358713969
	2	−0.027672093058			
	3	0.140995348426			
	4	0.511526483447			
	5	0.448290824189			
	6	0.011739461568			
	7	−0.123975681306			
	8	−0.014921249934			
	9	0.013816076478			

Table A.3: Scaling coefficients of Daubechies wavelets and symmlets of the order N

	n	$h_n/2$		n	$h_n/2$
$N=6$	0	0.078871216001	$N=6$	0	0.015646131090
Daubechies	1	0.349751907037		1	0.035319285399
	2	0.531131879940		2	0.011728127481
	3	0.222915661465		3	0.107133813002
	4	−0.159993299446		4	0.434688189148
	5	−0.091759032030		5	0.503849253613
	6	0.068944046487		6	0.088340371945
	7	0.019461604854		7	−0.177399845866
	8	−0.022331874165		8	−0.059071213109
	9	0.000391625576		9	0.034937515487
	10	0.003378031181		10	0.008668393443
	11	−0.000761766902		11	−0.003840021637
$N=6$	0	−0.027805493901	$N=6$	0	0.010892350163
	1	−0.054233997044	symmlet	1	0.002468306185
	2	0.143385273586		2	−0.083431607706
	3	0.503093999618		3	−0.034161560793
	4	0.456682437035		4	0.347228986479
	5	0.033355893026		5	0.556946391962
	6	−0.104305385873		6	0.238952185666
	7	0.025310862161		7	−0.051362484930
	8	0.036257717320		8	−0.014891875649
	9	−0.009687535129		9	0.031625281330
	10	−0.004214548168		10	0.001249961046
	11	0.002160777368		11	−0.005515933754

Table A.4: Scaling coefficients of Daubechies wavelets and symmlets of the order N

	n	$h_n/2$		n	$h_n/2$
$N=7$	0	0.055049715372	$N=7$	0	0.008496184454
Daubechies	1	0.280395641812		1	0.012171695109
	2	0.515574245818		2	−0.045896889461
	3	0.332186241105		3	−0.045347669909
	4	−0.101756911213		4	0.254712916415
	5	−0.158417505640		5	0.552902060972
	6	0.050423232504		6	0.341964557934
	7	0.057001722579		7	−0.040166830810
	8	−0.026891226294		8	−0.071425507119
	9	−0.011719970782		9	0.031637618715
	10	0.008874896189		10	0.014470379949
	11	0.000303757497		11	−0.012817445408
	12	−0.001273952359		12	−0.002321642172
	13	0.000250113426		13	0.001620571330
$N=7$	0	0.012286972695	$N=7$	0	0.007260697380
	1	0.040182583818	symmlet	1	0.002835671342
	2	0.041547646336		2	−0.076231935947
	3	0.097253249303		3	−0.099028353403
	4	0.360520275701		4	0.204091969862
	5	0.517283172606		5	0.542891354907
	6	0.182993243745		6	0.379081300981
	7	−0.191705016286		7	0.012332829744
	8	−0.126706547094		8	−0.035039145611
	9	0.044886120516		9	0.048007383967
	10	0.033023123202		10	0.021577726290
	11	−0.009020701114		11	−0.008935215825
	12	−0.003664714587		12	−0.000740612957
	13	0.001120591156		13	0.001896329267

Table A.5: Scaling coefficients of Daubechies wavelets and Symmlets of the order N

	n	$h_n/2$		n	$h_n/2$
$N = 8$ Daubechies	0	0.038477811054		8	−0.198719904453
	1	0.221233623576		9	0.028252026243
	2	0.477743075213		10	0.069248101198
	3	0.413908266211		11	−0.006329877093
	4	−0.011192867666		12	−0.014346730375
	5	−0.200829316390		13	0.002112292200
	6	0.000334097046		14	0.001415030545
	7	0.091038178423		15	−0.000338238082
	8	−0.012281950522	$N = 8$	0	−0.014059927334
	9	−0.031175103325		1	−0.047499279052
	10	0.009886079648		2	0.029308222110
	11	0.006184422409		3	0.337208867970
	12	−0.003443859628		4	0.541215484374
	13	−0.000277002274		5	0.259952909392
	14	0.000477614855		6	−0.105963690298
	15	−0.000083068630		7	−0.071651077867
$N = 8$	0	0.009449849797		8	0.069344490723
	1	0.039533768686		9	0.024448262275
	2	0.060849958763		10	−0.025854030635
	3	0.100630225409		11	−0.001820883647
	4	0.300329698686		12	0.006777463534
	5	0.498697376050		13	−0.000866133038
	6	0.271773995837		14	−0.000768012475
	7	−0.162557573414		15	0.000227333968

