Debt Contracts and Stochastic Default Barrier

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Abstract

This article presents an EBIT-based asset pricing model with stochastic interest rate and default barrier, that turns out to be a more realistic assumption than the constant default barrier used in the currently available literature. Further, with the use of game theory analysis, the parameters of an optimal capital structure and safety covenants are examined. This set-up is able to catch the different optimal capital structures in various business cycle periods, as well as bankruptcy decisions dependent on the state of the economy. The effects of an exogenous change in the risk-free interest rate on the asset value, probability of default, and optimal debt ratio are also explained. This is the main value added of the paper: it is able to catch the connection between the business cycle, the central bank's monetary policy, the commercial bank's lending willingness and the micro-level changes in individual firms' financial condition.

Keywords: credit contracts, stochastic default barrier, asset pricing, EBIT-based models, structural models

JEL classification: C73, G12, G32, G33

1 Introduction

In the past decades financial markets rapidly gain on complexity due to an increased demand for risk diversification and hedging. The pricing of these advanced instruments was not sufficiently accurate using the traditional asset pricing models, and therefore required a new approach.

A solution was proposed by Merton (1974), who set up a framework that regarded the claims issued by a firm (e.g. debt, equity) as contingent claims on the firm's asset value. Models building on this basic idea are called "structural models", reflecting the fact, that they take into account the structural properties of the underlying assets.

The original Merton (1974) model uses several assumptions that limit its practical implementability. One of the most unrealistic restrictions is the impossibility of default prior to maturity. To solve this problem Black & Cox (1976) assumed that default occurs if the firm's asset value (also called as unlevered equity value) touches a threshold level. This level is called the Default Barrier (DB), and generally can be a constant (Leland, 1994; Longstaff & Schwartz, 1995), a deterministic (Black & Cox, 1976; Leland & Toft, 1996) or a stochastic (Briys & de Varenne, 1997; Collin-Dufresne & Goldstein, 2001) function of time. Models with a DB not only explain early default, but are also able to produce a large variety of Recovery Rates (RRs) and therefore reflect more precisely factors as bond covenants, bankruptcy costs or taxes.

This work extends the existing literature of structural models by a stochastic interest rate environment and uses the produced framework to examine the effects of parameter settings in debt contracts. With the introduction of a non-deterministic interest rate, it is possible to consider the implications of the business cycle period on the optimal debt ratio, and—using stochastic default barrier—on the bankruptcy decision as well. In order to capture the strategic behaviour of the parties in interest, game theory is invoked. The starting-point of our paper is the Goldstein *et al.* (2001) model, that will be extended by the relaxation of the constant (or deterministic) interest rate requirement.

2 Goals and Set-up

There are two basic questions that should be answered by a structural asset pricing model:

a) what is the fair value of a firm, and

b) how this value should be divided among the stakeholders?

To answer these questions, some other issues need to be specified and solved as well, such as the question of default—its conditions, expected time, and implications—or the behaviour of the parties in interest.

To set up the cornerstone of our model (henceforth the model or just Model), we use two stylized facts about the modelled firms. **First,** they maintain some (non-zero) leverage, and **second,** they keep operating until default. Consequently their debt is never paid back, or alternatively, it is rolled over. This leads us to an infinite horizon asset pricing model with continuous interest and dividend payments. Since these payments are part of the firm's cash-flow, it seems reasonable to model the earnings as the source of these pay-outs. The advantages of such an approach will be explained in more detail later.

Summing up the previous thoughts, there are two important factors in the model: a) the firm's future earnings that will be the source of the generated income, and b) the future interest rate that discounts these earnings and determines the creditors nominal revenues.

Assumptions

Being specified the basic goals and the starting point, it is time to make the model's assumptions:

(i) There are only two parties of interest, the shareholders (also called equity holders) and the debt-holders (also creditors). The management fully represents the shareholders' interest.

(ii) The Absolute Priority Rule (APR) is never violated, i.e. the shareholders are paid out only if the creditors are fully compensated.

(iii) Asset sales are prohibited, interest payments are financed by earnings and equity dilution.

(iv) When the earnings exceed the paid interest, the difference is paid out as dividend.

(v) Paid interest is a tax deductible item, however no tax carry-back or carry-forward exists.

(vi) There is a sufficiently large number of investors, and only a limited amount of companies that could be invested in.

Assumption (i) specifies the parties of interest and assumptions (ii)-(iv) determine the way how the income is divided between these two parties. Assumptions (iii), (iv), and (v) imply the unimportance of the historical cash flow and assumption (vi) has the consequence that a creditor-provided loan is always fairly priced, since the investors perfectly compete with each other.

Next to these initial assumptions we will use further suppositions in the subsequent sections, particularly during the description of the stochastic evolution of the variables: the risk-free interest follows an Ornstein-Uhlenbeck process, the Earnings Before Interest and Taxes (EBIT) is supposed to follow a Geometric Brownian Motion (GBM), and so on.

Risk-free Interest Rate

Most of the structural models assume constant risk-free interest rate in order to simplify the calculation. However, in reality this interest rate does change in time, reflecting the situation of the overall economy. Modelling the interest rate stochastically allows us to include the possibility of a macro-level change and catch the correlation between the overall market and the modelled asset.

The risk-free interest rate $r(t)$ follows an Ornstein-Uhlenbeck process suggested by Vasicek (1977), and used—among others—in the Longstaff & Schwartz (1995) approach:

$$dr = \alpha(\gamma - r)dt + \sigma_r dW_t, \tag{1}$$

where $\alpha > 0$ indicates the force pulling the interest rate back to its long-term mean γ at speed $\alpha(\gamma - r)$ per unit of time. The stochastic element is a standard Wiener process W_t times the volatility σ_r.

The expected value and variance at time s given $r(t)$ are

$$
\begin{aligned}
E_t[r(s)] &= \gamma + (r(t) - \gamma)e^{-\alpha(s-t)}, \quad t \leq s \\
Var_t[r(s)] &= \frac{\sigma_r^2}{2\alpha}(1 - e^{-2\alpha(s-t)}), \quad t \leq s
\end{aligned} \tag{2}
$$

respectively. The distribution of $r(s)$ given $r(t)$, $t \leq s$ can be written as

$$
r(s) = r(t)e^{-\alpha(s-t)} + \gamma(1 - e^{-\alpha(s-t)}) + \frac{\sigma_r}{\sqrt{2\alpha}} W_t(e^{2\alpha(s-t)} - 1)e^{-\alpha(s-t)}
$$

Having the assumption of risk-neutral measure (i.e. the yield to maturity is not dependent on the maturity date and thus there is no risk premium), the value of \$1 received at time $s \geq t$ has the value of

$$
P(t, s) = E_t \left[\exp \left\{ - \int_t^s r(\tau) d\tau \right\} \right] \tag{3}
$$

received at t.[1]

Earnings Before Interest and Taxes

Traditional models—building on the basis of the Merton (1974) framework—take unlevered equity as primitive variable with log-normal dynamics. However, for some models it seems to be more straightforward to use earnings instead of unlevered equity. Mella-Barral & Perraudin (1997) considers a firm that produces output and sells it on the market, where the price of the sold product follows a geometric Brownian motion. Mello & Parsons (1992) use a similar framework with a mining company and stochastic commodity price movements. Graham (2000) models EBIT flow as a pseudo-random walk with drift, Goldstein *et al.* (2001) and Broadie *et al.* (2007) use geometric Brownian motion for the evolution of EBIT.

To see the advantages of such approach, we should review some of the main shortcomings of the traditional framework. **First**, unlevered equity ceases to exist as a traded asset when debt is issued. This problem is one of the motivating factors behind the frameworks of Kane *et al.* (1984, 1985) and Fischer *et al.* (1989). **Second**, they treat tax payments in a different fashion as they deal with cash flows to debt and equity holders. In fact, they count tax benefit as capital inflow instead of using it for reduction of outflows. This implicitly assumes that it is always possible to deduce fully the interest costs from the tax payments, however, this is not the case when the cost of debt service is higher than the current EBIT. Leland (1994) deals with this issue introducing an asset level under which there is zero deductibility. This is basically a hybrid approach that

[1] For a closed-form solution see Vasicek (1977).

converts firm value to current EBIT, however it ignores partial deductibility. Another problem with the tax benefit approach is, that it implies higher firm value through higher tax shield as the tax rate increases. This is not only contra-intuitive, but it has been also found to be invalid by Lang & Shackelford (2000), who investigated the stock price movements during a decision process of change in capital gains tax. **Third,** as Goldstein *et al.* (2001) noted, these models may significantly overestimate the risk-neutral drift, consequently underestimate the probability of bankruptcy and so the optimal leverage ratio.

