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DOCTORAL THESIS

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Topological and descriptive methods in the theory of function and Banach spaces

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I hereby declare that I have written this doctoral thesis on my own and that the references herein include all sources I have utilized.

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Prague   3rd February 2011   Miroslav Kačena

Názov práce: Topologické a deskriptívne metódy v teórii funkčných a Banachových priestorov
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Abstrakt: Práca pozostáva zo štyroch vedeckých článkov. Prvé tri sa zaoberajú Choquetovou teóriou funkčných priestorov. V kapitole 1 je rozvinutá teória o súčinoch a projektívnych limitách funkčných priestorov. Je ukázané, že súčin simpliciálnych priestorov je simpliciálny priestor. Stabilita priestoru maximálnych mier vzhľadom k spojitým afinným zobrazeniam sa skúma v kapitole 2. Tretia kapitola využíva výsledky predchádzajúcich kapitol ku konštrukcii príkladu funkčného priestoru, kde nie je riešiteľný abstraktný Dirichletov problém pre žiadnu triedu funkcií n–tej Baireovej triedy s $n \in \mathbb{N}$. Je ukázané, že podobný príklad sa nedá skonštruovať ako priestor harmonických funkcií. V poslednej kapitole sa vyšetruje nedávno zavedená trieda sekvenciálne Správnych Banachových priestorov. Sú ustanovené vzťahy k ďalším izomorfným vlastnostiam Banachových priestorov a podané viaceré charakterizácie.

Kľúčové slová: Choquetova teória, funkčný priestor, simplex, abstraktný Dirichletov problém, slabo kompaktný operátor

Title: Topological and descriptive methods in the theory of function and Banach spaces
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Abstract: The thesis consists of four research papers. The first three deal with the Choquet theory of function spaces. In Chapter 1, a theory on products and projective limits of function spaces is developed. It is shown that the product of simplicial spaces is a simplicial space. The stability of the space of maximal measures under continuous affine mappings is studied in Chapter 2. The third chapter employs results from the previous chapters to construct an example of a function space where the abstract Dirichlet problem is not solvable for any class of Baire–n functions with $n \in \mathbb{N}$. It is shown that such an example cannot be constructed via the space of harmonic functions. In the final chapter, the recently introduced class of sequentially Right Banach spaces is being investigated. Connections to other isomorphic properties of Banach spaces are established and several characterizations are given.

Keywords: Choquet theory, function space, simplex, abstract Dirichlet problem, weakly compact operator

Contents

Introduction

This thesis consists of the following four research papers:

- M. Kačena, 'Products and projective limits of function spaces', *Commentat. Math. Univ. Carol.* **49** (2008), no. 4, 547–578.
- M. Kačena and J. Spurný, 'Affine images of compact convex sets and maximal measures', *Bull. Sci. Math.* **133** (2009), no. 5, 493–500.
- M. Kačena and J. Spurný, 'Affine Baire functions on Choquet simplices', *Cent. Eur. J. Math.* **9(1)** (2011), 127–138.
- M. Kačena, 'On sequentially Right Banach spaces', submitted.

The principal areas of study are the Choquet theory of function spaces, in particular, the stability of several properties of function spaces under products, projective limits and continuous images, the abstract Dirichlet problem and isomorphic properties of Banach spaces.

In the first chapter, we develop a theory on products and projective limits of function spaces. By a function space we mean a subspace of the space of continuous functions $C(K)$ on a compact Hausdorff space K containing constants and separating points of K. An important special case is the space of all continuous affine functions on a compact convex set. A definition of a sensible 'product' is not entirely straightforward even in the convex case. Let us consider a simple example. A compact interval on the real line is a simplex. However, the cartesian product of the interval with itself is not. The first constructions of a simplex, which could be reasonably called a product of a family of simplices, were given independently by A. J. Lazar [5], E. B. Davies and G. F. Vincent-Smith [2]. They defined the product of a family of simplices $\{X_i\}_{i\in I}$ as the state space of the space of all continuous multiaffine functions on the cartesian product $\prod_{i\in I} X_i$ and showed that this set is indeed a simplex.

Our aim in this chapter is not merely to study properties of the natural generalization of this concept to the framework of function spaces but also to investigate all function spaces which can be reasonably called a product space. It turns out that there are several possible definitions yielding a space with desired properties, that there is the least and the greatest such space and we investigate all these spaces simultaneously.

There are three main results in this chapter. The first one characterizes the Choquet boundary of a product space as the cartesian product of Choquet boundaries of the original spaces (see Chapter 1, Theorem 3.42). More generally, it is shown that the cartesian product of extremal sets is extremal and that every projection of an extremal set is extremal. Certainly the most difficult is the result that a product of simplicial function spaces is simplicial (see Chapter 1, Theorem 3.52).

It follows from an approximation lemma which in turn is a generalized version of a non-trivial result of A. J. Lazar from [6]. As another consequence of this lemma, it is shown that in a certain sense the product of simplicial spaces is unique (see Chapter 1, Proposition 3.49). Finally, the Radon measures maximal with respect to the Choquet ordering are investigated. We prove that a measure on a product space is maximal if and only if every projection is maximal (see Chapter 1, Theorem 3.58). In particular, we get that the Radon product of maximal measures is a maximal measure. All these results converge to a simple theorem stating that the unique maximal measure representing a point in the product of simplicial function spaces is precisely the Radon product of the unique maximal measures representing each of its coordinates (see Chapter 1, Theorem 3.59). This theorem is used later, in Chapter 3, to construct a counterexample to a question concerning the abstract Dirichlet problem.

At the end of the first chapter we study projective limits of function spaces. Characterizations of the Choquet boundary, simplicial function spaces and maximal measures similar to those in the previous section are provided. Some of the results on maximal measures are postponed until the next chapter.

It was during the writing of the paper from the first chapter and my study of continuous images of maximal measures when Jiří Spurný found several examples contradicting results and conjectures of S. Teleman [11]. The object of Chapter 2 is to show that if $\varphi : X \to Y$ is a continuous affine mapping of a compact convex set X into a compact convex set Y then a certain behaviour of φ on the set of extreme points $\operatorname{ext} X$ of X does not imply the same behaviour of φ on the set of maximal measures $\mathcal{M}^1_{\max}(X)$ in general, but only under some additional assumptions. In particular, we show that if the set of extreme points $\operatorname{ext} Y$ of Y is a Lindelöf space or if Y is a simplex then $\varphi(\operatorname{ext} X) \subset \operatorname{ext} Y$ (and φ is injective on $\operatorname{ext} X$) if and only if $\varphi_\sharp(\mathcal{M}^1_{\max}(X)) \subset \mathcal{M}^1_{\max}(Y)$ (and $\varphi_\sharp$ is injective on $\mathcal{M}^1_{\max}(X)$). Here $\varphi_\sharp : \mathcal{M}^1(X) \to \mathcal{M}^1(Y)$ denotes the induced map on the set of all probability Radon measures on X (see Chapter 2 for more information). Subsequently, the following three examples are provided: an example of φ mapping $\operatorname{ext} X$ into $\operatorname{ext} Y$ injectively which does not map $\mathcal{M}^1_{\max}(X)$ into $\mathcal{M}^1_{\max}(Y)$ (see Chapter 2, Example 1.1), an example of φ mapping injectively $\mathcal{M}^1_{\max}(X)$ into $\mathcal{M}^1_{\max}(Y)$ which is not injective on all of X (see Chapter 2, Example 1.4) and an example of φ mapping $\mathcal{M}^1_{\max}(X)$ into $\mathcal{M}^1_{\max}(Y)$ which is injective on $\operatorname{ext} X$, but not injective on the set $\mathcal{M}^1_{\max}(X)$ (see Chapter 2, Example 1.5). My contribution to this paper was around 30%.

In Chapter 3 we study the abstract Dirichlet problem. That is, for a given metrizable simplex X and a bounded Borel function f on $\operatorname{ext} X$, a question of an affine extension of f to the whole set X that preserves the Baire class of the function f. By the minimum principle, the only affine extension of f is the function Tf given by

$$Tf(x) = \delta_x(f), \quad x \in X,$$

where δ_x is the unique maximal measure representing the point $x \in X$. The problem of continuous extensions was solved by H. Bauer. He showed in [1] that $T(C(\operatorname{ext} X)) \subset C(X)$ *if and only if* $\operatorname{ext} X$ *is closed.* An analogous question for Baire–one functions was solved by Jiří Spurný in [9], namely $T(\mathcal{B}^b_1(\operatorname{ext} X)) \subset \mathcal{B}^b_1(X)$

if and only if $\operatorname{ext} X$ *is an* F_σ *set.* It has turned out in [10] that such a characterization is impossible for functions of higher Baire classes.

Thus there exist simplices such that the operator T does not preserve continuous or Baire–one functions. By the result in [10] there is also a simplex such that T does not preserve Baire–two functions. On the other hand, it is not difficult to realize that $T(\mathcal{B}^b_\alpha(\operatorname{ext} X)) \subset \mathcal{B}^b_\alpha(X)$ for each metrizable simplex X and $\alpha \in [\omega_0, \omega_1)$ (see Chapter 3, Theorem 1.1(a)). The aim of our paper is to show that the shift of classes can occur for any finite Baire class. Precisely, we use results of Chapter 1 to construct a metrizable simplex X such that $T(\mathcal{B}^b_\alpha(\operatorname{ext} X)) \not\subset \mathcal{B}^b_\alpha(X)$ for each $\alpha \in [0, \omega_0)$ (see Chapter 3, Theorem 1.1(b)). In fact, this construction was the main motivation to develop the theory from Chapter 1 in the first place. The question that is left open is whether the shift of classes can stop exactly at a given $\alpha \in [0, \omega_0)$. We provide examples for $\alpha \in \{0, 1\}$ (see Chapter 3, Theorem 1.1(c)).

Based on Theorem 1.1(a) we further show a characterization of all functions of affine class α for $\alpha \in [\omega_0, \omega_1)$ (see Chapter 3, Theorem 1.2). Finally, it is proved that in harmonic spaces the abstract Dirichlet problem always has a solution for functions of Baire class 2 (see Chapter 3, Theorem 1.3).

Contributions of both authors to this paper are comparable. The idea of using products of function spaces to construct a counterexample from Theorem 1.1(b) comes from Jiří Spurný, the actual construction was done by myself. The section on spaces of harmonic functions is due to Jiří Spurný and partly Professor Wolfhard Hansen.

It is well-known that the space $A(X)$ of all affine continuous functions on a simplex X is an L^1-predual. By the result of W. B. Johnson and M. Zippin [4], L^1-preduals inherit many isomorphic properties of $C(K)$ spaces. In Chapter 4, we are particularly interested in so-called 'reciprocal Dunford-Pettis properties' possessed by every $C(K)$ space. The main object of study in this chapter is the recently introduced class of sequentially Right Banach spaces and its relations to other isomorphic properties of Banach spaces.

The definition comes from A. M. Peralta, I. Villanueva, J. D. M. Wright and K. Ylinen. They proved in [8] that for a given Banach space X there is a locally convex topology on X, called by them the 'Right topology', such that every operator T from X into a Banach space Y is weakly compact if and only if it is Right-to-norm continuous. This topology is obtained as the restriction of the Mackey topology $\tau(X^{**}, X^*)$ to X. It is the topology of uniform convergence on absolutely convex $\sigma(X^*, X^{**})$-compact subsets of X^*. In general, the Right topology is stronger than the weak topology and weaker than the norm topology, thus compatible with the dual pair $\langle X, X^* \rangle$. Every Right-to-norm continuous operator is surely Right-to-norm sequentially continuous. A simple look at the identity operator on ℓ_1 reveals, however, that the converse is not true. Authors in [8] call Right-to-norm sequentially continuous operators *pseudo weakly compact* and Banach spaces, on which every pseudo weakly compact operator is weakly compact, *sequentially Right*. They have shown that every Banach space possessing Pełczyński's property (V) is sequentially Right and they asked whether the converse holds.

We first introduce a stronger *property (RD)* which is an analogue of the *Dieudonné property* introduced by A. Grothendieck in [3] alongside the *Dunford-Pettis property* and the *Reciprocal Dunford-Pettis property.* (For precise definitions of these properties and the Pełczyński's property (V) see Chapter 4.) Then we investigate relations between these properties. We improve the result of [8] and show that property (V) actually implies property (RD). Several examples are provided to complete the picture. In particular, an example of a sequentially Right Banach space without property (V) is shown which answers a question from [8].

Further, dual characterizations of property (RD) and sequential Rightness are given. We use them to generalize a result of A. Pełczyński from [7] and show that every sequentially Right Banach space has weakly sequentially complete dual.

We also take an interest in topological behaviour of the Right topology. Two most important special cases are in the centre of our attention. It is shown that the sequential coincidence of the Right topology with the weak one is just another characterization of the Dunford-Pettis property. Multiple characterizations are also given for the sequential coincidence of the Right topology with the norm topology.

Finally, we show that if K is a scattered compact Hausdorff space, then $C(K, X)$, the Banach space of all continuous functions from K to a Banach space X, is sequentially Right (resp. has property (RD)) if and only if X has the same property. The chapter ends with some results on extensions of pseudo weakly compact operators on $C(K, X)$ spaces.

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Chapter 1

Products and projective limits of function spaces

M. Kačena, 'Products and projective limits of function spaces', *Commentat. Math. Univ. Carol.* **49** (2008), no. 4, 547–578. (original paper)

PRODUCTS AND PROJECTIVE LIMITS OF FUNCTION SPACES

MIROSLAV KAČENA

ABSTRACT. We introduce a notion of a product and projective limit of function spaces. We show that the Choquet boundary of the product space is the product of Choquet boundaries. Next we show that the product of simplicial spaces is simplicial. We also show that the maximal measures on the product space are exactly those with maximal projections. We show similar characterizations of the Choquet boundary and the space of maximal measures for the projective limit of function spaces under some additional assumptions and we prove that the projective limit of simplicial spaces is simplicial.

1. INTRODUCTION

Let $\{X_i\}_{i\in I}$ be a family of Choquet simplexes. We can construct a compact convex set X as the state space of the space of all continuous multiaffine functions on $\prod_{i\in I} X_i$. It has been shown in [6] and [16] that X itself is a simplex with extreme points being the evaluation functionals at the points $(x_i)_{i\in I} \in \prod_{i\in I} X_i$ with $x_i \in \operatorname{ext} X_i$ for every $i \in I$. Generalizations to products of arbitrary compact convex sets followed (see [11], [18]). Characterization of maximal measures on the product of two compact convex sets, as the measures whose every 'projection' is a maximal measure, appeared later in [3] and [2].

In Section 3 we transfer these results to the context of function spaces. We first introduce a notion of a product of function spaces with several special products. We compare these products and prove appropriate associative laws. Then we show that the Choquet boundary of a product space is the product of Choquet boundaries. We prove that the product is simplicial if and only if every of the original spaces is simplicial. Finally we show that maximal measures on the product of arbitrary many spaces are exactly those with maximal projections.

In Section 4 we transfer known results from [6] and [13] on projective limits of compact convex sets to function spaces. We use Grossman's definition of the projective limit of function spaces from [10] and prove that the projective limit of simplicial spaces is simplicial. We also derive similar characterizations of the Choquet boundary and maximal measures as in the case of product of function spaces.

2. PRELIMINARIES

Let K be a compact Hausdorff space. We denote by $\mathcal{C}(K)$ the space of all continuous functions on K, by $\mathcal{M}^+(K)$ the set of all positive Radon measures on K and by $\mathcal{M}^1(K)$ the set of all probability Radon measures on K. Let ε_x stand for the Dirac measure at $x \in K$. We say that a linear subspace $\mathcal{H}$ of $\mathcal{C}(K)$ is a *function space*, if it contains 1_K (the function identically 1 on K) and separates the points of K. Let $\mathcal{M}_x(\mathcal{H})$ be the set of all $\mathcal{H}$*-representing measures* for $x \in K$, i.e.,

$$\mathcal{M}_x(\mathcal{H}) := \{\mu \in \mathcal{M}^1(K) : h(x) = \int_K h\,d\mu \text{ for every } h \in \mathcal{H}\}.$$

The set $\mathrm{Ch}_{\mathcal{H}}\, K := \{x \in K : \mathcal{M}_x(\mathcal{H}) = \{\varepsilon_x\}\}$ is called the *Choquet boundary* of $\mathcal{H}$. It is a G_δ–set if K is metrizable (see [1, Corollary I.5.17]). We denote by $\nabla_{\mathcal{H}}\, K$ the *Šilov boundary* of $\mathcal{H}$ (see [1, p. 50] for definition) and we remark that $\nabla_{\mathcal{H}}\, K$ is equal to the closure of $\mathrm{Ch}_{\mathcal{H}}\, K$ (see [1, Theorem I.5.15] for the proof). A nonempty closed set $E \subset K$ is called $\mathcal{H}$*-extremal*, if $\operatorname{spt}\mu \subset E$ for every $x \in E$ and $\mu \in \mathcal{M}_x(\mathcal{H})$. Finally, for every $x \in K$ we denote $F_x(\mathcal{H}) := \bigcup\{\operatorname{spt}\mu : \mu \in \mathcal{M}_x(\mathcal{H})\}$.

We define the space $\mathcal{A}^c(\mathcal{H})$ of all continuous $\mathcal{H}$*-affine functions* as the space of all continuous functions on K satisfying the following formula:

$$f(x) = \int_K f\,d\mu \quad \text{for each} \quad x \in K \quad \text{and} \quad \mu \in \mathcal{M}_x(\mathcal{H})\,.$$

Clearly $\mathcal{A}^c(\mathcal{H})$ is a uniformly closed function space with $\mathcal{M}_x(\mathcal{H}) = \mathcal{M}_x(\mathcal{A}^c(\mathcal{H}))$ for every $x \in K$.

Here we recall main examples of function spaces:

(a) *Convex case* - Let X be a compact convex subset of a locally convex space and let $\mathcal{H}$ be the linear space $A(X)$ of all continuous affine functions on X. The Choquet boundary is the set $\operatorname{ext} X$ of all extreme points of X.

(b) *Harmonic case* - Let U be a bounded open subset of the Euclidean space $\mathbb{R}^n$ and let the corresponding function space $H(U)$ be the family of all continuous functions on $\overline{U}$ which are harmonic on U. The Choquet boundary coincides with the set $\partial_{\mathrm{reg}} U$ of all regular points.

An upper bounded Borel function f is called $\mathcal{H}$*-convex* if $f(x) \le \mu(f)$ for any $x \in K$ and $\mu \in \mathcal{M}_x(\mathcal{H})$. Let $\mathcal{K}^c(\mathcal{H})$ denote the family of all continuous $\mathcal{H}$-convex functions on K. Notice that the space $\mathcal{K}^c(\mathcal{H}) - \mathcal{K}^c(\mathcal{H})$ is uniformly dense in $\mathcal{C}(K)$ due to the lattice version of the Stone–Weierstrass theorem.

The convex cone $\mathcal{K}^c(\mathcal{H})$ determines a partial ordering $\preceq_{\mathcal{H}}$ (called the *Choquet ordering*) on the space $\mathcal{M}^+(K)$:

$$\mu \preceq_{\mathcal{H}} \nu \quad \text{if} \quad \mu(f) \le \nu(f) \quad \text{for each} \quad f \in \mathcal{K}^c(\mathcal{H}).$$

(If the space $\mathcal{H}$ is obvious, we simply write $\mu \preceq \nu$.)

We remark that $\mu \preceq \nu$ if and only if $\mu(f) \le \nu(f)$ for every $f \in \mathcal{W}(\mathcal{H})$, where $\mathcal{W}(\mathcal{H})$ is the smallest family of functions containing $\mathcal{H}$ and closed with respect to taking supremum of finite families.

For any measure $\mu \in \mathcal{M}^+(K)$ there exists a maximal measure ν with $\mu \preceq \nu$. In particular, for every $x \in K$ there exists a maximal $\mathcal{H}$-representing measure. This is the content of the Choquet–Bishop–de-Leeuw theorem [1, Theorem I.5.19].

If K is metrizable, then a measure $\mu \in \mathcal{M}^+(K)$ is maximal if and only if $\mu(K \setminus \mathrm{Ch}_{\mathcal{H}}\, K) = 0$. In nonmetrizable spaces every maximal measure μ satisfies $\mu(G) = 0$ for any G_δ–set disjoint from $\mathrm{Ch}_{\mathcal{H}}\, K$ (see [1, Proposition I.5.22]).

Theorem 2.1. *Let $\mu \in \mathcal{M}^+(K)$. Then the following assertions are equivalent:*

(i) *μ is maximal,*

(ii) *there exists a set $S \subset \mathcal{C}(K)$ separating points of K such that every function from S is constant on $F_x(\mathcal{H})$ for μ-a.e. $x \in K$,*

(iii) *every function from* $\mathcal{C}(K)$ *is constant on* $F_x(\mathcal{H})$ *for* μ*-a.e.* $x \in K$.

Proof. See [2, Proposition 2]. □

Proposition 2.2. *Let* $(K', \mathcal{G})$ *be a function space and* $\rho : K \to K'$ *a continuous mapping such that* $F_{\rho(x)}(\mathcal{G}) \subset \rho(F_x(\mathcal{H}))$ *for every* $x \in \overline{\operatorname{Ch}_{\mathcal{H}} K}$. *Then the image measure* $\rho\mu$ *is a maximal measure on* K' *for every maximal measure* μ *on* K.

Proof. See [2, Corollary 3]. □

If for every $x \in K$ the maximal $\mathcal{H}$-representing measure is uniquely determined, we say that $\mathcal{H}$ is *simplicial*. In the convex case it is equivalent to say that X is a *Choquet simplex*. We denote the unique maximal measure representing $x \in K$ by δ_x.

We say that $\mathcal{H}$ has the *weak Riesz interpolation property (W.R.I.P.)*, if for every $a_1, a_2, b_1, b_2 \in \mathcal{H}$ such that $a_i < b_j$, $i, j = 1, 2$, there exists $c \in \mathcal{H}$ such that $a_i < c < b_j$, $i, j = 1, 2$. It can be shown that $\mathcal{H}$ is simplicial if and only if $\mathcal{A}^c(\mathcal{H})$ has W.R.I.P. (see [1, Corollary II.3.11] or [4, Theorem 3.3]).

For a function $f : K \to \mathbb{R}$ we define the *upper envelope* f^* as

$$f^*(x) := \inf\{h(x) : h \geq f,\, h \in \mathcal{H}\}, \quad x \in K,$$

and the *lower envelope* as $f_* := -(-f)^*$. We denote $\widehat{\mathcal{H}} := \{f \in \mathcal{C}(K) : f_* = f^*\}$. It is true that $\mathcal{A}^c(\mathcal{H}) = \widehat{\mathcal{H}}$. By [1, Proposition I.5.9 and Corollary I.5.10], we have:

Proposition 2.3. *Let* $\mu \in \mathcal{M}^+(K)$. *Then the following statements are equivalent:*

(i) μ *is maximal,*
(ii) $\mu(f) = \mu(f^*)$ *for every* $f \in \mathcal{C}(K)$,
(iii) $\mu(k) = \mu(k^*)$ *for every* $k \in \mathcal{K}^c(\mathcal{H})$.

Corollary 2.4. *Let* $x \in K$. *Then the following statements are equivalent:*

(i) $x \in \operatorname{Ch}_{\mathcal{H}} K$,
(ii) $f(x) = f^*(x)$ *for every* $f \in \mathcal{C}(K)$,
(iii) $k(x) = k^*(x)$ *for every* $k \in \mathcal{K}^c(\mathcal{H})$.

If f and g are functions on K, we write $f \vee g$ for their pointwise maximum and $f \wedge g$ for minimum.

Now we introduce a notation concerning cartesian products: Let $\{E_i\}_{i \in I}$ be a family of topological spaces and let $E := \prod_{i \in I} E_i$ be their cartesian product with the usual topology. We use the convention $\prod_{i \in \emptyset} E_i := \{\emptyset\}$.

Let $J \subset I$. The natural projection from E onto $\prod_{i \in J} E_i$ is denoted by π_J. Let $A \subset E$ and $z \in \prod_{i \in I \setminus J} E_i$. We denote by $\pi_J^z(A)$ the set $\{x \in \prod_{i \in J} E_i : (x, z) \in A\}$.

We use a similar notation for functions. Let $f : E \to \mathbb{R}$ and $y \in \prod_{i \in I \setminus J} E_i$. Then $\pi_J^y(f) : \prod_{i \in J} E_i \to \mathbb{R}$ is defined as

$$\pi_J^y(f)(x) := f(x, y), \quad x \in \prod_{i \in J} E_i.$$

In case f is independent on y, we use notation $\pi_J(f)$.

Finally, for $f_1 : E_1 \to \mathbb{R}$ and $f_2 : E_2 \to \mathbb{R}$ we define $f_1 \otimes f_2 : E_1 \times E_2 \to \mathbb{R}$ by

$$(f_1 \otimes f_2)(x, y) = f_1(x) f_2(y), \quad x \in E_1,\, y \in E_2.$$

We conclude this section with known results on products of Radon measures: Let $\{(K_i, \mathcal{S}_i, \mu_i)\}_{i\in I}$ be a family of compact Hausdorff spaces with Radon probability measures. There exists a unique product measure μ on $\prod_{i\in I} K_i$ with $\mu(\prod_{i\in I} E_i) = \prod_{i\in I} \mu_i(E_i)$, whenever $E_i \in \mathcal{S}_i$ for each $i \in I$ and $E_i \neq K_i$ for finitely many $i \in I$ (see [12, Chapter VI, Theorem 5.3]). By [8, Theorem 417Q], μ can be uniquely extended to a Radon measure $\bigotimes_{i\in I} \mu_i$. We call this measure the *Radon product measure*. Radon products satisfy associative law (see [8, Theorem 417J]) and we can also use Fubini's theorem (see [8, Theorem 417H]). Finally we remark that if two Radon measures coincide on the cylinder sets $\prod_{i\in I} E_i$, where $E_i \subset K_i$ is Borel for each $i \in I$ and $E_i \neq K_i$ for finitely many $i \in I$, then they are equal (see [12, Chapter I, Proposition 5.3] and the proof of [8, Corollary 417F]).

3. Products of function spaces

3.1. Definitions and relations.

Definition 3.1. Let $\{(K_i, \mathcal{H}_i)\}_{i\in I}$ be a family of function spaces and let $K := \prod_{i\in I} K_i$. We define

(a) *algebraic tensor product* $\bigodot_{i\in I} \mathcal{H}_i$ as the linear span of the set

$$\{h_1 \otimes \ldots \otimes h_n \otimes 1_{\prod\{K_i : i\in I\setminus\{i_1,\ldots,i_n\}\}} : h_k \in \mathcal{H}_{i_k},\, i_k \in I,\, 1 \le k \le n,\, n \in \mathbb{N}\},$$

(b) *injective tensor product* $\bigotimes_{i\in I} \mathcal{H}_i$ as the closure of $\bigodot_{i\in I} \mathcal{H}_i$,

(c) *multiaffine product* by

$$\boxtimes_{i\in I} \mathcal{H}_i := \{f \in \mathcal{C}(K) : \pi_j^y(f) \in \mathcal{H}_j \text{ for all } j \in I \text{ and } y \in \prod_{i\in I\setminus\{j\}} K_i\}.$$

We say that a function space $\mathcal{H}$ on K is a *product* of function spaces $\mathcal{H}_i,\ i \in I$, if

$$\bigodot_{i\in I} \mathcal{H}_i \subset \mathcal{H} \subset \boxtimes_{i\in I} \mathcal{A}^c(\mathcal{H}_i).$$

In case I is an empty set, we put all products to be equal $\{\emptyset\}$.

Remark 3.2. It can be shown, that $\mathcal{H}_1 \odot \mathcal{H}_2$ is really the 'algebraic tensor product', and if $\mathcal{H}_1$ and $\mathcal{H}_2$ are closed, i.e., Banach spaces, then $\mathcal{H}_1 \otimes \mathcal{H}_2$ is their 'weak (injective) tensor product' (see [19, 20.5.5]). If $\mathcal{H}_i = A(X_i)$ for some compact convex sets X_i, $i \in I$, then $\boxtimes_{i\in I} \mathcal{H}_i$ is the space of all continuous multiaffine functions on K.

Example 3.3. Let $U_1 \subset \mathbb{R}^m$, $U_2 \subset \mathbb{R}^n$, be bounded open sets. We take $\mathcal{H}_i := H(U_i)$, $i = 1, 2$ (see Example (b) in Section 2). If $\mathcal{H}$ is a product of $\mathcal{H}_i$, $i = 1, 2$, then $\mathcal{H} \subset H(U_1 \times U_2)$. Indeed, choose $h \in \mathcal{H} \subset \mathcal{A}^c(\mathcal{H}_1) \boxtimes \mathcal{A}^c(\mathcal{H}_2) = H(U_1) \boxtimes H(U_2)$. Then we have

$$\Delta h(x_1, x_2) = \Delta\pi_1^{x_2}(h)(x_1) + \Delta\pi_2^{x_1}(h)(x_2) = 0, \quad x_1 \in U_1,\, x_2 \in U_2.$$

However, even the largest product does not have to contain all harmonic functions on the cartesian product. Consider $U_i := (0, 1) \subset \mathbb{R}$, $i = 1, 2$. Then $H(U_i) = A(\overline{U_i})$, $i = 1, 2$. So every product consists only of biaffine functions. Now take $f(x, y) := x^2 - y^2$ for $x, y \in [0, 1]$. Clearly, f is harmonic, but not biaffine.

Proposition 3.4. *The following assertions hold:*

(i) $\bigodot_{i\in I} \mathcal{H}_i \subset \boxtimes_{i\in I} \mathcal{H}_i$.

(ii) *If all* $\mathcal{H}_i$ *are closed, then* $\bigodot_{i\in I}\mathcal{H}_i \subset \bigotimes_{i\in I}\mathcal{H}_i \subset \boxtimes_{i\in I}\mathcal{H}_i$. *Moreover,* $\boxtimes_{i\in I}\mathcal{H}_i$ *is closed.*

(iii) *If* $\mathcal{H}_j$ *is not closed for some* $j \in I$*, then* $\bigodot_{i\in I}\mathcal{H}_i \subsetneq \bigotimes_{i\in I}\mathcal{H}_i$ *and* $\bigotimes_{i\in I}\mathcal{H}_i \not\subset \boxtimes_{i\in I}\mathcal{H}_i$.

Proof. Statement (i) and the first inclusion in (ii) are trivial. Since (i) holds, the second inclusion in (ii) will be proved if we show that $\boxtimes_{i\in I}\mathcal{H}_i$ is closed. So let $\{f_n\}_{n\in\mathbb{N}} \subset \boxtimes_{i\in I}\mathcal{H}_i$ be such that $f_n \rightrightarrows f \in \mathcal{C}(K)$. Further, let $j \in I$ and $y \in \prod_{i\in I\setminus\{j\}} K_i$. Then $\pi_j^y(f_n) \rightrightarrows \pi_j^y(f)$, and since $\pi_j^y(f_n) \in \mathcal{H}_j$ for each n and $\mathcal{H}_j$ is closed, we have $\pi_j^y(f) \in \mathcal{H}_j$. Thus $f \in \boxtimes_{i\in I}\mathcal{H}_i$.

Using previous inclusions, it suffices to find $f \in (\bigotimes_{i\in I}\mathcal{H}_i) \setminus (\boxtimes_{i\in I}\mathcal{H}_i)$ to prove (iii). Let $j \in I$ be such that $\mathcal{H}_j$ is not closed and put $K' := \prod_{i\in I\setminus\{j\}} K_i$. There are functions $\{h_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_j$ such that $h_n \rightrightarrows h \notin \mathcal{H}_j$. Then also $h_n \otimes 1_{K'} \rightrightarrows h \otimes 1_{K'}$. Since $h_n \otimes 1_{K'} \in \bigodot_{i\in I}\mathcal{H}_i$ for every $n \in \mathbb{N}$, we have $h \otimes 1_{K'} \in \bigotimes_{i\in I}\mathcal{H}_i$. But $\pi_j(h \otimes 1_{K'}) = h \notin \mathcal{H}_j$, therefore $h \otimes 1_{K'} \notin \boxtimes_{i\in I}\mathcal{H}_i$. □

Remark 3.5. Using previous proposition, we can see that all products defined in Definition 3.1 are indeed function spaces, since they are linear spaces and contain algebraic tensor product, which contains constants and separates points.

In the rest of this subsection we will show that the two inclusions in Proposition 3.4 (ii) may be proper.

Example 3.6. Let $K_i := [0,1] \subset \mathbb{R}$, $\mathcal{H}_i := \mathcal{C}(K_i)$, $i = 1, 2$, and denote $K := K_1 \times K_2$. The functions of $\mathcal{H}_1 \odot \mathcal{H}_2$ are of the form $\sum_{j=1}^n f_1^j \otimes f_2^j$, where $f_i^j \in \mathcal{C}(K_i)$, $i = 1, 2$, $j = 1, \dots, n$, $n \in \mathbb{N}$. Since $\mathcal{H}_1 \odot \mathcal{H}_2$ contains all polynomials, we have $\mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{C}(K)$. However $\mathcal{H}_1 \odot \mathcal{H}_2 \subsetneq \mathcal{C}(K)$, as can be seen by considering the function $f(x, y) := e^{xy}$, $x \in K_1, y \in K_2$.

This example also shows that algebraic tensor product of closed function spaces does not have to be closed.

Definition 3.7. A Banach space E is said to have the *approximation property*, if, for every compact set $C \subset E$ and every $\varepsilon > 0$, there is a continuous linear operator $T : E \to E$ of finite rank so that $\|Tx - x\| < \varepsilon$ for every $x \in C$.

(We refer the reader to [14, Chapter 7] for more information on the approximation property.)

Theorem 3.8 (Namioka-Phelps)**.** *The following statements are equivalent:*

(i) *For every two compact convex subsets* X_1, X_2 *of locally convex Hausdorff spaces is* $A(X_1) \otimes A(X_2) = A(X_1) \boxtimes A(X_2)$.

(ii) *Every Banach space has the approximation property.*

Proof. See [18, Theorem 2.4 and the subsequent remark]. □

Using Theorem 3.8 and Enflo's counterexample [7] of a Banach space not having the approximation property, we may state the following:

Corollary 3.9. *There exist compact convex sets* X_1 *and* X_2 *such that*

$$A(X_1) \otimes A(X_2) \subsetneq A(X_1) \boxtimes A(X_2).$$

3.2. **Associative laws.** In order to be able to use products defined above effectively, we need to establish 'associative laws' for them.

