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Věnováno mé mamince Iloně

a mému tatínkovi Václavovi

Poděkování

Tato práce vznikala za poněkud netradičních okolností. Během mého magisterského studia se mi poštěstilo nabýt příležitosti vycestovat za poznáním a zkušenostmi do zahraničí. Kombinací různých okolností, v nichž sehráli roli kvalitní reference naší fakulty v zahraničí, elementy náhody a snad i mé vlastní přičinění, se mi můj pobyt poněkud protáhl. Dané okolnosti mi ale nemohly zabránit v úmyslu dokončit můj program v Praze, což byl pro mě od prvního kontaktu s naší fakultou vyčkávaný moment zhodnocení mého úsilí, jež nebylo zanedbatelné. V tomto momentu musím upřímně poděkovat vedení fakulty, které mi v té době umožnilo absolvovat některé zbylé povinnosti se skromnější přítomností v Praze.

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Veronika Ročková
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Abstrakt:
Časové řady sestávající z nezáporných pozorování se hojně vyskytují v praxi napříč vědními disciplínami. Nezápornost daných pozorování lze využít k odvození speciálních metod odhadu, které mohou konvergovat rychleji než klasické silně konzistentní odhady. Metody odhadu v modelech nezáporných časových řad však musí zohlednit podmínky, za kterých daný model skutečně odpovídá nezáporným náhodným veličinám. Podmínky nezápornosti pak mohou sloužit kupříkladu při odvození omezujících podmínek popisujících obor přípustných řešení při optimizační úloze. V této práci jsou shrnuty podmínky nezápornosti pro $ARMA$ modely, které zahrnují jak výsledky už dřívé odvozené, tak nově formulované. V diskuzi se zaměřujeme především na jednorozměrné časové řady. Krátce je ale věnována pozornost i mnohorozměrným modelům.

Klíčová slova: Podmínky nezápornosti, $ARMA$ procesy, absolutně monotonní funkce

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Abstract:
Models for non-negative time series find their usefulness in many diverse areas of applications (hydrology, medicine, finance). The non-negative nature of the observations has been utilized for deriving estimators with superior asymptotic properties. For the purposes of estimation, it is necessary to recognize the situations when the estimated model indeed defines a non-negative time series. Such non-negativity conditions can then be used as a basis for constrained optimization. The main thrust of this work is to review the non-negativity conditions currently available for $ARMA$ models and, more importantly, to generalize the existing results for some models for which the explicit result was missing. We center our discussion mainly on univariate models. However, we note that the pursued ideas are directly applicable also for multivariate time series. This observation enables determination of some readily obtainable conditions for lower order vector valued Autoregressive Moving Average models.

Keywords: Conditions for nonnegativity, $ARMA$ processes, absolutely monotone functions

Contents

List of Symbols

Ω	The set of elementary events
A	Sigma-algebra of subsets of Ω
P	The probability measure defined on $\mathcal{A}$
Z	Set of integer numbers
N	Set of positive integer numbers
R	Set of real numbers
C	Set of complex numbers
card A	Number of elements in a set A
Re(x)	Real part of a complex number x
Im(x)	Imaginary part of a complex number x
E X	Expectation of a random variable X
Var X	Variance of a random variable X
`iid`	Independent and identically distributed
A$\geq$ 0	Matrix A with non-negative entries
$\mathrm{A}^{(q)}$	q-th power of matrix A
tr(A)	Trace of matrix A
det(A)	Determinant of matrix A
adj(A)	The adjugate of the matrix A
I_p	Identity matrix od dimension p

Chapter 1

Introduction

Many time series arising in practice are intrinsically non-negative, in the sense that the nature of observed phenomena does not allow negative values. Non-negative time series occur in many different domains of applications involving many seemingly distant disciplines. Increasing demand for statistical methodologies which can model adequately such observations has been observed in hydrological applications. Many aspects of hydrological cycle, e.g. rainfall runoff, precipitation or streamflow, have been subjected to non-negative time series models, see e.g. Hutton (1990). Another area of applications with cumulative occurence of non-negative time series is financial data, see Tsay (2005).

In the context of autoregressive moving average models, the established methodology based on the assumptions of normality (in the innovation sequence) is no longer applicable to adequately represent the non-negative time series. Instead, more suitable innovation distributions have been sucessively accommodated in the autoregressive/moving average schemes. The associated estimation theory have been developed by several authors: Anděl and Garrido (1991), Anděl (1988a), Anděl (1988b), Anděl (1990), Datta and McCormick (1995), Bell and Smith (1986).

Parameter estimation in non-negative autoregressive moving average models raises some additional challenges. Even under the assumption of positive innovations, the model parameters need to satisfy certain conditions so that the fitted model indeed defines a valid non-negative random process. Such non-negativity conditions can be then directly incorporated in the estimation scheme, in that the solution can be obtained by solving a constrained optimization problem. If not included in estimation, the conditions for non-negativity can be used for a post-hoc verification that our fitted model indeed corresponds to a non-negative time series. Our interest here will be centered almost exclusively on the non-negativity conditions. For an overview of the estimation techniques in non-negative time series we refer to the last chapter in this thesis and to the cited literature.

The practical utility of conditions for non-negativity is recognized mainly in analysis of financial time series. One important application is modeling volatility of as-set return. A popular econometric model for volatility modeling is the Generalized Autoregressive Conditional Heteroscedastic ($GARCH$) model (Engle (1982), Bollerslev (1986)). Recently, considerable effort has been made to identify conditions under which the process of conditional variances in the $GARCH$ model is non-negative almost surely. Nelson and Cao (1992) present a set of necessary and sufficient conditions for the non-negativity of a lower order $GARCH(p, q)$ process and a sufficient condition for the general $GARCH(p, q)$ model. Tsai and Chan (2008) showed that this conditions was also necessary. Recently, non-negativity conditions for hyperbolic $GARCH$ model were provided by Conrad (2010).

Despite the majority of the up-to-date literature devoted to the conditions for non-negativity in time series deals with the $GARCH$ model or some modification thereof, we believe that the conditions for Autoregressive Moving Average ($ARMA$) processes are equally important. However, the literature on the non-negativity conditions for $ARMA$ models is far more sparse.

In this thesis we give an overview of the state-of-the-art results on this topic. These involve mainly results for lower order univariate autoregressive and autoregressive moving average models based on the connection between the non-negativity of a kernel sequence and absolute monotonicity of its generating function. We aspired to extend the set of existing results by deriving conditions for higher order univariate $ARMA$ models. Our methodological strategy consists of two inferential approaches. The first one produces conditions expressed in terms of roots of autoregressive lag polynomial. This approach has been utilized to derive non-negativity conditions for lower order AR models, Tsai and Chan (2007). We demonstrate that this strategy enables derivation of tractable non-negativity conditions also for $ARMA$ models, namely $ARMA(3, 1)$ and $ARMA(3, 2)$. These two models are not the single ones for which the explicit result was missing. To the best of our knowledge, the conditions for $ARMA(2, 1)$ models have also not been derived yet. We present an explicit result for this model, which appears as a special case of the conditions we deduce for $ARMA(2, q)$ and finally also for $ARMA(p, q)$ models. The latter results were derived using the second approach, which rests purely on the similarity between $ARMA$ and $GARCH$ models. We exploited the existing results for $GARCH$ models to derive analogous conditions for the general $ARMA$ model.

Finally, we benefit from our experience acquaired in the univariate setting to elaborate on conditions in multivariate time series. These are based on the observation that the absolute monotonicity argument applies also for multivariate time series, just with the coefficients replaced by matrices. We present a set of necessary and sufficient conditions for two-variate $AR(1)$, $ARMA(1, 1)$ and $ARMA(q, 1)$ models. Whereas the result for $AR(1)$ model has been derived previously by Anděl (1992), just by different argument, the two later results are rather new.

We start our discussion with the theoretical introduction into time series methodology, Chapter 2. The univariate conditions for the non-negativity are dealt in Chapter 3. The treatment of multivariate conditions is postponed until Chapter 4. In Chapter 5 we briefly discuss estimation techniques in non-negative time series. We wrap up with a discussion on further research topics in Chapter 6.

Chapter 2

Theoretical Set up

In this section we introduce the basic theoretical background for (vector-valued) autoregressive moving average time series, following the line of the book of Brockwell and Davis (1986).

2.1 Univariate Stationary ARMA Processes

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $T \equiv \mathbb{Z}$ be an index set. Denote $X(t, \omega)$ a real valued function defined on $T \times \Omega$ such that for each given $t \in T$, $X(t, \omega)$ is a real random variable on $(\Omega, \mathcal{A}, P)$. Let X_t denote the random variable $X(t, \omega)$. Then a real valued **time series** is defined as a collection of random variables $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$. For a fixed $\omega \in \Omega$, $X(t, \omega)$ is a real valued function in t known as the **realization function**. Given $\omega \in \Omega$, the sequence $\{X(t, \omega) : t \in T\}$ is called a **realization of a time series** $\{X_t : t \in T\}$. In the following text we will use the term time series to characterize both the sequence of random variables and a sequence of realizations. We believe that the proper meaning will be clear from the context.

Two important probabilistic concepts are often distinguished in the time series literature: the strict stationarity and the weak stationarity.

Definition 2.1.1. For a given $n \in \mathbb{N}$ and a set of indices $i_1, \dots, i_n$ denote $F_{i_1,\dots,i_n}(\cdot)$ the joint distribution function of a random vector $(X_{i_1}, \dots, X_{i_n})'$. The time series $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ is said to be **strictly stationary** if

$$F_{i_1,\dots,i_n}(x_1, \dots, x_n) = F_{i_1+h,\dots,i_n+h}(x_1, \dots, x_n),$$

for all $n \in \mathbb{N}$, any set of integer indices $i_1, \dots, i_n$ and any real numbers $x_1, \dots, x_n$.

The weak stationarity relaxes the distributional assumptions as it operates only with first and second order moments.

Definition 2.1.2. The time series $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ is said to be **weakly stationary** if

(1) $\mathsf{E}X_t^2 < \infty \quad (t = 0, \pm 1, \pm 2, \dots)$,

(2) $\mathsf{E}X_t \equiv c \quad (t = 0, \pm 1, \pm 2, \dots)$,

(3) $\mathsf{E}(X_{t+h} - \mathsf{E}X_{t+h})(X_t - \mathsf{E}X_t)$ is independent of t for any choice of $h \in \mathbb{Z}$.

Remark 2.1.1. In what will follow, by a stationary random process we always refer to a weakly stationary random process.

One important class of time series appears as a stationary solution to a certain set of linear difference equations involving white noise.

Definition 2.1.3. A sequence of random variables $\{Z_t : t = 0, \pm 1, \pm 2, \dots\}$ is said to be **white noise** with mean μ and variance σ^2, written as $\{Z_t\} \sim \mathrm{WN}(\mu, \sigma^2)$, if and only if the variables $Z_t, t \in \mathbb{Z}$, are uncorrelated and have mean μ and variance σ^2.

Remark 2.1.2. The white noise sequence $\{Z_t : t = 0, \pm 1, \pm 2, \dots\}$ is sometimes called a sequence of inovations. We do use these two terms interchangeably.

Definition 2.1.4. The time series $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ is said to be an **autoregressive moving average process** of orders p and q (abbreviated as $ARMA(p, q)$) if it is stationary and if for every $t \in \mathbb{Z}$

$$X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}, \qquad (2.1.1)$$

where $\{Z_t : t = 0, \pm 1, \pm 2, \dots\}$ is the white noise with mean μ and variance σ^2 and $\phi_1, \dots, \phi_p$, $\theta_1, \dots, \theta_q$ are real coefficients.

Remark 2.1.3. It is customary to assume that the mean μ of the white noise sequence is equal to zero. Nevertheless, later in the text we will work with $ARMA$ processes with non-negative innovations. That is why we introduced a more general definition.

Remark 2.1.4. The **autoregressive process** of order p (abbreviated as $AR(p)$) refers to a special case of $ARMA(p, q)$ when $q = 0$. The **moving average process** of order q (abbreviated as $MA(q)$) refers to a special case of $ARMA(p, q)$ when $p = 0$.

The equation in (2.1.1) is usually written compactly using the back-shift operator B given by $B^j X_t = X_{t-j}$ and the lag polynomials

$$\boldsymbol{\phi}(z) = 1 - \phi_1 z - \dots - \phi_p z^p$$

and

$$\boldsymbol{\theta}(z) = 1 + \theta_1 z + \dots + \theta_p z^q$$

as follows:

$$\boldsymbol{\phi}(B)X_t = \boldsymbol{\theta}(B)Z_t, \quad t = 0, \pm1, \pm2, \ldots.$$

The polynomials $\boldsymbol{\phi}(\cdot)$ and $\boldsymbol{\theta}(\cdot)$ are called autoregressive and moving average lag polynomials respectively. In practice it is often possible to describe the behavior of the observed time series using elementary time series like white noise. These are known as linear time series, as they represent the series as an (infinite) linear combination of the white noise sequence. This property relates to the concept of causality of a stationary random process.

Definition 2.1.5. The $ARMA(p, q)$ process $\{X_t : t = 0, \pm1, \pm2, \ldots\}$ defined by (2.1.1) is said to be **causal** if there exists a sequence of constants $\{\psi_j\}_{j=0}^{\infty}$ such that $\sum_{j=0}^{\infty} |\psi_j| < \infty$ and

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \quad t = 0, \pm1, \pm2, \ldots. \tag{2.1.2}$$

The following theorem (Theorem 3.1.1 in Brockwell and Davis (1986), p. 85) gives a set of necessary and sufficient conditions for an $ARMA$ process to be causal.

Theorem 2.1.1. *Let $\{X_t : t = 0, \pm1, \pm2, \ldots\}$ be an $ARMA(p, q)$ process for which the polynomials $\boldsymbol{\phi}(\cdot)$ and $\boldsymbol{\theta}(\cdot)$ have no common zeroes. Then $\{X_t : t = 0, \pm1, \pm2, \ldots\}$ is causal if and only if $\boldsymbol{\phi}(z) \neq 0$ for all $z \in \mathbb{C}$ such that $|z| \leq 1$. The coefficients ψ_j in (2.1.2) are determined from the equation*

$$\boldsymbol{\psi}(z) = \sum_{j=0}^{\infty} \psi_j z^j = \frac{\theta(z)}{\phi(z)}, \quad |z| \leq 1. \tag{2.1.3}$$

Proof. We refer to Brockwell and Davis (1986), p. 85.

2.2 Multivariate Stationary ARMA Processes

In many practical situations the variable measured over time, a subject to a time series model, may be a part of a more complex system. Its behavior may be partially attributed to some other interacting variables. Modelling co-movements and interactions between these variables is then necessary to acquire understanding of the underlying mechanisms in such a system. Multivariate time series are uniquely suited to capture these complex relationships.

By m**-variate time series** $\{\boldsymbol{X}_t = (X_{t1}, \ldots, X_{tm})' : t = 0, \pm1, \pm2, \ldots\}$ we understand a collection of m scalar time series $\{X_{ti} : t = 0, \pm1, \pm2, \ldots\}, i = 1, \ldots, m$, which are observed in parallel and encapsulated for each $t \in \mathbb{Z}$ in a vector the possibly related random variables $\{X_{t1}, \ldots, X_{tm}\}$.

The theory on univariate time series can be extended in a natural way for the multivariate setting. The concepts of stationarity and causality are analogous.

Definition 2.2.1. The m-variate series $\{\boldsymbol{X}_t = (X_{t1}, \dots, X_{tm})' : t = 0, \pm 1, \pm 2, \dots\}$ is said to be **weakly stationary** if

(1) $\mathsf{E}X_{ti}^2 < \infty \quad (i = 1, \dots, m; t = 0, \pm 1, \pm 2, \dots)$,

(2) $\mathsf{E}\boldsymbol{X}_t \equiv (\mathsf{E}X_{t1}, \dots, \mathsf{E}X_{tm})' = \boldsymbol{\mu} \quad (t = 0, \pm 1, \pm 2, \dots)$,

(3) $\mathsf{E}[(\boldsymbol{X}_{t+h} - \mathsf{E}\boldsymbol{X}_{t+h})(\boldsymbol{X}_t - \mathsf{E}\boldsymbol{X}_t)']$ is independent of t for any choice of $h \in \mathbb{Z}$.

Definition 2.2.2. A sequence of random vectors $\{\boldsymbol{Z}_t = (Z_{t1}, \dots, Z_{tm})' : t = 0, \pm 1, \pm 2, \dots\}$ is said to be m**-variate white noise** with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, abbreviated as as $\{\boldsymbol{Z}_t\} \sim \mathrm{MWN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, if and only if the random vectors $\boldsymbol{Z}_t, t \in \mathbb{Z}$, have mean $\boldsymbol{\mu}$ and for $h \in \mathbb{N}$ it holds that

$$\mathsf{E}[(\boldsymbol{Z}_{t+h} - \boldsymbol{\mu})(\boldsymbol{Z}_t - \boldsymbol{\mu})'] = \begin{cases} \boldsymbol{\Sigma} & \text{if} \quad h = 0, \\ \boldsymbol{0}, & \text{otherwise,} \end{cases}$$

where $\boldsymbol{0}$ denotes a $m \times m$ matrix with zero entries and where the matrix $\boldsymbol{\Sigma}$ is positive definite.

Definition 2.2.3. The series $\{\boldsymbol{X}_t = (X_{t1}, \dots, X_{tm})', t = 0, \pm 1, \pm 2, \dots\}$ is said to be a **vector-valued autoregressive moving average** process of orders p and q (abbreviated as $VARMA(p, q)$) if it is weakly stationary and if for every $t \in \mathbb{Z}$

$$\boldsymbol{X}_t - \Phi_1 \boldsymbol{X}_{t-1} - \dots - \Phi_p \boldsymbol{X}_{t-p} = \boldsymbol{Z}_t + \Theta_1 \boldsymbol{Z}_{t-1} + \dots + \Theta_q \boldsymbol{Z}_{t-q}, \tag{2.2.4}$$

where $\{\boldsymbol{Z}_t = (Z_{t1}, \dots, Z_{tm})' : t = 0, \pm 1, \pm 2, \dots\}$ is a m-variate white noise sequence with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ and where $\Phi_1, \Phi_2, \dots, \Phi_p, \Theta_1, \Theta_2, \dots, \Theta_q$ are real $m \times m$ matrices.

Again, the equations in (2.2.4) can be rewritten using the back-shift operator $B(\cdot)$ and matrix-valued autoregressive polynomial

$$\boldsymbol{\Phi}(z) = \mathrm{I}_m - \Phi_1 z - \dots - \Phi_p z^p$$

and moving average polynomial

$$\boldsymbol{\Theta}(z) = \mathrm{I}_m + \Theta_1 z + \Theta_2 z^2 + \dots + \Theta_q z^q$$

as follows:

$$\boldsymbol{\Phi}(B)\boldsymbol{X}_t = \boldsymbol{\Theta}(B)\boldsymbol{Z}_t.$$

The definition of a causal multivariate process is practically the same as the Definition 2.1.5. The only difference is that the coefficients $\{\psi_j\}_{j=0}^{\infty}$ in the univariate setting are now replaced by matrices $\{\Psi_j\}_{j=0}^{\infty}$.

Definition 2.2.4. The vector-valued autoregressive moving average process $\{\boldsymbol{X}_t = (X_{t1}, \ldots, X_{tm})' : t = 0, \pm 1, \pm 2, \ldots\}$ defined by (2.2.4) is said to be **causal** if there exists a sequence of matrices $\left\{\Psi_k = \left(\psi_{ij}^k\right)_{i,j=1}^m\right\}_{k=0}^{\infty}$, which is absolutely summable, i.e. $\sum_{k=0}^{\infty} |\psi_{ij}^k| < \infty \quad (i, j = 1, \ldots, m)$, and such that

$$\boldsymbol{X}_t = \sum_{k=0}^{\infty} \Psi_k \boldsymbol{Z}_{t-k}, \quad t = 0, \pm 1, \pm 2, \ldots.$$

The necessary and sufficient condition for the $VARMA$ process to be causal is summarized in the following theorem.

Theorem 2.2.1. *Let $\{\boldsymbol{X}_t = (X_{t1}, \ldots, X_{tm})' : t = 0, \pm 1, \pm 2, \ldots\}$ be a $VARMA(p, q)$ process. Then $\{\boldsymbol{X}_t : t = 0, \pm 1, \pm 2, \ldots\}$ is causal if*

$$\det[\boldsymbol{\Phi}(z)] \neq 0 \quad \textit{for all} \quad z \in \mathbb{C} \quad \textit{such that} \quad |z| \leq 1.$$

The matrices Ψ_j are then determined uniquely from

$$\boldsymbol{\Psi}(z) = \sum_{j=0}^{\infty} \Psi_j z^j = \boldsymbol{\Phi}(z)^{-1} \boldsymbol{\Theta}(z), \quad |z| \leq 1.$$

Proof. The proof can be found in Brockwell and Davis (1986), p. 408.

Chapter 3

Non-negativity Conditions for Univariate ARMA Processes

The amount of literature on non-negativity conditions for $ARMA$ models is relatively modest. Relevant work on this topic has been done by Anděl (1991), who derived necessary and sufficient conditions for the non-negativity of $AR(2)$ and $AR(1)$ models. His work falls within the framework of conditions which are formulated naturally as a set of constraints on the autoregressive model parameters. In some situations, however, it may be more convenient to express the non-negativity conditions in terms of roots of autoregressive (and/or moving average) polynomials. This methodological framework builds on the theory of absolutely monotone functions and has been introduced by Tsai and Chan (2007). The ease of verifiability of the two types of conditions is model dependent.

This section provides a compact review of the two approaches and discloses some interesting connections between them. Some of the presented results are rather new, as we have not found any equivalent in the literature published to date. Our contributions can be summarized in the following points:

- Tsai and Chan (2007) derived sufficient and necessary conditions for $AR(1), AR(2)$, $AR(3), AR(4)$ models. We provide a set of sufficient and necessary conditions for an autoregressive model of a general order.

- Tsai and Chan (2007) derived sufficient and necessary conditions for $ARMA(1, q)$ models. We provide recipes for explicit results for $ARMA(2, 1), ARMA(3, 1)$ and $ARMA(3, 2)$.

- Tsai and Chan (2007) provided only necessary conditions for $ARMA(p, q)$ processes. We argue that the conditions of Tsai and Chan (2008) for $GARCH(p, q)$ processes can be applied with only slight modifications also for $ARMA(p, q)$ models. We take advantage of this similarity and formulate an analogous set of sufficient and necessary conditions for $ARMA$ models of general orders. We will see that these conditions are relatively easy to verify for $ARMA(2, q)$ models.

Our discussion will start with the conditions for non-negativity of a general linear process.

3.1 Non-negativity Conditions for Linear Processes

We have seen in Theorem 2.1.1 that under mild assumptions on the zeroes of the lag polynomials, the $ARMA$ process can be expressed as an infinite sum of weighted lagged inovations, i.e.

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \quad (t = 0, \pm 1, \pm 2, \dots). \tag{3.1.1}$$

A similar property holds also for the conditional variances in $GARCH$ models, where squared inovations are used instead. Clearly, if the sequence of weights in the infinite linear combination is non-negative, the process of conditional variances must be also non-negative. The same would apply for the $ARMA$ process if we assumed that the inovation sequence $\{Z_t : t = 0, \pm 1, \pm 2, \dots\}$ in (3.1.1) consists of non-negative random variables.

Definition 3.1.1. A sequence of random variables $\{Z_t^* : t = 0, \pm 1, \pm 2, \dots\}$ is said to be a **non-negative inovation sequence** if the random variables Z_t^* are uncorrelated, $\mathrm{P}(Z_t^* \geq 0) = 1$ and $0 < \operatorname{var} Z_t^* < \infty$, $\forall t \in \mathbb{Z}$.

The following result of Anděl (1991) shows that when the distribution of the non-negative innovation sequence satisfies certain conditions, the non-negativity of $\{\psi_j\}_{j=0}^{\infty}$ is also a necessary condition for the non-negativity of the resulting linear process.

Theorem 3.1.1. *Let $\{Z_t^* : t = 0, \pm 1, \pm 2, \dots\}$ be a non-negative inovation sequence. Assume that the inovations are* **iid** *random variables with a distribution function $F(\cdot)$. Assume that $F(d) - F(c) < 1$ for all $0 < c < d < \infty$. If there exists an index $k \in \mathbb{Z}$ such that $\psi_k < 0$, then with probability one $X_t < 0$ for infinitely many indices $t \in \mathbb{Z}$.*

Proof. A proof can be found in Anděl (1991).

From Theorem 3.1.1 and the preceding discussion it follows that it makes enough sense to investigate conditions under which the non-negativity of the sequence $\{\psi_j\}_{j=0}^{\infty}$ holds. As will be seen in a while, the non-negativity of the "kernel" sequence $\{\psi_j\}_{j=0}^{\infty}$ can be related to the absolute monotonicity property of its generating function. Let us first recall definitions of a generating function and an absolutely monotone function.

Definition 3.1.2. Let $\{\psi_j\}_{j=0}^{\infty}$ denote a sequence of real numbers $\psi_0, \psi_1, \psi_2 \dots$. If

$$\boldsymbol{\psi}(z) = \psi_0 + \psi_1 z + \psi_2 z^2 + \dots \tag{3.1.2}$$

converges in an interval $-z_0 < z < z_0$, where $z_0 \in \mathbb{R}^+$, then $\boldsymbol{\psi}(\cdot)$ is called the **generating function** of the sequence $\{\psi_j\}_{j=0}^{\infty}$.

Definition 3.1.3. A continuous function $f(\cdot)$ is said to be **absolutely monotone** in the interval $0 \leq z < 1$ if all the derivatives $f^{(n)}(z)$ $(n \in \mathbb{N})$ are non-negative for $0 < z < 1$.

The following theorem links the non-negativity property of the sequence of real numbers with the absolute monotonicity property of its generating function.

Theorem 3.1.2. *The sequence of real numbers $\{\psi_j\}_{j=0}^{\infty}$ is non-negative if and only if its generating function $\psi(z)$ is absolutely monotone in $0 \leq z < 1$.*

Proof. The proof can be found in Feller (1971), p. 232.

Remark 3.1.1. Feller (1979), p. 232, proves even stronger statement about absolutely monotone functions, namely the equivalence of the following conditions:

(1) a continuous function $f(z)$ defined on $0 \leq z < 1$ is absolutely monotone,

(2) a continuous function $f(z)$ defined on $0 \leq z < 1$ admits a power series representation (3.1.2) with non-negative coefficients.

The linkage between the non-negativity of the weights $\{\psi\}_{j=0}^{\infty}$ in the infinite moving average representation and the absolute monotonicity of its generating function allows derivation of several easily verifiable non-negativity conditions. The sustainability of absolute monotonicity with respect to multiplication is especially helpful, as will be seen later in this section.

Theorem 3.1.3. *A product of two absolutely monotone functions is absolutely monotone.*

Proof. A proof can be found in Widder (1946), p. 145.

3.1.1 The Implications for ARMA Processes

From the discussion above, we can already draw some important implications for the non-negativity of $ARMA$ models. These will be utilized throughout next sections. Therefore we find it convenient to introduce them already at this point.

From this point onwards, whenever we mention autoregressive moving average models, we implicitly assume that the polynomials $\boldsymbol{\theta}(z)$ and $\boldsymbol{\phi}(z)$ have no common roots.

Theorem 3.1.4. *The kernel sequence $\{\psi_j\}_{j=0}^{\infty}$ in the moving average representation of the general $ARMA(p,q)$ process is non-negative if and only if its generating function*

$$\boldsymbol{\psi}(z) = \frac{\boldsymbol{\theta}(z)}{\boldsymbol{\phi}(z)} \quad (|z| < 1)$$

is absolutely monotone in $0 \leq z < 1$.

Proof. A proof follows directly from Theorem 3.1.2.

The following necessary condition for the non-negativity of the kernel sequence will be repeatedly utilized in next sections.

Theorem 3.1.5. *Let $\lambda_1, \ldots, \lambda_p$ denote the roots of the autoregressive lag polynomial $\boldsymbol{\phi}(z) = 1 - \phi_1 z - \ldots \phi_p z^p$, such that $1 < |\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_p|$. Assume that these roots are distinct. If the kernel sequence $\{\psi\}_{j=0}^{\infty}$ of the ARMA(p,q) process is non-negative, then λ_1 is real and $\lambda_1 > 1$.*

Proof. The indication of the proof was given in Tsai and Chan (2007). We will discuss it in more detail.

By the equation (4.8) in Feller (1968), p. 276, we have

$$\psi_n = \sum_{i=1}^{p} \frac{r_i}{\lambda_1^{n+1}}, \quad n \geq \max(q-p, 0) + 1, \tag{3.1.3}$$

where $r_i = -\frac{\boldsymbol{\theta}(\lambda_i)}{\boldsymbol{\phi}^{(1)}(\lambda_i)}$. First assume that $\lambda_1 \in \mathbb{R}$. We can write

$$\psi_n \lambda_1^{n+1} = r_1 + \sum_{i=2}^{p} \left(\frac{\lambda_1}{\lambda_i}\right)^{n+1} r_i.$$

Assume that a root $\lambda_i, i \in \{2, \ldots, p\}$, is real. From the assumption $|\lambda_1| < |\lambda_i|$ it holds that

$$\lim_{n\to\infty} r_i \left(\frac{\lambda_1}{\lambda_i}\right)^{n+1} = 0.$$

If a root $\lambda_i, i \in \{2, \ldots, p-1\}$, is complex and $\lambda_{i+1} = \bar{\lambda}_i$, then we have

$$\frac{\lambda_1^{n+1} r_i}{\lambda_i^{n+1}} + \frac{\lambda_1^{n+1} r_{i+1}}{\lambda_{i+1}^{n+1}} = 2\mathrm{Re}(r_i \bar{\lambda}_i^{n+1}) \left(\frac{\lambda_1}{|\lambda_i|^2}\right)^{n+1}.$$

Since, $\left|2\mathrm{Re}(r_i \bar{\lambda}_i^{n+1}) \left(\frac{\lambda_1}{|\lambda_i|^2}\right)^{n+1}\right| \leq 2|r_i| \left(\frac{|\lambda_1|}{|\lambda_i|}\right)^{n+1}$ and $\lim_{n\to\infty} 2|r_i| \left(\frac{|\lambda_1|}{|\lambda_i|}\right)^{n+1}$ is zero, we have

$$\lim_{n\to\infty} r_i \left(\frac{\lambda_1}{\lambda_i}\right)^{n+1} + r_{i+1} \left(\frac{\lambda_1}{\lambda_{i+1}}\right)^{n+1} = 0.$$

This altogether gives

$$\lim_{n\to\infty} \sum_{i=2}^{p} \left(\frac{\lambda_1}{\lambda_i}\right)^{n+1} r_i = 0,$$

which implies that

$$\lim_{n\to\infty} \frac{\psi_n \lambda_1^{n+1}}{r_1} = 1. \tag{3.1.4}$$

Now assume that $\lambda_1 < -1$. The sequence $\left\{\frac{\psi_n \lambda_1^{n+1}}{r_1}\right\}_{i=1}^{\infty}$ is of oscilating sign, meaning that $\frac{\psi_n \lambda_1^{n+1}}{r_1}$ is negative for infinitely many $n \in \mathbb{N}$. Therefore, it is not possible that the limit of this sequence equals one. This implies that if $\lambda_1 \in \mathbb{R}$ and $|\lambda_1| > 1$, it must be greater than one.

