Charles University in Prague, Faculty of Arts
Department of Logic

Master Thesis

Intuitionistic Logic and Axiomatic Theories

(Intuicionistická logika a axiomatické teorie)

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I declare that the following Master thesis is my own work for which I used only the sources and literature mentioned.

Prohlašuji, že jsem diplomovou práci vypracoval samostatně a že jsem uvedl všechny použité prameny a literaturu.

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Abstract

This thesis explores some properties of elementary intuitionistic theories. We focus on the following theories: the theory of equality, linear order, dense linear order, the theory of a successor function, Robinson arithmetic and the theory of rational numbers with addition; moreover, we usually deal with two different formulations of the theories. As for the properties, our main interest is in the following four: coincidence with the classical version of a theory, saturation, De Jongh's theorem and decidability. The thesis draws especially from the results of C. Smorynski and D. de Jongh and tries to develop them. Some results known for Heyting arithmetic are proved for other theories. We also try to answer the question of what is the effect of replacing an axiom by a different (classically equivalent) axiom, or which properties a "good" intuitionistic theory should have.

Keywords: intuitionistic logic, elementary theories, saturation, decidability, De Jongh's theorem

Abstrakt

Práce zkoumá vlastnosti některých elementárních intuicionistických teorií. Vybrány jsou následující teorie: teorie rovnosti, lineárního uspořádání, hustého lineárního uspořádání, teorie následníka, Robinsonova aritmetika a teorie sčítání racionálních čísel; navíc téměř každou z těchto teorií formulujeme dvěma různými způsoby. Z vlastností teorií nás zajímají především následující čtyři: splývání s klasickou verzí teorie, saturovanost, platnost De Jonghovy věty a rozhodnutelnost. Diplomová práce vychází zejména z výsledků C. Smorynského a D. de Jongha a snaží se je rozvinout. Některé výsledky známé pro Heytingovu aritmetiku dokazuje i pro jiné teorie. Dále se pokouší odpovědět například na to, jaký vliv má záměna axiomu teorie za jiný (klasicky ekvivalentní) axiom nebo jaké vlastnosti by měla mít *dobrá* intuicionistická teorie.

Klíčová slova: intuicionistická logika, elementární teorie, saturovanost, rozhodnutelnost, De Jonghova věta

Contents

1

Introduction

As the title suggests, this thesis is concerned with intuitionistic theories. It may seem that we are going to embrace a very broad topic, but in fact, we are not so ambitious. The main aim of this work is exploring some properties of elementary theories over intuitionistic predicate logic.

One of the most important properties of classical theories is their completeness, but asking whether an intuitionistic theory is complete does not make much sense. Intuitionistic theories are usually incomplete, since the law of excluded middle is not an intuitionistic tautology and therefore, the existence of a sentence φ such that φ is not provable and $\neg\varphi$ is not provable is quite natural. Hence, we have to deal with more sensible properties of intuitionistic theories than the completeness.

The property that does not lose its relevance in intuitionistic logic and that should be focused on is decidability. The other properties under consideration are purely intuitionistic. We are going to investigate whether theories are saturated, whether they coincide with their classical extensions and whether De Jongh's theorem holds.

For our purposes, we chose some theories that we think are worthy of being inspected. The choice was made so that the theories have the following characteristics. First, they are simple—they have only a few symbols in their languages and few axioms or schemas. Second, their classical versions have well-known properties, and third, they are "reasonable" in the sense that they express some mathematical concepts. Moreover, our choice was also motivated by the fact that two classically equivalent sentences may be intuitionistically non-equivalent. Therefore, we are going to explore more versions of theories differing only by the formulation of an axiom. This should help us answer the question of what is the best formulation of a theory.

We emphasize that we not only focus on the concrete results (i.e. information that a theory is saturated, undecidable etc.), but we are also interested in methods that lead to the results. In fact, the methods often are the crucial part.

The main sources that we use are [Smo73a], [Smo73b] and [BJ05]. Sometimes, we are strongly inspired by the results obtained for Heyting arithmetic and we try to modify the methods used for Heyting arithmetic in order for them to be applicable to the theories in our scope.

The thesis is divided into seven chapters. After the introduction, we present some basic information on intuitionistic logic and the theories that we investigate. Chapter 3 looks into coincidence between the theories and their classical extensions. Chapter 4 investigates whether the theories are saturated or not. Chapter 5 is concerned with De Jongh's theorem, thereby contributing to the considerations about trivialization of intuitionistic logic that were initiated in Chapter 3. Chapter 6 explores decidability of the theories and we finish with the conclusion.

2

Preliminaries

2.1 Intuitionistic logic

In this section, we try to briefly describe some basic facts concerning intuitionistic logic. We do not provide any complete introduction to intuitionistic logic, instead, we focus on introducing the notation that we use in this thesis and on listing the definitions and theorems that we make use of later. If the reader wants to have a more extensive background, we recommend her the relevant chapters of [Dal01], [Dal08], [Mos10] or [Šve02].

Even though we inspect theories formulated in intuitionistic *predicate* logic, in this chapter, we mention even some basic facts about intuitionistic *propositional* logic, since especially in chapter 5, we deal with propositional models and formulas.

2.1.1 Propositional logic

The propositional language contains symbols $\neg$, $\&$, $\vee$, $\rightarrow$ and $\equiv$, where $\equiv$ is defined as the conjunction of two implications. The atoms are denoted by p with indices and formulas are denoted by $A, B, C, \ldots$. To begin with, we define Kripke semantics for intuitionistic propositional logic.

Definition 2.1. *Let K be a non-empty set. The elements of K are denoted by α, β, γ,... and called* nodes. *Let $\leq$ be a reflexive, transitive and antisymmetric relation on K and let $\Vdash$ be a relation between nodes and formulas. Then $\mathcal{K} = \langle K, \leq, \Vdash \rangle$ is called* Kripke model for propositional intuitionistic logic *iff for any $\alpha, \beta \in K$, any formulas A, B and any atom p:*

(i) if $\alpha \leq \beta$ and $\alpha \Vdash p$, then $\beta \Vdash p$,

(ii) $\alpha \Vdash \neg A$ iff for all $\beta \geq \alpha$, $\beta \nVdash A$,

(iii) $\alpha \Vdash A \& B$ iff $\alpha \Vdash A$ and $\alpha \Vdash B$,

(iv) $\alpha \Vdash A \vee B$ iff $\alpha \Vdash A$ or $\alpha \Vdash B$,

(v) $\alpha \Vdash A \rightarrow B$ iff for all $\beta \geq \alpha$, if $\beta \Vdash A$, then $\beta \Vdash B$.

The first condition is called persistence*.*

Definition 2.2. *Let $\mathcal{K} = \langle K, \leq, \Vdash \rangle$ be a Kripke model and Γ a set of formulas. We say that $\mathcal{K}$* is a model of Γ *and write $\mathcal{K} \Vdash \Gamma$ iff for any $\alpha \in K$ and any $A \in \Gamma$, $\alpha \Vdash A$. If $\Gamma = \{A\}$, we write $\mathcal{K} \Vdash A$ instead of $\mathcal{K} \Vdash \{A\}$.*

Now, we present a calculus for intuitionistic propositional logic and the deduction theorem, which can be proved in exactly the same way as in classical logic.

Definition 2.3. *A* Hilbert type calculus for intuitionistic propositional logic (IPC) *can be formulated by the following axioms:*

A1: $A \to (B \to A)$

A2: $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

A3: $(A \to B) \to ((A \to \neg B) \to \neg A)$

A4: $A \,\&\, B \to A$

A5: $A \,\&\, B \to B$

A6: $A \to (B \to C) \to (A \,\&\, B \to C)$

A7: $A \to A \vee B$

A8: $B \to A \vee B$

A9: $(A \to C) \to ((B \to C) \to (A \vee B \to C))$

A10: $A \to (\neg A \to B)$

and one rule:

MP: $A,\ A \to B\ /\ B.$

Let $\vdash_{\mathrm{IPC}}$ be defined in the obvious manner.

Definition 2.4. *Let Γ be a set of formulas. Kripke model $\mathcal{K}$ is a* counterexample *or a* countermodel *to $\Gamma \vdash_{\mathrm{IPC}} A$ iff $\mathcal{K}$ is a model of Γ, but $\mathcal{K} \nVdash A$.*

Theorem 2.5 (Deduction theorem)**.** *Let Γ be a set of formulas. If $\Gamma, A \vdash_{\mathrm{IPC}} B$, then $\Gamma \vdash_{\mathrm{IPC}} A \to B$.*

The completeness theorem not only connects Kripke semantics with IPC, but it also asserts that we can restrict our semantical considerations to finite tree models. The proof for sequent calculus can be found in [Šve02, pp. 377–378] and sequent calculus is equivalent to Hilbert type calculus (see [TS00]), thus we can formulate the completeness theorem as

Theorem 2.6 (Completeness theorem)**.** IPC *is complete for the class of finite tree models, i.e., $\Gamma \nvdash_{\mathrm{IPC}} A$ iff there exists a countermodel to $\Gamma \vdash_{\mathrm{IPC}} A$ in a finite tree.*

In chapter 5, it does not suffice to have finite models, we need to deal with special subset of finite models called *the modified Jaskowski trees.* Here is the definition and an important theorem taken from [Smo73a, pp. 349–352].

Definition 2.7. The modified Jaskowski trees *are finite trees* $J_1, J_2, \ldots$ *inductively defined as follows:*

(i) J_1 *is the tree with one node,*

(ii) J_{n+1} *is defined such that the origin of* J_{n+1} *has* $n+1$ *successors and at each of them, there is* J_n.

Theorem 2.8. *Let* $\{\langle K_n, \leq_n \rangle\}_n$ *be a sequence of finite trees with the property that every finite tree* $\langle K, \leq \rangle$ *can be embedded as a subtree to some* $\langle K_n, \leq_n \rangle$*. Then,* IPC *is complete for the sequence* $\{\langle K_n, \leq_n \rangle\}_n$*. Particularly,* IPC *is complete for the sequence of the modified Jaskowski trees.*

2.1.2 Predicate logic

The predicate language contains symbols $\neg, \&, \vee, \rightarrow, \equiv, \forall, \exists$. Formulas are denoted by $\varphi, \psi, \chi \ldots$ and the subscript *at* suggests that a formula is atomic (φ_{at}). Let Var denote a set of all variables $x, y, z, \ldots$. The definition of Kripke semantics for intuitionistic predicate logic follows.

Definition 2.9. *Let* L *be a language,* K *a non-empty set of nodes,* $\leq$ *a reflexive, transitive and antisymmetric relation on* K *and let* l *be a function from nodes to classical structures for* L *such that*

(i) if X *and* Y *are domains of* $l(\alpha)$ *and* $l(\beta)$ *respectively and* $\alpha \leq \beta$*, then* $X \subseteq Y$*,*

(ii) if $s^{l(\alpha)}$ *and* $s^{l(\beta)}$ *are realizations of a function or predicate symbol* s *in structures* $l(\alpha)$ *and* $l(\beta)$ *respectively and* $\alpha \leq \beta$*, then* $s^{l(\alpha)} \subseteq s^{l(\beta)}$.

Then, $\langle K, \leq, l \rangle$ *is called* Kripke structure for language L.

Remark 2.10. *For convenience, we write* $a \in l(\alpha)$ *instead of "a is an element of the domain of* $l(\alpha)$*".*

Definition 2.11. *Let* $\langle K, \leq, l \rangle$ *be a Kripke structure for language* L*,* $e : \mathrm{Var} \rightarrow l(\alpha)$ *an evaluation of variables and* $\Vdash$ *a relation between nodes, formulas and evaluations such that*

(i) $\alpha \Vdash \varphi_{at}[e]$ *iff* $l(\alpha) \models \varphi_{at}[e]$ *(in the sense of classical logic),*

(ii) $\alpha \Vdash \neg\varphi[e]$ *iff for all* $\beta \geq \alpha$*,* $\beta \nVdash \varphi[e]$*,*

(iii) $\alpha \Vdash (\varphi \,\&\, \psi)[e]$ *iff* $\alpha \Vdash \varphi[e]$ *and* $\alpha \Vdash \psi[e]$*,*

(iv) $\alpha \Vdash (\varphi \vee \psi)[e]$ *iff* $\alpha \Vdash \varphi[e]$ *or* $\alpha \Vdash \psi[e]$*,*

(v) $\alpha \Vdash (\varphi \rightarrow \psi)[e]$ *iff for all* $\beta \geq \alpha$*, if* $\beta \Vdash \varphi[e]$*, then* $\beta \Vdash \psi[e]$*,*

(vi) $\alpha \Vdash \exists x \varphi[e]$ *iff there exists an element* $a \in l(\alpha)$ *such that* $\alpha \Vdash \varphi[e(x/a)]$*,*

(vii) $\alpha \Vdash \forall x\varphi[e]$ *iff for all* $\beta \geq \alpha$ *and all* $b \in l(\beta)$, $\beta \Vdash \varphi[e(x/b)]$.

Then, $\mathcal{K} = \langle K, \leq, l, \Vdash \rangle$ *is called* Kripke model for predicate intuitionistic logic*.*

Definition 2.12. *Let* $\mathcal{K} = \langle K, \leq, l, \Vdash \rangle$ *be a Kripke model. We say that* φ *is* valid in $\mathcal{K}$ *and write* $\mathcal{K} \Vdash \varphi$ *iff for any* $\alpha \in K$ *and any* $e : \mathrm{Var} \to l(\alpha)$, $\alpha \Vdash \varphi[e]$.

Definition 2.13. *Let* $\mathcal{K}$ *be a Kripke model and* Γ *a set of formulas. We say that* $\mathcal{K}$ is a model of Γ *and write* $\mathcal{K} \Vdash \Gamma$ *iff for any* $\varphi \in \Gamma$, $\mathcal{K} \Vdash \varphi$.

Now, we define a calculus for predicate logic and state the deduction and completeness theorem. Note that in contrast to propositional logic, we cannot restrict models to finite trees. The proof of the completeness theorem can be found, e.g., in [Smo73a, pp. 329–332]. A direct consequence of the proof is Löwenheim-Skolem theorem.

Definition 2.14. *A* Hilbert type calculus for intuitionistic predicate logic (IQC) *has the propositional axioms A1–A10, the rule MP and the following axioms and rules:*

B1: $\forall x\varphi \to \varphi_x(t)$

B2: $\varphi_x(t) \to \exists x\varphi$

Gen-A: $\psi \to \varphi \;/\; \psi \to \forall x\varphi$

Gen-E: $\varphi \to \psi \;/\; \exists x\varphi \to \psi$,

where t *is a term substitutable for* x *in* φ *and* x *is not a free variable in* ψ.

Let $\vdash_{\mathrm{IQC}}$ *be defined in the obvious manner and let* $\mathrm{Thm}(\Gamma)$ *denote the set* $\{\, \varphi\,;\, \Gamma \vdash_{\mathrm{IQC}} \varphi \,\}$. *(We usually use* $\vdash$ *instead of* $\vdash_{\mathrm{IQC}}$*. When we need to distinguish propositional and predicate logic, we use* $\vdash_{\mathrm{IPC}}$ *and* $\vdash_{\mathrm{IQC}}$ *respectively. When we need to distinguish classical and intuitionistic logic, we use* $\vdash_{\mathrm{c}}$ *and* $\vdash_{\mathrm{i}}$ *respectively.)*

Definition 2.15. *Let* Γ *be a set of sentences. Kripke model* $\mathcal{K}$ *is a* counterexample *or a* countermodel *to* $\Gamma \vdash_{\mathrm{IQC}} \varphi$ *iff* $\mathcal{K}$ *is a model of* Γ, *but* $\mathcal{K} \nVdash \varphi$.

Theorem 2.16 (Deduction theorem)**.** *Let* ψ *be a sentence. If* $\Gamma, \psi \vdash_{\mathrm{IQC}} \varphi$, *then* $\Gamma \vdash_{\mathrm{IQC}} \psi \to \varphi$.