Our model assumes an EBIT evolution with log-normal dynamics, and therefore is able to address the mentioned issues. It abolishes the problem of unobservable and multiple (unlevered and levered) equity values, treats the different claims (coupon payments, dividends and taxes) in a self-consistent fashion and it is more flexible in implementing different set-ups, such as more sophisticated capital structures.

As mentioned, the evolution of the firm's instantaneous EBIT, δ_t is modeled using geometric Brownian motion with risk-neutral measure $\mathbb{Q}$, similarly as Broadie *et al.* (2007):

$$\frac{d\delta_t}{\delta_t} = \mu dt + \sigma dX_t(\mathbb{Q}), \tag{4}$$

where

$$X_t = \rho W_t + \sqrt{(1-\rho^2)} Z_t.$$

W_t is the same process as in (1), Z_t is a standard Wiener process and ρ is the correlation coefficient between the risk-free interest rate and EBIT.

If the δ_t is known at $t = 0$, the differential equation (4) has the solution

$$\delta_t = \delta_0 \cdot \exp\left\{ \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma X_t \right\} \tag{5}$$

Debt

In line with the goals and set-up described in the beginning of Section 2, we assume a perpetual debt, i.e. the principal is never paid back. Consequently the debt needs to be served regularly with respect to the actual interest rate. In order to fit our continuous-time model and instantaneous EBIT flow, we assume continuous interest payments as well. To catch constructions like a sinking fund, or alternatively a growth in debt principal, a parameter for a growth rate in the face value of debt (κ) is introduced as well. As a result, with $\kappa > 0$ firms are able to avoid the fast decline of debt to equity ratio due to increasing nominal earnings and constant face value of debt.

The debt is set up it the following way:

1. The rate of growth in face value of debt, κ is chosen
2. The borrower (i.e. the firm) chooses the initial face value of debt, FV_0

3. The lender calculates the fair value of this debt given FV_0, κ, and the current EBIT, and provides a transfer to the borrower equal to this fair value.

After receiving the funds, the borrower starts to serve the interest payments. The Face Value of debt (FV) at any point in time is given as:

$$FV_t = FV_0 \cdot e^{\kappa t}$$

Table 1: Notation

Symbol	Explanation	Base value
	Interest rate	
$r(t)$	Risk-free interest rate	$r(0) = \gamma$
γ	Long-term mean of risk-free interest rate	3%
α	Speed of expected risk-free interest rate convergence to γ	0.25
σ_r	The volatility of risk-free interest rate	0.5%
$P(t,s)$	The price of a $1 face value riskless zero-coupon bond at time t, maturing at time s	
	Firm	
δ_t	EBIT	$\delta_0 = 100$
μ	Drift of EBIT under $\mathbb{Q}$	0.01
σ	Volatility of EBIT	20%
ρ	Correlation coefficient between $r(t)$ and δ_t	0.2
V^0	Firm value with no leverage and the assumption of zero taxes	
T_C	Corporate tax rate	35%
	Debt	
FV_t	Face value of debt	
κ	Growth rate of the face value of debt FV_t	1%
$D(\delta_t)$	Debt value	
c_t	Coupon rate, equals to $FV_t \cdot r(t)$	
	Default	
DB_t	Default Barrier	
τ	Time of default	
RR	Recovery rate defined as a multiple of yearly EBIT	10×

The interest is continuously paid out at a rate $c_t = FV_t \cdot r(t)$ (coupon rate) with infinite time horizon. We assume $\kappa < \gamma$ for convergence.

The economic intuition behind this model is a floating coupon perpetual bond issue, where this corporate bond is (usually) sold below par.

Default

The event of default corresponds to the situation, when the firm does not meet its obligation on interest payments. We assume, that creditors take over the firm immediately after the default. Such default is associated with losses (due to liquidation, reorganization or other costs related to the takeover). Absolute priority rule is enforced, therefore after bankruptcy equity holders receive nothing, whereas creditors have to pay the bankruptcy costs.

As the state variable is the instantaneous EBIT, it is convenient to define the recovery value as a multiple of the EBIT at the moment of default. Since a firm effectively becomes unlevered after bankruptcy (as its debt holders become the new equity holders), and we calculate the unlevered value as well, this multiplier can be easily transformed to Loss Given Default (LGD)—a ratio that expresses the asset value lost due to bankruptcy.

Default Barrier

As we have an EBIT based model, DB will be defined for instantaneous earnings. Since the principal is never paid back, we take the DB as a function of instantaneous coupon rate instead of FV.

Such modification would imply a lower barrier in recession (low risk-free rate), and thus work counter-cyclically. There are several facts that support this design: in recession the number of bankruptcies increases (see, for example Altman *et al.*, 2005), thus banks experience losses in connection with other loans and might prefer immediate payments instead of triggering bankruptcy that yields uncertain income later. Furthermore as Altman *et al.* (2005) also showed, the recovery rate is significantly lower in recession. Exactly the opposite holds for economic boom and high interest rates, therefore higher default barrier is reasonable.

For the above mentioned reasons (even if the recovery rate is assumed to be constant and therefore independent from the risk-free interest rate) we decided to search optimal default barrier level as a linear function of the actual coupon rate c_t. To justify this decision, we have run simulations with a default barrier that is dependent only on the actual face value of debt, and therefore is not influenced by the interest rate. For the results of these simulations, see Section 5, where this deterministic default barrier is compared with the otherwise used stochastic barrier.

The Bankruptcy Decision

The decision of bankruptcy (i.e. the determination of the default barrier) can be made in several ways. The concrete realization is dependent on the transparency of the firm, on the credit contract, on the firm's structural parameters and also on the overall macro-economical situation.

When the state variable is not publicly observable, the firm's management (representing the equity holders' interests, see Assumption (i)) decides whether to default on interest payments—and therefore trigger bankruptcy—or keep the equity holders' option on firm's assets alive. Note, that if the EBIT is not sufficiently large to cover the interest payments, they need to be financed through equity dilution (as asset sales are prohibited) in order to avoid bankruptcy. This is modelled as negative dividend, since it effectively lowers equity holders' payoff by diluting their claim.

On the contrary, when the state variable is observable, bankruptcy condition can be determined within the credit contract, and therefore support more favourable debt financing. This is a safety covenant, that ensures the creditors the right to force bankruptcy if the firm performs poorly. This poor performance is indicated by crossing the DB.

3 Sensitivity analysis

As the model is so complex, that it is hardly possible to find closed form solutions to determine the values of the claims, the probability of default and other properties, we decided to use Monte-Carlo simulations in order to uncover the model's sensitivity on its parameters.[2]

These Monte-Carlo simulations are then used to investigate the reactions of the model to changes in different parameters. This is important for several reasons:

first, it helps to understand the model and its implications more properly,

second, effects of possible or expected changes in macro environment can be predicted, and

third, it facilitates to ascertain the equity holders' incentives to change these parameters if it rises their expected payoff. Therefore effective safety covenants can be introduced to avoid risk-shifting and incentive contracts can be developed that mitigate motives to such behaviour.

The Effects of Debt Face Value

The main questions addressed in the following lines are, whether—given our assumptions—it pays off to issue debt at all (as the empirical evidence suggests), if there is a maximal firm value for a given EBIT and if so, what level of FV corresponds to this maximum. Furthermore it is examined how is this optimal value dependent on the DB.[3]

Figure 1 illustrates the dependence of debt, equity and firm values on credit contracts with different face values. As it is visible, the results are in conformance with the theory known from corporate finance: the increasing tax shield is offset by the increasing bankruptcy costs at some point, producing one optimal debt ratio which maximizes the overall firm value.

[2] More details about the method of calculations can be found in Appendix C.

[3] At this point we do not concentrate on the problem how the DB is chosen; that issue will be covered in Section 4.

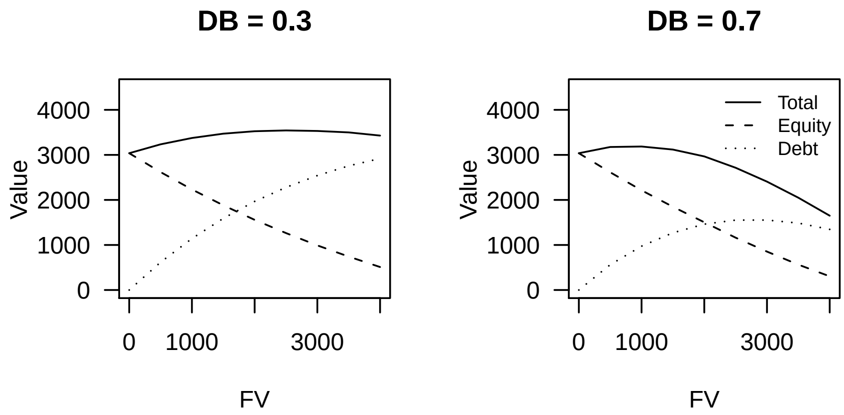

Figure 1: Debt, equity and total value with different face values of debt

With a low DB equal to 0.3[4] the firm value can reach 35 times the yearly EBIT if a debt is issued with face value between 20 and 30 yearly earnings. This means an optimal debt ratio of circa $60 - 80\%$. As the DB rises, this optimal ratio declines due to higher Probability of Default (PD): with $DB = 0.7$ the maximal firm value declines below 3200 (i.e. 32 times the yearly EBIT) with debt ratio of 30% only. The effects of changes in the DB are described in details in the following section.