Now assume that $\lambda_1 \in \mathbb{C}$ and $\lambda_2 = \bar{\lambda}_1$. We can write

$$\psi_n |\lambda_1^{n+1}| = \frac{2\mathrm{Re}(r_1 \bar{\lambda}_1^{n+1})}{|\lambda_1^{n+1}|} + \sum_{i=3}^{p} \left(\frac{|\lambda_1|}{\lambda_i}\right)^{n+1} r_i.$$

Similarly as in the previous case, the assumption $|\lambda_1| < |\lambda_i|$ implies

$$\lim_{n\to\infty} \sum_{i=3}^{p} \left(\frac{|\lambda_1|}{\lambda_i}\right)^{n+1} r_i = 0$$

and therefore

$$\lim_{n\to\infty} \frac{\psi_n |\lambda_1^{n+1}|^2}{2\mathrm{Re}(r_1 \bar{\lambda}_1^{n+1})} = 1. \tag{3.1.5}$$

We claim that there exists an infinite number of indices $n \in \mathbb{N}$ such that $2\mathrm{Re}(r_1 \bar{\lambda}_1^{n+1})$ is negative. We prove this claim by contradiction. Assume that $2\mathrm{Re}(r_1 \bar{\lambda}_1^{n+1})$ is non-negative $\forall n \in \mathbb{N}$. This assumption is without loss of generality, since we could (without much change in the proof) assume that $2\mathrm{Re}(r_1 \bar{\lambda}_1^{n+1}) \geq 0$ for all $n \in \mathbb{N}$, which are greater than or equal to some $n_0 \in \mathbb{N}$.

Denote $r_1 = A\mathrm{e}^{\mathrm{i}\alpha}$ and $\lambda_1 = B\mathrm{e}^{\mathrm{i}\beta}$. Then we have $2\mathrm{Re}(r_1 \bar{\lambda}_1^{n+1}) = 2\cos[\alpha + (n+1)\beta]$. Note that it is not possible that $2\cos[\alpha + (n+1)\beta] = 0$ for all $n \in \mathbb{N}$. Even if we could find $n \in \mathbb{N}$ such that $2\cos[\alpha + (n+1)\beta] = 0$, then $2\cos[\alpha + (n+2)\beta]$ would not be zero. This follows from the fact that $\beta \neq k\pi, k \in \mathbb{N}$ (because we assumed $\lambda_1 \in \mathbb{C}$). That means there exist infinitely many indices $n \in \mathbb{N}$ such that $2\mathrm{Re}(r_1 \bar{\lambda}_1^{n+1}) > 0$. For those $n \in \mathbb{N}$, two situations can occur

(i) $0 < [\alpha + (n+1)\beta] \bmod 2\pi < \pi/2$, or

(ii) $3\pi/2 < [\alpha + (n+1)\beta] \bmod 2\pi < 2\pi$.

Because $\beta \neq k\pi, k \in \mathbb{N}$, there must exist integer K such that $\pi/2 \leq (K_1\beta) \bmod 2\pi \leq \pi$.

Assume $n \in \mathbb{N}$ satisfies the condition in (i). Then it holds that $\pi/2 < [\alpha + (n+K+1)\beta] \bmod 2\pi < 3\pi/2$, which implies that $2\mathrm{Re}(r_1 \bar{\lambda}_1^{n+K+1})$ is negative for such $n \in \mathbb{N}$. For those $n \in \mathbb{N}$, which satisfy (ii), we have $\pi/2 < [\alpha + (n-K+1)\beta] \bmod 2\pi < 3\pi/2$. This implies that $2\mathrm{Re}(r_1 \bar{\lambda}_1^{n-K+1})$ is negative. One way or another, there exists an infinite number of indices $n \in \mathbb{N}$, such that $2\mathrm{Re}(r_1 \bar{\lambda}_1^{n+1})$ is negative. This, however, contradicts the fact that the limit in (3.1.5) is one. Therefore λ_1 cannot be a complex number. $\square$

Remark 3.1.2. The assumption that all the roots are distinct can be relaxed. The Theorem 3.1.5 still holds, provided that the root λ_1 is of the multiplicity one. If, say, root $\lambda_i, i \in \{2, \dots, p\}$, is of the multiplicity 2, the expansion in (3.1.3) will contain an additional term of the form $\frac{a(n+1)}{\lambda_i^{n+2}}$. This term has no impact of the limit behavior in (3.1.4) and (3.1.5). A similar property holds for multiplicities higher than 2.

From now onwards, we will assume that the noise sequence in the infinite moving average representation of a causal $ARMA$ process is the non-negative inovation sequence. The non-negativity of the series $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ with such a representation then boils down to the non-negativity of the sequence $\{\psi_j\}_{j=0}^{\infty}$. In the following sections we discuss separately some interesting cases of lower order $ARMA$ models as well as general conditions for $ARMA(p,q)$ models. We start with the discussion of AR models.

3.2 Autoregressive processes

Investigating the non-negativity of the kernel sequence $\{\psi_j\}_{j=0}^{\infty}$ via the absolute monotonicity property of its generating function $\boldsymbol{\psi}(z)$ in (2.1.3) has appeared to be a fruitfull idea. Tsai and Chan (2007) derived a variety of sufficient, necessary, and sufficient and necessary conditions for $ARMA$ processes. In this section we review the existing results for lower-order autoregressive processes and we suggest necessary and sufficient conditions for a general $AR(p)$ process. We start with the simplest possible model, the $AR(1)$ model.

3.2.1 AR(1)

The $AR(1)$ process $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ with non-negative innovations is defined as a stationary solution of the following stochastic difference equations

$$X_t = \phi_1 X_{t-1} + Z_t^* \quad (t = 0, \pm 1, \pm 2, \dots),$$

where $\{Z_t^* : t = 0, \pm 1, \pm 2, \dots\}$ is the non-negative inovation sequence from Definition 3.1.1. From the Theorem 2.1.1 we know that in order the process $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ to be causal it must hold that $|\phi_1| < 1$. The fulfillment of the causality condition allows the following linear representation

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}^* \quad (t = 0, \pm 1, \pm 2,),$$

where $\psi_j = \phi_1^j$. Clearly, the kernel sequence $\{\psi_j\}_{j=0}^{\infty}$ is non-negative if and only if $\phi_1 \geq 0$. The following theorem summarizes the non-negativity of the $AR(1)$ process formally.

Theorem 3.2.1. *Let $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ be $AR(1)$ process. Denote λ_1 the root of the autoregressive lag polynomial $\boldsymbol{\phi}(z) = 1 - \phi_1 z$. The sequence $\{\psi_j\}_{j=0}^{\infty}$ is non-negative if and only if λ_1 is real and $\lambda_1 > 1$.*

Proof. In order λ_1 to be well-defined, assume that $\phi_1 \neq 0$. The root λ_1 of the autoregressive polynomial $1 - \phi_1 z = 0$ is always real and equals $\lambda_1 = \frac{1}{\phi_1}$. In order to assure the non-negativity of $\{\psi_j\}_{j=0}^{\infty}$, it is necessary and sufficient that $0 < \phi_1 < 1$. This condition is equivalent to $\lambda_1 > 1$. □

Remark 3.2.1. Another way to prove the sufficy in Theorem 3.2.1 is by exploiting the properties of absolutely monotone functions. For the $AR(1)$ proces, the generating function is the following

$$\boldsymbol{\psi}(z) = \frac{1}{1 - \frac{z}{\lambda_1}}.$$

For $0 < z < 1$ and $\lambda_1 > 0$ the $n-$th derivative

$$\boldsymbol{\psi}^{(n)}(z) = \frac{n!}{\lambda_1^n}\left(1 - \frac{z}{\lambda_1}\right)^{-(n+1)}$$

is non-negative for every $n \geq 0$. In other words, for $\lambda_1 > 1$ the function $\boldsymbol{\psi}(z)$ is absolutely monotone in $0 \leq z < 1$. By Theorem 3.1.2, this is equivalent to the non-negativity of $\{\psi_j\}_{j=0}^{\infty}$.

Remark 3.2.2. The necessity of the condition $\lambda_1 > 1$ in Theorem 3.2.1 follows also from the Theorem 3.1.5.

The non-negativity conditions for $AR(1)$ process were rather trivial. The situations gets more complicated for $AR(2)$.

3.2.2 AR(2)

The $AR(2)$ process $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ with non-negative innovations is defined as a stationary solution of the following autoregressive equations

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t^* \quad (t = 0, \pm 1, \pm 2, \dots).$$

The causality condition from Theorem 2.1.1 demands that the roots of the autoregressive lag polynomial λ_1 and λ_2 are outside the unit circle. Said in terms of autoregressive parameters ϕ_1 and ϕ_2, this translates as follows:

$$\phi_1 + \phi_2 < 1, \quad \phi_2 - \phi_1 < 1, \quad \phi_2 > -1.$$

Anděl (1991) has shown that the non-negative parametric region for $AR(2)$ process is

$$\phi_1 + \phi_2 < 1, \quad \phi_1^2 + 4\phi_2 \geq 0, \quad \phi_1 \geq 0. \tag{3.2.6}$$

The same conclusion supported by a different argument was given also by Tsai and Chan (2007). They formulated the following non-negativity conditions.

Theorem 3.2.2. *Let $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ be an $AR(2)$ process. Denote λ_1, λ_2 the roots of the autoregressive lag polynomial $\boldsymbol{\phi}(z) = 1 - \phi_1 z - \phi_2 z^2$, such that $1 < |\lambda_1| \le |\lambda_2|$. Then the sequence $\{\psi_j\}_{j=0}^{\infty}$ is non-negative if and only if λ_1 and λ_2 are real, $\lambda_1 > 1$ and $\lambda_1^{-1} + \lambda_2^{-1} \ge 0$.*

Proof. We reproduce the proof as it was given by Tsai and Chan (2007). The necessity of the condition $\lambda_1 > 1$ follows from the Theorem 3.1.5. Note that roots of a real polynomial are either all real or come in conjugate imaginary pairs. The necessity of $\lambda_2 \in \mathbb{R}$ hence comes along with the necessary condition $\lambda_1 \in \mathbb{R}$. The necessity of $\lambda_1^{-1} + \lambda_2^{-1} \ge 0$ follows from two facts: (1) the coefficient ϕ_1 of the autoregressive lag polynomial $1 - \phi_1 z - \phi_2 z^2$ equals $\lambda_1^{-1} + \lambda_2^{-1}$, (2) from the equation (3.3.5) of Brockwell and Davis (1986) p. 91 we get $\psi_1 = \phi_1$.

To prove the sufficiency of given conditions, we first assume $\lambda_2 > 1$. The generating function of the sequence $\{\psi_j\}_{j=0}^{\infty}$ is

$$\boldsymbol{\psi}(z) = \frac{1}{(1 - \frac{z}{\lambda_1})(1 - \frac{z}{\lambda_2})}. \tag{3.2.7}$$

From Remark 3.2.1 we know that for $0 < z < 1$, $\lambda_1 > 1$ and $\lambda_2 > 1$, each of the two factors in (3.2.7) is an absolutely monotone function. According to Theorem 3.1.3, the function $\boldsymbol{\psi}(z)$ is also absolutely monotone. The non-negativity of the sequence $\{\psi_j\}_{j=0}^{\infty}$ then follows from Theorem 3.1.2.

Now we prove the sufficiency of $\lambda_1 > 0$ and $1/\lambda_1 - 1/\lambda_2 > 0$ for $\lambda_2 < -1$. According to equation (4.8) in Feller (1968) p. 276 it holds that

$$\psi_n = \frac{1}{\phi_2(\lambda_1 - \lambda_2)} \left(\frac{1}{\lambda_1^{n+1}} - \frac{1}{\lambda_2^{n+1}} \right). \tag{3.2.8}$$

The coefficient ϕ_2 equals $-\frac{1}{\lambda_1 \lambda_2}$. Provided that $\lambda_1 > 1$ and $\lambda_2 < -1$, we have $\phi_2 > 0$ and also $\lambda_1 - \lambda_2 > 0$. The non-negativity of (3.2.8) is then assured whenever $\frac{1}{\lambda_1^{n+1}} - \frac{1}{\lambda_2^{n+1}} \ge 0$. This expression is always non-negative for $n = 2k$. For $n = 2k + 1$ we can rewrite it as $\frac{1}{\lambda_1^{n+1}} - \frac{1}{(-\lambda_2)^{n+1}}$, which is non-negative because we assumed $\lambda_1^{-1} + \lambda_2^{-1} \ge 0$. □

Remark 3.2.3. The assumption $\lambda_1^{-1} + \lambda_2^{-1} \ge 0$ in Theorem 3.2.2 is somewhat redundant. For $\lambda_1 > 1$ and $\lambda_2 > 1$ it holds trivially and for $\lambda_1 > 1$ and $\lambda_2 < -1$ it translates as $\lambda_1 \le |\lambda_2|$, which we assumed anyway.

Remark 3.2.4. There is indeed a mutual correspondence between the Theorem 3.2.2 and the non-negativity parameter constraints in (3.2.6). Since $\phi_1 = \lambda_1^{-1} + \lambda_2^{-1}$, the condition $\lambda_1^{-1} + \lambda_2^{-1} \ge 0$ coincides with $\phi_1 \ge 0$. The condition $\lambda_1 > 1$ is contained within $\lambda_1^{-1} + \lambda_2^{-1} > 0$. If $\lambda_1 < -1$, we would have $0 \le \lambda_2 < -\lambda_1$. This would contradict our nontation $1 < |\lambda_1| \le |\lambda_2|$. Next, the two roots λ_1, λ_2 are real if and only if $\phi_1^2 + 4\phi_2 \ge 0$. The non-negative causal parametric region for $AR(2)$ process is depicted on Figure 3.1.

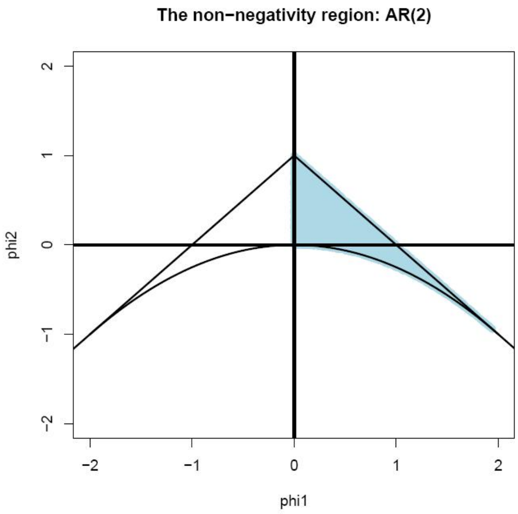


Figure 3.1: The $AR(2)$ non-negativity parametric region for ϕ_1 and ϕ_2

3.2.3 AR(3)

Now, we consider a causal $AR(3)$ process $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ with non-negative innovations, determined from the following stochastic difference equations

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \phi_3 X_{t-3} = Z_t^* \quad (t = 0, \pm 1, \pm 2, \dots),$$

where the roots of the autoregressive lag polynomial $\boldsymbol{\phi}(z) = 1 - \phi_1 z - \phi_2 z^2 - \phi_3 z^3$ lie outside the unit circle. We distinguish the following two situations: (1) all the roots of the autoregressive polynomial are real, (2) one root is real and the other two form a complex conjugate pair. In the first scenario, the conditions for the non-negativity are analogous to those for $AR(2)$ in Theorem 3.2.2. The proof rests purely on algebraic operations with inequalities and is rather lengthy. We do not reproduce it here and refer a reader to Tsai and Chan (2007).

Theorem 3.2.3. *Let $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ be an $AR(3)$ process. Denote $\lambda_1, \lambda_2, \lambda_3$ the roots of the autoregressive lag polynomial $\boldsymbol{\phi}(z) = 1 - \phi_1 z - \phi_2 z^2 - \phi_3 z^3$, such that $1 < |\lambda_1| \leq |\lambda_2| \leq |\lambda_3|$. Assume $\lambda_j \in \mathbb{R}, j = 1, \dots, 3$. Then $\{\psi_j\}_{j=0}^{\infty}$ is non-negative if and only if $\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} \geq 0$ and $\lambda_1 > 1$.*

Proof. A proof is given in Tsai and Chan (2007).

Now, consider the second situation. Denote again λ_1, λ_2 and λ_3 the the three roots of the autoregressive lag polynomial so that $1 < |\lambda_1| \leq |\lambda_2| \leq |\lambda_3|$. From Theorem 3.1.5 the conditions $\lambda_1 \in \mathbb{R}$ and $\lambda_1 > 1$ are necessary for the non-negativity of $\{\psi\}_{j=0}^{\infty}$. In the following, we assume they are satisfied. Denote further $\lambda_2 = \bar{\lambda}_3 = a + b\mathrm{i}$, where $a, b \in \mathbb{R}$ and $b > 0$. From the equation (4.8) in Feller (1968) p. 276 it holds that

$$\psi_n = \frac{r_1}{\lambda_1^{n+1}} + \frac{r_2}{\lambda_2^{n+1}} + \frac{r_3}{\lambda_3^{n+1}},$$

where $r_i = -\frac{1}{\phi^{(1)}(\lambda_i)}$, $i = 1, 2, 3$. More specifically,

$$r_1 = \frac{1}{\phi_3(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}, \quad r_2 = \frac{1}{\phi_3(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \quad \text{and} \quad r_3 = \frac{1}{\phi_3(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}.$$

Now denote $\psi_n^* = \psi_n|\lambda_1 - \lambda_2|^2 \lambda_1^{n+1} |\lambda_2|^{2n+2}$. Then we have

$$\psi_n^* = |\lambda_2|^{2n+2} + \frac{\lambda_1^{n+2}(\lambda_3^{n+1} - \lambda_2^{n+1})}{\lambda_3 - \lambda_2} - \frac{\lambda_1^{n+1}(\lambda_3^{n+2} - \lambda_2^{n+2})}{\lambda_3 - \lambda_2} \tag{3.2.9}$$

$$= |\lambda_2|^{2n+2} + \frac{\lambda_1^{n+2}|\lambda_2|^{n+1}\sin[(n+1)\theta]}{|\lambda_2|\sin\theta} - \frac{\lambda_1^{n+1}|\lambda_2|^{n+2}\sin[(n+2)\theta]}{|\lambda_2|\sin\theta} \tag{3.2.10}$$

$$= \lambda_1^{n+2}|\lambda_2|^n \left[\left|\frac{\lambda_2}{\lambda_1}\right|^{n+2} - \left|\frac{\lambda_2}{\lambda_1}\right| \frac{\sin[(n+2)\theta]}{\sin\theta} + \frac{\sin[(n+1)\theta]}{\sin\theta} \right]. \tag{3.2.11}$$

Denote

$$f_{n,\theta}(x) = x^{n+2} - x\frac{\sin[(n+2)\theta]}{\sin\theta} + \frac{\sin[(n+1)\theta]}{\sin\theta}. \tag{3.2.12}$$

It is not difficult to see that the following three conditions are equivalent:

(a) the single coefficient ψ_n is non-negative,

(b) the single coefficient ψ_n^* is non-negative,

(c) $f_{n,\theta}\left(\left|\frac{\lambda_2}{\lambda_1}\right|\right)$ is non-negative.

The function $f_{n,\theta}(\cdot)$ is increasing on $[1, \infty)$ for any $n \in \mathbb{N}$ and $\theta \in (0, \pi)$. This stems from the following fact: for $x \geq 1$ and $\theta \in (0, \pi)$ it holds

$$\begin{aligned} f'_{n,\theta}(x) &= (n+2)x^{n+1} - \frac{\sin[(n+2)\theta]}{\sin\theta} \\ &\geq \frac{(n+2)\sin\theta - \sin[(n+2)\theta]}{\sin\theta} \\ &> 0. \end{aligned}$$

We have utilized the following inequality $\sin[(n+2)\theta] < (n+2)\sin\theta$, for $\theta \in (0, \pi)$ and any $n \in \mathbb{N}$. A proof of this inequality can be found in Tsai and Chan (2007) Now, there are two possibilities:

(i) $f_{n,\theta}(1) \geq 0$, implying that the function $f_{n,\theta}(x)$ is non-negative on $[1, \infty)$,

(ii) $f_{n,\theta}(1) < 0$, implying that there must exist one point $x_{n,\theta}$ such that $f_{n,\theta}(x) > 0$ on the interval $(x_{n,\theta}, \infty)$.

For some choices of $\theta \in (0, \pi)$, namely $\theta = 2\pi/k\,(k = 3, 4, \dots)$, it is automatically assured that $f_{n,\theta}(1) \geq 0$ for all $n \in \mathbb{N}$. The condition for the non-negativity of the sequence $\{\psi_j\}_{j=0}^{\infty}$ then boils down to $|\lambda_2|/\lambda_1 \geq 1$.

For more "general" θ's the condition $|\lambda_2|/\lambda_1 \geq 1$ is not sufficient to assure the non-negativity of the whole sequence. Let us take one such $\theta^* \neq 2\pi/k\,(k = 3, 4, \dots)$. Then there exists a non-empty set of indices I_{θ^*} such that $\forall n \in I_{\theta^*}$, $f_{n,\theta^*}(1) < 0$. We know that for any $n \in I_{\theta^*}$, there exists a real number $x_{n,\theta^*} > 1$, such that $f_{n,\theta^*}(x) \geq 0$ for $x \geq x_{n,\theta^*}$. The non-negativity of the single coefficient $\psi_n, n \in I_{\theta^*}$, is then equivalent to the condition $|\lambda_2|/\lambda_1 \geq x_{n,\theta^*}$. In order the whole sequence $\{\psi_n\}_{n=0}^{\infty}$ to be non-negative, $|\lambda_2|/\lambda_1$ has to be at least as large as the maximum of the roots $x^*_{\theta^*} = \max_{n \in I_{\theta^*}} x_{n,\theta^*}$.

In summary, if the complex root λ_2 has the argument θ^* and an absolute value greater than or equal to $x^*_{\theta^*} \times \lambda_1$, then the sequence $\{\psi_j\}_{j=0}^{\infty}$ is non-negative. Tsai and Chan (2007) showed that $x^*_{\theta^*} = x_{n_0,\theta^*}$, where $n_0 = \min_{n \in I_{\theta^*}} n$. They also showed that $n \in I_{\theta^*}$ if and only if $\sin[(n+1)\theta^*] < 0$ and $\sin[(n+2)\theta^*] > 0$. Their result for $AR(3)$ process is summarized formally in the following theorem.

Theorem 3.2.4. *Let $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ be an $AR(3)$ process. Denote $\lambda_1, \lambda_2, \lambda_3$ the roots of the autoregressive lag polynomial $\boldsymbol{\phi}(z) = 1 - \phi_1 z - \phi_2 z^2 - \phi_3 z^3$, such that $1 < |\lambda_1| \leq |\lambda_2| \leq |\lambda_3|$. Assume $\lambda_2 = \bar{\lambda}_3 = |\lambda| \mathrm{e}^{\mathrm{i}\theta} = 1 + b\mathrm{i}$, where $a, b \in \mathbb{R}$, $b > 0$ and $0 < \theta < \pi$.*

(i) If $\theta = 2\pi/k$ for some integer $k \geq 3$, then $\{\psi_j\}_{j=0}^{\infty}$ is non-negative if and only if $|\lambda_2| \geq \lambda_1 > 1$.

(ii) If $\theta \notin \{2\pi/k | k = 3, 4, \dots\}$, then $\{\psi_j\}_{j=0}^{\infty}$ is non-negative if and only if $|\lambda_2|/\lambda_1 \geq x_{0,\theta} > 1$, where $x_{0,\theta}$ is the root of $f_{n_0,\theta}(x) = 0$, where

$$f_{n_0,\theta}(x) = x^{n_0+2} - x\frac{\sin[(n_0+2)\theta]}{\sin\theta} + \frac{\sin[(n_0+1)\theta]}{\sin\theta},$$

and n_0 is the smallest positive integer n such that $\sin[(n+1)\theta] < 0$ and $\sin[(n+2)\theta] > 0$.

(iii) If $a \geq \lambda_1 > 1$, then $\{\psi_j\}_{j=0}^{\infty}$ is non-negative.

Proof. A proof is given in Tsai and Chan (2007).

Remark 3.2.5. The conditions in Theorem 3.2.4 are slightly more difficult to verify, since they do not involve model parameters but roots of the autoregressive lag polynomial. A graphical representation of the conditions might help to clarify them. Let us suppose

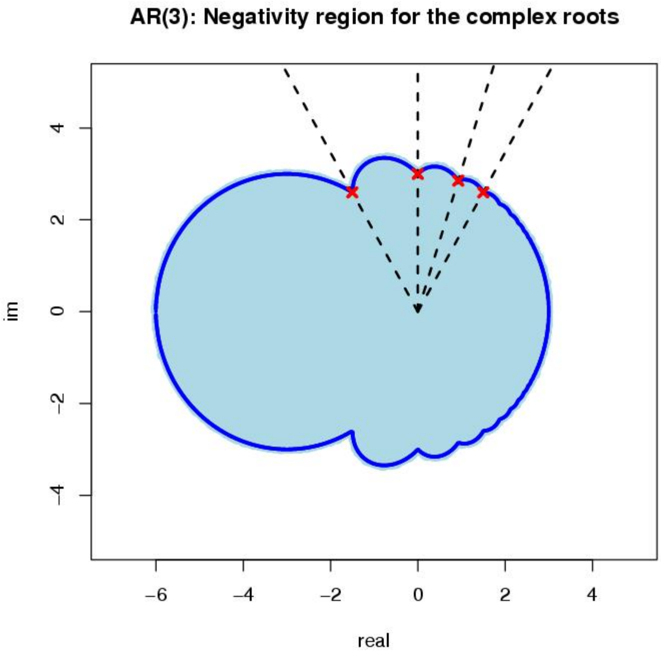


Figure 3.2: Regions of negativity and non-negativity of the sequence $\{\psi_j\}_{j=0}^{\infty}$ for $AR(3)$ and $\lambda_1 = 3$

that the real root λ_1 of the polynomial $1 - \phi_1 z - \phi_2 z^2 - \phi_3 z^3$ is equal to 3 and the other two roots are complex. The complex roots $\lambda_2 = \bar{\lambda}_3$ have to satisfy at least one of the conditions in the Theorem 3.2.4 to assure the non-negativity of the sequence $\{\psi_j\}_{j=0}^{\infty}$.

The plot in Figure 3.2 shows two regions in complex plane, divided by a closed curve (coloured with dark blue). The "blue" area represents those complex roots λ_2 and λ_3 for which at least one coefficient in $\{\psi_j\}_{j=0}^{\infty}$ is negative. Note that the roots λ_2 are depicted in the "positive hemisphere" (i.e. positive imaginary part), whereas the conjugates λ_3 lie in the negative hemisphere. The area complementary to the blue domain, is the region of the non-negativity described analytically in Theorem 3.2.4.

According to the part (i) in Theorem 3.2.4, the complex roots $\lambda_2 = \bar{\lambda}_3$ with argument $\theta = 2\pi/k, k = 3, 4, \ldots,$ need to have absolute value greater than 3 in order to assure the non-negativity. The black dashed lines in Figure 3.2 correspond to roots λ_2 with respective arguments $2\pi/3, 2\pi/4, 2\pi/5, 2\pi/6$. The red points then correspond to those roots which have the absolute value equal to $\lambda_1 = 3$. These points indeed lie on the border of the negativity region.

According to the part (ii) in Theorem 3.2.4, if $\theta \notin \{2\pi/k : k = 3, 4, \ldots\}$, in order the root λ_2 to fall into the non-negativity region, its absolute value must be greater than 3 times the "maximal root" $x_{0,\theta} > 1$. This explains the cloud-like shape of the negativity region.

According to the part (iii) in Theorem 3.2.4, if the real part a of the complex roots

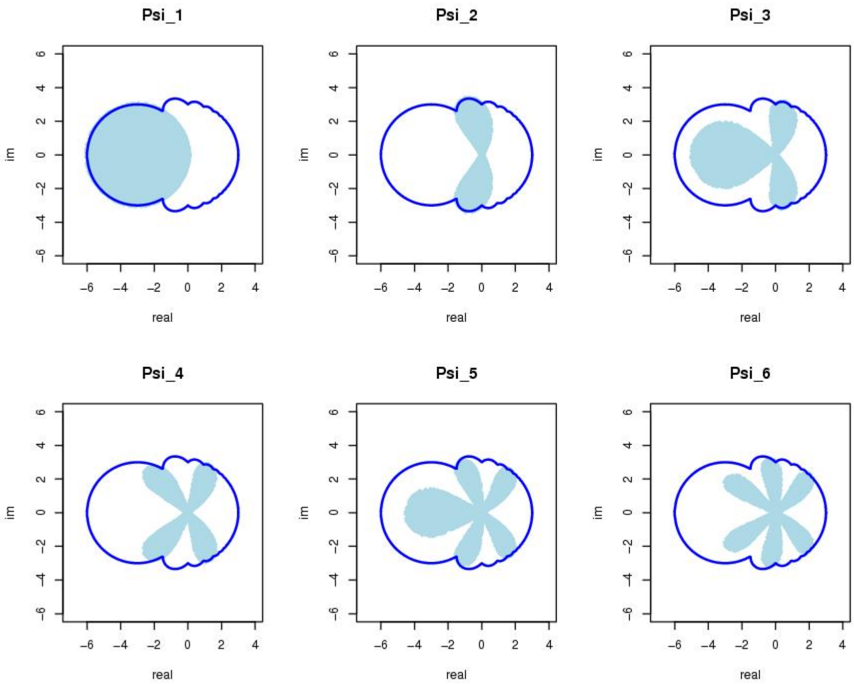


Figure 3.3: Regions of negativity (in blue) of the single coefficients $\psi_1, \dots, \psi_6$ for $AR(3)$ and $\lambda_1 = 3$, the blue curve corresponds to the boundary of the $AR(3)$ negativity region

λ_2 and λ_3 is greater than 3, the sequence $\{\psi_j\}_{j=0}^{\infty}$ is non-negative. We notice from the picture that the cloud is located to the left from all complex values with a real part greater than 3.