Definition 3.10. We say, that $\{J_\gamma\}_{\gamma\in\Gamma}$ is a *partition* of a set I, if $\bigcup_{\gamma\in\Gamma} J_\gamma = I$ and $J_\alpha \cap J_\beta = \emptyset$ for every $\alpha, \beta \in \Gamma$ such that $\alpha \neq \beta$.

To the end of this subsection, let $\{(K_i, \mathcal{H}_i)\}_{i\in I}$ be a family of function spaces and $\{J_\gamma\}_{\gamma\in\Gamma}$ a partition of I. In the following, we naturally identify spaces $\mathcal{C}(\prod_{i\in I} K_i)$ and $\mathcal{C}(\prod_{\gamma\in\Gamma}(\prod_{i\in J_\gamma} K_i))$.

Proposition 3.11. *The following assertions hold:*

(i) $\bigodot_{i\in I} \mathcal{H}_i = \bigodot_{\gamma\in\Gamma}(\bigodot_{i\in J_\gamma} \mathcal{H}_i)$,

(ii) $\mathcal{A}^c(\bigodot_{i\in I} \mathcal{H}_i) = \mathcal{A}^c(\bigodot_{\gamma\in\Gamma}(\bigodot_{i\in J_\gamma} \mathcal{H}_i))$.

Proof. To prove (i), it clearly suffices to show, that the generating functions of both spaces are the same. Function f is a generating function of $\bigodot_{i\in I} \mathcal{H}_i$, if

$$f = h_1^1 \otimes \ldots \otimes h_1^{m_1} \otimes \ldots \otimes h_n^1 \otimes \ldots \otimes h_n^{m_n} \otimes 1_{\prod\{K_i : i\in I\setminus\{i_1^1,\ldots,i_n^{m_n}\}\}},$$

for some $h_k^l \in \mathcal{H}_{i_k^l}$, $i_k^l \in J_{\gamma_k}$, $l = 1, \ldots, m_k$, $k = 1, \ldots, n$. Since

$$f_k := h_k^1 \otimes \ldots \otimes h_k^{m_k} \otimes 1_{\prod\{K_i : i\in J_{\gamma_k}\setminus\{i_k^1,\ldots,i_k^{m_k}\}\}} \in \bigodot_{i\in J_{\gamma_k}} \mathcal{H}_i \quad \text{for each } k = 1, \ldots, n,$$

we have

$$f = f_1 \otimes \ldots \otimes f_n \otimes 1_{\prod\{K_i : i\in I\setminus(J_{\gamma_1}\cup\ldots\cup J_{\gamma_n})\}},$$

which is a generating function of $\bigodot_{\gamma\in\Gamma}(\bigodot_{i\in J_\gamma} \mathcal{H}_i)$. Reverting the proof we obtain the converse inclusion.

Assertion (ii) follows from (i) and the fact that $\mathcal{A}^c(\mathcal{H}) = \widehat{\mathcal{H}}$. ∎

Proposition 3.12. *The following assertions hold:*

(i) $\bigotimes_{i\in I} \mathcal{H}_i = \bigotimes_{\gamma\in\Gamma}(\bigotimes_{i\in J_\gamma} \mathcal{H}_i)$,

(ii) $\mathcal{A}^c(\bigotimes_{i\in I} \mathcal{H}_i) = \mathcal{A}^c(\bigotimes_{\gamma\in\Gamma}(\bigotimes_{i\in J_\gamma} \mathcal{H}_i))$.

Proof. Using Proposition 3.11, we have

$$\bigotimes_{i\in I} \mathcal{H}_i = \overline{\bigodot_{i\in I} \mathcal{H}_i} = \overline{\bigodot_{\gamma\in\Gamma}(\bigodot_{i\in J_\gamma} \mathcal{H}_i)} \subset \overline{\bigodot_{\gamma\in\Gamma}(\bigotimes_{i\in J_\gamma} \mathcal{H}_i)} = \bigotimes_{\gamma\in\Gamma}(\bigotimes_{i\in J_\gamma} \mathcal{H}_i).$$

For the converse inclusion, it suffices to prove $\bigodot_{\gamma\in\Gamma}(\bigotimes_{i\in J_\gamma} \mathcal{H}_i) \subset \bigotimes_{i\in I} \mathcal{H}_i$, since the latter space is closed. Let f be a generating function of $\bigodot_{\gamma\in\Gamma}(\bigotimes_{i\in J_\gamma} \mathcal{H}_i)$. We can write

$$f = f_1 \otimes \ldots \otimes f_n \otimes 1_{\prod\{K_i : i\in I\setminus(J_{\gamma_1}\cup\ldots\cup J_{\gamma_n})\}},$$

where $f_i \in \bigotimes_{j\in J_{\gamma_i}} \mathcal{H}_j$, $i = 1, \ldots, n$. We may assume that $f_i > 0$, $i = 1, \ldots, n$ (otherwise we write $f_i = (\|f_i\| + 1) - (\|f_i\| + 1 - f_i)$ and use distributive law). Denote $M := \max_{i=1,\ldots,n} \|f_i\|$. Now choose $0 < \varepsilon < 1$ so that $f_i > \varepsilon$, $i = 1, \ldots, n$. For each f_i we can find $h_i \in \bigodot_{j\in J_{\gamma_i}} \mathcal{H}_j$ such that $f_i - \varepsilon < h_i < f_i$. We define

$$h := h_1 \otimes \ldots \otimes h_n \otimes 1_{\prod\{K_i : i\in I\setminus(J_{\gamma_1}\cup\ldots\cup J_{\gamma_n})\}} \in \bigodot_{i\in I} \mathcal{H}_i,$$

(we used Proposition 3.11) and compute

$$\begin{aligned}\|f-h\| &= \sup_{x_1\in\prod_{i\in J_{\gamma_1}}K_i}\dots\sup_{x_n\in\prod_{i\in J_{\gamma_n}}K_i}\Big(\prod_{i=1}^n f_i(x_i)-\prod_{i=1}^n h_i(x_i)\Big)\\ &< \sup_{x_1\in\prod_{i\in J_{\gamma_1}}K_i}\dots\sup_{x_n\in\prod_{i\in J_{\gamma_n}}K_i}\Big(\prod_{i=1}^n f_i(x_i)-\prod_{i=1}^n (f_i(x_i)-\varepsilon)\Big)\\ &= \sup_{x_1\in\prod_{i\in J_{\gamma_1}}K_i}\dots\sup_{x_n\in\prod_{i\in J_{\gamma_n}}K_i}\varepsilon\Big(\sum_{k=1}^n(-1)^{k-1}\varepsilon^{k-1}\sum_{|\alpha|=n-k}\prod_{i=1}^{n-k}f_{\alpha_i}(x_{\alpha_i})\Big)\\ &\le \varepsilon\Big(\sum_{k=1}^n\sum_{|\alpha|=n-k}\prod_{i=1}^{n-k}\|f_{\alpha_i}\|\Big)\le\varepsilon\Big(\sum_{k=0}^{n-1}\binom{n}{k}M^k\Big).\end{aligned}$$

Since ε is arbitrary, we conclude that $f\in\bigotimes_{i\in I}\mathcal{H}_i$.

Assertion (ii) follows from (i) and the fact that $\mathcal{A}^c(\mathcal{H})=\widehat{\mathcal{H}}$. □

Proposition 3.13. *The following assertions hold:*

(i) $\boxtimes_{i\in I}\mathcal{H}_i=\boxtimes_{\gamma\in\Gamma}(\boxtimes_{i\in J_\gamma}\mathcal{H}_i)$,

(ii) $\mathcal{A}^c(\boxtimes_{i\in I}\mathcal{H}_i)=\mathcal{A}^c(\boxtimes_{\gamma\in\Gamma}(\boxtimes_{i\in J_\gamma}\mathcal{H}_i))$.

Proof. Let $f\in\boxtimes_{i\in I}\mathcal{H}_i$. Pick $\gamma_0\in\Gamma$ and $k'\in\prod_{i\in I\setminus J_{\gamma_0}}K_i$. We want to prove that $\pi^{k'}_{J_{\gamma_0}}(f)\in\boxtimes_{i\in J_{\gamma_0}}\mathcal{H}_i$, i.e., that $\pi^{k''}_j(\pi^{k'}_{J_{\gamma_0}}(f))\in\mathcal{H}_j$ for every $j\in J_{\gamma_0}$ and $k''\in\prod_{i\in J_{\gamma_0}\setminus\{j\}}K_i$. But this is true, since $\pi^{k''}_j(\pi^{k'}_{J_{\gamma_0}}(f))=\pi_j^{(k',k'')}(f)\in\mathcal{H}_j$.
Conversely, let $f\in\boxtimes_{\gamma\in\Gamma}(\boxtimes_{i\in J_\gamma}\mathcal{H}_i)$. Pick $j\in I$ and $k\in\prod_{i\in I\setminus\{j\}}K_i$. Then $j\in J_{\gamma_0}$ for some $\gamma_0\in\Gamma$. Using the assumption, we have

$$\pi^k_j(f)=\pi_j^{\pi_{J_{\gamma_0}\setminus\{j\}}(k)}(\pi_{J_{\gamma_0}}^{\pi_{I\setminus J_{\gamma_0}}(k)}(f))\in\mathcal{H}_j.$$

Assertion (ii) follows from (i) and the fact that $\mathcal{A}^c(\mathcal{H})=\widehat{\mathcal{H}}$. □

From now on, we consider $(K,\mathcal{H})$ *to be a product of* $(K_i,\mathcal{H}_i)$, $i\in I$, *unless said otherwise.*

3.3. **Representing measures.**

Notation 3.14. Let $J\subset I$. We denote by $\mathcal{H}_J$ the space of all functions from $\mathcal{H}$ depending on coordinates from J, i.e.,

$$\mathcal{H}_J:=\{h\in\mathcal{H}:x,y\in K,\ \pi_J(x)=\pi_J(y)\Rightarrow h(x)=h(y)\},$$

and let $\mathcal{H}_f$ be the space of all functions from $\mathcal{H}$ depending on a finite number of coordinates, i.e.,

$$\mathcal{H}_f:=\{h\in\mathcal{H}:\exists J\subset I\text{ finite, so that }h\in\mathcal{H}_J\}.$$

Observation 3.15. *Using the above notation, we observe:*

(a) $I_1\subset I_2\subset I,\ h\in\mathcal{H}_{I_1}\ \Rightarrow\ h\in\mathcal{H}_{I_2}$,

(b) $h\in\mathcal{H}_J\ \Leftrightarrow\ h=\pi_J(h)\otimes 1_{\prod\{K_i:i\in I\setminus J\}}$,

(c) $\mu\in\mathcal{M}^+(K),\ h\in\mathcal{H}_J\ \Rightarrow\ \mu(h)=(\pi_J\mu)(\pi_J(h))$,

(d) $\mathcal{H}_f$ *is a product of* $\mathcal{H}_i$, $i\in I$.

Proposition 3.16. *Let us assume either*

(a) $\mathcal{H} \subset \bigotimes_{i\in I} \mathcal{H}_i$, *or*
(b) $\mathcal{H} = \boxtimes_{i\in I} \mathcal{H}_i$.

Then $\mathcal{H}_f$ is dense in $\mathcal{H}$.

Proof. Assuming (a), conclusion is trivial, since $\bigodot_{i\in I} \mathcal{H}_i \subset \mathcal{H}_f$. Assuming (b), we can use the same technique as in the proof of [16, Theorem 3.1] or [6, Lemma 4]. □

Corollary 3.17. *$\mathcal{C}_f(K)$ is dense in $\mathcal{C}(K)$.*

Proof. Notice that $\mathcal{C}(K) = \boxtimes_{i\in I} \mathcal{C}(K_i)$ and use Proposition 3.16 (b). □

Example 3.18. The conclusion of Proposition 3.16 does not have to be true for all products. Suppose we have $f \in (\boxtimes_{i\in I} \mathcal{H}_i) \setminus (\bigotimes_{i\in I} \mathcal{H}_i)$, which does not depend on finitely many coordinates. Let $\mathcal{H}$ be the linear span of $\bigodot_{i\in I} \mathcal{H}_i \cup \{f\}$. Then $\mathcal{H}_f = \bigodot_{i\in I} \mathcal{H}_i$, but $f \notin \overline{\mathcal{H}_f}$.

Now we construct such a function f. Let $(K_i, \mathcal{H}_i) := (X_i, A(X_i))$, $i = 1, 2$, be as in Corollary 3.9. Then there is $f_1 \in (\mathcal{H}_1 \boxtimes \mathcal{H}_2) \setminus (\mathcal{H}_1 \otimes \mathcal{H}_2)$. This function is not constant with respect to any of the two coordinates, since $f_1 \notin \mathcal{H}_1 \odot \mathcal{H}_2$. Set $\mathcal{H}_{2n+1} := \mathcal{H}_1$, $\mathcal{H}_{2n+2} := \mathcal{H}_2$, $n \in \mathbb{N}$, and let $f_{n+1} := f_1$ be the function from $(\mathcal{H}_{2n+1} \boxtimes \mathcal{H}_{2n+2}) \setminus (\mathcal{H}_{2n+1} \otimes \mathcal{H}_{2n+2})$ for every $n \in \mathbb{N}$. Set

$$f := \sum_{n=1}^{\infty} 2^{-n+1} f_n \otimes 1_{\prod\{K_i : i\in\mathbb{N}\setminus\{2n-1,2n\}\}}.$$

Obviously, f does not depend on finite number of coordinates and $f \in \boxtimes_{i\in\mathbb{N}} \mathcal{H}_i$ since this space is closed. Also $f \notin \bigotimes_{i\in\mathbb{N}} \mathcal{H}_i$. Indeed, if we suppose the contrary, then

$$f \in \bigotimes_{i\in\mathbb{N}} \mathcal{H}_i = (\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes (\bigotimes_{i=3}^{\infty} \mathcal{H}_i) \subset (\mathcal{H}_1 \otimes \mathcal{H}_2) \boxtimes (\bigotimes_{i=3}^{\infty} \mathcal{H}_i).$$

Thus, for $y \in \prod_{i=3}^{\infty} K_i$ is $\pi^y_{\{1,2\}}(f) \in \mathcal{H}_1 \otimes \mathcal{H}_2$. But $\pi^y_{\{1,2\}}(f) = f_1 + c$, where c is a constant, which is a contradiction, since $f_1 \notin \mathcal{H}_1 \otimes \mathcal{H}_2$.

Definition 3.19. Let $(K, \mathcal{H})$ be a product of $(K_i, \mathcal{H}_i)$, $i \in I$. For $J \subset I$ we define the projection of $\mathcal{H}$ by

$$\pi_J(\mathcal{H}) := \{f \in \mathcal{C}(\prod_{i\in J} K_i) : f \otimes 1_{\prod\{K_i : i\in I\setminus J\}} \in \mathcal{H}\}.$$

Observation 3.20. *The following assertions hold:*
(a) *$\pi_J(\mathcal{H})$ is a product of $\mathcal{H}_i$, $i \in J$,*
(b) $\pi_J(\bigodot_{i\in I} \mathcal{H}_i) = \bigodot_{i\in J} \mathcal{H}_i$,
(c) $\pi_J(\bigotimes_{i\in I} \mathcal{H}_i) = \bigotimes_{i\in J} \mathcal{H}_i$,
(d) $\pi_J(\boxtimes_{i\in I} \mathcal{H}_i) = \boxtimes_{i\in J} \mathcal{H}_i$.

Proposition 3.21. *Let $x \in K$, $\mu \in \mathcal{M}_x(\mathcal{H})$ and $J \subset I$. Then $\pi_J\mu \in \mathcal{M}_{\pi_J(x)}(\pi_J(\mathcal{H}))$.*

Proof. Let $h_J \in \pi_J(\mathcal{H})$ and define $h := h_J \otimes 1_{\prod\{K_i:\ i\in I\setminus J\}}$. Then $h \in \mathcal{H}$ and

$$h_J(\pi_J(x)) = h(x) = \mu(h) = (\pi_J\mu)(h_J).$$

□

Proposition 3.22. *Let $x = (x_i)_{i\in I} \in K$ and $\mu_i \in \mathcal{M}_{x_i}(\mathcal{H}_i)$ for every $i \in I$. Then $\mu := \bigotimes_{i\in I} \mu_i \in \mathcal{M}_x(\mathcal{H})$.*

Proof. It suffices to prove the assertion for $\mathcal{H} = \boxtimes_{i\in I} \mathcal{A}^c(\mathcal{H}_i)$.

(i) First, let $|I| = n \in \mathbb{N}$. Choose $h \in \mathcal{H}$. By Fubini's theorem,

$$\mu(h) = \int_K h\, d\mu = \int_{K_1} \dots \int_{K_n} h(y_1, \dots, y_n)\, d\mu_n(y_n) \dots d\mu_1(y_1).$$

Since the function $y_n \mapsto h(y_1, \dots, y_n)$ is in $\mathcal{A}^c(\mathcal{H}_n)$ and $\mu_n \in \mathcal{M}_{x_n}(\mathcal{H}_n)$, we have

$$\int_{K_n} h(y_1, \dots, y_{n-1}, y_n)\, d\mu_n(y_n) = h(y_1, \dots, y_{n-1}, x_n)$$

for every $(y_1, \dots, y_{n-1}) \in \prod_{i=1}^{n-1} K_i$. Using induction, we can see that $\mu(h) = h(x_1, \dots, x_n) = h(x)$. Therefore $\mu \in \mathcal{M}_x(\mathcal{H})$.

(ii) Now, let I be an arbitrary index set. Choose $h \in \mathcal{H}$ and $\varepsilon > 0$. By Proposition 3.16 (b), there is $g \in \mathcal{H}_J$ for some finite $J \subset I$ so that

$$\|g - h\| < \frac{\varepsilon}{2}.$$

Using the first part of the proof, we write

$$\mu(g) = \big(\bigotimes_{i \in J} \mu_i\big)(\pi_J(g)) = \pi_J(g)(\pi_J(x)) = g(x).$$

Let us estimate

$$|\mu(h) - h(x)| \le |\mu(h) - \mu(g)| + |\mu(g) - g(x)| + |g(x) - h(x)| < \varepsilon.$$

Since ε is arbitrary, $\mu(h) = h(x)$. Hence $\mu \in \mathcal{M}_x(\mathcal{H})$.

$\square$

Notation 3.23. Let $A_i \subset \mathcal{M}^1(K_i)$ for every $i \in I$. We denote $\bigotimes_{i \in I} A_i := \{\bigotimes_{i \in I} \mu_i : \mu_i \in A_i,\, i \in I\}$.

Example 3.24. If $|I| = 2$, Proposition 3.22 yields the inclusion

$$\overline{\text{co}}^{w^*}(\mathcal{M}_{x_1}(\mathcal{H}_1) \otimes \mathcal{M}_{x_2}(\mathcal{H}_2)) \subset \mathcal{M}_x(\mathcal{H}), \quad x = (x_1, x_2) \in K.$$

Now we show that the inclusion may be proper.

Let $K_i := \{r_i, s_i, t_i\}$, $\mathcal{H}_i := \{f \in \mathcal{C}(K_i) : f(s_i) = \frac{1}{2}(f(r_i) + f(t_i))\}$, $i = 1, 2$. Then $\mathcal{M}_{s_i}(\mathcal{H}_i) = \text{co}\,\{\varepsilon_{s_i}, \frac{\varepsilon_{r_i} + \varepsilon_{t_i}}{2}\}$. Suppose $(K, \mathcal{H})$ is a product of these two spaces. Denote

$$C := \text{co}\,\{\varepsilon_{s_1} \otimes \varepsilon_{s_2},\, \varepsilon_{s_1} \otimes \frac{\varepsilon_{r_2} + \varepsilon_{t_2}}{2},\, \frac{\varepsilon_{r_1} + \varepsilon_{t_1}}{2} \otimes \varepsilon_{s_2},\, \frac{\varepsilon_{r_1} + \varepsilon_{t_1}}{2} \otimes \frac{\varepsilon_{r_2} + \varepsilon_{t_2}}{2}\}.$$

We see that $\overline{\text{co}}^{w^*}(\mathcal{M}_{s_1}(\mathcal{H}_1) \otimes \mathcal{M}_{s_2}(\mathcal{H}_2)) = C$. Define

$$\mu := \frac{\varepsilon_{(s_1, t_2)}}{2} + \frac{\varepsilon_{(r_1, r_2)}}{4} + \frac{\varepsilon_{(t_1, r_2)}}{4}.$$

Obviously $\mu \in \mathcal{M}_{(s_1, s_2)}(\mathcal{H})$. For every $x \in K \setminus \{(s_1, t_2), (r_1, r_2), (t_1, r_2)\}$ is $\mu(\{x\}) = 0$. However, if μ was an element of C, then at least one of the points (s_1, s_2), (s_1, r_2), (r_1, s_2), (r_1, t_2) would have a non-zero measure.

Example 3.25. Let $x \in K$. Denote $\mathcal{M}_x^\pi(\mathcal{H})$ the set of all $\mu \in \mathcal{M}^1(K)$ such that $\pi_i(\mu) \in \mathcal{M}_{\pi_i(x)}(\mathcal{H}_i)$ for every $i \in I$. Proposition 3.21 yields

$$\mathcal{M}_x(\mathcal{H}) \subset \mathcal{M}_x^\pi(\mathcal{H}).$$

Once again, we show that the inclusion may be proper.

Let $(K_i, \mathcal{H}_i)$, $i = 1, 2$, be as in Example 3.24. Consider

$$\mu := \frac{\varepsilon_{(r_1,r_2)}}{2} + \frac{\varepsilon_{(t_1,t_2)}}{2}.$$

We see that $\pi_i(\mu) = \frac{\varepsilon_{r_i}}{2} + \frac{\varepsilon_{t_i}}{2} \in \mathcal{M}_{s_i}(\mathcal{H}_i)$, $i = 1, 2$. Thus $\mu \in \mathcal{M}^{\pi}_{(s_1,s_2)}(\mathcal{H})$. However $\mu \notin \mathcal{M}_{(s_1,s_2)}(\mathcal{H})$. Indeed, take $f_i \in \mathcal{H}_i$ such that $f_i(r_i) = 0$, $f_i(s_i) = 1$, $f_i(t_i) = 2$, for $i = 1, 2$. Define $f := f_1 \otimes f_2$. Then $f \in \mathcal{H}$, but

$$f(s_1, s_2) = 1 \neq 2 = \mu(f).$$

Question 3.26. Is there a way to characterize $\mathcal{M}_x(\mathcal{H})$ by $\mathcal{M}_{\pi_i(x)}(\mathcal{H}_i)$, $i \in I$?

Proposition 3.27. *Let $x = (x_i)_{i\in I} \in K$. Then $F_x(\mathcal{H}) = \prod_{i\in I} F_{x_i}(\mathcal{H}_i)$.*

Proof. First we show $F_x(\mathcal{H}) \subset \prod_{i\in I} F_{x_i}(\mathcal{H}_i)$. For each $\mu \in \mathcal{M}_x(\mathcal{H})$ and $i \in I$ we have $\pi_i(\operatorname{spt}\mu) = \operatorname{spt}\pi_i\mu$ and since, by Proposition 3.21, $\pi_i\mu \in \mathcal{M}_{x_i}(\mathcal{H}_i)$, we get $\pi_i(\operatorname{spt}\mu) \subset F_{x_i}(\mathcal{H}_i)$. Therefore $\pi_i(F_x(\mathcal{H})) \subset F_{x_i}(\mathcal{H}_i)$ for every $i \in I$.

Conversely, let $\mu_i \in \mathcal{M}_{x_i}(\mathcal{H}_i)$ for every $i \in I$. Proposition 3.22 yields $\bigotimes_{i\in I}\mu_i \in \mathcal{M}_x(\mathcal{H})$ and thus $\prod_{i\in I}\operatorname{spt}\mu_i = \operatorname{spt}\bigotimes_{i\in I}\mu_i \subset F_x(\mathcal{H})$. □

3.4. $\mathcal{H}$-affine functions.

Proposition 3.28. $\mathcal{A}^c(\mathcal{H}) \subset \boxtimes_{i\in I}\mathcal{A}^c(\mathcal{H}_i)$.

Proof. Choose $f \in \mathcal{A}^c(\mathcal{H})$, $j \in I$ and $y = (y_i) \in \prod_{i\in I\setminus\{j\}} K_i$. We prove that $f_j := \pi^y_j(f) \in \mathcal{A}^c(\mathcal{H}_j)$. Let $x_j \in K_j$ and $\mu_j \in \mathcal{M}_{x_j}(\mathcal{H}_j)$. Define $x := (x_j, y)$ and $\mu := \mu_j \otimes (\bigotimes_{i\in I\setminus\{j\}}\varepsilon_{y_i})$. According to Proposition 3.22, $\mu \in \mathcal{M}_x(\mathcal{H})$, so we have

$$f_j(x_j) = f(x) = \mu(f) = \mu_j(f_j).$$

Hence $f_j \in \mathcal{A}^c(\mathcal{H}_j)$. □

Lemma 3.29. *Let $|I| = 2$. Then $\mathcal{A}^c(\mathcal{H}_1) \otimes \mathcal{A}^c(\mathcal{H}_2) \subset \mathcal{A}^c(\mathcal{H})$.*

Proof. Consider $a_1 \in \mathcal{A}^c(\mathcal{H}_1)$, $a_2 \in \mathcal{A}^c(\mathcal{H}_2)$. We show that $a_1 \otimes a_2 \in \mathcal{A}^c(\mathcal{H})$ by using the characterization $\mathcal{A}^c(\mathcal{H}) = \widehat{\mathcal{H}}$.

First suppose that $a_1, a_2 \geq 0$. Choose $x = (x_1, x_2) \in K$ and $\varepsilon > 0$. Find $\delta > 0$ so that

$$\delta(a_1(x_1) + a_2(x_2) + \delta) < \varepsilon.$$

Since $a_i^* = a_i$, $i = 1, 2$, there are $h_1 \in \mathcal{H}_1$, $h_1 \geq a_1$ and $h_2 \in \mathcal{H}_2$, $h_2 \geq a_2$ such that

$$h_1(x_1) < a_1(x_1) + \delta \quad \text{and} \quad h_2(x_2) < a_2(x_2) + \delta.$$

Obviously $h_1 \otimes h_2 \in \mathcal{H}$, $h_1 \otimes h_2 \geq a_1 \otimes a_2$ and

$$\begin{aligned} a_1(x_1)a_2(x_2) &\leq h_1(x_1)h_2(x_2) < (a_1(x_1) + \delta)(a_2(x_2) + \delta) \\ &= a_1(x_1)a_2(x_2) + \delta(a_1(x_1) + a_2(x_2) + \delta) < a_1(x_1)a_2(x_2) + \varepsilon. \end{aligned}$$

Thus $(a_1 \otimes a_2)^* = a_1 \otimes a_2$.

Now suppose $a_1 \geq 0$ and a_2 is arbitrary. Then $a_2 + \|a_2\| \geq 0$. Since $f \mapsto f^*$ is a sublinear functional on $\mathcal{C}(K)$ and $(a_1 \otimes c)^* = a_1 \otimes c$ for every constant function c on K_2, we get

$$\begin{aligned} a_1 \otimes a_2 &\leq (a_1 \otimes a_2)^* = (a_1 \otimes (a_2 + \|a_2\| - \|a_2\|))^* \\ &= (a_1 \otimes (a_2 + \|a_2\|) - a_1 \otimes \|a_2\|)^* \\ &\leq (a_1 \otimes (a_2 + \|a_2\|))^* + (a_1 \otimes (-\|a_2\|))^* \\ &= a_1 \otimes (a_2 + \|a_2\|) + (a_1 \otimes (-\|a_2\|)) = a_1 \otimes a_2. \end{aligned}$$

For the lower envelope we have

$$(a_1 \otimes a_2)_* = -(a_1 \otimes (-a_2))^* = -(a_1 \otimes (-a_2)) = a_1 \otimes a_2.$$

Thus $a_1 \otimes a_2 \in \widehat{\mathcal{H}} = \mathcal{A}^c(\mathcal{H})$.

Finally, let a_1, a_2 be arbitrary. Then

$$a_1 \otimes a_2 = (a_1 + \|a_1\|) \otimes a_2 - \|a_1\| \otimes a_2 \in \mathcal{A}^c(\mathcal{H}).$$

Since $\mathcal{A}^c(\mathcal{H})$ is a closed linear space, the conclusion follows. □

Proposition 3.30. $\bigotimes_{i\in I} \mathcal{A}^c(\mathcal{H}_i) \subset \mathcal{A}^c(\mathcal{H})$.

Proof. It suffices to prove $\bigodot_{i\in I} \mathcal{A}^c(\mathcal{H}_i) \subset \mathcal{A}^c(\mathcal{H})$, since the latter space is closed.

(i) Assume first, that $|I| = n \in \mathbb{N}$ and the assertion holds for $|I| = n-1$. Using the assumption, previous Lemma 3.29 and the associative law, we get

$$\begin{aligned}\bigodot_{i=1}^{n} \mathcal{A}^c(\mathcal{H}_i) &= \Big(\bigodot_{i=1}^{n-1} \mathcal{A}^c(\mathcal{H}_i)\Big) \odot \mathcal{A}^c(\mathcal{H}_n) \subset \mathcal{A}^c\Big(\bigodot_{i=1}^{n-1} \mathcal{H}_i\Big) \odot \mathcal{A}^c(\mathcal{H}_n)\\ &\subset \mathcal{A}^c\Big(\Big(\bigodot_{i=1}^{n-1} \mathcal{H}_i\Big) \odot \mathcal{H}_n\Big) = \mathcal{A}^c\Big(\bigodot_{i=1}^{n} \mathcal{H}_i\Big) \subset \mathcal{A}^c(\mathcal{H}).\end{aligned}$$

(ii) Now, let I be an arbitrary index set. Choose $f \in \bigodot_{i\in I} \mathcal{A}^c(\mathcal{H}_i)$. Then there is a finite $J \subset I$ such that f depends only on coordinates from J. So, according to the first part of the proof, $\pi_J(f) \in \bigodot_{i\in J} \mathcal{A}^c(\mathcal{H}_i) \subset \mathcal{A}^c(\bigodot_{i\in J} \mathcal{H}_i)$. Since $f = \pi_J(f) \otimes 1_{\prod\{K_i : i\in I\setminus J\}}$, we have

$$\begin{aligned}f &\in \mathcal{A}^c\Big(\bigodot_{i\in J} \mathcal{H}_i\Big) \odot \mathcal{A}^c\Big(\bigodot_{i\in I\setminus J} \mathcal{H}_i\Big) \subset \mathcal{A}^c\Big(\Big(\bigodot_{i\in J} \mathcal{H}_i\Big) \odot \Big(\bigodot_{i\in I\setminus J} \mathcal{H}_i\Big)\Big)\\ &= \mathcal{A}^c\Big(\bigodot_{i\in I} \mathcal{H}_i\Big) \subset \mathcal{A}^c(\mathcal{H}).\end{aligned}$$

□

Corollary 3.31. *$\mathcal{A}^c(\mathcal{H})$ is a product of both $\mathcal{H}_i$, $i \in I$, and $\mathcal{A}^c(\mathcal{H}_i)$, $i \in I$.*

Proof. From Proposition 3.30 we have

$$\bigodot_{i\in I} \mathcal{H}_i \subset \bigodot_{i\in I} \mathcal{A}^c(\mathcal{H}_i) \subset \mathcal{A}^c(\mathcal{H}),$$

and from Proposition 3.28

$$\mathcal{A}^c(\mathcal{H}) \subset \boxtimes_{i\in I} \mathcal{A}^c(\mathcal{H}_i) = \boxtimes_{i\in I} \mathcal{A}^c(\mathcal{A}^c(\mathcal{H}_i)).$$

□

Proposition 3.32. *If $\mathcal{A}^c(\mathcal{H}) \subset \boxtimes_{i\in I} \mathcal{H}_i$, then $\mathcal{H}_i = \mathcal{A}^c(\mathcal{H}_i)$ for every $i \in I$.*

Proof. Choose $i \in I$. We prove that $\mathcal{A}^c(\mathcal{H}_i) \subset \mathcal{H}_i$. Pick $f_i \in \mathcal{A}^c(\mathcal{H}_i)$ and define $f := f_i \otimes 1_{\prod\{K_j : j\in I\setminus\{i\}\}}$. Choose $x = (x_j)_{j\in I} \in K$ and $\mu \in \mathcal{M}_x(\mathcal{H})$. From Proposition 3.21 we have $\mu_i := \pi_i\mu \in \mathcal{M}_{x_i}(\mathcal{H}_i)$, which implies

$$f(x) = f_i(x_i) = \mu_i(f_i) = \mu(f).$$

Thus $f \in \mathcal{A}^c(\mathcal{H}) \subset \boxtimes_{i\in I} \mathcal{H}_i$, so $f_i = \pi_i(f) \in \mathcal{H}_i$. □

Proposition 3.33. *Let $\mathcal{H} = \boxtimes_{i\in I} \mathcal{H}_i$. Then $\mathcal{H} = \mathcal{A}^c(\mathcal{H})$ if and only if $\mathcal{H}_i = \mathcal{A}^c(\mathcal{H}_i)$ for every $i \in I$.*

Proof. If $\mathcal{H} = \mathcal{A}^c(\mathcal{H})$, we use Proposition 3.32. Conversely, from Proposition 3.28 we have

$$\mathcal{H} \subset \mathcal{A}^c(\mathcal{H}) \subset \bigboxtimes_{i\in I} \mathcal{A}^c(\mathcal{H}_i) = \bigboxtimes_{i\in I} \mathcal{H}_i = \mathcal{H}.$$

□

Corollary 3.34. *There are function spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that*

$$\mathcal{A}^c(\mathcal{H}_1) \otimes \mathcal{A}^c(\mathcal{H}_2) \subsetneq \mathcal{A}^c(\mathcal{H}_1 \boxtimes \mathcal{H}_2).$$

Proof. By Corollary 3.9, there are $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H}_1 \otimes \mathcal{H}_2 \subsetneq \mathcal{H}_1 \boxtimes \mathcal{H}_2$ and $\mathcal{H}_i = \mathcal{A}^c(\mathcal{H}_i)$, $i = 1, 2$. Proposition 3.33 implies $\mathcal{H}_1 \boxtimes \mathcal{H}_2 = \mathcal{A}^c(\mathcal{H}_1 \boxtimes \mathcal{H}_2)$. Thus

$$\mathcal{A}^c(\mathcal{H}_1) \otimes \mathcal{A}^c(\mathcal{H}_2) = \mathcal{H}_1 \otimes \mathcal{H}_2 \subsetneq \mathcal{H}_1 \boxtimes \mathcal{H}_2 = \mathcal{A}^c(\mathcal{H}_1 \boxtimes \mathcal{H}_2).$$

□

Example 3.35. Example 3.6 shows there are function spaces such that

$$\mathcal{A}^c(\mathcal{H}_1) \odot \mathcal{A}^c(\mathcal{H}_2) \subsetneq \mathcal{A}^c(\mathcal{H}_1 \odot \mathcal{H}_2).$$

Question 3.36. Is $\bigotimes_{i\in I} \mathcal{A}^c(\mathcal{H}_i) = \mathcal{A}^c(\bigodot_{i\in I} \mathcal{H}_i)$?