Table A.6: Scaling coefficients of Daubechies wavelets and symmlets of the order N

	n	$h_n/2$		n	$h_n/2$
$N=8$	0	0.006545446801		8	−0.077725910029
	1	0.016570300394		9	0.028952344937
	2	−0.016369812240		10	0.043128446276
	3	−0.037109012273		11	−0.010610508133
	4	0.181117395608		12	−0.008716700365
	5	0.512912249947		13	0.003873144408
	6	0.428862348059		14	0.000825769287
	7	0.010077635671		15	−0.000587390020
	8	−0.128113129234	$N=8$	0	−0.003453008313
	9	0.007614506816		1	−0.006257669534
	10	0.036779991371		2	0.015012405552
	11	−0.013678236129		3	0.029564906605
	12	−0.010058470130		4	0.001691819165
	13	0.004100879620		5	0.118722122182
	14	0.001236229764		6	0.442771683734
	15	−0.000488324047		7	0.492963432967
$N=8$	0	0.005441527715		8	0.090185948015
	1	0.007649851559		9	−0.176089031859
	2	−0.047545455053		10	−0.061395024064
	3	−0.097149698475		11	0.049127103047
	4	0.117229476364		12	0.016863684627
	5	0.491130234915		13	−0.008956519759
	6	0.467362845805		14	−0.001677508717
	7	0.076742020809		15	0.000925656351

Table A.7: Scaling coefficients of Daubechies wavelets and symmlets of the order N

	n	$h_n/2$		n	$h_n/2$
$N=8$	0	0.001607510599	$N=8$	0	−0.002391729255
	1	0.001552001869	symmlet	1	−0.000383345448
	2	−0.005526502806		2	0.022411811521
	3	0.012603759865		3	0.005379305875
	4	0.054813799939		4	−0.101324327643
	5	0.021368805817		5	−0.043326807702
	6	0.022374570495		6	0.340372673594
	7	0.302897386357		7	0.549553315268
	8	0.541193606217		8	0.257699335187
	9	0.257699335187		9	−0.036731254380
	10	−0.113735920114		10	−0.019246760631
	11	−0.140952895689		11	0.034745232955
	12	−0.002646758114		12	0.002693194376
	13	0.025724590719		13	−0.010572843264
	14	0.001919693783		14	−0.000214197150
	15	−0.001988353343		15	0.001336396696

A.2 The Scaling Coefficients of Coiflets

Table A.8: Scaling coefficients of coiflets of order N, the length of filter $2M$ and the Sobolev exponent of smoothness γ