In fact, the blue region in Figure 3.2 is an union of infinitely many smaller regions, each representing the negativity of each of the coefficients ψ_j, see Figure 3.3. The area of negativity of ψ_1 is a circle centered at $[-\lambda_1, 0]$ with a diameter λ_1. A negativity region for ψ_2 consists of two "drops" connected at origin and so on.

3.2.4 AR(p)

In this section we discuss non-negativity conditions for higher order autoregressive processes. Recall that the $AR(p)$ process $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ is defined as a stationary solution of the following stochastic difference equations:

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \dots \phi_p X_{t-p} = Z_t^* \quad (t = 0, \pm 1, \pm 2, \dots).$$

Again we assume the process is causal and that the innovations are non-negative. One particular result for $AR(4)$ models has been derived by Tsai and Chan (2007). The proof is again a rather technical application of algebric operations and we omit it here.

Theorem 3.2.5. *Let $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ be an $AR(4)$ process. Denote $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ the roots of the autoregressive lag polynomial $\boldsymbol{\phi}(z) = 1 - \phi_1 z - \phi_2 z^2 - \phi_3 z^3 - \phi_4 z^4$, such that $1 < \lambda_1 \le |\lambda_2| \le |\lambda_3| \le |\lambda_4|$. Assume $\lambda_i \in \mathbb{R}, i = 1, \dots, 4$. Then the sequence $\{\psi_j\}_{j=0}^{\infty}$ is non-negative if and only if $\lambda_1 > 1$ and $\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} + \lambda_4^{-1} \ge 0$.*

Proof. A proof is given in Tsai and Chan (2007).

Remark 3.2.6. Given the Theorem 3.2.3 and Theorem 3.2.5, one would expect that conditions $\lambda_1 > 1$ and $\sum_{i=1}^{p} \lambda_i^{-1} \ge 0$ are sufficient and necessary for a general $AR(p)$ process to be non-negative, as long as the roots λ_i are all real. A proof of the necessity is not difficult.

Theorem 3.2.6. *Let $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ be an $AR(p)$ process. Denote $\lambda_1, \lambda_2, \dots, \lambda_p$ the roots of the autoregressive lag polynomial $\boldsymbol{\phi}(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$, such that $\lambda_1 \le |\lambda_2| \le \cdots \le |\lambda_p|$. If the sequence $\{\psi_j\}_{j=0}^{\infty}$ is non-negative, then $\sum_{i=1}^{p} \lambda_j^{-1} \ge 0$, λ_1 is real and greater than one.*

Proof. We reproduce the proof from Tsai and Chan (2007). The autoregressive lag polynomial can be expressed as follows:

$$\boldsymbol{\phi}(z) = 1 - \sum_{i=1}^{p} \phi_i z^i = \prod_{i=1}^{p} \left(1 - \frac{z}{\lambda_i}\right).$$

Comparing the coefficients on both the sides, we get $\phi_1 = \sum_{i=1}^{p} \lambda_i^{-1}$ From equations (3.3.5) in Brockwell and Davis (1986) p. 91 we know that $\psi_1 = \phi_1$. Therefore $\sum_{i=1}^{p} \lambda_i^{-1} \ge 0$ whenever $\{\psi_j\}_{j=0}^{\infty}$ is non-negative. The necessity of the condition $\lambda_1 \in \mathbb{R}$ and $\lambda_1 > 1$ follows from Theorem 3.1.5. □

Remark 3.2.7. The sufficiency of the conditions in Theorem 3.2.6 is more difficult to show. It has not been yet proven analytically in the literature.

In Section 3.3.7 we will derive a set of sufficient and necessary conditions for the non-negativity of $ARMA(p, q)$ processes. These conditions can be easily accommodated for $AR(p)$ processes. In contrast to what we had so far, these conditions utilize model parameters rather than the zeros of autoregressive lag polynomial. We formulate the result for $AR(p)$ now, but wait with the proof until the Section 3.3.7.

Theorem 3.2.7. *Let $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ be an $AR(p)$ process. Denote $\lambda_1, \lambda_2, \dots, \lambda_p$ the roots of the autoregressive lag polynomial $\boldsymbol{\phi}(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$, such that $|\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_p|$. Assuming that these roots are distinct, the conditions (1)-(2) are necessary and sufficient for $\{\psi_j\}_{j=0}^{\infty}$ to be non-negative:*

(1) λ_1 *is real and* $\lambda_1 > 1$,

(2) $\psi_k \ge 0$ *for* $k = 1, \dots, k^*$,

where k^* *is the smallest integer greater than on equal to* $\max\{0, \gamma\}$*, where*

$$\gamma = \frac{\log r_1 - \log[(p-1)r^*]}{\log|\lambda_1| - \log|\lambda_2|} - 1$$

with $r^* = \max_{2 \le j \le p} |r_j|$ *and* $r_j = -\frac{1}{\boldsymbol{\phi}^{(1)}(\lambda_j)}$, $1 \le j \le p$.

Proof. A proof follows from Theorem 3.3.18.

Remark 3.2.8. Assuming that the roots λ_i are distinct and real, the sufficiency of the conditions in Theorem 3.2.6 could be proven by showing that the γ in Theorem 3.2.7 equals 1.

Some other sufficient conditions for the non-negativity of $AR(p)$ processes can be readily obtained, again using the properties of absolutely monotone functions.

Theorem 3.2.8. *Let* $\{X_t : t = 0, \pm1, \pm2, \dots\}$ *be an* $AR(p)$ *process. Denote* $\lambda_1, \lambda_2, \dots, \lambda_p$ *the roots of the autoregressive lag polynomial* $\boldsymbol{\phi}(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$*, such that* $\lambda_1 \le |\lambda_2| \le \cdots \le |\lambda_p|$*. If all the* $\lambda_1, \dots, \lambda_p$ *are real and greater than one, then the sequence* $\{\psi_j\}_{j=0}^{\infty}$ *is non-negative.*

Proof. The generating function is now a product of p factors:

$$\boldsymbol{\psi}(z) = \prod_{i=1}^{p} \frac{1}{1 - \frac{z}{\lambda_1}}. \tag{3.2.13}$$

If all the roots λ_i are real and greater than one, each of the factors in (3.2.13) is absolutely monotone in $0 \le z < 1$, see Remark 3.2.1. Their product is absolutely monotone as well, implying that the sequence $\{\psi_j\}_{j=0}^{\infty}$ is non-negative. □

Remark 3.2.9. Another set of sufficient conditions can be obtained when the roots are not necessarily all real. Recall that roots of the real autoregressive lag polynomial $\boldsymbol{\phi}(z)$ are either all real or occur in conjugate imaginary pairs. If for each complex pair we can find a real root for which one of the conditions in Theorem 3.2.4 is satisfied, then the generating function $\boldsymbol{\psi}(z) = 1/\boldsymbol{\phi}(z)$ is absolutely monotone.

3.3 Autoregressive Moving Average Processes

In this section we concentrate on the conditions for the non-negativity in $ARMA$ time series models. Similarly as in the previous section, we will investigate situations, when the sequence of weights $\{\psi_j\}_{j=0}^{\infty}$ in the infinite moving average representation is non-negative. Some conditions for $ARMA$ models emerge as a natural extension of those we have seen for AR models. Let us start with some instructive examples of lower order models.

3.3.1 ARMA(1,1)

Assume a stationary $ARMA(1,1)$ process $\{X_t : t = 0, \pm1, \pm2, \dots\}$, which is determined from the following stochastic difference equations

$$X_t - \phi_1 X_{t-1} = Z_t^* + \theta_1 Z_{t-1}^* \quad (t = 0, \pm1, \pm2, \dots), \tag{3.3.14}$$

where $|\phi_1| < 1$ and the sequence $\{Z_t^* : t = 0, \pm1, \pm2, \dots\}$ is the non-negative innovation sequence. Such a process has a infinite moving average representation

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}^*,$$

where the weights $\psi_j, j \in \mathbb{N}$, are determined from

$$\boldsymbol{\psi}(z) = \sum_{j=0}^{\infty} \psi_j z^j = \frac{1 + \theta_1 z}{1 - \phi_1 z} \quad (0 \leq z < 1). \tag{3.3.15}$$

One sufficient condition for the non-negativity of $\{\psi_j\}_{j=0}^{\infty}$ readily follows from the factorization of (3.3.15) into two absolutely monotone functions. If $\theta_1 \geq 0$ the numerator is an absolutely monotone function. Moreover, we have shown in Remark 3.2.1 that if the coefficient ϕ_1 is non-negative, then the function $\frac{1}{1-\phi_1 z}$ is absolutely monotone in $0 \leq z < 1$. The property that a product of absolutely monotone functions is again absolutely monotone gives immediately the sufficiency of the conditions $\phi_1 \geq 0$ and $\theta_1 \geq 0$. However, these conditions are unnecessarily strict. The following theorem gives less stringent non-negativity restrictions on the two model parameters.

Theorem 3.3.1. *Let $\{X_t : t = 0, \pm1, \pm2, \dots\}$ be an $ARMA(1,1)$ process given in (3.3.14). The sequence $\{\psi_j\}_{j=0}^{\infty}$ is non-negative if and only if $\phi_1 \geq 0$ and $\phi_1 + \theta_1 \geq 0$.*

Proof. From the equations (3.3.3) and (3.3.4) in Brockwell and Davis (1986) p. 91 it follows that

$$\begin{aligned} \psi_0 &= 1, \\ \psi_1 &= \theta_1 + \phi_1, \\ \psi_k &= \phi_1 \psi_{k-1} \quad (k \geq 2). \end{aligned}$$

If $\psi_1 \geq 0$ and $\phi_1 \geq 0$, then the whole sequence $\{\psi_j\}_{j=0}^{\infty}$ is non-negative. The necessity follows also trivially. $\square$

3.3.2 ARMA(1,2)

Consider now the causal $ARMA(1,2)$ process $\{X_t : t = 0, \pm1, \pm2, \dots\}$ with non-negative innovations $\{Z_t^* : t = 0, \pm1, \pm2, \dots\}$, which is given by the set of equations

$$X_t - \phi_1 X_{t-1} = Z_t^* + \theta_1 Z_{t-1}^* + \theta_2 Z_{t-2}^* \quad (t = 0, \pm1, \pm2, \dots), \tag{3.3.16}$$

where $|\phi_1| < 1$. The generating function for the sequence $\{\psi_j\}_{j=0}^{\infty}$ is now the following

$$\boldsymbol{\psi}(z) = \frac{1 + \theta_1 z + \theta_2 z^2}{1 - \phi_1 z} \quad (0 \le z < 1).$$

Similarly as for the $ARMA(1,1)$, if all the coefficients θ_1, θ_2 and ϕ_1 are non-negative, the generating function is a product of absolutely monotone functions. The following theorem shows that we can suffice with milder restrictions.

Theorem 3.3.2. *Let $\{X_t : t = 0, \pm1, \pm2, \dots\}$ be an $ARMA(1,2)$ process given in (3.3.16). The sequence $\{\psi_j\}_{j=0}^{\infty}$ is non-negative if and only if $\phi_1 \ge 0$, $\phi_1 + \theta_1 \ge 0$ and $\theta_2 \ge -\phi_1(\theta_1 + \phi_1)$.*

Proof. The equations (3.3.3) and (3.3.4) in Brockwell and Davis (1979) for $ARMA(1,2)$ model are the following

$$\begin{aligned}
\psi_0 &= 1, \\
\psi_1 &= \theta_1 + \phi_1, \\
\psi_2 &= \theta_2 + \phi_1(\theta_1 + \phi_1), \\
\psi_k &= \phi_1 \psi_{k-1} \quad (k \ge 3).
\end{aligned}$$

The sufficiency and necessity then follows immediately. □

The trivial results for $ARMA(1,1)$ and $ARMA(1,2)$ suggest the following simple result for models $ARMA(1,q)$.

3.3.3 ARMA(1,q)

Consider the stationary $ARMA(1,q)$ process $\{X_t : t = 0, \pm1, \pm2, \dots\}$ defined from the autoregressive moving average equations

$$X_t - \phi_1 X_{t-1} = Z_t^* + \theta_1 Z_{t-1}^* + \theta_2 Z_{t-2}^* + \dots + \theta_q Z_{t-q}^* \quad (t = 0, \pm1, \pm2, \dots), \tag{3.3.17}$$

where $|\phi_1| < 1$ and the sequence of innovations $\{Z_t^* : t = 0, \pm1, \pm2, \dots\}$ is non-negative. The generating function of the $ARMA(1,q)$ model is

$$\boldsymbol{\psi}(z) = \frac{1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q}{1 - \phi_1 z} \quad (0 \le z < 1).$$

The non-negativity of the $ARMA(1,q)$ kernel sequence $\{\psi_j\}_{j=0}^{\infty}$ is summarized in the following theorem of Tsai and Chan (2007).

Theorem 3.3.3. *Let* $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ *be an* $ARMA(1, q)$ *process given in (3.3.17). The sequence* $\{\psi_j\}_{j=0}^{\infty}$ *is non-negative if and only if* $\phi_1 \geq 0$ *and* $\psi_j = \phi_1 \psi_{j-1} + \theta_j \geq 0$, $1 \leq j \leq q$.

Proof. Again, the necessity and sufficiency follows trivially from the equations (3.3) and (3.4) of Brockwell and Davis (1986) p. 91:

$$\begin{aligned}
\psi_0 &= 1 \\
\psi_1 &= \theta_1 + \phi_1, \\
\psi_2 &= \theta_2 + \phi_1 \psi_1, \\
&\vdots \\
\psi_q &= \theta_q + \phi_1 \psi_{q-1}, \\
\psi_k &= \phi_1 \psi_{k-1} \quad (k \geq q+1). \quad \square
\end{aligned}$$

The Theorem 3.3.3 shows that the infinite number of inequalities $\psi_j \geq 0 \, (j \geq 0)$ can be effectively reduced to a finite set of inequalities, which are expressed directly in terms of model parameters. Later we show a similar set of conditions for $ARMA(2, q)$ models. First, we investigate $ARMA(\cdot, 1)$ processes.

3.3.4 ARMA(2,1)

We consider the $ARMA(2, 1)$ process $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ determined by the stochastic difference equations

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t^* + \theta_1 Z_{t-1}^* \quad (t = 0, \pm 1, \pm 2, \dots), \tag{3.3.18}$$

where the roots $|\lambda_1| \leq |\lambda_2|$ of the polynomial $1 - \phi_1 z - \phi_2 z^2$ are outside the unit circle and again the innovations are non-negative. The generating function for the $ARMA(2, 1)$ kernel sequence $\{\psi_j\}_{j=0}^{\infty}$ is

$$\boldsymbol{\psi}(z) = \frac{1 + \theta_1 z}{1 - \phi_1 z - \phi_2 z^2} \quad (0 \leq z < 1). \tag{3.3.19}$$

We can again readily derive sufficient conditions for the non-negativity by using the factorization of the function (3.3.19) into absolutely monotone components. From Theorem 3.1.5 we know that the smallest root in absolute value, i.e. λ_1, has to be real and greater than one. We have also seen in Theorem 3.2.7 that if $\lambda_2 > 1$, the function $\frac{1}{1-\phi_1 z - \phi_2 z^2}$ is absolutely monotone. Together with the assumption $\theta_1 > 0$, we obtain immediately a set of sufficient conditions. However, it is possible to derive without substantial difficulties also a set of sufficient and necessary conditions. We have derived the following result.

Theorem 3.3.4. *Let* $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ *be* $ARMA(2,1)$ *process given by (3.3.18). Let* λ_1, λ_2 *be the two distinct roots of the autoregressive lag polynomial* $\boldsymbol{\phi}(z) = 1 - \phi_1 z - \phi_2 z^2$*, such that* $1 < |\lambda_1| \le |\lambda_2|$*. The sequence* $\{\psi_j\}_{j=0}^{\infty}$ *is non-negative if and only if*

(1) $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 > 1$,

(2) $\boldsymbol{\theta}(\lambda_1) > 0$,

(3) $\psi_j \ge 0$, $j = 1, 2$.

Proof. Let us start with the necessity. The necessity of the conditions $\lambda_1 \in \mathbb{R}$ and $\lambda_1 > 1$ follows from Theorem 3.1.5. We know that whenever $\lambda_1 \in \mathbb{R}$, the second root λ_2 has to be real as well. The necessity of (3) is trivial. The necessity of the condition (2) follows from Theorem 3.3.18 which we prove later in this section.

Now we prove the sufficiency. According to Feller (1968) the coefficients ψ_n are given by

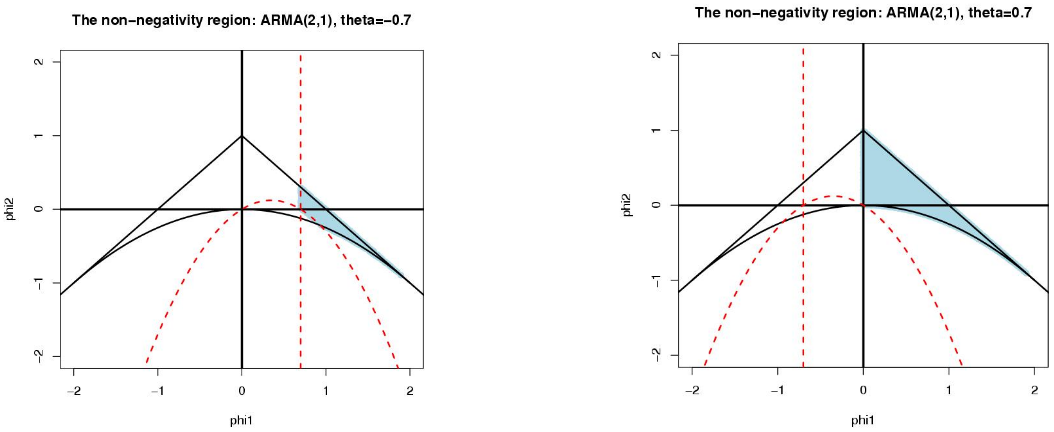
$$\psi_n = \frac{1}{\phi_2(\lambda_1 - \lambda_2)} \left(\frac{1 + \theta_1 \lambda_1}{\lambda_1^{n+1}} - \frac{1 + \theta_1 \lambda_2}{\lambda_2^{n+1}} \right), \quad n \ge 1.$$

We are assuming that $1 < \lambda_1 < |\lambda_2|$, which implies that $\phi_2(\lambda_1 - \lambda_2)$ is always non-negative. This follows from the following facts: (a) if $\lambda_2 < -\lambda_1 < -1$, the coefficient $\phi_2 = -\frac{1}{\lambda_1 \lambda_2}$ is positive and $\lambda_1 - \lambda_2 > 0$, (b) when $\lambda_2 > \lambda_1 > 1$ we have $\phi_2 < 0$ and $\lambda_1 - \lambda_2 < 0$. Next, we can write

$$\psi_n \phi_2 (\lambda_1 - \lambda_2) \lambda_1^{n+1} = 1 + \theta_1 \lambda_1 - \left(\frac{\lambda_1}{\lambda_2} \right)^{n+1} (1 + \theta_1 \lambda_2). \tag{3.3.20}$$

The non-negativity of a single coefficient ψ_n is then equivalent to the non-negativity of the expression in (3.3.20). From the assumption (2), the term $1 + \theta_1 \lambda_1$ is positive. Denote $A_n = \left(\frac{\lambda_1}{\lambda_2} \right)^{n+1} (1 + \theta_1 \lambda_2)$. This term is declining in magnitude as n approaches infinity and it has eventually oscilating sign. We now show that the non-negativity of the first two coefficients ψ_1 and ψ_2 will assure the non-negativity of the whole sequence $\{\psi_j\}_{j=0}^{\infty}$.

Let us first consider the case $\lambda_2 > \lambda_1 > 1$ and $1 + \theta_1 \lambda_2 > 0$. Then the term A_n is always positive with the largest value for $n = 1$. This implies that if the expression in (3.3.20) is non-negative for $n = 1$, then it is non-negative for any $n > 1$. Consider further the following case: $\lambda_2 > \lambda_1 > 1$ and $1 + \theta_1 \lambda_2 < 0$. The term A_n is now always negative and adds positively to $1 + \theta_1 \lambda_1$ giving a non-negative value of the whole expression in (3.3.20) for any $n \in \mathbb{N}$. Assuming that $\lambda_2 < -\lambda_1 < -1$ and $1 + \theta_1 \lambda_1 > 0$, the term A_n is negative for $n = 2k$ and positive for $n = 2k + 1$. For $n = 2k$ it adds positively, whereas for $n = 2k + 1$ it contributes negatively in (3.3.20). Nevertheless, because the term A_n diminishes in absolute value as $n \to \infty$, it holds that if the expression (3.3.20) is non-negative for $n = 1$, then it is non-negative for any $n = 2k+1$. Finally, let $\lambda_2 < -\lambda_1 < -1$ and $1 + \theta_1 \lambda_2 < 0$. Now the expression (3.3.20) is always non-negative for $n = 2k + 1$ and always positive for $n = 2k$. By the similar argument, if (3.3.20) is non-negative for $n = 2$ then it is non-negative for any $n = 2k$. □



(a) $\theta_1 = -0.7$ (b) $\theta_1 = 0.7$

Figure 3.4: The non-negativity parametric region for ϕ_1 and ϕ_2 for $ARMA(2,1)$ model for given parameter θ_1

Remark 3.3.1. The conditions in Theorem 3.3.4 can be expressed in terms of model parameters. We then obtain a similar non-negativity causality region as in (3.2.6) for the $AR(2)$ process.

(1) The assumption $1 < \lambda_1 < |\lambda_2|$ implies $\phi_1 = \lambda_1^{-1} + \lambda_2^{-1} > 0$.

(2) The roots λ_1, λ_2 are real whenever $\phi_1^2 + 4\phi_2 \geq 0$.

(3) The condition $\boldsymbol{\theta}(\lambda_1) > 0$ translates into $1 + \theta_1\lambda_1 > 0$. Assuming that $\lambda_1 > 1$, $1 + \theta_1\lambda_1$ holds trivially for $\theta_1 > 0$.

(4) The conditions $\psi_1 \geq 0$ and $\psi_2 \geq 0$ correspond to $\theta_1 + \phi_1 \geq 0$ and $\phi_2 + \phi_1(\phi_1 + \theta_1) \geq 0$, respectively.

Two examples of non-negative parametric regions for $ARMA(2,1)$ are given in Figure 3.4(a) and Figure 3.4(b). Both plots depict the parametric region for ϕ_1 and ϕ_2, keeping the third parameter θ_1 fixed. The vertical dashed line corresponds to the restriction $\phi_1 + \theta_1 \geq 0$. The dashed parabola corresponds to $\phi_2 + \phi_1(\phi_1 + \theta_1) = 0$. The parabola always passes through origin. For $\theta_1 > 0$, the vertex of the parabole is always located to the left from the origin. In Figure 3.4(b), we have $\theta_1 = 0.7$. This non-negative parametric region coincides with the one for $AR(2)$ in Figure 3.1.

3.3.5 ARMA(3,1)

The sufficient and necessary non-negativity conditions for $ARMA(2,1)$ were derived without the notion of absolute monotonicity. By essentially the same approach we will later in this chapter derive the necessary and sufficient conditions for a general $ARMA(p,q)$ model. One might wonder, whether it is really necessary to continue scrutinizing separately some other lower order $ARMA$ models. Our reason for doing so is that we are curious to see whether a different approach would lead to conditions that are more easily verifiable. That is why we try to generalize the result of Tsai and Chan (2007) derived for $AR(3)$ models also for $ARMA(3,1)$ models. Such conditions would be expressed in terms of moving average coefficients and roots of the autoregressive polynomial. We discuss what the additional complications are and provide some explicit results.

We consider the $ARMA(3,1)$ process $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$, which is determined from the following autoregressive moving average equations

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \phi_3 X_{t-3} = Z_t^* + \theta_1 Z_{t-1}^* \quad (t = 0, \pm 1, \pm 2, \dots),$$

where $\{Z_t^* : t = 0, \pm 1, \pm 2, \dots\}$ is the non-negative innovation sequence. We assume that the process is causal and therefore admits the infinite moving average representation.

The generating function for $ARMA(3,1)$ process takes the following form

$$\psi(z) = \frac{1 + \theta_1 z}{1 - \phi_1 z - \phi_2 z^2 - \phi_3 z^3}, \quad 0 \le z < 1. \tag{3.3.21}$$

Noting that (3.3.21) is a product of a generating function for the $AR(3)$ process and the moving average lag polynomial, one sufficient condition for the non-negativity immediately follows.

Theorem 3.3.5. *Let λ_1, λ_2 and λ_3 be the roots of the autoregressive lag polynomial $\boldsymbol{\phi}(z) = 1 - \phi_1 z - \phi_2 z^2 - \phi_3 z^3$, such that $1 < |\lambda_1| \le |\lambda_2| \le |\lambda_3|$. Suppose that $\theta_1 \ge 0$. Assume further that either the conditions in Theorem 3.2.3 hold, or at least one of the conditions (i), (ii), (iii) in Theorem 3.2.4 is satisfied. Then the sequence $\{\psi_j\}_{j=0}^{\infty}$ is non-negative.*

Proof. A proof follows directly from Theorem 3.1.3. □

Using the fact that the absolute monotonicity property retains with multiplication, the non-negativity of any higher order autoregressive moving average model could be split into two or more lower order problems. In Theorem 3.3.5, the non-negativity of $ARMA(3,1)$ has been factorized into two separate tasks: the non-negativity of a corresponding $AR(3)$ process and the absolute monotonicity of the moving average lag polynomial. However, this sort of factorization will provide only a set of sufficient conditions for the non-negativity. In order to obtain necessary, or sufficient and necessary conditions, we need to adopt a different approach. We will investigate the case when $\lambda_2, \lambda_3 \in \mathbb{C}$.

According to the equation (4.8) in Feller (1968) p. 276 the coefficients ψ_j are given by

$$\psi_n = \frac{r_1}{\lambda_1^{n+1}} + \frac{r_2}{\lambda_2^{n+1}} + \frac{r_3}{\lambda_3^{n+1}},$$

where $r_i = -\theta(\lambda_i)/\phi^{(1)}(\lambda_i)$, $i = 1, 2, 3$. More specifically,

$$r_1 = \frac{1+\theta_1\lambda_1}{\phi_3(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}, \quad r_2 = \frac{1+\theta_1\lambda_2}{\phi_3(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} \quad \text{and} \quad r_3 = \frac{1+\theta_1\lambda_3}{\phi_3(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}.$$

Now, denote $\psi_n^{**} = \phi_3|\lambda_1-\lambda_2|^2\lambda_1^{n+1}|\lambda_2|^{2n+2}\psi_n$. Then we have

$$\begin{aligned}
\psi_n^{**} =& (1+\theta_1\lambda_1)|\lambda_2|^{2n+2} + \frac{\lambda_1^{n+1}\lambda_3^{n+1}(\lambda_3-\lambda_1)(1+\theta_1\lambda_2)}{(\lambda_2-\lambda_3)} + \frac{\lambda_1^{n+1}\lambda_2^{n+1}(\lambda_2-\lambda_1)(1+\theta_1\lambda_3)}{\lambda_3-\lambda_2}\\
=& |\lambda_2|^{2n+2} + \frac{\lambda_1^{n+2}(\lambda_3^{n+1}-\lambda_2^{n+1})}{\lambda_3-\lambda_2} - \frac{\lambda_1^{n+1}(\lambda_3^{n+1}-\lambda_2^{n+1})}{\lambda_3-\lambda_2}\\
&+ \theta_1\lambda_1|\lambda_2|^{2n+2} + \theta_1\frac{\lambda_1^{n+2}|\lambda_2|^2(\lambda_3^{n}-\lambda_2^{n})}{\lambda_3-\lambda_2} - \theta_1\frac{\lambda_1^{n+1}|\lambda_2|^2(\lambda_3^{n+1}-\lambda_2^{n+1})}{\lambda_2-\lambda_3}.
\end{aligned}$$

We can write

$$\psi_n^{**} = \psi_n^* + \theta_1\lambda_1|\lambda_2|^2\psi_{n-1}^*, \tag{3.3.22}$$

where

$$\psi_n^* = |\lambda_2|^{2n+2} + \frac{\lambda_1^{n+2}(\lambda_3^{n+1}-\lambda_2^{n+1})}{\lambda_3-\lambda_2} - \frac{\lambda_1^{n+1}(\lambda_3^{n+2}-\lambda_2^{n+2})}{\lambda_3-\lambda_2},$$

which we already defined in (3.2.9). In Section 3.2.3 we have shown that the non-negativity of ψ_n^* relates to the absolute monotonicity of a function $\frac{1}{1-\psi_1 z-\psi_2 z^2-\psi_3 z^3}$. As we know, this is a generating function corresponds of the $AR(3)$ kernel sequence. We have also derived that

$$\begin{aligned}
\psi_n^* &= |\lambda_2|^{2n+2} + \frac{\lambda_1^{n+2}|\lambda_2|^{n+1}\sin[(n+1)\theta]}{|\lambda_2|\sin\theta} - \frac{\lambda_1^{n+1}|\lambda_2|^{n+2}\sin[(n+2)\theta]}{|\lambda_2|\sin\theta}\\
&= \lambda_1^{n+2}|\lambda_2|^n\left\{\left|\frac{\lambda_2}{\lambda_1}\right|^{n+2} - \left|\frac{\lambda_2}{\lambda_1}\right|\frac{\sin[(n+2)\theta]}{\sin\theta} + \frac{\sin[(n+1)\theta]}{\sin\theta}\right\}.
\end{aligned}$$

Plugging the expression for ψ_n^* into (3.3.22) yields

$$\begin{aligned}
\frac{\psi_n^{**}}{\lambda_1^{n+2}|\lambda_2|^n} =& \left|\frac{\lambda_2}{\lambda_1}\right|^{n+2}(1+\theta_1\lambda_1) - \left|\frac{\lambda_2}{\lambda_1}\right|^2\theta_1\lambda_1\frac{\sin[(n+1)\theta]}{\sin\theta} - \left|\frac{\lambda_2}{\lambda_1}\right|\left\{\frac{\sin[(n+2)\theta]}{\sin\theta} - \theta_1\lambda_1\frac{\sin n\theta}{\sin\theta}\right\}\\
&+ \frac{\sin[(n+1)\theta]}{\sin\theta}.
\end{aligned}$$

For $0 < \theta < \pi$ and $n \in \mathbb{N}$, denote

$$g_{n,\theta}(x) = x^{n+2}(1+\theta_1\lambda_1) - x^2\theta_1\lambda_1\frac{\sin[(n+1)\theta]}{\sin\theta} - x\left\{\frac{\sin[(n+2)\theta]}{\sin\theta} - \theta_1\lambda_1\frac{\sin n\theta}{\sin\theta}\right\} + \frac{\sin[(n+1)\theta]}{\sin\theta}.$$

Then we can write

$$\psi_n^{**} = \lambda_1^{n+2}|\lambda_2|^n g_{n,\theta}\left(\left|\frac{\lambda_2}{\lambda_1}\right|\right). \qquad (3.3.23)$$

Note the similarity between the expression in (3.3.23) and $\psi_n^* = \lambda_1^{n+2}|\lambda_2|^n f_{n,\theta}\left(\left|\frac{\lambda_2}{\lambda_1}\right|\right)$, where the function $f_{n,\theta}(\cdot)$ was defined in (3.2.12). Similarly as in Section 3.2.3, assuming that $\lambda_1 \in \mathbb{R}$ and $\lambda_1 > 1$, we have the equivalence between the following conditions

(a) a single coefficient ψ_n is non-negative,

(b) a single coefficient ψ_n^{**} is non-negative,

(c) $g_{n,\theta}\left(\left|\frac{\lambda_2}{\lambda_1}\right|\right)$ is non-negative.