Question 3.37. Is $\mathcal{A}^c(\bigodot_{i\in I} \mathcal{H}_i) = \mathcal{A}^c(\boxtimes_{i\in I} \mathcal{H}_i)$?

Question 3.38. Is $\mathcal{A}^c(\boxtimes_{i\in I} \mathcal{H}_i) = \boxtimes_{i\in I} \mathcal{A}^c(\mathcal{H}_i)$?

3.5. **$\mathcal{H}$-extremal sets.**

Proposition 3.39. *Let $E \subset K$ be an $\mathcal{H}$-extremal set. Let $J \subset I$, $y \in \prod_{i\in I\setminus J} K_i$, and let $\mathcal{G}$ be a product of $\mathcal{H}_i$, $i \in J$. Then $\pi_J^y(E)$ is either empty or a $\mathcal{G}$-extremal set.*

Proof. Suppose $E^y := \pi_J^y(E)$ is non-empty and not $\mathcal{G}$-extremal. Then there is $x \in E^y$ and $\mu_J \in \mathcal{M}_x(\mathcal{G})$ so that $\operatorname{spt} \mu_J \not\subset E^y$. According to Proposition 3.21, $\mu_i := \pi_i \mu_J \in \mathcal{M}_{\pi_i(x)}(\mathcal{H}_i)$ for every $i \in J$. Since $\operatorname{spt} \mu_J \subset \operatorname{spt} \bigotimes_{i\in J} \mu_i$, we can see that $\operatorname{spt} \bigotimes_{i\in J} \mu_i \not\subset E^y$. Define

$$\mu := \Big(\bigotimes_{i\in J} \mu_i\Big) \otimes \Big(\bigotimes_{i\in I\setminus J} \varepsilon_{\pi_i(y)}\Big).$$

Hence, we have $(x, y) \in E$ and by Proposition 3.22 also $\mu \in \mathcal{M}_{(x,y)}(\mathcal{H})$. But $\operatorname{spt} \mu \not\subset E$, which is a contradiction. □

The next two propositions are generalizations of Proposition 4.1 and Theorem 4.2 from [16] to function spaces:

Proposition 3.40. *Let $E \subset K$ be an $\mathcal{H}$-extremal set. Let $\emptyset \neq J \subset I$ and let $\mathcal{G}$ be a product of $\mathcal{H}_i$, $i \in J$. Then $\pi_J(E)$ is a $\mathcal{G}$-extremal set.*

Proof. Let $x \in \pi_J(E)$ and $\mu \in \mathcal{M}_x(\mathcal{G})$. Then there is $y \in \prod_{i\in I\setminus J} K_i$ such that $(x, y) \in E$, i.e., $x \in \pi_J^y(E)$. By Proposition 3.39, $\pi_J^y(E)$ is a $\mathcal{G}$-extremal set, therefore $\operatorname{spt} \mu \subset \pi_J^y(E) \subset \pi_J(E)$. □

Proposition 3.41. *Let $E_i \subset K_i$ be an $\mathcal{H}_i$-extremal set for every $i \in I$. Then $E := \prod_{i\in I} E_i$ is an $\mathcal{H}$-extremal set.*

Proof. Obviously, E is a closed set.

(i) Assume $|I| = 2$. Suppose there is $x = (x_1, x_2) \in E$ and $\mu \in \mathcal{M}_x(\mathcal{H})$ so that $\mu(K \setminus E) > 0$. Denote $\mu_1 := \pi_1\mu$ and $\mu_2 := \pi_2\mu$. Since

$$K \setminus E = ((K_1 \setminus E_1) \times K_2) \cup (K_1 \times (K_2 \setminus E_2)),$$

we have

$$0 < \mu(K \setminus E) \leq \mu_1(K_1 \setminus E_1) + \mu_2(K_2 \setminus E_2).$$

We may assume $\mu_1(K_1 \setminus E_1) > 0$. By Proposition 3.21, $\mu_1 \in \mathcal{M}_{x_1}(\mathcal{H}_1)$. But this is a contradiction, because $x_1 \in E_1$.

We proceed similarly for arbitrary finite products.

(ii) Now, let I be infinite. Suppose there is $x = (x_i)_{i\in I} \in E$ and $\mu \in \mathcal{M}_x(\mathcal{H})$ so that $\mu(K \setminus E) > 0$. Then there is some $g \in \mathcal{C}(K)$ such that $g = 0$ on E and $\mu(g) > 0$. Choose $\varepsilon > 0$. According to Corollary 3.17, there is $f \in \mathcal{C}_J(K)$, where $J \subset I$ is finite and $\|g - f\| < \varepsilon$. Then $\pi_J\mu \in \mathcal{M}_{\pi_J(x)}(\pi_J(\mathcal{H}))$ and by the first part of the proof, $\pi_J(x)$ is an element of the $\pi_J(\mathcal{H})$-extremal set $E_J := \prod_{i\in J} E_i$. Thus $\operatorname{spt} \pi_J\mu \subset E_J$ and $|\pi_J(f)| < \varepsilon$ on E_J. Therefore

$$|\mu(f)| = |(\pi_J\mu)(\pi_J(f))| \leq \int_{E_J} |\pi_J(f)|\, d(\pi_J\mu) + \int_{(\prod_{i\in J} K_i)\setminus E_J} |\pi_J(f)|\, d(\pi_J\mu) < \varepsilon.$$

Hence we get

$$0 < |\mu(g)| \leq |\mu(g) - \mu(f)| + |\mu(f)| < 2\varepsilon,$$

which is a contradiction, since ε is arbitrary.

□

Using previous results, we can derive the main theorem of this subsection (cf. also [6, Lemma 5], [16, Theorem 3.2] and [10, Lemma 5.11]):

Theorem 3.42. $\operatorname{Ch}_{\mathcal{H}} K = \prod_{i\in I} \operatorname{Ch}_{\mathcal{H}_i} K_i$.

Proof. Immediately follows from Propositions 3.40 and 3.41. □

Corollary 3.43. $\nabla_{\mathcal{H}} K = \prod_{i\in I} \nabla_{\mathcal{H}_i} K_i$.

Proof. Using Theorem 3.42 we can write

$$\nabla_{\mathcal{H}} K = \overline{\operatorname{Ch}_{\mathcal{H}} K} = \overline{\prod_{i\in I} \operatorname{Ch}_{\mathcal{H}_i} K_i} = \prod_{i\in I} \overline{\operatorname{Ch}_{\mathcal{H}_i} K_i} = \prod_{i\in I} \nabla_{\mathcal{H}_i} K_i.$$

□

Remarks 3.44. As has been shown by Grossman [9], the characterizations of Choquet and Šilov boundary hold also for the space $\mathcal{H}_1 + \mathcal{H}_2$ defined by

$$\mathcal{H}_1 + \mathcal{H}_2 := \{h_1 + h_2 : h_1 \in \mathcal{H}_1,\, h_2 \in \mathcal{H}_2\}, \quad \text{where}$$
$$[h_1 + h_2](x, y) = h_1(x) + h_2(y), \quad (x, y) \in K_1 \times K_2.$$

It is clear that $\mathcal{H}_1 + \mathcal{H}_2$ does not have to be a product, since the inclusion $\mathcal{H}_1 \odot \mathcal{H}_2 \subset \mathcal{H}_1 + \mathcal{H}_2$ does not have to hold.

Versions of Theorem 3.42 for various tensor products of compact convex sets has been proved by I. Namioka and R.R. Phelps in [18].

Example 3.45. In Example 3.3 we have shown that the space of all harmonic functions on a cartesian product does not have to be a product of harmonic spaces. Moreover, it is not even possible to extend the notion of a product so that the product of harmonic spaces would be a harmonic space and Theorem 3.42 would still hold. Indeed, consider the sets from Example 3.3 and denote $U := U_1 \times U_2$. Then

$$\mathrm{Ch}_{H(U)}\,\overline{U} = \partial_{\mathrm{reg}} U = \partial U \neq \{0,1\} \times \{0,1\} = \mathrm{Ch}_{H(U_1)}\,\overline{U}_1 \times \mathrm{Ch}_{H(U_2)}\,\overline{U}_2.$$

3.6. **Approximation in product spaces.** In the following, we will need some results on approximation of functions in simplicial spaces. So we first state here results that are adaptation of Section 2 from [17].

Definition 3.46. Let $(K, \mathcal{H})$ be a function space. A collection of nonnegative functions $\{\psi_j\}_{j=1}^m \subset \mathcal{H}$ is called a *partition of unity* on K, if $\sum_{j=1}^m \psi_j = 1_K$.

Lemma 3.47. *Let $(K, \mathcal{H})$ be a simplicial function space. Let $\{f_i\}_{i=1}^n \subset \mathcal{A}^c(\mathcal{H})$ and $\varepsilon > 0$. Suppose that $\{\phi_j\}_{j=1}^m$ are nonnegative functions defined on $\mathrm{Ch}_{\mathcal{H}}\, K$, $\{k_l\}_{l=1}^m \subset \mathrm{Ch}_{\mathcal{H}}\, K$ and $\{\alpha_{ij} : 1 \leq i \leq n,\ 1 \leq j \leq m\}$ are real numbers such that*

(i) $\sum_{j=1}^m \phi_j = 1$,
(ii) $\phi_j(k_l) = \delta_{jl}, \quad 1 \leq j, l \leq m$,
(iii) $|f_i(k) - \sum_{j=1}^m \alpha_{ij}\phi_j(k)| \leq \varepsilon, \quad k \in \mathrm{Ch}_{\mathcal{H}}\, K,\ 1 \leq i \leq n.$

Then there exists a partition of unity $\{\psi_j\}_{j=1}^m \subset \mathcal{A}^c(\mathcal{H})$ such that

(iv) $\psi_j(k_l) = \delta_{jl}, \quad 1 \leq j, l \leq m$,
(v) $|f_i(k) - \sum_{j=1}^m \alpha_{ij}\psi_j(k)| \leq \varepsilon, \quad k \in K,\ 1 \leq i \leq n.$

Proof. See [17, Corollary 2.2]. □

The proof of the next lemma is based on the proof of [17, Lemma 2.4]:

Lemma 3.48. *Let $(K_1, \mathcal{H}_1)$ and $(K_2, \mathcal{H}_2)$ be two function spaces, where $\mathcal{H}_1$ is simplicial. Suppose that $\{f_i\}_{i=1}^n \subset \mathcal{A}^c(\mathcal{H}_1) \boxtimes \mathcal{H}_2$ and $\varepsilon > 0$. Then there is a partition of unity $\{\psi_j\}_{j=1}^m \subset \mathcal{A}^c(\mathcal{H}_1)$, $\{k_l\}_{l=1}^m \subset \mathrm{Ch}_{\mathcal{H}_1}\, K_1$ and $\{y_{ij}\} \subset \mathcal{H}_2$, $1 \leq i \leq n$, $1 \leq j \leq m$, so that*

(i) $\psi_j(k_l) = \delta_{jl}, \quad 1 \leq j, l \leq m$,
(ii) $\|f_i - \sum_{j=1}^m \psi_j \otimes y_{ij}\| < \varepsilon, \quad 1 \leq i \leq n.$

Proof. Denote by $\mathcal{H}_2^n$ the n-tuple cartesian product of $\mathcal{H}_2$ with the maximum norm, i.e.,

$$\|y\|_{\max} = \max_{1 \leq i \leq n} \|\pi_i(y)\| \quad \text{for all } y \in \mathcal{H}_2^n,$$

where π_i is the projection to the i-th coordinate. We denote by $B_r(x)$ the open ball with center x and radius $r > 0$.

Let f be a function from K_1 to $\mathcal{H}_2^n$ defined by

$$f(k) := (\pi_2^k(f_1), \ldots, \pi_2^k(f_n)), \quad k \in K_1.$$

Since $\pi_i \circ f$ is a continuous function for every $i = 1, \ldots, n$ (we use the fact that $\mathcal{C}(K_1 \times K_2)$ is isometric to $\mathcal{C}(K_1, \mathcal{C}(K_2))$), f is also a continuous function on K_1.

For each $y \in \mathcal{H}_2^n$ set

$$U_y := \left\{ k \in K_1 : \|y - f(k)\|_{\max} < \frac{\varepsilon}{3} \right\}. \tag{1}$$

The family $\{U_y\}_{y\in\mathcal{H}_2^n}$ is an open covering of K_1. Let $U_{y_1},\dots,U_{y_p}$ be a finite subcovering. Define

$$V_{y_j} := U_{y_j} \cap \mathrm{Ch}_{\mathcal{H}_1}\ K_1, \quad 1 \le j \le p.$$

Without loss of generality we may assume that there is $m \le p$ such that $\{V_{y_l}\}_{l=1}^m$ is an open covering of $\mathrm{Ch}_{\mathcal{H}_1}\ K_1$ and for every $l \in \{1,\dots,m\}$ there exists $k_l \in V_{y_l}$ such that $k_l \notin V_{y_j}$ for $j \ne l$, $1 \le j \le m$.

Denote

$$C := \{y_1,\dots,y_p\} - \mathrm{co}(y_1,\dots,y_m),$$
$$D := C + B_{\frac{\varepsilon}{3}}(0).$$

Choose $i \in \{1,\dots,n\}$. Since C is a compact subset of $\mathcal{H}_2^n$, also $\pi_i(C)$ is a compact subset of $\mathcal{H}_2$. By Arzelà-Ascoli's theorem, the set $\pi_i(C)$ is equicontinuous. Therefore, for each $\xi \in K_2$ we can find its open neighbourhood W_ξ such that $\mathrm{osc}_{W_\xi} h < \frac{\varepsilon}{3}$ for every $h \in \pi_i(C)$. From the open covering $\{W_\xi\}_{\xi\in K_2}$ we choose a finite subcovering $\{W_{\xi_{ir}}\}_{r=1}^{q_i}$. For every $h \in \pi_i(C)$ there is $x_h \in K_2$ such that $|h(x_h)| = \|h\|$. The point x_h is an element of some $W_{\xi_{ir}}$ and so $\|h\| - \frac{\varepsilon}{3} < |h(\xi_{ir})|$. Thus

$$\|h\| - \frac{\varepsilon}{3} < \max_{1\le r\le q_i} |h(\xi_{ir})| \le \|h\|, \quad h \in \pi_i(C),$$

and since $\pi_i(D) \subset \pi_i(C) + B_{\frac{\varepsilon}{3}}(0)$, also

$$\|h\| - \frac{2}{3}\varepsilon < \max_{1\le r\le q_i} |h(\xi_{ir})| \le \|h\|, \quad h \in \pi_i(D). \tag{2}$$

Let $\Gamma_{ir} \in (\mathcal{H}_2^n)^*$ be a continuous linear functional defined by

$$\Gamma_{ir}(y) := \pi_i(y)(\xi_{ir}), \quad 1 \le i \le n,\ 1 \le r \le q_i,\ y \in \mathcal{H}_2^n.$$

From (2) we can write

$$\|h\|_{\max} - \frac{2}{3}\varepsilon < \max_{\substack{1\le i\le n\\ 1\le r\le q_i}} |\Gamma_{ir}(h)| \le \|h\|_{\max}, \quad h \in D. \tag{3}$$

Set

$$\phi_j(k) := \begin{cases} 1, & \text{if } j = \min\{l : k \in V_{y_l},\ 1 \le l \le m\}, \\ 0, & \text{otherwise,} \end{cases} \qquad 1 \le j \le m,\ k \in \mathrm{Ch}_{\mathcal{H}_1}\ K_1.$$

Clearly $\phi_j \ge 0$, $\phi_j(k_l) = \delta_{jl}$, $1 \le j,l \le m$, and $\sum_{j=1}^m \phi_j = 1$. Moreover, for every $k \in \mathrm{Ch}_{\mathcal{H}_1}\ K_1$ there is a unique index j_k so that $\phi_{j_k}(k) \ne 0$. For this index is $k \in V_{y_{j_k}}$. Thus, from (1) we have

$$\|f(k) - y_{j_k}\|_{\max} < \frac{\varepsilon}{3}.$$

We can rewrite this inequality as

$$\|f(k) - \sum_{j=1}^m \phi_j(k) y_j\|_{\max} < \frac{\varepsilon}{3}, \quad k \in \mathrm{Ch}_{\mathcal{H}_1}\ K_1. \tag{4}$$

Since $f(k) - \sum_{j=1}^m \phi_j(k)y_j \in D$, using (3) and (4) we have

$$\begin{aligned}|\Gamma_{ir}(f(k)) - \sum_{j=1}^m \phi_j(k)\Gamma_{ir}(y_j)| &= |\Gamma_{ir}(f(k) - \sum_{j=1}^m \phi_j(k)y_j)| \\ &\le \|f(k) - \sum_{j=1}^m \phi_j(k)y_j\|_{\max} \\ &< \frac{\varepsilon}{3}, \quad 1 \le i \le n,\ 1 \le r \le q_i,\ k \in \mathrm{Ch}_{\mathcal{H}_1}\ K_1.\end{aligned}$$

Lemma 3.47 yields a partition of unity $\{\psi_j\}_{j=1}^m \subset \mathcal{A}^c(\mathcal{H}_1)$ such that

$$(5)\qquad |\Gamma_{ir}(f(k)) - \sum_{j=1}^m \psi_j(k)\Gamma_{ir}(y_j)| \le \frac{\varepsilon}{3}, \quad 1 \le i \le n,\ 1 \le r \le q_i,\ k \in K_1,$$

$$(6)\qquad \psi_j(k_l) = \delta_{jl}, \quad 1 \le j, l \le m.$$

Since $f(k) - \sum_{j=1}^m \psi_j(k)y_j \in D$ for every $k \in K_1$, using (3) and (5) we get

$$\begin{aligned}\|f(k) - \sum_{j=1}^m \psi_j(k)y_j\|_{\max} - \frac{2}{3}\varepsilon &< \max_{\substack{1\le i\le n\\ 1\le r\le q_i}} |\Gamma_{ir}(f(k) - \sum_{j=1}^m \psi_j(k)y_j)| \\ &= \max_{\substack{1\le i\le n\\ 1\le r\le q_i}} |\Gamma_{ir}(f(k)) - \sum_{j=1}^m \psi_j(k)\Gamma_{ir}(y_j)| \\ &\le \frac{\varepsilon}{3}, \quad k \in K_1.\end{aligned}$$

Hence

$$(7)\qquad \|f(k) - \sum_{j=1}^m \psi_j(k)y_j\|_{\max} < \varepsilon, \quad k \in K_1.$$

Finally, define $y_{ij} := \pi_i(y_j) \in \mathcal{H}_2,\ 1 \le i \le n,\ 1 \le j \le m$. Assertion (i) then follows from (6) and (ii) follows from (7). □

3.7. **Products of simplicial spaces.**

Proposition 3.49. *Suppose that at most one of the spaces $\mathcal{H}_i$, $i \in I$, is not simplicial. Then*

$$\bigotimes_{i\in I} \mathcal{A}^c(\mathcal{H}_i) = \mathcal{A}^c(\mathcal{H}) = \boxtimes_{i\in I} \mathcal{A}^c(\mathcal{H}_i).$$

Proof. Due to Propositions 3.30 and 3.28, it suffices to prove $\boxtimes_{i\in I} \mathcal{A}^c(\mathcal{H}_i) \subset \bigotimes_{i\in I} \mathcal{A}^c(\mathcal{H}_i)$.

(i) First we prove the assertion for finite products. Let $|I| = n \ge 2$ and suppose that $\mathcal{H}_1, \dots, \mathcal{H}_{n-1}$ are simplicial. We repeatedly use associative laws and Lemma 3.48 to get

$$\begin{aligned}\boxtimes_{i=1}^{n}\mathcal{A}^c(\mathcal{H}_i) &= \mathcal{A}^c(\mathcal{H}_1)\boxtimes(\mathcal{A}^c(\mathcal{H}_2)\boxtimes(\dots(\mathcal{A}^c(\mathcal{H}_{n-1})\boxtimes\mathcal{A}^c(\mathcal{H}_n))\dots))\\ &\subset \mathcal{A}^c(\mathcal{H}_1)\otimes(\mathcal{A}^c(\mathcal{H}_2)\boxtimes(\dots(\mathcal{A}^c(\mathcal{H}_{n-1})\boxtimes\mathcal{A}^c(\mathcal{H}_n))\dots))\\ &\subset\dots\subset \mathcal{A}^c(\mathcal{H}_1)\otimes(\mathcal{A}^c(\mathcal{H}_2)\otimes(\dots(\mathcal{A}^c(\mathcal{H}_{n-1})\otimes\mathcal{A}^c(\mathcal{H}_n))\dots))\\ &= \bigotimes_{i=1}^{n}\mathcal{A}^c(\mathcal{H}_i).\end{aligned}$$

(ii) Now, let I be infinite. Choose $f \in \boxtimes_{i\in I}\mathcal{A}^c(\mathcal{H}_i)$ and $\varepsilon > 0$. According to Proposition 3.16 (b), there is some $h \in \boxtimes_{i\in I}\mathcal{A}^c(\mathcal{H}_i)$ depending on finitely many coordinates J such that $\|f-h\| < \varepsilon$. From the first part of the proof we have $\pi_J(h) \in \boxtimes_{i\in J}\mathcal{A}^c(\mathcal{H}_i) \subset \bigotimes_{i\in J}\mathcal{A}^c(\mathcal{H}_i)$. Thus, $h \in \bigotimes_{i\in I}\mathcal{A}^c(\mathcal{H}_i)$. Since the space is closed, we get $f \in \bigotimes_{i\in I}\mathcal{A}^c(\mathcal{H}_i)$.

□

Example 3.50. The assumption on the number of simplicial spaces in Proposition 3.49 may not be weakened. We show that for every index set I with $|I| \geq 2$ there is a family of function spaces $\mathcal{H}_i$, $i \in I$, with two non-simplicial spaces, which does not satisfy the equality in Proposition 3.49. Once again, we use Corollary 3.9 to construct a counterexample.

Let $\mathcal{H}_i$, $i \in I$, be a family of function spaces such that there are $i_1, i_2 \in I$ so that $\mathcal{H}_{i_1}, \mathcal{H}_{i_2}$ are as in Corollary 3.9. Thus, there is $f' \in (\mathcal{A}^c(\mathcal{H}_{i_1}) \boxtimes \mathcal{A}^c(\mathcal{H}_{i_2})) \setminus (\mathcal{A}^c(\mathcal{H}_{i_1}) \otimes \mathcal{A}^c(\mathcal{H}_{i_2}))$. Using Proposition 3.49, we can see that neither of the two spaces is simplicial. Now, define

$$f := f' \otimes 1_{\prod\{K_i : i\in I\setminus\{i_1,i_2\}\}}.$$

We have $f \in \boxtimes_{i\in I}\mathcal{A}^c(\mathcal{H}_i)$, but $f \notin \bigotimes_{i\in I}\mathcal{A}^c(\mathcal{H}_i)$.

Lemma 3.51. *Let $|I| = 2$ and suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are simplicial. Then $\mathcal{H}$ is simplicial.*

Proof. It is sufficient to show that $\mathcal{A}^c(\mathcal{H})$ has W.R.I.P. Let a, b, c, d be functions from $\mathcal{A}^c(\mathcal{H}) = \mathcal{A}^c(\mathcal{H}_1) \boxtimes \mathcal{A}^c(\mathcal{H}_2)$ such that $a \vee b < c \wedge d$. By Lemma 3.48, there is a partition of unity $\{\psi_j\}_{j=1}^m \subset \mathcal{A}^c(\mathcal{H}_1)$, $\{k_l\}_{l=1}^m \subset \operatorname{Ch}_{\mathcal{H}_1} K_1$ and functions $\{a_j, b_j, c_j, d_j\}_{j=1}^m \subset \mathcal{A}^c(\mathcal{H}_2)$ so that

$$\psi_j(k_l) = \delta_{jl}, \quad 1 \leq j, l \leq m, \tag{8}$$

and for

$$\begin{aligned} a' &:= \sum_{j=1}^m \psi_j \otimes a_j, \quad b' := \sum_{j=1}^m \psi_j \otimes b_j,\\ c' &:= \sum_{j=1}^m \psi_j \otimes c_j, \quad d' := \sum_{j=1}^m \psi_j \otimes d_j,\end{aligned} \tag{9}$$

is

$$a \vee b < a' \vee b' < c' \wedge d' < c \wedge d. \tag{10}$$

Then also

$$\pi_2^k(a') \vee \pi_2^k(b') < \pi_2^k(c') \wedge \pi_2^k(d'), \quad k \in K_1. \tag{11}$$

For every $j = 1, \ldots, m$, we get from (8) and (9)

$$\pi_2^{k_j}(a') = a_j, \quad \pi_2^{k_j}(b') = b_j, \quad \pi_2^{k_j}(c') = c_j, \quad \pi_2^{k_j}(d') = d_j,$$

and from (11)

$$a_j \vee b_j < c_j \wedge d_j.$$

Since $\mathcal{A}^c(\mathcal{H}_2)$ has W.R.I.P., there are $h_j \in \mathcal{A}^c(\mathcal{H}_2)$, $j = 1, \ldots, m$, such that

$$a_j \vee b_j < h_j < c_j \wedge d_j. \tag{12}$$

Define $h := \sum_{j=1}^m \psi_j \otimes h_j \in \mathcal{A}^c(\mathcal{H}_1) \otimes \mathcal{A}^c(\mathcal{H}_2) = \mathcal{A}^c(\mathcal{H})$. The non-negativity of $\{\psi_j\}_{j=1}^m$ and inequalities (12) and (10) imply

$$a \vee b < h < c \wedge d.$$

Hence $\mathcal{A}^c(\mathcal{H})$ has W.R.I.P. and the proof is complete. □

Now we may prove the theorem, which is a generalization of [6, Theorem 11] and [16, Theorem 3.1]:

Theorem 3.52. *Suppose that $\mathcal{H}_i$ is simplicial for each $i \in I$. Then $\mathcal{H}$ is simplicial.*

Proof. First we prove the theorem for finite I. By Lemma 3.51, theorem holds for $|I| = 2$. Suppose that $|I| = n > 2$ and the theorem holds for $|I| < n$. Clearly $\boxtimes_{i=1}^n \mathcal{H}_i = (\boxtimes_{i=1}^{n-1} \mathcal{H}_i) \boxtimes \mathcal{H}_n$ is simplicial and $\mathcal{A}^c(\mathcal{H}) = \mathcal{A}^c(\boxtimes_{i=1}^n \mathcal{H}_i)$ has W.R.I.P. Therefore $\mathcal{H}$ is simplicial.

Now, let I be infinite. Choose a, b, c, d from $\mathcal{A}^c(\mathcal{H}) = \boxtimes_{i \in I} \mathcal{A}^c(\mathcal{H}_i)$ such that $a \vee b < c \wedge d$. According to Proposition 3.16 (b), there are

$$a' \in \left[\boxtimes_{i \in I} \mathcal{A}^c(\mathcal{H}_i)\right]_{I_a}, \quad b' \in \left[\boxtimes_{i \in I} \mathcal{A}^c(\mathcal{H}_i)\right]_{I_b},$$
$$c' \in \left[\boxtimes_{i \in I} \mathcal{A}^c(\mathcal{H}_i)\right]_{I_c}, \quad d' \in \left[\boxtimes_{i \in I} \mathcal{A}^c(\mathcal{H}_i)\right]_{I_d},$$

so that

$$a \vee b < a' \vee b' < c' \wedge d' < c \wedge d$$

and $J := I_a \cup I_b \cup I_c \cup I_d$ is a finite subset of I. Then also

$$\pi_J(a') \vee \pi_J(b') < \pi_J(c') \wedge \pi_J(d').$$

From the first part of the proof we know that $\boxtimes_{i \in J} \mathcal{H}_i$ is simplicial, so we can find $h' \in \mathcal{A}^c(\boxtimes_{i \in J} \mathcal{H}_i) = \boxtimes_{i \in J} \mathcal{A}^c(\mathcal{H}_i)$ such that

$$\pi_J(a') \vee \pi_J(b') < h' < \pi_J(c') \wedge \pi_J(d').$$

Function $h := h' \otimes 1_{\prod\{K_i : i \in I \setminus J\}} \in \boxtimes_{i \in I} \mathcal{A}^c(\mathcal{H}_i) = \mathcal{A}^c(\mathcal{H})$ clearly satisfies

$$a \vee b < h < c \wedge d.$$

Hence $\mathcal{A}^c(\mathcal{H})$ has W.R.I.P. and $\mathcal{H}$ is simplicial. □

The converse, whose special case has been proved in [18, Proposition 2.10], is also valid:

Theorem 3.53. *Suppose that $\mathcal{H}$ is simplicial. Then $\mathcal{H}_i$ is simplicial for each $i \in I$.*

Proof. We use the W.R.I.P. property of simplicial spaces again. Choose $j \in I$. Let $a_j, b_j, c_j, d_j \in \mathcal{A}^c(\mathcal{H}_j)$ be such that $a_j \vee b_j < c_j \wedge d_j$. Denote $K' := \prod_{i \in I \setminus \{j\}} K_i$. According to Proposition 3.30,

$$a := a_j \otimes 1_{K'}, \quad b := b_j \otimes 1_{K'}, \quad c := c_j \otimes 1_{K'}, \quad d := d_j \otimes 1_{K'},$$

are elements of $\mathcal{A}^c(\mathcal{H})$. Moreover, $a \vee b < c \wedge d$. Using simpliciality of $\mathcal{H}$, there exists $h \in \mathcal{A}^c(\mathcal{H})$ so that

$$a \vee b < h < c \wedge d.$$

Pick $y \in K'$. By Proposition 3.28, $h \in \boxtimes_{i \in I} \mathcal{A}^c(\mathcal{H}_i)$, therefore $\pi_j^y(h) \in \mathcal{A}^c(\mathcal{H}_j)$. Since

$$a_j \vee b_j < \pi_j^y(h) < c_j \wedge d_j,$$

we conclude that the space $\mathcal{A}^c(\mathcal{H}_j)$ has W.R.I.P. □

Example 3.54. The space $\mathcal{H}_1 + \mathcal{H}_2$, defined by Grossman (see Remarks 3.44), does not have to be simplicial, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are simplicial. Indeed, let $K_1 = K_2 = [0,1] \subset \mathbb{R}$ and $\mathcal{H}_1 = \mathcal{H}_2 = A([0,1])$. Obviously, $\mathcal{H}_1$ and $\mathcal{H}_2$ are simplicial spaces. Denote $K := K_1 \times K_2$. It is easy to prove that $\mathcal{H}_1 + \mathcal{H}_2 = A(K)$. However, K is not a simplex, which is the sought contradiction.

3.8. **Maximal measures.** We start with two propositions, which are analogies of [2, Theorem 4]:

Proposition 3.55. *Let $\mu \in \mathcal{M}^+(K)$ be $\mathcal{H}$-maximal. Let $J \subset I$ and let $\mathcal{G}$ be a product of $\mathcal{H}_i$, $i \in J$. Then $\pi_J \mu$ is a $\mathcal{G}$-maximal measure.*

Proof. According to Proposition 2.2, it suffices to show $F_{\pi_J(x)}(\mathcal{G}) \subset \pi_J(F_x(\mathcal{H}))$ for every $x \in K$. Using Proposition 3.27, we have

$$F_{\pi_J(x)}(\mathcal{G}) = \prod_{i \in J} F_{x_i}(\mathcal{H}_i) = \pi_J(\prod_{i \in I} F_{x_i}(\mathcal{H}_i)) = \pi_J(F_x(\mathcal{H})),$$

for every $x = (x_i)_{i \in I} \in K$. □

Lemma 3.56. *Let $\mu, \nu \in \mathcal{M}^+(K)$ be such that $\mu \preceq_{\mathcal{H}} \nu$. Then for every $J \subset I$ is $\pi_J \mu \preceq_{\pi_J(\mathcal{H})} \pi_J \nu$.*

Proof. Choose $w_J \in \mathcal{W}(\pi_J(\mathcal{H}))$. Then $w := w_J \otimes 1_{\prod\{K_i : i \in I \setminus J\}} \in \mathcal{W}(\mathcal{H})$. Thus $\mu(w) \leq \nu(w)$, and we get

$$(\pi_J \mu)(w_J) = \mu(w) \leq \nu(w) = (\pi_J \nu)(w_J).$$

Since w_J is arbitrary, we have $\pi_J \mu \preceq \pi_J \nu$. □

Proposition 3.57. *Let $|I| = 2$ and let $\mu \in \mathcal{M}^+(K)$ be such that $\pi_i \mu$ is an $\mathcal{H}_i$-maximal measure for $i = 1, 2$. Then μ is $\mathcal{H}$-maximal.*

In particular, if $\mu_i \in \mathcal{M}^1(K_i)$ is an $\mathcal{H}_i$-maximal measure for $i = 1, 2$, then $\mu_1 \otimes \mu_2$ is $\mathcal{H}$-maximal.

Proof. We may proceed exactly as in the second part of the proof of [2, Theorem 4] to show that for every $h \in \mathcal{H}$ and μ-almost all $x \in K$ is

$$h(x_1, x_2) = h(\pi_1(x), \pi_2(x)), \quad x_1 \in F_{\pi_1(x)}(\mathcal{H}_1),\ x_2 \in F_{\pi_2(x)}(\mathcal{H}_2).$$

According to Proposition 3.27, $F_x(\mathcal{H}) = F_{\pi_1(x)}(\mathcal{H}_1) \times F_{\pi_2(x)}(\mathcal{H}_2)$ for every $x \in K$. Therefore h is constant on $F_x(\mathcal{H})$ for μ-almost all $x \in K$. As follows from Theorem 2.1, μ is an $\mathcal{H}$-maximal measure. □

Theorem 3.58. *Let $\mu \in \mathcal{M}^+(K)$ be such that $\pi_i\mu$ is an $\mathcal{H}_i$-maximal measure for every $i \in I$. Then μ is $\mathcal{H}$-maximal.*

In particular, if $\mu_i \in \mathcal{M}^1(K_i)$ is an $\mathcal{H}_i$-maximal measure for every $i \in I$, then $\bigotimes_{i\in I} \mu_i$ is $\mathcal{H}$-maximal.