	n	$h_n/2$		n	$h_n/2$
$N=1, 2M=2$	0	$\frac{1}{2}$		1	$\frac{7+\sqrt{7}}{16}$
Haar wavelet	1	$\frac{1}{2}$		2	$\frac{1+\sqrt{7}}{16}$
$N=1, 2M=4$	-1	$\frac{3}{8}-\frac{\sqrt{3}}{8}$		3	$\frac{-3-\sqrt{7}}{32}$
$\gamma=0.604$	0	$\frac{3}{8}+\frac{\sqrt{3}}{8}$	$N=2, 2M=6$	-2	$\frac{1-\sqrt{7}}{32}$
	1	$\frac{1}{8}+\frac{\sqrt{3}}{8}$	$\gamma=1.022$	-1	$\frac{5+\sqrt{7}}{32}$
	2	$\frac{1}{8}-\frac{\sqrt{3}}{8}$	most symmetrical	0	$\frac{7+\sqrt{7}}{16}$
$N=1, 2M=4$	-1	$\frac{3}{8}+\frac{\sqrt{3}}{8}$		1	$\frac{7-\sqrt{7}}{16}$
$\gamma=0.050$	0	$\frac{3}{8}-\frac{\sqrt{3}}{8}$		2	$\frac{1-\sqrt{7}}{16}$
	1	$\frac{1}{8}-\frac{\sqrt{3}}{8}$		3	$\frac{-3+\sqrt{7}}{32}$
	2	$\frac{1}{8}+\frac{\sqrt{3}}{8}$	$N=3, 2M=8$	-1	$\frac{15}{64}+\frac{3\sqrt{1495}}{1664}$
$N=2, 2M=6$	-1	$\frac{9+\sqrt{15}}{32}$	$\gamma=0.147$	0	$\frac{59}{128}-\frac{\sqrt{1495}}{832}$
$\gamma=0.041$	0	$\frac{13-\sqrt{15}}{32}$		1	$\frac{15}{64}-\frac{9\sqrt{1495}}{1664}$
	1	$\frac{3-\sqrt{15}}{16}$		2	$\frac{15}{128}+\frac{3\sqrt{1495}}{832}$
	2	$\frac{3+\sqrt{15}}{16}$		3	$\frac{5}{64}+\frac{9\sqrt{1495}}{1664}$
	3	$\frac{1+\sqrt{15}}{32}$		4	$-\frac{15}{64}-\frac{3\sqrt{1495}}{832}$
	4	$\frac{-3-\sqrt{15}}{32}$		5	$-\frac{3}{64}-\frac{3\sqrt{1495}}{1664}$
$N=2, 2M=6$	-1	$\frac{9-\sqrt{15}}{32}$		6	$\frac{5}{128}+\frac{\sqrt{1495}}{832}$
$\gamma=1.232$	0	$\frac{13+\sqrt{15}}{32}$	$N=3, 2M=8$	-1	$\frac{15}{64}-\frac{3\sqrt{1495}}{1664}$
	1	$\frac{3+\sqrt{15}}{16}$	$\gamma=1.775$	0	$\frac{59}{128}+\frac{\sqrt{1495}}{832}$
	2	$\frac{3-\sqrt{15}}{16}$		1	$\frac{15}{64}+\frac{9\sqrt{1495}}{1664}$
	3	$\frac{1-\sqrt{15}}{32}$		2	$\frac{15}{128}-\frac{3\sqrt{1495}}{832}$
	4	$\frac{-3+\sqrt{15}}{32}$		3	$\frac{5}{64}-\frac{9\sqrt{1495}}{1664}$
$N=2, 2M=6$	-2	$\frac{1+\sqrt{7}}{32}$		4	$-\frac{15}{64}+\frac{3\sqrt{1495}}{832}$
$\gamma=0.590$	-1	$\frac{5-\sqrt{7}}{32}$		5	$-\frac{3}{64}+\frac{3\sqrt{1495}}{1664}$
	0	$\frac{7-\sqrt{7}}{16}$		6	$\frac{5}{128}-\frac{\sqrt{1495}}{832}$

Table A.9: Scaling coefficients of coiflets of order N, the length of filter $2M$ and the Sobolev exponent of smoothness γ