Similarly as in Section 3.2.3 we need to show that for each $n \in \mathbb{N}$ and $\theta \in (0,\pi)$ such that $g_{n,\theta}(1) < 0$ there exists one and only one root $x_{n,\theta} > 1$ of the equation $g_{n,\theta}(x) = 0$. The convenient monotonicity property on the interval $[1,\infty)$ of the functions $f_{n,\theta}(\cdot)$ made the situation for the $AR(3)$ process much easier. Unfortunatelly, such property is no longer valid for the functions $g_{n,\theta}(\cdot)$ (at least not in general), see Figure 3.5. We circumvent this difficulty by showing that each function $g_{n,\theta}(\cdot)$ such that $g_{n,\theta}(1) < 0$ is increasing on some interval $(x_{n,\theta}^*, \infty), x_{n,\theta}^* > 1$, and negative on interval $(1, x_{n,\theta}^*)$. In order to demonstrate such property, we need to prove several auxiliary lemmas. We split our considerations into two cases: $\theta_1 \geq 0$ and $\theta_1 < 0$.

(1) $\theta_1 \geq 0$

The assumption $\theta_1 \geq 0$ simplifies matters a great deal. We first show some auxiliary statements about the functions $g_{n,\theta}(\cdot)$. Without noting explicitly, throughout this section we assume that $1 < \lambda_1 \leq |\lambda_2| \leq |\lambda_3|$.

Lemma 3.3.6. *For a given $\theta \in (0,\pi)$ and $n \in \mathbb{N}$, let $g_n(\cdot)$ denote $g_{n,\theta}(\cdot)$. Suppose that $\theta_1 \geq 0$. Then $\forall n \in \mathbb{N}$ and $\forall\, x \geq 1$, the following statement holds:*

$$\textit{if} \quad g_n(x) \geq 0 \quad \textit{then} \quad g_n'(x) \geq 0.$$

Proof. We prove this claim by contradiction, that is we assume that for any $x \geq 1$ the statement $g_n(x) \geq 0$ and $g_n'(x) < 0$ leads to a contradictory conclusion. Denote

$$A = x^{n+1}(1+\theta_1\lambda_1) - x\,\theta_1\lambda_1\frac{\sin[(n+1)\theta]}{\sin\theta} - \frac{\sin[(n+2)\theta]}{\sin(\theta)} + \theta_1\lambda_1\frac{\sin(n\theta)}{\sin\theta}.$$

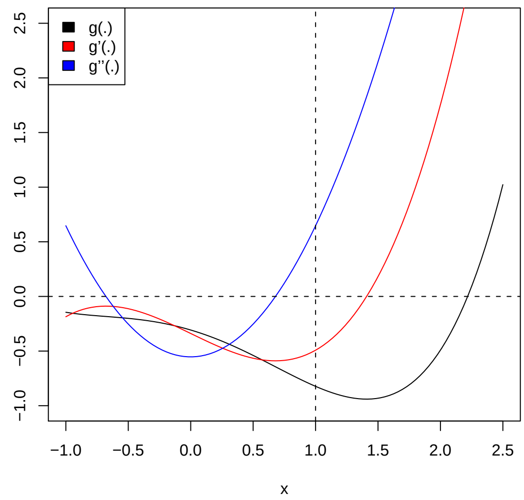


Figure 3.5: Plot of $g_{n,\theta}(x)$ (black line), $g'_{n,\theta}(x)$ (red line) and $g''_{n,\theta}(x)$ (blue line) for: $\theta = 2, \theta_1 = -0.3, \lambda_1 = 3$ and $n = 2$

Then, the inequality $g_n(x) \geq 0$ is equivalent to

$$Ax \geq -\frac{\sin[(n+1)\theta]}{\sin\theta}. \tag{3.3.24}$$

The first derivative $g'_n(\cdot)$ is given by a formula

$$g'_n(x) = x^{n+1}(1+\theta_1\lambda_1)(n+2) - 2x\theta_1\lambda_1\frac{\sin[(n+1)\theta]}{\sin\theta} - \frac{\sin[(n+2)\theta]}{\sin\theta} + \theta_1\lambda_1\frac{\sin(n\theta)}{\sin\theta}.$$

The statement $g'_n(x) < 0$ is then equivalent to an inequality

$$A < -(n+1)x^{n+1}(1+\theta_1\lambda_1) + x\theta_1\lambda_1\frac{\sin[(n+1)\theta]}{\sin\theta},$$

in particular (for $x \geq 1$)

$$Ax < -(n+1)x^{n+2}(1+\theta_1\lambda_1) + x^2\theta_1\lambda_1\frac{\sin[(n+1)\theta]}{\sin\theta}. \tag{3.3.25}$$

From (3.3.24) and (3.3.25), we obtain

$$(n+1)x^{n+2}(1+\theta_1\lambda_1) < \frac{\sin[(n+1)\theta]}{\sin\theta}(x^2\theta_1\lambda_1 + 1). \tag{3.3.26}$$

Because we assumed $\theta_1 \geq 0$, it holds that $0 < 1 + x^2\theta_1\lambda_1 \leq x^2(1+\theta_1\lambda_1)$ for $x \geq 1$. According to the inequality $\sin[(n+1)\theta] < (n+1)\sin\theta$, $\theta \in (0,\pi)$, we have:

$$(n+1)x^{n+2}(1+\theta_1\lambda_1) < (n+1)(x^2\theta_1\lambda_1 + 1) \leq (n+1)x^2(\theta_1\lambda_1 + 1).$$

Because $1 + \theta_1\lambda_1 > 0$, we have

$$(n+1)x^n < n+1,$$

which is contradictory, as we assumed $x \geq 1$. $\square$

Lemma 3.3.7. *For a given $\theta \in (0, \pi)$ and $n \in \mathbb{N}$, let $g_n(\cdot)$ denote $g_{n,\theta}(\cdot)$. Assume that $\theta_1 \geq 0$. Then $g_n''(x) > 0$, $\forall n \in \mathbb{N}$ and $\forall x \geq 1$.*

Proof. The second derivative equals

$$g_n''(x) = (n+2)(n+1)(1+\theta_1\lambda_1)x^n - 2\theta_1\lambda_1 \frac{\sin[(n+1)\theta]}{\sin\theta}.$$

We can see that the second derivative is a monotone function, increasing whenever $1 + \lambda_1\theta_1 > 0$. For $\theta_1 \geq 0$, we can write

$$g_n''(x) \geq g_n''(1) \geq (n+2)(n+1)(1+\theta_1\lambda_1) - 2\theta_1\lambda_1(n+1).$$

The expression on the right hand side can be rewritten as

$$2(n+1) + (n+1)n(1+\theta_1\lambda_1) > 0. \quad \square$$

Lemma 3.3.8. *For a given $\theta \in (0, \pi)$ and $n \in \mathbb{N}$, let $g_n(\cdot)$ denote $g_{n,\theta}(\cdot)$. Suppose $\theta_1 \geq 0$. If $g_{n,\theta}(1) < 0$, then there exists one and only one $x_n > 1$ such that $g_n(x_n) = 0$ and $g_n(x) \geq 0$, $\forall x \geq x_n$.*

Proof. First assume that $g_n'(1) > 0$. From Lemma 3.3.7 we know that $g_n''(x) > 0, \forall n \in \mathbb{N}$ and $x \geq 1$, which implies that $g_n'(x)$ is increasing for $x \geq 1$ and therefore positive on $(1, \infty)$. From the positivity of the derivative $g_n'(\cdot)$ on $(1, \infty)$, it follows that the function $g_n(\cdot)$ is increasing for $x > 1$. Therefore, there must exist some $x_n > 1$ such that $g_n(x_n) = 0$ and $g_n(x) > 0$, $\forall x > x_n$.

Now, assume that $g_n'(1) \leq 0$. From the fact that the second derivative $g_n''(1)$ is positive, the derivative $g_n'(\cdot)$ is increasing on $(1, \infty)$, which implies that there must be one point $x_n' > 1$ such that the function $g_n(\cdot)$ is decreasing in $1 \leq x < x_n'$ and increasing for $x \geq x_n'$. Conclusively, there must be some point $x_n > x_n'$ such that $g_n(x_n) = 0$ and $g_n(x) > 0$, $\forall x > x_n$. $\square$

Assume that $\theta \in (0, \pi)$ is given. For those $n \in \mathbb{N}$, for which $g_{n,\theta}(1) \geq 0$, we have $\psi_n \geq 0$. This follows from the same chain of arguments as for the $AR(3)$ model. The function $g_{n,\theta}(\cdot)$ is non-negative on $(1, \infty)$, Lemma 3.3.6 and Lemma 3.3.7, and therefore $g_{n,\theta}\left(\frac{|\lambda_2|}{\lambda_1}\right) \geq 0$. For $n \in \mathbb{N}$, such that $g_{n,\theta}(1) < 0$, it follows from Lemma 3.3.8 that there exists one and only one root $x_{n,\theta} > 1$ of the equation $g_{n,\theta}(x) = 0$ and that the function $g_{n,\theta}(\cdot)$ is non-negative on $(x_{n,\theta}, \infty)$. It then follows that if $|\lambda_2|/\lambda_1 \geq x_{n,\theta} > 1$, then a single coefficient ψ_n is non-negative.

Similarly as in Section 3.2.3, we can show that for certain θ's in $(0, \pi)$ it holds that $g_{n,\theta}(1) \geq 0, \forall n \in \mathbb{N}$. The non-negativity of the sequence $\{\psi_j^{**}\}_{j=0}^{\infty}$ is then equivalent to the statement $|\lambda_2|/\lambda_1 \geq 1$. This situation is summarized in the following theorem.

Theorem 3.3.9. *Suppose that* $\lambda_1 \in R$ *and* $\lambda_2 = \bar{\lambda}_3 = |\lambda_2|\mathrm{e}^{\mathrm{i}\theta} = a + b\mathrm{i}$, *where* $a, b \in R$ *and* $0 < \theta < \pi$. *If* $\theta_1 \geq 0$ *and* $\theta = 2\pi/k$ *for some integer* $k \geq 3$, *then the sequence* $\{\psi_j\}_{j=0}^{\infty}$ *is nonnegative if and only if* $|\lambda_2| \geq \lambda_1 > 1$.

Proof. It suffices to show that $g_{n,\theta}(1) \geq 0$, $\forall n \geq 0$, whenever $\theta = 2\pi/k$ for some integer $k \geq 3$. We know from Theorem 3.2.4 that this condition is sufficient for the functions

$$f_{n,\theta}(x) = x^{n+2} - x\frac{\sin[(n+2)\theta]}{\sin\theta} + \frac{\sin[(n+1)\theta]}{\sin\theta} \quad (n \geq 0)$$

to be non-negative on $[1, \infty)$. It can be easily verified that the function $g_{n,\theta}(\cdot)$ is a linear combination of the functions $f_{n,\theta}(\cdot)$ and $f_{n-1,\theta}(\cdot)$, namely

$$g_{n,\theta}(x) = f_{n,\theta}(x) + x\,\theta_1\lambda_1 f_{n-1,\theta}(x) \quad (n \geq 0),$$

where $f_{-1,\theta}(\cdot)$ is defined as a zero constant function. Then for $\lambda_1 > 1$ and $\theta_1 \geq 0$ we obtain $g_{n,\theta}(1) \geq 0$, $\forall n \in \mathbb{N}$. Therefore the condition $|\lambda_2|/\lambda_1 \geq 1$ is sufficient and necessary for the non-negativity of the sequence $\{\psi_j^{**}\}_{j=0}^{\infty}$, which is equivalent to the non-negativity of $\{\psi_j\}_{j=0}^{\infty}$. $\square$

Remark 3.3.2. In Theorem 3.3.5 we already stated that when $\theta_1 \geq 0$ and $\theta = 2\pi/k, k = 3, 4, \ldots$, the condition $|\lambda_2|/\lambda_1 \geq 1$ was sufficient for the non-negativity of $\{\psi_j^{**}\}_{j=0}^{\infty}$. In Theorem 3.3.9, we proved that this condition was also necessary.

Similarly as in Theorem 3.2.4, we would assume that if $\theta \notin \{2\pi/k : k = 3, 4, \ldots\}$, the condition $|\lambda_2|/\lambda_1 \geq 1$ is not sufficient to assure that the whole sequence $\{\psi_j\}_{j=0}^{\infty}$ is non-negative. As we will see in the following theorem, this condition is still sufficient when $\theta_1\lambda_1 = 1$.

Theorem 3.3.10. *Suppose that* $\lambda_1 \in R$, $\theta_1 \geq 0$ *and* $\lambda_2 = \bar{\lambda}_3 = |\lambda_2|\mathrm{e}^{\mathrm{i}\theta} = a + b\mathrm{i}$, *where* $a, b \in R$ *and* $0 < \theta < \pi$. *If* $\lambda_1 > 1$ *and* $\theta_1\lambda_1 = 1$, *then the sequence* $\{\psi_j\}_{j=0}^{\infty}$ *is non-negative.*

Proof. It suffices to show that under the given assumptions, $g_{n,\theta}(1) \geq 0$ for all $n \in \mathbb{N}$ and for every $\theta \in (0, \pi)$. Note that when $\theta_1\lambda_1 = 1$ we have $g_{n,\theta}(1) = 2 - \frac{\sin[(n+2)\theta]}{\sin\theta} + \frac{\sin(n\theta)}{\sin\theta}$. Then we have

$$\sin(n\theta) - \sin[(n+2)\theta] = -2\sin\theta\cos[(n+1)\theta] \geq -2\sin\theta.$$

Since $\sin\theta > 0$, this is equivalent to

$$\frac{\sin(n\theta) - \sin[(n+2)\theta]}{\sin\theta} \geq -2.$$

Then we have

$$g_{n,\theta}(1) \geq 0, \quad \forall n \in \mathbb{N} \quad \text{and} \quad \theta \in (0,\pi). \quad \square$$

Now assume that $\theta \notin \{2\pi/k : k = 3,4,\dots\}$ and $\theta_1\lambda_1 \neq 1$. Similarly as in Section 3.2.3 we denote I_θ a set of indices $n \in \mathbb{N}$, such that $g_{n,\theta}(1) < 0$. We hypothesize that this set is non-empty under the given assumptions. According to Theorem 3.3.8 we can find for each $n \in I_\theta$ a single root $x_{n,\theta} > 1$ of the equation $g_{n,\theta}(x) = 0$. Again, if we denote $x_\theta^* = \max_{n\in I_\theta} x_{n,\theta}$, then $g_{n,\theta}(x) \geq 0$ for $x \geq x_\theta^*$ and $\forall n \in \mathbb{N}$. For the $AR(3)$ process, the situation was much easier, because the maximal root x_θ^* was actually the "first" root $x_{n_0,\theta}$, where $n_0 = \min_{n\in I_\theta} n$. This is no longer true for $ARMA(3,1)$, see Figure 3.6(b).

The blue dots in Figure 3.6(a) and Figure 3.6(b) represent the roots (that are greater than one) of the equations $g_{n,\theta}(x) = 0$ for $n \in I_\theta, n \leq 50$, where the argument θ was chosen equal to 2. The parameter θ_1 is also kept fixed. On the left we have $\theta_1 = 0$ and on the right $\theta_1 = 2$. The dashed line represents a function of a continuous variable n defined as a constant 1 if $g_{n,\theta}(1) \geq 0$ and $x_{n,\theta}$ if $g_{n,\theta}(1) < 0$. The Figure 3.6(a) actually corresponds to the roots of a function $f_{n,\theta}(\cdot)$. We can see that the first root is indeed the largest one. According to the Theorem 3.2.4 we know that this holds in general. On the other hand, the first root of the function $g_{n,\theta}(\cdot)$ is not the maximal one, Figure 3.6(b). However, we observe, that the maximal root is contained within first, say k, roots with the smallest indices $n_1 < n_2 < \dots < n_k, n_i \in I_\theta, i = 1,\dots,k$. The question is how to find the appropriate k in practice. The guideline is given in the following theorem.

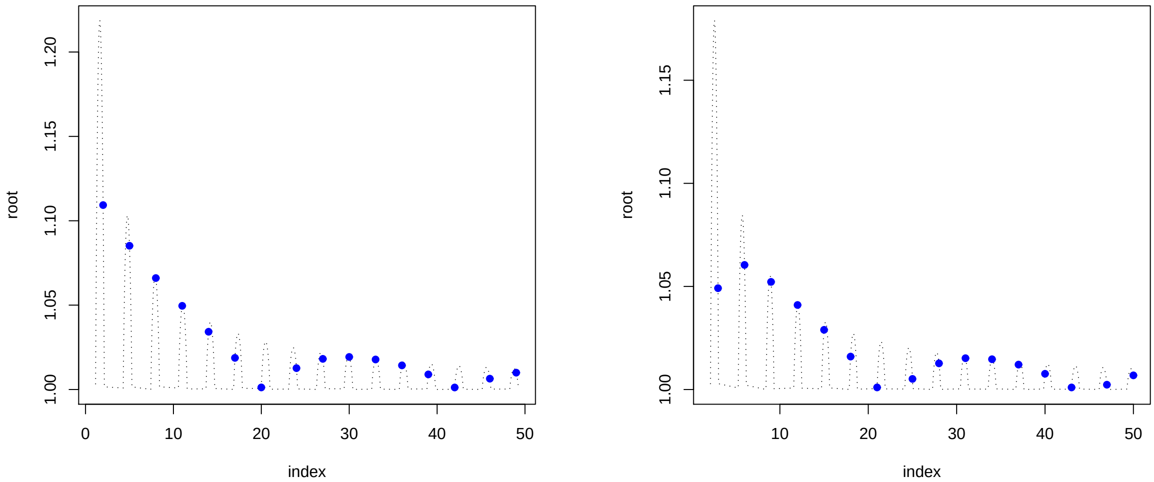
Theorem 3.3.11. *Suppose that $\lambda_1 \in R$ and $\lambda_2 = \bar{\lambda}_3 = |\lambda_2|\mathrm{e}^{\mathrm{i}\theta} = a + b\mathrm{i}$, where $a, b \in R$ and $0 < \theta < \pi$. Assume $\theta_1 \geq 0$, $\theta_1\lambda_1 \neq 1$ and $\theta \notin \{2\pi/k : k = 3,4,\dots\}$. For the given $\theta \in (0,\pi)$ and an integer number a^*, denote I_{θ,a^*} a set of indices $n \leq a^*, n \in \mathbb{N}$, so that $g_{n,\theta}(1) < 0$ for $n \in I_{\theta,a^*}$. Let a' be defined as the smallest positive integer number a^* such that*

$$a^* \geq \frac{\log 2}{\log \tilde{x}_{\theta,a^*}} \quad \text{and} \quad \min\left[\left(\frac{\theta a^*}{2}\right) \bmod \pi, \pi - \left(\frac{\theta a^*}{2}\right) \bmod \pi\right] \leq \frac{\sin\theta}{4},$$

where $\tilde{x}_{\theta,a^*} = \max_{n\in I_{\theta,a^*}} x_{n,\theta}$ and $x_{n,\theta}$ is a root of the equation $g_{n,\theta}(x) = 0$. Then the sequence $\{\psi_j\}_{j=0}^\infty$ is non-negative if and only if $|\lambda_2|/\lambda_1 \geq \tilde{x}_{\theta,a'} > 1$.

Proof. Assume that for some $\theta \in (0,\pi)$ and $n \in \mathbb{N}$ it holds that $g_{n,\theta}(1) < 0$. Lemma 3.3.8 gives the existence of one particular root $x_{n,\theta} > 1$ of the equation $g_{n,\theta}(x) = 0$. Furthermore, the function $g_{n,\theta}(x)$ is increasing and therefore non-negative on (x_n, ∞). The condition $|\lambda_2|/\lambda_1 \geq x_{n,\theta}$ is then equivalent to the non-negativity of a single coefficient ψ_n^{**}, resp. coefficient ψ_n. To prove the theorem, we need to show that for a given $\theta \in (0,\pi)$ it holds that $g_{n,\theta}(x) \geq 0, n \in \mathbb{N}$, whenever $x \geq \tilde{x}_{\theta,a'}$. Denote

$$m_{\theta,x}(n) = -x^2\theta_1\lambda_1\frac{\sin[(n+1)\theta]}{\sin\theta} - x\left\{\frac{\sin[(n+2)\theta]}{\sin\theta} - \theta_1\lambda_1\frac{\sin n\theta}{\sin\theta}\right\} + \frac{\sin[(n+1)\theta]}{\sin\theta}.$$



(a) Roots of the function $f_{n,\theta(\cdot)}, \theta = 2$ (b) Roots of the function $g_{n,\theta}(\cdot), \theta_1 = 2, \theta = 2$

Figure 3.6: The dashed line is the function $h(n) = \begin{cases} 1 & \text{if} \quad g_{n,\theta}(1) \geq 0 \\ x_{n,2} & \text{if} \quad g_{n,\theta}(1) < 0 \end{cases}$, the blue dots are the roots $x_{i,\theta}, i \in I_\theta, i \leq 50, \theta = 2$

Then we can write

$$g_{n,\theta}(x) = x^{n+2}(1 + \theta_1\lambda_1) + m_{\theta,x}(n).$$

The function $m_{\theta,x}(n)$ as a continuous function in n is periodic. Its period $2\pi/\theta$ is not an integer number for $\theta \notin \{2\pi/k : k = 3, 4, \dots\}$. However, we might find some "approximate period", say a^*, such that $|m_{\theta,x}(n + a^*) - m_{\theta,x}(n)|$ is sufficiently small for any n. For a^* large enough, the difference between the leading terms $x^{n+a^*}(1+\theta_1\lambda_1)$ and $x^n(1 + \theta_1\lambda_1)$ will be sufficiently big to assure that $g_{n+a^*,\theta}(x) - g_{a^*,\theta}(x) \geq 0$, $\forall n \in \mathbb{N}$ and for all x greater than some x^*. We will now show that the integer $a^* = a'$ and $x^* = \tilde{x}_{\theta,a'}$ from our theorem satisfy these requirements.

Note that $\forall n \in \mathbb{N}$ and $\forall a \in \mathbb{N}$ it holds that

$$\begin{aligned} |\sin[(n+a)\theta - \sin(n\theta)]| &= \left| 2\cos\left(\frac{2n+a}{2}\theta\right)\sin\left(\frac{a}{2}\theta\right)\right| \\ &\leq 2\min\left[\left(\frac{\theta a}{2}\right) \bmod \pi, \pi - \left(\frac{\theta a}{2}\right) \bmod \pi\right]. \end{aligned}$$

Denote $\Delta = 2\min\left[\left(\frac{\theta a'}{2}\right) \bmod \pi, \pi - \left(\frac{\theta a'}{2}\right) \bmod \pi\right]$. From our assumptions it holds that $2\Delta \leq \sin\theta$. Then for $x \geq 1$ and $\theta_1 \geq 0$, we have

$$\begin{aligned}|m_{\theta,x}(n+a') - m_{\theta,x}(n)|\sin\theta &\leq x^2\theta_1\lambda_1\Delta + x\Delta + x\theta_1\lambda_1\Delta + \Delta \\ &= \Delta(1+x)(1+x\theta_1\lambda_1) \leq 2\Delta x^2(1+\theta_1\lambda_1) \\ &\leq x^2(1+\theta_1\lambda_1)\sin\theta.\end{aligned}$$

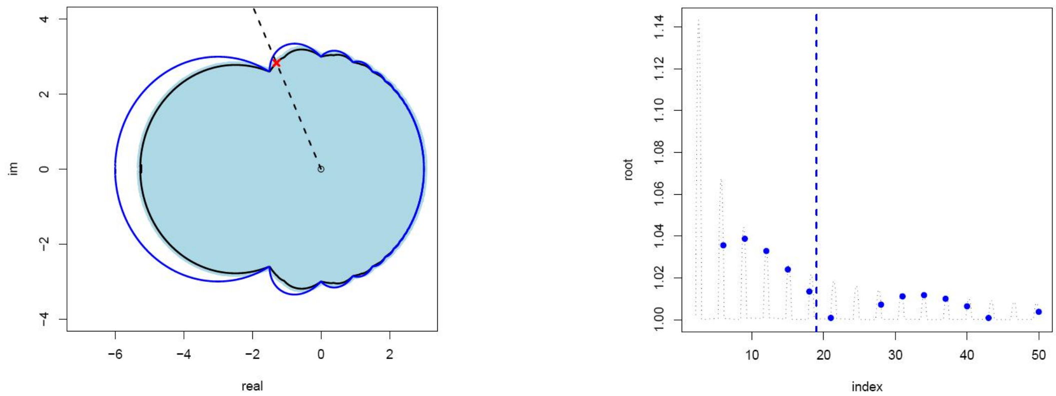
Because $\sin\theta > 0$, we have $|m_{\theta,x}(n+a') - m_{\theta,x}(n)| \leq x^2(1+\theta_1\lambda_1)$.

Conclusively, we obtain

$$\begin{aligned}g_{n+a',\theta}(x) - g_{n,\theta}(x) &\geq x^{n+a'+2}(1+\theta_1\lambda_1) - x^{n+2}(1+\theta_1\lambda_1) - x^2(1+\theta_1\lambda_1) \\ &= x^2(1+\theta_1\lambda_1)(x^{n+a'} - x^n - 1).\end{aligned}$$

From the assumption $a' \geq \frac{\log 2}{\log \tilde{x}_{\theta,a'}}$ it holds that $\tilde{x}_{\theta,a'}^{n+a'} - \tilde{x}_{\theta,a'}^{n} - 1 \geq 0$. Therefore $g_{n+a',\theta}(\tilde{x}_{\theta,a'}) - g_{n,\theta}(\tilde{x}_{\theta,a'}) \geq 0, \forall n \in \mathbb{N}$. From the definition of $\tilde{x}_{\theta,a'}$ it holds that $g_{n,\theta}(\tilde{x}_{\theta,a'}) \geq 0$ for all $n \leq a'$. We have shown above that $g_{n+a',\theta}(\tilde{x}_{\theta,a'}) \geq g_{n,\theta}(\tilde{x}_{\theta,a'}) \geq 0, \forall n \in \mathbb{N}$. This means that $g_{n+a',\theta}(\tilde{x}_{\theta,a'}) \geq 0, \forall n \in \mathbb{N}$. From Lemma 3.3.6 and Lemma 3.3.7, we obtain that $g_{n,\theta}(\cdot)$ is increasing on $[\tilde{x}_{\theta,a'}, \infty)$ and therefore $g_{n,\theta}(x) \geq 0, \forall x \geq \tilde{x}_{\theta,a'}$. □

Example 3.3.1. We illustrate the application of Theorem 3.3.11 on an example. Assume that $\lambda_1 = 3, \theta_1 = 2$ and $\theta = 2$. We need to find the maximal root $\max_{n\in I_2} x_{n,2}$. If we can find an integer number a' which satisfies the two requirements in the Theorem 3.3.11, we know that the maximal root will be contained within first $k \leq a'$ roots. To find a' which fulfills the first requirement, let us take the smallest integer a such that $\min[a \bmod \pi, \pi - a \bmod \pi] < \frac{\sin 2}{4} = 0.227$. This integer equals 3. The set of indices $I_{\theta=2,a=3}$ is an empty set, since the smallest integer n such that $g_{n,2}(1) < 1$ is 6. The second smallest integer a which satisfies the condition $\min[a \bmod \pi, \pi - a \bmod \pi] < 0.227$ is 19. Then we have $I_{2,19} = \{6, 9, 12, 15, 18\}$. The corresponding roots are 1.036, 1.039, 1.033, 1.024 and 1.013. The maximum of these roots $\tilde{x}_{2,19}$ equals 1.039. The second requirement $a' \geq \frac{\log 2}{\log \tilde{x}_{2,a'}}$ is satisfied for $a' = 19$, indeed $19 \geq \frac{\log 2}{\log 1.039} = 18.25$. According to the Theorem 3.3.11, the sequence $\{\psi_j\}_{j=0}^{\infty}$ is non-negative if and only if $|\lambda_2|/3 \geq 1.039$. This condition is depicted graphically in Figure 3.7(a). The light-blue region again represents those complex roots λ_2 and λ_3 for which at least one coefficient $\psi_j, j \in \mathbb{N}$, is negative. The dark blue envelope is the boundary of the negativity region of $AR(3)$ process. The dashed line connects all the complex roots with the argument equal to 2. We have just derived that the complex number with argument 2 has to be at least at a distance of 3×1.039 from the origin to assure the non-negativity of $\{\psi_j\}_{j=0}^{\infty}$. A complex root with such a distance is marked by the red point. The Figure 3.7(b) depicts the roots of the equations $g_{n,2}(x) = 0, n \leq 50, n \in I_2$. The dashed vertical line corresponds to the approximate



(a) The negativity region for $ARMA(3,1)$ (b) The roots of the function $g_{n,2}(\cdot)$

Figure 3.7: On the left, the light blue area is the region of negativity for $ARMA(3,1)$ with $\lambda_1 = 3$ and $\theta_1 = 2$, the dark blue closed curve corresponds to the boundary of the negativity region for $AR(3)$; on the right, the roots of the function $g_{n,2}(\cdot)$ with $\theta_1 = 2$ and $\lambda_1 = 3$, the dashed vertical line corresponds to $n = 19$

period $a' = 19$. We observe that indeed the largest root, being the second one in a row, is contained within the first $\text{card} I_{2,19} = 5$ roots.