Proof. It suffices to show that μ is a $(\bigodot_{i\in I} \mathcal{H}_i)$-maximal measure.

First we prove the assertion for finite products. Suppose that it holds for $|I| \le n$ and let $|I| = n+1$. We know that $\pi_{n+1}\mu$ is an $\mathcal{H}_{n+1}$-maximal measure. Since $\pi_i(\pi_{\{1,\dots,n\}}\mu) = \pi_i\mu$ is an $\mathcal{H}_i$-maximal measure for every $i = 1,\dots,n$, the induction hypothesis implies that $\pi_{\{1,\dots,n\}}\mu$ is a $(\bigodot_{i=1}^n \mathcal{H}_i)$-maximal measure. Thus, both projections are maximal measures and Proposition 3.57 implies that μ is a $((\bigodot_{i=1}^n \mathcal{H}_i) \odot \mathcal{H}_{n+1})$-maximal measure, therefore also $(\bigodot_{i=1}^{n+1} \mathcal{H}_i)$-maximal measure.

Now, let I be infinite. According to Choquet-Bishop-de Leeuw's theorem, there exists a $(\bigodot_{i\in I} \mathcal{H}_i)$-maximal measure $\nu \in \mathcal{M}^+(K)$ such that $\mu \preceq_{\bigodot_{i\in I} \mathcal{H}_i} \nu$. Suppose $J \subset I$ is finite. By Lemma 3.56, $\pi_J\mu \preceq_{\bigodot_{i\in J} \mathcal{H}_i} \pi_J\nu$. From the first part of the proof is $\pi_J\mu$ a $(\bigodot_{i\in J} \mathcal{H}_i)$-maximal measure and therefore $\pi_J\mu = \pi_J\nu$. Hence, for every finite subset $J \subset I$ and every $E = \prod_{i\in I} E_i$, where E_i is a Borel subset of K_i for each $i \in I$ and $E_i = K_i$ for $i \in I \setminus J$,

$$\mu(E) = (\pi_J\mu)(\prod_{i\in J} E_i) = (\pi_J\nu)(\prod_{i\in J} E_i) = \nu(E).$$

Since μ and ν coincide on the Borel cylinder sets, they must coincide as Radon measures. Therefore μ is a $(\bigodot_{i\in I} \mathcal{H}_i)$-maximal measure. □

Theorem 3.59. *Suppose that $\mathcal{H}_i$ is simplicial for each $i \in I$. Then $\delta_x = \bigotimes_{i\in I} \delta_{x_i}$ for every $x = (x_i)_{i\in I} \in K$.*

Proof. From Proposition 3.22 we have $\bigotimes_{i\in I} \delta_{x_i} \in \mathcal{M}_x(\mathcal{H})$ and by Theorem 3.58, this measure is $\mathcal{H}$-maximal. Since $\mathcal{H}$ is simplicial, according to Theorem 3.52, we get $\delta_x = \bigotimes_{i\in I} \delta_{x_i}$. □

At the end of this section we investigate relationship between maximal measures in product spaces and Radon products of maximal measures. We denote by $\mathcal{Z}^1(\mathcal{H})$ the set of $\mathcal{H}$-maximal measures from $\mathcal{M}^1(K)$. Let $\varepsilon_{\mathrm{Ch}_\mathcal{H} K} := \{\varepsilon_x : x \in \mathrm{Ch}_\mathcal{H} K\}$ and let $\mathcal{D}(\mathcal{H})$ denote the linear span of $\mathcal{C}(K) \cup \{f^* : f \in \mathcal{C}(K)\}$. We denote by τ the weak topology on $\mathcal{M}^1(K)$ generated by $\mathcal{D}(\mathcal{H})$. Then we have:

Proposition 3.60. *The following assertions hold:*

(a) $\operatorname{co} \varepsilon_{\mathrm{Ch}_\mathcal{H} K} \subset \mathcal{Z}^1(\mathcal{H}) \subset \overline{\operatorname{co}}^{w^*} \varepsilon_{\mathrm{Ch}_\mathcal{H} K}$,

(b) $\mathcal{Z}^1(\mathcal{H}) = \overline{\operatorname{co}}^{\tau} \varepsilon_{\mathrm{Ch}_\mathcal{H} K}$.

Proof.

(a) The first inclusion is obvious. The second follows from the fact that
$$\overline{\operatorname{co}}^{w^*} \varepsilon_{\mathrm{Ch}_\mathcal{H} K} = \mathcal{M}^1(\overline{\mathrm{Ch}_\mathcal{H} K})$$
and all maximal measures are supported by $\overline{\mathrm{Ch}_\mathcal{H} K}$.

(b) We may proceed as in the proof of [1, Theorem I.6.14] to show that $\mathcal{Z}^1(\mathcal{H})$ is a τ-closed set and that for every $\mu \in \mathcal{Z}^1(\mathcal{H}) \setminus \overline{\operatorname{co}}^{\tau} \varepsilon_{\mathrm{Ch}_\mathcal{H} K}$, there are $f \in \mathcal{C}(K)$ and $\alpha \in \mathbb{R}$ such that
$$\sup_{x\in \mathrm{Ch}_\mathcal{H} K} \varepsilon_x(f) = \alpha < \mu(f).$$

Therefore $f(x) \le \alpha$ for every $x \in \overline{\mathrm{Ch}_{\mathcal{H}} K}$. But since $\operatorname{spt}\mu \subset \overline{\mathrm{Ch}_{\mathcal{H}} K}$, also $\mu(f) \le \alpha$, which is a contradiction.

□

Example 3.61. From Theorem 3.58 we have $\operatorname{co} \bigotimes_{i\in I} \mathcal{Z}^1(\mathcal{H}_i) \subset \mathcal{Z}^1(\mathcal{H})$. By Proposition 3.60 (a), $\mathcal{Z}^1(\mathcal{H}) \subset \overline{\operatorname{co}}^{w^*} \varepsilon_{\mathrm{Ch}_{\mathcal{H}} K}$. Since $\varepsilon_{\mathrm{Ch}_{\mathcal{H}} K} = \bigotimes_{i\in I} \varepsilon_{\mathrm{Ch}_{\mathcal{H}_i} K_i} \subset \bigotimes_{i\in I} \mathcal{Z}^1(\mathcal{H}_i)$, we get

$$\operatorname{co} \bigotimes_{i\in I} \mathcal{Z}^1(\mathcal{H}_i) \subset \mathcal{Z}^1(\mathcal{H}) \subset \overline{\operatorname{co}}^{w^*} \bigotimes_{i\in I} \mathcal{Z}^1(\mathcal{H}_i).$$

Now we show that both inclusions may be proper:

Let $K_i := [0,2] \subset \mathbb{R}$, $\mathcal{H}_i := \{f \in \mathcal{C}(K_i) : f(1) = \frac{f(0)+f(2)}{2}\}$, $i = 1,2$. Then $\mathrm{Ch}_{\mathcal{H}_i} K_i = [0,1) \cup (1,2]$, $i = 1,2$. Choose $\{x_n\}_{n\in\mathbb{N}} \subset [0,1) \cup (1,2]$ so that $x_n \to 1$ and let $(K,\mathcal{H})$ be a product of $(K_i,\mathcal{H}_i)$, $i = 1,2$.

(a) Define $\mu := \sum_{n=1}^{\infty} 2^{-n} \varepsilon_{(x_n,x_n)}$. Clearly $\mu \in \mathcal{Z}^1(\mathcal{H})$, since it is supported by $\mathrm{Ch}_{\mathcal{H}} K$. However, $\mu \notin \operatorname{co}(\mathcal{Z}^1(\mathcal{H}_1) \otimes \mathcal{Z}^1(\mathcal{H}_2))$. Indeed, μ is supported by the diagonal Δ of K, but the only measures of $\mathcal{Z}^1(\mathcal{H}_1) \otimes \mathcal{Z}^1(\mathcal{H}_2)$ supported by Δ are ε_x, $x \in \Delta$. Thus, μ would be supported by a finite set.

(b) Obviously $\varepsilon_{(1,1)} \notin \mathcal{Z}^1(\mathcal{H})$. However, $\varepsilon_{(x_n,x_n)} \overset{w^*}{\to} \varepsilon_{(1,1)}$. Thus, $\varepsilon_{(1,1)} \in \overline{\operatorname{co}}^{w^*}(\mathcal{Z}^1(\mathcal{H}_1) \otimes \mathcal{Z}^1(\mathcal{H}_2))$.

Proposition 3.62. $\mathcal{Z}^1(\mathcal{H}) = \overline{\operatorname{co}}^{\tau} \bigotimes_{i\in I} \mathcal{Z}^1(\mathcal{H}_i)$.

Proof. Using Proposition 3.60 (b) and Theorems 3.42 and 3.58, we can write

$$\mathcal{Z}^1(\mathcal{H}) = \overline{\operatorname{co}}^{\tau} \varepsilon_{\mathrm{Ch}_{\mathcal{H}} K} = \overline{\operatorname{co}}^{\tau} \bigotimes_{i\in I} \varepsilon_{\mathrm{Ch}_{\mathcal{H}_i} K_i} \subset \overline{\operatorname{co}}^{\tau} \bigotimes_{i\in I} \mathcal{Z}^1(\mathcal{H}_i) \subset \mathcal{Z}^1(\mathcal{H}).$$

□

4. Projective limits of function spaces

Definition 4.1. Let $(K_1,\mathcal{H}_1)$ and $(K_2,\mathcal{H}_2)$ be function spaces. We say that a continuous surjection $\varphi : K_2 \to K_1$ is an *admissible map*, if $\mathcal{H}_1 \circ \varphi := \{h \circ \varphi : h \in \mathcal{H}_1\} \subset \mathcal{H}_2$.

Let I be an up-directed index set. We say that $((K_i,\mathcal{H}_i),\pi_{ij})_{i,j\in I}$ is a *projective system* of function spaces, if every $\pi_{ij} : K_j \to K_i$, $i \le j$, is an admissible map such that

(i) π_{ii} is the identity on K_i for each i,
(ii) $\pi_{ij} \circ \pi_{jk} = \pi_{ik}$ for all $i \le j \le k$.

Projective limit, denoted by $\varprojlim ((K_i,\mathcal{H}_i),\pi_{ij})_{i,j\in I}$, of this projective system is the function space $(K,\mathcal{H})$, where

$$K := \{(x_i)_{i\in I} \in \prod_{i\in I} K_i : x_i = \pi_{ij}(x_j) \text{ for every } i \le j,\ i,j \in I\}$$

and $\mathcal{H}$ is the restriction to K of the function space $\bigcup_{i\in I} \mathcal{H}_i \circ \pi_i$ with π_i the i-th projection map.

It follows from standard results on projective limits of compact Hausdorff spaces (see e.g. [5]), that K is a non-empty compact Hausdorff space, if K_i is non-empty for every $i \in I$, and that each π_i is a surjection. Notice that $\pi_{ij} \circ \pi_j = \pi_i$ for every $i \le j$. Clearly, $\mathcal{H}$ contains constant functions and separates points of K. If

$h = h_i \circ \pi_i \in \mathcal{H}$ for some $i \in I$, then also $\alpha h \in \mathcal{H}$ for every $\alpha \in \mathbb{R}$, since $\alpha h_i \in \mathcal{H}_i$. Now, let $h_1, h_2 \in \mathcal{H}$. Suppose $h_1 = h_{i_1} \circ \pi_{i_1}, h_2 = h_{i_2} \circ \pi_{i_2}$ for some $i_1, i_2 \in I$ and $h_{i_1} \in \mathcal{H}_{i_1}, h_{i_2} \in \mathcal{H}_{i_2}$. Let $j \in I$ be such that $i_1, i_2 \leq j$. Then $h_1 = h_{i_1} \circ \pi_{i_1 j} \circ \pi_j$ and $h_2 = h_{i_2} \circ \pi_{i_2 j} \circ \pi_j$ where $h_{i_1} \circ \pi_{i_1 j}, h_{i_2} \circ \pi_{i_2 j} \in \mathcal{H}_j$. Now it is easy to see that $h_1 + h_2 \in \mathcal{H}$, since $\mathcal{H}_j$ is a linear space. Thus $\mathcal{H}$ is a function space with each π_i being an admissible map.

Remark 4.2. If $(K_i, \mathcal{H}_i) = (X_i, A(X_i))$ where X_i is a compact convex set for every $i \in I$, then the projective limit defined above is dense in $A(K)$ as shown in [13].

Lemma 4.3. *Let $(K_i, \mathcal{H}_i)$, $i = 1, 2$, be function spaces, $\varphi : K_2 \to K_1$ admissible map and $x \in K_2$. If $\mu \in \mathcal{M}_x(\mathcal{H}_2)$, then $\varphi\mu \in \mathcal{M}_{\varphi(x)}(\mathcal{H}_1)$.*

Proof. Choose $h \in \mathcal{H}_1$. Then

$$(\varphi\mu)(h) = \mu(h \circ \varphi) = (h \circ \varphi)(x) = h(\varphi(x)),$$

since $h \circ \varphi \in \mathcal{H}_2$. □

Observation 4.4. *If $\mu \in \mathcal{M}^+(K)$, then $(\pi_i\mu, \pi_{ij})_{i,j \in I}$ forms a projective system of measures.*

Theorem 4.5. *Let $(\mu_i, \pi_{ij})_{i,j \in I}$ be a projective system of measures with $\mu_i \in \mathcal{M}^1(K_i)$ for each $i \in I$. Then there is a unique measure $\mu = \varprojlim \mu_i \in \mathcal{M}^1(K)$ such that $\pi_i\mu = \mu_i$ for every $i \in I$.*

Proof. See [8, Theorem 418M and Proposition 418O]. □

Proposition 4.6. *Let $x = (x_i)_{i \in I} \in K$ and $\mu \in \mathcal{M}^1(K)$. Then $\mu \in \mathcal{M}_x(\mathcal{H})$ if and only if $(\pi_i\mu, \pi_{ij})_{i,j \in I}$ is a projective system of measures with $\pi_i\mu \in \mathcal{M}_{x_i}(\mathcal{H}_i)$ for each $i \in I$.*

Proof. First assume $\mu \in \mathcal{M}_x(\mathcal{H})$. It follows from Lemma 4.3 that $\pi_i\mu \in \mathcal{M}_{x_i}(\mathcal{H}_i)$ for each $i \in I$, since each π_i is admissible, and from Observation 4.4 that this system is projective.

On the contrary, suppose $\pi_i\mu \in \mathcal{M}_{x_i}(\mathcal{H}_i)$, $i \in I$. Let $h \in \mathcal{H}$. Then $h = h_j \circ \pi_j$ with $h_j \in \mathcal{H}_j$ for some $j \in I$. Thus

$$\mu(h) = \mu(h_j \circ \pi_j) = (\pi_j\mu)(h_j) = h_j(x_j) = h(x).$$

□

Corollary 4.7. *Let $x = (x_i)_{i \in I} \in K$ and let $(\mu_i, \pi_{ij})_{i,j \in I}$ be a projective system of measures with $\mu_i \in \mathcal{M}_{x_i}(\mathcal{H}_i)$ for each $i \in I$. Then $\mu := \varprojlim \mu_i \in \mathcal{M}_x(\mathcal{H})$.*

Lemma 4.8. *Let $\varphi : (K_2, \mathcal{H}_2) \to (K_1, \mathcal{H}_1)$ be an admissible map. Then $\mathcal{K}^c(\mathcal{H}_1) \circ \varphi \subset \mathcal{K}^c(\mathcal{H}_2)$.*

Proof. Let $k \in \mathcal{K}^c(\mathcal{H}_1)$. Choose $x \in K_2$ and $\mu \in \mathcal{M}_x(\mathcal{H}_2)$. Since $\varphi\mu \in \mathcal{M}_{\varphi(x)}(\mathcal{H}_1)$, we have

$$(k \circ \varphi)(x) = k(\varphi(x)) \leq (\varphi\mu)(k) = \mu(k \circ \varphi).$$

Thus $k \circ \varphi \in \mathcal{K}^c(\mathcal{H}_2)$. □

Lemma 4.9. *Let $\varphi : (K_2, \mathcal{H}_2) \to (K_1, \mathcal{H}_1)$ be an admissible map. Then $\mathcal{A}^c(\mathcal{H}_1) \circ \varphi \subset \mathcal{A}^c(\mathcal{H}_2)$. In particular, $\bigcup_{i \in I} \mathcal{A}^c(\mathcal{H}_i) \circ \pi_i \subset \mathcal{A}^c(\mathcal{H})$.*

Proof. Follows from Lemma 4.8, since $\mathcal{A}^c(\mathcal{H}_i) = \mathcal{K}^c(\mathcal{H}_i) \cap (-\mathcal{K}^c(\mathcal{H}_i))$, $i = 1, 2$. □

Proposition 4.10. *If $\mathcal{H}_i$ is simplicial for every $i \in I$, then $\bigcup_{i\in I} \mathcal{A}^c(\mathcal{H}_i) \circ \pi_i$ is dense in $\mathcal{A}^c(\mathcal{H})$.*

Proof. Let $a \in \mathcal{A}^c(\mathcal{H})$ and $\varepsilon > 0$. Since $a \in \widehat{\mathcal{H}}$, for every $x \in K$ there are $h_x^-, h_x^+ \in \mathcal{H}$ such that $h_x^- < a < h_x^+$ and

$$a(x) - \varepsilon < h_x^-(x) < a(x) < h_x^+(x) < a(x) + \varepsilon.$$

These inequalities hold on some open neighbourhood U_x of x. By compactness, we can choose $U_{x_1}, \dots, U_{x_n}$ covering K. Suppose that $h_{x_m}^-$ and $h_{x_m}^+$ depend on coordinates $i_m^-, i_m^+ \in I$, respectively, for $m = 1, \dots, n$. Let $j \in I$ be an upper bound of the set $\{i_m^-, i_m^+\}_{m=1}^n$. Denote $h^- := h_{x_1}^- \vee \dots \vee h_{x_n}^-$ and $h^+ := h_{x_1}^+ \wedge \dots \wedge h_{x_n}^+$. Now we have $h^- < a < h^+$ and $\|a - h^-\|, \|a - h^+\| < \varepsilon$. Since both h^-, h^+ depend on coordinate j, using W.R.I.P. for $\mathcal{H}_j$ we find $a_j \in \mathcal{A}^c(\mathcal{H}_j)$ such that $h^- < a_j \circ \pi_j < h^+$. Hence $a_j \circ \pi_j \in \bigcup_{i\in I} \mathcal{A}^c(\mathcal{H}_i) \circ \pi_i$ and $\|a - a_j \circ \pi_j\| < \varepsilon$. □

Theorem 4.11. *If $\mathcal{H}_i$ is simplicial for every $i \in I$, then $\mathcal{H}$ is simplicial.*

Proof. We show that $\mathcal{A}^c(\mathcal{H})$ has W.R.I.P. Let $a_1, \dots, a_4 \in \mathcal{A}^c(\mathcal{H})$ be such that $a_1 \vee a_2 < a_3 \wedge a_4$. By Proposition 4.10, we may assume that $a_1, \dots, a_4 \in \bigcup_{i\in I} \mathcal{A}^c(\mathcal{H}_i) \circ \pi_i$ with a_m depending on coordinate i_m, $m = 1, \dots, 4$. Let $j \in I$ be an upper bound of $i_1, \dots, i_4$. Since a_m, $m = 1, \dots, 4$, depend on coordinate j, from W.R.I.P. for $\mathcal{H}_j$ there is $a_j \in \mathcal{A}^c(\mathcal{H}_j)$ such that

$$a_1 \vee a_2 < a_j \circ \pi_j < a_3 \wedge a_4.$$

By Lemma 4.9, $a_j \circ \pi_j \in \mathcal{A}^c(\mathcal{H})$, which completes the proof. □

Proposition 4.12. *Let $\varphi : (K_2, \mathcal{H}_2) \to (K_1, \mathcal{H}_1)$ be an admissible map. Then $\varphi(\mathrm{Ch}_{\mathcal{H}_2} K_2) \supset \mathrm{Ch}_{\mathcal{H}_1} K_1$.*

Proof. See [1, Proposition I.5.20]. □

Proposition 4.13. *Let $\varphi : (K_2, \mathcal{H}_2) \to (K_1, \mathcal{H}_1)$ be an admissible map, where $\mathcal{H}_1$ is simplicial. Then the following assertions are equivalent:*

(i) $\varphi(\mathrm{Ch}_{\mathcal{H}_2} K_2) = \mathrm{Ch}_{\mathcal{H}_1} K_1$,
(ii) $(k \circ \varphi)^* = k^* \circ \varphi$ *for every* $k \in \mathcal{K}^c(\mathcal{H}_1)$,
(iii) φ *maps* $\mathcal{H}_2$*-maximal measures onto* $\mathcal{H}_1$*-maximal measures.*

Proof. The proof of (i) ⇒ (ii) is included in the proof of [15, Theorem 1.3]. Moreover, the proof mentioned above shows that (ii) is a sufficient condition for φ to map maximal measures onto maximal measures. The last implication (iii) ⇒ (i) is immediate. □

A convex versions of the next theorem can be found in [6, Theorem 14] and [13, Theorem 2]. A proof for closed function spaces has been given in [10, Corollary 4.13]. For the sake of completeness we include the proof using different approach:

Theorem 4.14. *Let $x = (x_i)_{i\in I} \in K$. The following assertions hold:*

(i) *If $x_i \in \mathrm{Ch}_{\mathcal{H}_i} K_i$ for every $i \in I$, then $x \in \mathrm{Ch}_{\mathcal{H}} K$.*
(ii) *Suppose that $\mathcal{H}_i$ is simplicial for every $i \in I$ and $\pi_{ij}(\mathrm{Ch}_{\mathcal{H}_j} K_j) \subset \mathrm{Ch}_{\mathcal{H}_i} K_i$ for every $i \le j$, $i, j \in I$. Then $x \in \mathrm{Ch}_{\mathcal{H}} K$ if and only if $x_i \in \mathrm{Ch}_{\mathcal{H}_i} K_i$ for every $i \in I$.*

Proof. First assume $x_i \in \mathrm{Ch}_{\mathcal{H}_i} K_i$ for every $i \in I$. Let $\mu \in \mathcal{M}_x(\mathcal{H})$. According to Proposition 4.6, $(\pi_i\mu, \pi_{ij})_{i,j\in I}$ is a projective system of measures with $\pi_i\mu \in \mathcal{M}_{x_i}(\mathcal{H}_i)$ for each $i \in I$. Thus $\pi_i\mu = \varepsilon_{x_i}$ for each $i \in I$ and from the uniqueness of the projective limit of measures we see that $\mu = \varprojlim (\varepsilon_{x_i}, \pi_{ij})_{i,j\in I} = \varepsilon_x$.

Now assume $x \in \mathrm{Ch}_{\mathcal{H}} K$ and the conditions of (ii) are satisfied. Choose $i \in I$. According to Corollary 2.4, it is enough to prove that $k_i(x_i) = k_i^*(x_i)$ for every $k_i \in \mathcal{K}^c(\mathcal{H}_i)$. So let $k_i \in \mathcal{K}^c(\mathcal{H}_i)$ and $\varepsilon > 0$. Denote $k := k_i \circ \pi_i \in \mathcal{K}^c(\mathcal{H})$. Since $x \in \mathrm{Ch}_{\mathcal{H}} K$, there is some $h \in \mathcal{H}$ such that $k \le h$ and $k(x) \le h(x) < k(x) + \varepsilon$. Without loss of generality suppose that $h = h_j \circ \pi_j$ for some $j \ge i$, $j \in I$, and $h_j \in \mathcal{H}_j$. Then $(k_i \circ \pi_{ij})(x_j) \le h_j(x_j) < (k_i \circ \pi_{ij})(x_j) + \varepsilon$. Using these inequalities and Proposition 4.13 we get

$$(k_i^* \circ \pi_{ij})(x_j) = (k_i \circ \pi_{ij})^*(x_j) \le (k_i \circ \pi_{ij})(x_j) + \varepsilon.$$

Hence $k_i^*(x_i) \le k_i(x_i) + \varepsilon$. Since ε is arbitrary, we conclude that $k_i(x_i) = k_i^*(x_i)$. □

Example 4.15. This example shows that the characterization in Theorem 4.14 (ii) does not have to hold, if we omit the assumption of simpliciality, and also that the converse to Theorem 4.11 is not valid.

Choose a sequence $\{q_n\}_{n\in\mathbb{N}} \subset (0,1)$ of real numbers such that $q_n \to 0$. For every $i \in \mathbb{N}$ set $K_i := \{0\} \cup \{-q_n, q_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and

$$\mathcal{H}_i := \{f \in \mathcal{C}(K_i) : f(0) = \frac{f(-q_n) + f(q_n)}{2},\ n \ge i,\ n \in \mathbb{N}\}.$$

Let $(K, \mathcal{H}) := \varprojlim((K_i, \mathcal{H}_i), \mathrm{Id}_{ij})_{i,j\in\mathbb{N}}$, where $\mathrm{Id}_{ij} : K_j \to K_i$ denotes the identity map. Clearly $\mathrm{Ch}_{\mathcal{H}_i} K_i = K_i \setminus \{0\}$ for every $i \in \mathbb{N}$. We claim that $x := (0, 0, \ldots) \in \mathrm{Ch}_{\mathcal{H}} K$. Indeed, choose $\mu \in \mathcal{M}_x(\mathcal{H})$. By Proposition 4.6, $\pi_i\mu \in \mathcal{M}_0(\mathcal{H}_i)$ for every $i \in \mathbb{N}$ and $(\pi_i\mu, \mathrm{Id}_{ij})_{i,j\in\mathbb{N}}$ is a projective system, so $\pi_i\mu = \pi_j\mu$ for every $i, j \in \mathbb{N}$. But the only measure representing 0 in all spaces $(K_i, \mathcal{H}_i)$, $i \in \mathbb{N}$, is ε_0. Hence $\mu = \varprojlim (\varepsilon_0, \mathrm{Id}_{ij})_{i,j\in\mathbb{N}} = \varepsilon_x$, which proves the claim. Using Theorem 4.14 (i) we conclude that $\mathrm{Ch}_{\mathcal{H}} K = K$.

Therefore the conclusion of Theorem 4.14 (ii) does not hold and we also see that the projective limit of non-simplicial spaces may be simplicial.

Example 4.16. Now we show that we cannot take the restriction of a product space from Section 3 as the definition of the projective limit of function spaces, if we want Theorem 4.14 to hold.

Let $K_i := [-1, 1] \subset \mathbb{R}$ and $\mathcal{H}_i := A(K_i)$ for $i = 1, 2$. Let K stand for the topological projective limit of the projective system $(K_i, \mathrm{Id}_{ij})_{i,j=1,2}$ (i.e., the diagonal of $K_1 \times K_2$) and define $\mathcal{H} := (\mathcal{H}_1 \odot \mathcal{H}_2) \restriction_K$.

Clearly all conditions of Theorem 4.14 (ii) are satisfied. However, we can see that $0 \notin \mathrm{Ch}_{\mathcal{H}_i} K_i$, $i = 1, 2$, but $(0, 0) \in \mathrm{Ch}_{\mathcal{H}} K$, since $f_1 \otimes f_2 \in \mathcal{H}$ is an exposing function of $(0, 0)$, where $f_i(x) = x$, $x \in K_i$, $i = 1, 2$. The point $(0, 0)$ is also in the Choquet boundary of the restriction of any other product space, since $\mathcal{H}_1 \odot \mathcal{H}_2$ is the smallest product.

Lemma 4.17. *Let $\varphi : (K_2, \mathcal{H}_2) \to (K_1, \mathcal{H}_1)$ be an admissible map and let $\mu, \nu \in \mathcal{M}^+(K_2)$ be such that $\mu \preceq \nu$. Then $\varphi\mu \preceq \varphi\nu$.*

Proof. Let $k \in \mathcal{K}^c(\mathcal{H}_1)$. Since $k \circ \varphi \in \mathcal{K}^c(\mathcal{H}_2)$, we have

$$(\varphi\mu)(k) = \mu(k \circ \varphi) \le \nu(k \circ \varphi) = (\varphi\nu)(k).$$

□

Proposition 4.18. *Suppose $\mathcal{H}_i$ is simplicial for every $i \in I$ and $\pi_{ij}(\mathrm{Ch}_{\mathcal{H}_j} K_j) \subset \mathrm{Ch}_{\mathcal{H}_i} K_i$ for every $i \leq j$, $i, j \in I$. Let $\mu \in \mathcal{M}^1(K)$. Then μ is $\mathcal{H}$-maximal if and only if $\pi_i\mu$ is $\mathcal{H}_i$-maximal for every $i \in I$.*

Proof. First assume that μ is maximal and choose $i \in I$. According to Theorem 4.14, $\pi_i(\mathrm{Ch}_{\mathcal{H}} K) = \mathrm{Ch}_{\mathcal{H}_i} K_i$. Using Proposition 4.13 we conclude that $\pi_i\mu$ is maximal.

Conversely, let $\pi_i\mu$ be maximal for every $i \in I$. Let $\nu \in \mathcal{M}^1(K)$ be such that $\mu \preceq \nu$. By Lemma 4.17, $\pi_i\mu \preceq \pi_i\nu$ for every $i \in I$. Therefore $\pi_i\mu = \pi_i\nu$ for every $i \in I$ and from the uniqueness of the projective limit $\mu = \nu$. □

Definition 4.19. We say that $J \subset I$ is *cofinal*, if for every $i \in I$ there is $j \in J$ such that $i \leq j$.

Proposition 4.20. *Let $J \subset I$ be cofinal and let $(K', \mathcal{H}') := \varprojlim ((K_i, \mathcal{H}_i), \pi_{ij})_{i,j \in J}$. Then*

(a) *there is a homeomorphism $\phi : K \to K'$,*
(b) *$\mathcal{H}$ is isometrically isomorphic to $\mathcal{H}'$,*
(c) *$\phi(\mathrm{Ch}_{\mathcal{H}} K) = \mathrm{Ch}_{\mathcal{H}'} K'$,*
(d) *$\mu \in \mathcal{M}^+(K)$ is maximal if and only if $\phi\mu$ is maximal,*
(e) *$\mathcal{H}$ is simplicial if and only if $\mathcal{H}'$ is simplicial.*

In particular, if there is the greatest element $m \in I$, then previous statements hold with $(K_m, \mathcal{H}_m)$ in place of $(K', \mathcal{H}')$.

Proof.

(a) The canonical bijection $\phi : (x_i)_{i \in I} \mapsto (x_i)_{i \in J}$ is a homeomorphism by standard results (see e.g. [5]).
(b) Mapping $\Phi : f \mapsto f \circ \phi$ is an isometrical isomorphism of $\mathcal{C}(K')$ onto $\mathcal{C}(K)$. Let us denote by π_i projections on K and by π_i' projections on K'. Suppose $h = h_j \circ \pi_j' \in \mathcal{H}'$ for some $h_j \in \mathcal{H}_j$ and $j \in J$. Then $\Phi(h) = h \circ \phi = h_j \circ \pi_j' \circ \phi = h_j \circ \pi_j \in \mathcal{H}$. Conversely, let $h = h_i \circ \pi_i \in \mathcal{H}$ for some $h_i \in \mathcal{H}_i$ and $i \in I$. Choose $j \in J$ such that $i \leq j$ and denote $h_j := h_i \circ \pi_{ij} \in \mathcal{H}_j$. Then $\Phi^{-1}(h) = h \circ \phi^{-1} = h_j \circ \pi_j \circ \phi^{-1} = h_j \circ \pi_j' \in \mathcal{H}'$.
(c) Notice that the mapping Φ above is also order preserving. The statement follows easily from the characterization of the Choquet boundary (Corollary 2.4) and properties of Φ.
(d) Since ϕ is a homeomorphism, $\phi : \mathcal{M}^+(K) \to \mathcal{M}^+(K')$ is a bijection. Now we use Proposition 2.3. Suppose μ is $\mathcal{H}$-maximal and let $k \in \mathcal{K}^c(\mathcal{H}')$. From the proof of (b) we can see that ϕ is admissible map and $(k \circ \phi)^* = k^* \circ \phi$. Thus
$$(\phi\mu)(k) = \mu(k \circ \phi) = \mu((k \circ \phi)^*) = \mu(k^* \circ \phi) = (\phi\mu)(k^*).$$
Since k is arbitrary, maximality of $\phi\mu$ follows. Converse is analogical.
(e) Let $x \in K$. We claim that ϕ maps $\mathcal{M}_x(\mathcal{H})$ onto $\mathcal{M}_{\phi(x)}(\mathcal{H}')$. Indeed, suppose $\mu \in \mathcal{M}_x(\mathcal{H})$ and let $h \in \mathcal{H}'$ be arbitrary. Now $(\phi\mu)(h) = \mu(\Phi(h)) = \Phi(h)(x) = h(\phi(x))$. Therefore $\phi\mu \in \mathcal{M}_{\phi(x)}(\mathcal{H}')$. Converse is analogical. Hence, using statement (d), ϕ maps maximal representing measures onto maximal representing measures and the conclusion follows.

□

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Chapter 2

Affine images of compact convex sets and maximal measures

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AFFINE IMAGES OF COMPACT CONVEX SETS AND MAXIMAL MEASURES

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Abstract. Let $\varphi\colon X \to Y$ be an affine continuous mapping of a compact convex set X onto a compact convex set Y. We show that the induced mapping $\varphi_\sharp$ need not map maximal measures on X to maximal measures on Y even in case φ maps extreme points of X to extreme points of Y. This disproves Théorème 6 of [17]. We prove the statement of Théorème 6 under an additional assumption that $\operatorname{ext} Y$ is Lindelöf or Y is a simplex. We also show that under either of these two conditions injectivity of φ on $\operatorname{ext} X$ implies injectivity of $\varphi_\sharp$ on maximal measures. A couple of examples illustrate the results.

Résumé. Soit $\varphi\colon X \to Y$ une application affine et continue d'un compact convexe X sur un compact convexe Y. Nous montrons que l'image d'une mesure maximale par l'application induite $\varphi_\sharp$ n'est pas nécessairement une mesure maximale, même pas, si les images des points extrémaux sont des points extrémaux. Ceci réfute Théorème 6 dans [17]. Nous prouvons l'énoncé de ce théorème sous l'hypothèse supplémentaire que $\operatorname{ext} Y$ est Lindelöf ou Y est un simplex. En plus, nous démontrons que, en supposant l'une ou l'autre de ces deux propriétés, l'injectivité de φ sur $\operatorname{ext} X$ implique l'injectivité de φ pour les mesures maximales. Quelques exemples explicitent les résultats.