Theorem 2.17 (Completeness theorem)**.** *Let* Γ *be a set of sentences. Then,* IQC *is complete for the class of predicate Kripke models, i.e.,* $\Gamma \nvdash_{\mathrm{IQC}} \varphi$ *iff there exists a countermodel to* $\Gamma \vdash_{\mathrm{IQC}} \varphi$.

Theorem 2.18 (Löwenheim-Skolem theorem)**.** *Let* T *be a consistent theory in language* L. *Then, there exists a Kripke model* $\mathcal{K}$ *of* T *such that for every* $\alpha \in K$, $|l(\alpha)| \leq \aleph_0 + |L|$.

There are some notions that are commonly used, but might be a bit confusing. We explain them in the following definition.

Definition 2.19. *We say that theory* T *is* decidable *iff* $\mathrm{Thm}(T)$ *is a recursive set. In contrast, we say that formula* φ *is* decidable in a theory T *iff* $T \vdash \varphi \vee \neg\varphi$. *Formula* φ *is* stable in a theory T *iff* $T \vdash \neg\neg\varphi \to \varphi$.

As it is shown in [Dal08, pp. 160–162], classical logic can be embedded into intuitionistic logic, when we interpret the classical disjunction and existence quantifier in a weak sense.

Definition 2.20. Gödel translation ($\varphi \mapsto \varphi^{\mathrm{g}}$) *is a mapping of formulas to formulas such that*

(i) $\varphi_{at}^{\mathrm{g}} = \neg\neg\varphi_{at}$*,*

(ii) $(\neg\varphi)^{\mathrm{g}} = \neg\varphi^{\mathrm{g}}$*,*

(iii) $(\varphi \,\&\, \psi)^{\mathrm{g}} = \varphi^{\mathrm{g}} \,\&\, \psi^{\mathrm{g}}$*,*

(iv) $(\varphi \vee \psi)^{\mathrm{g}} = \neg(\neg\varphi^{\mathrm{g}} \,\&\, \neg\psi^{\mathrm{g}})$*,*

(v) $(\varphi \to \psi)^{\mathrm{g}} = \varphi^{\mathrm{g}} \to \psi^{\mathrm{g}}$*,*

(vi) $(\forall x\varphi)^{\mathrm{g}} = \forall x\varphi^{\mathrm{g}}$*,*

(vii) $(\exists x\varphi)^{\mathrm{g}} = \neg\forall x\neg\varphi^{\mathrm{g}}$*.*

We define $\Gamma^{\mathrm{g}} = \{\, \varphi^{\mathrm{g}} \,;\, \varphi \in \Gamma \,\}$*.*

Theorem 2.21. $\Gamma \vdash_{\mathrm{c}} \varphi$ *iff* $\Gamma^{\mathrm{g}} \vdash_{\mathrm{i}} \varphi^{\mathrm{g}}$*.*

2.2 Elementary theories

The aim of this section is to list the elementary theories which we study in this thesis. We introduce the notation of axioms which is consistently used in the following chapters. Although this work deals with elementary *intuitionistic* theories, we find it useful to mention some significant characteristics of the theories when the underlying logic is *classical*. This is because we are also interested in comparing classical and intuitionistic theories. We take the familiar characteristics of classical theories mainly from [Šve02]. A common property of all the theories below is their consistency. In the following list of theories, we suppose that equality is a non-logical symbol (i.e., the underlying logic is either classical or intuitionistic first order logic *without* equality).

The theory of equality (E) E has the language $\{=\}$ and the following axioms:

E1: $\forall x(x = x)$

E2: $\forall x\forall y(x = y \to y = x)$

E3: $\forall x\forall y\forall z(x = y \,\&\, y = z \to x = z)$

Classical version of E is incomplete (an independent sentence is, for example, $\forall x\forall y(x = y)$) and decidable [Mon76, p. 234].

The theories with function or predicate symbols will possess two more schemas of equality[1]:

E4: $\forall \underline{x} \forall \underline{y}(x_1 = y_1 \ \& \ \ldots \ \& \ x_n = y_n \rightarrow F(\underline{x}) = F(\underline{y}))$, for any function symbol F

E5: $\forall \underline{x} \forall \underline{y}(x_1 = y_1 \ \& \ \ldots \ \& \ x_n = y_n \rightarrow P(\underline{x}) \equiv P(\underline{y}))$, for any predicate symbol P

The theory of linear order (LO) LO has the language $\{=, <\}$ and the following axioms:

E1–E3, E5

LO1: $\forall x \forall y \forall z(x < y \ \& \ y < z \rightarrow x < z)$

LO2: $\forall x \neg(x < x)$

LO3: $\forall x \forall y(x < y \lor x = y \lor y < x)$

The classical version of LO is incomplete, $\forall x \exists y(x < y)$ is an independent sentence.

The weaker theory of linear order (wLO) Theory wLO is intuitionistically weaker version of LO where LO3 is replaced by the following two axioms:

wLO3: $\forall x \forall y \forall z(x < y \rightarrow x < z \lor z < y)$

AP: $\forall x \forall y(\neg(x < y \lor y < x) \rightarrow x = y)$

An intuitionistically equivalent theory to wLO can be obtained by dropping LO2 and formulating AP as equivalence instead of implication. The classical version of wLO is incomplete, since LO and wLO are classicaly equivalent.

The theory of dense linear order (DNO) DNO is LO enriched with the axioms of density:

DN1: $\forall x \forall y(x < y \rightarrow \exists z(x < z \ \& \ z < y))$

DN2: $\forall x \exists y(x < y)$

DN3: $\forall x \exists y(y < x)$

Classical DNO is complete and thus decidable.

The weaker theory of dense linear order (wDNO) Theory wDNO is given by the axioms of wLO plus DN1, DN2 and DN3. Obviously, the classical versions of wDNO and DNO coincide. Therefore, the classical wDNO is complete and decidable.

The theory of a successor function ($\mathbf{SUCC}^{\vee}$) $\mathrm{SUCC}^{\vee}$ has a language with one constant (0), one unary function (S) and one binary relation (=) symbol. The axioms of

[1] $\underline{x}$ is a shorthand for $x_1, x_2, \ldots x_n$.

$\mathrm{SUCC}^{\vee}$ are the following:

E1–E4

Q1: $\forall x \forall y(\mathrm{S}(x) = \mathrm{S}(y) \rightarrow x = y)$

Q2: $\forall x \neg(\mathrm{S}(x) = 0)$

$\mathrm{Q3}^{\vee}$: $\forall x(x = 0 \vee \exists y(x = \mathrm{S}(y))$

L^{n}: $\forall x \neg(\mathrm{S}^{n}(x) = x)$, for all $n \geq 1$

The weaker theory of a successor function ($\mathbf{SUCC}^{\rightarrow}$) $\mathrm{SUCC}^{\rightarrow}$ is a slightly modified version of $\mathrm{SUCC}^{\vee}$. Here, $\mathrm{Q3}^{\vee}$ is replaced by intuitionistically weaker axiom

$\mathrm{Q3}^{\rightarrow}$: $\forall x(\neg(x = 0) \rightarrow \exists y(x = \mathrm{S}(y))$

In classical logic, $\mathrm{SUCC}^{\vee}$ and $\mathrm{SUCC}^{\rightarrow}$ coincide and both are complete and decidable.

Robinson arithmetic ($\mathbf{Q}^{\vee}$) $\mathrm{Q}^{\vee}$ has the language $\{0, \mathrm{S}, +, \cdot, =, \leq, <\}$ and the following axioms:

E1–E5

Q1, Q2, $\mathrm{Q3}^{\vee}$

Q4: $\forall x(x + 0 = x)$

Q5: $\forall x \forall y(x + \mathrm{S}(y) = \mathrm{S}(x + y))$

Q6: $\forall x(x \cdot 0 = 0)$

Q7: $\forall x \forall y(x \cdot \mathrm{S}(y) = x \cdot y + x)$

Q8: $\forall x \forall y(x \leq y \equiv \exists v(v + x = y))$

Q9: $\forall x \forall y(x < y \equiv \exists v(\mathrm{S}(v) + x = y))$

The classical version of $\mathrm{Q}^{\vee}$ is incomplete and undecidable as a result of Gödel's first incompleteness theorem.

The weaker version of Robinson arithmetic ($\mathbf{Q}^{\rightarrow}$) $\mathrm{Q}^{\rightarrow}$ is a classically equivalent theory to $\mathrm{Q}^{\vee}$. The only difference resides in formulating axiom Q3. Here, $\mathrm{Q3}^{\vee}$ is replaced by $\mathrm{Q3}^{\rightarrow}$.

In the following chapters, Q (without superscript) refers to the cases where it does not make any difference whether we formulate Q3 as a disjunction or an implication (we also use Q3 without superscript). The meaning of SUCC is analogous.

The theory of rational numbers with addition (RNA) RNA has a language with two constants (0, 1), one binary function symbol (+) and two binary relation symbols

$(=, <)$. The axioms of RNA are:

E1–E5

LO1–LO3

DN1–DN3

RN1: $\forall x \forall y (x + y = y + x)$

RN2: $\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$

RN3: $\forall x (x + 0 = x)$

RN4: $\forall x \exists y (x + y = 0)$

RN5: $\forall x \forall y \forall z (x < y \to x + z < y + z)$

RN6: $0 < 1$

RN7: $\forall x \exists y (ny = x)$, for $n \geq 2$, where ny means $y + y + \ldots + y$ (n times)

The classical version of RNA is complete and decidable. The sense of RNA is that it embraces the concept of the addition of rational numbers and at the same time it has a name for every natural number.

The weaker theory of rational numbers with addition (wRNA) The only difference between RNA and wRNA is that the latter contains axioms wLO3 and AP instead of LO3. If brief, RNA is based on LO while wRNA is based on wLO. It follows from the previous paragraphs that wRNA is classically equivalent to RNA.

3

Coincidence with classical theories

3.1 A coincidence criterion

In the introduction, we mentioned that we are concerned with properties of intuitionistic theories. The first question that naturally arises is "Does certain intuitionistic theory differ from its classical version at all?" Without thinking twice, we would answer: "Of course, there must be a difference, since the underlying logic is different." Nevertheless, we show in this section that some of the theories listed above are the same regardless of the underlying logic. If such a phenomenon occurs, we say that the intuitionistic theory *coincides with* its classical version or that the theory *trivializes* intuitionistic logic. It happens if the axioms of certain intuitionistic theory are so strong that all the classically logically valid formulas can be proved.

If we are able to prove $\varphi \vee \neg\varphi$ for all φ in some theory, then it is the same as if we added schema $A \vee \neg A$ to the logic, thereby changing intuitionistic logic into classical. However, in some theories, $\varphi \vee \neg\varphi$ can be proved merely for formulas φ without quantifiers or for formulas whose atomic subformulas are of a specific form; see the next section.

In this section, we present a useful coincidence criterion which was suggested in [Smo73b, pp. 110–111]. The method is model-theoretic, but it uses some significant results of classical versions of the theories in scope. We suppose that the reader is familiar with basic properties of embeddings and elementary embeddings. Before presenting a theorem of Smorynski, we mention a proposition and a lemma that are used in the proof of the theorem and its corollaries.

Definition 3.1. *Classical theory T is* model complete *if every embedding between models of T is elementary embedding.*

Proposition 3.2. *Every (classical) theory T which admits quantifier elimination is model complete.*

Proof. Let $\mathbb{A}$ and $\mathbb{B}$ be two models of T and let $f : \mathbb{A} \to \mathbb{B}$ be an embedding between them. We want to show that f is an elementary embedding. Suppose $\varphi(\underline{x})$ is a formula in the language of T and $a_1, \ldots, a_n$ are elements of the domain of $\mathbb{A}$. There exists a quantifier-free formula $\psi(\underline{x})$ such that $\mathbb{A} \models \forall\underline{x}(\varphi(\underline{x}) \equiv \psi(\underline{x}))$, for T admits quantifier elimination. Satisfiability of quantifier-free formulas is preserved by embeddings, thus $\mathbb{A} \models \psi(\underline{a})$ iff $\mathbb{B} \models \psi(\underline{f(a)})$. Hence, we have $\mathbb{A} \models \varphi(\underline{a})$ iff $\mathbb{A} \models \psi(\underline{a})$ iff $\mathbb{B} \models \psi(\underline{f(a)})$ iff $\mathbb{B} \models \varphi(\underline{f(a)})$. Consequently, f is an elementary embedding. $\square$

Lemma 3.3. *If for all atomic formulas φ_{at}, $T \vdash \varphi_{at} \vee \neg\varphi_{at}$, then for all quantifier-free formulas φ, $T \vdash \varphi \vee \neg\varphi$.*

Proof. By induction on φ.

(i) φ is $\neg\psi$. By induction hypothesis, we have $T \vdash \psi \vee \neg\psi$. Since $\psi \to \neg\neg\psi$ is intuitionistically logically valid, we have $T \vdash \neg\neg\psi \vee \neg\psi$. Thus, $T \vdash \neg\varphi \vee \varphi$.

(ii) φ is $\psi \,\&\, \chi$. By induction hypothesis, we have $T \vdash \psi \vee \neg\psi$ and $T \vdash \chi \vee \neg\chi$ which entails

$$T \vdash ((\psi \vee \neg\psi) \,\&\, \chi) \vee ((\psi \vee \neg\psi) \,\&\, \neg\chi)$$

$$T \vdash (\psi \,\&\, \chi) \vee (\neg\psi \,\&\, \chi) \vee (\psi \,\&\, \neg\chi) \vee (\neg\psi \,\&\, \neg\chi)$$

It suffices to check that for each disjunct D of the previous disjunction

$$T \vdash D \to (\psi \,\&\, \chi) \vee \neg\psi \vee \neg\chi$$

holds. Due to the fact that

$$\vdash \neg\psi \vee \neg\chi \to \neg(\psi \,\&\, \chi),$$

we finally obtain

$$T \vdash (\psi \,\&\, \chi) \vee \neg(\psi \,\&\, \chi).$$

(iii) φ is $\psi \vee \chi$. The proof is analogous to the previous case. Similarly, we use the fact that

$$\vdash \neg\psi \,\&\, \neg\chi \to \neg(\psi \vee \chi).$$

(iv) φ is $\psi \to \chi$. Again, similar to the previous case. We use the fact that

$$\vdash \neg\psi \vee \chi \vee (\psi \,\&\, \neg\chi) \to (\psi \to \chi) \vee \neg(\psi \to \chi).$$

$\square$

Now, let us show Smorynski's result.

Lemma 3.4 ([Smo73b, p. 110])**.** *Suppose that $\mathfrak{M}$ is a set of classical models such that every embedding between the elements of $\mathfrak{M}$ is elementary embedding. Suppose further that $\mathcal{K}$ is a Kripke model obtained by associating with every $\alpha \in K$ some model $\mathbb{D}_\alpha \in \mathfrak{M}$ and forcing at α*

$$\alpha \Vdash \varphi_{at}[e] \quad \text{iff} \quad \mathbb{D}_\alpha \models \varphi_{at}[e], \tag{3.1}$$

for every atomic formula φ_{at} and every $e : \mathrm{Var} \to D_\alpha$. Then, (3.1) holds for any formula φ.

Proof. By induction on φ.

(i) φ is $\neg\psi$. The following lines are equivalent:

$$\begin{aligned}
& & \alpha \Vdash \neg\psi[e] & \\
&\text{for all } \beta \geq \alpha, & \beta \not\Vdash \psi[e] & \qquad (3.2)\\
&\text{for all } \beta \geq \alpha, & \mathbb{D}_\beta \not\models \psi[e] & \qquad (3.3)\\
&\text{for all } \beta \geq \alpha, & \mathbb{D}_\beta \models \neg\psi[e] & \qquad (3.4)\\
& & \mathbb{D}_\alpha \models \neg\psi[e] & \qquad (3.5)
\end{aligned}$$

(3.2) $\Leftrightarrow$ (3.3) holds due to induction hypothesis. A remarkable case is (3.5) $\Rightarrow$ (3.4). It is true, since $\mathbb{D}_\beta$ is an elementary extension of $\mathbb{D}_\alpha$.