Similarly as for the $AR(3)$ process, we can prove that when the real part of the conjugate complex roots λ_2, λ_3 is greater than λ_1, then the non-negativity of the sequence $\{\psi_j\}_{j=1}^{\infty}$ is always assured.

Theorem 3.3.12. *Suppose that $\lambda_1 \in R$, $\theta_1 \geq 0$ and $\lambda_2 = \bar{\lambda}_3 = |\lambda_2|e^{i\theta} = a + bi$, where $a, b \in R$ and $0 < \theta < \pi$. If $a \geq \lambda_1 > 1$, then $\{\psi_j\}_{j=0}^{\infty}$ is non-negative.*

Proof. Note that $a \geq \lambda_1 > 1$ implies $|\lambda_2| \cos\theta \geq \lambda_1 > 1$, or equivalently

$$\left|\frac{\lambda_2}{\lambda_1}\right| \geq \frac{1}{\cos\theta} > 1.$$

We need to prove that

$$g_n(x) \geq 0, \quad \text{for} \quad x \geq \frac{1}{\cos\theta} > 1.$$

Note that it suffices to show that $g_n\left(\frac{1}{\cos\theta}\right) \geq 0$. From Lemma 3.3.6 we then readily obtain that $g_n'\left(\frac{1}{\cos\theta}\right) \geq 0$. From Lemma 3.3.7 we know that the second derivative is positive on $[1, \infty)$. It then follows that $g'(x) \geq 0$ for $x \geq \frac{1}{\cos\theta} > 1$ and therefore $g(x) \geq 0$

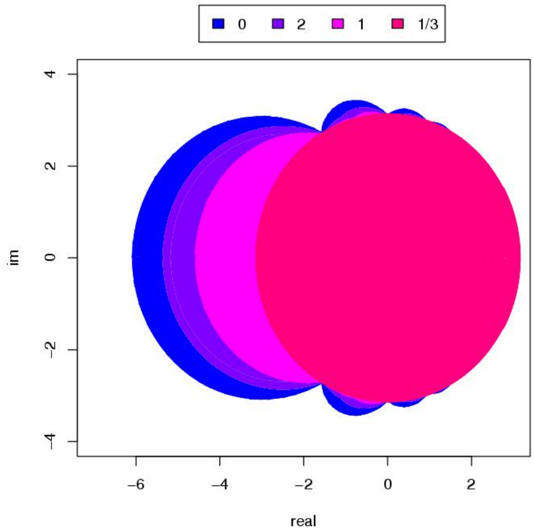


Figure 3.8: Negativity regions for $ARMA(3,1)$ models with $\lambda_1 = 3$ and $\theta_1 = 0$ (dark blue), $\theta_1 = 2$ (violet), $\theta_1 = 1$ (pink), $\theta_1 = 1/3$ (red)

for $x \geq \frac{1}{\cos\theta} > 1$. Now we show that that $g_n\left(\frac{1}{\cos\theta}\right) > 0$. We have

$$
\begin{aligned}
g_{n,\theta}\left(\frac{1}{\cos\theta}\right) &= \frac{1 - \cos^{n+1}\theta\cos[(n+1)\theta] + \theta_1\lambda_1 - \theta_1\lambda_1\cos^n\theta\cos(n\theta)}{\cos^{n+2}\theta} \\
&= \frac{1 + \theta_1\lambda_1 - \cos^n\theta[\theta_1\lambda_1\cos(n\theta) + \cos\theta\cos[(n+1)\theta]]}{\cos^{n+2}\theta} \\
&\geq \frac{1 + \theta_1\lambda_1 - \cos^n\theta(1 + \theta_1\lambda_1)}{\cos^{n+2}\theta} \\
&= \frac{(1 + \theta_1\lambda_1)(1 - \cos^n\theta)}{\cos^{n+2}\theta} \geq 0. \quad \square
\end{aligned}
$$

Remark 3.3.3. The Figure 3.7(a) pictures the negativity/non-negativity region for one special $ARMA(3,1)$ model. In order to get an idea about the influence of the coefficient θ_1 on the size of the region, we depicted several regions for several choices of θ_1, see Figure 3.8. The dark blue region corresponds to the choice $\theta_1 = 0$, which is the $AR(3)$ negativity region we already saw in Figure 3.2. We observe that the other three regions, for $\theta_1 = 2$ (violet), $\theta_1 = 1$ (pink) and $\theta_1 = 1/3$ (red), are nested within each other. The smallest one is a circle centered at zero with a diameter $1/\lambda_1 = 1/3$. This corresponds to the Theorem 3.3.10. The principal observation is that with increasing θ_1 which is greater than $1/\lambda_1$, the negativity region stretches, but according to Theorem 3.3.5 never exceeds the $AR(3)$ region. A similar stretching tendency can be observed also for $\theta_1 < 1/\lambda_1$ which are approaching zero. The second observation is that the negativity region never includes roots with a real part greater than $\lambda_1 = 3$. This is in accordance with Theorem 3.3.12.

(b) $\theta_1 < 0$

A similar result as in Theorem 3.3.11 can be obtained also when $\theta_1 < 0$. Again we will need to prove some auxiliary statements about the functions $g_{n,\theta}(\cdot)$.

Lemma 3.3.13. *For a given $\theta \in (0, \pi)$ and $n \in \mathbb{N}$, let $g_n(\cdot)$ denote $g_{n,\theta}(\cdot)$. Suppose that $\theta_1 < 0$ and $1 + \theta_1\lambda_1 \geq 0$. Then $\forall n \in \mathbb{N}$ and $\forall\, x \geq 1$, the following statement holds:*

$$\textit{if} \quad g_n(1) \geq 0 \quad \textit{then} \quad g_n'(1) \geq 0.$$

Proof. We have

$$g_{n,\theta}'(1) - g_{n,\theta}(1) = (n+1)(1+\theta_1\lambda_1) - \frac{\sin[(n+1)\theta]}{\sin\theta}(1+\theta_1\lambda_1).$$

Since $1 + \theta_1\lambda_1 \geq 0$ and $\frac{\sin[(n+1)\theta]}{\sin\theta} < n+1$ it holds that $g_{n,\theta}'(1) - g_{n,\theta}(1) \geq 0$, $\forall n \in \mathbb{N}$ and $\forall \theta \in (0, \pi)$. □

Lemma 3.3.14. *For a given $\theta \in (0, \pi)$ and $n \in \mathbb{N}$, let $g_n(\cdot)$ denote $g_{n,\theta}(\cdot)$. Suppose $\theta_1 < 0$ and $1 + 2\,\theta_1\lambda_1 \geq 0$. If $g_{n,\theta}(1) < 0$, then there exists one and only one $x_n > 1$ such that $g_n(x_n) = 0$ and $g_n(x) \geq 0$, $\forall x \geq x_n$.*

Proof. First we show that the assumption $1 + 2\,\theta_1\lambda_1 > 0$ implies that $\forall n \in \mathbb{N}$ and any $\theta \in (0, \pi)$ it holds $g_{n,\theta}''(x) \geq 0, x \geq 1$. For $\theta_1 < 0$ and $x \geq 1$ we have

$$g_n''(x) \geq g_n''(1) \geq (n+2)(n+1)(1+\theta_1\lambda_1) + 2\theta_1\lambda_1(n+1).$$

The expression on the right hand side can be rewritten as

$$n(n+1)(1+\theta_1\lambda_1) + 2(n+1)(1+2\theta_1\lambda_1),$$

which is non-negative for any $n \in \mathbb{N}$, assuming $1{+}2\,\theta_1\lambda_1 \geq 0$. Now assume that for a given $\theta \in (0, \pi)$ and $n \in \mathbb{N}$, $g_{n,\theta}(1) < 0$ and $g_n'(1) \geq 0$. We know that under the assumption $1 + 2\,\theta_1\lambda_1 \geq 0$, the second derivative is positive on $[0, \infty)$ whenever $1 + \theta_1\lambda_1 > 0$, which is satisfied because we assume $1 + 2\,\theta_1\lambda_1 \geq 0$. This implies that $g_n'(x)$ is increasing for $x \geq 1$ and therefore positive on $(1, \infty)$. From the positivity of the derivative $g_n'(\cdot)$ on $(1, \infty)$, it follows that the function $g_n(\cdot)$ is increasing for $x > 1$. Therefore, there must exist some $x_n > 1$ such that $g_n(x_n) = 0$ and $g_n(x) > 0$, $\forall x > x_n$.

Now, assume that $g_{n,\theta}(1) < 0$ and $g_n'(1) < 0$. We know that the second derivative $g_n''(1)$ is positive on $[1, \infty)$ and therefore the derivative $g_n'(\cdot)$ is increasing on $[1, \infty)$, which implies that there must be one point $x_n' > 1$ such that the function $g_n(\cdot)$ is decreasing in $1 \leq x < x_n'$ and increasing for $x \geq x_n'$. Conclusively, there must be some point $x_n > x_n'$ such that $g_n(x_n) = 0$ and $g_n(x) > 0$, $\forall x > x_n$. □

Remark 3.3.4. The assumption $1 + 2\,\theta_1\lambda_1 \geq 0$ in Lemma 3.3.14 might be replaced by $1 + \theta_1\lambda_1 > 0$. This observation has been supported by empirical evidence, but appeared slightly difficult to prove analytically. The key would be to show that if $1 + \theta_1\lambda_1 > 0$, then the non-negativity of the first derivative implies the non-negativity of the second derivative on the interval $[1, \infty)$.

In the previous section we distinguished between two situations, where the argument θ could or could not be expressed as $2\pi/k$ for some $3 \leq k \in \mathbb{N}$. For $\theta_1 < 0$ we will do practically the same thing. Previously, if $\theta \in \{2\pi/k : k = 3, 4, \dots\}$ the sequence was automatically non-negative as long as $1 < \lambda_1 \leq |\lambda_2|$ was satisfied. For those θ's which could not be expressed this way, the ratio $|\lambda_2|/\lambda_1$ had to be greater than or equal to the maximal root, which is contained within several first roots with the lowest indices. When $\theta_1 < 0$ the results change slightly. Even if $\theta \in \{2\pi/k : k = 3, 4, \dots\}$, the condition $|\lambda_2|/\lambda_1 \geq 1$ is not enough to assure the non-negativity. Again this ratio needs to be greater than or equal to the largest root. Similarly, we show that the largest root is contained within first, say j, roots. The advantage now is that it is straightforward to determine the upper bound for j. We show that the maximal root lies within first $j \leq k$ roots, where k is determined from the relation $\theta = 2\pi/k$.

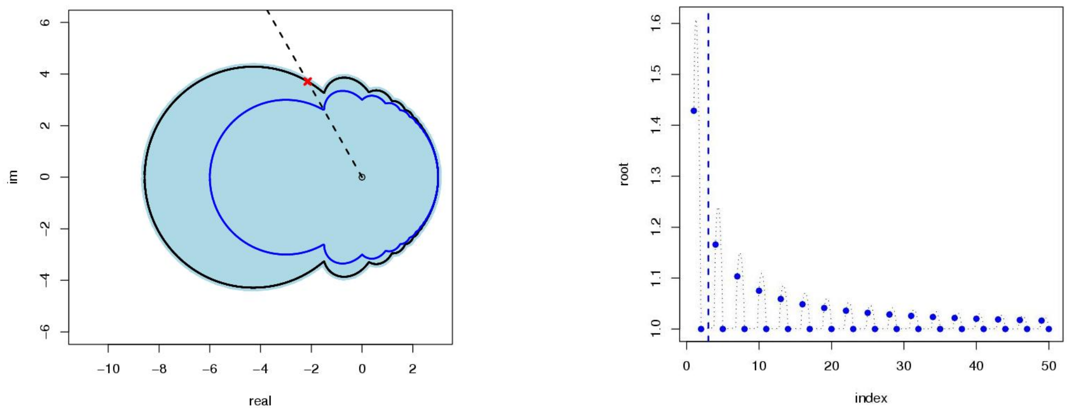
Theorem 3.3.15. *Suppose that $\lambda_1 \in R$ and $\lambda_2 = \bar{\lambda}_3 = |\lambda_2|e^{i\theta} = a + bi$, where $a, b \in R$ and $0 < \theta < \pi$. Assume $\theta_1 < 0$, $1 + 2\,\theta_1\lambda_1 \geq 0$ and $\theta \in \{2\pi/k : k = 3, 4, \dots\}$. For the given θ and an integer number a^*, denote I_{θ,a^*} a set of indices $n \leq a^*, n \in \mathbb{N}$, so that $g_{n,\theta}(1) < 0$ for $n \in I_{\theta,a^*}$ and denote $\tilde{x}_{\theta,a^*} = \max_{n \in I_{\theta,a^*}} x_{n,\theta}$, where $x_{n,\theta}$ is a root of equation $g_{n,\theta}(x) = 0$. Then the sequence $\{\psi_j\}_{j=0}^{\infty}$ is non-negative if and only if $|\lambda_2|/\lambda_1 \geq \tilde{x}_{\theta,k} > 1$.*

Proof. We know that for those indices $n \in \mathbb{N}$ for which $g_{n,\theta}(1) \geq 0$, it is always assured that $g_{n,\theta}(x) \geq 0, x \geq 1$. This follows from Lemma 3.3.13 and the fact that the second derivative is non-negative on $[1, \infty)$. Now, assume that $g_{n,\theta}(1) < 0$. Lemma 3.3.14 gives the existence of one particular root x_n of an equation $g_{n,\theta}(x) = 0$ on an interval $(1, \infty)$, such that $g_{n,\theta}(x)$ is increasing and therefore non-negative on (x_n, ∞). The condition $|\lambda_2|/\lambda_1 \geq x_{n,\theta}$ is then equivalent to the non-negativity of a single coefficient ψ_n.

To complete the proof of the theorem we need to show that if $x \geq \tilde{x}_{\theta,k}$, then $g_n(x) \geq 0$, $\forall n \geq 0$. Recall the definition of the function $m_{\theta,x}(\cdot)$:

$$m_{\theta,x}(n) = -x^2\theta_1\lambda_1 \frac{\sin[(n+1)\theta]}{\sin\theta} - x\left\{\frac{\sin[(n+2)\theta]}{\sin\theta} - \theta_1\lambda_1 \frac{\sin n\theta}{\sin\theta}\right\} + \frac{\sin[(n+1)\theta]}{\sin\theta}.$$

The function $m_{\theta,x}(n)$ is periodic with a period $2\pi/\theta$, i.e. $m_{\theta,x}(n) = m_{\theta,x}(n + 2\pi/\theta), \forall n \in \mathbb{N}$. The period equals k if $\theta = 2\pi/k$. From the definition of $\tilde{x}_{\theta,k}$ it follows that $g_{n,\theta}(x) \geq 0$ for all $x \geq \tilde{x}_{\theta,k}$ and $n = 0, \dots, k$. Because $g_{n,\theta}(x) = x^{n+2}(1 + \theta_1\lambda_1) + m_{\theta,x}(n)$ and $m_{\theta,x}(n) = m_{\theta,x}(n + k), n \in \mathbb{N}$, it holds that $g_{n+k,\theta}(x) > g_{n,\theta}(x)$ for any $n \in \mathbb{N}$ and $x \geq \tilde{x}_{\theta,k}$, as long as $1 + \theta_1\lambda_1 > 0$. The condition $1 + \theta_1\lambda_1 > 0$ follows from the



(a) The negativity region for $ARMA(3,1)$ (b) The roots $g_{n,2\pi/3}(\cdot)$

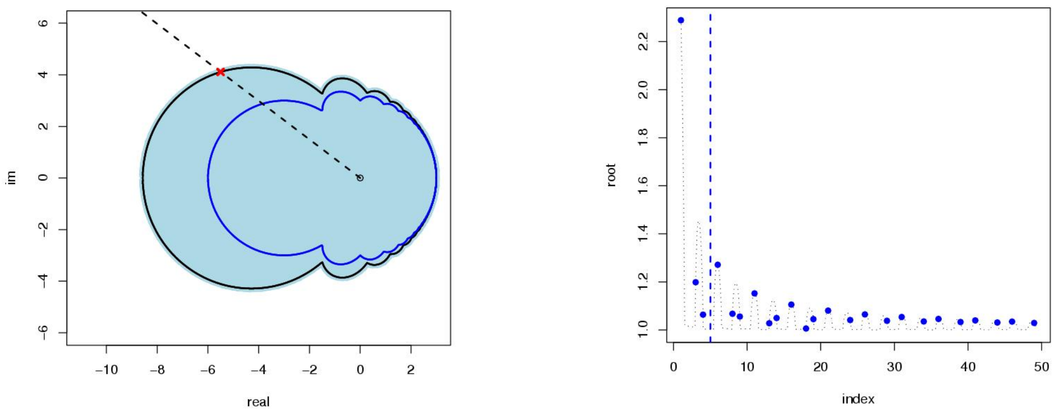
Figure 3.9: On the left, the light blue area is the region of negativity for $ARMA(3,1)$ with $\lambda_1 = 3$ and $\theta_1 = -0.1$, the dark blue closed curve corresponds to the boundary of the negativity region for $AR(3)$, dashed black line connects complex roots with an argument $2\pi/3$, the red cross corresponds to the root with argument $2\pi/3$ and absolute value 4.287; on the right, the roots of the function $g_{n,2\pi/3}(\cdot)$ with $\theta_1 = -0.1$ and $\lambda_1 = 3$, the dashed vertical line corresponds to $n = 3$

assumptions $1 + 2\,\theta_1\lambda_1 \geq 0, \theta_1 < 0$. From all the considerations, it easily follows that $g_{n,\theta}(x) \geq 0, \forall n \in \mathbb{N}$ and $x \geq \tilde{x}_{\theta,k}$. □

Example 3.3.2. We again illustrate the use of the Theorem 3.3.15 on the example. Suppose that $\theta_1 = -0.1, \lambda_1 = 3$ and $\theta = 2\pi/3$, i.e. $k = 3$. We know that the non-negativity of $\{\psi_j\}_{j=0}^{\infty}$ is equivalent to the condition $|\lambda_2| \geq 3 \times \tilde{x}_{2\pi/3}$, where $\tilde{x}_{2\pi/3}$ is the maximum of the roots $x_{n,2\pi/3}$ of the equations $g_{n,2\pi/3}(x) = 0, n \in I_{2\pi/3}$. From the Theorem 3.3.15 we know that this maximal root is contained within first card $I_{2\pi/3,3} \leq 3$ roots. The roots $x_{n,2\pi/3}$ for $n \in \mathrm{I}_{2\pi/3}, n \leq 50$, are depicted on Figure 3.9(b). There are only two indices $n \leq 3$ so that $g_{n,2\pi/3}(1) < 1$, i.e. card $I_{2\pi/3,3} = 2$. The two roots corresponding to these indices are 1.429 and 1.18. According to the Theorem 3.3.15. The sequence $\{\psi_j\}_{j=0}^{\infty}$ is non-negative if and only if $|\lambda_2| \geq 1.429 \times 3 = 4.287$. A complex root with an argument $2\pi/3$ and absolute value 4.287 is marked with a red cross on Figure 3.9(a). The light blue region is the negativity region for complex roots λ_2 and λ_3 for the given $\theta_1 = -0.1, \lambda_1 = 3$. This region contains the $AR(3)$ region (bounded with dark blue).

Now we focus on the case when $\theta \notin \{2\pi/k : k = 3, 4, \dots\}$.

Theorem 3.3.16. *Suppose that $\lambda_1 \in R$ and $\lambda_2 = \bar{\lambda}_3 = |\lambda_2|\mathrm{e}^{\mathrm{i}\theta} = a + b\mathrm{i}$, where $a, b \in R$ and $0 < \theta < \pi$. Assume $\theta_1 < 0$, $1 + 2\,\theta_1\lambda_1 > 0$ and $\theta \notin \{2\pi/k : k = 3, 4, \dots\}$. For the given θ and an integer number a^*, denote I_{θ,a^*} a set of indices $n \leq a^*, n \in \mathbb{N}$, so that $g_{n,\theta}(1) < 0$ for $n \in I_{\theta,a^*}$. Let a' be defined as the smallest positive integer number a^**



(a) The negativity region for $ARMA(3,1)$

(b) The roots of the function $g_{n,2.5}(\cdot)$

Figure 3.10: On the left, the light blue area is the region of negativity for $ARMA(3,1)$ with $\lambda_1 = 3$ and $\theta_1 = -0.1$, the dark blue closed curve corresponds to the boundary of the negativity region for $AR(3)$, dashed black line connects complex roots with an argument 2.5, the red cross corresponds to the root with argument 2.5 and absolute value 6.87. On the right, the roots of the function $g_{n,2.5}(\cdot)$ with $\theta_1 = -0.1$ and $\lambda_1 = 3$, the dashed vertical line corresponds to $n = 3$

such that

$$a^* \geq \frac{\log 2}{\log \tilde{x}_{\theta,a^*}} \quad \textit{and} \quad \min\left[\left(\frac{\theta a^*}{2}\right) \bmod \pi, \pi - \left(\frac{\theta a^*}{2}\right) \bmod \pi\right] \leq \frac{(1+\theta_1\lambda_1)\sin\theta}{4(1-\theta_1\lambda_1)},$$

where $\tilde{x}_{\theta,a^*} = \max_{n \in I_{\theta,a^*}} x_{n,\theta}$ *and* $x_{n,\theta}$ *is a root of equation* $g_{n,\theta}(x) = 0$. *Then the sequence* $\{\psi_j\}_{j=0}^{\infty}$ *is non-negative if and only if* $|\lambda_2|/\lambda_1 \geq \tilde{x}_{\theta,a'} > 1$.

Proof. The construction of the proof is analogous to the one in Theorem 3.3.11. Denote now $\Delta = 2\min\left[\left(\frac{\theta a^*}{2}\right) \bmod \pi, \pi - \left(\frac{\theta a^*}{2}\right) \bmod \pi\right]$. For $x \geq 1, \theta_1 < 0$ and $1 + 2\theta_1\lambda_1 \geq 0$, we now have

$$\begin{aligned} |m_{\theta,x}(n+a') - m_{\theta,x}(n)| \sin\theta \leq & -x^2\theta_1\lambda_1\Delta + x\Delta - x\theta_1\lambda_1\Delta + \Delta \\ = & \Delta(1+x)(1-x\theta_1\lambda_1) \leq \Delta 2x^2(1-\theta_1\lambda_1). \end{aligned}$$

From the assumptions it follows that $2\Delta(1-\theta_1\lambda_1) \leq (1+\theta_1\lambda_1)\sin\theta$. Therefore, $|m_{\theta,x}(n+a') - m_{\theta,x}(n)|\sin\theta \leq x^2(1+\theta_1\lambda_1)\sin\theta$. Because $\sin\theta > 0$, we have $|m_{\theta,x}(n+a') - m_{\theta,x}(n)| \leq x^2(1+\theta_1\lambda_1)$. The rest of the proof is again the same as in Theorem 3.3.11. □

Example 3.3.3. Let us suppose that $\theta_1 = -0.1, \lambda_1 = 3$ and $\theta = 2.5$. The minimal integer number a for which $\min\left[\left(\frac{2.5\,a}{2}\right) \bmod \pi, \pi - \left(\frac{2.5\,a}{2}\right) \bmod \pi\right] \leq \frac{(1+\theta_1\lambda_1)\sin\theta}{4(1-\theta_1\lambda_1)} = 0.08$ is 5. The set $I_{2.5,5}$ contains three indices $1, 3, 4$. The corresponding roots are 2.29, 1.19 and 1.06. Because it holds that $5 > \frac{\log 2}{\log 2.29} = 0.8370$, according to Theorem 3.3.16, 2.29 is the

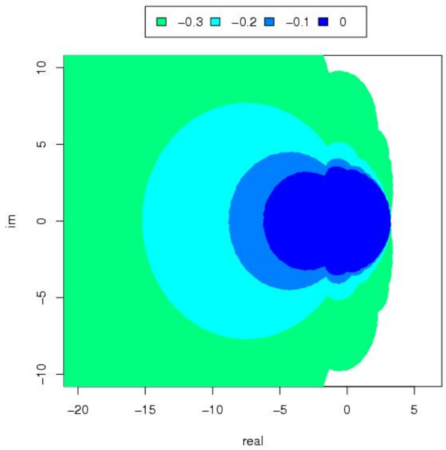


Figure 3.11: Negativity regions for $ARMA(3,1)$ models with $\lambda_1 = 3$ and $\theta_1 = -0.3$ (green), $\theta_1 = -0.2$ (light-blue), $\theta_1 = -0.2$ (marine blue), $\theta_1 = 0$ (dark blue)

overally maximal root. From the plot of roots on Figure 3.10(b) we see that it is indeed the case. Also note the irregular behavior of the roots when compared to Figure 3.9(b). This artefact is caused by the non-integer period of the function $m_{2.5,x}(n)$. To conclude, the sequence $\{\psi_j\}_{j=0}^{\infty}$ is then non-negative if and only if $|\lambda_2| \geq 2.29 \times 3 = 6.87$. Again, the complex root with an argument 2.5 and an absolute value 6.87 is marked by a red cross on Figure 3.10(a).

Remark 3.3.5. In order to see the effect of θ_1, the negativity regions were plotted in one figure for various choices of $\theta_1 \leq 0$, Figure 3.11. The value λ_1 was chosen equal to 3. Again, the regions are nested within each other. This time the $AR(3)$ region corresponding to $\theta_1 = 0$ is the smallest one. With a decreasing θ_1, the region stretches. Note that the results for $\theta_1 < 0$ were derived under assumption $1 + 2\,\theta_1\lambda_1 > 0$. This condition is not satisfied for $\theta_1 = -0.2$ and $\theta_2 = -0.3$. However, as noted in Remark 3.3.4 we believe the given results would hold also if we assumed only $1 + \theta_1\lambda_1 > 0$. It would suffice to show that the functions $g_{n,\theta}(\cdot)$, for which $g_{n,\theta}(1) < 0$, have also one unique root $x_{n,\theta}$. Inspecting the plots functions $g_{n,\theta}(\cdot)$ and their derivatives for different choices of parameters, we believe that this is true. We are convinced that the negativity regions for $\theta_1 = -0.2$ and $\theta_1 = -0.3$ can be parametrized similarly as those under the assumption $1 + 2\,\theta_1\lambda_1 > 0$. This conjecture was supported empirically by simulating the conjugate roots λ_2, λ_3 and plotting those, for which at least one of the first, say 1 000, coefficients in the sequence $\{\psi_j\}_{j=0}^{\infty}$ was negative. These simulated values clustered in the regions whose borders correspond to $\lambda_1 \times \max_{n\in I_\theta} x_{n,\theta}$.

Remark 3.3.6. Looking at the negativity regions in Figure 3.11, we see that regardless the θ_1, the complex roots with a real part greater than $\lambda_1 = 3$ lie always outside the negativity region. This property was also observed for $\theta_1 \geq 0$. In order to prove this for $\theta_1 < 0$ we might suffice to show e.g. that $g_{n,\theta}(x) \geq 0$ implies $g'_{n,\theta}(x) \geq 0$ for any $n \in \mathbb{N}, \theta \in (0, \pi)$ and $x \geq 1$, which was not that straightforward task.

3.3.6 ARMA(3,2)

Consider the causal $ARMA(3,2)$ process $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ with non-negative innovations, determined from the following stochastic difference equations:

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \phi_3 X_{t-3} = Z_t^* + \theta_1 Z_{t-1}^* + \theta_2 Z_{t-2}^* \quad (t = 0, \pm 1, \pm 2, \dots),$$

where the roots of the autoregressive polynomial $1 - \phi_1 z - \phi_2 z^2 - \phi_3 z^3$ lie outside the unit circle. The generating function for the $ARMA(3,2)$ kernel sequence $\{\psi_j\}_{j=0}^{\infty}$ is

$$\boldsymbol{\psi}(z) = \frac{1 + \theta_1 z + \theta_2 z^2}{1 - \phi_1 z - \phi_2 z^2 - \phi_3 z^3} \quad (0 \leq z < 1). \tag{3.3.27}$$

A similar chain of reasoning to the one we used in previous section could be entertained to derive a set of necessary and sufficient conditions for $ARMA(3,2)$ models. Note that this time the coefficients ψ_n are obtained from the following relations:

$$\psi_n = \frac{r_1}{\lambda_1^{n+1}} + \frac{r_2}{\lambda_2^{n+1}} + \frac{r_3}{\lambda_3^{n+1}} \quad (n \geq 1),$$

where $r_i = -\theta(\lambda_i)/\psi^{(1)}(\lambda_i)$, $i = 1, 2, 3$. More specifically,

$$r_1 = \frac{1 + \theta_1\lambda_1 + \theta_2\lambda_1}{\phi_3(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}, \quad r_2 = \frac{1 + \theta_1\lambda_2 + \theta_2\lambda_2}{\phi_3(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \quad \text{and} \quad r_3 = \frac{1 + \theta_1\lambda_3 + \theta_2\lambda_3}{\phi_3(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}.$$

Now, denote $\psi_n^{***} = \phi_3 |\lambda_1 - \lambda_2|^2 \lambda_1^{n+1} |\lambda_2|^{2n+2} \psi_n$. Then we have

$$\begin{aligned}
\psi_n^{***} =& (1 + \theta_1\lambda_1 + \theta_2\lambda_1)|\lambda_2|^{2n+2} + \frac{\lambda_1^{n+1}\lambda_3^{n+1}(\lambda_3 - \lambda_1)(1 + \theta_1\lambda_2 + \theta_2\lambda_2)}{(\lambda_2 - \lambda_3)} \\
&+ \frac{\lambda_1^{n+1}\lambda_2^{n+1}(\lambda_2 - \lambda_1)(1 + \theta_1\lambda_3 + \theta_2\lambda_3)}{\lambda_3 - \lambda_2} \\
=& |\lambda_2|^{2n+2} + \frac{\lambda_1^{n+2}(\lambda_3^{n+1} - \lambda_2^{n+1})}{\lambda_3 - \lambda_2} - \frac{\lambda_1^{n+1}(\lambda_3^{n+1} - \lambda_2^{n+1})}{\lambda_3 - \lambda_2} \\
&+ \theta_1\lambda_1|\lambda_2|^2 \left[|\lambda_2|^{2n} + \frac{\lambda_1^{n+1}(\lambda_3^n - \lambda_2^n)}{\lambda_3 - \lambda_2} - \frac{\lambda_1^n(\lambda_3^n - \lambda_2^n)}{\lambda_2 - \lambda_3} \right] \\
&+ \theta_2\lambda_1^2|\lambda_2|^4 \left[|\lambda_2|^{2n-2} + \frac{\lambda_1^n(\lambda_3^{n-1} - \lambda_2^{n-1})}{\lambda_3 - \lambda_2} - \frac{\lambda_1^{n-1}(\lambda_3^{n-1} - \lambda_2^{n-1})}{\lambda_2 - \lambda_3} \right].
\end{aligned}$$

Note that we can write

$$\psi_n^{***} = \psi_n^* + \theta_1\lambda_1|\lambda_2|^2\psi_{n-1}^* + \theta_2\lambda_1|\lambda_2|^4\psi_{n-2}^*,$$

where ψ_n^* were defined in (3.2.9). For $0 < \theta < \pi$ and $n \in \mathbb{N}$, denote

$$\begin{aligned} h_{n,\theta}(x) =& x^{n+2}(1+\theta_1\lambda_1+\theta_2\lambda_1) - x^3\theta_2\lambda_1\frac{\sin(n\theta)}{\sin\theta} - x^2\left[\theta_1\lambda_1\frac{\sin[(n+2)\theta]}{\sin\theta} + \theta_2\lambda_1\frac{\sin[(n-1)\theta]}{\sin\theta}\right] \\ &- x\left[\frac{\sin[(n+2)\theta]}{\sin\theta} - \theta_1\lambda_1\frac{\sin(n\theta)}{\sin\theta}\right] + \frac{\sin[(n+1)\theta]}{\sin\theta}. \end{aligned}$$

It can be easily verified that

$$\psi_n^{***} = \lambda_1^{n+2}|\lambda_2|^n h_{n,\theta}\left(\left|\frac{\lambda_2}{\lambda_1}\right|\right). \tag{3.3.28}$$

The expression in (3.3.28) is more than familiar. We have already seen similar ones before: $\psi_n^* = \lambda_1^{n+2}|\lambda_2|^n f_{n,\theta}\left(\left|\frac{\lambda_2}{\lambda_1}\right|\right)$, $AR(3)$ models, $\psi_n^{**} = \lambda_1^{n+2}|\lambda_2|^n g_{n,\theta}\left(\left|\frac{\lambda_2}{\lambda_1}\right|\right)$, $ARMA(3,1)$ models. This immediately suggests that the same strategy used previously for $AR(3)$ and $ARMA(3,1)$ can be applied. Again, the following statements are equivalent

(a) a single coefficient ψ_n is non-negative,

(b) a single coefficient ψ_n^{***} is non-negative,

(c) $h_{n,\theta}\left(\left|\frac{\lambda_2}{\lambda_1}\right|\right)$ is non-negative.