1. Introduction

All topological spaces are considered to be Hausdorff. If X is a compact convex subset of a real locally convex space, we write $\operatorname{ext} X$ for the set of *extreme points* of X and $\mathcal{M}^1_{max}(X)$ for the set of all *maximal probability Radon measures* on X (see [1, Chapter I, §3], we also refer the reader to [5, Chapter 6], [9, Sections 1–3], [2, Chapter 1], [14] and [12, Chapter 7]). If $\varphi\colon X \to Y$ is a continuous mapping of a compact space X to a compact set Y, it induces a continuous mapping $\varphi_\sharp\colon \mathcal{M}^1(X) \to \mathcal{M}^1(Y)$ from the set of all probability Radon measures on X to the set of all probability Radon measures on Y by the formula $\varphi_\sharp\mu = \mu \circ \varphi^{-1}$ (see [10, Theorem 418I]). The induced mapping $\varphi_\sharp$ is surjective if φ is surjective.

For any $\mu \in \mathcal{M}^1(X)$ we write $r(\mu)$ for the *barycenter* of μ (see [1, Chapter I, §2]). If $x \in X$, we write $\mathcal{M}_x$ for the set of all measures $\mu \in \mathcal{M}^1(X)$ satisfying $r(\mu) = x$. We recall that a set $F \subset X$ is *extremal* if $x, y \in F$ whenever $x, y \in X$, $\alpha \in (0,1)$ and $\alpha x + (1-\alpha)y \in F$. It is a *face* if F is a convex extremal set. We also mention the well–known fact that $\operatorname{ext} F = F \cap \operatorname{ext} X$ for any face F.

Let $\varphi\colon X \to Y$ be a continuous affine mapping of a compact convex set X to a compact convex set Y. If $\varphi\colon X \to Y$ is surjective, it is easy to see that $\varphi(\operatorname{ext} X) \supset \operatorname{ext} Y$ and $\varphi_\sharp(\mathcal{M}^1_{max}(X)) \supset \mathcal{M}^1_{max}(Y)$. In order to ensure the reverse inclusion

$\varphi_\sharp(\mathcal{M}^1_{max}(X)) \subset \mathcal{M}^1_{max}(Y)$, it is necessary to assume that $\varphi(\operatorname{ext} X) \subset \operatorname{ext} Y$. This observation prompts the following two questions.

Question. *Let $\varphi\colon X \to Y$ be a continuous affine mapping of a compact convex X to a compact convex set Y.*

(1) If $\varphi(\operatorname{ext} X) \subset \operatorname{ext} Y$, does it imply that $\varphi_\sharp(\mathcal{M}^1_{max}(X)) \subset \mathcal{M}^1_{max}(Y)$?
(2) If $\varphi(\operatorname{ext} X) \subset \operatorname{ext} Y$ and φ is injective on $\operatorname{ext} X$, does it imply that $\varphi_\sharp$ is injective on $\mathcal{M}^1_{max}(X)$?

If Y is a simplex (see [1, Chapter II, §3]), both questions were answered affirmatively in [7, Corollary 2 and 3]. For X and Y being simplices, the result can be found in [6, Lemma 6] and [11, Theorem 1]. It is claimed in [17, Théorème 6] without a proof that Question 1 has the affirmative answer without any restrictions. The author also suggests to study Question 2 in [17, Conjecture].

Unfortunately, the answer to Question 1 is in general negative as the following example shows.

Example 1.1. *There exists a continuous affine surjection φ of a simplex X onto a compact convex set Y and a measure $\mu \in \mathcal{M}^1_{max}(X)$ such that*

- *$\varphi(\operatorname{ext} X) = \operatorname{ext} Y$ and φ is injective on $\operatorname{ext} X$,*
- *$\varphi_\sharp\mu \notin \mathcal{M}^1_{max}(Y)$.*

Nevertheless, we prove in Theorem 1.2 that the answer to both questions is positive if we assume that $\operatorname{ext} Y$ is a Lindelöf space (see [8, Section 3.8]).

Theorem 1.2. *Let $\varphi\colon X \to Y$ be a continuous affine map of a compact convex set X to a compact convex set Y and let $\operatorname{ext} Y$ be a Lindelöf space.*

(a) Then the following assertions are equivalent:
  (i) $\varphi(\operatorname{ext} X) \subset \operatorname{ext} Y$,
  (ii) $\varphi_\sharp(\mathcal{M}^1_{max}(X)) \subset \mathcal{M}^1_{max}(Y)$.
(b) Further, the following assertions are equivalent:
  (i') $\varphi(\operatorname{ext} X) \subset \operatorname{ext} Y$ and φ is injective on $\operatorname{ext} X$,
  (ii') $\varphi_\sharp(\mathcal{M}^1_{max}(X)) \subset \mathcal{M}^1_{max}(Y)$ and $\varphi_\sharp$ is injective on $\mathcal{M}^1_{max}(X)$.

We also provide in Theorem 1.3(a) a slightly different proof of [7, Corollary 2]. The case of injectivity is described in Theorem 1.3(b), where the proof is based upon the results of E.A. Reznichenko from [15]. We indicate in Remark 2.4 an alternative proof of this assertion that uses a notion of induced measures on the set of extreme points, which is a technique developed by S. Teleman and C.J.K. Batty in [18] and [3].

Theorem 1.3. *Let $\varphi\colon X \to Y$ be a continuous affine map of a compact convex set X to a simplex Y.*

(a) Then the following assertions are equivalent:
  (i) $\varphi(\operatorname{ext} X) \subset \operatorname{ext} Y$,
  (ii) $\varphi_\sharp(\mathcal{M}^1_{max}(X)) \subset \mathcal{M}^1_{max}(Y)$,
  (iii) $\varphi(F)$ is a face for each closed face $F \subset X$,
  (iv) $\varphi(F)$ is a closed extremal set for each closed extremal $F \subset X$.
(b) Further, the following assertions are equivalent:
  (i') $\varphi(\operatorname{ext} X) \subset \operatorname{ext} Y$ and φ is injective on $\operatorname{ext} X$,
  (ii') $\varphi_\sharp(\mathcal{M}^1_{max}(X)) \subset \mathcal{M}^1_{max}(Y)$ and $\varphi_\sharp$ is injective on $\mathcal{M}^1_{max}(X)$,
  (iii') φ is a homeomorphism onto $\varphi(X)$.

The following example shows that Theorem 1.3(b) need not hold if we omit the condition imposed on Y.

Example 1.4. *There exists a continuous affine surjection φ of a metrizable simplex X onto a compact convex set Y such that*

- *φ is injective on* $\operatorname{ext} X$*,*
- *$\varphi_\sharp(\mathcal{M}^1_{max}(X)) \subset \mathcal{M}^1_{max}(Y)$ and $\varphi_\sharp$ is injective on $\mathcal{M}^1_{max}(X)$,*
- *φ is not injective on X.*

Our last example shows that even if $\varphi_\sharp$ maps maximal measures to maximal measures and φ is injective on $\operatorname{ext} X$, the induced mapping need not be injective on $\mathcal{M}^1_{max}(X)$.

Example 1.5. *There exists a continuous affine surjection φ of a simplex X onto a compact convex set Y such that*

- *φ is injective on* $\operatorname{ext} X$*,*
- *$\varphi_\sharp(\mathcal{M}^1_{max}(X)) \subset \mathcal{M}^1_{max}(Y)$,*
- *$\varphi_\sharp$ is not injective on $\mathcal{M}^1_{max}(X)$.*

2. Proofs of the positive results

If $f : X \to \mathbb{R}$ is a function on a compact convex set X, we recall the definition from [1, p. 4] of the *upper enevelope* f^* of f defined as

$$f^*(x) = \inf\{h(x) : h \geq f, h \text{ continuous affine on } X\}, \quad x \in X.$$

Before embarking on the proof of the main theorems, we need a couple of auxiliary results.

Proposition 2.1. *Let f, g, be upper semicontinuous real functions on X such that f is concave, g is convex and $f \geq g$ on* $\operatorname{ext} X$*. Then $f \geq g$ on X.*

Proof. Given f and g as in the premise, let x be a point of X. We fix $\varepsilon > 0$ and use [1, Corollary I.1.3] to find a concave continuous function f' such that $f' \geq f$ and $f(x) \geq f'(x) - \varepsilon$.

Then $f' - g$ is a lower semicontinuous concave function on X such that $f' - g \geq 0$ on $\operatorname{ext} X$. According to Bauer's minimum principle [1, Theorem I.5.3], $f' - g \geq 0$ on X. Thus

$$g(x) \leq f'(x) \leq f(x) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we are done. □

Proposition 2.2. *Let* $\operatorname{ext} X$ *be Lindelöf and $\mu \in \mathcal{M}^1(X)$. Then the following assertions are equivalent:*

(i) $\mu \in \mathcal{M}^1_{max}(X)$,
(ii) $\mu_*(X \setminus \operatorname{ext} X) = 0$ *(here μ_* stands for the inner measure induced by μ).*

Proof. Let $\mu \in \mathcal{M}^1_{max}(X)$ be given and $F \subset X \setminus \operatorname{ext} X$ be an arbitrary closed set. For any point $x \in \operatorname{ext} X$ we can find a cozero set U_x such that $x \in U_x \subset X \setminus F$. (We recall that a subset of a normal space is cozero if and only if it is an open F_σ set, see [8, p. 42].) By the Lindelöf property of $\operatorname{ext} X$, there exists a cozero set U such that

$$\operatorname{ext} X \subset U \subset X \setminus F.$$

According to [5, Theorem 27.11], $\mu(U) = 1$ and hence $\mu(F) = 0$. Thus $\mu_*(X \setminus \operatorname{ext} X) = 0$ and (i) $\Longrightarrow$ (ii).

For the proof of (ii) $\Longrightarrow$ (i), let μ satisfy (ii). For any continuous function f on X, [1, p. 32] yields

$$\operatorname{ext} X \subset \{x \in X \colon f^*(x) = f(x)\}.$$

Hence $\mu(\{x \in X \colon f^*(x) = f(x)\}) = 1$ and $\mu(f^*) = \mu(f)$. By [1, Proposition I.4.5], $\mu \in \mathcal{M}^1_{max}(X)$. □

Proof of Theorem 1.2. For the proof of (a) we first notice that the implications (ii) $\Longrightarrow$ (i) and (ii') $\Longrightarrow$ (i') are obvious. We start the proof of the converse implications by showing (i) $\Longrightarrow$ (ii). To this end, let $\mu \in \mathcal{M}^1_{max}(X)$ be given. We fix an arbitrary closed set $F \subset Y \setminus \operatorname{ext} Y$. Since $\operatorname{ext} Y$ is Lindelöf, there exists a countable family of cozero sets $\{U_n \colon n \in \mathbb{N}\}$ in Y such that

$$\operatorname{ext} Y \subset \bigcup_{n=1}^{\infty} U_n \subset Y \setminus F.$$

Then $G = \varphi^{-1}(\bigcup_{n=1}^{\infty} U_n)$ is an F_σ set. By the assumptions, $\operatorname{ext} X \subset G$ and hence $\mu(G) = 1$. Thus

$$(\varphi_\sharp \mu)(\bigcup_{n=1}^{\infty} U_n) = \mu(G) = 1,$$

and hence $\mu(F) = 0$.

Thus $(\varphi_\sharp \mu)_*(Y \setminus \operatorname{ext} Y) = 0$, and $\varphi_\sharp \mu \in \mathcal{M}^1_{max}(Y)$ by virtue of Proposition 2.2.

We proceed with the proof of (i') $\Longrightarrow$ (ii'). We start by proving

$$\varphi(X \setminus \operatorname{ext} X) \subset Y \setminus \operatorname{ext} Y. \tag{1}$$

Indeed, given $y \in \operatorname{ext} Y \cap \varphi(X)$, the set $\varphi^{-1}(y)$ is a closed face. Since

$$\varphi^{-1}(y) = \overline{\operatorname{co}}(\operatorname{ext} \varphi^{-1}(y)) = \overline{\operatorname{co}}(\varphi^{-1}(y) \cap \operatorname{ext} X),$$

the assumption yields that $\varphi^{-1}(y)$ is a singleton. Hence (1) follows.

Let $\mu \in \mathcal{M}^1_{max}(X)$ be given. For any set $F \subset X \setminus \operatorname{ext} X$, inclusion (1) gives

$$\varphi(F) \subset Y \setminus \operatorname{ext} Y.$$

This along with Proposition 2.2 and the first part of the proof yields

$$(\varphi_\sharp \mu)(\varphi(F)) = 0, \quad F \subset X \setminus \operatorname{ext} X \text{ closed}.$$

Hence

$$\mu(F) \le \mu(\varphi^{-1}(\varphi(F))) = (\varphi_\sharp \mu)(\varphi(F)) = 0, \quad F \subset X \setminus \operatorname{ext} X \text{ closed},$$

and thus

$$\mu(\varphi^{-1}(\varphi(F))) = \mu(F), \quad F \subset X \text{ closed}. \tag{2}$$

If $\mu, \nu \in \mathcal{M}^1_{max}(X)$ are measures with $\varphi_\sharp \mu = \varphi_\sharp \nu$, then (2) yields

$$\begin{aligned} \mu(F) &= \mu(\varphi^{-1}(\varphi(F))) = (\varphi_\sharp \mu)(\varphi(F)) \\ &= (\varphi_\sharp \nu)(\varphi(F)) = \nu(\varphi^{-1}(\varphi(F))) = \nu(F) \end{aligned}$$

for any closed $F \subset X$. Hence $\mu = \nu$ and $\varphi_\sharp$ is injective on $\mathcal{M}^1_{max}(X)$. □

Remark 2.3. It can be easily verified that the mapping $\varphi\colon X \to Y$ is a homeomorphism of $\operatorname{ext} X$ onto $\varphi(\operatorname{ext} X)$ if $\varphi(\operatorname{ext} X) \subset \operatorname{ext} Y$ and φ is injective on $\operatorname{ext} X$.

Indeed, since

$$\varphi(\operatorname{ext} X) \subset \operatorname{ext} Y \quad \text{and} \quad \varphi(X \setminus \operatorname{ext} X) \subset Y \setminus \operatorname{ext} Y,$$

it is not difficult to realize that $\varphi(F \cap \operatorname{ext} X) = \varphi(F) \cap \operatorname{ext} Y$ for any $F \subset X$. Hence $\varphi\colon \operatorname{ext} X \to \operatorname{ext} Y$ is a closed mapping, and thus a homeomorphism on $\operatorname{ext} X$.

Hence we obtain that $\operatorname{ext} X$ is a Lindelöf space if $\operatorname{ext} Y$ is Lindelöf and φ as above.

Proof of Theorem 1.3. For the proof of (a), we first verify (i) $\Longrightarrow$ (ii). To this end, let μ be a maximal probability measure on X. To show that $\varphi_\sharp\mu$ is maximal on Y, we use Mokobodzki's maximality test [1, Proposition I.4.5].

Let g be a convex continuous function on Y. Since Y is a simplex, g^* is an affine function (see [1, Theorem II.3.7]). By the assumption and [1, Proposition I.4.1],

$$g^* \circ \varphi = (g \circ \varphi)^* \quad \text{on } \operatorname{ext} X.$$

By Proposition 2.1, $g^* \circ \varphi \le (g \circ \varphi)^*$ on X.

On the other hand, given $x \in X$, there exists a measure $\lambda \in \mathcal{M}_x$ such that $\lambda(g \circ \varphi) = (g \circ \varphi)^*(x)$ (see [1, Proposition I.3.5]). Then $\varphi_\sharp\lambda \in \mathcal{M}_{\varphi(x)}$ and

$$(g \circ \varphi)^*(x) = \lambda(g \circ \varphi) = (\varphi_\sharp\lambda)(g) \le g^*(\varphi(x)).$$

Hence $g^* \circ \varphi = (g \circ \varphi)^*$ on X.

Thus the equality

$$(\varphi_\sharp\mu)(g) = \mu(g \circ \varphi) = \mu((g \circ \varphi)^*) = \mu(g^* \circ \varphi) = (\varphi_\sharp\mu)(g^*)$$

shows that $\varphi_\sharp\mu$ is a maximal measure on Y.

We proceed with the proof by showing (ii) $\Longrightarrow$ (iii). Let $F \subset X$ be a closed face. Since $\varphi(F)$ is obviously convex, we need to check its extremality.

Let $\nu \in \mathcal{M}^1_{max}(Y)$ satisfy $r(\nu) \in \varphi(F)$. We find a point $x \in F$ with $\varphi(x) = r(\nu)$ and select a measure $\mu \in \mathcal{M}^1_{max}(X)$ such that $r(\mu) = x$. Since F is a closed face, $\mu \in \mathcal{M}^1(F)$. Then $\varphi_\sharp\mu$ is supported by $\varphi(F)$ and by the assumption, $\varphi_\sharp\mu$ is maximal. Since

$$r(\varphi_\sharp\mu) = \varphi(r(\mu)) = r(\nu)$$

and Y is a simplex, $\varphi_\sharp\mu = \nu$. Hence $\nu \in \mathcal{M}^1(\varphi(F))$.

Let now an arbitrary $\nu' \in \mathcal{M}^1(Y)$ satisfy $r(\nu') \in \varphi(F)$. We find a maximal measure $\nu \in \mathcal{M}^1_{max}(Y)$ such that $\nu' \preceq \nu$ (here $\preceq$ is the Choquet ordering, see [1, Chapter I, §3] and [1, Lemma I.4.7]). Since $r(\nu) = r(\nu')$, ν is supported by $\varphi(F)$ according to the paragraph above. Since it is easy to see that $\operatorname{spt}\nu' \subset \overline{\operatorname{co}}\operatorname{spt}\nu$, the measure ν' is supported by $\varphi(F)$ as well. Thus $\varphi(F)$ is a face as needed.

Since a closed set is extremal if and only if it is a union of closed faces (see [13, §4, Theorem 7]), we get (iii) $\Longrightarrow$ (iv). We proceed to the proof of (iv) $\Longrightarrow$ (i). But this is immediate, because a set $\{x\}$ is extremal if and only if $x \in \operatorname{ext} X$. This concludes the proof of (a).

We start the proof of (b) by showing (i') $\Longrightarrow$ (iii'). We know from the part (a) that $\varphi(X)$ is a face of Y and hence a simplex. Since $\operatorname{ext}\varphi(X) = \varphi(X) \cap \operatorname{ext} Y$, we may assume from now on that φ is a surjective mapping onto a simplex Y.

Thus we may use [15, Proposition 1.6] to get that φ is a simplicial map, that is, the function

$$\tilde{a}(y) = \inf\{a(x)\colon x \in \varphi^{-1}(y)\}, \quad y \in Y,$$

is affine on Y for any continuous affine function a on X (see [15, Definition 1.3]). Since φ is injective on $\operatorname{ext} X$, [15, Theorem 1.5] yields that φ is a homeomorphism.

Since the remaining implications are obvious, the proof is finished. □

Remark 2.4. We remark that Theorem 1.3(b) can be deduced from results of S. Teleman and C.J.K. Batty on maximal measures.

For the proof of (i') $\Longrightarrow$ (ii') we realize that $F = \varphi^{-1}(\varphi(F))$ for any closed face $F \subset X$ and hence also for any closed extremal set $F \subset X$. It is shown in [3, Section 6] or in [18, Theorem 5.2] and [19, Theorem 6] that

$$\mu(B) = \sup\{\mu(F) \colon F \subset B \text{ is closed extremal}\}, \quad B \subset X \text{ Baire},$$

for any measure $\mu \in \mathcal{M}^1_{max}(X)$. From this fact we get that $\varphi_\sharp$ is injective on $\mathcal{M}^1_{max}(X)$.

To verify (ii') $\Longrightarrow$ (iii'), it is enough to check injectivity of φ on X. Let $x_1, x_2 \in X$ satisfy $y = \varphi(x_1) = \varphi(x_2)$. For $i = 1, 2$, we find a maximal measure $\mu_i \in \mathcal{M}_{x_i}$. Then the measure $\varphi_\sharp\mu_i \in \mathcal{M}_y$, $i = 1, 2$, and thus $\varphi_\sharp\mu_1 = \varphi_\sharp\mu_2$ (we remind that Y is a simplex). By the assumption, $\mu_1 = \mu_2$ and thus $x_1 = x_2$.

Obviously, (iii') $\Longrightarrow$ (i') which finishes this remark.

3. Construction of examples

All the construction are based upon the notion of a *function space* $\mathcal{H}$, which is a subspace of the space $\mathcal{C}(K)$ of all continuous functions on a compact space K such that $\mathcal{H}$ contains constant functions and separates points of K. Then the *state space*

$$X = \{\xi \in \mathcal{H}^* \colon \xi \geq 0, \xi(1) = 1\}$$

endowed with the weak* topology is a convex compact set that inherits many properties from $\mathcal{H}$. The mapping $\phi \colon K \to X$, where $\phi(x)$ is the evaluation mapping at a point $x \in K$, is a homeomorphic embedding. (We refer the reader to [14, Chapter 6], [5, Chapter 6, §29] and [16] for a detailed information on the issue.)

Construction of Example 1.1. Let $K_1 = [0,1] \times \{-1, 0, 1\}$ with the "porcupine" topology (see [4, Section VII] or [1, Proposition I.4.15]) and let K_2 be the quotient of K_1 where all points of $[0,1] \times \{0\}$ are identified with the point $(0,0)$ (see [8, Section 2.4]). We write $q \colon K_1 \to K_2$ for the quotient mapping and take

$$\mathcal{H}_1 = \{f \in \mathcal{C}(K_1) \colon 2f((t,0)) = f((t,-1)) + f((t,1)), t \in [0,1]\} \quad \text{and}$$
$$\mathcal{H}_2 = \{f \in \mathcal{C}(K_2) \colon 2f((0,0)) = f((t,-1)) + f((t,1)), t \in [0,1]\}.$$

Let X, Y be the state space of $\mathcal{H}_1$, $\mathcal{H}_2$, respectively, and ϕ_1, ϕ_2 be the respective embeddings. Then $\operatorname{ext} X = \phi_1(K_1 \setminus ([0,1] \times \{0\}))$ and $\operatorname{ext} Y = \phi_2(K_2 \setminus \{(0,0)\})$. We denote by $\varphi \colon X \to Y$ the restriction of the adjoint operator $h \mapsto h \circ q$, $h \in \mathcal{H}_2$. Then X is a simplex and $\phi_\sharp\lambda \in \mathcal{M}^1(X)$ is maximal for any continuous measure $\lambda \in \mathcal{M}^1([0,1] \times \{0\})$, eventhough $\phi_\sharp\lambda$ is supported by a compact set disjoint with $\operatorname{ext} X$ (see [1, Chapter I, §4, p. 42]). (We recall that λ is continuous if $\lambda(\{x\}) = 0$ for each $x \in X$.)

Then $\varphi(\operatorname{ext} X) = \operatorname{ext} Y$ and φ is even injective on $\operatorname{ext} X$. On the other hand, if λ is any continuous probability measure on $\phi_1([0,1] \times \{0\})$, then λ is maximal on X, yet the measure $\varphi_\sharp\lambda$ equals the Dirac measure at the point $\phi_2((0,0))$, and hence $\varphi_\sharp\lambda$ is not maximal. □

Construction of Example 1.4. Let $K_1 = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and K_2 be the quotient of K_2, if we identify y_2 with x_2. Again we denote by $q\colon K_1 \to K_2$ the quotient mapping. Let

$$\mathcal{H}_1 = \{f \in \mathcal{C}(K_1)\colon 2f(x_2) = f(x_1) + f(x_3), 2f(y_2) = f(y_1) + f(y_3)\} \quad \text{and}$$
$$\mathcal{H}_2 = \{f \in \mathcal{C}(K_2)\colon f(x_1) + f(x_3) = 2f(x_2) = f(y_1) + f(y_3)\}.$$

We take X, Y, ϕ_1, ϕ_2 and $\varphi\colon X \to Y$ as above. Then X is a simplex, $\operatorname{ext} X = \phi_1(K_1 \setminus \{x_2, y_2\})$, $\operatorname{ext} Y = \phi_2(K_2 \setminus \{x_2\})$, $\varphi\colon \operatorname{ext} X \to \operatorname{ext} Y$ is a bijection, yet φ is not injective on X. Obviously, $\varphi_\sharp$ maps injectively maximal measures to maximal measures. □

Construction of Example 1.5. Let $K_1 = [0,1] \cup [2,3] \times \{-1, 0, 1\}$ endowed again with the "porcupine" topology and let K_2 be the quotient of K_1 after identifying points $(t+2, 0)$ with $(t, 0)$, $t \in [0,1]$. Let

$$\mathcal{H}_1 = \{f \in \mathcal{C}(K_1)\colon 2f((t+i, 0)) = f((t+i, -1)) + f((t+i, 1)), t \in [0,1], i = 0, 2\},$$
$$\mathcal{H}_2 = \{f \in \mathcal{C}(K_2)\colon 2f((t, 0)) = f((t+i, -1)) + f((t+i, 1)), t \in [0,1], i = 0, 2\},$$

and let X, Y, ϕ_1, ϕ_2 and φ be as above.

Then

$$\operatorname{ext} X = \phi_1(K_1 \setminus ([0,1] \cup [2,3] \times \{0\})), \quad \operatorname{ext} Y = \phi_2(K_2 \setminus ([0,1] \times \{0\})),$$

and φ maps injectively $\operatorname{ext} X$ onto $\operatorname{ext} Y$.

We claim that $\varphi_\sharp(\mathcal{M}^1_{max}(X)) \subset \mathcal{M}^1_{max}(Y)$. Indeed, a probability measure λ is maximal on X if and only if $\lambda = (\phi_1)_\sharp \mu$ for some measure $\mu \in \mathcal{M}^1(K_1)$ that is continuous on $[0,1] \cup [2,3] \times \{0\}$. Similarly, any maximal probability measure on Y is of the form $(\phi_2)_\sharp \mu$ for some measure $\mu \in \mathcal{M}^1(K_2)$ that is continuous on $[0,1] \times \{0\}$. From these observations the claim follows.

Finally, if we take the Lebesgue measure λ_1 on $[0,1] \times \{0\}$ and λ_2 on $[2,3] \times \{0\}$, then

$$\varphi_\sharp((\phi_1)_\sharp \lambda_1) = \varphi_\sharp((\phi_1)_\sharp \lambda_2).$$

Hence $\varphi_\sharp$ is not injective on $\mathcal{M}^1_{max}(X)$. □

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Chapter 3

Affine Baire functions on Choquet simplices

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AFFINE BAIRE FUNCTIONS ON CHOQUET SIMPLICES

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Abstract. We construct a metrizable simplex X such that for each $n \in \mathbb{N}$ there exists a bounded function f on $\operatorname{ext} X$ of Baire class n that cannot be extended to a strongly affine function of Baire class n. We show that such an example cannot be constructed via the space of harmonic functions.

1. Introduction

Throughout the paper, we consider a *function space* $\mathcal{H}$ on a compact Hausdorff topological space K. By this we mean a subspace $\mathcal{H}$ of the space $\mathcal{C}(K)$ of all continuous functions on K that contains constant functions and separates points of K. We focus on a particular class of function spaces, so-called simplicial function spaces (see the definitions below), which can be viewed as a more general version of spaces of affine continuous functions on simplices and, from the point of view of Banach space theory, come under the theory of L_1-preduals (see [14, p. 59]). Abstract affine classes of functions with respect to $\mathcal{H}$ (defined below) coincide with the so-called *intrinsic Baire classes* and *Baire classes* of the Banach space $\overline{\mathcal{H}}$ as defined in [2, p. 1044] and thus our results aim to provide a better understanding of these classes within the framework of L_1-preduals.

If K is a compact space, we write $\mathcal{M}(K)$ for the space of all signed Radon measures on K. By a (positive) Radon measure we mean a complete measure that is inner regular with respect to compact sets and is defined on a σ-algebra including all Borel subsets of K. A signed measure is Radon if the total variation $|\mu|$ of μ is a Radon measure. We refer the reader to [13, Section 416] for more information on Radon measures. Let $\mathcal{M}^1(K)$ denote the set of all Radon probability measures on K. We always consider $\mathcal{M}(K)$ endowed with the weak* topology. We say that a function $f : K \to \mathbb{R}$ is *universally measurable* if f is μ-measurable for every $\mu \in \mathcal{M}^1(K)$. Let $\mathcal{M}^+(K)$ and $\mathcal{M}^1(K)$ stand for the set of all positive and probability Radon measures on K, respectively. We write $\operatorname{spt} \mu$ for the support of a measure $\mu \in \mathcal{M}^+(K)$.

If $\mathcal{F}$ is a set of functions, we inductively define the following sets of functions: we set $\mathcal{F}_0 = \mathcal{F}$ and having $\mathcal{F}_\beta$, $\beta < \alpha$, already defined for an ordinal number $\alpha \in (0, \omega_1)$, we define $\mathcal{F}_\alpha$ to be the space of all pointwise limits of bounded sequences of functions from $\bigcup_{\beta<\alpha} \mathcal{F}_\beta$. We write $\mathcal{B}^b(K)$ for the space of all bounded Borel functions on K and $\mathcal{B}^b_\alpha(K) = (\mathcal{C}(K))_\alpha$ for the space of all bounded Baire functions of class α, $\alpha \in [0, \omega_1)$.

If $\mathcal{F}$ is a set of bounded universally measurable functions on K, we write $\mathcal{F}^\perp$ for the space of all measures $\mu \in \mathcal{M}(K)$ with $\mu(f) = 0$ for each $f \in \mathcal{F}$, and $\mathcal{F}^{\perp\perp}$ for the space of all bounded universally measurable functions f satisfying $\mu(f) = 0$ for each $\mu \in \mathcal{F}^\perp$.

For any $x \in K$, let $\mathcal{M}_x(\mathcal{H})$ denote the set of all $\mathcal{H}$–*representing* measures, i.e., $\mu \in \mathcal{M}_x(\mathcal{H})$ if and only if $\mu \in \mathcal{M}^+(K)$ and $\mu(h) = h(x)$ for each $h \in \mathcal{H}$. Let $\operatorname{Ch}_{\mathcal{H}} K = \{x \in K : \mathcal{M}_x(\mathcal{H}) = \{\varepsilon_x\}\}$ stand for the *Choquet boundary* of K. It follows from the Choquet theorem that for every $x \in K$ there exists $\mu \in \mathcal{M}_x(\mathcal{H})$ such that $\mu(\operatorname{Ch}_{\mathcal{H}} K) = 1$ (see [1, Theorem I.5.19]). We denote by $\mathcal{A}(\mathcal{H})$ the space of all $\mathcal{H}$–*affine functions*, i.e., the space of all bounded universally measurable functions f on K satisfying $\mu(f) = f(x)$ for every $x \in K$ and $\mu \in \mathcal{M}_x(\mathcal{H})$. The space of all continuous elements in $\mathcal{A}(\mathcal{H})$ is denoted as $\mathcal{A}_c(\mathcal{H})$.

A function space $\mathcal{H}$ on a metrizable compact space K is *simplicial* if for every $x \in K$ there exists a unique measure $\delta_x \in \mathcal{M}^1(K)$ carried by $\operatorname{Ch}_{\mathcal{H}} K$ that $\mathcal{H}$–represents x. If K is not metrizable, the set $\operatorname{Ch}_{\mathcal{H}} K$ need not be a measurable set and thus simpliciality of $\mathcal{H}$ has to be defined differently. One way to do this is to say that $\mathcal{H}$ is simplicial if $\mathcal{A}_c(\mathcal{H})$ has the *Riesz interpolation property* (see [1, Corollary II.3.11]), which means that, for every quadruple $f_1, f_2, g_1, g_2 \in \mathcal{A}_c(\mathcal{H})$ satisfying $f_i \le g_j$, $i, j \in \{1, 2\}$, there exists a function $h \in \mathcal{A}_c(\mathcal{H})$ with $f_i \le h \le g_j$, $i, j \in \{1, 2\}$.

A detailed survey of properties of function spaces can be found in any of the following sources: [1, Chapter I, §5], [11, Chapter 6], [19, Section 6], [17, Chapter 7], [7, Chapter I], [9] or [21, Section 2].

For a simplicial function space $\mathcal{H}$, we define an operator $T : \mathcal{B}^b(K) \to (\mathcal{A}_c(\mathcal{H}))^{\perp\perp}$ as

$$Tf(x) := \delta_x(f), \quad x \in K,\ f \in \mathcal{B}^b(K). \tag{1}$$

We refer the reader Proposition 2.2 below for the proof of the fact that $Tf \in (\mathcal{A}_c(\mathcal{H}))^{\perp\perp}$ for any bounded Borel function f on K. If f is a bounded Borel function defined on a Borel subset F of K, we set $Tf = T\widehat{f}$, where $\widehat{f} = f$ on F and 0 elsewhere.

If X is a compact convex subset of a locally convex space and $\mathcal{H}$ equals the space $\mathfrak{A}(X)$ of all continuous affine functions, then $\operatorname{Ch}_{\mathfrak{A}(X)} X = \operatorname{ext} X$ (the set of extreme points of X) and $\mathfrak{A}(X)$ is simplicial if and only if X is a *simplex* (see [1, Chapter II, §3], [4, Chapter 3]). The functions contained in $\mathfrak{A}(X)^{\perp\perp}$ are called *strongly affine* in [24, Introduction] or the functions *satisfying the barycentric formula*.

Given a simplicial function space $\mathcal{H}$ on a metrizable compact space K and a bounded Borel function f on $\operatorname{Ch}_{\mathcal{H}} K$, we may consider the *abstract Dirichlet problem*, i.e., the question of an $\mathcal{H}$–affine extension of f to the whole set K that preserves topological properties of f. By the minimum principle (see e.g. [20, Proposition 3.6]), the only $\mathcal{H}$–affine extension of f is the function Tf. The problem of continuous $\mathcal{H}$–affine extensions was solved by H. Bauer. He showed in [5, Satz 13] that $T(\mathcal{C}(K)) \subset \mathcal{C}(K)$ *if and only if* $\operatorname{Ch}_{\mathcal{H}} K$ *is closed*. An analogous question for Baire–one functions was solved in [20, Theorem 3.1], namely $T(\mathcal{B}_1^b(K)) \subset \mathcal{B}_1^b(K)$ *if and only if* $\operatorname{Ch}_{\mathcal{H}} K$ *is an* F_σ *set.* It has turned out in [23, Theorem 1.1] that such a characterization is impossible for functions of higher Baire classes.