The other FV level, that is interesting (next to the firm value maximization) is the one at which the debt value is maximal: this is the highest possible amount of money that could be borrowed. Consequently this is the maximum reasonable FV of the debt contract, as higher values would increase the interest payments and the PD, but it would cut back the amount of money received.

The Effects of Default Barrier Level

Next, we should explore how the output variables react on different levels of default barriers. To do so, we have plotted our basic calculation,[5] where no extreme values distort the picture. Figure 2 shows how the level of default barrier affects the equity, debt and overall firm value.

The overall firm value has the most unequivocal trend: it is declining as the barrier rises: the FV affects only the slope, not the tendency. Intuitively, setting the DB lower implies a drop in the number of bankruptcies, later occurrence of the expected bankruptcy, and shrink of the LGD in absolute terms. Recall that the expected costs of bankruptcy equal to the product of these three factors: PD, LGD and the discount.

[4] Recall that a default barrier of 0.3 means triggering default when the instantaneous earnings are at 30% of the coupon rate.

[5] That is the one with parameters set to their base levels.

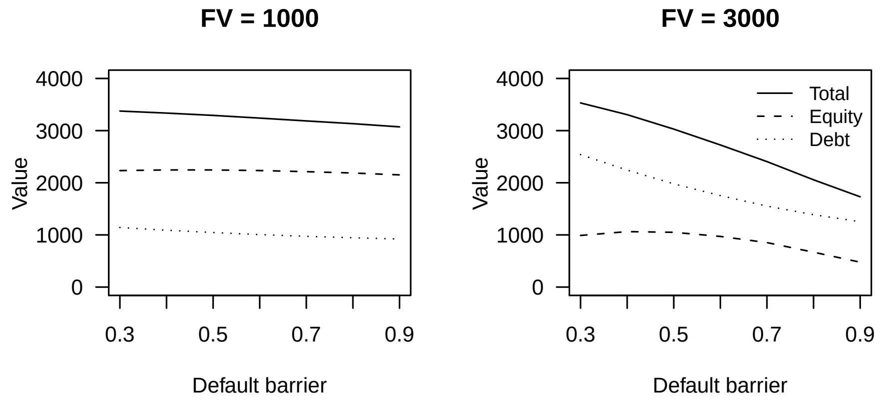

Figure 2: Debt, equity and total value dependence on the DB with FV 1000 and 3000

The value of debt is declining with higher DB level. Again, this is intuitive, since default occurs sooner, therefore less money flows to creditors through equity dilution.

The equity value curve is somewhat different: it has a “quadratic” shape with a maximum around 0.5. This means that, from the equity holders’ point of view, there exists an optimal non-zero default decision. This result is highly important for our game theory analysis in Section 4, where we examine the rational behaviour of the involved parties. This conclusion, as well as the results related to the firm and debt values, is in line with Ziegler’s (2004) findings derived using closed form calculations in constant interest rate environment.

Growth Rate in Face Value of Debt

The growth rate in face value of debt is denoted as the κ parameter, and next to the face value is one of the two exogenously set parameters that determine the interest payments and therefore their present value. The influence of κ on the coupon payments’ growth rate can be derived from the formula that determines these coupon payments:

$$c(t) = FV_t \cdot r(t) = FV_0 \cdot e^{\kappa t} \cdot r(t)$$

The only stochastic variable in this equation is $r(t)$, therefore using (2) we can express the expected value of $c(t)$ as

$$E[c(t)] = FV_0 \cdot (\gamma + (r(0) - \gamma)e^{-\alpha t}) \cdot e^{\kappa t}$$

and so

$$\lim_{t \to \infty} E[c(t)] = FV_0 \cdot \gamma \cdot e^{\kappa t}$$

The growth rate in expected values of interest payments thus converges to κ as time passes, however in the early years it is dependent on the initial risk-free interest rate level: with rate below the long term average (γ) the growth in expected interest payments is higher and vice versa. A precise calculation can be found in Appendix A.

The value of κ also influences the probability of default: it occurs when

$$\delta_t \leq DB \cdot FV_0 \cdot e^{\kappa t} \cdot r(t)$$

holds for the first time (as both sides are continuous in t, equality can be used as well). Assuming constant $r(t) = r$, the probability of default is[6]:

$$P\{\tau_b < \infty\} = \begin{cases} 1 & \text{if } \mu - \frac{\sigma^2}{2} \leq \kappa \\ \exp\left[-2\hat{\mu}\hat{b}\right] & \text{if } \mu - \frac{\sigma^2}{2} > \kappa \end{cases} \tag{6}$$

where

$$\hat{b} = -\frac{\ln(DB \cdot FV_0 \cdot r/\delta_0)}{\sigma}$$
$$\hat{\mu} = \frac{\left(\mu - \frac{\sigma^2}{2} - \kappa\right)}{\sigma}$$

and τ_b is the time of default. As equation (6) shows, default is sure if $\mu - \frac{\sigma^2}{2} \leq \kappa$ and its probability is otherwise increasing in the DB, initial face value of debt, volatility and growth rate in face value of debt, κ.

On the other hand it seems to be reasonable to keep κ above some level: as the EBIT and the total firm value are supposed to growth at a rate μ, the leverage ratio is expected to decline for $\kappa < \mu$. Since the equity value, debt value and total firm value are homogeneous function of degree one with respect to the instantaneous EBIT, the optimal proportion of EBIT to debt face value is constant. The question is, what is the breakpoint of κ at which the gain from smaller expected distance to optimal leverage in the future is offset by increased probability of bankruptcy.

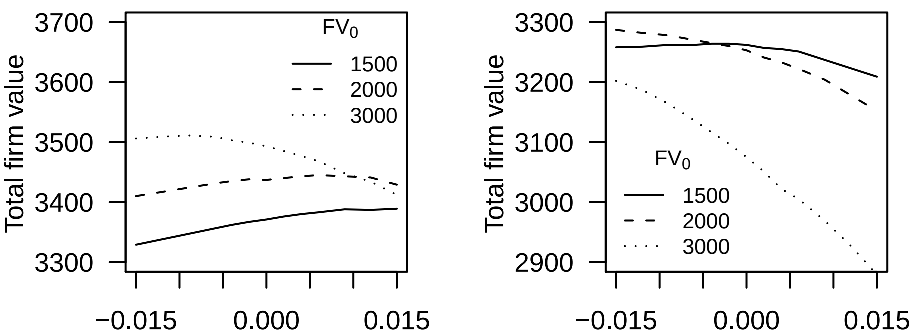

Figure 3: Firm value dependence on κ with DB=0.3 (left) and DB=0.5 (right)

We have run simulations in order to see the effect of changes in the κ parameter in our model. The results are consistent with the theoretical calculations

[6] For the derivation of this result see Section A

and our expectations. Figure 3 illustrates the evolution of the firm value as κ changes for two different DBs and three different initial FVs. As it noticeable, the κ maximizing total firm value is lower for higher DB and higher initial FV. This means that when the other parameters are increasing the probability of default it is optimal to offset this by lower κ value. Consequently, for companies with high probability of default (due to high leverage, default barrier, volatility or any other factors) it pays off to establish a sinking fund.

Tax Rate

In our model, debt financing exists only because of the presence of a positive corporate tax:[7] interest payments are not taxed, and therefore debt financing creates a tax shield that increases the value of the firm. Consequently a higher tax rate implies an incentive for higher debt issue in order to reduce tax payments. The other natural effect of a higher tax is decrease in overall firm value.