(ii) φ is $\psi \to \chi$. Similarly, the following lines are equivalent:

$$\begin{aligned}
& & \alpha \Vdash (\psi \to \chi)[e] \\
&\text{for all } \beta \geq \alpha, & \text{if } \beta \Vdash \psi[e], \text{ then } \beta \Vdash \chi[e] \\
&\text{for all } \beta \geq \alpha, & \text{if } \mathbb{D}_\beta \models \psi[e], \text{ then } \mathbb{D}_\beta \models \chi[e] \\
&\text{for all } \beta \geq \alpha, & \mathbb{D}_\beta \models (\psi \to \chi)[e] \\
& & \mathbb{D}_\alpha \models (\psi \to \chi)[e]
\end{aligned}$$

(iii) φ is $\exists x\psi$.[1]

$$\begin{aligned}
& & \alpha \Vdash \exists x\psi[e] \\
&\text{there exists } a \in l(\alpha) \text{ such that} & \alpha \Vdash \psi[e(x/a)] \\
&\text{there exists } a \in D_\alpha \text{ such that} & \mathbb{D}_\alpha \models \psi[e(x/a)] \\
& & \mathbb{D}_\alpha \models \exists x\psi[e]
\end{aligned}$$

(iv) φ is $\forall x\psi$.

$$\begin{aligned}
& & \alpha \Vdash \forall x\psi[e] \\
&\text{for all } \beta \geq \alpha \text{ and for all } a \in l(\beta), & \beta \Vdash \psi[e(x/a)] \\
&\text{for all } \beta \geq \alpha \text{ and for all } a \in D_\beta, & \mathbb{D}_\beta \models \psi[e(x/a)] \\
&\text{for all } \beta \geq \alpha, & \mathbb{D}_\beta \models \forall x\psi[e] \\
& & \mathbb{D}_\alpha \models \forall x\psi[e]
\end{aligned}$$

The other cases are left to the reader. □

Theorem 3.5 ([Smo73b, p. 111])**.** *Let T be an intuitionistic theory and T^c its classical extension, let Γ be a set of prenex axioms for T^c and suppose the following are satisfied:*

(i) T^c is model complete

[1] D_α denotes the domain of $\mathbb{D}_\alpha$.

(ii) $T \vdash \varphi_{at} \vee \neg\varphi_{at}$*, for all atomic* φ_{at}

(iii) $T \vdash \varphi$*, for all* $\varphi \in \Gamma$*.*

Then T *and* T^{c} *coincide.*

Proof. Let $\mathcal{K}$ be an arbitrary Kripke model of T and ψ an arbitrary formula in the language of T. We want to prove that for every $\alpha \in K$ and every $e : \mathrm{Var} \to D_\alpha$, $\alpha \Vdash (\psi \vee \neg\psi)[e]$.

We define a classical model $\mathbb{D}_\alpha$ for $\alpha \in K$ by putting the domain of $\mathbb{D}_\alpha$ to be the domain of $l(\alpha)$ and letting $\mathbb{D}_\alpha \models \varphi_{at}[e]$ iff $\alpha \Vdash \varphi_{at}[e]$, for atomic φ_{at}. In order to verify that every $\mathbb{D}_\alpha$ is a classical model of T^{c}, we need to show that $\alpha \Vdash \chi[e]$ implies $\mathbb{D}_\alpha \models \chi[e]$, for any $\chi \in \Gamma$. We give an example: Assume that χ is $\exists x \forall y \forall z \exists w \varphi(x, y, z, w)$ where φ is a quantifier-free formula. Then[2],

$$
\begin{aligned}
& \alpha \Vdash \exists x \forall y \forall z \exists w \varphi(x, y, z, w) \\
\Leftrightarrow\ & \exists a \in l(\alpha) \forall \beta \geq \alpha \forall b \in l(\beta) \forall \gamma \geq \beta \forall c \in l(\gamma) \exists d \in l(\gamma)\ \gamma \Vdash \varphi(a, b, c, d) \\
\Rightarrow\ & \exists a \in l(\alpha) \forall b \in l(\alpha) \forall c \in l(\alpha) \exists d \in l(\alpha)\ \alpha \Vdash \varphi(a, b, c, d) && (3.6) \\
\Rightarrow\ & \exists a \in D_\alpha \forall b \in D_\alpha \forall c \in D_\alpha \exists d \in D_\alpha\ \mathbb{D}_\alpha \models \varphi(a, b, c, d) && (3.7) \\
\Rightarrow\ & \mathbb{D}_\alpha \models \exists x \forall y \forall z \exists w \varphi(x, y, z, w)
\end{aligned}
$$

Note that (3.6) $\Rightarrow$ (3.7) is only possible due to assumption (ii) and Lemma 3.3. If it were not for them, e.g., $\alpha \Vdash \neg\neg R(a, b, c, d)$, but $\mathbb{D}_\alpha \not\models \neg\neg R(a, b, c, d)$ could happen. Furthermore, dropping off assumption (iii) could also lead to undesirable results, e.g., $\alpha \Vdash \neg\neg(\forall x \exists y R(x, y))$, but $\mathbb{D}_\alpha \not\models \neg\neg(\forall x \exists y R(x, y))$.

So far, we have shown that every $\mathbb{D}_\alpha$ is a classical model of T^{c} and $\mathbb{D}_\alpha \models \varphi_{at}[e]$ iff $\alpha \Vdash \varphi_{at}[e]$. Moreover, T^{c} is model complete. Thus, all the assumptions of Lemma 3.4 are satisfied and we obtain (3.1) for any φ. Particularly, $\alpha \Vdash (\psi \vee \neg\psi)[e]$ for any α, ψ and e. $\square$

Now, we can apply the theorem to some of the theories in our scope. The following corollaries demonstrate that DNO and RNA coincide with their classical extensions.

Corollary 3.6. DNO *trivializes intuitionistic logic.*

Proof. We verify the assumptions of the theorem.

(i) Classical DNO does not admit elimination of quantifiers, but if we add two propositional constants, $\top$ and $\bot$, to the language of DNO, it does (see [KK67, pp. 51–53]). Let us look at the proof of Proposition 3.2 and modify it in order to be applicable to DNO. Suppose that T is DNO. The modified proof is almost the same as the original one; the only difference is that formula ψ is not formulated in the language $\{<\}$, but in $\{<, \top, \bot\}$. Immediately, classical DNO is model complete.

(ii) The decidability of atomic formulas is shown in the next section by Proposition 3.10.

[2] We use $\forall$ and $\exists$ also as metamathematical symbols and write $\alpha \Vdash \varphi(a, b)$ instead of $\alpha \Vdash \varphi(x, y)[(x/a)(y/b)]$.

(iii) The only task is to demonstrate that DNO $\vdash \forall x \forall y \exists z(x < y \rightarrow x < z \ \& \ z < y)$. We use DN1 and the fact that DNO $\vdash x < y \lor \neg x < y$. The following formulas are provable in DNO:

$$\begin{aligned} \exists z(x < z \ \& \ z < y) &\rightarrow \exists z(x < y \rightarrow x < z \ \& \ z < y) \\ x < y &\rightarrow \exists z(x < y \rightarrow x < z \ \& \ z < y) \\ \neg(x < y) &\rightarrow \exists z(x < y \rightarrow x < z \ \& \ z < y) \end{aligned}$$

Thus, DNO $\vdash \forall x \forall y \exists z(x < y \rightarrow x < z \ \& \ z < y)$.

□

Corollary 3.7. RNA *trivializes intuitionistic logic.*

Proof. We verify the assumptions of the theorem.

(i) Classical RNA admits elimination of quantifiers (it follows from the slightly modified proof of quantifier elimination for theory DOS [Šve02, p. 234]). By Proposition 3.2, classical RNA is model complete.

(ii) The decidability of atomic formulas is shown in the next section by Proposition 3.10.

(iii) All the axioms of RNA, except for DN1, are in prenex form. Prenex form of DN1 is proved by the same means as in the proof of the previous corollary.

□

3.2 Incoincidence

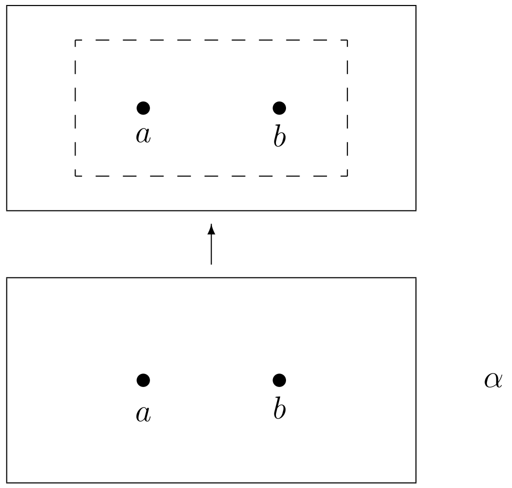
In the shadow of theorems and proofs, we would like to consider rather more general question. What are the characteristics of a "good" intuitionistic theory? So far, we have shown that some theories coincide with their classical extensions. This property must be totally positive for someone who finds similarities between intuitionistic and classical versions of theories. By the way, we also make use of this fact by asserting that DNO is complete and decidable. On the other hand, if we are more enthusiastic about the distinctions between theories, we would privilege the theories which do not trivialize intuitionistic logic. The theories that coincide with their classical versions fail to be the "genuine" intuitionistic theories.

In the following paragraphs, we show that all the theories introduced in section 2.2 except for DNO and RNA do not trivialize intuitionistic logic. We start with E.

Proposition 3.8. *Atomic formula $x = y$ is not decidable in* E.

Proof. Figure 3.1 shows a model of E where $\alpha \not\Vdash a = b \lor \neg(a = b)$. Equality relation is represented by the dashed rectangle (obvious equality axiom E1 is assumed but not drawn in the picture). □

Figure 3.1: A model of E where $\alpha \not\Vdash a = b \vee \neg(a = b)$.



Proposition 3.8 immediately leads to the following

Corollary 3.9. E *does not trivialize intuitionistic logic.*

We move on to theory LO which will be shown to *partly* trivialize intuitionistic logic, but not to coincide with its classical version. The partial trivialization is demonstrated in the following two propositions.

Proposition 3.10. $\mathrm{LO} \vdash \varphi \vee \neg\varphi$*, for all* φ *in which no quantifier occurs.*

Proof. It suffice to prove that atomic formulas are decidable in LO. The rest is a consequence of Lemma 3.3.

(i) Let φ be $x < y$. There is a proof of $\mathrm{LO} \vdash x < y \vee \neg(x < y)$:

$$\begin{array}{lll} \mathrm{LO} \vdash x = y \rightarrow \neg(x < y) & & ;\text{ from E5 and LO2} \\ \mathrm{LO} \vdash y < x \rightarrow \neg(x < y) & & ;\text{ from LO1 and LO2} \\ \mathrm{LO} \vdash x = y \vee x < y \vee y < x \rightarrow x < y \vee \neg(x < y) & & \\ \mathrm{LO} \vdash x < y \vee \neg(x < y) & & ;\text{ from LO3} \end{array}$$

(ii) Let φ be $x = y$. The proof of $\mathrm{LO} \vdash x = y \vee \neg(x = y)$ is similar to the previous one:

$$\begin{array}{lll} \mathrm{LO} \vdash x < y \rightarrow \neg(x = y) & & ;\text{ from E5 and LO2} \\ \mathrm{LO} \vdash y < x \rightarrow \neg(x = y) & & ;\text{ from E5 and LO2} \\ \mathrm{LO} \vdash x = y \vee x < y \vee y < x \rightarrow x = y \vee \neg(x = y) & & \\ \mathrm{LO} \vdash x = y \vee \neg(x = y) & & ;\text{ from LO3} \end{array}$$

□

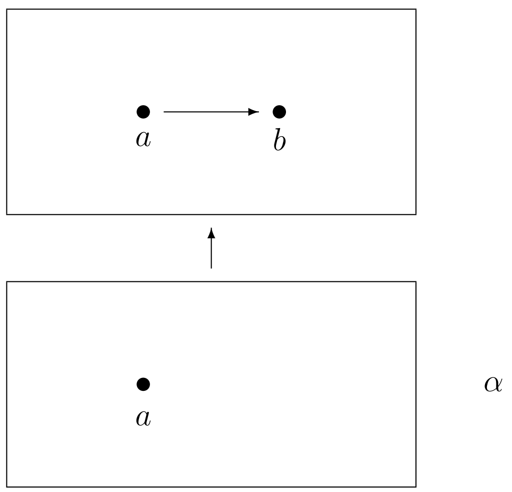
Proposition 3.11. LO *does not trivialize intuitionistic logic. In particular,*

$$\mathrm{LO} \nvdash \forall x(\exists y(x < y) \ \vee \ \neg\exists y(x < y)).$$

Proof. Figure 3.2 shows a model of LO and a node α in the model such that

$$\alpha \not\Vdash \forall x(\exists y(x < y) \ \vee \ \neg\exists y(x < y)).$$

Figure 3.2: A model of LO where $\alpha \not\Vdash \forall x(\exists y(x < y) \vee \neg\exists y(x < y))$. (Relation $<$ is depicted by arrows between elements of domain.)



□

Decidability of atomic formulas brings us a bit closer to classical logic. Among other things, it has an effect on Gödel translation which may be simplified by putting $\varphi^{\mathrm{g}}_{at} = \varphi_{at}$ instead of $\varphi^{\mathrm{g}}_{at} = \neg\neg\varphi_{at}$.

Now we move on to the theory wLO. Proposition 3.12 shows one of the differences between LO and wLO.

Proposition 3.12. *Atomic formulas $x = y$ and $x < y$ are not decidable in* wLO.

Proof. (i) Let φ_{at} be $x = y$. Figure 3.3 shows a model of wLO which consists of three nodes with constant domains $\{a, b\}$. As in Figure 3.2, relation $<$ is depicted by arrows between elements of domain and the dashed rectangle represents equality relation. Then, $\alpha \not\Vdash a = b \ \vee \ \neg(a = b)$.

(ii) Let φ_{at} be $x < y$. We take a model of wLO from Figure 3.4. Then, $\alpha \not\Vdash a < b \ \vee \ \neg(a < b)$.

□

An immediate consequence of Proposition 3.12 is

Figure 3.3: A model of wLO where $\alpha \not\Vdash a = b \vee \neg(a = b)$.