	n	$h_n/2$		n	$h_n/2$
$N = 3, 2M = 8$	-2	$\frac{21}{640} - \frac{3\sqrt{31}}{320}$	$N = 3, 2M = 8$	-3	$-\frac{1}{32} - \frac{\sqrt{7}}{128}$
$\gamma = 1.422$	-1	$\frac{51}{320} + \frac{3\sqrt{31}}{640}$	$\gamma = 1.773$	-2	$-\frac{3}{128}$
	0	$\frac{257}{640} + \frac{9\sqrt{31}}{320}$	most symmetrical	-1	$\frac{9}{32} + \frac{3\sqrt{7}}{128}$
	1	$\frac{147}{320} - \frac{9\sqrt{31}}{640}$		0	$\frac{73}{128}$
	2	$\frac{63}{640} - \frac{9\sqrt{31}}{640}$		1	$\frac{9}{32} - \frac{3\sqrt{7}}{128}$
	3	$\frac{-47}{320} + \frac{9\sqrt{31}}{320}$		2	$-\frac{9}{128}$
	4	$\frac{-21}{640} + \frac{3\sqrt{31}}{320}$		3	$-\frac{1}{32} + \frac{\sqrt{7}}{128}$
	5	$\frac{9}{320} - \frac{3\sqrt{31}}{640}$		4	$\frac{3}{128}$
$N = 3, 2M = 8$	-2	$\frac{21}{640} + \frac{3\sqrt{31}}{320}$	$N = 4, 2M = 12$	-5	$\frac{7}{1024} + \frac{\sqrt{31}}{1024} - \frac{\sqrt{336+82\sqrt{31}}}{2048}$
$\gamma = 0.936$	-1	$\frac{51}{320} - \frac{3\sqrt{31}}{640}$	$\gamma = 1.707$	-4	$\frac{7}{2048} - \frac{3\sqrt{31}}{2048}$
	0	$\frac{257}{640} - \frac{9\sqrt{31}}{320}$		-3	$-\frac{53}{1024} - \frac{3\sqrt{31}}{1024} + \frac{5\sqrt{336+82\sqrt{31}}}{2048}$
	1	$\frac{147}{320} + \frac{9\sqrt{31}}{640}$		-2	$-\frac{39}{2048} + \frac{11\sqrt{31}}{2048}$
	2	$\frac{63}{640} + \frac{9\sqrt{31}}{640}$		-1	$\frac{151}{512} + \frac{\sqrt{31}}{512} - \frac{5\sqrt{336+82\sqrt{31}}}{1024}$
	3	$\frac{-47}{320} - \frac{9\sqrt{31}}{320}$		0	$\frac{555}{1024} - \frac{7\sqrt{31}}{1024}$
	4	$\frac{-21}{640} - \frac{3\sqrt{31}}{320}$		1	$\frac{151}{512} + \frac{\sqrt{31}}{512} + \frac{5\sqrt{336+82\sqrt{31}}}{1024}$
	5	$\frac{9}{320} + \frac{3\sqrt{31}}{640}$		2	$-\frac{47}{1024} + \frac{3\sqrt{31}}{1024}$
$N = 3, 2M = 8$	-3	$-\frac{1}{32} + \frac{\sqrt{7}}{128}$		3	$-\frac{53}{1024} - \frac{3\sqrt{31}}{1024} - \frac{5\sqrt{336+82\sqrt{31}}}{1024}$
$\gamma = 1.464$	-2	$-\frac{3}{128}$		4	$\frac{51}{2048} + \frac{\sqrt{31}}{2048}$
	-1	$\frac{9}{32} - \frac{3\sqrt{7}}{128}$		5	$\frac{7}{1024} + \frac{\sqrt{31}}{1024} + \frac{\sqrt{336+82\sqrt{31}}}{1024}$
	0	$\frac{73}{128}$		6	$-\frac{11}{2048} - \frac{\sqrt{31}}{2048}$
	1	$\frac{9}{32} + \frac{3\sqrt{7}}{128}$			
	2	$-\frac{9}{128}$			
	3	$-\frac{1}{32} - \frac{\sqrt{7}}{128}$			
	4	$\frac{3}{128}$			

Table A.10: Scaling coefficients of coiflets of order N, the length of filter $2M$ and the Sobolev exponent of smoothness γ

	n	$h_n/2$
$N = 4, 2M = 12$	-5	$\frac{7}{1024} + \frac{\sqrt{31}}{1024} + \frac{\sqrt{336+82\sqrt{31}}}{2048}$
$\gamma = 2.174$	-4	$\frac{7}{2048} - \frac{3\sqrt{31}}{2048}$
	-3	$-\frac{53}{1024} - \frac{3\sqrt{31}}{1024} - \frac{5\sqrt{336+82\sqrt{31}}}{2048}$
	-2	$-\frac{39}{2048} + \frac{11\sqrt{31}}{2048}$
	-1	$\frac{151}{512} + \frac{\sqrt{31}}{512} + \frac{5\sqrt{336+82\sqrt{31}}}{1024}$
	0	$\frac{555}{1024} - \frac{7\sqrt{31}}{1024}$
	1	$\frac{151}{512} + \frac{\sqrt{31}}{512} - \frac{5\sqrt{336+82\sqrt{31}}}{1024}$
	2	$-\frac{47}{1024} + \frac{3\sqrt{31}}{1024}$
	3	$-\frac{53}{1024} - \frac{3\sqrt{31}}{1024} + \frac{5\sqrt{336+82\sqrt{31}}}{1024}$
	4	$\frac{51}{2048} + \frac{\sqrt{31}}{2048}$
	5	$\frac{7}{1024} + \frac{\sqrt{31}}{1024} - \frac{\sqrt{336+82\sqrt{31}}}{1024}$
	6	$-\frac{11}{2048} - \frac{\sqrt{31}}{2048}$