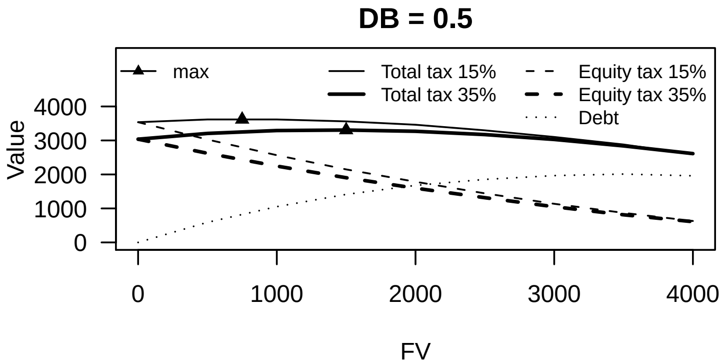

Figure 4: Firm value dependence on tax rate

Figure 4 depicts the sensitivity of firm, debt and equity values on the rate of corporate tax. The chart suggests, that the results are in line with the theoretical expectations. The FV maximizing the firm value is increasing in the tax rate, whilst the maximal firm value decreases with higher tax. However, the bankruptcy decision is not affected by the tax burden.[8] Consequently the probability of default is not affected by a change in corporate tax rate, and therefore leaves the debt value unaffected.

[7] However, there might be other reasons for preferring debt financing to equity issue. An important example is the situation when the current owner wants to keep his full control over the company, however he has not enough funds to finance the ongoing or new projects.

[8] The increased tax rate only scales down the equity value, it does not change its sign. Consequently equity dilution works independently on the tax burden.

Recovery Rate

Usually the Recovery Rate (RR) represents the fraction of the firm's asset value that remains to the owners after the costs of the bankruptcy are paid. In our case, however, it is more convenient to define RR as a multiple of the yearly EBIT at the moment of default in order to simplify the calculations. This can be done without the loss of generality, as our RR can be easily transformed to the classical one: the asset value is equal to the value of an unlevered firm, what is calculated in our simulations as an EBIT multiple. For example, our basic set-up has unlevered equity value 30 yearly EBITs (see Table 3), therefore $RR = 5$, $RR = 10$, and $RR = 20$ corresponds to "classical" recovery rate of 17%, 33%, and 66% respectively.

Since we assume no APR violations, the bankruptcy costs are born solely by the debt holders, similarly as in Leland's (1994) model. Consequently the value of the shareholders' claim is independent on the RR whereas the debt and so the total value are increasing in RR.

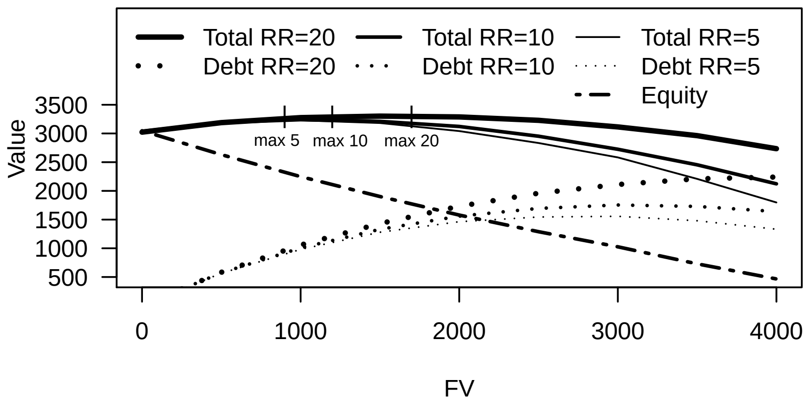

Figure 5: Firm value dependence on Recovery rate, $DB = 0.6$

Since equity holder's bankruptcy decision is not affected by the RR, we do not need to examine different DB levels. On the contrary, the firm value is dependent on the RR, and so FV that maximizes it might be sensitive as well. To find out how the optimal debt ratio is dependent on the RR, we have simulated firms with three different (5,10, and 20) recovery rates, all other variables leaving unchanged. The obtained values are plotted on Figure 5, with maximal firm values visualised. According to the calculated results, the optimal debt ratio increases with higher RR. This is an intuitive outcome: the expected costs of default are decreasing in the RR, and therefore the optimal FV shifts to higher levels.

4 Agency Costs

Agency costs refer to the costs incurred by the firm in question due to divergent objectives of parties in interest. If information asymmetry is present, it prevents the possibility of signing an enforceable agreement that could avoid these losses. As the parties (in this case the shareholders and the creditors) follow their interest in order to maximize their payoff in several steps, the proper tool for the analysis is game theory.

We will use backward induction with a game tree of three levels representing the three stages of a loan agreement: 1) before signing the contract, when the shareholders make decision about the debt parameters (FV_0 and κ), 2) signing the contract and providing the loan, when the creditor calculates the fair price, and 3) managing the leveraged firm. Figure 6 illustrates these three stages.

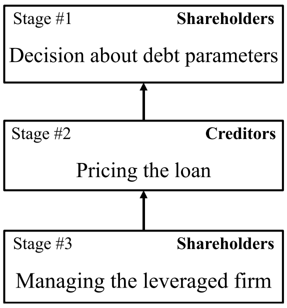

Figure 6: The game tree of the loan agreement

Observable Actions

With observable actions, the creditor is able to control the parameters that affect the probability distribution of the EBIT flow, most importantly σ, which is determined by the riskiness of the firm's projects. This situation significantly simplifies the arrangement of the credit contract, since the lender does not need to study the set of possible actions that might be done by the debtor. In other words, the probability distribution of the payoffs is given, and therefore risk-shifting is not possible.[9]

Observable State Variable

The simplest situation is, when the firm is completely transparent, and therefore the creditor can observe both the management's actions and the state of the firm. In this case a debt contract can be signed with such covenants that enforce

[9] More about risk shifting in the next section, where—in contrast with the present situation—it is possible.

an agreed volatility and defines a default barrier at which bankruptcy will be triggered.

Table 2: Total firm values - Basic parameters

	0.3	**0.4**	**0.5**	0.6	0.7	0.8	0.9
0	3037	3037	3037	3037	3037	3037	3037
500	3234	3222	3207	3191	3175	3156	3137
1000	3375	3336	3292	3240	3186	3133	3072
1500	3471	3393	**3306**	3211	3118	3010	2897
2000	3525	**3404**	3270	3122	2966	2778	2582
2500	**3543**	3375	3173	2952	2714	2463	2179
3000	3531	3306	3029	2725	2406	2057	1731
3500	3499	3203	2839	2454	2049	1632	1203
4000	3429	3045	2613	2123	1650	1147	1000

Default barrier on the X-axis and debt face value on the Y-axis

In this case such a combination of debt face value and default barrier will be chosen that maximizes firm value (in our basic calculation with results printed in Table 2, this corresponds to the setting $DB = 0.3$[10], $FV = 2500$, with total firm value of circa 35.5 yearly EBITs). This leads to a highly leveraged firm (to maximize the value of tax shield), and to a low default barrier (to minimize the bankruptcy costs). Note, that it might be impossible to specify an arbitrarily low DB: when the EBIT decreases so drastically, that the equity becomes worthless, it is not possible to finance the interest payments trough equity dilution. In a stock company the shareholders cannot be forced to transfer additional funds to the distressed firm. In contrast, when the considered firm is owned by a parent company, the interest payments can be guaranteed by the mother.

Not Observable State Variable

Similarly as in the previous case, actions are observable, and therefore risk shifting is not possible. However, as the state variable is not followed by the creditor, a bankruptcy barrier as safety covenant can not be included in the credit contract, because it would be impossible to enforce it. Consequently the debtor will choose the default barrier in a way that maximizes its equity holders' value under the given circumstances. This decision is the bottom level of the game tree (Stage #3), and therefore it determines the expected payoffs under certain credit contract parameters. Table 3 shows an equity value matrix for several debt face values calculated using the base parameter setting.[11] As it can

[10] We did not calculate cases with even lower barrier. These would have produced higher total values, however it is hard to imagine that the firm would be kept alive with extremely low earnings. Furthermore there are usually some fixed assets owned by the company (immovable property, etc.) that cannot lose their values completely. Consequently the RR might be higher for firms with extremely low EBIT flow. Since we assume constant RR, we decided to leave out these extreme cases.

[11] See Table 1

be seen, the equity holders will choose to default on interest payments when the EBIT will be between 40 and 50% of the coupon rate (bold values in Table 3).

Table 3: Equity values - Basic parameters

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	3037	3037	3037	3037	3037	3037	3037
500	2618	**2623**	**2623**	2619	2614	2604	2593
1000	2233	**2246**	**2246**	2234	2213	2187	2151
1500	1881	**1904**	**1903**	1881	1848	1792	1727
2000	1558	**1595**	**1595**	1562	1504	1404	1281
2500	1264	**1319**	**1316**	1257	1162	1037	865
3000	990	**1063**	**1050**	971	853	669	477
3500	742	**837**	**813**	725	563	354	121
4000	510	**623**	**610**	483	304	73	0

Default barrier on the X-axis and debt face value on the Y-axis

In Stage #2 the creditor anticipates the hareholders' behaviour in the bankruptcy triggering decision and he prices the loan according to this action.