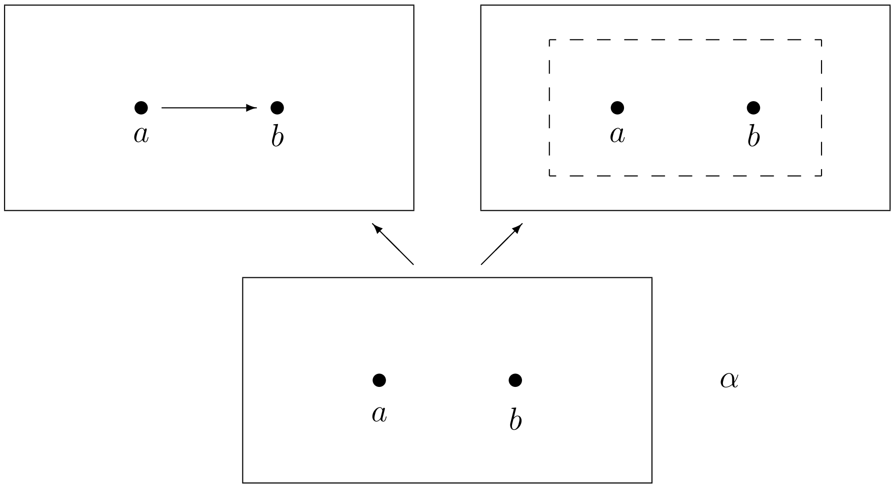
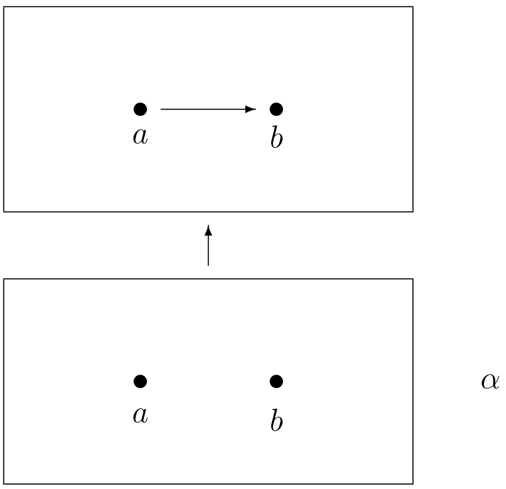


Figure 3.4: A model of wLO where $\alpha \not\Vdash a < b \vee \neg(a < b)$.



Corollary 3.13. wLO *does not trivialize intuitionistic logic.*

Corollaries 3.6 and 3.7 showed the coincidence of DNO and RNA with their classical versions. It is a legitimate question to ask whether we could weaken these theories so that they would not trivialize intuitionistic logic, but at the same time would provide some sensible concept of dense linear order and rational numbers with addition. wDNO and wRNA may be appropriate candidates for such weaker theories. Despite they differ from DNO and RNA only slightly, the following paragraphs demonstrate that they do not coincide with their classical versions. Moreover, atomic formulas are not decidable either in wDNO or in wRNA.

Proposition 3.14. *Atomic formulas $x < y$ and $x = y$ are not decidable in* wDNO*.*

Proof. (i) We proof that wDNO $\nvdash x < y \vee \neg(x < y)$ by constructing an appropriate model $\mathcal{K}_1$. Let $\mathcal{K}_1$ have two nodes—bottom node α and its successor β. As for the domains[3], $D_\alpha = D_\beta = \{\, a_q \,;\, q \in \mathsf{Q} \setminus \{0\} \,\} \cup \{\, b_q \,;\, q \in \mathsf{Q} \,\}$. Now define [4]

$$\begin{aligned} \alpha \Vdash a_q < a_r \quad &\text{iff} \quad q < r \\ \alpha \Vdash a_q < b_r \quad &\text{iff} \quad q < 0 \\ \alpha \Vdash b_q < a_r \quad &\text{iff} \quad r > 0 \\ \beta \Vdash b_q < b_r \quad &\text{iff} \quad q < r. \end{aligned}$$

Elements b_q are incomparable in α, but not in β. Equality is defined in the obvious manner (i.e., every element is equal to itself and to nothing else). We easily check that $\mathcal{K}_1$ is a model of wDNO (e.g., $\alpha \Vdash$ AP because $\beta \Vdash (x < y \vee y < x)$). Atomic formulas are not decidable, for $\alpha \nVdash b_q < b_r \vee \neg(b_q < b_r)$, where $q < r$.

(ii) The proof of wDNO $\nvdash x = y \vee \neg(x = y)$ necessitates constructing a slightly more complicated model than $\mathcal{K}_1$. Let $\mathcal{K}_2$ be a model of wDNO which contains three nodes—bottom node α and its two successors β and γ. As in the previous case, let $D_\alpha = D_\beta = D_\gamma = \{\, a_q \,;\, q \in \mathsf{Q} \setminus \{0\} \,\} \cup \{\, b_q \,;\, q \in \mathsf{Q} \,\}$. Now extend the definition of $\Vdash$ from the previous case by putting

$$\gamma \Vdash b_q = b_r \quad \text{for all } q, r \in \mathsf{Q}.$$

The essential point of the construction is dealing with elements b_q. They are incomparable in α, ordered in β, whereas merged in γ. Node β is needed in order to obtain $\alpha \nVdash b_q = b_r$ for all $q \neq r$, since it follows from axiom AP that incomparables which are not ordered in any successor node must be equal. Node γ is necessary for proving $\alpha \nVdash \neg(b_q = b_r)$. Hence, we have $\alpha \nVdash b_q = b_r \vee \neg(b_q = b_r)$, for all $q \neq r$. □

Corollary 3.15. wDNO *does not trivialize intuitionistic logic.*

[3] Q is the set of rationals.

[4] We write, as in the previous section, $\alpha \Vdash a < b$ instead of $\alpha \Vdash x < y\ [(x/a)(y/b)]$ and we use $<$ and other symbols ambiguously both in *object* and *meta*language.

Proposition 3.16. *(i) Formulas $x < y$, $0 < x$, $x < 0$, $1 < x$, $x < 1$ are not decidable in* wRNA*. (ii) Formulas $x = y$, $x = 0$, $x = 1$ are not decidable in* wRNA*.*

Proof. (i) We constuct a model of wRNA, $\mathcal{K}_1$, with two nodes α and β (α is a bottom node and β its successor). We define $D_\alpha = D_\beta = \{\, a_{q_1,q_2} \,;\, q_1, q_2 \in \mathsf{Q} \,\}$ and

$$\begin{array}{lll} \alpha & \Vdash & 0 = a_{0,0} \\ \alpha & \Vdash & 1 = a_{1,0} \\ \alpha & \Vdash & a_{q_1,q_2} + a_{r_1,r_2} = a_{q_1+r_1,q_2+r_2} \\ \alpha & \Vdash & a_{q_1,q_2} < a_{r_1,r_2} \quad \text{iff} \quad q_1 < r_1. \end{array}$$

Equality is defined in the obvious manner. In β, $\Vdash$ is extended by putting

$$\beta \Vdash a_{q_1,q_2} < a_{r_1,r_2} \quad \text{iff} \quad q_1 < r_1 \text{ or } (q_1 = r_1 \text{ and } q_2 < r_2).$$

It is not difficult to verify that we obtained a model of wRNA; we leave it to the reader. The construction of the model proves the undecidability of atomic formula $x < y$ because for all $q, q_1, q_2 \in \mathsf{Q}$ such that $q_1 < q_2$,

$$\alpha \not\Vdash a_{q,q_1} < a_{q,q_2} \ \vee\ \neg(a_{q,q_1} < a_{q,q_2}).$$

Particularly, for all $r, s \in \mathsf{Q}$ such that $r > 0$ and $s < 0$,

$$\begin{array}{l} \alpha \not\Vdash 0 < a_{0,r} \ \vee\ \neg(0 < a_{0,r}) \\ \alpha \not\Vdash a_{0,s} < 0 \ \vee\ \neg(a_{0,s} < 0) \\ \alpha \not\Vdash 1 < a_{1,r} \ \vee\ \neg(a_{1,r} < 1) \\ \alpha \not\Vdash a_{1,s} < 1 \ \vee\ \neg(1 < a_{1,s}) \end{array}$$

(ii) We use the same idea as in the proof of Proposition 3.14. We obtain the counterexample to the decidability of formula $x = y$ by adding one node into model $\mathcal{K}_1$. We construct model $\mathcal{K}_2$ with a bottom node α and its two successors β and γ. Let $D_\alpha = D_\beta = D_\gamma = \{\, a_{q_1,q_2} \,;\, q_1, q_2 \in \mathsf{Q} \,\}$. The definition of $\Vdash$ in nodes α and β is the same as it was described in (i). In node γ, however, all the elements that do not differ in the first index, merge:

$$\gamma \Vdash a_{q,q_1} = a_{q,q_2}, \quad \text{for all } q, q_1, q_2 \in \mathsf{Q}.$$

Now, for all $q, q_1, q_2 \in \mathsf{Q}$ such that $q_1 \neq q_2$,

$$\alpha \not\Vdash a_{q,q_1} = a_{q,q_2} \ \vee\ \neg(a_{q,q_1} = a_{q,q_2}).$$

Particularly, for all $r \in \mathsf{Q}$ such that $r \neq 0$,

$$\begin{array}{l} \alpha \not\Vdash a_{0,r} = 0 \ \vee\ \neg(a_{0,r} = 0) \\ \alpha \not\Vdash a_{1,r} = 1 \ \vee\ \neg(a_{1,r} = 1). \end{array}$$

$\square$

The other theories in our scope are two versions of the theory of a successor function. Either $\mathrm{SUCC}^{\vee}$ or $\mathrm{SUCC}^{\rightarrow}$ does not coincide with its classical extension and furthermore, $x = y$ is not a decidable formula. Still, there is a difference in the decidability of $x = 0$. The following propositions clarify our considerations.

Proposition 3.17. $\mathrm{SUCC}^{\rightarrow} \nvdash x = \mathrm{S}^n(0) \ \vee \ \neg(x = \mathrm{S}^n(0))$, *where* $n \in \mathsf{N}$.[5] *Particularly,* $\mathrm{SUCC}^{\rightarrow} \nvdash x = 0 \ \vee \ \neg(x = 0)$.

Proof. The model of $\mathrm{SUCC}^{\rightarrow}$ which demonstrates the result is comprised of two nodes $\alpha < \beta$ with the same domains $D_\alpha = D_\beta = \{ a_n \ ; \ n \in \mathsf{N} \} \cup \{ b_n \ ; \ n \in \mathsf{N} \}$. The successor function and 0 are defined as follows:

$$\begin{array}{lll} \alpha & \Vdash & a_{n+1} = \mathrm{S}(a_n), \quad \text{for all } n \in \mathsf{N} \\ \alpha & \Vdash & b_{n+1} = \mathrm{S}(b_n), \quad \text{for all } n \in \mathsf{N} \\ \alpha & \Vdash & a_0 = 0. \end{array}$$

Every element of D_α is defined to be equal to itself. However, $\Vdash$ in β is extended by putting

$$\beta \Vdash a_n = b_n, \quad \text{for all } n \in \mathsf{N}.$$

Thus, we obtained the model of $\mathrm{SUCC}^{\rightarrow}$ such that for every $n \in \mathsf{N}$,

$$\alpha \nVdash b_n = \mathrm{S}^n(0) \ \vee \ \neg(b_n = \mathrm{S}^n(0)).$$

□

Proposition 3.18. $\mathrm{SUCC}^{\vee} \vdash x = \mathrm{S}^n(0) \ \vee \ \neg(x = \mathrm{S}^n(0))$, *where* $n \in \mathsf{N}$. *Particularly,* $\mathrm{SUCC}^{\vee} \vdash x = 0 \ \vee \ \neg(x = 0)$.

Proof. By induction on n.

(i) $n = 0$. We use axiom $\mathrm{Q3}^{\vee}$. If $x = 0$, then $x = 0 \ \vee \ \neg(x = 0)$ holds. If $\exists y(x = \mathrm{S}(y))$, then we will prove that $\neg(x = 0)$ as follows: Let $x = 0$. Then, $\exists y(0 = \mathrm{S}(y))$, but from Q2 we may conclude $\neg\exists y(0 = \mathrm{S}(y))$, hence $\exists y(x = \mathrm{S}(y)) \rightarrow \neg(x = 0)$. Consequently, $\exists y(x = \mathrm{S}(y)) \rightarrow x = 0 \ \vee \ \neg(x = 0)$ and finally $x = 0 \ \vee \ \neg(x = 0)$.

(ii) Let $\mathrm{SUCC}^{\vee} \vdash x = \mathrm{S}^n(0) \ \vee \ \neg(x = \mathrm{S}^n(0))$. By equality axioms and Q1, we obtain

$$\mathrm{SUCC}^{\vee} \vdash \mathrm{S}(x) = \mathrm{S}^{n+1}(0) \ \vee \ \neg(\mathrm{S}(x) = \mathrm{S}^{n+1}(0)).$$

From this, it can be easily proved

$$\mathrm{SUCC}^{\vee} \vdash y = \mathrm{S}(x) \ \rightarrow \ (y = \mathrm{S}^{n+1}(0) \ \vee \ \neg(y = \mathrm{S}^{n+1}(0))$$

and subsequently,

$$\mathrm{SUCC}^{\vee} \vdash \exists x(y = \mathrm{S}(x)) \ \rightarrow \ (y = \mathrm{S}^{n+1}(0) \ \vee \ \neg(y = \mathrm{S}^{n+1}(0)).$$

[5] N is the set of natural numbers.

Axiom Q2 proves $\neg(0 = \mathrm{S}^{n+1}(0))$. Thus,

$$\mathrm{SUCC}^{\vee} \vdash y = 0 \rightarrow y = \mathrm{S}^{n+1}(0) \vee \neg(y = \mathrm{S}^{n+1}(0)).$$

Now, due to Q3,

$$\mathrm{SUCC}^{\vee} \vdash y = \mathrm{S}^{n+1}(0) \vee \neg(y = \mathrm{S}^{n+1}(0)).$$

□

Proposition 3.19. *Formula $x = y$ is not decidable either in $\mathrm{SUCC}^{\vee}$ or in $\mathrm{SUCC}^{\rightarrow}$. Furthermore, $x = y$ is not stable either in $\mathrm{SUCC}^{\vee}$ or in $\mathrm{SUCC}^{\rightarrow}$.*

Proof. We suggest constructing the model of $\mathrm{SUCC}^{\vee}$ (and $\mathrm{SUCC}^{\rightarrow}$) with two nodes $\alpha < \beta$ such that $\alpha \not\Vdash x = y \vee \neg(x = y)$. The nodes have constant domains $D_\alpha = D_\beta = \{ a_n ; n \in \mathsf{N} \} \cup \{ b_z ; z \in \mathsf{Z} \} \cup \{ c_z ; z \in \mathsf{Z} \}$.[6] The successor function and 0 are defined as follows:

$$\begin{aligned}
\alpha &\Vdash a_0 = 0 \\
\alpha &\Vdash a_{n+1} = \mathrm{S}(a_n), \quad \text{for all } n \in \mathsf{N} \\
\alpha &\Vdash b_{z+1} = \mathrm{S}(b_z), \quad \text{for all } z \in \mathsf{Z} \\
\alpha &\Vdash c_{z+1} = \mathrm{S}(c_z), \quad \text{for all } z \in \mathsf{Z}
\end{aligned}$$

Every element of D_α is defined to be equal to itself. However, in β, $\Vdash$ is extended by putting

$$\beta \Vdash b_z = c_z, \qquad \text{for all } z \in \mathsf{Z}.$$

Now, $\alpha \not\Vdash b_z = c_z \vee \neg(b_z = c_z)$ and $\alpha \not\Vdash \neg\neg(b_z = c_z) \rightarrow b_z = c_z$, for all $z \in \mathsf{Z}$. □

Corollary 3.20. *Neither $\mathrm{SUCC}^{\rightarrow}$ nor $\mathrm{SUCC}^{\vee}$ trivializes intuitionistic logic.*

Let's try to sum up the results of the last three propositions. We saw that SUCC theories do not coincide with their classical extensions. To a certain extent, they even deal with atomic formulas quite similarly. The difference was found in proving the decidability of atomic formulas of the form $x = \mathrm{S}^n(0)$. In $\mathrm{SUCC}^{\vee}$, each standard element s (i.e., 0 or $\mathrm{S}^n(0)$) possesses the classical property that every element is either equal to s, or is not equal to s. In contrast, in models of $\mathrm{SUCC}^{\rightarrow}$, there can be elements which are neither equal to s nor unequal to s.

The last two theories we are going to investigate are two versions of Q. The following paragraphs show that the relation between these two versions is similar to the relation between SUCCs.

Proposition 3.21. $\mathrm{Q}^{\rightarrow} \nvdash x = \mathrm{S}^n(0) \vee \neg(x = \mathrm{S}^n(0))$, *where $n \in \mathsf{N}$.*

[6] Z is the set of integers.