Thus there exist simplicial function spaces such that the operator T does not preserve continuous or Baire–one functions. On the other hand, it is not difficult to realize that $T(\mathcal{B}_\alpha^b(K)) \subset \mathcal{B}_\alpha^b(K)$ for each $\alpha \in [\omega_0, \omega_1)$ (see Theorem 1.1(a) below). The aim of our paper is to show that the shift of classes can occur for any finite Baire class. Precisely we get the following results.

Theorem 1.1.

(a) *If $\mathcal{H}$ is a simplicial function space on a metrizable compact space K, then $T(\mathcal{B}^b_\alpha(K)) \subset (\mathcal{A}_c(\mathcal{H}))_\alpha$ for each $\alpha \in [\omega_0, \omega_1)$.*

(b) *There exists a metrizable simplex X such that $T(\mathcal{B}^b_\alpha(X)) \not\subset \mathcal{B}^b_\alpha(X)$ for each $\alpha \in [0, \omega_0)$.*

(c) *If $n \in \{0,1\}$, then there exists a metrizable simplex X such that $T(\mathcal{B}^b_n(X)) \not\subset \mathcal{B}^b_n(X)$ and $T(\mathcal{B}^b_\alpha(X)) \subset \mathcal{B}^b_\alpha(X)$ for each $\alpha \in (n, \omega_1)$.*

If X is a compact convex set and $\alpha < \omega_1$, functions $\mathfrak{A}_\alpha(X)$ of *affine class* α are introduced in [10] as functions from $(\mathfrak{A}(X))_\alpha$. By a theorem of G. Choquet and G. Mokobodzki, any affine Baire–one function on X is in $\mathfrak{A}_1(X)$ (see [1, Theorem I.2.6] or [2, Theorem II.1.2(a)]). On the other hand, by a result of M. Talagrand in [24, Theorem], there exists a metrizable compact convex set X and a function $f \in \mathcal{B}^b_2(X) \cap (\mathfrak{A}(X))^{\perp\perp}$ such that $f \notin \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha(X)$. If X is a simplex, M. Capon showed in [10, Théorème 2] that $\mathcal{B}^b_\alpha(X) \cap (\mathfrak{A}(X))^{\perp\perp} \subset \mathfrak{A}_{\alpha+1}(X)$ for any $\alpha < \omega_1$. By combining the method of a separable reduction from [10] and Theorem 1.1(a) we get the following improvement.

Theorem 1.2.

(a) *If X is a simplex, then $\mathcal{B}^b_\alpha(X) \cap (\mathfrak{A}(X))^{\perp\perp} = \mathfrak{A}_\alpha(X)$ for any $\alpha \in [\omega_0, \omega_1)$.*

(b) *If $\mathcal{H}$ is a simplicial function space on a compact space K, then $\mathcal{B}^b_\alpha(K) \cap \mathcal{A}(\mathcal{H}) = (\mathcal{A}_c(\mathcal{H}))_\alpha$ for each $\alpha \in [\omega_0, \omega_1)$.*

If U is an open bounded subset of the Euclidean space $\mathbb{R}^d$, we get a particular example of a simplicial function space $\mathbf{H}(U)$ of all continuous functions on $\overline{U}$ that are harmonic on U. In this case, $\delta_x = \varepsilon_x^{\mathbb{R}^d \setminus U}$, $x \in \overline{U}$ (see [8, Theorem 3.3 and Theorem 4.1] or [7, Proposition 5.6]). Then T need not preserve continuous or Baire–one functions. It turns out that the shift of classes ceases to exist for $\alpha \geq 2$ as the following result shows.

Theorem 1.3. *Let $U \subset \mathbb{R}^d$ be a bounded open set and $\mathbf{H}(U)$ be the space of all continuous functions on $\overline{U}$ harmonic on U. Then $T(\mathcal{B}^b_\alpha(\overline{U})) \subset \mathcal{B}^b_\alpha(\overline{U})$ for each $\alpha \in [2, \omega_1)$.*

2. Auxiliary results on simplicial spaces and state spaces

In the sequel we will need the following results on function spaces. The first one is proved in [18, Theorem 5.1].

Proposition 2.1. *Let $\mathcal{H}$ be a function space on a compact space K. Then $\mathcal{B}^b_1(K) \cap \mathcal{H}^{\perp\perp} = \mathcal{H}_1$.*

Proposition 2.2. *Let $\mathcal{H}$ be a simplicial function space on a metrizable compact space K and let T be the operator defined by* (1).

(a) *For every $f \in \mathcal{B}^b(K)$, $Tf \in (\mathcal{A}_c(\mathcal{H}))^{\perp\perp}$.*

(b) *We have $T(\mathcal{C}(K)) \subset (\mathcal{A}_c(\mathcal{H}))_1$.*

(c) *We have $\mathcal{A}(\mathcal{H}) = (\mathcal{A}_c(\mathcal{H}))^{\perp\perp}$.*

Here assertions (a) and (b) follow from Corollary 6.2, Proposition 6.1 and Theorem 6.3 in [18]. Assertion (c) can be found in [21, Thereom 2.6(b2)].

If $\mathcal{H}$ is a function space on a compact space K, we consider its *state space* $\mathbf{S}(\mathcal{H})$ defined as

$$\mathbf{S}(\mathcal{H}) = \{s \in \mathcal{H}^* : s \geq 0, s(1) = 1\}.$$

Then $\mathbf{S}(\mathcal{H})$, endowed with the weak* topology, is a compact convex set. The mapping $\phi : K \to \mathbf{S}(\mathcal{H})$ defined as $\phi(x)(h) = h(x)$, $h \in \mathcal{H}$, is a homeomorphic embedding of K to $\mathbf{S}(\mathcal{H})$ that maps $\operatorname{Ch}_{\mathcal{H}} K$ onto $\operatorname{ext}\mathbf{S}(\mathcal{H})$ (see [1, Theorem II.2.1]). By the Hahn-Banach theorem, the restriction mapping $\pi : \mathcal{M}^1(K) \to \mathbf{S}(\mathcal{H})$ is an affine continuous surjection. According to [21, Theorem 2.5], the formula

$$(2) \qquad If(s) = \mu(f), \quad \mu \in \mathcal{M}^1(K), \pi(\mu) = s, \qquad s \in \mathbf{S}(\mathcal{H}),$$

defines an isometric isomorphism $I : \mathcal{H}^{\perp\perp} \to (\mathfrak{A}(\mathbf{S}(\mathcal{H})))^{\perp\perp}$, that preserves natural order of functions. Moreover,

$$(3) \qquad I(\mathcal{H}_\alpha) = \mathfrak{A}_\alpha(\mathbf{S}(\mathcal{H})) \quad \text{and} \quad I\left(\mathcal{B}^b_\alpha(K) \cap \mathcal{H}^{\perp\perp}\right) = \mathcal{B}^b_\alpha(X) \cap (\mathfrak{A}(X))^{\perp\perp}$$

for every $\alpha \in [0, \omega_1)$. Its inverse $I^{-1}\colon\ (\mathfrak{A}(\mathbf{S}(\mathcal{H})))^{\perp\perp} \to \mathcal{H}^{\perp\perp}$ is given by the formula

$$I^{-1}\widehat{f} = \widehat{f} \circ \phi, \quad \widehat{f} \in (\mathfrak{A}(\mathbf{S}(\mathcal{H})))^{\perp\perp}.$$

Proposition 2.3. *Let $\mathcal{H}$ be a function space on a compact space K and let $X := \mathbf{S}(\mathcal{A}_c(\mathcal{H}))$ be the state space of the function space $\mathcal{A}_c(\mathcal{H})$. Let $\phi : K \to X$ denote the embedding defined above.*

(a) The space $\mathcal{H}$ is simplicial if and only if X is a simplex.

(b) Assume that $\mathcal{H}$ is simplicial and K is metrizable. Let $\widehat{T} : \mathcal{B}^b(X) \to (\mathfrak{A}(X))^{\perp\perp}$ be the operator from (1) considered for the function space $\mathfrak{A}(X)$. If f is a bounded Borel function on K, let

$$\widehat{f}(s) := \begin{cases} f(x), & s = \phi(x), \\ 0, & s \in X \setminus \phi(K). \end{cases}$$

Then

$$\widehat{T}\widehat{f}(\phi(x)) = Tf(x), \quad x \in K.$$

Proof. Assertion (a) is a content of [6, Theorem]. To prove (b), let f be a bounded Borel function on K and let $\widehat{f}$ be defined as in (b). For any $x \in K$, we need to show that

$$\delta_{\phi(x)}(\widehat{f}) = \delta_x(f).$$

Since $\delta_{\phi(x)}$ is supported by $\overline{\operatorname{ext} X} \subset \phi(K)$, we can write

$$\delta_{\phi(x)}(\widehat{f}) = \delta_{\phi(x)}(f \circ \phi^{-1}) = (\phi^{-1}\delta_{\phi(x)})(f).$$

Thus it is enough to show that $\phi(\delta_x) = \delta_{\phi(x)}$. (Here $\phi^{-1}\delta_{\phi(x)}$ and $\phi(\delta_x)$ denote the image measures under the mappings ϕ^{-1} and ϕ, respectively.)

A straightforward verification shows that $\phi(\delta_x)$ $\mathfrak{A}(X)$–represents $\phi(x)$. Since $\phi(\delta_x)$ is carried by $\phi(\operatorname{Ch}_{\mathcal{H}} K) = \operatorname{ext} X$ and X is a simplex by (a), $\phi(\delta_x) = \delta_{\phi(x)}$ as required. This concludes the proof. □

3. Auxiliary results on Borel sets and products of function spaces

Before the proofs of the main results, we recall several facts from the descriptive set theory. If X is a Polish (i.e., separable completely metrizable) space, we write $\mathbf{\Sigma}^0_\alpha(X), \mathbf{\Pi}^0_\alpha(X)$ for the additive and multiplicative classes of Borel subsets of X (see [16, p. 68]). The following classical result can be found in [16, Theorem 24.3 and Theorem 24.10].

Theorem 3.1. *Let f be a function on a Polish space X and $\alpha \in [0, \omega_1)$. Then f is of Baire class α if and only if $f^{-1}(U) \in \mathbf{\Sigma}^0_{\alpha+1}(X)$ for every open set $U \subset \mathbb{R}$. In particular, $\bigcup_{\alpha<\omega_1} \mathcal{B}^b_\alpha(X)$ is the class of all bounded Borel functions.*

If A is a subset of a set X, we write χ_A for the characteristic function of A. From the preceding theorem we see that $\chi_A \in \mathcal{B}^b_\alpha(X)$ if and only if $A, X \setminus A \in \mathbf{\Sigma}^0_{\alpha+1}(X)$. The following proposition is proved in [16, Theorem 22.10, Exercise 23.3 and Exercise 24.20].

Proposition 3.2. *Let $\alpha \in [1, \omega_1)$ and let X_n, $n \in \mathbb{N}$, be Polish spaces. Let $A_n \subset X_n$ satisfy $A_n \in \mathbf{\Sigma}^0_\alpha(X_n) \setminus \mathbf{\Pi}^0_\alpha(X_n)$, $n \in \mathbb{N}$, and let $X = \prod_{n\in\mathbb{N}} X_n$. Then*

$$\prod_{n\in\mathbb{N}} A_n \in \mathbf{\Pi}^0_{\alpha+1}(X) \setminus \mathbf{\Sigma}^0_{\alpha+1}(X).$$

Our construction of simplicial spaces is based upon the notion of *products* of function spaces. We briefly recall the construction and the properties relevant for our purposes. We consider $\{(K_i, \mathcal{H}_i)\}_{i\in I}$ a family of function spaces and we denote by $K = \prod_{i\in I} K_i$ the cartesian product of spaces K_i. For any $f \in \mathcal{C}(K)$, $j \in I$ and $y \in \prod_{i\in I\setminus\{j\}} K_i$ we define a function f^y on K_j by $f^y(x) = f(x, y)$, $x \in K_j$. We say that the function space

$$\mathcal{H} := \{f \in \mathcal{C}(K) : f^y \in \mathcal{H}_j \text{ for all } j \in I \text{ and } y \in \prod_{i\in I\setminus\{j\}} K_i\}$$

is the *product* of $\mathcal{H}_i$, $i \in I$.

If $\mu_i \in \mathcal{M}^1(K_i)$, $i \in I$, we denote by $\bigotimes_{i\in I} \mu_i$ the unique Radon measure extending the ordinary product measure (see [13, Theorem 417Q]).

Theorem 3.3. *Let $\mathcal{H}$ be the product of $\mathcal{H}_i$, $i \in I$. Then $\operatorname{Ch}_{\mathcal{H}} K = \prod_{i\in I} \operatorname{Ch}_{\mathcal{H}_i} K_i$. Further, $\mathcal{H}$ is simplicial if and only if $\mathcal{H}_i$ is simplicial for every $i \in I$. In that case we have $\delta_x = \bigotimes_{i\in I} \delta_{x_i}$ for each $x = (x_i)_{i\in I} \in K$.*

For a proof see [15, Theorems 3.42, 3.52, 3.53 and 3.59].

4. Baire solutions of the abstract Dirichlet problem

Lemma 4.1. *Let $\mathcal{H}$ be a simplicial function space on a metrizable compact space K, let $\alpha_0 \in [0, \omega_1)$ and let $\mathcal{F}$ be a set of functions on K. If $T(\mathcal{B}^b_{\alpha_0}(K)) \subset \mathcal{F}_{\alpha_0}$, then $T(\mathcal{B}^b_\alpha(K)) \subset \mathcal{F}_\alpha$ for every $\alpha \in [\alpha_0, \omega_1)$.*

Proof. We proceed by induction. We fix $\alpha > \alpha_0$ and suppose that $T(\mathcal{B}^b_\zeta(K)) \subset \mathcal{F}_\zeta$ for every $\alpha_0 \leq \zeta < \alpha$. Choose $f \in \mathcal{B}^b_\alpha(K)$. Then there is a uniformly bounded sequence of functions $f_n \in \mathcal{B}^b_{\alpha_n}(K)$, $\alpha_0 \leq \alpha_n < \alpha$, $n \in \mathbb{N}$, such that $f_n \to f$. Since $Tf_n \in \mathcal{F}_{\alpha_n}$ for every $n \in \mathbb{N}$ and, by Lebesgue's dominated convergence theorem, $Tf_n \to Tf$, we get $Tf \in \mathcal{F}_\alpha$. ☐

Proposition 4.2. *Let $\mathcal{H}$ be a simplicial function space on a metrizable compact space K and let $\alpha \in [0, \omega_1)$. If $f \in \mathcal{B}^b_\alpha(K)$, then $Tf \in (\mathcal{A}_c(\mathcal{H}))_{\alpha+1}$.*

Proof. It follows e.g. from [1, Proposition II.3.14] or [18, Proposition 6.1 and Theorem 6.3] that $T(\mathcal{C}(K)) \subset \mathcal{B}^b_1(K) \cap (\mathcal{A}_c(\mathcal{H}))^{\perp\perp} = (\mathcal{A}_c(\mathcal{H}))_1$. The assertion now follows from Lemma 4.1 with $\mathcal{F} = (\mathcal{A}_c(\mathcal{H}))_1$ and $\alpha_0 = 0$. ☐

Proof of Theorem 1.1(a). According to Lemma 4.1, it is sufficient to prove the assertion for $\alpha = \omega_0$. Hence, choose $f \in \mathcal{B}^b_{\omega_0}(K)$. There is a uniformly bounded sequence of functions $f_n \in \mathcal{B}^b_{\alpha_n}(K)$, $\alpha_n, n \in \mathbb{N}$, such that $f_n \to f$. By Proposition 4.2, $Tf_n \in (\mathcal{A}_c(\mathcal{H}))_{\alpha_n+1}$ for every $n \in \mathbb{N}$ and by the dominated convergence theorem $Tf_n \to Tf$. Therefore $Tf \in (\mathcal{A}_c(\mathcal{H}))_{\omega_0}$. □

Theorem 4.3. *There exists a simplicial function space $\mathcal{H}$ on a metrizable compact space K such that $\mathcal{A}_c(\mathcal{H}) = \mathcal{H}$ and $T(\mathcal{B}^b_\alpha(K)) \not\subset \mathcal{B}^b_\alpha(K)$ for every $\alpha \in [0, \omega_0)$.*

Proof. Step 1. First we prove the following claim:

For every $0 \le \alpha < \omega_0$ there exists a simplicial function space $\mathcal{H}_\alpha$ on a metrizable compact space K_α and subsets $A_\alpha, D_\alpha \subset K_\alpha$ such that:

(i) $A_\alpha \in \mathbf{\Pi}^0_{\alpha+1}(K_\alpha) \setminus \mathbf{\Sigma}^0_{\alpha+1}(K_\alpha)$,
(ii) $D_0 \in \mathbf{\Pi}^0_1(K_0) \cap \mathbf{\Sigma}^0_1(K_0)$ *and* $D_\alpha \in \mathbf{\Pi}^0_\alpha(K_\alpha)$ *if* $\alpha \ge 1$,
(iii) if $x \in A_\alpha$, *then* $\operatorname{spt}\delta_x \subset D_\alpha$,
(iv) if $x \notin A_\alpha$, *then* $\operatorname{spt}\delta_x \subset K_\alpha \setminus D_\alpha$,
(v) $\mathcal{A}_c(\mathcal{H}_\alpha) = \mathcal{H}_\alpha$.

We construct such a space $(K_\alpha, \mathcal{H}_\alpha)$ and sets A_α, D_α for each $\alpha \in [0, \omega_0)$ by induction:

Choose $q \in (0,1) \subset \mathbb{R}$ and a sequence $\{s_n\}_{n\in\mathbb{N}} \subset (0,1) \setminus \{q\}$ such that $s_n \to q$. Define $K_0 := \{0,1,q\} \cup \{s_n\}_{n\in\mathbb{N}}$ and

$$\mathcal{H}_0 := \{f \in \mathcal{C}(K_0) : f(q) = \frac{1}{2}(f(0) + f(1))\}.$$

The space $(K_0, \mathcal{H}_0)$ is obviously simplicial with $\operatorname{Ch}_{\mathcal{H}_0} K_0 = K_0 \setminus \{q\}$ and $\delta_q = \frac{1}{2}(\varepsilon_0 + \varepsilon_1)$. We take $A_0 := \{0,1,q\}$ and $D_0 := \{0,1\}$. It is easy to verify that conditions (i)–(v) above are satisfied.

Now suppose we have $(K_{\alpha-1}, \mathcal{H}_{\alpha-1}), A_{\alpha-1}, D_{\alpha-1}$ satisfying the conditions, where $\alpha \in \mathbb{N}$. Let $(K_\alpha, \mathcal{H}_\alpha)$ be the product of countably many copies of the space $(K_{\alpha-1}, \mathcal{H}_{\alpha-1})$. Define $A_\alpha := \prod(K_{\alpha-1} \setminus A_{\alpha-1})$ and $D_\alpha := \prod(K_{\alpha-1} \setminus D_{\alpha-1})$. Then all the required conditions are satisfied:

- From the assumption (i) on $A_{\alpha-1}$ and Proposition 3.2 we have $A_\alpha \in \mathbf{\Pi}^0_{\alpha+1}(K_\alpha) \setminus \mathbf{\Sigma}^0_{\alpha+1}(K_\alpha)$.
- Using the assumption (ii) on $D_{\alpha-1}$, it is easy to see that $D_\alpha \in \mathbf{\Pi}^0_\alpha(K_\alpha)$ for $\alpha > 1$. If $\alpha = 1$, notice that $K_0 \setminus D_0$ is a compact set, so D_1 is compact.
- Suppose $x = (x_i)_{i\in\mathbb{N}} \in A_\alpha$. Then $x_i \notin A_{\alpha-1}$ for each $i \in \mathbb{N}$. Using Theorem 3.3 and the assumption (iv) we see that
$$\operatorname{spt}\delta_x = \prod_{i\in\mathbb{N}} \operatorname{spt}\delta_{x_i} \subset \prod_{i\in\mathbb{N}}(K_{\alpha-1} \setminus D_{\alpha-1}) = D_\alpha.$$
- Conversely, let $x = (x_i)_{i\in\mathbb{N}} \notin A_\alpha$. Then there is some $j \in \mathbb{N}$ such that $x_j \in A_{\alpha-1}$ and from the assumption (iii) $\operatorname{spt}\delta_{x_j} \subset D_{\alpha-1}$. Thus, we get
$$\operatorname{spt}\delta_x = \prod_{i\in\mathbb{N}} \operatorname{spt}\delta_{x_i} \subset K_\alpha \setminus (\prod_{i\in\mathbb{N}}(K_{\alpha-1} \setminus D_{\alpha-1})) = K_\alpha \setminus D_\alpha.$$
- Condition (v) follows from [15, Proposition 3.33].

Step 2. Now we show that for each $\alpha \in [0, \omega_0)$ the function space $(K_\alpha, \mathcal{H}_\alpha)$ constructed above satisfies $T(\mathcal{B}^b_\alpha(K)) \not\subset \mathcal{B}^b_\alpha(K)$. So let $\alpha \in [0, \omega_0)$. Define $f_\alpha := \chi_{D_\alpha}$. Clearly, $f_\alpha \in \mathcal{B}^b_\alpha(K_\alpha)$. If $x \in A_\alpha$, then $Tf_\alpha(x) = \delta_x(f_\alpha) = 1$, since $\operatorname{spt}\delta_x \subset$

D_α. On the other hand, if $x \notin A_\alpha$, then $Tf_\alpha(x) = 0$. Therefore $Tf_\alpha = \chi_{A_\alpha} \notin \mathcal{B}^b_\alpha(K_\alpha)$ and so $T(\mathcal{B}^b_\alpha(K_\alpha)) \not\subset \mathcal{B}^b_\alpha(K_\alpha)$.

Step 3. Finally, we show that $(K_1, \mathcal{H}_1)$ is the sought function space. There is a homeomorphism $\phi : K_1 \to K_2$ such that $\phi(\delta_x) = \delta_{\phi(x)}$ for every $x \in K_1$. Indeed, let ϕ be a natural bijection between $K_1 = \prod_{i\in\mathbb{N}} K_0$ and $K_2 = \prod_{j\in\mathbb{N}} \prod_{k\in\mathbb{N}} K_0$. Using Theorem 3.3 and an associative law for Radon product measures (see [13, Theorem 417J]) we arrive at

$$\phi(\delta_x) = \phi(\bigotimes_{i\in\mathbb{N}} \delta_{x_i}) = \bigotimes_{j\in\mathbb{N}} \bigotimes_{k\in\mathbb{N}} \delta_{x_{j,k}} = \delta_{\phi(x)}$$

for every $x = (x_i)_{i\in\mathbb{N}} \in K_1$ with $\phi(x) = ((x_{j,k})_{k\in\mathbb{N}})_{j\in\mathbb{N}} \in K_2$. Now suppose $\alpha \in \mathbb{N}$. If $f_\alpha \in \mathcal{B}^b_\alpha(K_2)$, then $f_\alpha \circ \phi \in \mathcal{B}^b_\alpha(K_1)$, since ϕ is a homeomorphism. If $Tf_\alpha \notin \mathcal{B}^b_\alpha(K_2)$, then $T(f_\alpha \circ \phi) \notin \mathcal{B}^b_\alpha(K_1)$, because

$$T(f_\alpha \circ \phi)(x) = \delta_x(f_\alpha \circ \phi) = (\phi\delta_x)(f_\alpha) = \delta_{\phi(x)}(f_\alpha) = Tf_\alpha(\phi(x)), \quad x \in K_1.$$

By induction and Step 2, for every $\alpha \in \mathbb{N}$ there is $f_\alpha \in \mathcal{B}^b_\alpha(K_1)$ with $Tf_\alpha \notin \mathcal{B}^b_\alpha(K_1)$. If $\alpha = 0$, the existence of such a function is ensured by Lemma 4.1. □

Theorem 4.4. *Let $n \in \{0, 1\}$. Then there exists a simplicial function space $\mathcal{H}$ on a metrizable compact space K such that $\mathcal{A}_c(\mathcal{H}) = \mathcal{H}$, $T(\mathcal{B}^b_n(K)) \not\subset \mathcal{B}^b_n(K)$ and $T(\mathcal{B}^b_\alpha(K)) \subset \mathcal{B}^b_\alpha(K)$ for each $\alpha \in (n, \omega_1)$.*

Proof. If $n = 0$, it is enough to take any simplicial function space $\mathcal{H}$ on a metrizable compact K such that $\mathcal{A}_c(\mathcal{H}) = \mathcal{H}$ and $\operatorname{Ch}_{\mathcal{H}} K$ is a non-closed F_σ-subset of K (take e.g. $(K_0, \mathcal{H}_0)$ from the proof of Theorem 4.3). Then there exists a continuous function f such that Tf is not continuous and $T(\mathcal{B}^b_1(K)) \subset \mathcal{B}^b_1(K)$.

If $n = 1$, let $\mathcal{H}$ be the function space constructed in [20, Example 3.10]. We briefly recall this construction. Let $\{q_n\}$ be an enumeration of rational numbers contained in $[0, 1]$. We define a subset $K \subset \mathbb{R}^2$ as follows

$$K := ([0, 1] \times \{0\}) \cup \{(q_n, n^{-1}), (q_n, -n^{-1}) : n \in \mathbb{N}\}\ .$$

(We write (a, b) for a point in $\mathbb{R}^2$ with the coordinates a and b.) Obviously, K is a compact set in $\mathbb{R}^2$ (considered with the usual Euclidean topology). Let

$$\mathcal{H} := \{f \in \mathcal{C}(K) : f(q_n, 0) = \frac{1}{2}\left(f(q_n, -n^{-1}) + f(q_n, n^{-1})\right), n \in \mathbb{N}\}\ .$$

Then $\mathcal{H}$ is a simplicial function space with $\operatorname{Ch}_{\mathcal{H}} K = K \setminus \{(q_n, 0) \in K : n \in \mathbb{N}\}$. Obviously,

$$\delta_{(x,y)} = \begin{cases} \frac{1}{2}\left(\varepsilon_{(q_n,n^{-1})} + \varepsilon_{(q_n,-n^{-1})}\right), & x = q_n \text{ for some } n \in \mathbb{N} \text{ and } y = 0, \\ \varepsilon_{(x,y)}, & \text{otherwise.} \end{cases}$$

Hence $T(\chi_{[0,1]\times\{0\}})$ has no point of continuity on $[0, 1] \times \{0\}$, and thus $T(\mathcal{B}^b_1(K)) \not\subset \mathcal{B}^b_1(K)$. On the other hand, if f is any bounded function of Baire class α, $\alpha \in [2, \omega_1)$, then the set

$$\{x \in K : Tf(x) \neq f(x)\}$$

is at most countable. Hence $T(\mathcal{B}^b_\alpha(K)) \subset \mathcal{B}^b_\alpha(K)$ for each $\alpha \in [2, \omega_1)$. □

Proof of Theorem 1.1(b),(c). For the proof of (b), let $\mathcal{H}$ be the function space on the compact space K constructed in Theorem 4.3 and let X be the state space of $\mathcal{H}$. By Proposition 2.3, X is a metrizable simplex. Let $\phi : K \to X$ be the homeomorphic embedding and $\widehat{T} : \mathcal{B}^b(X) \to (\mathfrak{A}(X))^{\perp\perp}$ be the operator from (1) considered for the simplicial function space $\mathfrak{A}(X)$.

For any $\alpha \in \mathbb{N}$, let $f \in \mathcal{B}^b_\alpha(K)$ be such that $Tf \notin \mathcal{B}_\alpha(K)$. Then $\widehat{f}$ defined as

$$\widehat{f}(s) := \begin{cases} f(x), & s = \phi(x), \\ 0, & s \in X \setminus \phi(K), \end{cases}$$

is in $\mathcal{B}^b_\alpha(X)$ and satisfies $T\widehat{f} \notin \mathcal{B}^b_\alpha(X)$. Indeed, Proposition 2.3(b) gives

$$Tf = \widehat{T}\widehat{f} \circ \phi = I^{-1}\widehat{T}\widehat{f},$$

from which the conclusion follows because I^{-1} preserves Borel classes. For $\alpha = 0$ we deduce the existence of a function $\widehat{f} \in \mathcal{C}(X)$ satisfying $\widehat{T}\widehat{f} \notin \mathcal{C}(X)$ from Lemma 4.1.

For the verification of (c) we use the same method as in (b), the required simplices are taken to be the state spaces of the function spaces constructed in Theorem 4.4. This concludes the proof. □

Proof of Theorem 1.2. For the proof of (a), let $f \in (\mathfrak{A}(X))^{\perp\perp}$ be a Baire–α function on a simplex X for some $\alpha \in [\omega_0, \omega_1)$. By the method of the proof of [10, Théorème 2], there exists a metrizable simplex Y, a continuous affine surjective mapping $\varphi : X \to Y$ and a Baire function g on Y such that $f = g \circ \varphi$. By [22, Examples 2.4], $g \in \mathcal{B}^b_\alpha(Y) \cap (\mathfrak{A}(Y))^{\perp\perp}$ as well. Since $g \in (\mathfrak{A}(Y))^{\perp\perp}$, $Tg = g$. Hence $g \in \mathfrak{A}_\alpha(Y)$ by Theorem 1.1(a). It follows that $f \in \mathfrak{A}_\alpha(X)$ as well.

For the proof of (b), if $\mathcal{H}$ is a simplicial function space on a compact space K, we know from Proposition 2.2 that $\mathcal{A}(\mathcal{H}) = (\mathcal{A}_c(\mathcal{H}))^{\perp\perp}$. Further, the state space $X = \mathbf{S}(\mathcal{A}_c(\mathcal{H}))$ is a simplex by Proposition 2.3(a). Let $I : (\mathcal{A}_c(\mathcal{H}))^{\perp\perp} \to \mathfrak{A}(X)$ be the mapping defined by (2). If $\alpha \in [\omega_0, \omega_1)$ and

$$f \in \mathcal{B}^b_\alpha(K) \cap \mathcal{A}(\mathcal{H}) = \mathcal{B}^b_\alpha(K) \cap (\mathcal{A}_c(\mathcal{H}))^{\perp\perp},$$

then

$$If \in \mathcal{B}^b_\alpha(X) \cap (\mathfrak{A}(X))^{\perp\perp} = \mathfrak{A}_\alpha(X)$$

by (3) and assertion (a). Hence

$$f \in (\mathcal{A}_c(\mathcal{H}))_\alpha,$$

again by (3). This finishes the proof. □

5. Spaces of harmonic functions

Lemma 5.1. *Let $\mathcal{H}$ be a simplicial function space on a metrizable compact space K and $F \subset \operatorname{Ch}_{\mathcal{H}} K$ be compact. Then $T(\mathcal{B}^b_\alpha(F)) \subset (\mathcal{A}_c(\mathcal{H}))_\alpha$, $\alpha \in [1, \omega_1)$.*

Proof. Let $F \subset \operatorname{Ch}_{\mathcal{H}} K$ be a compact set. First we prove that $T(\mathcal{B}^b_1(F)) \subset (\mathcal{A}_c(\mathcal{H}))_1$. If f is a bounded Baire–one function on F, $Tf \in (\mathcal{A}_c(\mathcal{H}))^{\perp\perp}$ by Proposition 2.2(a). Further, we want to show that $Tf \in \mathcal{B}^b_1(K)$. According to [16, Theorem 24.3], Tf is a Baire–one function if and only if both $\{x \in X : Tf(x) > c\}$ and $\{x \in X : Tf(x) < c\}$ are sets of type F_σ for each $c \in \mathbb{R}$.

If $f = \chi_A$ for $A \subset F$ closed, then Tf is Baire–one. This follows from the fact that

$$T\chi_A = (\chi_A)^* \ (= \inf\{h \in \mathcal{H} \colon h \geq \chi_A\})$$

is upper semicontinuous and thus Baire–one (see [1, Theorem II.3.7 and Theorem II.3.8]).

If $f = \chi_A$, where $A \subset K$ is both F_σ and G_δ set, then $T\chi_A$ is also a Baire–one function. To verify this, we write $A = \bigcup_{n=1}^\infty F_n$, where the sets F_n are closed in K. Then for each $c \in \mathbb{R}$, the set

$$\{x \in X : T\chi_A(x) > c\} = \bigcup_{n=1}^\infty \{x \in X : T\chi_{F_n}(x) > c\}$$

is of type F_σ according to the reasoning above. Similarly we get that $\{x \in X : T\chi_{K\setminus A} > c\}$ is of type F_σ for every $c \in \mathbb{R}$.

Thus for arbitrary $c \in \mathbb{R}$, the set

$$\{x \in X : T\chi_A(x) < c\}$$
$$= \bigcup_{\substack{q_1 - q_2 < c \\ q_1, q_2 \in \mathbb{Q}}} \{x \in X : T\chi_K(x) < q_1\} \cap \{x \in X : T\chi_{K\setminus A}(x) > q_2\}$$

is of type F_σ. Thus the function $T\chi_A$ is Baire–one.

Hence $T(\sum_{n=1}^k c_n \chi_{A_n})$ is Baire–one whenever the sets A_n are both F_σ and G_δ in F and c_n, $n = 1, \dots, k$, are real numbers. Since functions of this type are uniformly dense in the space of all bounded Baire–one functions on F and Baire–one functions are stable with respect to uniform convergence, $T(\mathcal{B}_1^b(F)) \subset \mathcal{B}_1^b(K)$.

Since

$$\mathcal{B}_1^b(K) \cap (\mathcal{A}_c(\mathcal{H}))^{\perp\perp} = (\mathcal{A}_c(\mathcal{H}))_1$$

by Proposition 2.1, $T(\mathcal{B}_1^b(F)) \subset (\mathcal{A}_c(\mathcal{H}))_1$.