Finally we get to the initial Stage #1, where the shareholders are supposed to choose the credit contract that maximizes their pay-off given the loan's pricing method. This pay-off has two components: first, they receive the granted amount of money (the loan), and second, they maintain a residual claim on the firms earnings. This adds up to the total firm value (the initial debt value plus the value of the residual claim, i.e. the equity value), and so they will choose a loan that maximizes the firm value. Table 2 gives the valuation to this stage of the game: the creditor offers loans priced according to the equity holders's default decision, therefore the equity holders can choose total firm value only within the column specified by the planned (by shareholders) respectively assumed (by bondholders) DB. In this case the optimal face value of debt is 2000 for $DB = 0.4$ and 1500 for $DB = 0.5$. The corresponding firm values are 3400 and 3300 respectively.[12] The resulting total value (33–34 yearly EBITs) is significantly higher than the unlevered value with 30 EBITs only. On the other hand, the maximally possible 35.5 EBITs is not reached due to agency costs caused by asymmetric information.

Paradoxically, the equity holders' ex post effort to increase the value of their claim decreases the total firm value (and so their total payoff) ex ante. This problem can be solved if they manage to ensure the lender, that they will default on their payments when the EBIT truly crosses the agreed DB. Such contract requires monitoring with some associated costs, however if these expenses are below the agency costs then monitoring should be introduced.

[12] All these values are rounded: as we want to illustrate the decision process, the accurate numbers are not important. In real the DB is one number (between the mentioned 0.4 and 0.5) not an interval, and the FV that corresponds to the maximal firm value given this DB is determined unambiguously as well.

Hidden Actions

When the management's actions are not observable, the debtor is able to modify the parameters driving the EBIT flow (in Stage #3), and so to change the expected payoffs of the involved parties. More specifically, he is able to shift the risk to the creditor, and consequently to enhance the value of his claim on the creditor's costs.

To find out whether risk-shifting appears in our model, and if it does, what are the consequences, we have run simulations[13] with several different EBIT volatility parameters. With higher σ values we observed the following (see Figure 7):

Equity value was rising, with steeper slopes for lower DB settings. In consequence the equity holders try to increase the EBIT volatility as much as they can, however they have a lower incentive to do so when the DB is higher. This means that if there are some additional costs of higher volatility paid by the equity holders[14], than they will not set the volatility to such high levels as they would so with lower DB.

Debt value was declining, however this decline was moderate for high DB settings. There are two reasons that support lower losses in debt value: First, and most importantly, default occurs at higher firm value, and therefore the firm has higher residual value after the bankruptcy that is transferred to the creditor. Second, default occurs earlier, therefore the asset value received has a smaller discount.

Probability of default rose.

Total firm value was decreasing due to increased PD.

Default barrier chosen by the equity holders was decreasing: their option on the firm's assets become more valuable with the increased volatility.

All of these observations are in line with the conclusions of Ziegler (2004), who based his analysis on game theory and gave closed-form results for his model with constant risk-free interest rate. Next we examine how the observability of the instantaneous EBIT affects the credit contract's design and the behaviour of the involved parties.

Observable State Variable

If the state variable is observable, it is feasible to mitigate the equity holders' risk-shifting incentive by setting a sufficiently high DB as a safety covenant. This is intuitive, if we extend Merton's idea: instead of a simple European call, we represent the equity value as a down-and-out call barrier option. Such an

[13] For some of the results, see Section B.
[14] This could be lower expected EBIT growth, or some risk of being exposed, for example.

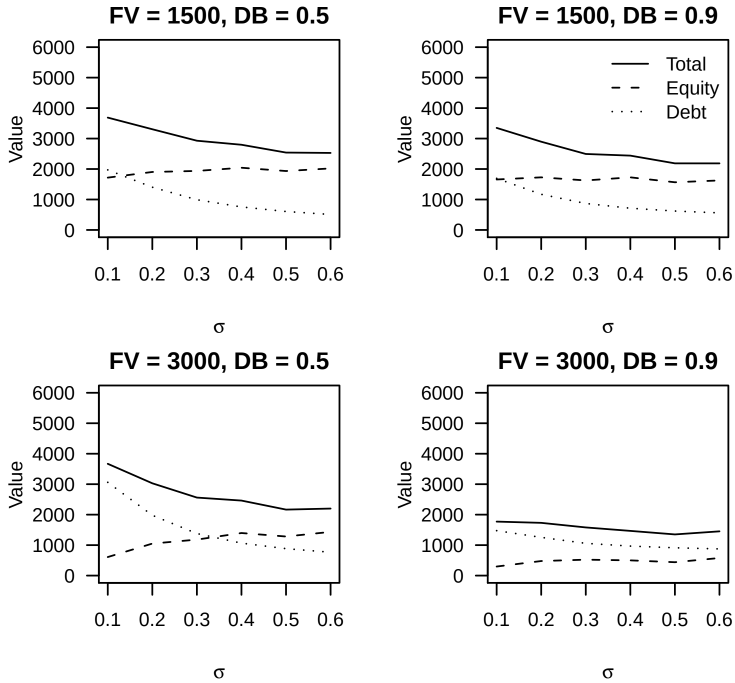

Figure 7: Firm value dependence on σ

option is less dependent on the underlying asset's volatility than the standard European call, especially when the stock price is close to the barrier.

Our model shows a similar behaviour: when the DB is high (80–90% of the coupon rate), the equity value is not increasing significantly with higher volatility. This implies a loan with low FV (about 5 yearly EBITs in our basic setting; recall Figure 1), and consequently results a total firm value of only circa 3150 (31.5 yearly EBITs). Comparing this number with the theoretical maximum of a fully transparent firm (3550), the losses caused by risk-shifting are equal to the firm's four yearly earnings. Similarly as in the case of not observable state variable, it might pay off to introduce monitoring on the management's actions, and therefore to avoid risk-shifting.

Not Observable State Variable

If the state variable is not observable, equity holders will increase the EBIT volatility and default on interest payments later (Stage #3). Since the creditor anticipates such behaviour (Stage #2), he prices the loan with respect to higher expected volatility. Consequently the resulting firm value (as it is depicted in Figure 7) is lower than the value of the unlevered firm. The shareholders' ex-post behaviour therefore disables debt financing, and hence making the possible tax benefits unavailable.

5 The Contribution of the Stochastic environment

Up to this point, the presented model provided similar results and conclusions as the existing literature. It was shown, that it is compatible with the referred articles, at so it is their legitimate extension. In this section we present the contribution of the stochastic interest rate assumption and the reasonability of the stochastic Default Barrier (DB).

Initial Interest Rate Level

An important advantage of the introduced mean-reverting interest rate environment is, that it can deal with a risk-free interest rate that is not on its long-term average (γ). In such case the rate is expected to return to γ, however, this takes some (random) time. In models with constant interest rate it is not possible to cover this situation. With a stochastic interest rate model though, it is just a question of different initial value $r(0)$ in the Stochastic Differential Equation (SDE) (1). Furthermore, the effects of exogenous changes in this initial level can be examined. These exogenous changes in the risk-free interest rate correspond to the decisions of the central bank, and therefore we are able to predict the effects of the monetary policy on microeconomical level.

To see the effects of changes in the initial interest rate, we have run calculations with $r(0) = 1\%$, $r(0) = 3\%$, and $r(0) = 5\%$. Figure 8 demonstrates the obtained results for two different FVs. The tick lines show the total firm value dependence on the DB for three different initial interest rate levels. The gap between these lines represents the loss—ceteris paribus—when the interest rate suddenly increases to the next examined level. This drop in firm value is caused by two factors: higher discount for all future earnings and increased PD due to higher interest payments.[15] The mentioned gap is a sum of declines in equity and debt value, and therefore we can divide this area to distinguish the losses of the two involved parties.[16]

For a better understanding of the forces driving these changes, we have plotted the PD in the first 50 years and the mean times of defaults happened before year 50 ($E[\,\tau \mid \tau < 50\,]$, where τ is the time of default, as usually). As it can be seen on Figure 9, both the PD and the expected time of default seem to be insensitive to changes in the initial interest rate, when the FV and the DB are low.[17] On contrary, when the default probability is high due to other parameters, both PD and $E[\tau]$ become sensitive to initial interest rate movements.

A larger fraction of the firm losses is booked by the equity holders (recall Figure 8). Their claim is depreciated by the factors that affect the firm value

[15] Higher interest payments imply higher DB in absolute terms. The DB of the x axis on Figure 8 is a ratio of the instantaneous interest payments.

[16] For the calculated debt and equity values, see Section B.