Proof. The idea is completely due to V. Švejdar. We use the model introduced in Proposition 3.17 and show that if we enrich it with the definition of $\cdot$ and $+$, we obtain a model of $\mathrm{Q}^{\rightarrow}$. The operations $\cdot$ and $+$ are defined by putting

$$\begin{array}{lll} \alpha & \Vdash & a_n + a_m = a_{n+m} \\ \alpha & \Vdash & b_n + a_m = b_{n+m} \\ \alpha & \Vdash & a_n + b_m = a_{n+m} \\ \alpha & \Vdash & b_n + b_m = a_{n+m} \\ \alpha & \Vdash & c_n \cdot d_m = a_{n \cdot m}, \qquad \text{where } c, d \in \{a, b\}, \end{array}$$

for all $n, m \in \mathsf{N}$.[7] Now, we should verify that axioms Q4–Q7 are valid in the model. We demonstrate only Q7 and leave Q4–Q6 to the reader.

(i) $a_n \cdot \mathrm{S}(a_m) = a_n \cdot a_{m+1} = a_{n\cdot(m+1)} = a_{n\cdot m+n} = a_{n\cdot m} + a_n = a_n \cdot a_m + a_n$

(ii) $a_n \cdot \mathrm{S}(b_m) = a_n \cdot b_{m+1} = a_{n\cdot(m+1)} = a_{n\cdot m+n} = a_{n\cdot m} + a_n = a_n \cdot b_m + a_n$

(iii) $b_n \cdot \mathrm{S}(a_m) = b_n \cdot a_{m+1} = a_{n\cdot(m+1)} = a_{n\cdot m+n} = a_{n\cdot m} + b_n = b_n \cdot a_m + b_n$

(iv) $b_n \cdot \mathrm{S}(b_m) = b_n \cdot b_{m+1} = a_{n\cdot(m+1)} = a_{n\cdot m+n} = a_{n\cdot m} + b_n = b_n \cdot b_m + b_n$

□

Proposition 3.22. $\mathrm{Q}^{\vee} \vdash x = \mathrm{S}^n(0) \ \vee \ \neg(x = \mathrm{S}^n(0))$*, where* $n \in \mathsf{N}$*.*

Proof. The proof is obtained by a mere copying of the proof of Proposition 3.18 and ensuring that axiom L^n was not used. □

Proposition 3.23. *Formula* $x = y$ *is not decidable either in* $\mathrm{Q}^{\vee}$ *or in* $\mathrm{Q}^{\rightarrow}$*. Furthermore,* $x = y$ *is not stable either in* $\mathrm{Q}^{\vee}$ *or in* $\mathrm{Q}^{\rightarrow}$*.*

Proof. We construct a model of $\mathrm{Q}^{\vee}$ (and $\mathrm{Q}^{\rightarrow}$) that is a counterexample to $\mathrm{Q} \vdash x = y \ \vee \ \neg(x = y)$. There are two nodes α, β in the model such that $\alpha < \beta$. We define $D_\alpha = D_\beta = \{\, a_n \,;\, n \in \mathsf{N} \,\} \cup \{\, b_n \,;\, n \in \mathsf{N} \,\}$ and for all $n, m \in \mathsf{N}$,

$$\begin{array}{lll} \alpha & \Vdash & a_0 = 0 \\ \alpha & \Vdash & a_{n+1} = \mathrm{S}(a_n) \\ \alpha & \Vdash & b_n = \mathrm{S}(b_n) \\ \alpha & \Vdash & a_n + a_m = a_{n+m} \\ \alpha & \Vdash & a_n + b_m = b_{m+1} \\ \alpha & \Vdash & b_n + a_m = b_n \\ \alpha & \Vdash & b_n + b_m = b_{m+1} \\ \alpha & \Vdash & a_n \cdot a_m = a_{n\cdot m} \end{array}$$

[7] To avoid confusion, we point out that $+$ and $\cdot$ used in the indices of elements of the domain are *meta*mathematical symbols with their natural meanings. They are not elements of the language.

$$\begin{array}{lll} \alpha & \Vdash & a_n \cdot b_m = b_0 \\ \alpha & \Vdash & b_n \cdot 0 = 0 \\ \alpha & \Vdash & b_n \cdot a_m = b_{n+1}, \qquad \text{where } m > 0 \\ \alpha & \Vdash & b_n \cdot b_m = b_{n+1}. \end{array}$$

In addition, every element of the domain is equal to itself. If we think of α as it was a classical structure, it is the model of classical Q taken from [Šve02, p. 284]. In β, $\Vdash$ is extended by

$$\beta \quad \Vdash \quad b_n = b_m, \quad \text{for all } n, m \in \mathsf{N}.$$

The construction can be easily verified to be a model of $\mathrm{Q}^{\vee}$. (Merging of elements b_n in β does not affect the validity of Q4–Q7 and all the other axioms also remain valid in β.) Furthermore,

$$\alpha \quad \nVdash \quad b_n = b_m \ \vee \ \neg(b_n = b_m), \quad \text{for any } n, m \in \mathsf{N}.$$

□

Corollary 3.24. *Neither* $\mathrm{Q}^{\rightarrow}$ *nor* $\mathrm{Q}^{\vee}$ *trivializes intuitionistic logic.*

At the end of this chapter, let us try to look on the results from a rather different point of view. So far, we declared that we merely inspected properties of some intuitionistic theories, but at the same time, we could say that we searched for the "genuine" formulation of the theories. From the intuitionistic perspective, DNO and RNA were badly formulated because they enable non-constructive sentences to be proved. However, we have found "better" formulations of dense linear order and addition of rational numbers. Even LO, which does not coincide with its classical extension, can be formulated more constructively.

We may assert that wDNO, wRNA and wLO are the "genuine" theories, but the qualification of different versions of SUCC and Q is much more doubtable. It depends on whether we prefer the recognition of non-zero elements (i.e., $x = 0 \ \vee \ \neg(x = 0)$) or keeping far from non-constructive sentences.

4

Saturated theories

Saturated sets have shown their importance in the proof of the completeness theorem (see [Smo73a, pp. 329–332]) and we believe that saturation is one of good candidates for metamathematical property which might substitute for completeness. In this chapter, we try to find some criterion that would help us in deciding whether certain intuitionistic theory is saturated or not. Subsequently, we apply the criterion to the theories in scope and in the last section, we compare saturated theories with theories of Harrop formulas and present another criterion of saturation.

Now, it is high time that we introduced the definition of saturated theories.

Definition 4.1. *Let Γ be a set of sentences in language L. Γ is L-*saturated *iff the following conditions are satisfied:*

1. *Γ is consistent,*
2. *if $\Gamma \vdash \varphi \lor \psi$, then $\varphi \in \Gamma$ or $\psi \in \Gamma$, for any sentences φ and ψ* *(Γ is* d-complete*),*
3. *if $\Gamma \vdash \exists x \chi$, then there exists a closed term t such that $\chi_x(t) \in \Gamma$, for any formula χ with one free variable x* *(Γ is* e-complete*).*

Lemma 4.2. *Let Γ be L-saturated. Then, for all φ, if $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$ (Γ is deductively closed).*

Proof. If $\Gamma \vdash \varphi$, then $\Gamma \vdash \varphi \lor \varphi$ and by the d-completeness, $\varphi \in \Gamma$. $\square$

Definition 4.3. *Let T be a theory with language L and a set Γ of axioms. Then, T is* saturated *iff* $\mathrm{Thm}(\Gamma)$ *is L-saturated.*

4.1 The Aczel slash

The aim of this section is to introduce a metamathematical relation called the *Aczel slash* and to demonstrate its usefulness in deciding whether a theory is saturated or not. Our definition of the Aczel slash extends the one presented for propositional logic in [BJ05, p. 21] and slightly differs from the syntactical characterization mentioned in [Smo73a, p. 333].

Definition 4.4. *Let Γ be a set of sentences. The* Aczel slash, $|$, *is inductively defined as follows:*

1. *If φ is a closed formula, then*

   (i) if φ is atomic, then $\Gamma \mid \varphi$ iff $\Gamma \vdash \varphi$,

   (ii) $\Gamma \mid \neg\varphi$ iff $\Gamma \vdash \neg\varphi$ and $\Gamma \nmid \varphi$,

   (iii) $\Gamma \mid \varphi \mathbin{\&} \psi$ iff $\Gamma \mid \varphi$ and $\Gamma \mid \psi$,

   (iv) $\Gamma \mid \varphi \vee \psi$ iff $\Gamma \mid \varphi$ or $\Gamma \mid \psi$,

   (v) $\Gamma \mid \varphi \to \psi$ iff $\Gamma \vdash \varphi \to \psi$ and ($\Gamma \nmid \varphi$ or $\Gamma \mid \psi$),

   (vi) $\Gamma \mid \exists x\varphi$ iff $\Gamma \mid \varphi_x(t)$ for some closed term t,

   (vii) $\Gamma \mid \forall x\varphi$ iff $\Gamma \vdash \forall x\varphi$ and $\Gamma \mid \varphi_x(t)$ for all closed terms t.

2. *If φ is not closed and $\forall\varphi$ is the universal closure of φ, then $\Gamma \mid \varphi$ iff $\Gamma \mid \forall\varphi$.*

If T is a theory with language L and a set Γ of axioms, we write $T \mid \varphi$ instead of $\Gamma \mid \varphi$.

The importance of defining the Aczel slash for all formulas, not just for sentences, will turn out in the following proofs of lemmas and theorems.

In the rest of this section, we suppose that a language with at least one constant is at our disposal. It means that it will not be possible to apply the following results to the theory of equality or to the theories of order. However, it is nothing to worry about. It does not make much sense to speak about saturated theories when they do not possess any constant in their language. According to Definition 4.1, no such theory in which, for example, $\exists x(x = x)$ is provable, is saturated.

The following two lemmas show some basic characteristics of the Aczel slash.

Lemma 4.5. *If $\Gamma \mid \varphi$, then $\Gamma \vdash \varphi$.*

Proof. By induction on φ. □

Lemma 4.6. *If Γ is inconsistent, then $\Gamma \mid \varphi$ for all φ.*

Proof. 1. Let φ be a sentence. We proceed by induction on φ.

   (i) Let $\Gamma \nmid \varphi_{at}$. Then by Definition 4.4, $\Gamma \nvdash \varphi_{at}$ and thus, Γ is consistent.

   (ii) Let $\Gamma \nmid \exists x\varphi$. Then by Definition 4.4, for all closed terms t, $\Gamma \nmid \varphi_x(t)$. By induction hypothesis, Γ is consistent.

   (iii) Let $\Gamma \nmid \forall x\varphi$. Then by Definition 4.4, there are two cases:

      i. There exists a closed term t such that $\Gamma \nmid \varphi_x(t)$. By induction hypothesis, Γ is consistent.

ii. $\Gamma \nvdash \forall x\varphi$. Then immediately, Γ is consistent.

The other cases are left to the reader

2. Let $\Gamma \nmid \varphi$ for a formula φ with one free variable x. By Definition 4.4, $\Gamma \nmid \varphi$ iff $\Gamma \nmid \forall x\varphi$ which passes to 1.(iii). The case that φ is a formula with more free variables is similar.

$\square$

In the following paragraphs, a key role is played by the set $\mathrm{Acz}(\Gamma) = \{\varphi ; \Gamma \mid \varphi\}$. Our considerations lead to the investigation of the relation between $\mathrm{Acz}(\Gamma)$ and $\mathrm{Thm}(\Gamma)$.

Lemma 4.7. $\mathrm{Acz}(\Gamma)$ *is closed under deduction. That is, if* $\mathrm{Acz}(\Gamma) \vdash \varphi$*, then* $\Gamma \mid \varphi$*.*

Proof. We verify all the logical axioms and rules. At each step, we deal with sentences first and then with non-sentences. The proof is easy but rather tedious. We use Definition 4.4.

1. Suppose that instances of the axioms and rules have no free variables except for the instances of Gen-A and Gen-E which contain formula φ with one free variable.

   (i) Axiom A10. $\Gamma \mid A \to (\neg A \to B)$ iff $\Gamma \vdash A \to (\neg A \to B)$ and [if $\Gamma \mid A$, then $\Gamma \mid (\neg A \to B)$]. $\Gamma \vdash A \to (\neg A \to B)$ is valid and the latter conjunct is equivalent to [if $\Gamma \mid A$, then $\Gamma \vdash (\neg A \to B)$ and (if $\Gamma \mid \neg A$, then $\Gamma \mid B$)]. Suppose $\Gamma \mid A$. By Lemma 4.5, $\Gamma \vdash A$ which implies $\Gamma \vdash (\neg A \to B)$. The last step is to reason (if $\Gamma \mid \neg A$, then $\Gamma \mid B$). $\Gamma \mid \neg A$ iff $\Gamma \nmid A$ and $\Gamma \vdash \neg A$. Thus, we obtained $\Gamma \mid A$ and $\Gamma \nmid A$ which implies $\Gamma \mid B$.

   (ii) Rule MP. Let $\Gamma \mid A$ and $\Gamma \mid A \to B$. If $\Gamma \mid A \to B$, then $\Gamma \nmid A$ or $\Gamma \mid B$. Hence, $\Gamma \mid B$.

   (iii) Axiom B1. As it was assumed, no free variables occur in the formula $\forall x\varphi \to \varphi_x(t)$, especially, t is a closed term. $\Gamma \mid \forall x\varphi \to \varphi_x(t)$ iff $\Gamma \vdash \forall x\varphi \to \varphi_x(t)$ and ($\Gamma \nmid \forall x\varphi$ or $\Gamma \mid \varphi_x(t)$). $\Gamma \vdash \forall x\varphi \to \varphi_x(t)$ holds. Suppose that $\Gamma \mid \forall x\varphi$. Then by the definition, for all closed terms t, $\Gamma \mid \varphi_x(t)$.

   (iv) Axiom B2. $\Gamma \mid \varphi_x(t) \to \exists x\varphi$ iff $\Gamma \vdash \varphi_x(t) \to \exists x\varphi$ and ($\Gamma \nmid \varphi_x(t)$ or $\Gamma \mid \exists x\varphi$). $\Gamma \vdash \varphi_x(t) \to \exists x\varphi$ holds. Let us assume that $\Gamma \mid \varphi_x(t)$. Then directly by the definition, $\Gamma \mid \exists x\varphi$.

   (v) Rule Gen-A. Assume that $\Gamma \mid \psi \to \varphi$, x is free in φ but not in ψ. By the definition, $\Gamma \mid \psi \to \varphi$ iff $\Gamma \mid \forall x(\psi \to \varphi)$. It means that for every closed term t,
   $$\Gamma \mid \psi \to \varphi_x(t). \tag{4.1}$$
   We want to prove that $\Gamma \mid \psi \to \forall x\varphi$. It happens if and only if $\Gamma \vdash \psi \to \forall x\varphi$ and ($\Gamma \nmid \psi$ or $\Gamma \mid \forall x\varphi$). The proof of $\Gamma \vdash \psi \to \forall x\varphi$ is immediate due to Lemma 4.5 and rule Gen-A. Let $\Gamma \mid \psi$. Then, $\Gamma \vdash \psi$ and by modus ponens, $\Gamma \vdash \forall x\varphi$. To prove $\Gamma \mid \forall x\varphi$, we have to show that for each closed term t, $\Gamma \mid \varphi_x(t)$. It is obtained due to (ii) and (4.1).

(vi) Rule Gen-E. Assume that $\Gamma \mid \varphi \to \psi$, x is free in φ but not in ψ. By the definition, $\Gamma \mid \forall x(\varphi \to \psi)$. It means that for every closed term t,

$$\Gamma \mid \varphi_x(t) \to \psi. \tag{4.2}$$

We want to prove that $\Gamma \mid \exists x\varphi \to \psi$. Due to Lemma 4.5, we have $\Gamma \vdash \varphi \to \psi$ and therefore, $\Gamma \vdash \exists x\varphi \to \psi$. Suppose $\Gamma \mid \exists x\varphi$. It means that there exists a closed term t such that $\Gamma \mid \varphi_x(t)$. Now we use (ii), (4.2) and conclude with $\Gamma \mid \psi$.