The rest of the proof now follows from Lemma 4.1. □

Theorem 5.2. *Let $\mathcal{H}$ be a simplicial function space on a metrizable compact space K. Assume that there exist compact sets $K_n \subset \operatorname{Ch}_{\mathcal{H}} K$, $n \in \mathbb{N}$, such that $\delta_x(K \setminus \bigcup_{n=1}^\infty K_n) = 0$ for each $x \in K \setminus \operatorname{Ch}_{\mathcal{H}} K$. Then $T(\mathcal{B}_\alpha^b(K)) \subset \mathcal{B}_\alpha^b(K)$, $\alpha \in [2, \omega_1)$.*

Proof. Given a sequence $\{K_n\}$ of compact sets in $\operatorname{Ch}_{\mathcal{H}} K$ as in the premise, we assume without loss of generality that $\{K_n\}$ is increasing and denote

$$H_1 := \bigcup_{n=1}^\infty K_n, \quad H_2 := \operatorname{Ch}_{\mathcal{H}} K \setminus \bigcup_{n=1}^\infty K_n.$$

Let f be a bounded Baire–two function on K. We find a bounded sequence $\{f_n\}$ of Baire–one functions converging to f. According to Lemma 5.1,

$$T(f_n \chi_{K_n}) \in (\mathcal{A}_c(\mathcal{H}))_1 \subset \mathcal{B}_1^b(K), \quad n \in \mathbb{N}.$$

It is easy to see that

$$T(f_n \chi_{K_n}) \to T(f\chi_{H_1}).$$

Since

$$Tf = T(f\chi_{H_1}) + T(f\chi_{H_2}) = T(f\chi_{H_1}) + f\chi_{H_2}$$

and χ_{H_2} is a Baire–two function, $Tf \in \mathcal{B}_2^b(K)$ as well. Since the assertion for higher Baire classes follows by Lemma 4.1, the proof is finished. □

Proof of Theorem 1.3. Throughout the proof, we write B^c for the set $\mathbb{R}^d \setminus B$ whenever $B \subset \mathbb{R}^d$. By Theorem 5.2, it is enough to find compact sets $K_n \subset \partial_{\mathrm{reg}} U$, $n \in \mathbb{N}$, such that

$$\delta_x(\bigcup_{n=1}^\infty K_n) = \varepsilon_x^{U^c}(\bigcup_{n=1}^\infty K_n) = 1$$

for each $x \in \overline{U} \setminus \partial_{\mathrm{reg}} U$. We divide the proof in a couple of steps.

Step 1. There exist compact sets $K_n \subset \partial_{\mathrm{reg}} U$, $n \in \mathbb{N}$, such that $\varepsilon_x^{U^c}(\bigcup_{n=1}^\infty K_n) = 1$ for each $x \in U$.

To verify this, let $\{x_n : n \in \mathbb{N}\}$ be a dense subset of U. For each $n \in \mathbb{N}$ we select compact sets $L_{nk} \subset \partial_{\mathrm{reg}} U$, $k \in \mathbb{N}$, such that $\varepsilon_{x_n}^{U^c}(\bigcup_{k=1}^\infty L_{nk}) = 1$. If we enumerate the sets $\{L_{nk} : n, k \in \mathbb{N}\}$ into a single sequence $\{K_n\}$, we get the desired sets.

Indeed, writing A for the set $\bigcup_{n=1}^\infty K_n$, let h denote the PWB–solution of the generalized Dirichlet problem on U with the boundary function χ_A (see [7, Chapter VII, Section 2] or [3, Chapter 6]). Since $1 = h(x_n) = \varepsilon_{x_n}^{U^c}(A)$ for each $n \in \mathbb{N}$ and h is a harmonic function on U, $h = 1$ on U. Hence $\varepsilon_x^{U^c}(A) = h(x) = 1$ for all $x \in U$.

Step 2. If A is chosen as above, then $\varepsilon_x^{U^c}(A) = 1$ for every $x \in \overline{U} \setminus \partial_{\mathrm{reg}} U$.

By the previous paragraph, the claim holds true for every point $x \in U$. Let now x be an irregular point of the boundary of U.

Let $\varepsilon > 0$ be given. Since $\varepsilon_z^{U^c}(\{x\}) = 0$, there exists an open neighborhood V of x such that $\varepsilon_x^{U^c}(V) < \varepsilon$. We select an increasing sequence of compact sets $\{F_n\}$ in U such that $\bigcup_{n=1}^\infty F_n = U$. Then $\varepsilon_x^{F_n \cup U^c} \to \varepsilon_x$ in the weak*–topology by [7, Chapter VI, Corollary 10.3]. Hence there exists a compact set $L \subset U$ so that the measure $\mu = \varepsilon_x^{L \cup U^c}$ satisfies $\mu(V) > 1 - \varepsilon$. Since μ is supported by $U^c \cup L$, using [7, Chapter VI, Proposition 9.4] we get

$$\varepsilon_x^{U^c} = \mu|_{U^c} + (\mu|_L)^{U^c}. \tag{4}$$

(Here $\mu|_B$ denotes the restriction of the measure μ to a set B.) By our choice of A,

$$(\mu|_L)^{U^c}(A^c) = \int_L \varepsilon_t^{U^c}(A^c)\, d\mu(t) = 0. \tag{5}$$

Since

$$\mu(V^c) = \mu(\mathbb{R}^d) - \mu(V) \leq 1 - (1 - \varepsilon) = \varepsilon,$$

and by (4)

$$\mu(U^c \cap V) \leq \varepsilon_x^{U^c}(V) < \varepsilon,$$

we get $\mu(U^c) < 2\varepsilon$. Thus by (4)

$$\varepsilon_x^{U^c}(A^c) = \mu(U^c \cap A^c) + (\mu|_L)^{U^c}(A^c) \leq 2\varepsilon.$$

This concludes the proof of the second step as well as the proof of the theorem. □

6. Open problems

Question 6.1. We do not know any general construction that would lead, for a given $n \in \mathbb{N}$, to a simplicial function space $\mathcal{H}$ on a compact space K such that $T(\mathcal{B}_n^b(K)) \not\subset \mathcal{B}_n^b(K)$ and $T(\mathcal{B}_\alpha^b(K)) \subset \mathcal{B}_\alpha^b(K)$ for each $\alpha \in (n, \omega_1)$.

Question 6.2. Let $\mathcal{H}$ be a simplicial function space on a metrizable compact space K. By [18, Theorem 6.3], $\mathcal{B}_1^b(K) \cap (\mathcal{A}_c(\mathcal{H}))^{\perp\perp} = (\mathcal{A}_c(\mathcal{H}))_1$. It is witnessed by the function space constructed in [21] that this equality does not hold for higher classes, precisely it is shown that $\mathcal{B}_2^b(K) \cap (\mathcal{A}_c(\mathcal{H}))^{\perp\perp} \not\subset (\mathcal{A}_c(\mathcal{H}))_2$. It might be interesting to know whether such an example can be of type $\mathbf{H}(U)$ for some bounded open set $U \subset \mathbb{R}^d$.

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Chapter 4

On sequentially Right Banach spaces

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ON SEQUENTIALLY RIGHT BANACH SPACES

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Abstract. In this paper, we study the recently introduced class of sequentially Right Banach spaces. We introduce a stronger property (RD) and compare these two properties with other well-known isomorphic properties of Banach spaces such as property (V) or the Dieudonné property. In particular, we show that there is a sequentially Right Banach space without property (V). This answers a question of A.M. Peralta, I. Villanueva, J.D.M. Wright and K. Ylinen. We also generalize a result of A. Pełczyński and prove that every sequentially Right Banach space has weakly sequentially complete dual. Finally, it is shown that if K is a scattered compact Hausdorff space then the space $C(K, X)$ of X-valued continuous functions on K is sequentially Right (resp. has property (RD)) if and only if X has the same property.

1. Introduction

In [23], A.M. Peralta, I. Villanueva, J.D.M. Wright and K. Ylinen proved that for a given Banach space X there is a locally convex topology on X, called by them the 'Right topology', such that every operator T from X into a Banach space Y is weakly compact if and only if it is Right-to-norm continuous. This topology is obtained as the restriction of the Mackey topology $\tau(X^{**}, X^*)$ to X. It is the topology of uniform convergence on absolutely convex $\sigma(X^*, X^{**})$-compact subsets of X^*. In general, the Right topology is stronger than the weak topology and weaker than the norm topology, thus compatible with the dual pair $\langle X, X^* \rangle$. Every Right-to-norm continuous operator is surely Right-to-norm sequentially continuous. A simple look at the identity operator on ℓ_1 reveals, however, that the converse is not true. Authors in [23] call Right-to-norm sequentially continuous operators *pseudo weakly compact* and Banach spaces, on which every pseudo weakly compact operator is weakly compact, *sequentially Right*. They have shown that every Banach space possessing property (V) is sequentially Right (see [23, Corollary 15]) and in the subsequent papers [22] and [35] they asked whether the converse holds. We provide a negative answer to this question.

In fact, we study relations of pseudo weakly compact operators and sequentially Right Banach spaces with respect to several other well-known classes of operators and isomorphic properties of Banach spaces. Among these properties are the *Dunford-Pettis property*, the *Reciprocal Dunford-Pettis property*, the *Dieudonné property* and the aforementioned *Pełczyński's property (V)*. We also introduce a new *property (RD)* which is an analogue of the Dieudonné property and is (at least formally) stronger than the property of being sequentially Right. A Banach space X is said to have property (RD) if every operator T from X into a Banach space Y which maps Right-Cauchy sequences into Right-convergent sequences is weakly compact. We improve the result of [23] and show that property (V) actually implies property (RD). Characterizations of property (RD) and sequential Rightness are provided. We generalize the result of A. Pełczyński [21, Corollary 5] and show that every sequentially Right Banach space has weakly sequentially complete dual.

We also take an interest in topological behaviour of the Right topology. Two most important special cases are in the centre of our attention. It is shown that the sequential coincidence of the Right topology with the weak one is just another characterization of the Dunford-Pettis property. Multiple characterizations are also given for the sequential coincidence of the Right topology with the norm topology.

Finally, we show that if K is a scattered compact Hausdorff space, then $C(K, X)$, the Banach space of all continuous functions from K to a Banach space X, is sequentially Right (resp. has property (RD)) if and only if X has the same property.

2. Preliminaries

Throughout this paper, we follow standard notation as in [8] or [17]. The term *operator* means a bounded linear map, all Banach spaces are over real numbers. For a Banach space X, we denote by B_X its closed unit ball.

Let X be a Banach space. Given a Banach space Y, an operator $T : X \to Y$ is called *completely continuous* (*cc*) if it maps weakly Cauchy sequences into norm convergent sequences. Banach space X has the *Dunford-Pettis property (DP)* if, for any Banach space Y, every weakly compact operator $T : X \to Y$ is completely continuous. This is equivalent to saying that for any weakly null sequences (x_n) and (x_n^*) in X and X^*, respectively, $\lim_n x_n^*(x_n) = 0$ (see, e.g., [7, Theorem 1]). X is said to have the *Reciprocal Dunford-Pettis property (RDP)* if, for any Banach space Y, every completely continuous operator $T : X \to Y$ is weakly compact. Examples of Banach spaces with (RDP) trivially include all reflexive spaces while, on the other hand, an infinite-dimensional reflexive space can never possess (DP). $C(K)$ spaces are known to enjoy both (DP) and (RDP). We refer to [7] for more information on the Dunford-Pettis property.

We say that an operator $T : X \to Y$ is *weakly completely continuous* (*wcc*) if it sends weakly Cauchy sequences into weakly convergent sequences. Let us denote by $\mathcal{B}_1(X)$ the subspace of X^{**} formed by all $\sigma(X^{**}, X^*)$-limits of weakly Cauchy sequences in X. In case X is a $C(K)$ space, $\mathcal{B}_1(X)$ is precisely the space of all bounded Baire-one functions on K ([14, p. 160]). X is said to have the *Dieudonné property (D)* if, for any Banach space Y, every *wcc* operator $T : X \to Y$ is weakly compact. This happens if and only if every operator $T : X \to Y$, such that $T^{**}(\mathcal{B}_1(X)) \subset Y$, satisfies $T^{**}(X^{**}) \subset Y$ (see, e.g., [11, Proposition 9.4.9]). Clearly, any weakly compact operator is *wcc* and also any *cc* operator is *wcc*. So the Dieudonné property implies (RDP). It follows from Rosenthal's ℓ_1-theorem ([26]) that all spaces not containing ℓ_1 have property (D). The identity operator on $L_1([0, 1])$ is an example of a *wcc* operator which is not *cc*, since L_1 is weakly sequentially complete space without the Schur property (see, e.g., [17, pp. 16–18]). To the best of our knowledge, it is still unknown, whether (D) and (RDP) are equivalent.

A series $\sum_n x_n$ in X is called *weakly unconditionally Cauchy* (*wuC*) if $\sum_n |x^*(x_n)| < \infty$ for every $x^* \in X^*$. We say that an operator $T : X \to Y$ is *unconditionally converging* (*uc*) if it sends every *wuC* series into an unconditionally convergent series. This is the same as saying that X does not contain a subspace isomorphic to c_0 on which T is an isomorphism (see, e.g., [7, p. 37]). Banach space X is said to have *Pełczyński's property (V)* if, for any Banach space Y, every *uc* operator $T : X \to Y$ is weakly compact. Using the Orlicz-Pettis theorem ([8, p. 24]), it is easy to see that every *wcc* operator is *uc*. Therefore, every Banach space with property (V) has property (D). The converse does not hold generally (see, e.g., Example 3.32 below). Examples of Banach spaces with property (V) include all reflexive spaces, $C(K)$ spaces ([21, Theorem 1]), L_1-preduals ([18]) and C^*-algebras ([24]). For more information on these and other isomorphic properties of Banach spaces, we refer to [27].

The relative topology induced on X by restricting the Mackey topology $\tau(X^{**}, X^*)$ will be termed the *Right$_X$ topology* (or simply Right if the space X is obvious). Let us recall that the Mackey topology $\tau(X^{**}, X^*)$ is the finest locally convex topology for the dual pair $\langle X^{**}, X^* \rangle$. It is the topology of uniform convergence on absolutely convex $\sigma(X^*, X^{**})$-compact subsets of X^*. Since X^* is a Banach space, it follows from the theorem of Krein (see, e.g., [28, Chapter IV, Theorem 11.4]) that the closed absolutely convex hull of a relatively weakly compact subset of X^* is weakly compact. So $\tau(X^{**}, X^*)$ can also be viewed as the topology of uniform convergence on relatively $\sigma(X^*, X^{**})$-compact subsets of X^*. The space X^{**} is complete in the $\tau(X^{**}, X^*)$-topology (see [29, Proposition 1.1]). In reflexive spaces, $\tau(X^{**}, X^*)$-topology agrees with the norm topology. For more information on topological vector spaces, we refer to [11] or [28].

A linear map between Banach spaces is bounded if and only if it is Right-to-Right continuous ([23, Lemma 12]). An operator $T : X \to Y$ is called *pseudo weakly compact* (*pwc*) if it transforms Right-null sequences into norm-null sequences. Banach space X is said to be *sequentially Right (SR)* if, for any Banach space Y, every *pwc* operator $T : X \to Y$ is weakly compact. The following theorem has been proved in [23].

Theorem 2.1 ([23, Corollary 5])**.** *Let $T : X \to Y$ be an operator. Then the following assertions are equivalent:*

(i) *T is Right-to-norm continuous,*
(ii) *$T \restriction_{B_X}$ is Right-to-norm continuous,*
(iii) *T is weakly compact,*
(iv) *$T^{**} : X^{**} \to Y^{**}$ is $\tau(X^{**}, X^*)$-to-norm continuous.*

Clearly, every weakly compact operator is *pwc*. The converse does not hold, as the identity operator on ℓ_1 shows ([23, Example 8]). In fact, no infinite-dimensional Schur space can be sequentially Right. Since every *pwc* operator is *uc* ([23, Proposition 14]), every Banach space with property (V) is sequentially Right ([23, Corollary 15]).

We say that an operator $T : X \to Y$ is *Right completely continuous* (*Rcc*) if it maps Right-Cauchy sequences into Right-convergent sequences. Let us denote by $\mathcal{R}_1(X)$ the subspace of X^{**} formed by all $\tau(X^{**}, X^*)$-limits of Right-Cauchy sequences in X. Clearly, $\mathcal{R}_1(X) \subset \mathcal{B}_1(X)$. We call a set $K \subset X^*$ an *R-set* if for any Right-null sequence (x_n) in X one has $\lim_n \sup_{x^* \in K} x^*(x_n) = 0$. Banach space X is said to have the *Right Dieudonné property (RD)* if, for any Banach space Y, every *Rcc* operator $T : X \to Y$ is weakly compact.

3. Main results

For better clarity, we start with the scheme of classification of operators we will shortly establish:

$$\begin{array}{ccccc} \text{weakly compact} & \searrow\ \nearrow & pwc & \searrow & \\ & & & & Rcc \to uc \\ cc & \nearrow\ \searrow & wcc & \nearrow & \end{array}$$

For Banach space properties we will have:

$$\begin{array}{ccccc} & \nearrow & \text{(SR)} & \searrow & \\ \text{(V)} \to \text{(RD)} & & & & \text{(RDP)} \\ & \searrow & \text{(D)} & \nearrow & \end{array}$$

The following lemma will be used implicitly throughout this paper without further mentioning.

Lemma 3.1. *Let (X, τ_X) and (Y, τ_Y) be two topological vector spaces. Then a linear map $T : X \to Y$ maps τ_X-null sequences into τ_Y-null sequences if and only if T maps τ_X-Cauchy sequences into τ_Y-Cauchy sequences.*

In particular, if τ_1 and τ_2 are two vector topologies on X, then every τ_1-null sequence in X is τ_2-null if and only if every τ_1-Cauchy sequence in X is τ_2-Cauchy.

Proof. Notice first that a sequence (x_n) in a topological vector space is Cauchy if and only if for every increasing sequence of natural numbers $j_n < k_n < j_{n+1}$, the sequence $(x_{k_n} - x_{j_n})$ converges to zero. Suppose T maps τ_X-null sequences into τ_Y-null sequences. Let (x_n) be a τ_X-Cauchy sequence in X. If $j_n < k_n < j_{n+1}$ is an arbitrary increasing sequence of natural numbers, then $(x_{k_n} - x_{j_n})$ converges to zero in X and hence $(T(x_{k_n}) - T(x_{j_n}))$ converges to zero in Y. It follows that $(T(x_n))$ is a τ_Y-Cauchy sequence.

On the other hand, suppose T maps τ_X-Cauchy sequences into τ_Y-Cauchy sequences and let (x_n) be τ_X-null. Then the sequence $0, x_1, 0, x_2, \dots$ is also τ_X-null, hence it is τ_X-Cauchy and so the sequence $0, T(x_1), 0, T(x_2), \dots$ is τ_Y-Cauchy. By the observation in the beginning of the proof, $(T(x_n) - 0) = (T(x_n))$ is τ_Y-null.

The special case follows by considering the identity map $T : (X, \tau_1) \to (X, \tau_2)$. □

Proposition 3.2. *Let X, Y be Banach spaces and $T : X \to Y$ an operator. Then the following assertions hold:*

(i) If T is completely continuous, then it is pseudo weakly compact.
(ii) If T is pseudo weakly compact, then it is Right completely continuous.

Proof. Assertion (i) is trivial. As for (ii), since T is *pwc*, every Right-Cauchy sequence in X is mapped into norm-Cauchy and therefore norm-convergent sequence in Y. Since Right$_Y$ topology is weaker than norm, the assertion follows. □

Corollary 3.3. *Every Banach space with property (RD) is sequentially Right. Every sequentially Right Banach space has property (RDP).*

Proposition 3.4. *Let X be a Banach space. The following assertions hold:*

(a) *For any Banach space Y, an operator $T : X \to Y$ is Rcc if and only if $T^{**}(\mathcal{R}_1(X)) \subset Y$.*
(b) *X has property (RD) if and only if, for any Banach space Y, any operator $T : X \to Y$ such that $T^{**}(\mathcal{R}_1(X)) \subset Y$, satisfies $T^{**}(X^{**}) \subset Y$.*

Proof. This proposition is a special case of more general [11, Proposition 9.4.9]. We refer the reader to its proof. ☐

Corollary 3.5. *Every wcc operator is Rcc. Every Banach space with property (RD) has property (D).*

Proof. The assertions follow from the characterizations given in Proposition 3.4, since $\mathcal{R}_1(X) \subset \mathcal{B}_1(X)$ and T being a *wcc* operator is equivalent to $T^{**}(\mathcal{B}_1(X)) \subset Y$. ☐

Corollary 3.6. *Let X be a Banach space such that $\mathcal{R}_1(X) = X^{**}$. Then the space X has property (RD).*

Proof. Follows immediately from Proposition 3.4(b). ☐

Remark 3.7. The condition in Corollary 3.6 is satisfied if, for example, the unit ball $B_{X^{**}}$ is metrizable in the $\tau(X^{**}, X^*)$ topology. Characterizations of such Banach spaces can be found in [29].

Proposition 3.8. *Let X and Y be two Banach spaces. If Y is a quotient space of X (in particular, if Y is a complemented subspace of X or if Y is isomorphic to X) and X is sequentially Right (resp. has property (RD)), then Y has the same property.*

Proof. Let $q : X \to Y$ be the quotient map from X to Y. Suppose X is sequentially Right (resp. has property (RD)). Then for any *pwc* (resp. *Rcc*) operator $T : Y \to Z$, where Z is a Banach space, $T \circ q$ is a *pwc* (resp. an *Rcc*) and thus a weakly compact operator on X. Since q is a quotient map, by the open mapping theorem T is weakly compact. ☐

Proposition 3.9. *Let X be a Banach space and let $Y \subset X$ be its closed subspace. Then Right$_Y$ is finer than Right$_X \restriction_Y$ and both topologies coincide if there is a weakly continuous extension map $T : Y^* \to X^*$, i.e., a weakly continuous map T such that for every $y^* \in Y^*$ one has $T(y^*) \restriction_Y = y^*$ (in particular, if Y is complemented in X or if $X = Y^{**}$).*

Proof. We denote by $i : Y \to X$ the natural inclusion. Since every operator is Right-to-Right continuous by [23, Lemma 12], i is Right$_Y$-to-Right$_X$ continuous. Hence, Right$_Y$ is finer than Right$_X \restriction_Y$.

Suppose there is a weakly continuous extension map $T : Y^* \to X^*$. Let $(y_\alpha) \subset Y$ be a Right$_X$-null net and let K be an arbitrary weakly compact subset of Y^*. Then $T(K)$ is weakly compact in X^* and we have

$$\lim_\alpha \sup_{y^* \in K} |y^*(y_\alpha)| = \lim_\alpha \sup_{y^* \in K} |T(y^*)(y_\alpha)| = \lim_\alpha \sup_{x^* \in T(K)} |x^*(y_\alpha)| = 0.$$

Hence, (y_α) is Right$_Y$-null, which was to show.

If Y is complemented in X, then an extension operator is given by $T : y^* \mapsto y^* \circ p$, $y^* \in Y^*$, where $p : X \to Y$ is a continuous projection of X onto Y. If $X = Y^{**}$, then the natural inclusion of Y^* into Y^{***} provides the required extension. ☐

Corollary 3.10. *Let X, Y be Banach spaces and let $T : X \to Y$ be a pwc (resp. Rcc) operator. Then for every closed subspace $Z \subset X$, $T \restriction_Z : Z \to Y$ is pwc (resp. Rcc).*

Proof. Let (z_n) be a Right$_Z$-Cauchy sequence in Z. Then, by Proposition 3.9, (z_n) is Right$_X$-Cauchy and the assumption of the operator T finishes the proof. □

Corollary 3.11. *A Banach space X is sequentially Right (resp. has property (RD)) if every separable subspace has the same property.*

Proof. Let $T : X \to Y$ be a *pwc* (resp. *Rcc*) operator and let (x_n) be a sequence in B_X. We need to show that there is a subsequence such that $(T(x_{n_k}))$ is weakly convergent in Y. Put $Z := \overline{\text{span}}\{x_n : n \in \mathbb{N}\}$. Then Z is a separable subspace of X and by Corollary 3.10, $T\restriction_Z$ is *pwc* (resp. *Rcc*). Using the assumption, $T\restriction_Z$ is weakly compact and therefore there exists the sought subsequence. It follows that T is weakly compact. □

Remark 3.12. Corollary 3.11 cannot be reversed. Indeed, consider ℓ_1 as a subspace of $C([0, 1])$. By [21, Theorem 1], $C(K)$ spaces have property (V). Corollary 3.18 below shows that property (V) implies property (RD). However, ℓ_1 does not even possess property (RDP).

Lemma 3.13. *Let X be a Banach space and (x_n^{**}) a w^*-null sequence in X^{**}. The following assertions are equivalent:*

(i) $x_n^{**} \to 0$ *in the* $\tau(X^{**}, X^*)$ *topology.*
(ii) $\lim_n x_n^{**}(x_n^*) = 0$ *for every weakly null sequence* (x_n^*) *in* X^**.*
(iii) $\lim_n x_n^{**}(x_n^*) = 0$ *for every weakly Cauchy sequence* (x_n^*) *in* X^**.*
(iv) *The operator* $T : X^* \to c_0$ *given by* $T(x^*) = (x_n^{**}(x^*))_{n\in\mathbb{N}}$ *is completely continuous.*

In case $(x_n^{**}) \subset X$*, the statements above are equivalent to* $x_n^{**} \overset{\text{Right}_X}{\to} 0$ *and the operator* T *in* (iv) *moreover satisfies* $T^*(\ell_1) \subset X$*.*

Proof. Suppose (i) holds and let (x_n^*) be a weakly null sequence in X^*. Since $\{x_n^* : n \in \mathbb{N}\}$ is a relatively weakly compact subset of X^*, (x_n^{**}) converges to zero uniformly on $\{x_n^* : n \in \mathbb{N}\}$. This proves (ii).

For (ii) ⇒ (iii), let (x_n^*) be a weakly Cauchy sequence in X^*. If (iii) does not hold then by passing to a subsequence if necessary we may assume that $|x_n^{**}(x_n^*)| > \varepsilon$ for some $\varepsilon > 0$ and all $n \in \mathbb{N}$. Since (x_n^{**}) is w^*-null, there is an increasing sequence of natural numbers (k_n) such that $|x_{k_n}^{**}(x_{k_{n-1}}^*)| < \frac{\varepsilon}{2}$. Now $(x_{k_n}^* - x_{k_{n-1}}^*)$ is weakly null in X^*, but

$$|x_{k_n}^{**}(x_{k_n}^* - x_{k_{n-1}}^*)| = |x_{k_n}^{**}(x_{k_n}^*) - x_{k_n}^{**}(x_{k_{n-1}}^*)| > \frac{\varepsilon}{2},$$

which is a contradiction.

Let T be defined as in (iv). Assuming (iii), it is easy to show that (x_n^{**}) converges to zero uniformly on every weakly Cauchy sequence in X^*. Let (x_n^*) be such a sequence. Then $\lim_n \sup_k |x_n^{**}(x_k^*)| = 0$. A quick computation now shows that $(T(x_n^*))$ is norm Cauchy in c_0. Thus T is completely continuous.

Finally, to prove (iv) ⇒ (i), let K be a weakly compact subset of X^* and let T be as in (iv). Since T is completely continuous, $T(K)$ is norm compact in c_0. By a well-known characterization of compact sets in c_0, $\lim_{n\to\infty} \sup_{x^*\in K} |(T(x^*))_n| = 0$. So x_n^{**} converges uniformly to zero on K. Assertion (i) now follows.

The last statement follows immediately from the definition of the Right$_X$ topology and the fact that the operator T^* maps $(t_n) \in \ell_1$ to $\sum_n t_n x_n^{**}$. □

For any Banach space X, the Right_X topology is the weakest locally convex topology τ that makes every weakly compact operator, with X as its domain, τ-to-norm continuous. Indeed, suppose τ is a locally convex topology on X that is strictly weaker than Right_X. Then there is a semi-norm p on X which is continuous with respect to the Right_X topology, but not with respect to τ (see, e.g., [28, p. 48]). According to [22, Proposition 2.2], we can assume p is of the form $p(x) = \|T(x)\|$, $x \in X$, where $T : X \to Y$ is an operator into a reflexive space Y. Clearly, T is weakly compact, but not τ-to-norm continuous. For sequential continuity we can state the following.

Proposition 3.14. *Let X be a Banach space and let τ be a locally convex topology on X compatible with the duality $\langle X, X^*\rangle$ and weaker than Right_X. Then the following assertions are equivalent:*

- *(i) For any Banach space Y, every weakly compact operator $T : X \to Y$ is τ-to-norm sequentially continuous.*
- *(ii) Topologies τ and Right_X coincide sequentially on X.*

Proof. By [16, Theorem 2], every weakly compact operator $T : X \to Y$ is τ-to-norm sequentially continuous if and only if for any weakly null sequence (x_n^*) in X^* and any τ-null sequence (x_n) in X we have $\lim_n x_n^*(x_n) = 0$. Using Lemma 3.13 and the fact that τ is stronger than $\sigma(X, X^*)$, it is the same as saying that every τ-null sequence is Right-null. This completes the proof. □

Proposition 3.15 (cf. [14, Proposition 1 bis])**.** *For a Banach space X, the following assertions are equivalent:*

- *(i) X has the Dunford-Pettis property.*
- *(ii) Topologies $\sigma(X, X^*)$ and Right_X coincide sequentially.*
- *(iii) Every (relatively) $\sigma(X, X^*)$-compact subset of X is (relatively) Right_X-compact.*
- *(iv) For any Banach space Y, every pseudo weakly compact operator $T : X \to Y$ is completely continuous.*

Proof. The equivalence (i) ⇔ (ii) is just a restatement of Proposition 3.14 with $\tau = \sigma(X, X^*)$. Equivalence of (ii) and (iii) follows from the fact that both topologies $\sigma(X, X^*)$ and Right_X are angelic ([25, Definition 0.2]), in particular, from the fact that every subset of X is (relatively) compact if and only if it is (relatively) sequentially compact in the respective topologies (see [25, Theorem 1.2]). Trivially, (ii) ⇒ (iv) and using Theorem 2.1, (iv) ⇒ (i). □

As a direct consequence we have:

Corollary 3.16. *Let X be a Banach space with the Dunford-Pettis property. Then the following assertions hold:*

- (a) *For any Banach space Y, an operator $T : X \to Y$ is pseudo weakly compact if and only if it is completely continuous.*
- (b) *X is sequentially Right if and only if it has property (RDP).*
- (c) *For any Banach space Y, an operator $T : X \to Y$ is Right completely continuous if and only if it is weakly completely continuous.*
- (d) *X has property (RD) if and only if it has property (D).*
- (e) *$\mathcal{R}_1(X) = \mathcal{B}_1(X)$.*

Remark 3.17. While condition (a) of Corollary 3.16 actually implies the Dunford-Pettis property (see Proposition 3.15(iv)), this is not true for conditions (b)–(e). Indeed, just consider an arbitrary infinite-dimensional reflexive space.

The next corollary improves [23, Proposition 14 and Corollary 15].

Corollary 3.18. *Every Rcc operator is uc. Every Banach space with property (V) has property (RD).*

Proof. Let $T : X \to Y$ be an *Rcc* operator between two Banach spaces. Suppose T is not unconditionally converging. Then there is an injection $I : c_0 \to X$ such that $T \circ I$ is an isomorphism (see, e.g., [8, p. 54]). Let us denote by (e_n) the unit vector basis in c_0. The sequence $(\sum_{k=1}^{n} e_k)_n$ is weakly Cauchy but not weakly convergent in c_0. So the isomorphism $T \circ I$ is not a *wcc* operator. Since c_0 has the Dunford-Pettis property (see, e.g., [8, p. 113]), by Corollary 3.16(c) $T \circ I$ is not an *Rcc* operator. This, however, contradicts the assumption. The second statement is immediate. □

Proposition 3.19. *Let X be a Banach space and let $K \subset X^*$ be a bounded subset. The following assertions are equivalent:*

(i) K is an R-set.
(ii) The $\sigma(X^*, X)$-closed absolutely convex hull of K is an R-set.
(iii) Every completely continuous operator $T : X^* \to c_0$ such that $T^*(\ell_1) \subset X$ maps K into a relatively compact subset of c_0.
(iv) For every $\varepsilon > 0$ there is an R-set $K_\varepsilon \subset X^*$ such that

$$K \subset K_\varepsilon + \varepsilon B_{X^*}.$$

Proof. Let us start with (i) ⇒ (ii). We denote by A the $\sigma(X^*, X)$-closed absolutely convex hull of K. It is easily seen that A is an R-set if and only if every countable subset of A is an R-set. It is also easy to see that an absolutely convex hull of an R-set is an R-set. Without loss of generality, we may assume that K is absolutely convex. Suppose (ii) does not hold. Then there is a Right-null sequence (x_n) in X and a sequence (x_n^*) in A such that $x_n^*(x_n) > \varepsilon$ for all $n \in \mathbb{N}$ and some $\varepsilon > 0$. For every n, since x_n^* is in the w^*-closure of K, there is $y_n^* \in K$ such that $y_n^*(x_n) > \varepsilon$. Since $\{y_n^* : n \in \mathbb{N}\}$ is not an R-set, neither is K. Converse implication (ii) ⇒ (i) is trivial.

(i) ⇔ (iii): Lemma 3.13 shows there is one to one correspondence between Right-null sequences in X and completely continuous operators $T : X^* \to c_0$ with $T^*(\ell_1) \subset X$. Indeed, if (x_n) is a Right-null sequence in X, then the corresponding operator T is defined as in Lemma 3.13(iv). Conversely, if T is such an operator and (e_n) the unit basis in ℓ_1 then $(T^*(e_n))$ defines the Right-null sequence in X corresponding to T (again by Lemma 3.13).

If K is an R-set and T as in (iii), then by the observation above $(T^*(e_n))$ is a Right-null sequence in X. Hence

$$0 = \lim_n \sup_{x^* \in K} |\langle T^*(e_n), x^* \rangle| = \lim_n \sup_{x^* \in K} |\langle e_n, T(x^*) \rangle|.$$

By the well-known characterization of compact sets in c_0, $T(K)$ is relatively compact.

If, on the other hand, we suppose (iii) is true and (x_n) is a Right-null sequence in X, then the corresponding operator T maps K into a relatively compact set.

Again, the characterization of compact subsets of c_0 gives the uniform convergence of (x_n) to zero on K. Thus (iii) $\Rightarrow$ (i).

(iv) $\Rightarrow$ (iii): Suppose (iv) holds. Let T be as in (iii). Then, for every $\varepsilon > 0$,

$$T(K) \subset T(K_\varepsilon) + \varepsilon T(B_{X^*}) \subset T(K_\varepsilon) + \varepsilon \|T\| B_{c_0},$$

and $T(K_\varepsilon)$ is relatively compact. Hence, $T(K)$ is relatively compact (see, e.g., [12, p. 275]).

The implication (i) $\Rightarrow$ (iv) is obvious. □

Proposition 3.20. *Let X and Y be Banach spaces and let $T : X \to Y$ be an operator. The following assertions are equivalent:*

(i) T is pseudo weakly compact.

(ii) $T^*(B_{Y^*})$ is an R-set.