[17] A more precise description would be, that the difference of these values is below the level of significance.

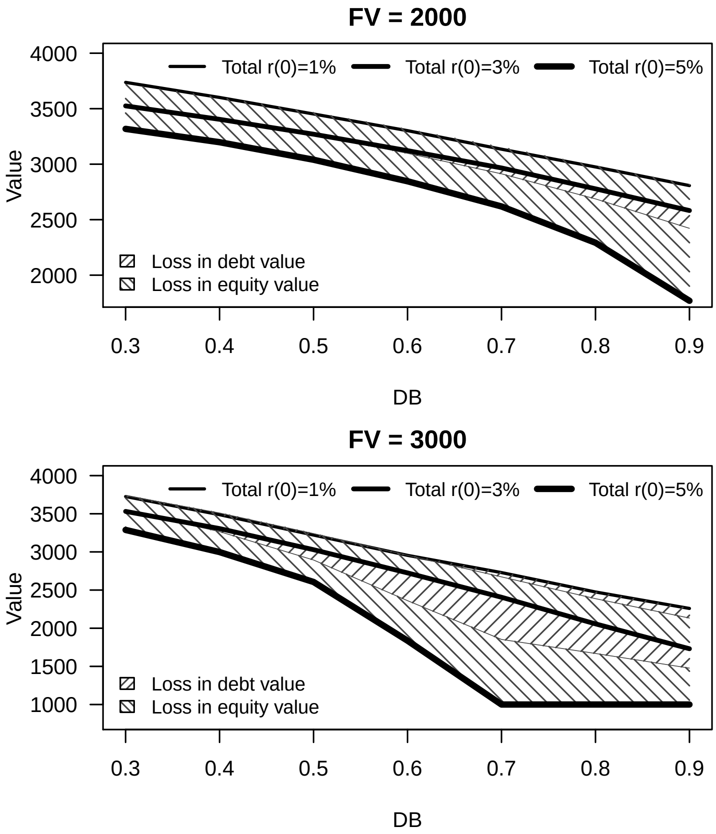

Figure 8: Firm value dependence on initial interest rate

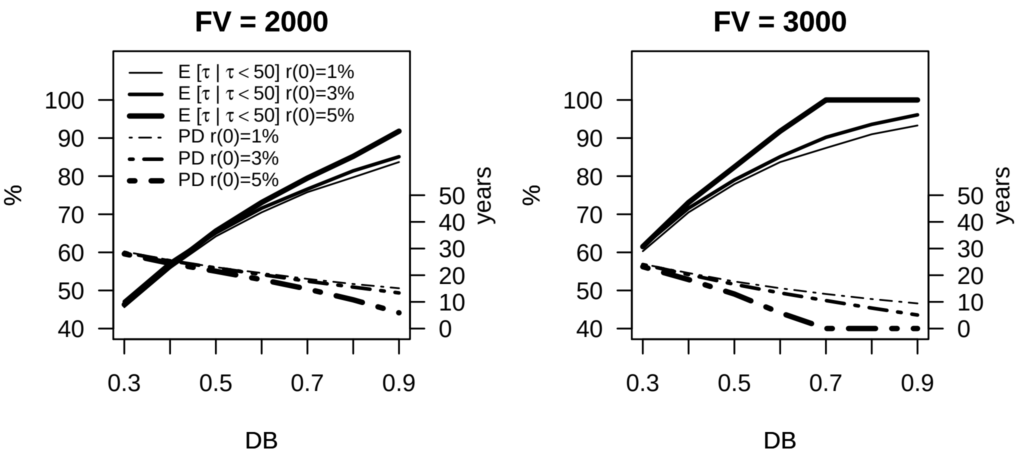

Figure 9: PD and default time dependence on initial interest rate

(i.e. higher discount of future income and increased PD), and also by one additional: higher interest paid out to debt holders.

We can see that the debt value is insensitive to changes in initial interest rate, when the probability of early default is close to zero due to low FV and DB. Our conclusion is, that increased coupon payments perfectly offset higher discount on future cash flows.[18] Consequently the only factor that decreases the bond's value is the increased default probability and its earlier expected occurrence.

Note, that this section explains how the central bank's interventions work. In economical downturn the monetary policy can support the companies by targeting a lower short-term rate. This increases the value of both traded and non-traded assets, reduces the number of defaults, and supports debt financing through the decrease of interest paid on the outstanding principal. The latter is favoured by two factors: the risk-free interest is low, and the risk-premium drops due to lower PD. On the contrary, an overheated economy can be cooled down with higher risk-free interest.

Comparison of Stochastic and Deterministic Default Barrier

Stochastic risk-free interest rate and DB are the two features of our model that distinguish it from other EBIT-based works (Goldstein *et al.*, 2001; Broadie *et al.*, 2007). The contribution of a stochastic interest rate is intuitive: a constant or deterministic risk-free rate is hardly acceptable. Its usefulness was presented also in Section 5, where our model has easily dealt with different initial interest rate levels and it was able to predict the implications of such macro-level shocks. The benefits of a stochastic DB were however not proved. In the description of the DB for our model (see Section 2) we mentioned why banks might prefer a DB that is dependent on the interest rate. We saw however, that it is not the bank who sets the default triggering level: it is the debtor or it is specified in the debt contract, that is designed by both parties.

In order to examine whether it is correct to base our model on stochastic DB we simulated two firms with identical parameters,[19] but different DB settings: one stochastic, driven by the instantaneous risk-free interest rate, and one deterministic DB, dependent only on FV_t.

The default triggering levels were therefore set to $FV_t \cdot r(t) \cdot DB$ in the stochastic case and to $FV_t \cdot \gamma \cdot DB$ in the deterministic case, where $DB > 0$ is the same variable in both cases. Figure 10 visualizes the comparison of results obtained by stochastic and deterministic DB setting. For the first sight it is apparent that the total firm value is higher when the DB is defined as a deterministic function.

[18] For $\kappa = 0$ this is intuitive: the defaultable corporate bond can be represented as a risk-free bond with the same parameters minus the expected losses caused by default. Since the price of a riskless bond that pays continuous interest is always equal to its face value, it is not dependent on the current interest rate.

[19] These parameters were the same as in the basic setting, with the exception of lower recovery rate (5 yearly EBITs), and higher correlation between the EBIT and interest rate processes ($\rho = 0.5$). These modifications were made in order to make the results more sensible on the selection of the DB.

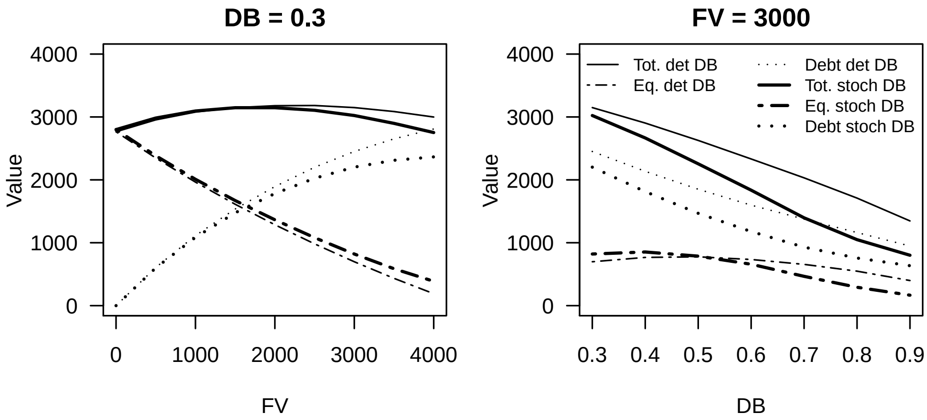

Figure 10: Stochastic vs. Deterministic DB

To understand the reason of this better performance, let us take an example macro-level shock. Assume a firm with our basic parameters, that is near the defined DB with the parameter $DB = 0.3$: set the EBIT to $\delta_t = 100$, the Face Value of debt to $FV_t = 10,000$ and the risk-free interest rate to $r(t) = 3\%$. The default triggering level is in both cases $10,000 \cdot 0.03 \cdot 0.3 = 90$, as $r(t) = \gamma = 0.03$. Now suppose a jump in the W Wiener process, so that $dW_t = 1$ for a very short time frame (i.e. dt is close to zero). Using (1) and (4) we can calculate the changes in the earnings and the interest payments:

$$dr = \alpha(\gamma - r)dt + \sigma_r dW_t = 0 + 0.005 \cdot 1 = 0.005$$

and

$$d\delta_t = \delta_t(\mu dt + \sigma dX_t) = 100 \cdot (0 + 0.2 \cdot dX_t),$$

as $dt \approx 0$. Using $X_t = \rho W_t + \sqrt{(1-\rho^2)}Z_t$ we can write

$$d\delta_t = 100 \cdot 0.2 \cdot (\rho dW_t + \sqrt{(1-\rho^2)}dZ_t).$$

Knowing that $E[dZ_t] = 0$, we obtain the expected change in the EBIT process:

$$E[d\delta_t] = 100 \cdot 0.2 \cdot 0.2 \cdot 1 = 4$$

If this shock was positive, the new interest rate is 3.5%, and therefore the stochastic default triggering level increases to $10,000 \cdot 0.035 \cdot 0.3 = 105$. The EBIT grows to 104,[20] as we calculated, and consequently default occurs with stochastic DB, whereas it does not occur with deterministic DB that remains at level 90, independently on the interest rate.