All the other axioms are left to the reader.

2. Suppose that the instances of the axioms and rules have some free variables.

(i) Axiom B1. Suppose that term t contains one free variable y. Then, $\Gamma \mid \forall x\varphi \to \varphi_x(t(y))$ iff $\Gamma \mid \forall y(\forall x\varphi \to \varphi_x(t(y)))$. This is if and only if $\Gamma \vdash \forall y(\forall x\varphi \to \varphi_x(t(y)))$ and for all closed terms s, $\Gamma \mid \forall x\varphi \to (\varphi_x(t(y)))_y(s)$. The former is logically valid and the latter was proved in 1.(iii).

(ii) The general proof for instances with free variables can be outlined in this way: We want to prove $\Gamma \mid \varphi(\underline{x})$. This is if and only if $\Gamma \mid \forall \underline{x}\varphi(\underline{x})$. To prove it, we show that $\Gamma \vdash \forall \underline{x}\varphi(\underline{x})$ (it is trivial) and that for all closed terms $\underline{t}$, $\Gamma \mid \varphi_{\underline{x}}(\underline{t})$ which passes to point 1.

$\square$

The following theorem was proved by C. Smorynski in [Smo73a, p. 332]. Smorynski's proof and also his definition of the Aczel slash are semantical. In contrast, our considerations are syntactical and we draw from the propositional version of the theorem mentioned in [BJ05, p. 21].

Theorem 4.8. $\mathrm{Acz}(\Gamma)$ *is maximal d-complete, e-complete and deductively closed subset of* $\mathrm{Thm}(\Gamma)$.

Proof. (i) $\mathrm{Acz}(\Gamma)$ is closed under deduction. From Lemma 4.7.

(ii) $\mathrm{Acz}(\Gamma)$ is d-complete. Suppose $\mathrm{Acz}(\Gamma) \vdash \varphi \vee \psi$. From Lemma 4.7, $\Gamma \mid \varphi \vee \psi$ and by Definition 4.4, $\varphi \in \mathrm{Acz}(\Gamma)$ or $\psi \in \mathrm{Acz}(\Gamma)$.

(iii) $\mathrm{Acz}(\Gamma)$ is e-complete. Suppose $\mathrm{Acz}(\Gamma) \vdash \exists x\varphi$. From Lemma 4.7, we have $\Gamma \mid \exists x\varphi$ and by Definition 4.4, there exists a closed term t such that $\varphi_x(t) \in \mathrm{Acz}(\Gamma)$.

(iv) Maximality. Suppose that Δ is a d-complete and e-complete set of formulas which is closed under deduction and

$$\mathrm{Acz}(\Gamma) \subseteq \Delta \subseteq \mathrm{Thm}(\Gamma). \tag{4.3}$$

Let us take a formula $\psi \in \Delta$. By induction on ψ, we show that $\psi \in \mathrm{Acz}(\Gamma)$.

(a) If $\psi_{at} \in \Delta$, then from (4.3), $\Gamma \vdash \psi_{at}$ and by Definition 4.4, $\Gamma \mid \psi_{at}$.

(b) ψ is $\chi \,\&\, \eta$. Δ is deductively closed, hence $\chi \in \Delta$ and $\eta \in \Delta$. By induction hypothesis, $\Gamma \mid \chi$ and $\Gamma \mid \eta$, thus $\Gamma \mid \chi \,\&\, \eta$.

(c) ψ is $\exists x\chi$. Δ is e-complete, thus there exists a closed term t such that $\chi_x(t) \in \Delta$. Now we use induction hypothesis and get $\Gamma \mid \chi_x(t)$. By Definition 4.4, $\Gamma \mid \exists x\chi$.

(d) ψ is $\forall x\chi$. $\Gamma \vdash \forall x\chi$ is immediate due to (4.3). Let t be a closed term. Δ is closed under deduction so $\chi_x(t) \in \Delta$ and by induction hypothesis, we have $\Gamma \mid \chi_x(t)$. Hence, $\Gamma \mid \forall x\chi$.

The other cases are left to the reader.

□

Corollary 4.9. *If* $\Gamma \mid \chi$ *for all* $\chi \in \Gamma$, *then* $\mathrm{Acz}(\Gamma) = \mathrm{Thm}(\Gamma)$.

Proof. Suppose $\mathrm{Acz}(\Gamma) \subset \mathrm{Thm}(\Gamma)$ and $\varphi \in \mathrm{Thm}(\Gamma)\backslash\mathrm{Acz}(\Gamma)$. $\Gamma \vdash \varphi$, thus $\mathrm{Acz}(\Gamma) \vdash \varphi$ and by Theorem 4.8, $\varphi \in \mathrm{Acz}(\Gamma)$. A contradiction. □

Corollary 4.10. *Let* T *be a consistent theory.* $T \mid \varphi$ *for all* $\varphi \in T$ *iff* T *is saturated.*

Proof. $\Rightarrow$ The previous corollary proves that $\mathrm{Acz}(T) = \mathrm{Thm}(T)$ and by Theorem 4.8, $\mathrm{Thm}(T)$ is d-complete and e-complete.

$\Leftarrow$ Suppose that $\mathrm{Thm}(T)$ is a saturated set. According to Theorem 4.8, $\mathrm{Acz}(T)$ is maximal d,e-complete and deductively closed subset of $\mathrm{Thm}(T)$, hence $\mathrm{Acz}(T) = \mathrm{Thm}(T)$. Consequently, $T \mid \varphi$ for all $\varphi \in T$.

□

4.2 Applications of the Aczel slash results

In the following paragraphs, we apply the theoretical results from the previous section to the theories in our scope. As it was already mentioned, the results cannot be used for inspecting theories without any constant in their language. Nevertheless, to have a complete list of properties for each theory, we state the following

Proposition 4.11. E, LO, wLO, DNO, wDNO *are not saturated.*

Proof. All these theories prove $\exists x(x = x)$, but do not have any closed term. Thus, e-completeness does not hold. □

All the propositions that follow are nothing more than a direct application of Corollary 4.10. It confirms that the previous section gave us a strong instrument for deciding whether a theory is saturated.

The simplest case is taking an empty set of axioms which result in Proposition 4.12.

Proposition 4.12. *The predicate calculus is saturated, assuming the language has at least one constant.*

Proof. A direct use of Corollary 4.10. □

We move on to theories SUCC and Q. The only task is to verify that $T \mid \varphi$ for all axioms φ of theory T.

Proposition 4.13. *Both versions of* SUCC *are saturated.*

Proof. We verify only few axioms and leave the rest to the reader. We use the definition of the Aczel slash and Corollary 4.10.

(i) Axiom E2.

$$\mathrm{SUCC} \mid \forall x \forall y(x = y \rightarrow y = x) \quad \text{iff}$$
$$\{\text{for all closed terms } t,\ \mathrm{SUCC} \mid \forall y(t = y \rightarrow y = t) \quad \text{and} \tag{4.4}$$
$$\mathrm{SUCC} \vdash \forall x \forall y(x = y \rightarrow y = x)\}. \tag{4.5}$$

(4.5) holds. (4.4) iff

$$\text{for all closed terms } t, u,\ \mathrm{SUCC} \mid t = u \rightarrow u = t \quad \text{and} \tag{4.6}$$
$$\mathrm{SUCC} \vdash \forall y(t = y \rightarrow y = t). \tag{4.7}$$

(4.7) holds. (4.6) iff

$$\text{for all closed terms } t, u,\ \mathrm{SUCC} \vdash t = u \rightarrow u = t \quad \text{and} \tag{4.8}$$
$$\text{if } \mathrm{SUCC} \mid t = u, \text{ then } \mathrm{SUCC} \mid u = t \tag{4.9}$$

(4.8) holds. (4.9) iff

$$\text{for all closed terms } t, u, \text{ if } \mathrm{SUCC} \vdash t = u, \text{ then } \mathrm{SUCC} \vdash u = t \tag{4.10}$$

(4.10) holds. We shall not be so accurate in the proof of the other cases, instead, we focus on the important steps.

(ii) Axiom Q2. The essential part is to prove that for all closed terms t, $\mathrm{SUCC} \nmid \mathrm{S}(t) = 0$, that is, $\mathrm{SUCC} \nvdash \mathrm{S}(t) = 0$. It is true because SUCC is consistent and from axiom Q2, we know that $\mathrm{SUCC} \vdash \neg(\mathrm{S}(t) = 0)$. More worthwhile is checking axioms Q3.

(iii) Axiom $\mathrm{Q3}^{\vee}$. Analogous steps that were made in (i) lead to the following result: $\mathrm{SUCC}^{\vee} \mid \forall x(x = 0 \vee \exists y(x = \mathrm{S}(y)))$ iff for all closed terms t, ($\mathrm{SUCC}^{\vee} \vdash t = 0$ or there exists a closed term u such that $\mathrm{SUCC}^{\vee} \vdash t = \mathrm{S}(u)$). All the closed terms in the language of SUCC are $0,\ \mathrm{S}(0),\ \mathrm{S}(\mathrm{S}(0)), \dots$. If t is 0, then $\mathrm{SUCC}^{\vee} \vdash t = 0$. If t has a form $\mathrm{S}^{n+1}(0)$, then there exists a term $u = \mathrm{S}^{n}(0)$ such that $\mathrm{SUCC}^{\vee} \vdash t = \mathrm{S}(u)$.

(iv) Axiom $\mathrm{Q3}^{\rightarrow}$. $\mathrm{SUCC}^{\rightarrow} \mid \forall x(\neg(x = 0) \rightarrow \exists y(x = \mathrm{S}(y)))$ iff for all closed terms t, if $\mathrm{SUCC}^{\rightarrow} \nvdash t = 0$, then there exists a closed term u such that $\mathrm{SUCC}^{\rightarrow} \vdash t = \mathrm{S}(u)$. Again, if t is 0, then $\mathrm{SUCC}^{\rightarrow} \vdash t = 0$. If t has a form $\mathrm{S}^{n+1}(0)$, then there exists a term $u = \mathrm{S}^{n}(0)$ such that $\mathrm{SUCC}^{\rightarrow} \vdash t = \mathrm{S}(u)$.

□

The proof that Q is saturated is not a mere extension of the previous one. Although some of the axioms are the same as in SUCC, there is a difference in the set of closed terms. Much more manifold closed terms can be created in Q, e.g., $0{\cdot}(0{+}0{\cdot}0)$, $S(S(0{\cdot}(0{+}0)){\cdot}0)\dots$. The following lemma will be helpful.

Lemma 4.14. *If t is a closed term in the language of* Q*, then* $\mathrm{Q} \vdash t = 0$ *or* $\mathrm{Q} \vdash t = \mathrm{S}^n(0)$*, for some* $n \in \mathsf{N} \setminus \{0\}$.

Proof. By induction on t.

(i) t is 0. Then, $\mathrm{Q} \vdash t = 0$.

(ii) t is $\mathrm{S}(u)$. If $\mathrm{Q} \vdash u = 0$, then $\mathrm{Q} \vdash t = \mathrm{S}(0)$. If $\mathrm{Q} \vdash u = \mathrm{S}^n(0)$, then $\mathrm{Q} \vdash t = \mathrm{S}^{n+1}(0)$.

(iii) t is $u_1 + u_2$. If $\mathrm{Q} \vdash u_2 = 0$, then, by Q4, $\mathrm{Q} \vdash u_1 + u_2 = u_1$ and by induction hypothesis, $\mathrm{Q} \vdash u_1 + u_2 = 0$ or $\mathrm{Q} \vdash u_1 + u_2 = \mathrm{S}^n(0)$. If $\mathrm{Q} \vdash u_2 = \mathrm{S}^n(0)$, then, by Q5, $\mathrm{Q} \vdash u_1 + u_2 = \mathrm{S}(u_1 + \mathrm{S}^{n-1}(0))$. By induction hypothesis applied on u_1 and by axiom Q5, the right side of the equation can be easily modified so that $\mathrm{Q} \vdash u_1 + u_2 = \mathrm{S}^m(0)$, for some $m \in \mathsf{N}$.

(iv) t is $u_1 \cdot u_2$. If $\mathrm{Q} \vdash u_2 = 0$, then, by Q6, $\mathrm{Q} \vdash u_1 \cdot u_2 = 0$. If $\mathrm{Q} \vdash u_2 = \mathrm{S}^n(0)$, then, by Q7, $\mathrm{Q} \vdash u_1 \cdot u_2 = u_1 \cdot \mathrm{S}^{n-1}(0) + u_1$. By axiom Q7, the right side of the equation can be modified so that $\mathrm{Q} \vdash u_1 \cdot u_2 = 0 + u_1 + \ldots + u_1$. By induction hypothesis applied on u_1 and by (iii), $\mathrm{Q} \vdash u_1 \cdot u_2 = 0$ or $\mathrm{Q} \vdash u_1 \cdot u_2 = \mathrm{S}^m(0)$, for some $m \in \mathsf{N}$.

□

Proposition 4.15. *Both versions of* Q *are saturated.*

Proof. The only non-trivial cases are axioms Q3, Q8 and Q9.

(i) Axiom Q3. Similarly to the proof of Proposition 4.13, we need to verify that for all closed terms t, $\mathrm{Q} \vdash t = 0$ or there exists a closed term u such that $\mathrm{Q} \vdash t = \mathrm{S}(u)$. The previous lemma showed that $\mathrm{Q} \vdash t = 0$ or $\mathrm{Q} \vdash t = \mathrm{S}^n(0)$, for some $n \in \mathsf{N} \setminus \{0\}$. The latter implies that there exists a closed term $u = \mathrm{S}^{n-1}(0)$ such that $\mathrm{Q} \vdash t = \mathrm{S}(u)$.

   Verification of axioms Q1, Q2, Q4–Q7 is left to the reader.

(ii) Axioms Q8 and Q9. We have to show that for all closed terms t_1, t_2, if $\mathrm{Q} \vdash t_1 \leq t_2$, then there exists a closed term u such that $\mathrm{Q} \vdash u + t_1 = t_2$. From Lemma 4.14, we know that t_1 and t_2 are equal to one of the values 0 or $\mathrm{S}^n(0)$. $\mathrm{Q} \vdash t_1 \leq t_2$ implies that the value of t_1 is not greater than the value of t_2. If both t_1 and t_2 are equal to the same value, say $\mathrm{S}^k(0)$, then there exists term $u = 0$ such that $\mathrm{Q} \vdash u + t_1 = t_2$. If t_1 has a value $\mathrm{S}^l(0)$ and t_2 has a value $\mathrm{S}^m(0)$, where $l < m$, then there exists term $u = \mathrm{S}^{m-l}(0)$ such that $\mathrm{Q} \vdash u + t_1 = t_2$. Axiom Q9 is dealt similarly.

□

So far, we have demonstrated the use of Corollary 4.10 in a positive sense. Anyway, the following paragraph points out that there are some theories which are, by the corollary, shown not to be saturated.

Proposition 4.16. *Neither* RNA *nor* wRNA *is saturated.*

Proof. The proof is the same for both theories. Due to axioms RN1 and RN3, all the closed terms are equal to $0, 1$ or $(1+1+\ldots+1)$. Let us check whether RNA slashes axiom DN1. According to Definition 4.4, we have to verify that for all closed terms t_1, t_2, if $\Gamma \vdash t_1 < t_2$, then there exists a closed term u such that RNA $\vdash t_1 < u$ and RNA $\vdash u < t_2$. But if we put $t_1 = 0$ and $t_2 = 1$, there does not exist any appropriate term u. $\square$

The question whether the theories in our scope are saturated was successfully answered. There are some more notes concerning the results. We could see that the different formulation of axioms Q3 and LO3 does not have any influence on saturation of the theories. The form and richness of closed terms seem to be much more relevant. Nevertheless, we do not claim that substituting axioms by their classically equivalent versions never affect the saturation.