Also note the following relationship between functions $h_{n,\theta}(\cdot), g_{n,\theta}(\cdot)$ and $f_{n,\theta}(\cdot)$:

$$h_{n,\theta}(x) = f_{n,\theta}(x) + x\theta_1\lambda_1 f_{n-1,\theta}(x) + x^2\theta_2\lambda_1 f_{n-2,\theta}(x) = g_{n,\theta}(x) + x^2\theta_2\lambda_1 f_{n-2,\theta}(x),$$

where $f_{-1}(\cdot)$ and $f_{-2}(\cdot)$ are again defined as constant functions equal to zero. An analogy of the Theorem 3.3.9 can be obtained readily also for $ARMA(3,2)$ processes.

Theorem 3.3.17. *Suppose that* $\lambda_1 \in R$ *and* $\lambda_2 = \bar{\lambda}_3 = |\lambda_2|e^{i\theta} = a + bi$, *where* $a, b \in R$ *and* $0 < \theta < \pi$. *If* $\theta_1 \geq 0, \theta_2 \geq 0$ *and* $\theta = 2\pi/k$ *for some integer* $k \geq 3$, *then the sequence* $\{\psi_j\}_{j=0}^{\infty}$ *is nonnegative if and only if* $|\lambda_2| \geq \lambda_1 > 1$.

Proof. The proof follows directly from the facts that: (a) $h_{n,\theta}(1)$ is a non-negative linear combination of $f_n(1), f_{n-1}(1)$ and $f_{n-2}(1)$, i.e.

$$h_{n,\theta}(1) = f_{n,\theta}(1) + \theta_1\lambda_1 f_{n-1,\theta}(1) + \theta_2\lambda_1 f_{n-2,\theta}(1),$$

where $\theta_1\lambda_1 \geq 0$ and $\theta_2\lambda_1 \geq 0$, (b) the values $f_n(1)$ are under the assumption $\theta = 2\pi/k$ always non-negative, Theorem 3.2.4. □

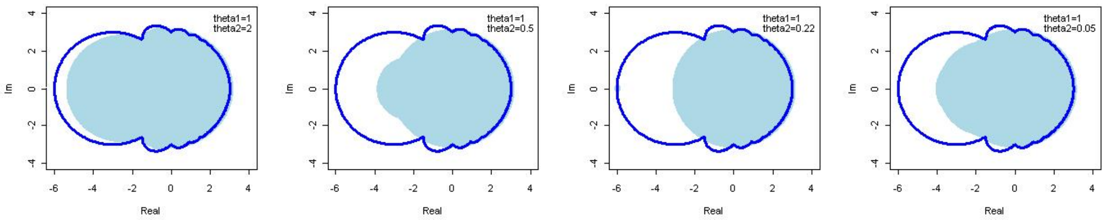


Figure 3.12: Negativity regions for $ARMA(3,2)$ models for various choices of θ_2 and fixed $\theta_1 = 1$

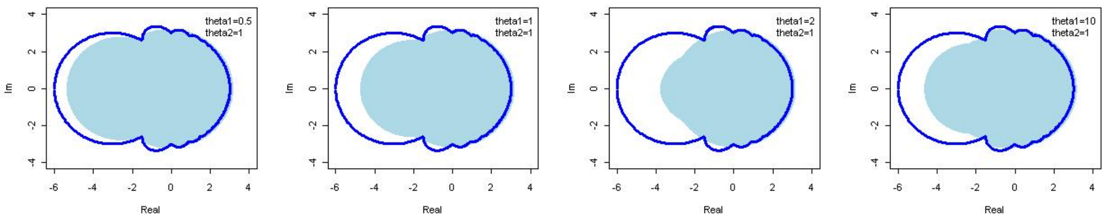


Figure 3.13: Negativity regions for $ARMA(3,2)$ models for various choices of θ_1 and fixed $\theta_2 = 1$

We anticipate that analogous conditions which we derived for $ARMA(3,1)$ processes in Theorem 3.3.11, Theorem 3.3.15 and Theorem 3.3.16 could be obtained also for $ARMA(3,2)$. However, the labor in studying monotonicity properties of the functions $h_{n,\theta}(\cdot)$ and their derivatives would be more involved. That is why we provide only some empirical evidence. Roots λ_2, λ_3 of the autoregressive polynomial were randomly generated and those roots for which at least one of the first 1 000 coefficients in the sequence $\{\psi_j\}_{j=0}^{\infty}$ was negative were plotted in the complex plane. These values tend to cluster within regions similar to those we saw previously for $AR(3)$ or $ARMA(3,1)$ processes. We wondered whether the borders of these regions can be parametrized as $\lambda_1 \times x_\theta^*$, where $x_\theta^* = \max_{n \in I_\theta} x_{n,\theta}$, I_θ again denotes the set of indices for which $h_{n,\theta}(1) < 0$ and $x_{n,\theta}, n \in I_\theta$, denotes a root of the equation $h_{n,\theta}(x) = 0$. Indeed, after plotting the complex roots with absolute value equal to $\lambda_1 \times x_\theta^*$ and with argument θ for a fine grid of values $\theta \in (0, \pi)$, the curve that resulted by joining these points nicely circumscribed the simulated values.

These regions are depicted for varying positive coefficients θ_2 and fixed positive θ_1 on Figure 3.12, varying positive coefficients θ_1 and fixed positive θ_2 on Figure 3.13. The blue closed curve again frames the negativity region for $AR(3)$. The borders of $AR(3)$ region and $ARMA(3,2)$ regions touch each other for $\theta = 2\pi/k, k = 3, 4, \ldots$. Again, this corresponds to the Theorem 3.3.17.

3.3.7 ARMA(p,q)

Consider the causal $ARMA(p,q)$ process $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ with non-negative innovations $\{Z_t^* : t = 0, \pm 1, \pm 2, \dots\}$, which is determined from the following stochastic difference equations

$$X_1 - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \cdots - \phi_p X_{t-p} = Z_t^* + \theta_1 Z_{t-1}^* + \theta_2 Z_{t-2}^* + \cdots + \theta_q Z_{t-q}^* \quad (t = 0, \pm 1, \pm 2, \dots).$$

The generating function of the kernel sequence $\{\psi_j\}_{j=0}^{\infty}$ is given by

$$\boldsymbol{\psi}(z) = \frac{1 + \theta_1 z + \theta_2 z^2 + \theta_q z^q}{1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p} \quad (0 \leq z < 1).$$

Some necessary conditions for the non-negativity of the sequence $\{\psi_j\}_{j=0}^{\infty}$ were already given in Theorem 3.1.5. A set of necessary and sufficient conditions has not been presented yet in the literature. However, Tsai and Chan (2008) investigated conditions for the non-negativity of $GARCH(p,q)$ time series. The generating function of the sequence $\{\psi_j\}_{j=0}^{\infty}$ for an $ARMA(p,q)$ process shares a lot of similarities with the one of $GARCH(p,q)$ process. Therefore, their result can be applied directly also for $ARMA(p,q)$ models. We now reformulate and prove the result for the $ARMA(p,q)$ process with only minor modifications.

Theorem 3.3.18. *Let $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ be an $ARMA(p,q)$ process. Denote $\lambda_1, \lambda_2, \dots, \lambda_p$ the roots of the autoregressive lag polynomial $\boldsymbol{\phi}(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$, such that $|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_p|$. Assuming that these roots are distinct, the conditions (1)-(3) are necessary and sufficient for $\{\psi_j\}_{j=0}^{\infty}$ to be non-negative:*

$$\begin{aligned} &(1) \quad \lambda_1 \text{ is real and } \lambda_1 > 1, \\ &(2) \quad \boldsymbol{\theta}(\lambda_1) > 0, \\ &(3) \quad \psi_k \geq 0, \text{ for } k = 1, \dots, k^*, \end{aligned}$$

where k^* is the smallest integer greater than or equal to $\max[\max(0, q-p) + 1, \gamma]$, where

$$\gamma = \frac{\log r_1 - \log[(p-1) r^*]}{\log |\lambda_1| - \log |\lambda_2|} - 1$$

with $r^* = \max_{2 \leq j \leq p} |r_j|$ and $r_j = -\frac{\boldsymbol{\theta}(\lambda_j)}{\boldsymbol{\phi}^{(1)}(\lambda_j)}$, $1 \leq j \leq p$.

Proof. Let us start with the necessity. The necessity of the condition (3) is obvious. The necessity of the condition (1) has been shown in Theorem 3.1.5. The necessity of (2) proceeds as follows: by equations (4.8) and (4.9) in Feller (1968) p. 276 it holds that

$$\psi_n = \sum_{i=1}^{p} r_i \lambda_i^{-(n+1)}, \quad n \geq \max(q-p, 0) + 1.$$

This can be rewritten as

$$\psi_n \lambda_1^{n+1} = r_1 + \sum_{i=2}^{p} r_i \left(\frac{\lambda_1}{\lambda_i}\right)^{n+1}. \tag{3.3.29}$$

In the proof of Theorem 3.1.5, we have already shown that under the assumption $1 < \lambda_1 < |\lambda_i|, i = 2, \dots, p$,

$$\lim_{n\to\infty} r_i \left(\frac{\lambda_1}{\lambda_i}\right)^{n+1} = 0.$$

This means, as n tends to infinity, the first element r_1 prevails the expansion in (3.3.29). The coefficients ψ_n can be for large n approximated by $r_1 \lambda_1^{-(n+1)}$. In other words

$$\lim_{n\to\infty} \frac{\psi_n \lambda_1^{n+1}}{r_1} = 1. \tag{3.3.30}$$

Assuming that the sequence $\{\psi_j\}_{j=0}^{\infty}$ is non-negative, we have shown that $\lambda_1 \in \mathbb{R}$ and $\lambda_1 > 1$, Theorem 3.1.5. In order the limit in (3.3.30) to be one, it is not possible that r_1 is negative. Let us now look more closely at the term $r_1 = -\frac{\boldsymbol{\theta}(\lambda_1)}{\boldsymbol{\phi}^{(1)}(\lambda_1)}$. It holds that

$$\boldsymbol{\phi}(z) = 1 - \sum_{i=1}^{p} \phi_i z^i = \prod_{i=1}^{p} \left(1 - \frac{z}{\lambda_i}\right).$$

The derivative evaluated at $\lambda_1 > 1$ equals

$$\boldsymbol{\phi}^{(1)}(\lambda_1) = -\frac{1}{\lambda_1} \prod_{i=2}^{p} \left(1 - \frac{\lambda_1}{\lambda_i}\right).$$

Assume that a root $\lambda_i, i \in \{2, \dots, p\}$ is real. If $\lambda_i < -\lambda_1 < -1$, the element $1 - \frac{\lambda_1}{\lambda_i}$ is always positive. If λ_i is positive, the property $1 - \frac{\lambda_1}{\lambda_i} > 0$ follows from the assumption that $\lambda_i > \lambda_1 > 1$. Now assume that a root λ_i is complex and that $\lambda_{i+1} = \bar{\lambda}_i$. Then we have

$$\left(1 - \frac{\lambda_1}{\lambda_i}\right)\left(1 - \frac{\lambda_1}{\lambda_{i+1}}\right) = \frac{|\lambda_i - \lambda_1|^2}{|\lambda_i|^2} > 0.$$

This altogether gives that $\prod_{i=2}^{p}\left(1 - \frac{\lambda_1}{\lambda_i}\right)$ is always positive and therefore $-\frac{1}{\boldsymbol{\phi}^{(1)}(\lambda_1)}$ is positive. Since we assume that $\boldsymbol{\theta}(z)$ and $\boldsymbol{\phi}(z)$ have no common roots, r_1 cannot be equal to zero. Moreover, the positivity of r_1 is equivalent to the positivity of $\boldsymbol{\theta}(\lambda_1)$. This proves the necessity of the condition (2).

Now, we prove the sufficiency. First suppose that γ is negative, i.e.

$$\frac{\log r_1 - \log[(p-1)r^*]}{\log|\lambda_1| - \log|\lambda_2|} < 1.$$

This is equivalent to

$$\log r_1 - \log[(p-1)r^*] > \log|\lambda_1| - \log|\lambda_2|. \tag{3.3.31}$$

We have

$$\begin{aligned}\psi_n &= \sum_{i=1}^{p} \frac{r_i}{\lambda_i^{n+1}} \geq \frac{r_1}{\lambda_1^{n+1}} - \left|\sum_{i=2}^{p} \frac{r_i}{\lambda_i^{n+1}}\right| \geq \frac{r_1}{\lambda_1^{n+1}} - \sum_{i=1}^{p}\left|\frac{r_i}{\lambda_i^{n+1}}\right| \geq \frac{r_1}{\lambda_1^{n+1}} - \sum_{i=1}^{p} \frac{r^*}{|\lambda_i^{n+1}|} \\ &\geq \frac{r_1}{\lambda_1^{n+1}} - \frac{(p-1)r^*}{|\lambda_2|^{n+1}}.\end{aligned}$$

The inequality (3.3.31) is equivalent to $r_1 > (p-1)r^* \frac{|\lambda_1|}{|\lambda_2|}$ and therefore , we obtain

$$\lambda_1^{n+1}\psi_n \geq r_1 - (p-1)r^* \frac{\lambda_1^{n+1}}{|\lambda_2|^{n+1}} > (p-1)r^* \frac{|\lambda_1|}{|\lambda_2|}\left(1 - \frac{|\lambda_1|^n}{|\lambda_2|^n}\right) \geq 0.$$

This implies that $\psi_n \geq 0, n \geq \max(q-p, 0) + 1$, whenever $\gamma < 0$. Now, suppose $\gamma \geq 0$. It still holds that

$$\lambda_1^{n+1}\psi_n \geq r_1 - (p-1)r^* \frac{|\lambda_1|^{n+1}}{|\lambda_2|^{n+1}}. \tag{3.3.32}$$

The first term r_1 is positive from the assumption (2). The second term monotonously decreases in magnitude as n tends to infinity. This means that if there exists n^* such that the term on the right hand side of (3.3.32) is non-negative, then the term remains non-negative for all $n > n^*$. The right hand side in (3.3.32) is non-negative whenever

$$n + 1 \geq \frac{\log r_1 - \log[(p-1)r^*]}{\log|\lambda_1| - \log|\lambda_2|}$$

or equivalently when $n \geq \gamma$. This completes the proof. □

In the following, we show that the conditions in Theorem 3.3.18 are relatively easy to verify for $ARMA(2, q)$ models.

Theorem 3.3.19. *Let $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ be $ARMA(2, q)$ process. Denote λ_1, λ_2 the roots of the autoregressive lag polynomial $\boldsymbol{\phi}(z) = 1 - \phi_1 z - \phi_2 z^2$, such that $|\lambda_1| \leq |\lambda_2|$. Assuming that these roots are distinct, the conditions (1)-(3) are necessary and sufficient for $\{\psi_j\}_{j=0}^{\infty}$ to be non-negative:*

(1) λ_1 *is real and* $\lambda_1 > 1$,

(2) $\boldsymbol{\theta}(\lambda_1) > 0$,

(3) $\psi_k \geq 0, \quad$ *for* $\quad k = 1, \dots, q$,

Proof. We prove only the sufficiency, since the necessity follows from Theorem 3.3.18. The equations (4.3) and (4.4) in Feller (1968) p. 276 for $ARMA(2,q)$ process take the following form:

$$\psi_n = \frac{1}{\phi_2(\lambda_1 - \lambda_2)} \left(\frac{\boldsymbol{\theta}(\lambda_1)}{\lambda_1^{n+1}} - \frac{\boldsymbol{\theta}(\lambda_2)}{\lambda_2^{n+1}} \right), \quad n \geq q-1.$$

We can write

$$\psi_n \frac{\lambda_1^{n+1}}{\phi_2(\lambda_1 - \lambda_2)} = \boldsymbol{\theta}(\lambda_1) - \left(\frac{\lambda_1}{\lambda_2} \right)^{n+1} \boldsymbol{\theta}(\lambda_2), \quad n \geq q-1. \tag{3.3.33}$$

Similarly as in the Theorem 3.3.4 we consider the following 4 cases: (i) $1 < \lambda_1 < \lambda_2$ and $\boldsymbol{\theta}(\lambda_2) \geq 0$, (ii) $1 < \lambda_1 < \lambda_2$ and $\boldsymbol{\theta}(\lambda_2) < 0$, (iii) $-1 > -\lambda_1 > \lambda_2$ and $\boldsymbol{\theta}(\lambda_2) > 0$, (iv) $-1 > -\lambda_1 > \lambda_2$ and $\boldsymbol{\theta}(\lambda_2) < 0$.

In the first case the second summand $\left(\frac{\lambda_1}{\lambda_2}\right)^{n+1} \boldsymbol{\theta}(\lambda_2)$ is positive and contributes a negative value to (3.3.33). It is monotonously decreasing in magnitude. If (3.3.33) is non-negative for $n = q-1$, then it is non-negative for all $n \geq q$. In the second case, the summand is always negative and therefore contributes positively in (3.3.33). In that case $\psi_n \geq 0, n \geq q-1$. In the third case, the summand has oscilating sign and monotonously decreases in absolute value. For $n = 2k-1$ the summand is positive. In case q is an even number, it suffices that (3.3.33) is non-negative for $n = q-1$. For odd q, it suffices to assume the non-negativity of (3.3.33) for $n = q$. If $n = 2k$, the summand is always negative and therefore (3.3.33) always non-negative for $n \geq q-1$. The case (iv) follows analogously as (iii). Still it is sufficient to assume that (3.3.33) is non-negative for $n = q-1$ or $n = q$, depending whether or not q is odd. □

Remark 3.3.7. The Theorem 3.3.4 is in fact as a special case of Theorem 3.3.19, where the expansion (3.3.29) is valid already for $n \geq 1$. We then needed to assume the non-negativity of just the first two coefficients. The same holds for $ARMA(2,2)$. However, for $ARMA(2,q)$, where $q > 2$, the expansion can be applied only for $n \geq q-1$. Therefore we need to assume the non-negativity of the first q coefficients.

Remark 3.3.8. Note the similarity between Theorem 3.3.19 and Theorem 3.3.3. The Theorem 3.3.19 contains the additional assumption $\boldsymbol{\theta}(z) \geq 0$.

Chapter 4

Conditions for Non-negativity in Multivariate Time Series

Many phenomena observed in practice arise as a result of many underlying processes that interact with each other. Simultaneous measurements on more than one such process are then demanded to aid understanding of such phenomenon. For example, in water resource streamflow modeling, in order to understand the dynamics of the streamflow within a water basin, it is usual to measure streamflow intensities at different sites of the basin. Joint modeling of these measurements requires taking into account possible spatial correlations between neighboring sites as well as correlations induced by time. Multivariate time series models are capable of modeling data with these characteristics.

Many observed multivariate time series are allowed to take only non-negative values. Examples can be found not only in hydrology but also other fields like medicine or finance. Describing the non-negative multivariate time series by a statistical model then again requires special modeling or parameter estimation techniques and, implicitly, a specification of the non-negativity conditions. Conditions for the non-negativity for multivariate processes have not been investigated as thoroughly as the univariate conditions. To our knowledge, the only note on this problematics was given by Anděl (1992), who derived a necessary and sufficient condition for a p-dimensional $AR(1)$ process. In this section we review this existing result and we discuss a different approach based on the absolute monotonicity argument, generalized to the multivariate case. We focus on the $VAR(1), VARMA(1,1)$ and $VARMA(1,q)$ models.

4.1 Conditions for Non-negativity

In Section 2.2 we established that if a multivariate (p-variate) $ARMA$ series $\{\boldsymbol{X}_t = (X_{t1}, \ldots, X_{tp})' : t = 0, \pm 1, \pm 2, \ldots\}$ satisfies the causality criterion $\det[\boldsymbol{\Phi}(z)] \neq 0, |z| \leq 1$, it can be represented as a infinite matrix-valued linear combination of the vector white noise sequence $\{\boldsymbol{Z}_t = (Z_{t1}, \dots, Z_{tp})' : t = 0, \pm 1, \pm 2, \dots\}$, i.e.

$$\boldsymbol{X}_t = \sum_{j=1}^{\infty} \Psi_j \boldsymbol{Z}_{t-j}, \quad t = 0, \pm 1, \pm 2, \dots. \tag{4.1.1}$$

Assuming that the multivariate white noise comprises of non-negative random variables and that all the elements of the matrices $\{\Psi_j\}_{j=0}^{\infty}$ are non-negative, it trivially follows that the vector process $\{\boldsymbol{X}_t : t = 0, \pm 1, \pm 2, \dots\}$ is also non-negative. It follows from Theorem 3.1 and Corollary 3.2 in Anděl (1992) that if the elements of the multivariate white noise are independent, the non-negativity of all the matrix components in the sequence $\{\Psi_j\}_{j=0}^{\infty}$ is also a necessary condition for the non-negativity of the process $\{\boldsymbol{X}_t : t = 0, \pm 1, \pm 2, \dots\}$.

Theorem 4.1.1. *Let $\boldsymbol{Z}_t = (Z_{t1}, \dots, Z_{tp})', t \in \mathbb{Z}$, be **iid** random vectors. Assume that $Z_{t1}, \dots, Z_{tp}, t \in \mathbb{N}$, are independent non-negative random variables. Assume $\mathsf{P}(Z_{ti} = 0) < 1$ and $\mathsf{P}(Z_{ti} < \varepsilon) > 0$ for all $i = 1, \dots p$, $t \in \mathbb{Z}$ and every $\varepsilon > 0$. Let $\{\boldsymbol{X}_t : t = 0, \pm 1, \pm 2, \dots\}$ be a p-dimensional ARMA process given by (4.1.1) and denote $\Psi_n = \left(\psi_{ij}^n\right)_{i,j=1}^p$. Assume that there exist $n \in \mathrm{N}$ and $i, j \in \{1, \dots, p\}$ such that $\psi_{ij}^n < 0$. Then there are infinitely many indices $t \in \mathbb{Z}$ such that $X_{ti} < 0$.*

Proof. From our assumptions it holds that $\mathsf{P}\left(\sum_{m=1}^p c_m Z_{tm} < \varepsilon\right) > 0$, $\forall \varepsilon > 0$, for any $(c_1, \dots, c_p)' \in \mathbb{R}^p$ and $t \in \mathbb{Z}$. Furthermore, from the assumptions $\mathsf{P}(Z_{ti} = 0) < 1$ and $\mathsf{P}(Z_{ti} \geq 0) = 1$, there must exist $c > 0$ such that

$$\mathsf{P}(\psi_{ij}^n Z_{tj} < -2c) > 0, \quad t \in \mathbb{Z}.$$

Then we have

$$\mathsf{P}\left(\sum_{m=1}^{p} \psi_{im}^n Z_{tm} < -c\right) \geq \mathsf{P}\left(\psi_{ij}^n Z_{tj} < -2c\right) \mathsf{P}\left(\sum_{m \neq j} \psi_{im}^n Z_{tm} < c\right) > 0. \tag{4.1.2}$$

The fulfillment of the condition in (4.1.2) together with Theorem 3.1 in Anděl (1992) gives that with probability 1 there exist infinitely many subscripts t such that $X_{ti} < 0$. $\square$

From now on we assume that the components of the vector valued strict white noise sequence are independent and non-negative random variables. We will investigate situations, when the elements of the matrix sequence $\{\Psi_j\}_{j=0}^{\infty}$ are non-negative. First, we establish concepts of matrix-valued generating and absolutely monotone functions.

Definition 4.1.1. Let $\{\Psi_n\}_{n=0}^{\infty}$ denote a sequence of real $p \times p$ matrices $\Psi_0, \Psi_1, \dots$. Denote

$$\boldsymbol{\Psi}(z) = [\psi_{ij}(z)]_{i,j=1}^p = \Psi_0 + \Psi_1 z + \Psi_2 z^2 + \dots$$

a matrix valued power series function. If each function $\psi_{ij}(z), i, j = 1, \dots p$, converges in some interval $-z_0 < z < z_0$, then $\boldsymbol{\Psi}(z)$ is said to be a **generating function** of the sequence $\{\Psi_n\}_{n=0}^{\infty}$.

Definition 4.1.2. A matrix-valued generating function $\mathbf{\Psi}(z) = \Psi_0 + \Psi_1 z + \Psi_2 z^2 + \dots$, where Ψ_i are $p \times p$ real matrices, is said to be **absolutely monotone**, if all its elements $\psi_{ij}(z) = \psi_{ij}^0 + \psi_{ij}^1 z + \psi_{ij}^2 z^2 + \dots$, $i, j = 1, \dots, p$, are absolutely monotone functions.

Definition 4.1.3. A real valued $p \times p$ matrix A is said to be **non-negative** if all its entries $a_{ij}, i, j = 1, \dots, p$, are non-negative.

The following theorem is a matrix-valued version of the Theorem 3.1.2.

Theorem 4.1.2. *The sequence of matrices* $\{\Psi_j\}_{j=0}^{\infty}$ *is non-negative if and only if its generating function* $\mathbf{\Psi}(z) = \Psi_0 + \Psi_1 z + \Psi_2 z^2 + \dots$ *is absolutely monotone on* $0 \leq z < 1$.

Proof. The theorem is an immediate consequence of Theorem 3.1.2.

Note that each of the functions $\psi_{ij}(\cdot)$ is a generating function of the sequence $\{\psi_{ij}^n : n \in \mathbb{N}\}$. This suggests that the non-negativity of the sequence of matrices $\{\Psi_j\}_{j=0}^{\infty}$ can be explored component-wise. For each sequence $\{\psi_{ij}^n : n \in \mathbb{N}\}$, we can utilize the duality between the non-negativity of a sequence of real numbers and the absolute monotonicity of its generating function.

From now on, we adopt the following assumptions:

(1) let $\boldsymbol{Z}_t = (Z_{t1}, \dots, Z_{tp})', t \in \mathbb{Z}$, be `iid` random vectors,

(2) the random variables $Z_{t1}, \dots, Z_{tp}, t \in \mathbb{N}$, are independent and non-negative,

(3) $\mathsf{P}(Z_{ti} = 0) < 1$ and $\mathsf{P}(Z_{ti} < \varepsilon) > 0$ for all $i = 1, \dots p$, $t \in \mathbb{Z}$ and every $\varepsilon > 0$.

A series of vector valued innovations $\{\boldsymbol{Z}_t = (Z_{t1}, \dots, Z_{tp})' : t = 0, \pm 1, \pm 2, \dots\}$ with properties (1), (2) and (3) will be further referred to as a **p-dimensional non-negative innovation sequence** and denoted $\{\boldsymbol{Z}_t^* = (Z_{t1}^*, \dots, Z_{tp}^*)' : t = 0, \pm 1, \pm 2, \dots\}$.