A macro-shock with the same magnitude, but opposite direction produces $\delta_t = 96$ and stochastic default barrier level of 75, using similar calculations as

[20] It is expected to grow to 104. However, as the considered time interval approaches to zero, the grow will converge to 4 independently on the realization of Z_t.

above. Therefore there is no default neither with stochastic nor with deterministic DB.

A deterministic DB therefore softens the default triggering bound, and hence increases the firm value. The problem is however, that when the primitive variable is not observable,[21] default is triggered by the equity holders in a way to maximize the value of their claim. Recall Figure 10: a stochastic DB bears higher equity value for barrier ratios below 0.5. Since the equity-maximizing DB is below 0.5 (as we have seen in Section 3 and Section 4), the equity holders will prefer triggering default according to a stochastic barrier. In fact this is a logical conclusion: the situation of the overall economy, as well as the size of the interest payments is taken into account.

6 Conclusion

Our work extends the available literature of structural asset pricing models by an Earnings Before Interest and Taxes (EBIT) based model with stochastic interest rate. This framework is able to price equity and debt in a way consistent with the cash flow of the firm, and therefore to address some defects of the current frameworks. It solves the "delicate" issue of Leland (1994), that the unlevered firm value might not be a traded asset, and deals with the problem of partial tax deductibility. The stochastic interest rate assumption contributes the possibility of analysing the effects of changes in the central bank's monetary policy, and it is able to answer the question how the macroeconomical situation affects the optimal capital structure. The default is triggered using a stochastic interest barrier, that is shown to be more accurate then its deterministic equivalent.

We also analyse the design of credit contracts, focusing on the finding of firm-value maximizing parameters and safety covenants. With the help of the game theory apparatus, actions taken by the involved parties can be predicted. Using this scheme the agency costs arising due to asymmetric information are computed, and methods are suggested for the minimization of these losses.

Since we use numerical calculations, the model can be easily extended and modified in many aspects. A natural candidate is a more complex capital structure, with several debt classes, contracts with finite horizon and absolute priority violations. Also, following Broadie *et al.* (2007) it would be fruitful to examine a two-barrier model, where reorganization and liquidation are distinguished.

A weak point in our design is the assumption that the EBIT process is driven by a GBM, and therefore it cannot handle negative earnings. It might be argued that employing arithmetic Brownian motion would be a better choice for this reason, however it should be noted that our model has an infinite time horizon. As the prices of the commodities grow exponentially, it is hard to accept a linear model for the EBIT evolution. Finding better alternatives for the EBIT process will be the subject of further research. A promising idea is to model the earnings

[21] As it was discussed in Section 4, observable primitive variable implies low default triggering level. Consequently there is insignificant difference in the values produced by the two DB types.

as a difference of two correlated GBMs (representing revenues and expenses): it has a clear economic intuition, it is able to produce negative values, has an exponential expected evolution, and works with observable figures.

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A Calculations

Growth Rate of Interest Payments

We know, that

$$E[c(t)] = FV_0 \cdot (\gamma + (r(0) - \gamma)e^{-\alpha t}) \;\cdot\; e^{\kappa t},$$

therefore the growth rate can be calculated as

$$\frac{\frac{\partial E[c(t)]}{\partial t}}{E[c(t)]} = \frac{FV_0 \cdot (\gamma\kappa + (r(0) - \gamma)(\kappa - \alpha)e^{-\alpha t}) \;\cdot\; e^{\kappa t}}{FV_0 \cdot (\gamma + (r(0) - \gamma)e^{-\alpha t}) \;\cdot\; e^{\kappa t}} =$$

$$\frac{\gamma\kappa + (r(0) - \gamma)(\kappa - \alpha)e^{-\alpha t}}{\gamma + (r(0) - \gamma)e^{-\alpha t}} = \kappa + \frac{-\alpha(r(0) - \gamma) \cdot e^{-\alpha t}}{\gamma + (r(0) - \gamma) \cdot e^{-\alpha t}}.$$

Probability of Default as a Function of κ

Recall the definition of processes involved:

Risk-free interest rate:

$$dr = \alpha(\gamma - r)dt + \sigma_r dW_t$$

$$r(t) = r(0)e^{-\alpha t} + \gamma(1 - e^{-\alpha t}) + \frac{\sigma_r}{\sqrt{2\alpha}} W_t(e^{2\alpha t} - 1)e^{-\alpha t}$$

EBIT:

$$\frac{d\delta_t}{\delta_t} = \mu dt + \sigma dX_t$$

$$X_t = \rho W_t + \sqrt{(1 - \rho^2)} Z_t$$

$$\delta_t = \delta_0 \cdot \exp\left\{ \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma X_t \right\}$$

Default occurs at:

$$\delta_t = DB \cdot FV_0 \cdot e^{\kappa t} \cdot r(t)$$

$$\delta_0 \cdot \exp\left\{ \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma X_t \right\} = DB \cdot FV_0 \cdot e^{\kappa t} \cdot r(t)$$

$$\exp\left\{ \left(\mu - \frac{\sigma^2}{2} - \kappa\right) t + \sigma X_t \right\} = \frac{DB \cdot FV_0 \cdot r(t)}{\delta_0}$$

$$\left(\mu - \frac{\sigma^2}{2} - \kappa\right) t + \sigma X_t = \ln\left(\frac{DB \cdot FV_0 \cdot r(t)}{\delta_0}\right)$$

$$X_t = \frac{\ln\left(DB \cdot FV_0 \cdot r(t)/\delta_0\right)}{\sigma} - \frac{\left(\mu - \frac{\sigma^2}{2} - \kappa\right)}{\sigma} \cdot t$$

Note that the first term is negative as the initial EBIT is supposed to be higher than the initial level of the DB (otherwise default would occur immediately). Now assume a constant $r(t) = r$, and see the probability that this equality will hold within a finite time horizon. As X_t is a standard Wiener process, and so has a symmetric probability density function with respect to the origin, we can multiply the right side by -1 without changing the calculated probability. Therefore the first constant term will be positive.

Denote

$$\hat{b} = -\frac{\ln\left(DB \cdot FV_0 \cdot r/\delta_0\right)}{\sigma}$$

$$\hat{\mu} = \frac{\left(\mu - \frac{\sigma^2}{2} - \kappa\right)}{\sigma}$$

and the first time of reaching the barrier as $\tau_b = \inf\left\{t : X_t = \hat{b} + \hat{\mu}t\right\}$. We want to calculate the probability

$$P\left\{\tau_b < \infty\right\}.$$

This is a simple boundary crossing problem, and has the following solution:

$$P\left\{\tau_b < \infty\right\} = \begin{cases} 1 & \text{if } \hat{\mu} \leq 0, \text{ i.e.; } \mu - \frac{\sigma^2}{2} \leq \kappa \\ \exp\left[-2\hat{\mu}\hat{b}\right] & \text{if } \hat{\mu} > 0, \text{ i.e.; } \mu - \frac{\sigma^2}{2} > \kappa \end{cases}$$

B Simulations

The simulations were run in the GNU R software environment on several computers with Gentoo Linux operating system. The number of iterations was set in a way to produce stable (and therefore significant) results, and it was typically 5,000. In simulations with higher asset volatility (i.e. $\sigma > 0.2$) the number of necessary iterations was higher: for $\sigma = 0.6$ we iterated 120,000 times.

The tables present the referred values for different FV and DB settings. The rows represent the different FVs (see the first column), whereas the columns correspond to different DB ratios (see the first row).

Basic Setting

Parameters:
$\rho = 0.2$, $\sigma = 0.2$, $\mu = 0.01$, $\kappa = 0.01$, $RR = 10$, $T_C = 35\%$,
$\alpha = 0.25$, $\gamma = 0.03$, $\sigma_r = 0.005$, $r(0) = \gamma$
For equity and total values, see Table 3 and Table 2.