It seems that in the case of saturation, RNA and wRNA behave alike. Neither of them is saturated, but the reason lies in axiom DN1 and the set of closed terms. If we added for each "standard" element one constant (i.e. constants for all rationals), both RNA and wRNA would be saturated, since DN1 would satisfy the criterion and there is no problem with the remaining axioms.

4.3 Other criteria of saturation

4.3.1 Harrop formulas

In this short subsection, we proceed with considerations that concern deciding whether a theory is saturated or not. The previous sections of this chapter showed a useful criterion—it suffices to look at the set Γ of axioms and check whether $\Gamma \mid \gamma$ for all $\gamma \in \Gamma$. However, it is not the only criterion. As it is demonstrated in [TS00, pp. 106–107], if theory T contains merely Harrop formulas in its set of axioms, then T is saturated. Our interest, now, is in comparing these two criteria.

Initially, we put the following definition:

Definition 4.17. Harrop formula *is inductively defined as follows:*

(i) Every atomic formula is Harrop.

(ii) $\varphi_1 \,\&\, \varphi_2$ *is Harrop provided that both* φ_1 *and* φ_2 *are Harrop.*

(iii) Every formula of the form $\neg\psi$ *is Harrop.*

(iv) $\psi \to \varphi$ *is Harrop provided that* φ *is Harrop.*

(v) $\forall x \varphi$ *is Harrop provided that* φ *is Harrop.*

(vi) No other formula is Harrop.

In brief, a formula is called *Harrop* iff all the occurrences of $\vee$ and $\exists$ lie in the antecedent of an implication or are in a scope of $\neg$.

The following proposition describes the relation between a set of Harrop axioms and a set of slashed formulas.

Proposition 4.18. *Let T be a consistent theory with a language containing at least one constant. Suppose that all the axioms of T are Harrop formulas. Then, $T \mid \varphi$, for all $\varphi \in T$.*

Proof. $\mathrm{Thm}(T)$ is saturated; see [TS00, pp. 106–107]. By Corollary 4.10, $T \mid \varphi$, for all $\varphi \in T$. $\square$

A natural question arises whether the proposition holds vice versa. That is, provided that we have a set of slashed formulas, does it necessarily mean that all such formulas are equivalent to Harrop formulas? The previous section has demonstrated that the answer is no. SUCC and Q are theories with slashed, but not Harrop axioms, e.g., Q3 is not Harrop.

4.3.2 Semantical criteria of saturation

Employing Theorem 4.8 and its corollaries and checking whether axioms are Harrop formulas are not the only ways how to prove that a theory is saturated. In this section, we want to introduce a semantical criterion that was presented in [Smo73a, pp. 334–335] and investigate whether it is applicable to the theories in our scope.

We start with the definitions of operations $\sum$ and $'$.

Definition 4.19. *Let $\mathfrak{F} = \{\, \mathcal{K}_n \,;\, n \in \mathsf{N} \,\}$ be a family of Kripke models. The* disjoint sum *of models in $\mathfrak{F}$, $\sum \mathfrak{F}$, is a Kripke model $\mathcal{K}$ defined by:*

(i) $K = \bigcup_{n \in \mathsf{N}} K_n \times \{n\}$

(ii) $\langle \alpha, n \rangle \leq \langle \beta, m \rangle$ *iff* $n = m$ *and* $\alpha \leq_n \beta$

(iii) $l(\langle \alpha, n \rangle) = l_n(\alpha)$

(iv) $\langle \alpha, n \rangle \Vdash \varphi_{at}$ *iff* $\alpha \Vdash_n \varphi_{at}$*, for atomic formula* φ_{at}*.*

Roughly speaking, $\sum \mathfrak{F}$ just puts together the models from $\mathfrak{F}$. The relation (iv) in the definition holds for any formula φ. It can be easily verified by induction on φ.

Definition 4.20. *Let $\mathcal{K}$ be a Kripke model and let a language L with at least one constant be given. Then, $\mathcal{K}' = \langle K', \leq', l', \Vdash' \rangle$ denotes any model obtained by adding a new node α_0 to K such that*

(i) $\alpha_0 \leq' \alpha$*, for all* $\alpha \in K$*,*

(ii) $l'(\alpha_0)$ *contains the realizations of all closed terms in* L *and nothing else,*

(iii) if $\alpha_0 \Vdash' \varphi_{at}$*, then* $\alpha \Vdash \varphi_{at}$*, for all* $\alpha \in K$*.*

It will be useful to apply operations $\sum$ and $'$ successively. First, we put models from $\mathfrak{F}$ together, and second, we connect them by adding the origin. The following theorem presents a semantical criterion of saturation.

Theorem 4.21 ([Smo73a, p. 335])**.** *Let* T *be a theory. If* $(\sum \mathfrak{F})'$ *exists for any class* $\mathfrak{F}$ *of models of* T*, and* $(\sum \mathfrak{F})'$ *is a model of* T*, then* T *is saturated.*

Proof. (i) Let φ and ψ be sentences and let $T \nvdash \varphi$ and $T \nvdash \psi$. Then, there exist Kripke models $\mathcal{K}_\varphi$ and $\mathcal{K}_\psi$ with origins α_φ and α_ψ respectively such that $\alpha_\varphi \nVdash \varphi$ and $\alpha_\psi \nVdash \psi$. Let $\mathfrak{F} = \{\mathcal{K}_\varphi, \mathcal{K}_\psi\}$ and let α_0 be the origin of $(\sum \mathfrak{F})'$. If $\alpha_0 \Vdash \varphi \vee \psi$, then $\alpha_0 \Vdash \varphi$ or $\alpha_0 \Vdash \psi$. But $\alpha_0 \Vdash \varphi$ is a contradiction to $\alpha_\varphi \nVdash \varphi$ and $\alpha_0 \Vdash \psi$ is a contradiction to $\alpha_\psi \nVdash \psi$. Hence, T is d-complete.

(ii) Let χ be a formula with one free variable x and let for each closed term t, $T \nvdash \chi_x(t)$. Then, for each closed term t, we can find a Kripke model $\mathcal{K}_t$ with origin α_t such that $\alpha_t \nVdash \chi_x(t)$. Let $\mathfrak{F} = \{\, \mathcal{K}_t \,;\, t$ is a closed term $\}$, and let α_0 be the origin of $(\sum \mathfrak{F})'$. If $\alpha_0 \Vdash \exists x \chi$, then for some $a \in l'(\alpha_0)$, $\alpha_0 \Vdash \chi[(x/a)]$. But from the definition of $'$, a is the realization of a closed term s, thus $\alpha_0 \Vdash \chi_x(s)$. But it is a contradiction to $\alpha_s \nVdash \chi_x(s)$. Hence, T is e-complete.

$\square$

Note that Theorem 4.21 cannot be formulated as equivalence. The following example shows that there exists a saturated theory such that some of its models cannot be joined by operation $\sum'$.

Example 4.22. Let $L = \{c_1, c_2\}$ be a language with two constants and let T be a theory in L with no axiom, i.e. merely predicate logic. We define Kripke model $\mathcal{K}_1 = \langle K_1, \leq_1, l_1, \Vdash_1 \rangle$ as follows: $K_1 = \{\alpha\}$, $l_1(\alpha) = \{a\}$, $c_1^{l_1(\alpha)} = a$, $c_2^{l_1(\alpha)} = a$, and Kripke model $\mathcal{K}_2 = \langle K_2, \leq_2, l_2, \Vdash_2 \rangle$ as follows: $K_2 = \{\alpha\}$, $l_2(\alpha) = \{a, b\}$, $c_1^{l_2(\alpha)} = a$, $c_2^{l_2(\alpha)} = b$. Both $\mathcal{K}_1$ and $\mathcal{K}_2$ are models of predicate logic, but they cannot be joined by $\sum'$ because of condition (ii) of Definition 2.9.

Let us briefly check whether the semantical criterion corresponds with the results obtained by means of the Aczel slash. SUCCs and Qs were proved to be saturated and also the semantical criterion is applicable to them, since every model of SUCC or Q contains the block of (standard) elements which are the realizations of closed terms. Theories RNA and wRNA were proved not to be saturated. If we try to apply the semantical criterion, we find out that no model created from a class of models of (w)RNA by $\sum'$ is a model of (w)RNA. The reason is that all the elements of the origin of the joined model are the realizations of terms $0, 1, (1+1), \ldots, (1+\ldots+1)$, since the language of (w)RNA does not allow to create closed terms of other values than $0, 1, (1+1), \ldots, (1+\ldots+1)$. But the axioms of (w)RNA imply, e.g, that there is no minimal element. Hence, the origin does not force all the axioms of (w)RNA.

Our considerations of finding other semantical criteria were inspired by the following criterion which holds for propositional logic.

Lemma 4.23 ([BJ05, p. 21]). *Let T be an intuitionistic propositional theory. Then, T is saturated*[1] *iff for all rooted models of T, $\mathcal{K}_1$ and $\mathcal{K}_2$, there exists a model $\mathcal{K}$ of T such that $\mathcal{K}_1$ and $\mathcal{K}_2$ are generated subframes of $\mathcal{K}$.*

Proof. $\Rightarrow$ Suppose that $\mathrm{Thm}(T)$ is a saturated set and $\mathcal{K}_1$ and $\mathcal{K}_2$ are models of T with origins α_1 and α_2 respectively. We use the canonical model, $\mathcal{K}$, constructed in the completeness proof of intuitionistic propositional logic. The nodes of $\mathcal{K}$ are all saturated supersets of $\mathrm{Thm}(T)$ and $A \in \Gamma$ iff $\Gamma \Vdash_{\mathrm{IPC}} A$. But $\alpha_1 \Vdash_{\mathrm{IPC}} \mathrm{Thm}(T)$ and $\alpha_2 \Vdash_{\mathrm{IPC}} \mathrm{Thm}(T)$, thus $\alpha_1 \in K$ and $\alpha_2 \in K$. Hence, $\mathcal{K}_1$ and $\mathcal{K}_2$ are generated subframes of $\mathcal{K}$.

$\Leftarrow$ Suppose that every two models of T can be extended into one model of T. Suppose next that $T \vdash A \vee B$, $T \nvdash A$ and $T \nvdash B$. Then, there exist models of T, $\mathcal{K}_1$ and $\mathcal{K}_2$, with origins α_1 and α_2 respectively such that $\alpha_1 \nVdash A$ and $\alpha_2 \nVdash B$. We extend the models into a model $\mathcal{K}$ of T. $\mathcal{K} \Vdash A \vee B$, thus $\mathcal{K} \Vdash A$ or $\mathcal{K} \Vdash B$, which is a contradiction to $\alpha_1 \nVdash A$ or $\alpha_2 \nVdash B$ respectively.

□

An interesting remark inspired by the previous lemma is that there are saturated propositional theories that have models which cannot be joined by adding only *one* node. An example is the theory with one axiom $\neg r \to p \vee q$.

We tried to restate the criterion presented in Lemma 4.23 for predicate logic, but surprisingly, it does not hold, as it was shown in Example 4.22. The reason is that the canonical model created in the proof of the completeness theorem for intuitionistic predicate logic has nodes with domains in which different constants are realized as different elements (for the details, see [Smo73a, p. 330]). Thus, if we take a model that does not satisfy this condition, e.g. model $\mathcal{K}_1$ from Example 4.22, it cannot be a submodel of the canonical model.

The last remark concerns the following question: Why do not we employ the criterion that arose from the proof of the completeness theorem, i.e. why do not we say that theory T is saturated iff there exists a model $\mathcal{K}$ with node α such that $\mathrm{Thm}(T) = \{\, \varphi\, ;\, \alpha \Vdash \varphi \,\}$? Let L be a language of T. By the completeness theorem, if $\mathrm{Thm}(T)$ is L-saturated set, then there really exists a model $\mathcal{K}$ with node α such that $\mathrm{Thm}(T) = \{\, \varphi\, ;\, \alpha \Vdash \varphi \,\}$. But conversely, the existence of the canonical model $\mathcal{K}$, where $\mathrm{Thm}(T) = \{\, \varphi\, ;\, \alpha \Vdash \varphi \,\}$ only implies that $\mathrm{Thm}(T)$ is L^*-saturated, where L^* is L extended by constants to include names for all elements of $l(\alpha)$. There is no guarantee that $\mathrm{Thm}(T)$ is L-saturated.

Let us sum up the whole chapter. We showed that the considerations concerning the Aczel slash lead to a very useful criterion that decides whether a theory is saturated or not. The strength of the criterion resides in the fact that it is formulated as equivalence.

[1] A propositional theory T is saturated iff it is consistent and $\mathrm{Thm}(T)$ is d-complete.

We demonstrated that there are some theories with some non-Harrop axioms that are saturated. In the last subsection, we presented a semantical criterion (that is not so strong) and verified that we obtain the same results as we did by the Aczel slash criterion.

There is a summary of practical results: Theories without any constant in their language are not saturated. Theories SUCCs and Qs were demonstrated to be saturated, but neither RNA nor wRNA is saturated. It is obvious that saturation of a theory depends on the number and type of closed terms that can be created in the language of the theory. For example, if we added infinitely many constants to the language of (w)RNA so that every rational number had its name, (w)RNA would be saturated.

5

De Jongh's theorem

In chapter 3, we discussed coincidences between intuitionistic theories and their classical extensions. We concluded that some theories of our scope coincide with their classical versions, some of them do not coincide and that there are theories which trivialize intuitionistic logic only *partly*. This chapter presents a finer criterion of "constructiveness" of theories than the coincidence; the criterion is called De Jongh's theorem. If De Jongh's theorem does not hold for some theory, it means, roughly, that the theory is so strong that it contains a propositional axiom which is not an intuitionistic tautology. We shall especially focus on theories that do not coincide with their classical extensions, but, at the same time, fail to satisfy De Jongh's theorem. Such theories might be seen as *not constructive enough*, despite they are not classical.

De Jongh's theorem was originally stated for Heyting arithmetic, but we formulate it more generally.

Definition 5.1. *Let $A(p_1,\dots,p_n)$ be a propositional formula with variables $p_1,\dots,p_n$. Intuitionistic theory T* satisfies De Jongh's theorem *iff the following statement holds. If $\nvdash_{\mathrm{IPC}} A(p_1,\dots,p_n)$, then $T \nvdash_{\mathrm{IQC}} A(\varphi_1,\dots,\varphi_n)$, for some sentences $\varphi_1,\dots,\varphi_n$ in the language of T. ($A(\eta_1,\dots,\eta_n)$ is a propositional combination of formulas $\eta_1,\dots,\eta_n$.)*

The proof that Heyting arithmetic satisfies De Jongh's theorem can be found in [Smo73a, p. 354]. We changed the proof so that it yields the result that theories E, LO and wLO satisfy De Jongh's theorem.

Lemma 5.2 ([Smo73a, p. 352])**.** *Let $\alpha_1,\dots,\alpha_k$ be the terminal nodes of a modified Jaskowski tree*[1] *K such that $\mathcal{K} = \langle K, \leq, \Vdash \rangle$ is a propositional Kripke model. Suppose that for each $i \in \{1,\dots,k\}$, there is a sentence ψ_i such that $\alpha_j \Vdash \psi_i$ iff $i = j$. Then, if X is a set of nodes of K such that $\alpha \in X$ and $\alpha \leq \beta$ imply $\beta \in X$, there exists a sentence φ constructed from sentences ψ_i for which $X = \{\, \alpha \,;\, \alpha \Vdash \varphi \,\}$. In particular, for any $\alpha \in K$ there is a sentence φ_α such that $\{\, \beta \,;\, \beta \geq \alpha \,\} = \{\, \beta \,;\, \beta \Vdash \varphi_\alpha \,\}$.*

[1] Modified Jaskowski tree is, besides other things, a finite tree which has a property that each node, except for terminal nodes, has at least two successors. For more information on modified Jaskowski trees see section 2.1.