4.2 Vector-valued AR(1)

A stationary p-dimensional $AR(1)$ process $\{\boldsymbol{X}_t = (X_{t1}, \dots, X_{tp})' : t = 0, \pm 1, \pm 2, \dots\}$ with non-negative innovation vectors is determined from the following system of difference equations:

$$\boldsymbol{X}_t - \Phi_1 \boldsymbol{X}_{t-1} = \boldsymbol{Z}_t^*, \quad t = 0, \pm 1, \pm 2, \dots, \tag{4.2.3}$$

where the eigenvalues of the matrix Φ_1 are assumed to lie inside the unit circle. The fulfillment of the causality condition enables to represent the series $\{\boldsymbol{X}_t : t = 0, \pm 1, \pm 2, \dots\}$ as an infinite matrix-valued moving average combination of the multivariate innovations, i.e.

$$\boldsymbol{X}_t = \sum_{j=0}^{\infty} \Psi_j \boldsymbol{Z}_{t-j}^*, \tag{4.2.4}$$

where the coefficient matrices are determined from the entity $\boldsymbol{\Psi}(z) = \sum_{j=0}^{\infty} \Psi_j z^j = (\mathrm{I}_p - \Phi_1 z)^{-1}$. According to the equations 11.3.12 in Brockwell and Davis (1986), the matrices Ψ_j can be obtained from the following recursive equations

$$\begin{aligned} \Psi_0 &= \mathrm{I}_p \\ \Psi_1 &= \Phi_1 \\ \Psi_k &= \Phi_1 \Psi_{k-1}, \quad k \geq 2. \end{aligned}$$

Under the assumptions of Theorem 4.1.1 it holds that the non-negativity of all the matrices Ψ_j is a necessary and sufficient condition for the non-negativity of the series $\{\boldsymbol{X}_t : t = 0, \pm 1, \pm 2, \dots\}$. For the p-variate $AR(1)$ process it holds that $\Psi_j = \Phi_1^j$. It then easily follows, that the condition $\Phi_1 \geq 0$ is a sufficient and necessary condition for the non-negativity of the sequence $\{\Psi_j\}_{j=0}^{\infty}$. The non-negativity of the process $VARMA(1)$ is summarized in the following theorem.

Theorem 4.2.1. *Let $\{\boldsymbol{X}_t : t = 0, \pm 1, \pm 2, \dots\}$ be a p-dimensional $AR(1)$ process given by (4.2.3). Then the non-negativity of the matrix Φ_1 is a sufficient and necessary condition for the non-negativity of the components in the process $\{\boldsymbol{X}_t : t = 0, \pm 1, \pm 2, \dots\}$.*

Proof. A proof follows from the Theorem 4.1.1, or from Corollary 3.2. in Anděl (1992).

Remark 4.2.1. To derive Theorem 4.2.1, we did not need to investigate the absolute monotonicity of the function $\Psi(z)$. However, it might be interesting to see whether the absolute monotonicity argument would lead to the same conclusion. We confine ourselves only to 2-dimensional $AR(1)$ process.

4.2.1 Two-dimensional AR(1)

Let us rewrite the system of stochastic difference equations as follows:

$$\begin{pmatrix} X_{t,1} \\ X_{t,2} \end{pmatrix} - \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} X_{t-1,1} \\ X_{t-1,2} \end{pmatrix} = \begin{pmatrix} Z_{t,1}^* \\ Z_{t,2}^* \end{pmatrix}, \quad t = 0, \pm 1, \pm 2, \dots.$$

Denote μ_1 and μ_2 the two eigenvalues of the matrix $\Phi_1 = (\phi_{ij})_{i,j=1}^2$. The fulfillment of the causality criterion $|\mu_1| < 1$ and $|\mu_2| < 1$ allows for the vector-valued infinite moving average representation with matrix coefficients, which are determined from the following representation

$$\boldsymbol{\Psi}(z) = \sum_{j=0}^{\infty} \Psi_j z^j = (\mathrm{I}_2 - \Phi_1 z)^{-1} = \begin{pmatrix} 1 - \phi_{11} z & -\phi_{12} z \\ -\phi_{21} z & 1 - \phi_{22} z \end{pmatrix}^{-1}.$$

$\psi_{22}(z)$		$\psi_{11}(z)$	
(1)	$\lambda_1, \lambda_2 \in \mathbb{R}, \quad \lambda_1 > 1,$	$(1)^*$	$\lambda_1, \lambda_2 \in \mathbb{R}, \quad \lambda_1 > 1,$
(2)	$1 - \phi_{11}\lambda_1 > 0,$	$(2)^*$	$1 - \phi_{22}\lambda_1 > 0,$
(3)	$\phi_{22} \geq 0,$	$(3)^*$	$\phi_{11} \geq 0$
(4)	$\phi_{22}^2 + \phi_{12}\phi_{21} \geq 0$	$(4)^*$	$\phi_{11}^2 + \phi_{12}\phi_{21} \geq 0.$

Table 4.1: Conditions for the absolute monotonicity of the functions $\psi_{11}(\cdot)$ and $\psi_{22}(\cdot)$

$\psi_{12}(z)$		$\psi_{21}(z)$	
(1)	$\lambda_1, \lambda_2 \in \mathbb{R}, \quad \lambda_1 > 1,$	$(1)^*$	$\lambda_1, \lambda_2 \in \mathbb{R}, \quad \lambda_1 > 1,$
(2)	$\phi_{12}\lambda_1 > 0,$	$(2)^*$	$\phi_{21}\lambda_1 > 0,$
(3)	$\phi_{12} \geq 0,$	$(3)^*$	$\phi_{21} \geq 0$
(4)	$\phi_{12}\mathrm{tr}(\Phi_1) \geq 0$	$(4)^*$	$\phi_{21}\mathrm{tr}(\Phi_1) \geq 0$

Table 4.2: Conditions for the absolute monotonicity of the functions $\psi_{12}(\cdot)$ and $\psi_{21}(\cdot)$

We can write

$$\mathbf{\Psi}(z) = \begin{pmatrix} \psi_{11}(z) & \psi_{12}(z) \\ \psi_{21}(z) & \psi_{22}(z) \end{pmatrix} = \frac{1}{(1 - \phi_{11}z)(1 - \phi_{22}z) - \phi_{12}\phi_{21}z^2} \begin{pmatrix} 1 - \phi_{22}z & \phi_{12}z \\ \phi_{21}z & 1 - \phi_{11}z \end{pmatrix}.$$

We know from Theorem 4.1.2 that the non-negativity of the sequence $\{\Psi_j\}_{j=0}^{\infty}$ is equivalent to the absolute monotonicity of $\mathbf{\Psi}(z)$, which is from our definition equivalent to the absolute monotonicity of each of the functions $\psi_{ij}(z)$. Note that each component $\psi_{ij}(z), i, j = 1, 2$, is a ratio of two polynomials and is similar in resemblance to the generating functions of the kernel sequences for univariate time series. We can therefore apply the results we already obtained in previous chapter.

Denote λ_1 and λ_2 roots of the polynomial $\det(\mathrm{I}_2 - \Phi_1 z)$. Without loss of generality assume that $|\lambda_1| \leq |\lambda_2|$. Note that λ_1^{-1} and λ_2^{-1} are eigenvalues of the matrix Φ_1 and from the causality it must hold that $1 < |\lambda_1| \leq |\lambda_2|$. Assume that the numerators and denominators of the functions in $\Psi(z)$ have distinct roots. The functions $\psi_{ij}(z)$ are ratios of two polynomials, where the numerator polynomial is of degree one and the denominator polynomial degree two. This suggests that we can apply results for univariate $ARMA(2,1)$ models. The numerator of the diagonal functions is of the form $1 + az$, which can be regarded as a moving-average polynomial of a degree one. Take for example the function $\psi_{11}(z)$. The denominator function is $1 - b_1 z - b_2 z^2$ with $b_1 = \mathrm{tr}\,\Phi_1$ and $b_2 = -\det\Phi_1$ and in the numerator we have is $1 + az$ with $a = -\phi_{22}$. According to the Theorem 3.3.4, the absolute monotonicity of the function $\psi_{11}(z)$ is equivalent to

$$\begin{aligned} \lambda_1, \lambda_2 \in \mathbb{R} \quad \text{and} \quad \lambda_1 &> 1, \\ 1 - \phi_{22}\lambda_1 &> 0, \\ \psi_{11}^1 &\geq 0, \\ \psi_{11}^2 &\geq 0. \end{aligned}$$

According to the equations (3.3.5) in Brockwell and Davis (1986), p. 91, we have $\psi_{11}^1 = a + b_1 = -\phi_{22} + \mathrm{tr}\Phi_1 = \phi_{11}$ and $\psi_{11}^2 = b_2 + b_1 a + b_1^2 = -\det \Phi_1 - \phi_{22}\mathrm{tr}\, \Phi_1 + (\mathrm{tr}\, \Phi_1)^2 = \phi_{11}^2 + \phi_{12}\phi_{21}$. The conditions for the absolute monotonicity of the functions $\psi_{11}(z)$ and $\psi_{22}(z)$ are summarized in Table 4.1.

The numerator of the off-diagonal functions $\psi_{ij}(\cdot), i \neq j$, does not take the form of a moving-average polynomial. Nevertheless, we can still apply the Theorem 3.3.4 or 3.3.19. The only difference is that the coefficients ψ_{ij}^1 and ψ_{ij}^2,$i \neq j$, follow from recursive formulas different from the ones in Brockwell and Davis (1986), p. 91. Let

$$\boldsymbol{\psi}(z) = \sum_{j=0}^{\infty} \psi_j z^j = \frac{\boldsymbol{\alpha}(z)}{\boldsymbol{\beta}(z)}, \quad 0 < z < 1, \tag{4.2.5}$$

where $\boldsymbol{\alpha}(z) = \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_{q^*} z^{q^*}$, $\boldsymbol{\beta}(z) = 1 - \beta_1 z - \beta_2 z^2 - \cdots - \beta_{p^*} z^{p^*}$ and $\boldsymbol{\alpha}(z), \boldsymbol{\beta}(z)$ have no common zeroes. Denote $r = \max\{q^*, p^*\}$ and define $\alpha_j = 0$ for $j > q^*$, $\beta_j = 0$ for $j > p^*$. Now we have

$$\psi_0 = 0, \tag{4.2.6}$$
$$\psi_1 = \alpha_1, \tag{4.2.7}$$
$$\psi_2 = \beta_1\psi_1 + \alpha_2, \tag{4.2.8}$$
$$\psi_3 = \beta_1\psi_2 + \beta_2\psi_1 + \alpha_3, \tag{4.2.9}$$
$$\vdots \tag{4.2.10}$$
$$\psi_r = \beta_1\psi_{r-1} + \beta_2\psi_{r-2} + \cdots + \beta_{r-1}\psi_0 + \alpha_r, \tag{4.2.11}$$
$$\psi_n = \beta_1\psi_{n-1} + \beta_2\psi_{n-2} + \cdots + \beta_r\psi_{n-r}, \quad (n \geq r+1). \tag{4.2.12}$$

The functions $\psi_{12}(z)$ (resp. $\psi_{21}(z)$) are of the form (4.2.5) with $q^* = 1$ and $p^* = 2$. We apply the Theorem 3.3.18 with $\boldsymbol{\theta}(z) = \phi_{12}\, z$ (resp. $\boldsymbol{\theta}(z) = \phi_{21}\, z$) and the conditions $\psi_{ij}^1 \geq 0, \psi_{ij}^2 \geq 0, i \neq j$, relating to the equations (4.2.7) and (4.2.8). The conditions for the absolute monotonicity of the functions $\psi_{12}(z)$ and $\psi_{21}(z)$ are summarized in Table 4.2.

Inverse spectrum problem

In Theorem 4.2.1 we derived the non-negativity conditions which utilize only model parameters gathered in the matrix Φ_1. We have seen in the preceding paragraph that the absolute monotonicity approach leads to conditions that operate rather with eigenvalues of the matrix Φ_1. There is a connection between the two approaches. In fact, the conditions in Table 4.1 and Table 4.2 should together be equivalent to the non-negativity of the matrix Φ_1. In this sense, these conditions relate to the inverse eigenvalue problem for non-negative matrices, that is a problem of determination sufficient and necessary conditions for a p-tuple of complex numbers to be a spectrum of a real valued $p \times p$ non-negative matrix. One immediate necessary condition is that the sum of the eigenvalues, being the trace of the matrix, needs to be non-negative. Another necessary condition is given in the following theorem.

Theorem 4.2.2. *If A is a non-negative $p \times p$ matrix, then it has a non-negative eigenvalue $\mu \in \mathbb{R}$ that is at least as large as the absolute value of any eigenvalue of A.*

Proof. A proof is given in Minc (1988) p. 14.

The necessary condition in Theorem 4.2.2 implies that the smallest root (in absolute value) of the autoregressive polynomial, here λ_1, has to be real and non-negative. Together with the causality condition, this means that λ_1 needs to be greater than one. The necessity of conditions (4) and $(4)^*$ for the non-negativity of Φ_1 is trivial. In fact, the conditions (3) and $(3)^*$ together translate as $\Psi_1 = \Phi_1 \geq 0$ and $\Psi_2 = \Phi_1^2 \geq 0$. The conditions (2) and $(2)^*$ can be for $\lambda_1 > 0$ together restated as

$$\operatorname{adj}(\mathrm{I}_2 - \Phi_1\lambda_1) > 0 \quad \text{or} \quad \operatorname{adj}\left(\frac{1}{\lambda_1}\mathrm{I}_2 - \Phi_1\right) > 0. \tag{4.2.13}$$

Assumming that the matrix Φ_1 is irreducible, the condition (4.2.13) holds (Corollary 4.1 in Minc (1988), p. 16).

4.3 Vector-valued ARMA(1,1)

We consider a causal p-dimensional $ARMA(1,1)$ process $\{\boldsymbol{X}_t : t = 0, \pm 1, \pm 2, \dots\}$ with non-negative innovation sequence, which is determined from the following system of difference equations:

$$\boldsymbol{X}_t - \Phi_1 \boldsymbol{X}_{t-1} = \boldsymbol{Z}_t^* + \Theta_1 \boldsymbol{Z}_{t-1}^*, \quad t = 0, \pm 1, \pm 2, \dots. \tag{4.3.14}$$

The matrix-valued generating function $\boldsymbol{\Psi}(z)$ takes the following form

$$\boldsymbol{\Psi}(z) = (\mathrm{I}_p - \Phi_1 z)^{-1}(\mathrm{I}_p + \Theta_1 z), \quad 0 \leq z < 1.$$

A set of sufficient conditions for the non-negativity of the matrix series $\{\Psi_j\}_{j=0}^{\infty}$ for the p-variate $ARMA(1,1)$ can be readily obtained from the equations (11.3.12) in Brockwell and Davis (1986).

Theorem 4.3.1. *Let* $\{\boldsymbol{X}_t : t = 0, \pm 1, \pm 2, \dots\}$ *be a* p*-dimensional* $ARMA(1,1)$ *process in (4.3.14). If* $\Phi_1 \geq 0$ *and* $\Phi_1 + \Theta_1 \geq 0$ *then the sequence of matrices* $\{\Psi_j\}_{j=0}^{\infty}$ *is non-negative.*

Proof. The equations 11.3.12 in Brockwell and Davis (1986) give the following representation of the matrices Ψ_j:

$$\begin{aligned}
\Psi_0 &= \mathrm{I}_p \\
\Psi_1 &= \Phi_1 + \Theta_1 \\
\Psi_k &= \Phi_1 \Psi_{k-1}, \quad k \geq 2.
\end{aligned}$$

The sufficiency follows immediately. □

In order to derive a set of sufficient and necessary conditions, we employ the absolute monotonicity approach. We confine ourselves to only a two-dimensional time series.

4.3.1 Two-dimensional ARMA(1,1)

Recall that the two dimensional $AR(1)$ time series appears as a stationary solution to the following set of equations:

$$\begin{pmatrix} X_{t,1} \\ X_{t,2} \end{pmatrix} - \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} X_{t-1,1} \\ X_{t-1,2} \end{pmatrix} = \begin{pmatrix} Z^*_{t,1} \\ Z^*_{t,2} \end{pmatrix} + \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} Z^*_{t-1,1} \\ Z^*_{t-1,2} \end{pmatrix}, \quad t = 0, \pm 1, \pm 2, \dots.$$

Assume that the eigenvalues $\mu_1 = 1/\lambda_1$ and $\mu_2 = 1/\lambda_2$ of the matrix Φ_1 are distinct and smaller than one in absolute value. The fulfillment of the causality criterion allows the matrix-valued infinite moving average representation, where the matrix coefficients are determined from the matrix-valued power series expansion of the generating function $\boldsymbol{\Psi}(z)$, i.e.

$$\boldsymbol{\Psi}(z) = \sum_{j=0}^{\infty} \Psi_j z^j = (\boldsymbol{I} - \Phi_1 z)^{-1} (\boldsymbol{I} + \Theta_1 z).$$

Denote

$$\Psi(z) = \begin{pmatrix} \psi_{11}(z) & \psi_{12}(z) \\ \psi_{21}(z) & \psi_{22}(z) \end{pmatrix}.$$

Then we have

$$\psi_{11}(z) = \frac{z^2(\phi_{12}\theta_{21} - \phi_{22}\theta_{11}) + z(\theta_{11} - \phi_{22}) + 1}{(1-\phi_{11}z)(1-\phi_{22}z) - \phi_{12}\phi_{21}z^2},$$
$$\psi_{12}(z) = \frac{z^2(\phi_{12}\theta_{22} - \phi_{22}\theta_{12}) + z(\phi_{12} + \theta_{12})}{(1-\phi_{11}z)(1-\phi_{22}z) - \phi_{12}\phi_{21}z^2},$$
$$\psi_{21}(z) = \frac{z^2(\theta_{11}\phi_{21} - \theta_{21}\phi_{11}) + z(\phi_{21} + \theta_{21})}{(1-\phi_{11}z)(1-\phi_{22}z) - \phi_{12}\phi_{21}z^2},$$
$$\psi_{22}(z) = \frac{z^2(\phi_{21}\theta_{12} - \phi_{11}\theta_{22}) + z(\theta_{22} - \phi_{11}) + 1}{(1-\phi_{11}z)(1-\phi_{22}z) - \phi_{12}\phi_{21}z^2}.$$

For a two-dimensional $ARMA(1,1)$ process, the functions $\psi_{ij}(\cdot)$ are ratios of two polynomials, both of a degree two. This naturally suggests that a result for univariate $ARMA(2,2)$ processes could be applied for each of the functions $\psi_{ij}(\cdot)$. For the two diagonal functions $\psi_{ii}(\cdot)(i = 1,2)$, the denominator again takes directly the form of a moving average lag polynomial. The absolute monotonicity conditions then directly follow from Theorem 3.3.19. For the ease of notation, we can rewrite the functions $\psi_{11}(\cdot)$ and $\psi_{22}(\cdot)$ as follows:

$$\psi_{11}(z) = \frac{1 + \operatorname{tr}(A)z + \det(A)z^2}{1 - \operatorname{tr}(\Phi_1)z + \det(\Phi_1)z^2}, \quad \psi_{22}(z) = \frac{1 + \operatorname{tr}(B)z + \det(B)z^2}{1 - \operatorname{tr}(\Phi_1)z + \det(\Phi_1)z^2},$$

where

$$\text{A}=\begin{pmatrix} \theta_{11} & -\phi_{12} \\ \theta_{21} & -\phi_{22} \end{pmatrix}, \quad \text{B}=\begin{pmatrix} \theta_{22} & -\phi_{21} \\ \theta_{12} & -\phi_{11} \end{pmatrix}.$$

Take for instance the function $\psi_{11}(\cdot)$. According to Theorem 3.3.19 and Remark 3.3.7 we know that the absolute monotonicity of $\psi_{11}(\cdot)$ on $[0,1)$ is equivalent to

$$\begin{aligned} \lambda_1, \lambda_2 \in \mathbb{R} \quad \text{and} \quad \lambda_1 > 1, \\ 1 + \operatorname{tr}(A)\lambda_1 + \det(A)\lambda_1^2 > 0, \\ \psi_{11}^1 = \operatorname{tr}(A) + \operatorname{tr}(\Phi_1) \geq 0, \\ \psi_{11}^2 = -\det(\Phi_1) + \operatorname{tr}(A)\operatorname{tr}(\Phi_1) + \operatorname{tr}(\Phi_1)^2 \geq 0. \end{aligned}$$

Conditions for the absolute monotonicity of functions $\psi_{11}(\cdot)$ and $\psi_{22}(\cdot)$ are summarized in Table 4.3. The off-diagonal functions $\psi_{12}(\cdot)$ and $\psi_{21}(\cdot)$ admit the slightly different expansion in (4.2.5) with $p^* = 2$ and $q^* = 2$. The Theorem 3.3.19 can be again applied, just with the minor modification that the coefficients $\psi_{ij}^1(i \neq j; i,j = 1,2)$ and $\psi_{ij}^2(i \neq j; i,j = 1,2)$ are now determined from (4.2.7) and (4.2.8), respectively. The conditions are summarized in Table 4.4

	$\psi_{11}(z)$		$\psi_{22}(z)$
(1)	$\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 > 1,$	$(1)^*$	$\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 > 1,$
(2)	$\lambda_1^2 \det(A) + \lambda_1 \operatorname{tr}(A) + 1 > 0,$	$(2)^*$	$\lambda_1^2 \det(B) + \lambda_1 \operatorname{tr}(B) + 1 > 0,$
(3)	$\theta_{11} + \phi_{11} \geq 0,$	$(3)^*$	$\theta_{22} + \phi_{22} \geq 0$
(4)	$\phi_{11}(\phi_{11} + \theta_{11}) + \phi_{12}(\phi_{21} + \theta_{21}) \geq 0,$	$(4)^*$	$\phi_{22}(\phi_{22} + \theta_{22}) + \phi_{21}(\phi_{12} + \theta_{12}) \geq 0.$

Table 4.3: Conditions for absolute monotonicity of functions $\psi_{11}(\cdot)$ and $\psi_{22}(\cdot)$

	$\psi_{12}(z)$		$\psi_{21}(z)$
(1)	$\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 > 1,$	$(1)^*$	$\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 > 1,$
(2)	$\lambda_1^2(\phi_{12}\theta_{22} - \phi_{22}\theta_{12}) + \lambda_1(\phi_{12} + \theta_{12}) > 0,$	$(2)^*$	$\lambda_1^2(\phi_{12}\theta_{22} - \phi_{22}\theta_{12}) + \lambda_1(\phi_{21} + \theta_{21}) > 0,$
(3)	$\phi_{12} + \theta_{12} \geq 0,$	$(3)^*$	$\phi_{21} + \theta_{21} \geq 0$
(4)	$(\phi_{12} + \theta_{12})\operatorname{tr}(\Phi_1) + \theta_{11}\phi_{12} - \theta_{21}\phi_{11} \geq 0,$	$(4)^*$	$(\phi_{21} + \theta_{21})\operatorname{tr}(\Phi_1) + \phi_{12}\theta_{22} - \phi_{22}\theta_{12} \geq 0,$

Table 4.4: Conditions for absolute monotonicity of functions $\psi_{12}(\cdot)$ and $\psi_{21}(\cdot)$

The conditions in Table 4.3 and Table 4.4 form together a set of sufficient and necessary conditions for the non-negativity of the kernel sequence $\{\Psi_j\}_{j=0}^{\infty}$. Note that the conditions $(3), (3)^*, (4), (4)^*$ are equivalent to $\Psi_1 = \Theta_1 + \Phi_1 \geq 0$ and $\Psi_2 = \Phi_1(\Theta_1 + \Phi_1) \geq 0$. The conditions (2) and (2^*) translate as $[\operatorname{adj}(\mathrm{I}_2 - \Phi_1\lambda_1)](\mathrm{I}_2 + \Theta_1\lambda_1) > 0$. This leads to the following theorem

Theorem 4.3.2. *Let $\{\boldsymbol{X}_t : t = 0, \pm1, \pm2, \dots\}$ be a bivariate $ARMA(1,1)$ process given in (4.3.14). Let μ_1, μ_2 denote eigenvalues of the matrix Φ_1, such that $|\mu_2| \leq |\mu_1| < 1$. Assume that the two eigenvalues are distinct. Then the sequence of matrices $\{\Psi_j\}_{j=0}^{\infty}$ is non-negative, if and only if the following conditions hold*

$$\begin{aligned}
&(1) \quad \mu_1, \mu_2 \in \mathbb{R}, \quad \textit{and} \quad 0 < \mu_1 < 1, \\
&(2) \quad \Phi_1 + \Theta_1 \geq 0, \\
&(3) \quad \Phi_1(\Phi_1 + \Theta_1) \geq 0, \\
&(4) \quad [\operatorname{adj}(\mu_1 \mathrm{I}_2 - \Phi_1)](\mu_1 \mathrm{I}_2 + \Theta_1) > 0
\end{aligned}$$

Proof. A proof follows from the discussion above. □

4.4 Vector-valued ARMA(1,q)

Finally, we consider a p-dimensional $ARMA(1,q)$ process $\{\boldsymbol{X}_t : t = 0, \pm1, \pm2, \dots\}$ with non-negative innovation vectors, which determined from the following system of difference equations:

$$\boldsymbol{X}_t - \Phi_1 \boldsymbol{X}_{t-1} = \boldsymbol{Z}_t^* + \Theta_1 \boldsymbol{Z}_{t-1}^* + \cdots + \Theta_p \boldsymbol{Z}_{t-1}^*, \quad t = 0, \pm 1, \pm 2, \ldots. \tag{4.4.15}$$

The matrix-valued generating function $\boldsymbol{\Psi}(z)$ now takes the following form

$$\boldsymbol{\Psi}(z) = (\mathrm{I}_p - \Phi_1 z)^{-1}(\mathrm{I}_p + \Theta_1 z + \cdots + \Theta_p z^q), \quad 0 \leq z < 1.$$

Assuming that all the matrices $\Theta_i, i = 1, \ldots, q$ and Φ_1 are non-negative, the non-negativity of the series $\{\boldsymbol{X}_t : t = 0, \pm 1, \pm 2, \ldots\}$ follows directly from the equations (11.3.12) of Brockwell and Davis (1986), p. 409:

$$\begin{aligned}
\Psi_0 &= \mathrm{I}_p, \\
\Psi_1 &= \Phi_1 + \Theta_1, \\
\Psi_2 &= \Phi_1(\Phi_1 + \Theta_1) + \Theta_2, \\
&\vdots \\
\Psi_q &= \Phi_1 \Psi_{p-1} + \Theta_q, \\
\Psi_k &= \Phi_1 \Psi_{k-1}, \quad k \geq q+1.
\end{aligned}$$

Similarly as in the previous section we derive a set of sufficient and necessary conditions using the absolute monotonicity approach. Again, we confine ourselves only to two dimensional time series.

4.4.1 Two-dimensional ARMA(1,q)

The components of a matrix-valued generating function of the sequence $\{\Psi_j\}_{j=0}^{\infty}$ are ratios of two polynomials. The denumerator is of a degree two whereas in the numerator we have polynomials of a degree $q+1$. For all the functions we can directly apply Theorem 3.3.19. Again the off-diagonal functions admit the expansion (4.2.5) with slightly different coefficients.

According to Theorem 3.3.19 applied to all the functions $\psi_{ij}(z), i, j = 1, \ldots, 2$, the conditions for the series of matrices $\{\Psi_j\}_{j=0}^{\infty}$ to be non-negative are the following:

(a) the roots μ_1, μ_2 of the matrix Φ_1 have to be real and the maximal root in absolute value, say μ_1, is positive (smaller than one from the causality),

(b) the numerators of the functions $\psi_{ij}(z)$ evaluated at $1/\mu_1$ are non-negative,

(c) first $q+1$ coefficient matrices $\Psi_1, \ldots, \Psi_{q+1}$ in the sequence $\{\Psi_j\}_{j=0}^{\infty}$ are non-negative.

These conditions are summarized in the following theorem.

Theorem 4.4.1. *Let* $\{\boldsymbol{X}_t : t = 0, \pm 1, \pm 2, \dots\}$ *be the bivariate* $ARMA(1,q)$ *process given in (4.4.15). Let* μ_1, μ_2 *denote eigenvalues of the matrix* Φ_1*, such that* $|\mu_2| \leq |\mu_1| < 1$*. Assume that the two eigenvalues are distinct. The sequence of matrices* $\{\Psi_j\}_{j=0}^{\infty}$ *is non-negative, if and only if the following conditions hold*

$$\begin{aligned}
&(1) \quad \mu_1, \mu_2 \in \mathbb{R}, \quad \textit{and} \quad 0 < \mu_1 < 1, \\
&(2) \quad [\mathrm{adj}(\mu_1 \mathrm{I}_2 - \Phi_1)]\,(\mu_1 \mathrm{I}_2 + \Theta_1) > 0, \\
&(3) \quad \Psi_j \geq 0, j = 1, \dots, q+1.
\end{aligned}$$

Proof. A proof follows from the discussion above.

Chapter 5

Parameter Estimation in Non-negative Time Series

Finding parsimonious, yet well-fitting, representation of the observed data by a statistical model usually requires several interconnected modeling steps. In the context of autoregressive (moving average) models, these are basically twofold: (1) determination of a dimensionality of the model (orders p and q of the autoregressive and moving average polynomials), (2) estimation of the corresponding coefficients $\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q$ and eventually mean and/or variance parameters of the innovation sequence. The estimation of the orders of autoregressive moving average models constitutes a challenge on its own. We will confine ourselves only to the discussion on the estimation of model parameters of a given dimension.

The customized approaches for parameter estimation involve methods based on Yule-Walker equations, (conditional) Gaussian maximum likelihood or least squares, Brockwell and Davis (1986) Chapter 8. However, the rate of convergence of such estimators, when utilized for non-negative time series models, is not the best we could accomplish. Improvements with respect to the convergence speed might be achieved by exploiting the non-negative nature of the observations. This was one of the motivations for developing special estimation techniques for non-negative time series models. The second reason is that the classical estimation methods might lead to a model that does not define a non-negative time series. That is why a special care needs to be exercised to make sure that the estimates satisfy the non-negativity conditions.

The methodology considered for discussion here builds on the following frameworks: (conditional) maximum likelihood estimation, eventually leading to the linear programming problem, and the Bayesian approach. We concentrate on lower order autoregressive and moving average models (eventually nonlinear and/or multi-dimensional).

5.1 AR(1)

Consider the causal $AR(1)$ process $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ which satisfies the difference equation

$$X_t = \phi_1 X_{t-1} + Z_t^*, \quad t = 0, \pm 1, \pm 2, \dots, \tag{5.1.1}$$

where the sequence $\{Z_t^* : t = 0, \pm 1, \pm 2, \dots\}$ consists of `iid` non-negative random variables distributed according to a distribution function $F(\cdot)$. Assuming that the variance of Z_t^* is finite, it can be shown that the first order sample autocorrelation estimate

$$\widehat{\phi}_1 = \frac{\sum_{t=1}^{n-1}(X_t - \bar{X})(X_{t+1} - \bar{X})}{\sum_{t=1}^{n}(X_t - \bar{X})^2}, \quad \text{where} \quad \bar{X} = \frac{\sum_{t=1}^{n} X_t}{n},$$

is asymptotically normally distributed with mean ϕ_1 and variance $n^{-1}(1 - \phi_1^2)$. In other words

$$\sqrt{n}(\widehat{\phi}_1 - \phi) \xrightarrow{\text{d}} \text{N}(0, 1 - \phi^2).$$

The rate of the convergence $n^{1/2}$ is the same as for the Yule-Walker estimators, Brockwell and Davis (1986) p. 233. The asymptotic properties can be improved, if the estimator takes into account the non-negativity of the observations.