Table 4: Debt value

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0	0	0	0	0	0	0
500	616	599	584	572	561	552	544
1000	1142	1090	1046	1006	973	946	921
1500	1590	1489	1403	1330	1270	1218	1170
2000	1967	1809	1675	1560	1462	1374	1301
2500	2279	2056	1857	1695	1552	1426	1314
3000	2541	2243	1979	1754	1553	1388	1254
3500	2757	2366	2026	1729	1486	1278	1082
4000	2919	2422	2003	1640	1346	1074	1000

Table 5: Debt ratio (Debt value/Total value)

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.00	0.00	0.00	0.00	0.00	0.00	0.00
500	0.19	0.19	0.18	0.18	0.18	0.17	0.17
1000	0.34	0.33	0.32	0.31	0.31	0.30	0.30
1500	0.46	0.44	0.42	0.41	0.41	0.40	0.40
2000	0.56	0.53	0.51	0.50	0.49	0.49	0.50
2500	0.64	0.61	0.59	0.57	0.57	0.58	0.60
3000	0.72	0.68	0.65	0.64	0.65	0.67	0.72
3500	0.79	0.74	0.71	0.70	0.73	0.78	0.90
4000	0.85	0.80	0.77	0.77	0.82	0.94	1.00

Table 6: Percentage defaulted

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
500	11.0	16.6	21.2	25.6	29.9	33.8	37.4
1000	25.6	33.8	40.8	47.1	52.7	57.3	61.6
1500	37.4	47.1	55.0	61.6	66.8	71.6	75.5
2000	47.1	57.3	65.1	71.6	76.6	81.4	85.1
2500	55.0	65.1	73.1	79.0	84.1	88.5	92.2
3000	61.6	71.6	79.0	85.1	90.2	93.6	96.1
3500	66.8	76.6	84.1	90.2	94.0	96.8	99.2
4000	71.6	81.4	88.5	93.6	96.8	99.4	100.0

Table 7: Average time of defaultduring the first 50 years

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0							
500	37.9	36.7	35.2	34.2	33.2	32.2	31.1
1000	34.2	32.2	30.1	28.5	26.9	25.4	24.0
1500	31.1	28.5	26.1	24.0	22.1	20.2	18.3
2000	28.5	25.4	22.7	20.2	17.8	15.6	13.4
2500	26.1	22.7	19.5	16.6	13.9	11.5	9.1
3000	24.0	20.2	16.6	13.4	10.5	7.7	5.1
3500	22.1	17.8	13.9	10.5	7.2	4.2	1.4
4000	20.2	15.6	11.5	7.7	4.2	1.0	0.0

Modified Parameters

These calculations generally differ from the basic setting in one parameter only. For the sake of simplicity, we will note explicitly only this one different parameter.

Different Initial Interest Rate

Initial level 1% $r(0) = 0.01$

Table 8: Debt value

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0	0	0	0	0	0	0
500	625	608	593	579	567	557	548
1000	1162	1104	1059	1019	983	951	924
1500	1604	1506	1411	1329	1262	1210	1160
2000	1986	1810	1663	1548	1445	1365	1293
2500	2287	2038	1847	1681	1548	1447	1366
3000	2530	2221	1960	1753	1609	1476	1380
3500	2728	2339	2028	1805	1609	1479	1374
4000	2894	2420	2062	1792	1600	1451	1334

Table 9: Equity value

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	3249	3249	3249	3249	3249	3249	3249
500	2824	2829	2829	2825	2817	2807	2794
1000	2430	2442	2442	2432	2409	2379	2342
1500	2074	2099	2098	2075	2035	1980	1905
2000	1751	1791	1789	1754	1690	1610	1514
2500	1460	1517	1513	1461	1381	1283	1181
3000	1196	1267	1261	1202	1120	999	879
3500	958	1043	1039	982	871	771	665
4000	736	846	847	778	688	589	487

Initial level 5% $r(0) = 0.05$

Table 10: Debt value

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0	0	0	0	0	0	0
500	608	592	580	568	559	550	543
1000	1135	1089	1047	1012	981	952	928
1500	1587	1493	1409	1335	1264	1206	1151
2000	1971	1813	1669	1536	1409	1283	1141
2500	2295	2054	1827	1598	1329	1017	1000
3000	2554	2203	1844	1390	1000	1000	1000
3500	2739	2241	1627	1000	1000	1000	1000
4000	2864	2132	1049	1000	1000	1000	1000

Table 11: Equity value

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	2820	2820	2820	2820	2820	2820	2820
500	2408	2412	2412	2408	2404	2395	2385
1000	2025	2036	2036	2027	2010	1980	1935
1500	1674	1698	1698	1666	1615	1536	1435
2000	1347	1385	1371	1310	1210	1009	628
2500	1034	1079	1057	954	657	53	0
3000	734	795	762	447	-0	0	0
3500	470	559	425	-0	0	0	0
4000	231	341	28	0	0	0	0

High Asset Volatility

40% volatility $\sigma = 0.4$

Table 12: Results

(a) Debt value

	0.5	0.9
0	0	0
1500	755	713
3000	1065	968

(b) Equity value

	0.5	0.9
0	2901	2901
1500	2042	1727
3000	1397	499

(c) Total value

	0.5	0.9
0	2901	2901
1500	2797	2440
3000	2462	1467

(d) Debt ratio

	0.5	0.9
0	0.00	0.00
1500	0.27	0.29
3000	0.43	0.66

(e) % defaulted

	0.5	0.9
0	0.0	0.0
1500	92.6	96.5
3000	97.1	99.3

(f) Avg. def. time

	0.5	0.9
0		
1500	14.3	9.2
3000	8.2	2.4

60% volatility $\sigma = 0.6$

Table 13: Results

(a) Debt value

	0.5	0.9
0	0	0
1500	508	560
3000	765	876

(b) Equity value

	0.5	0.9
0	2682	2682
1500	2020	1625
3000	1434	576

(c) Total value

	0.5	0.9
0	2682	2682
1500	2528	2185
3000	2199	1452

(d) Debt ratio

	0.5	0.9
0	0.00	0.00
1500	0.21	0.26
3000	0.36	0.61

(e) % defaulted

	0.5	0.9
0	0.0	0.0
1500	99.1	99.6
3000	99.6	99.9

(f) Avg. def. time

	0.5	0.9
0		
1500	8.1	5.1
3000	4.5	1.4

C Method of Calculations

Since the model is so complex, that it is hardly possible to find closed form solutions to determine the values of the claims, the probability of default and other properties, we decided to use numerical calculations in order to uncover the model's sensitivity on its parameters. The core of the Monte-Carlo simulations is the following: after the parameters are set (see Table 1 for their base values) a large number[22] of iterations is run. Every iteration calculates a randomly[23] chosen EBIT trajectory and a correlated interest rate evolution. Following the realizations, the discounted sum of cash and asset flows is calculated for both debt and equity holders. In order to observe the payoffs' sensitivity on the DB and FV, several combinations of these parameters are examined in each iteration. Consequently every iteration produces matrices, where every matrix contains the result of one output parameter:[24] different rows correspond to different debt face values while different columns correspond to different default barrier levels. The generated matrices are then averaged and so the expected values are obtained. These results are then used as payoff valuation for game trees analysed in Section 4. A sample result matrix for equity values can be found in Table 3, for the complete output see Section B.

Since our model has infinite time horizon, that cannot be calculated with the numerical approach, we had to approximate the results using finite number of years considered. We decided to encounter 150 years in our calculations, as the earnings in these first 150 years represent approximately 99% of the firm value.[25]

Because simulating 150 years would require time-consuming computations,

[22] The number of iterations is set in a way to produce stable results. It is typically between 5,000 and 120,000, depending mainly on σ, the variance of the EBIT process.

[23] The probability distributions that drive the simulated random values are described in equations (1) and (4).

[24] These output parameters are: Debt payoff, Equity payoff, Total payoff, Debt ratio, and Default time (zero indicates no default).

[25] The discount of 150 years with constant 3% continuously compounded interest rate is $1/\exp(0.03 \cdot 150) \approx 0.011$. This is a rough estimate only, as the EBIT is expected to grow, and on the other hand default in the first 150 years is possible. Considering the calculated default rate, that is above 30% in the first 50 years even for firms with low leverage, the time horizon of 150 years is sufficiently high.

we divided this time period into two parts: while the first 50 years are computed using high-precision simulations[26], the last 100 years are calculated using lower precision and then added to the first 50. Such division is faster for a given number of iterations, and produces results with smaller deviation, consequently a lower amount of iterations is sufficient.

[26]Here precision refers to the sampling frequency of the generated Wiener processes. "High-precision" calculations are sampled every trading day (i.e. 250 times a year), "lower precision" calculations are sampled once per ten days (i.e. 25 times a year). The two methods produce similar results with small differences in the produced output.