Proof. Assume (i) holds. Let (x_n) be a Right-null sequence in X. Then

$$\lim_n \sup_{x^* \in T^*(B_{Y^*})} |\langle x^*, x_n \rangle| = \lim_n \sup_{y^* \in B_{Y^*}} |\langle y^*, T(x_n) \rangle| = \lim_n \|T(x_n)\| = 0.$$

This implies (ii). The argument above can be reversed to obtain (ii) $\Rightarrow$ (i). □

Remark 3.21. An analogue of the Gantmacher's theorem (see, e.g., [19, Theorem 3.5.13]) does not hold for pseudo weakly compact operators. Consider the identity operator $i : c_0 \to c_0$. The space c_0 and all of its duals have the Dunford-Pettis property (see, e.g., [7, p. 19]). Using Corollary 3.16(a), the identity operator on a Banach space with the Dunford-Pettis property is *pwc* if and only if the space is Schur. Thus we see immediately that i is not *pwc*, while $i^* : \ell_1 \to \ell_1$ is, and again both i^{**} and i^{***} are not *pwc*. We remark that the space ℓ_∞^* is not a Schur space, because its predual contains ℓ_1 (see [7, p. 23]). The only conclusion in this direction is a consequence of Corollary 3.10: *An operator T is pwc if T^{**} is pwc.*

A set U in a Hausdorff topological vector space (X, τ) is called sequentially open if for every sequence $(x_n) \subset X$ converging to a point $x \in U$, x_n belongs to U eventually. I.e., if the complement of U is sequentially closed. The space X is said to be *C-sequential* if every convex sequentially open subset of X is open. We refer to [32] and [34] for more information on C-sequential spaces.

We say that the topological vector space X is a *Ck-space* if for each convex set $A \subset X$, the set A is open in X provided that $A \cap K$ is open in K for any compact subset K of X.

Lemma 3.22. *Let (X, τ) be a Hausdorff topological vector space such that the class of compact subsets of X coincides with the class of sequentially compact subsets of X. Then X is C-sequential if and only if X is a Ck-space.*

Proof. Suppose first that X is C-sequential. Let A be a convex subset of X such that $A \cap K$ is open in K for every compact $K \subset X$. To prove that A is open, it suffices to show that A is sequentially open. Let $(x_n) \subset X$ be a sequence converging to some $x \in A$. Then the set $L = \{x, x_1, x_2, \ldots\}$ is compact in X. By the assumption, $A \cap L$ is open in L and thus there is $n_0 \in \mathbb{N}$ such that $x_n \in A$ for all $n \geq n_0$. So A is sequentially open and hence open in X. This shows that X is a Ck-space.

Assume now that X is a Ck-space. Let U be a convex sequentially open subset of X. Consider a compact subset $K \subset X$ such that $K \not\subset U$. We want to show that $U \cap K$ is open in K, or equivalently, that $(X \setminus U) \cap K$ is closed in K. Let (x_n)

be a sequence in $(X \setminus U) \cap K$. Since K is sequentially compact in X, there is a subsequence (x_{n_k}) converging to a point $x \in K$. Since the set $X \setminus U$ is sequentially closed, $x \in X \setminus U$. This shows that $(X \setminus U) \cap K$ is sequentially compact in X and hence compact in X. Since X is a Hausdorff space, $(X \setminus U) \cap K$ is a closed set in X (see, e.g., [12, Theorem 3.1.8]). So we have shown that $U \cap K$ is open in K for every compact set $K \subset X$. Since X is a Ck-space, U is open. □

Theorem 3.23. *Let X be a Banach space. The following assertions are equivalent:*

(i) *X is sequentially Right.*
(ii) *Every pseudo weakly compact operator $T : X \to \ell_\infty$ is weakly compact.*
(iii) *Every R-subset of X^* is relatively $\sigma(X^*, X^{**})$-compact.*
(iv) (X, Right_X) *is C-sequential.*
(v) (X, Right_X) *is a Ck-space.*

Proof. (ii) ⇒ (i): Suppose there is a Banach space Y and an operator $T : X \to Y$ which is *pwc* but not weakly compact. Then there is an operator $U : Y \to \ell_\infty$ such that $U \circ T$ is not weakly compact (see [8, Chapter VII, Exercise 6]). Obviously, $U \circ T$ is *pwc*. But this contradicts (ii).

(i) ⇒ (iii): Let K be an R-set in X^*. Denote by $B(K)$ the space of all bounded real valued functions on K with the norm $\|f\| = \sup_{x^* \in K} |f(x^*)|$. The operator $T : X \to B(K)$, defined by $Tx(x^*) = x^*(x)$, for any $x \in X$ and $x^* \in K$, is easily seen to be *pwc*, since K is an R-set. By the assumption (i), T is weakly compact, and therefore also T^* is weakly compact (see, e.g., [19, Theorem 3.5.13]). For any $x^* \in K$, if we define $F \in B(K)^*$ by $F(f) = f(x^*)$, then $\|F\| = 1$ and $T^*(F) = x^*$. So $K \subset T^*(B_{B(K)^*})$, but the latter set is relatively weakly compact. Hence, K is relatively weakly compact. (Cf. the proof of [21, Proposition 1].)

(iii) ⇒ (ii): Let $T : X \to \ell_\infty$ be a *pwc* operator. By Proposition 3.20, $T^*(B_{\ell_\infty^*})$ is an R-set, and so by (iii) it is relatively weakly compact. Hence T^*, and therefore T, is weakly compact.

The equivalence (i) ⇔ (iv) follows from Theorem 2.1 and the fact that a topological vector space X is C-sequential if and only if, for any Banach space Y, every sequentially continuous operator $T : X \to Y$ is continuous (see [32, Theorem 2]).

The equivalence (iv) ⇔ (v) is a consequence of Lemma 3.22 and the fact that the topology Right_X is angelic (see [25, Theorem 1.2]). □

The next corollary generalizes [21, Corollary 5] stating that every Banach space with property (V) has weakly sequentially complete dual.

Corollary 3.24. *If X is a sequentially Right Banach space, then X^* is weakly sequentially complete.*

Proof. Let (x_n^*) be a weakly Cauchy sequence in X^*. Using Lemma 3.13, it is easy to show that any Right-null sequence in X converges to zero uniformly on $K := \{x_n^* : n \in \mathbb{N}\}$. Thus K is an R-set. Since X is sequentially Right, by Theorem 3.23, K is relatively weakly compact. Hence (x_n^*) is weakly convergent in X^*. This shows that X^* is weakly sequentially complete. □

In the previous paragraphs we have seen that coinciding of the Right topology with the weak one sequentially is just another characterization of the Dunford-Pettis property. Now we take a look at the other extreme: the norm topology. Let X be a Banach space. Using Theorem 2.1, we can clearly see by looking at the identity operator that the Right topology coincides with the norm toplogy on X if and only if X is reflexive. According to J. Borwein ([4]), Banach space X is called *sequentially reflexive* provided the Mackey topology $\tau(X^*, X)$ coincides sequentially with the norm topology on X^*. A result of P. Ørno ([20]) says that X *is sequentially reflexive if and only if X contains no copy of ℓ_1.*

Proposition 3.25. *A Banach space X is reflexive if and only if it is sequentially Right and X^* is sequentially reflexive.*

Proof. The necessity is trivial. Let us show sufficiency. From the definition, if X^* is sequentially reflexive then the Right_X topology coincides sequentially with the norm topology. Hence the identity operator on X is *pwc*. Since X is also sequentially Right, the identity is weakly compact and therefore X is reflexive. □

In spite of the fact that, by Rosenthal's ℓ_1-theorem ([26]), the next corollary is a weaker version of Corollary 3.24, we demonstrate an alternative proof using Proposition 3.25.

Corollary 3.26. *Let X be a sequentially Right Banach space. Then either*

(i) X is reflexive, or
(ii) X^* contains a copy of ℓ_1.

Proof. By Proposition 3.25, a non-reflexive sequentially Right Banach space cannot have sequentially reflexive dual. Using the result of P. Ørno [20], X^* must contain a copy of ℓ_1. □

Example 3.27. Although non-containment of ℓ_1 in X^* characterizes sequential coincidence of $\tau(X^{**}, X^*)$ and the norm topology on X^{**}, this condition is too strong to characterize sequential coincidence of the Right_X and the norm topology on X. This example shows there is a Banach space which contains (even complemented) copy of ℓ_1 in its dual, yet the Right and norm topologies coincide sequentially.

Indeed, the first Bourgain-Delbaen space X constructed in [5] is a non-reflexive Schur space whose dual is weakly sequentially complete (and as such contains ℓ_1 by [26]). In fact, the dual space X^* is isomorphic to $M([0, 1])$, the Banach space of Radon measures on $[0, 1]$.

Since, of course, X as a Schur space is not sequentially Right, this also shows that Corollary 3.24 cannot be reversed.

Remark 3.28. There is, in general, no connection between 'sequential Rightness' of a Banach space X and its bidual X^{**}. The classical chain of sequence spaces $c_0, \ell_1, \ell_\infty, \ell_\infty^*$, shows that both can have the same sequential Rightness. The space from Example 3.27 is not sequentially Right, but its bidual is isomorphic to a $C(K)$ space (see, e.g., [17, p. 20]) and thus has even property (V) ([21, Theorem 1]). On the other hand, the Banach space $X = (\sum \oplus \ell_1^n)_{c_0}$ has property (V) though its bidual $X^{**} = (\sum \oplus \ell_1^n)_{\ell_\infty}$ contains a complemented copy of ℓ_1 (see [27, p. 389]).

Proposition 3.29. *Let X be a Banach space. The following assertions are equivalent:*

(i) The Right_X topology coincides sequentially with the norm topology on X.
(ii) Every (relatively) Right_X-compact subset of X is (relatively) norm-compact.
(iii) For any Banach space Y, every operator $T : X \to Y$ is pseudo weakly compact.

(iv) *B_{X^*} is an R-set.*
(v) *Every bounded subset of X^* is an R-set.*
(vi) *For any Right-null sequence (x_n) in X and any bounded sequence (x_n^*) in X^* one has $\lim_n x_n^*(x_n) = 0$.*
(vii) *Every completely continuous operator $T : X^* \to c_0$ such that $T^*(\ell_1) \subset X$ is compact.*

Proof. (i) $\Leftrightarrow$ (ii): Assume (i). If $K \subset X$ is a (relatively) Right$_X$-compact set, then, by [25, Theorem 1.2], K is (relatively) sequentially Right$_X$-compact. Using the assumption (i), K is (relatively) sequentially norm-compact and hence (relatively) norm-compact. The converse is obvious, since every Right$_X$-null sequence is relatively Right$_X$-compact.

The implication (i) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (iv): If (iii) holds and we consider the identity operator on X then, by Proposition 3.20, B_{X^*} is an R-set.

Since every subset of an R-set is an R-set, the implication (iv) $\Rightarrow$ (v) is obvious.

(v) $\Rightarrow$ (vi): Assume (v) and let (x_n) and (x_n^*) be as in (vi). Then

$$\lim_n |x_n^*(x_n)| \leq \lim_n \sup_{k\in\mathbb{N}} |x_k^*(x_n)| = 0,$$

since $\{x_n^* : n \in \mathbb{N}\}$ is a bounded set in X^* and so, by the assumption, an R-set.

(vi) $\Rightarrow$ (i): If (i) does not hold, then there is a Right-null sequence (x_n) in X which does not converge to zero in norm. Hence there is a sequence (x_n^*) in B_{X^*} such that $(x_n^*(x_n))$ does not converge to zero. This contradicts (vi).

The equivalence of (iv) and (vii) follows from Proposition 3.19, where we put $K := B_{X^*}$. □

Corollary 3.30. *A Banach space X is a Schur space if and only if X has the Dunford-Pettis property and B_{X^*} is an R-set.*

Proof. The space X is Schur if and only if the weak and norm topologies coincide sequentially on X, i.e., if and only if the Right$_X$ topology coincides with both the weak and the norm topology sequentially. Combining Proposition 3.15 with Proposition 3.29 yields the requested equivalence. □

Remark 3.31. Let us only remark that X^* *is a Schur space if and only if X has the Dunford-Pettis property and X contains no copy of ℓ_1* (see [7, p. 23]).

Concerning compactness, it follows from [30, Proposition 3.1] that $B_{X^{**}}$ *is $\tau(X^{**}, X^*)$-compact if and only if X^* is a Schur space.* The situation is different for the Right$_X$ topology. Indeed, if B_X is Right$_X$-compact, then it is weakly compact and so X must be reflexive. Since in reflexive spaces the Right$_X$ and norm topologies coincide, X is necessarily finite-dimensional.

Now we return back to the classification of operators and Banach spaces. Here, for the convenience of the reader, we summarize the relations we have already established. There is generally no connection between weakly compact and *cc* operators. The identity operator on ℓ_2 is an example of a weakly compact operator that is not *cc*, the identity on ℓ_1 is a non-weakly compact *cc* operator. That weakly compact operators are both *pwc* and *wcc* has been mentioned in Preliminaries. Every *cc* operator is trivially *wcc*. By Proposition 3.2, every *cc* operator is *pwc* and all *pwc* operators are *Rcc*. Corollary 3.5 states that all *wcc* operators are *Rcc* and Corollary 3.18 that every *Rcc* is *uc*. The identity on L_1 provides an example of a *wcc* operator which is not *pwc*, since L_1 has the Dunford-Pettis property and so we can use Corollary 3.16. What remains is to show that there is a *pwc* operator which is not *wcc* (see Example 3.33 below) and a *uc* operator that is not *Rcc* (see Example 3.32 below).

As for Banach space properties, (V) $\Rightarrow$ (RD) is shown in Corollary 3.18, (RD) $\Rightarrow$ (SR) and (SR) $\Rightarrow$ (RDP) in Corollary 3.3 and (RD) $\Rightarrow$ (D) in Corollary 3.5. (D) $\Rightarrow$ (RDP) is mentioned in Preliminaries. Examples 3.32 and 3.33 below show (RD) $\not\Rightarrow$ (V) and (D) $\not\Rightarrow$ (SR), respectively.

Example 3.32. Let Y be the second Bourgain-Delbaen space constructed in [5]. It is a non-reflexive Banach space with the Dunford-Pettis property that does not contain c_0 or ℓ_1 and its dual is isomorphic to ℓ_1.

Since Y does not contain ℓ_1, it has the Dieudonné property. As a Dunford-Pettis space, it has also property (RD) by Corollary 3.16(d). However, since Y is not reflexive and does not contain c_0, it cannot possess property (V) (see [21, Proposition 8]). This answers the question raised in [22] and [35] whether every sequentially Right Banach space has property (V).

The identity operator $i : Y \to Y$ is clearly *uc*, since Y does not contain a copy of c_0. Since Y is not reflexive and does not contain ℓ_1, it cannot be weakly sequentially complete (by [26]). Hence, i is not *wcc*. By Corollary 3.16(c), i is not *Rcc*.

Example 3.33. In [15], R.C. James constructed a separable non-reflexive Banach space X isomorphic to its bidual. In particular, since X^{**} is separable, neither X nor X^* contains an isomorphic copy of ℓ_1.

Since X does not contain ℓ_1, X has property (D). By Corollary 3.26, X cannot be sequentially Right.

Since the dual space X^* does not contain ℓ_1, the Right$_X$ topology coincides with the norm topology on X sequentially (see the comments preceding Proposition 3.25). The identity operator $i : X \to X$ is therefore *pwc*. However, since X is neither reflexive nor contains ℓ_1, X is not weakly sequentially complete and hence i is not *wcc*.

Remark 3.34. The only loose end left is an example for (SR) $\not\Rightarrow$ (D). As far as we know, the implication (RDP) $\not\Rightarrow$ (D) has been an open problem ever since it was introduced by A. Grothendieck in [14]. The implication (SR) $\not\Rightarrow$ (RD) seems to be analogical.

4. Vector-valued continuous functions

For a compact Hausdorff space K and a Banach space X we denote by $C(K, X)$ the Banach space of all X-valued continuous functions defined on K, endowed with the supremum norm. It is a long-standing open problem whether the space $C(K, X)$ has property (V) (resp. (D), (RDP)) whenever X has the same property (see [27]). For the Dunford-Pettis property this has been shown to be false by M. Talagrand (see [33]). However, if the compact space K is scattered, then $C(K, X)$ *has property (V) (resp. (D), (RDP), (DP)) if and only if X has the same property* (see [6]). Recall that a compact space K is scattered if every subset A of K has a point relatively isolated in A. The aim of this section is to show that the equivalence above holds also for properties (RD) and (SR). We use the same ideas and techniques as in [6].

Let K be a compact Hausdorff space and X a Banach space. We denote by $\mathcal{B}$ the σ-algebra of Borel subsets of K. It is well-known that the dual space $C(K,X)^*$ is isometrically isomorphic to the Banach space $M(K,X^*)$ of all regular countably additive X^*-valued measures of bounded variation defined on the σ-algebra $\mathcal{B}$ and equipped with the variation norm $\|m\| = |m|(K)$. In fact, for any Banach space Y and any operator $T : C(K,X) \to Y$, there is a finitely additive set function $m : \mathcal{B} \to L(X, Y^{**})$, from $\mathcal{B}$ to the space of all operators from X to Y^{**}, having finite semi-variation $\widehat{m}(K)$ with $\widehat{m}(K) = \|T\|$ such that

$$T(f) = \int_K f\,dm \quad \text{for every } f \in C(K,X)$$

(see, e.g., [9, p. 182]). This set function m is called the *representing measure of* T. We recall that the semi-variation of m is defined by

$$\widehat{m}(E) = \sup\Big\{\Big\|\sum_{i=1}^{n} m(E_i)(x_i)\Big\| : E_i \in \mathcal{B},\ E_i \subset E,\ \{E_i\}_{i=1}^{n} \text{ pairwise disjoint,}$$
$$x_i \in B_X,\ i = 1,\dots,n,\ n \in \mathbb{N}\Big\}, \quad E \in \mathcal{B}$$

(see [2, p. 217]). The semi-variation $\widehat{m}$ is said to be continuous at $\emptyset$ if $\lim_{n\to\infty} \widehat{m}(E_n) = 0$ for every decreasing sequence $E_n \searrow \emptyset$ in $\mathcal{B}$, or equivalently, if there exists a control measure for $\widehat{m}$, that is, a positive countably additive regular Borel measure λ on K such that $\lim_{\lambda(E)\to 0} \widehat{m}(E) = 0$.

The representing measure m determines an extension $\widehat{T} : B(\mathcal{B}, X) \to Y^{**}$ of T, where $B(\mathcal{B}, X)$ denotes the Banach space of all strongly measurable functions on $\mathcal{B}$ with values in X, i.e., the Banach space of all functions $g : K \to X$ which are the uniform limit of a sequence of $\mathcal{B}$-simple functions, endowed with the supremum norm, given by

$$\widehat{T}(g) = \int_K g\,dm, \quad g \in B(\mathcal{B}, X),$$

with $\|\widehat{T}\| = \|T\|$ (see [2, Theorem 1]). This extension is just the restriction to $B(\mathcal{B}, X)$ of the biadjoint T^{**} of T.

It has been shown in [10, Theorem 3] that if T is unconditionally converging then m is $L(X,Y)$-valued and $\widehat{m}$ is continuous at $\emptyset$. In this case, by [2, Theorem 2], the extension $\widehat{T}$ maps $B(\mathcal{B}, X)$ into Y.

In the following we consider a compact Hausdorff space K and Banach spaces X, Y.

Proposition 4.1. *Let K be metrizable. Then an operator $T : C(K,X) \to Y$ is Rcc if and only if its extension $\widehat{T} : B(\mathcal{B}, X) \to Y^{**}$ is Rcc.*

Proof. Let $T : C(K,X) \to Y$ be an *Rcc* operator. Then, by Corollary 3.18, T is *uc* and so m is $L(X,Y)$-valued with a control measure λ and $\widehat{T}$ is Y-valued. Let (g_n) be a Right-Cauchy sequence in $B(\mathcal{B}, X)$ and let $y^{**} \in Y^{**}$ be the $\tau(Y^{**}, Y^*)$-limit of $(\widehat{T}(g_n))$ (recall that by [23, Lemma 12] every operator is Right-Right continuous and so the sequence $(\widehat{T}(g_n))$ is Right-Cauchy in Y, hence $\tau(Y^{**}, Y^*)$-convergent in Y^{**}).

Suppose, for contradiction, that $y^{**} \notin Y$. Since y^{**} is not $\sigma(Y^*, Y)$-continuous, by Grothendieck's completeness theorem ([28, Chapter IV, Theorem 6.2]) it is not

$\sigma(Y^*, Y)$-continuous on B_{Y^*}. Hence there exist $\varepsilon > 0$ and a net $(y^*_\alpha) \subset B_{Y^*}$ which is $\sigma(Y^*, Y)$-convergent to zero such that

$$|y^{**}(y^*_\alpha)| > \varepsilon \text{ for all } \alpha. \tag{1}$$

Choose $\delta > 0$, $\delta < \lambda(K)$, so that

$$\widehat{m}(E) < \frac{\varepsilon}{4 \sup \|g_n\|} \text{ for each } E \in \mathcal{B} \text{ with } \lambda(E) < \delta.$$

According to Lusin's theorem, for every $n \in \mathbb{N}$, there exists a compact set $K_n \subset K$ such that $\lambda(K \setminus K_n) < \frac{\delta}{2^n}$ and the restriction $g_n \restriction_{K_n}$ is continuous. Put $K_0 := \bigcap_{n=1}^\infty K_n$. Then $\lambda(K \setminus K_0) < \delta$ and $K_0 \neq \emptyset$ since $\delta < \lambda(K)$. Let us denote $f_n := g_n \restriction_{K_0}$ for every $n \in \mathbb{N}$.

We show that (f_n) is Right-Cauchy in $C(K_0, X)$. Consider the restriction operator $r : B(\mathcal{B}, X) \to B(\mathcal{B} \restriction_{K_0}, X)$. Since (g_n) is Right-Cauchy in $B(\mathcal{B}, X)$, (f_n) is Right-Cauchy in $B(\mathcal{B} \restriction_{K_0}, X)$. Every measure $\mu \in M(K_0, X^*) = C(K_0, X)^*$ can be naturally extended to an element of $B(\mathcal{B} \restriction_{K_0}, X)^*$. Using Proposition 3.9, (f_n) is Right-Cauchy in $C(K_0, X)$.

By the Borsuk-Dugundji theorem (see, e.g., [31, Theorem 21.1.4]), there is an extension operator $S : C(K_0, X) \to C(K, X)$, with $\|S\| = 1$, so that $S(f) \restriction_{K_0} = f$ for every $f \in C(K_0, X)$. Since $T \circ S$ is an *Rcc* operator, $(TS(f_n))$ is Right$_Y$-convergent to an element $y \in Y$. Since (y^*_α) is $\sigma(Y^*, Y)$-convergent to zero there exists an index α_0 so that

$$|y^*_\alpha(y)| < \frac{\varepsilon}{6} \text{ for all } \alpha \geq \alpha_0.$$

Let $\alpha \geq \alpha_0$. There is $n \in \mathbb{N}$ verifying

$$|\langle \widehat{T}(g_n) - y^{**}, y^*_\alpha \rangle| < \frac{\varepsilon}{6} \text{ and } |\langle TS(f_n) - y, y^*_\alpha \rangle| < \frac{\varepsilon}{6}.$$

Thus we have

$$\begin{aligned}
|y^{**}(y^*_\alpha)| &\leq |\langle y^{**} - \widehat{T}(g_n), y^*_\alpha \rangle| + |\langle \widehat{T}(g_n) - TS(f_n), y^*_\alpha \rangle| \\
&\quad + |\langle TS(f_n) - y, y^*_\alpha \rangle| + |\langle y, y^*_\alpha \rangle| \\
&< \frac{\varepsilon}{2} + \|y^*_\alpha\| \|\widehat{T}(g_n) - TS(f_n)\| \\
&\leq \frac{\varepsilon}{2} + \Big\| \int_{K \setminus K_0} g_n - S(f_n) \, dm \Big\| \\
&\leq \frac{\varepsilon}{2} + 2\|g_n\| \widehat{m}(K \setminus K_0) < \varepsilon.
\end{aligned}$$

But this contradicts (1).

Conversely, if $\widehat{T} : B(\mathcal{B}, X) \to Y^{**}$ is *Rcc* then, by Corollary 3.10, $T : C(K, X) \to Y^{**}$ is *Rcc*. Hence, every Right-Cauchy sequence (f_n) in $C(K, X)$ is mapped into a Right-convergent sequence in Y^{**}. Since Right$_{Y^{**}}$-topology is compatible with the norm topology and $(T(f_n))$ is contained in the closed convex set $Y \subset Y^{**}$, the limit point y of $(T(f_n))$ must be a member of Y (see, e.g., [28, Chapter IV, 3.1]). Now, Proposition 3.9 implies that $T(f_n) \to y$ in the Right$_Y$-topology. □

Proposition 4.2. *Let K be metrizable. Then an operator $T : C(K, X) \to Y$ is pseudo weakly compact if and only if its extension $\widehat{T} : B(\mathcal{B}, X) \to Y^{**}$ is pseudo weakly compact.*

Proof. Let $T : C(K,X) \to Y$ be a *pwc* operator. By Proposition 3.2(ii) and Corollary 3.18, T is *uc*. Let m and λ be as in the proof of Proposition 4.1. Let (g_n) be a Right-null sequence in $B(\mathcal{B},X)$. Suppose, for contradiction, that $\widehat{T}$ is not *pwc*. Without loss of generality we may assume that there is $\varepsilon > 0$ so that

$$\|\widehat{T}(g_n)\| > \varepsilon \text{ for all } n \in \mathbb{N}. \tag{2}$$

Choose $\delta > 0$, $\delta < \lambda(K)$, verifying

$$\widehat{m}(E) < \frac{\varepsilon}{4 \sup \|g_n\|} \text{ for each } E \in \mathcal{B} \text{ with } \lambda(E) < \delta.$$

Reasoning as in the proof of Proposition 4.1, there exist a non-empty compact set $K_0 \subset K$ with $\lambda(K \setminus K_0) < \delta$ such that $f_n = g_n \restriction_{K_0}$ is continuous for all $n \in \mathbb{N}$ and an isometric extension operator $S : C(K_0,X) \to C(K,X)$. By the same argument as in the proof of Proposition 4.1, (f_n) is Right-null in $C(K_0,X)$. So $TS(f_n) \to 0$ in Y and there exists $n_0 \in \mathbb{N}$ such that

$$\|TS(f_n)\| < \frac{\varepsilon}{2} \text{ for all } n \geq n_0.$$

Thus if $n \geq n_0$ one has

$$\begin{aligned}
\|\widehat{T}(g_n)\| &\leq \|\widehat{T}(g_n) - TS(f_n)\| + \|TS(f_n)\| \\
&< \left\| \int_{K \setminus K_0} g_n - S(f_n)\, dm \right\| + \frac{\varepsilon}{2} \\
&\leq 2\|g_n\| \widehat{m}(K \setminus K_0) + \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

But this contradicts (2).

The converse follows from Corollary 3.10. □

Lemma 4.3 ([6, Lemma 6])**.** *Let K be a metrizable scattered compact space and let $T : C(K,X) \to Y$ be an operator whose representing measure m verifies*

(i) $m(\mathcal{B}) \subset L(X,Y)$,
(ii) $m(E) : X \to Y$ is weakly compact for each $E \in \mathcal{B}$,
(iii) $\widehat{m}$ is continuous at $\emptyset$.

Then T is weakly compact.

Theorem 4.4. *Suppose that K is scattered. Then $C(K,X)$ is sequentially Right (resp. has property (RD)) if and only if X has the same property.*

Proof. The necessity follows from Proposition 3.8, since X can be identified with a complemented subspace of $C(K,X)$.

For the sufficiency, assume that X is sequentially Right (resp. has property (RD)) and $T : C(K,X) \to Y$ is a *pwc* (resp. *Rcc*) operator.

(A) Suppose first that K is metrizable. Since T is *uc*, by [10, Theorem 3] its representing measure m satisfies conditions (i) and (iii) of Lemma 4.3. For each $E \in \mathcal{B}$ we define an operator $\Phi_E : X \to B(\mathcal{B},X)$ by $\Phi_E(x) = x\chi_E$, $x \in X$, where χ_E is the characteristic function of E on K. It follows from Proposition 4.2 (resp. 4.1) that the operator $m(E) = \widehat{T} \circ \Phi_E : X \to Y$ is *pwc* (resp. *Rcc*) and so, since X is sequentially Right (resp. has property (RD)), $m(E)$ is weakly compact. Therefore, all conditions of Lemma 4.3 are satisfied and thus T is weakly compact.

(B) For a general K, let (f_n) be an arbitrary sequence in the unit ball of $C(K,X)$. The method used in [2, p. 236] shows there is a subspace H of $C(K,X)$ such that

$(f_n) \subset H$ and H is isometric to some $C(L, X)$, where L is a compact metric space and a quotient space of K. Since a metrizable quotient space of a scattered space is scattered (see [31, Proposition 8.5.3]), L is scattered. Corollary 3.10 in conjunction with the part (A) of this proof shows that $T \restriction_H$ is weakly compact. So there is a subsequence (f_{n_k}) of (f_n) such that $(T(f_{n_k}))$ is weakly convergent in Y. This shows that T is weakly compact. □

Analogues of Propositions 4.1 and 4.2 for *cc*, *wcc* and *uc* operators and *general* compact Hausdorff space K have been shown in [3]. The arguments of [3] cannot be employed here, since, unlike the weak topology, the Right topology is not preserved under subspaces in general. We do not know whether the metrizability assumption in Propositions 4.1 and 4.2 can be dropped completely. In the rest of this paper we show, however, that it is possible under the *Continuum Hypothesis (CH)* or if the weight of K is at most $\aleph_1$.

Let M and N be arbitrary Hausdorff topological spaces and let F be a map from M to non-empty subsets of N. We say that F is *upper semi-continuous* (*usc*) if $\{m \in M : F(m) \cap C \neq \emptyset\}$ is closed for every closed subset C of N. A map $f : M \to N$ is called a selection for F if $f(m) \in F(m)$ for all $m \in M$. The weight $w(M)$ of the topological space M is the smallest cardinality of a base for the topology of M. We denote by $\mathcal{B}(M)$ the σ-algebra of Borel subsets of M. If M is completely regular in addition then $\mathcal{B}_0(M)$ will be the σ-algebra of Baire subsets of M, i.e., the σ-algebra generated by the zero-sets of continuous functions on M. We recall that if M is a normal space then the zero-sets of continuous functions on M are precisely the closed G_δ-subsets of M and if M is a metric space then $\mathcal{B}_0(M) = \mathcal{B}(M)$ (see, e.g., [31, Proposition 6.5.2]).

Lemma 4.5. *Let (g_n) be a Right-null sequence in $B(\mathcal{B}(K), X)$. Let K_0 be a compact subset of K such that $g_n \restriction_{K_0} \in C(K_0, X)$ for all $n \in \mathbb{N}$. Assume (CH) or $w(K_0) \leq \aleph_1$. Then there is a Right-null sequence $(\tilde{f}_n)$ in $C(K, X)$ such that $\|\tilde{f}_n\| \leq \|g_n\|$ and $\tilde{f}_n(t) = g_n(t)$ for every $t \in K_0$ and $n \in \mathbb{N}$.*

Proof. Put $f_n := g_n \restriction_{K_0}$ for all $n \in \mathbb{N}$. We have already shown in the proof of Proposition 4.1 that (f_n) is Right-null in $C(K_0, X)$.

We will continue by employing the method from [2, p. 236]. Let us define the pseudo-metric p (see, e.g., [1, p. 15] for the definition of pseudo-metric) on K_0 by

$$p(t, t') = \sum_{n=1}^{\infty} 2^{-n} \|f_n(t) - f_n(t')\|, \quad t, t' \in K_0.$$

Let L be the set of equivalence classes τ of K_0 under the relation: $t \sim s$ if and only if $p(t, s) = 0$. The continuous mapping $\phi : t \mapsto \tau$ of a point $t \in K_0$ into its equivalence class is a continuous mapping from K_0 onto L and thus L is a compact metric space equipped with the metric $\rho(\tau, \tau') = p(t, t')$, $t \in \tau$, $t' \in \tau'$. The mapping $i : h \mapsto h \circ \phi$ defines an isometric embedding of $C(L, X)$ into $C(K_0, X)$. We denote by H the image of $C(L, X)$ in $C(K_0, X)$ under i. Clearly, $f_n \in H$ for all $n \in \mathbb{N}$.

Now we show that (f_n) is Right-null in H. Consider the multi-valued map $F : L \to 2^{K_0}$ defined by $F(\tau) = \phi^{-1}(\tau)$, $\tau \in L$. Since ϕ is continuous, F is compact-valued and *usc*. If we assume (CH) (resp. $w(K_0) \leq \aleph_1$) then, by [13, Theorem 7] (resp. [13, Theorem 3]), there exists a $\mathcal{B}(L)$-$\mathcal{B}_0(K_0)$-measurable selection φ for F. It is easy to verify that every continuous function $f \in C(K_0, X)$ is a uniform limit of $\mathcal{B}_0(K_0)$-simple functions. Since $\Phi : g \mapsto g \circ \varphi$ defines a bounded linear map from the normed vector space of all $\mathcal{B}_0(K_0)$-simple functions to $B(\mathcal{B}(L), X)$, extending Φ by continuity to all of $B(\mathcal{B}_0(K_0), X)$ and then restricting to $C(K_0, X)$ provides an operator, denoted again by Φ, from $C(K_0, X)$ into $B(\mathcal{B}(L), X)$ such that $\Phi(f) \in C(L, X)$ and $i(\Phi(f)) = f$ for every $f \in H$. Hence, $(\Phi(f_n))$ is Right-null in $B(\mathcal{B}(L), X)$ and so, by the same argument as in the proof of Proposition 4.1, Right-null in $C(L, X)$. Since $f_n = i(\Phi(f_n))$ for all $n \in \mathbb{N}$, (f_n) is Right-null in H.

The theorem of Arens [1, Theorem 4.2] (put $A := K_0, X := K, F := B_H, K := B_X, L := X$ and $q := p$) yields an extension operator $S : H \to C(K, X)$ with $\|S\| = 1$. Defining $\tilde{f_n} := S(f_n)$, $n \in \mathbb{N}$, finishes the proof. □

Proposition 4.6. *Assume (CH) or $w(K) \leq \aleph_1$. Then Propositions 4.1 and 4.2 hold without the metrizability assumption.*

Proof. The only reason for the metrizability asumption on K in Propositions 4.1 and 4.2 was the Borsuk-Dugundji theorem. We used this theorem only to obtain the conclusion of Lemma 4.5. □

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