An alternative estimator emerges naturally from the observation that in the model (5.1.1) it holds that

$$\phi_1 \le \frac{X_t}{X_{t-1}}, \quad t \in \mathbb{Z}.$$

Suppose that a segment of realizations $X_1, \dots, X_n$ has been observed. The so called **extreme value estimator** of the "correlation parameter" ϕ_1 is given by

$$W_n = \min_{2 \le t \le n} \frac{X_t}{X_{t-1}}.$$

The statistical relevance of this intuitive estimator can be justified by several arguments. First, Bell and Smith (1986) who were the first to consider this estimator proved its strong consistency under general assumptions.

Theorem 5.1.1. *Let $X_1 \ge 0, Z_t^* \ge 0\,(t \ge 2)$ and $\phi_1 \ge 0$. If $F(d) - F(c) < 1$ for all $0 < c < d < \infty$, then $W_n \to \phi_1$ almost surely.*

Proof. Bell and Smith (1986)

Second, the extreme value estimator is the conditional maximum likelihood estimator, assuming that X_1 is known and that the innovation sequence Z_t^* is distributed according to exponential distribution with the expectation λ (we utilize the following parametrization of the exponential density $f(z) = \frac{1}{\lambda}\exp(-\frac{z}{\lambda}))$. More precisely, provided that $X_1 = x_1$ the likelihood of $X_2, \ldots, X_n$ is given by

$$\frac{1}{\lambda^{n-1}} \exp\left(-\sum_{t=2}^{n} \frac{x_t - \phi_1 x_{t-1}}{\lambda}\right), \quad \text{provided that} \quad x_t - \phi_1 x_{t-1} \geq 0 \quad (t = 2, \ldots, n),$$

and zero otherwise. The expression inside the exponential function is increasing in ϕ_1, which implies that the conditional maximum likelihood estimate is the maximal value ϕ_1 that satisfies the given constraints. This is exactly the value $w_n = \min_{2\leq t\leq n} \frac{x_t}{x_{t-1}}$.

Third, the extreme value estimators appears to be equivalent in large samples to the Bayesian "posterior mean estimator" resulting from a certain uninformative prior coupled with an approximate likelihood function. More specifically, it is known that under the exponentially distributed innovations (with a mean λ), the expectation of the stationary distribution of the process in (5.1.1) equals $\frac{\lambda}{1-\phi_1}$, Anděl (1988b). Since the stationary distribution is mathematically intractable, Anděl (1988b) suggests to utilize a crude approximation of the distribution of X_1. This approximation is obtained by matching the stationary distribution with an exponential distribution of the same expectation. The "approximate" likelihood function of $X_1, \ldots, X_n$ then equals

$$\begin{aligned} L(\phi_1, \lambda; X_1 = x_1, \ldots, X_n = x_n) &= \frac{(1-\phi_1)}{\lambda^n} \exp\left\{-\frac{1}{\lambda}\left[(1-\phi_1)x_1 + \sum_{t=2}^{n}(x_t - \phi_1 x_{t-1})\right]\right\} \mathrm{I}_A(\boldsymbol{x}) \\ &= \frac{(1-\phi_1)}{\lambda^n} \exp\left\{-\frac{1}{\lambda}\left[n\bar{x} - \phi_1 S\right]\right\} \mathrm{I}_A(\boldsymbol{x}), \end{aligned}$$

where

$$\begin{aligned} &\boldsymbol{x} = (x_1, \ldots, x_n)', \\ &A = \{\boldsymbol{x} \in \mathbb{R}^n : x_1 > 0, x_t \geq \phi_1 x_{t-1}, t = 2, \ldots, n\}, \\ &\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_n, \\ &S = 2x_1 + \sum_{i=2}^{n-1} x_i. \end{aligned}$$

Anděl (1988b) then shows that the maximum likelihood estimators are

$$\begin{aligned} \widehat{\phi}_1^{ML} &= \min\left(W_n, \frac{S-1}{S}\right) \\ \widehat{\lambda}^{ML} &= \bar{x} - \widehat{\phi}_1^{ML}\frac{S}{n}. \end{aligned}$$

Turkman (1990) uses the likelihood construction of Anděl (1988b) and matches it with a non-informative improper prior. This prior is assumed to be proportional to the mean value of the stationary distribution, i.e.

$$p(\phi_1, \lambda) \propto \frac{\lambda}{1-\phi_1} \mathrm{I}_{[0,1)\times(0,\infty)}(\phi_1, \lambda).$$

The advantage of this type of prior is ease of analytical tractability. More general prior specification has been considered by Ibazizen and Fellag (2003), namely

$$p(\phi_1, \lambda) \propto \lambda \frac{\phi_1^{\beta-1}}{1-\phi_1} \mathrm{I}_{[0,1)\times(0,\infty)}(\phi_1, \lambda) \quad \text{for} \quad \beta > 0.$$

The authors utilized this prior in order to assess the sensitivity on the choice of β and thereby stability of the posterior inference under varying prior assumptions.

Under the prior of Turkman (1990), the joint posterior distribution of the parameters ϕ_1, λ following from the Bayes theorem is

$$p(\phi_1, \lambda | X_1 = x_1, \dots, X_n = x_n) = \frac{C}{\lambda^{n-1}} \exp\left\{ -\frac{1}{\lambda} \left[n\bar{x} - \phi_1 S \right] \right\} \mathrm{I}_{[0,\min(1,w_n)]}(\phi_1) \mathrm{I}_{(0,\infty)}(\lambda),$$

where C is the normalizing constant. The marginal posterior distribution of ϕ_1 then equals

$$p(\phi_1 | X_1 = x_1, \dots, X_n = x_n) = \frac{C\Gamma(n-2)}{(n\bar{x} - \phi_1 S)^{n-2}} \mathrm{I}_{[0,\min(1,w_n)]}(\phi_1).$$

Turkman (1990) analytically derived the posterior mean estimator for ϕ_1 under given prior assumptions, which is

$$\phi_1^B = \mathsf{E}(\phi_1 | X_1 = x_1, \dots, X_n = x_n) = \frac{n-1}{n\bar{x}r} \frac{1-r^n}{1-r^{n-1}},$$

where $r = 1 - w_n \frac{S}{n\bar{x}}$. She further shows that this Bayesian estimator is strongly consistent and that for large n it can be closely approximated by the extreme value estimator W_n. The extreme value estimator is known to be biased upwards, Bell and Smith (1986). It has been shown by means of simulations that the Bayesian estimator has little bias compared to the extreme value estimator, Turkman (1990). However, how well would the Bayesian estimator perform when compared to the bias corrected version of W_n introduced by Anděl and Zvára (1988) was not investigated.

The distributional properties of the extreme value estimator W_n have been studied by several authors. Anděl (1988b) derives exact distribution of W_n under the assumption of exponential innovations and $X_1 \sim \mathrm{Exp}\left(\frac{\lambda}{1-\phi_1}\right)$.

Theorem 5.1.2. *Let* $X_1 \sim \text{Exp}\left(\frac{\lambda}{1-\phi_1}\right)$ *and* $Z_t^* \sim \text{Exp}(\lambda)$*. Then the distribution of* W_n *is given by* $\mathsf{P}(W_n < w) = 1 - G(w)$*, where*

$$\begin{aligned} G(w) = &(1-\phi_1)\{[w+(1-\phi_1)] \times [w^2+(1-\phi_1)(1+w)] \times \dots \\ &[w^{n-2}+(1-\phi_1)(1+w+\dots+w^{n-3})]\times \\ &[w^{n-1}+(1-\phi_1)(1+w+\dots+w^{n-2})-\phi_1]\}^{-1} \end{aligned}$$

for $w \geq \phi_1$*, and* $G(v) = 1$ *for* $w < \phi_1$*.*

Proof. Anděl (1988b)

Furthermore, Anděl and Zvára (1988) derived $\mathsf{E}\,W_n$ and $(\mathsf{Var}\;W_n)^{1/2}$ which enables to quantify the bias. Davis and McCormick (1989) determined the limiting distribution under the assumption of regular variation of the distribution $F(\cdot)$ and that the distribution of innovations satisfies a suitable moment condition.

Theorem 5.1.3. *Let* Z_t^* *be distributed according to the distribution function* $F(\cdot)$*, which satisfies* $F(0^-) = 0$ *and*

$$\lim_{t\to 0^+} \frac{F(tx)}{F(t)} = x^\alpha \quad \textit{for all} \quad x > 0.$$

Assume further that

$$\int x^\beta \mathrm{d}\,\mu_F(x) < \infty \quad \textit{for some} \quad \beta > \alpha.$$

Then

$$\mathsf{P}\{c_\alpha a_n^{-1}(W_n - \phi_1) \leq x\} \to 1 - \exp(-x^\alpha) \quad \textit{for} \quad x > 0,$$

where

$$a_n = \inf\{x : F(x) \geq n^{-1}\} \quad \textit{and} \quad c_\alpha = (\mathsf{E}X_1^\alpha)^{1/\alpha}.$$

Proof. Davis and McCormick (1989)

In order to apply this asymptotic result, the innovation distribution $F(\cdot)$ is needed to be known in order to compute the index of regular variation α.

The extreme value estimator W_n offers improvements with respect to the rate of convergence. As we have seen, this rate depends on the behavior of $F(\cdot)$ at zero, more specifically, on the coefficient of regular variation α. For instance, if $F(x) \sim cx^\alpha$ as $x \to 0$, then W_n converges to ϕ_1 at rate $n^{1/\alpha}$. Assuming the exponential innovations, i.e. $F(x) = 1 - \mathrm{e}^{-x} \sim x$ as $x \to 0$, then the rate equals n.

It is possible to generalize the extreme value estimator also for non-linear $AR(1)$ models. Let X_1 be a assumed known and $g(\cdot)$ be some measurable function. The nonlinear $AR(1)$ process $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ is defined by the equations

$$X_t = \phi_1 g(X_{t-1}) + Z_t^* \quad (t \geq 2).$$

Assuming that the innovation sequence is non-negative and $X_1 > 0, \phi_1 > 0$ and $g(x) > 0$, Anděl (1988a) considers the following estimator

$$\widetilde{W}_n = \min_{2 \le t \le n} \frac{X_t}{g(X_{t-1})}.$$

He proves the following property:

Theorem 5.1.4.

(i) Let $F(d) - F(c) < 1$ for all $0 < c < d < \infty$. If $g(\cdot)$ is nondecreasing and $g(x) \to \infty$ as $x \to \infty$, then $\widetilde{W}_n \to \phi_1$ almost surely.

(ii) Let $F(c) > 0$ and $F(c) < 1$ for all $c > 0$. If $g(\cdot)$ is nonincreasing, $g(x) \to 0$ as $x \to \infty$ and $g(x) \to \infty$ as $x \to 0^+$, then $\widetilde{W}_n \to \phi_1$ almost surely.

Proof. Anděl (1988a)

5.2 Vector-valued AR(1)

The extreme value estimator for univariate $AR(1)$ model can be naturally extended to the multivariate setting. Consider the stationary $p-$dimensional autoregressive process of the first order $\{\boldsymbol{X}_t = (X_{t1}, \ldots, X_{tp})' : t = 0, \pm 1, \pm 2, \ldots\}$ given by the relation

$$\boldsymbol{X}_t = \Phi_1 \boldsymbol{X}_t + \boldsymbol{Z}_t^*, \quad t = 0, \pm 1, \pm 2, \ldots, \tag{5.2.2}$$

where the innovation vectors are independent and identically distributed according to a distribution function $F(\cdot)$ and $\Phi_1 = \left(\phi_1^{ij}\right)_{i,j=1}^p$ is a $p \times p$ matrix whose roots lie inside the unit circle. Assume that a finite set of realizations $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ arising from a model (5.2.2) in which all the elements of the matrix Φ_1 are non-negative and the vectors Z_t^* have only non-negative components has been observed. Anděl (1992) proposed estimator $\widehat{\Phi}_1$ of the matrix Φ_1, whose elements $\widehat{\phi}_1^{ij}$take the form

$$\widehat{\phi}_1^{ij} = \min_{2 \le t \le n} \frac{X_{ti}}{X_{t-1,j}}, \quad i, j \in \{1, \ldots, p\}.$$

Under mild assumptions on the distribution of the elements of the innovation vectors, Anděl (1992) proves that each estimator $\widehat{\phi}_1^{ij}$ is strongly consistent.

Theorem 5.2.1. *Assume that $P(Z_{t1}^* < z, \ldots, Z_{tn}^* < z) > 0$ for all $z > 0$ and that for every $\nu > 0$ and for each $i \in \{1, \ldots, p\}$ there exists a number $\gamma > 0$, such that*

$$\mathsf{P}(Z_{t1}^* < \nu, \ldots, Z_{t,i-1}^* < \nu, Z_{t,i}^* > \gamma, Z_{t,i+1}^* < \nu, \ldots, Z_{t,p}^* < \nu) > 0.$$

Then $\widehat{\phi}_1^{ij} \to \phi_1^{ij}$ almost surely as $n \to \infty$ for each $i, j \in \{1, \ldots, p\}$.

Proof. Anděl (1992)

The convergence of this estimator is quite slow, as noted by Anděl (1992). He proposed an alternative estimator based on the approximation of the solution to the conditional maximum likelihood, assuming independent exponentially distributed elements of the innovation random vectors. He exemplified the procedure on two-dimensional $AR(1)$ process. Let $Z_{t1} \sim \text{Exp}(\lambda_1)$ and $Z_{t2} \sim \text{Exp}(\lambda_2)$, the conditional likelihood of $\boldsymbol{X}_2, \dots, \boldsymbol{X}_n$, given $\boldsymbol{X}_1$ equals

$$\lambda_1^{-n+1} \exp\left\{ -\sum_{t=2}^{n} (X_{t1} - \phi_{11} X_{t-1,1} - \phi_{12} X_{t-1,2})/\lambda_1 \right\} \times$$
$$\lambda_2^{-n+1} \exp\left\{ -\sum_{t=2}^{n} (X_{t2} - \phi_{21} X_{t-1,1} - \phi_{22} X_{t-1,2})/\lambda_2 \right\},$$

subject to

$$X_{t1} - \phi_{11} X_{t-1,1} - \phi_{12} X_{t-1,2} \geq 0,$$
$$X_{t2} - \phi_{21} X_{t-1,1} - \phi_{22} X_{t-1,2} \geq 0,$$

for $t = 2, \dots, n$. The maximization of the likelihood boils down to finding a maximum of these two objective functions

$$\phi_{11} \sum_{t=2}^{n} X_{t-1,1} + \phi_{12} \sum_{t=2}^{n} X_{t-1,2} \quad \text{and} \quad \phi_{21} \sum_{t=2}^{n} X_{t-1,1} + \phi_{22} \sum_{t=2}^{n} X_{t-1,2}.$$

Since for n large enough the difference between $\sum_{t=2}^{n} X_{t-1,i}$ and $\sum_{t=1}^{n} X_{t,i}$ will be small (respectively for $i = 1, 2$), it is reasonable to assume that the solution to the original constrained optimization problem will be close to the solution of a linear program

$$\max \left(\phi_{i1} \sum_{t=1}^{n} X_{t,1} + \phi_{i2} \sum_{t=1}^{n} X_{t,2} \right)$$

under constraints

$$X_{ti} - \phi_{i1} X_{t-1,1} - \phi_{i2} X_{t-1,2} \geq 0 \quad (t = 2, \dots, n), \quad \phi_{i1} \geq 0, \phi_{i2} \geq 0,$$

for $i = 1, 2$. Anděl (1992) shows that the solution to this linear program gives strongly consistent estimators of ϕ_{ij} $(i, j = 1, 2)$.

5.3 AR(2)

Consider the causal autoregressive process $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ of the second order, which is given by the equations

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t^*, \quad t = 0, \pm 1, \pm 2, \dots, \tag{5.3.3}$$

where $\{Z_t^* : t = 0, \pm 1, \pm 2, \dots\}$ is an `iid` sequence of non-negative random variables with a common distribution function $F(\cdot)$. Anděl (1989) generalized the extreme value estimator for second order autoregressive model. Denote

$$\phi_1' = \min_{3 \le t \le n} \frac{X_t}{X_{t-1}} \quad \text{and} \quad \phi_2' = \min_{3 \le t \le n} \frac{X_t}{X_{t-2}}.$$

Anděl (1989) showed the strong consistency of these estimators.

Theorem 5.3.1. *Let $F(d) - F(c) < 1$ for all $0 < c < d < \infty$. Then $\phi_1' \to \phi_1$ and $\phi_2' \to \phi_2 + \phi_1^2$ almost surely.*

Proof. The proof is given in Anděl (1989) The motivation for the estimators ϕ_1' and ϕ_2' follows from two principal observations. The first one is that

$$\frac{X_t}{X_{t-1}} = \phi_1 + \frac{\phi_2 X_{t-2} + Z_t^*}{X_{t-1}}, \quad t = 2, 3, \dots.$$

Anděl (1989) shows that

$$\min_{3 \le t \le n} \left(\frac{\phi_2 X_{t-2} + Z_t^*}{X_{t-1}} \right) \to 0 \quad \text{almost surely.}$$

This implies that ϕ_1' is a strongly consistent estimator for ϕ_1. The second observation is that

$$\frac{X_t}{X_{t-2}} = \frac{\phi_1 X_{t-1} + Z_t^*}{X_{t-2}} + \phi_2 = \phi_2 + \phi_1^2 + \frac{\phi_1 \phi_2 X_{t-3} + \phi_1 Z_{t-1}^* + Z_t^*}{X_{t-2}}, \quad t = 3, 4, \dots.$$

Similarly, Anděl (1989) shows that

$$\min_{3 \le t \le n} \left(\frac{\phi_1 \phi_2 X_{t-3} + \phi_1 Z_{t-1}^* + Z_t^*}{X_{t-2}} \right) \to 0 \quad \text{almost surely,}$$

which implies the strong consistency of ϕ_2' in the estimation of $\phi_2 + \phi_1^2$. This altogether gives that $\phi_2' - \phi_1'^2$ is a strongly consistent estimator for ϕ_2.

Similarly as for the extreme value estimators in multivariate $AR(1)$ process, the convergence is quite slow. Anděl (1989) therefore proposed another estimators that might have superior asymptotic behavior. Such estimators follow from the maximum likelihood principle, assuming that X_2 and X_1 are assumed known. The conditional likelihood, assuming the exponential distribution Exp(λ) of the innovations, given X_1 and X_2 equals

$$\lambda^{-n+2} \exp\left\{-\sum_{t=3}^{n} \frac{x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2}}{\lambda}\right\},$$

provided that

$$x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} \geq 0 \quad (t = 3, \ldots, n), \tag{5.3.4}$$

and zero otherwise. The likelihood will attain its maximum for those values ϕ_1 and ϕ_2 which maximize $\phi_1 \sum_{t=3}^{n} X_{t-1} + \phi_2 \sum_{t=3}^{n} X_{t-2}$ under the restrictions in (5.3.4) and assuming $\theta_1 \geq 0, \theta_2 \geq 0$, which complete the set of sufficient conditions that assure the non-negativity of the series $\{X_t : t = 0, \pm 1, \pm 2, \ldots\}$. For n large enough the quantities $\sum_{t=3}^{n} X_{t-1}$ and $\sum_{t=3}^{n} X_{t-2}$ will be sufficiently close and therefore the objective function can be approximated by $(\phi_1 + \phi_2) \sum_{t=3}^{n} X_{t-1}$. The conditional maximum likelihood estimates will be then approximately determined by solving the linear program with the objective function $\phi_1 + \phi_2$. Anděl (1989) proves that such solution provides strongly consistent estimators.

Theorem 5.3.2. *Let $F(d) - F(c) < 1$ for all $0 < c < d < \infty$. Let ϕ_1^*, ϕ_2^* be values that maximize $\phi_1 + \phi_2$ under the conditions $\phi_1 \geq 0, \phi_2 \geq 0, X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} \geq 0\,(t = 3, \ldots, n)$. Then $\phi_1^* \to \phi_1$ and $\phi_2^* \to \phi_2$ almost surely.*

Proof. Anděl (1989)

Anděl (1989) found by simulation that these estimators converge at a faster rate then the Yule-Walker estimators. The probabilistic justification of this observation was given by Feigin and Resnick (1992) who established the rate of the convergence of these estimators.

Feigin and Resnick (1994) formalized the linear programming estimators for autoregressive models of a general order p. Based on the observed values on $X_1, \ldots, X_n$ these are defined as

$$\widehat{\phi}_{LP} = \arg\max_{\boldsymbol{\phi} \in D_n} (\phi_1 + \cdots + \phi_n),$$

where the feasible region is defined as

$$D_n = \left\{\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)' \in \mathbb{R}^p : X_t - \sum_{i=1}^{p} \phi_i X_{t-i} \geq 0, t = p+1, \ldots, n\right\}.$$

Whereas Anděl (1989) provided the motivation for the linear programming estimator as an approximate maximum likelihood estimators for $AR(2)$ models, Feigin and Resnick (1994) pointed out the theoretical connection to the generalized martingale estimating equations. Assuming regular variation of the innovation distribution function $F(\cdot)$, they show that there exist $r(n) \to \infty$ such that $r(n)[\hat{\boldsymbol{\phi}}_{LP} - \boldsymbol{\phi}]$ has a limiting distribution. The linear programming estimators often provide superior convergence rate to the Yule-Walker estimators, Feigin and Resnick (1994).

The extreme value, as well as linear programming, estimators were accommodated for nonlinear $AR(2)$ process by Anděl (1990). Assuming that $\phi_1 \geq 0, \phi_2 \geq 0$ and $h_1(\cdot), h_2(\cdot)$ are non-decreasing positive functions, the positive nonlinear $AR(2)$ process is defined as

$$X_t = \phi_1 h_1(X_{t-1}) + \phi_2 h_2(X_{t-2}) + Z_t^*, \quad t = 3, 4, \ldots,$$

where X_1, X_2 are given random variables. Anděl (1990) considered following estimators based on the finite realization $X_1, \ldots, X_n$

$$\phi_1^* = \min_{3 \leq t \leq n} \frac{X_t}{h_1(X_{t-1})} \quad \text{and} \quad \phi_2^* = \min_{3 \leq t \leq n} \frac{X_t}{h_2(X_{t-2})}.$$

He shows that under certain assumptions on the functions $h_1(\cdot), h_2(\cdot)$ (concerning monotonicity and limiting behavior) the two estimators are strongly consistent. Similarly, he shows that the linear program estimators defined as

$$\arg\max_{\phi \in D}(\phi_1 + \phi_2),$$

where $D = \{(\phi_1, \phi_2)' : \phi_1 \geq 0, \phi_2 \geq 0, \phi_1 h_1(X_{t-1}) + \phi_2 h_2(X_{t-2}) \leq X_t, t = 3, \ldots, n\}$ are strongly consistent.

The Bayesian approach to parameter estimation in $AR(2)$ has been dealt by Anděl and Garrido (1991). They considered an analogous prior specification as Turkman (1990) did for $AR(1)$. Again, the likelihood function is approximated, assuming that X_1 and X_2 are independent, exponentially distributed with a mean equal to the mean of the stationary distribution, which is $\frac{\lambda}{1-\phi_1-\phi_2}$. The approximate likelihood function based on the observed random variables $X_1, \ldots, X_n$ is

$$L(\phi_1, \phi_2, \lambda; X_1 = x_1, \ldots, X_n = x_n) = (1-\phi_1-\phi_2)^2 \lambda^{-n} \exp\{-\lambda^{-1}[(1-\phi_1-\phi_2)(x_1+x_2)]\} \times \exp\left\{\lambda^{-1}\left[\sum_{t=3}^{n}(x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2})\right]\right\} I_B(\boldsymbol{x}),$$

where

$$B = \{\boldsymbol{x} = (x_1, \ldots, x_n)' : x_1 \geq 0, x_2 \geq 0, x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} \geq 0, t = 3, \ldots, n\}.$$

Anděl and Garrido (1991) considered the following joint prior density for the parameters ϕ_1, ϕ_2 and λ

$$p(\lambda, \phi_1, \phi_2) = \frac{1}{\lambda(1-\phi_1-\phi_2)^2} \mathrm{I}_{C \times (0,\infty)}(\phi_1, \phi_2, \lambda),$$

where

$$C = \{(\phi_1, \phi_2)' : \phi_1 \geq 0, \phi_2 \geq 0, z^2 - \phi_1 z - \phi_2 \neq 0 \quad \text{for} \quad |z| \geq 1\}.$$

This prior is again improper and simplifies analytical derivation of the posterior densities. The joint posterior density from the Bayes theorem equals

$$p(\phi_1, \phi_2, \lambda | X_1 = x_1, \dots, X_n = x_n) = C' \lambda^{-n-1} \exp\left\{ -\frac{S - S_1\phi_1 - S_2\phi_2}{\lambda} \right\} \mathrm{I}_((0,\infty))(\lambda) \mathrm{I}_M(\phi_1, \phi_2),$$

where

$$S = \sum_{t=1}^{n} x_t, \qquad S_1 = x_2 + \sum_{t=1}^{n-1} x_t, \qquad S_2 = x_1 + x_2 + \sum_{t=1}^{n-2} x_t,$$

C' is the normalizing constant and

$$M = C \cap \{(\phi_1, \phi_2)' : \phi_1 \geq 0, \phi_2 \geq 0, X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} \geq 0, t = 3, \dots, n\}.$$

The calculation of posterior (marginal) moments is more involved. If we denote $I_{ij} = \iint_M \phi_1^i \phi_2^j (1 - s_1\phi_1 - s_2\phi_2)^{-n} \mathrm{d}\,\phi_1 \mathrm{d}\,\phi_2$, then it can be derived that

$$\mathsf{E}(\phi_1^i \phi_2^j | \boldsymbol{x}) = I_{00}\, I_{ij}, \quad i \geq 0, j \geq 0.$$

Anděl and Garrido (1991) show that the posterior expectation is strongly consistent estimator for the parameter vector $(\phi_1, \phi_2)'$. However, the calculation of the integrals involved in the posterior expectations is prohibitive for larger values of n. That is why they suggested an analytical approximation of the integrals I_{ij}, which involved both approximation of the set M and approximation of the integrand. By a simulation study they demonstrate that the bias reduction that applied nicely for the $AR(1)$ Bayes estimator, no longer characterizes the Bayesian $AR(2)$ estimator.

5.4 MA(1)

Less statistical theory has been developed for estimating parameters in non-negative moving average models. Anděl (1994) employed the extreme value idea to the $MA(1)$ and $MA(2)$ models.

Let the process $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ be an invertible non-negative $MA(1)$ process, i.e. it satisfies the equations

$$X_t = Z_t^* + Z_{t-1}^* \theta_1, \quad t = 0, \pm 1, \pm 2, \dots,$$

where $\{Z_t^* : t = 0, \pm 1, \pm 2, \dots\}$ is a sequence of `iid` random variables with non-negative support and a common distribution function $F(\cdot)$ and $0 < \theta_1 < 1$. Anděl (1994) defines the following estimator of the parameter θ_1

$$\theta_1' = \min_{2 \leq t \leq n-1} \frac{X_{t-1} + X_{t+1}}{X_t}.$$

He also shows its strong consistency.

Theorem 5.4.1. *Assume that* $F(d) - F(c) < 1$ *for all* $0 < c < d < \infty$. *Then* $\theta_1' \to \theta$ *almost surely, as* $n \to \infty$.

Anděl (1994) shows by simulation that this estimator suffers from severe positive bias. In order to correct for the bias, it would be handy to know the distribution of this estimator. However, the distributional properties are not easily obtainable. Anděl (1994) derived approximation of the expectation $\mathsf{E}\,\theta_1'$ under the assumption of exponentially distributed innovations.

5.5 MA(2)

The second order invertible non-negative moving average sequence $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ is given by the set of equations

$$X_t = Z_t^* + \theta_1 Z_{t-1}^* + \theta_2 Z_{t-2}^*, \quad t = 0, \pm 1, \pm 2, \dots,$$

where, again, the innovations constitute a sequence of `iid` non-negative random variables with a distribution function $F(\cdot)$ and the polynomial $1 + \theta_1 z + \theta_2 z^2$ has no complex roots inside the unit circle $|z| \leq 1$. Assume further that $(\theta_1, \theta_2)'$ fulfill the non-negativity conditions $\theta_1 \geq 0, \theta_2 \geq 0$, that are sufficient to assure that $\{X_t : t = 0, \pm 1, \pm 2, \dots\}$ is non-negative. Anděl (1994) suggested the following estimators

$$\theta_1' = \min_{2 \leq t \leq n-1} \frac{X_{t+1} + 3X_{t-1}}{X_t}, \quad \theta_2' = \min_{3 \leq t \leq n-2} \frac{X_{t+2} + 2X_{t-1} + X_{t-2}}{X_t}$$

and proved that they are strongly consistent.

Theorem 5.5.1. *Assume that* $F(d) - F(c) < 1$ *for all* $0 < c < d < \infty$. *Then* $\theta_1' \to \theta_1$ *and* $\theta_2' \to \theta_2$ *almost surely, as* $n \to \infty$.

These estimators again suffer from substantial bias. The linear programming estimators for moving average models were considered by Feigin et al. (1996).

Chapter 6

Discussion

This thesis summarizes theoretical developments associated with non-negativity conditions in (vector-valued) autoregressive and moving average models. These involve conditions already established and but also newly formulated. The majority of the existing conditions build extensively on the theory of absolutely monotone functions, Tsai and Chan (2007), but has the scope limited to only autoregressive (moving average) models of lower orders. We have attempted to broaden this scope by deriving new conditions for models for which an explicit result was missing. We provided conditions for $AR(p), ARMA(2,1)$, $ARMA(3,1)$, $ARMA(2,q)$ and $ARMA(p,q)$ models. By empirical demonstration, we also provided a direction for derivation of non-negativity conditions for $ARMA(3,2)$. We also presented illuminative graphical representation of the results for $ARMA(3,1), ARMA(3,2)$ and $AR(3)$.

The obtained conditions for $ARMA(p,q)$ served as a basis for derivation sufficient and necessary conditions for multivariate autoregressive moving average models. We confined ourselves to only two dimensional $AR(1), ARMA(1,1)$ and $ARMA(1,q)$. However, this is surely not an exhaustive list of models, for which such conditions are feasible. The non-negativity of practically any multivariate $ARMA(p,q)$ model could be conveniently described with the aid of the conditions for univariate $ARMA(p,q)$.

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