A.3 The Scaling Coefficients of the Generalized Coiflets

Table A.11: Scaling coefficients of generalized coiflets of order N

	n	$h_n/2$		n	$h_n/2$
$N=1$	0	0.5	$N=4$	0	-0.008998637357
	1	0.5		1	-0.020545524662
$N=2$	0	$(1-\sqrt{3})/8$		2	0.220209921146
	1	$(3-\sqrt{3})/8$		3	0.570191446584
	2	$(3+\sqrt{3})/8$		4	0.342257796831
	3	$(1+\sqrt{3})/8$		5	-0.073006459213
$N=3$	0	-0.063353596125		6	-0.053469080619
	1	0.212519317935		7	0.023418670209
	2	0.600215242237	$N=5$	0	0.015724263258
	3	0.298469414114		1	-0.053039808852
	4	-0.036861646111		2	-0.016279352032
	5	-0.010988732050		3	0.410584172809
$N=4$	0	-0.039527851223		4	0.550111767301
	1	0.127103128167		5	0.150228384526
	2	0.532338906605		6	-0.053037013367
	3	0.440002251136		7	-0.008804533899
	4	-0.005981694132		8	0.003480334839
	5	-0.071201321367		9	0.001031785416
	6	0.013170638749			
	7	0.004095942063			

Table A.12: Scaling coefficients of generalized coiflets of order N

	n	$h_n/2$		n	$h_n/2$
$N=6$	0	0.000833054168	$N=7$	0	−0.001229804614
	1	0.002413563827		1	0.000417033769
	2	−0.009748571974		2	0.000366250695
	3	−0.035304213365		3	−0.030003690931
	4	0.126090646773		4	−0.069399580261
	5	0.496542562554		5	0.248298668316
	6	0.472173309036		6	0.572679370746
	7	0.025277244581		7	0.318850697917
	8	−0.108927679059		8	−0.030241076190
	9	0.017828719771		9	−0.040473002488
	10	0.019579241056		10	0.009913842985
	11	−0.006757877370		11	0.003364498540
$N=6$	0	0.009021821692		12	−0.000154023551
	1	−0.030693789784		13	−0.000454205123
	2	−0.025260295870	$N=7$	0	−0.004510498758
	3	0.312059330579		1	0.017497183819
	4	0.573795952629		2	0.000366250695
	5	0.253252564505		3	−0.094620688793
	6	−0.076101860819		4	0.113198326551
	7	−0.037164474481		5	0.529271695399
	8	0.019885670571		6	0.438723157606
	9	0.002940613850		7	0.048971525837
	10	−0.001341288203		8	−0.053466870695
	11	−0.000394244669		9	−0.001907259332
				10	0.006184247263
				11	0.000915046434
				12	−0.000494612663
				13	−0.000127503364

Table A.13: Scaling coefficients of generalized coiflets of order N

	n	$h_n/2$		n	$h_n/2$
$N = 8$	0	-0.000693539333	$N = 8$	0	-0.002454686577
	1	0.000249009917		1	0.009724710268
	2	0.011360395257		2	0.002992842085
	3	-0.017470413296		3	-0.067549133162
	4	-0.057000918918		4	0.067510394700
	5	0.177711570080		5	0.469021298425
	6	0.539584849716		6	0.503091912735
	7	0.407748625352		7	0.096723115420
	8	-0.005458076983		8	-0.088759287900
	9	-0.082290227531		9	-0.007533195353
	10	0.016368182040		10	0.019784899511
	11	0.001531303121		11	-0.000111292609
	12	-0.004229838998		12	-0.002353718355
	13	-0.001453626681		13	-0.000322867559
	14	0.000068947220		14	0.000187643801
	15	0.000192030942		15	0.000047364570

Table A.14: Scaling coefficients of generalized coiflets of order N

	n	$h_n/2$		n	$h_n/2$
$N = 8$	0	-0.000302425414	$N = 8$	0	0.002060169947
	1	-0.000113684438		1	-0.007250578942
	2	0.002995064855		2	-0.004120350628
	3	0.005325787840		3	0.043320188994
	4	-0.030084447812		4	-0.022482398606
	5	-0.023301394608		5	-0.125140534930
	6	0.260786442748		6	0.159909798341
	7	0.558664656170		7	0.538777664267
	8	0.327663781277		8	0.401516074378
	9	-0.066832045963		9	0.055067457485
	10	-0.072162612014		10	-0.041535405748
	11	0.034084741144		11	-0.005690153045
	12	0.011325115575		12	0.005011506202
	13	-0.008415753464		13	0.001018073874
	14	-0.000220919215		14	-0.000359393886
	15	0.000587693320		15	-0.000102117705