Proof. Let $\varphi_\alpha \stackrel{\text{def}}{=} \bigwedge_{\alpha_i \not\geq \alpha} \neg\psi_i$ for every $\alpha \neq \alpha_0$ and let $\varphi_{\alpha_0} \stackrel{\text{def}}{=} \psi_1 \to \psi_1$. We must show that

$$\beta \Vdash \varphi_\alpha \quad \text{iff} \quad \beta \geq \alpha.$$

For α_0, it is immediate. Let $\alpha \neq \alpha_0$.

(i) Suppose that $\beta \Vdash \varphi_\alpha$. If $\beta \not\geq \alpha$, then there is a terminal node α_i such that $\alpha_i \geq \beta$ and $\alpha_i \not\geq \alpha$, since K is a modified Jaskowski tree. Thus, $\neg\psi_i$ is one of the conjuncts in φ_α. But $\beta \nVdash \neg\psi_i$, which is a contradiction to $\beta \Vdash \varphi_\alpha$.

(ii) Suppose that $\beta \geq \alpha$. If $\beta \nVdash \varphi_\alpha$, then for some i such that $\alpha_i \not\geq \alpha$, $\beta \nVdash \neg\psi_i$, thus for some $\alpha_j \geq \beta$, $\alpha_j \Vdash \psi_i$. $\alpha_i \not\geq \alpha$ implies $\alpha_i \not\geq \beta$, whence $i \neq j$. This is a contradiction to the assumption of the lemma that $\alpha_j \Vdash \psi_i$ iff $i = j$.

Finally, let $\varphi \stackrel{\text{def}}{=} \bigvee_{\alpha \in X} \varphi_\alpha$. □

Proposition 5.3. E *satisfies De Jongh's theorem.*

Proof. Let $\nvdash_{\mathrm{IPC}} A(p_1, \ldots, p_n)$. There exists a modified Jaskowski tree[2] K with an origin α_0 such that $\mathcal{K}^* = \langle K, \leq, \Vdash^* \rangle$ is a propositional Kripke model and $\alpha_0 \nVdash^* A(p_1, \ldots, p_n)$. Suppose the terminal nodes of K are $\alpha_1, \ldots, \alpha_k$. First, we define formulas $\chi_2, \chi_3, \ldots, \chi_{k+1}$ as follows:

$$\begin{aligned}
\chi_2 &\stackrel{\text{def}}{=} \forall x \exists y \neg(x = y) \\
\chi_3 &\stackrel{\text{def}}{=} \forall x_1 \forall x_2 \exists y (\neg(x_1 = y) \ \& \ \neg(x_2 = y)) \\
\chi_4 &\stackrel{\text{def}}{=} \forall x_1 \forall x_2 \forall x_3 \exists y (\neg(x_1 = y) \ \& \ \neg(x_2 = y) \ \& \ \neg(x_3 = y)) \\
&\vdots \\
\chi_{k+1} &\stackrel{\text{def}}{=} \forall x_1 \ldots \forall x_k \exists y (\neg(x_1 = y) \ \& \ \ldots \ \& \ \neg(x_k = y)).
\end{aligned}$$

Note that each χ_i expresses the existence of at least i elements. Second, we define formulas $\psi_1, \ldots, \psi_k$ as follows:

$$\begin{aligned}
\psi_1 &\stackrel{\text{def}}{=} \chi_2 \ \& \ \neg\chi_3 \\
&\vdots \\
\psi_{k-1} &\stackrel{\text{def}}{=} \chi_k \ \& \ \neg\chi_{k+1} \\
\psi_k &\stackrel{\text{def}}{=} \chi_{k+1}.
\end{aligned}$$

Third, we extend $\mathcal{K}^*$ into a predicate Kripke model $\mathcal{K} = \langle K, \leq, l, \Vdash \rangle$ as follows: For every non-terminal node α, let $l(\alpha)$ contain one element. For each terminal node α_i, let $l(\alpha_i)$ contain $i+1$ elements ($i \in \{1, \ldots, k\}$). $\Vdash$ is defined so that in every node of K, every element is equal to itself and to nothing else.

$\mathcal{K}$ is a model of E and for each terminal node α_j, $\alpha_j \Vdash \psi_i$ iff $i = j$. For each $i \in \{1, \ldots, n\}$, we define X_i to be $\{\, \beta \,;\, \beta \Vdash^* p_i \,\}$. Now, all the assumptions of Lemma 5.2

[2] Since propositional intuitionistic logic is complete for modified Jaskowski trees (see section 2.1).

are satisfied and we obtain the result that for each $i \in \{1, \dots, n\}$, there exists a sentence φ_i such that $X_i = \{\beta;\, \beta \Vdash \varphi_i\}$. It means that for every $\beta \in K$, $\beta \Vdash^* p_i$ iff $\beta \Vdash \varphi_i$. Hence, by induction on propositional formula B, $\beta \Vdash^* B(p_1, \dots, p_n)$ iff $\beta \Vdash B(\varphi_1, \dots, \varphi_n)$. Finally, $\alpha_0 \nVdash A(\varphi_1, \dots, \varphi_n)$. $\square$

Proposition 5.4. LO *and* wLO *satisfy De Jongh's theorem.*

Proof. We use the same proof as in the previous proposition. The only difference is that we extend $\Vdash$ to deal with formulas where $<$ occur. Let α_i be a terminal node with elements $a_1, \dots, a_l$. We define $\alpha_i \Vdash a_u < a_v$ iff $u < v$. Now, $\mathcal{K}$ is a model of both LO and wLO. $\square$

The following lemma shows that De Jongh's theorem is a finer criterion of constructiveness than coincidence.

Lemma 5.5. *If T coincide with its classical extension, then T does not satisfy De Jongh's theorem.*

Proof. $\nvdash_{\mathrm{IPC}} p \vee \neg p$, but for any sentence φ, $T \vdash \varphi \vee \neg\varphi$. $\square$

The following proposition is an immediate consequence of the lemma and section 3.1.

Proposition 5.6. *Neither* DNO *nor* RNA *satisfies De Jongh's theorem.*

In the following paragraphs, we demonstrate that De Jongh's theorem holds neither for SUCCs nor for wDNO and wRNA. This result yields an example of theories that do not coincide with their classical versions, but yet, they *partly* trivialize intuitionistic logic.

Proposition 5.7. *Neither of* SUCC*s satisfy De Jongh's theorem.*

Proof. Assume that $\nvdash_{\mathrm{IPC}} A(p_1, \dots, p_n)$. There exists a modified Jaskowski model $\mathcal{K}$ with an origin α_0 such that $\alpha_0 \nVdash A(p_1, \dots, p_n)$. According to formula A, $\mathcal{K}$ contains terminal nodes of unboundedly large finite number. It means that if we wanted to prove that De Jongh's theorem holds, we would need an infinite number of models of SUCC, $\mathcal{K}_1, \dots, \mathcal{K}_\infty$, and an infinite number of sentences $\psi_1, \dots, \psi_\infty$ such that

$$\mathcal{K}_j \Vdash \psi_i \quad \text{iff} \quad i = j. \tag{5.1}$$

We want to show that SUCC can have only finite number of models such that every two of them differ in forcing a sentence.

When we look at the axioms of $\mathrm{SUCC}^\vee$, we perceive that any model of $\mathrm{SUCC}^\vee$ must contain a block of "standard" elements, i.e. zero, the successor of zero, the successor of the successor of zero, etc., in all universes. Besides, in some universes, there may be blocks of "non-standard" elements, i.e. countably many elements with a successor function realized as in integers. We do not have to take uncountable domains into account, since Löwenheim-Skolem theorem guarantees that we do without them. Blocks of non-standard elements can merge as it similarly happens in model of $\mathrm{SUCC}^\rightarrow$ constructed in the proof of Proposition 3.17 and new blocks of non-standard elements can be added. We restrict our considerations to tree models, since every Kripke model is equivalent to a tree Kripke model, as it is proved in [Kri65, s. 1.2].

We deal with models of $\mathrm{SUCC}^{\vee}$ that are lineary ordered and show that there are at most four types of these models such that models of the same type cannot be distinguished by the use of sentences (i.e., none of the models of the same type is uniquely characterized by a sentence ψ as in (5.1)). Non-linear tree models comprise of linear models and thus, if there are not infinitely many linear models that are uniquely characterized by a sentence, then there are not such tree models either.

Here we put the four types of linear models mentioned.

(i) Models with added elements. For all nodes $\beta \geq \alpha$, $|l(\beta)| \geq |l(\alpha)|$ and $=$ is realized so that no elements merge.

(ii) Models with merged elements. For all nodes $\beta \geq \alpha$, $|l(\beta)| = |l(\alpha)|$ and for some α, in the successor of α some blocks of elements merge.

(iii) An infinite model in which for every α exists a $\beta \geq \alpha$ such that in $l(\beta)$, there are blocks of elements that are not merged, but in some $\gamma > \beta$, these blocks merge.

(iv) Other models combining merging and adding new elements.

It is not important whether models of every two different types can be distinguished as in (5.1)—however, at least the models of types (i) and (iii) can be uniquely characterized by sentences $\forall x \forall y(x = y \ \vee\ \neg(x = y))$ and $\neg\forall x \forall y(x = y \ \vee\ \neg(x = y))$ respectively—but the essential thing is that models of the same type *cannot* be uniquely distinguished. The reasons are that, first, we cannot express the number of blocks of non-standard elements, and second, according to the Kripke semantics, the validity of a sentence is expressed by *meta*quantifiers (e.g., $\forall\alpha\exists\beta \geq \alpha \dots$) and with this apparatus, we cannot express that something holds in the successor of α.

$\mathrm{SUCC}^{\rightarrow}$ is dealt analogously to $\mathrm{SUCC}^{\vee}$. The only difference is that in domains, there may be blocks of non-standard elements that look like natural numbers, but these blocks need to be merged with standard elements, since axiom $\mathrm{Q3}^{\rightarrow}$ must hold. $\square$

Proposition 5.8. wRNA *does not satisfy De Jongh's theorem.*

Proof. The proof is analogous to the previous one. The only difference is that standard and non-standard elements in models of wRNA look different. The axioms of wRNA imply that there is a block of rationals in every domain of a model of wRNA with $=$, $<$ and $+$ realized as in the rational numbers (we call that block "standard elements" and denote the elements as $\{\, a_q\,;\, q \in \mathsf{Q}\,\}$). Besides, there may be block(s) of "non-standard" elements $\{\, a_{q_1 q_2}\,;\, q_1, q_2 \in \mathsf{Q}\,\}$, $\{\, a_{q_1 q_2 q_3}\,;\, q_1, q_2, q_3 \in \mathsf{Q}\,\}$, etc. such that $a_{\mathbf{q}}$ is always identified with $a_{\mathbf{q}0}$.[3] Note that domains of a model of wRNA are, in fact, the sets $\mathsf{Q}, \mathsf{Q}^2, \mathsf{Q}^3, \dots$ with $+$ realized normally.

More variability is in defining $=$ and $<$. Similarly to the models of SUCC, blocks of non-standard elements can merge. The necessary condition for merging of elements is

[3] $\mathbf{q}$ is a sequence of rationals that has a certain length n.

that they were not comparable with each other (in the sense of relation $<$), otherwise, antireflexivity of $<$ would be violated. Then, $\{ a_{q_1,q_2} ; q_1, q_2 \in \mathsf{Q} \}$ can be merged with $\{ a_{q_1} ; q_1 \in \mathsf{Q} \}$ in such a way that $a_{q_1,q_2} = a_{q_1}$ for every $q_1, q_2 \in \mathsf{Q}$. To put one more example, $\{ a_{q_1,q_2,q_3,q_4} ; q_1, q_2, q_3, q_4 \in \mathsf{Q} \}$ can be merged with $\{ a_{q_1,q_2} ; q_1, q_2 \in \mathsf{Q} \}$ in such a way that $a_{q_1,q_2,q_3,q_4} = a_{q_1,q_2}$ for every $q_1, q_2, q_3, q_4 \in \mathsf{Q}$.

As for the realization of $<$, every two incomparable elements from $l(\alpha)$ must be ordered or merged in some $l(\beta)$, where $\beta \geq \alpha$ in order to satisfy axiom AP. Furthermore, if $a_{\mathbf{q_1} q_1}$ and $a_{\mathbf{q_2} q_2}$ are comparable, then every $a_{\mathbf{q_3}}$ and $a_{\mathbf{q_4}}$ must be also comparable, since it is demanded by axiom wLO3. By other words, any two elements can be ordered only if all the elements with shorter sequence of indices are already ordered.

Two models of wRNA are constructed in the proof of Proposition 3.16. They are quite simple, but they illustrate well how models of wRNA can look like. As in the previous proof, linear models of wRNA can be sorted into several types:

(i) Models with added elements. For all nodes $\beta \geq \alpha$, $|l(\beta)| \geq |l(\alpha)|$ and $=$ is realized so that no elements merge.

   a. Models in which all elements in all nodes are comparable or equal.

   b. Models with incomparables that are not equal.

(ii) Models with merged elements. For all nodes $\beta \geq \alpha$, $|l(\beta)| = |l(\alpha)|$ and for some α, in the successor of α some blocks of elements merge.

(iii) Infinite models in which for every α exists a $\beta \geq \alpha$ such that in $l(\beta)$, there are blocks of elements that are not comparable, but in some $\gamma > \beta$, these elements are ordered.

(iv) Infinite models in which for every α exists a $\beta \geq \alpha$ such that in $l(\beta)$, there are blocks of elements that are not merged, but in some $\gamma > \beta$, these blocks merge.

(v) Other models combining merging, ordering and adding new elements.

We claim that models of one of the six listed types *cannot* be uniquely distinguished by a sentence (i.e., they do not satisfy condition (5.1)). The reason is the same as in the previous proof. $\square$

Corollary 5.9. wDNO *does not satisfy De Jongh's theorem.*

Proof. Let $\nvdash_{\mathrm{IPC}} A(p_1, \ldots, p_n)$. By the previous proposition, for all sentences $\varphi_1, \ldots, \varphi_n$ in the language of wRNA, wRNA $\vdash_{\mathrm{IQC}} A(\varphi_1, \ldots, \varphi_n)$. Particularly, for all sentences $\varphi_1, \ldots, \varphi_n$ in the language of wDNO, wRNA $\vdash_{\mathrm{IQC}} A(\varphi_1, \ldots, \varphi_n)$. But wRNA is a conservative extension of wDNO[4], thus wDNO $\vdash_{\mathrm{IQC}} A(\varphi_1, \ldots, \varphi_n)$. $\square$

The last remark is that De Jongh's theorem holds for both versions of Robinson arithmetic, because the proof for Heyting arithmetic ([Smo73a, pp. 352–354]) does not use anything that could not be said for Robinson arithmetic (especially the Rosser's version of Gödel's first incompleteness theorem).

[4]It will be shown in Proposition 6.16.

To sum up, we could divide the theories into three groups according to the results obtained in this chapter. In the first group, there are theories DNO and RNA which do not satisfy De Jongh's theorem trivially, since they coincide with their classical extensions. In the second group, there are theories that satisfy De Jongh's theorem, viz. E, LO, wLO, $Q^{\vee}$ and $Q^{\rightarrow}$. Note that theory LO satisfies De Jongh's theorem, in spite of the fact that atomic formulas are decidable in LO (it was demonstrated in chapter 3). In the third group, there are theories that do not satisfy De Jongh's theorem, but they still do not coincide with their classical extensions, viz. $\mathrm{SUCC}^{\vee}$, $\mathrm{SUCC}^{\rightarrow}$, wDNO and wRNA. One could interpret these results by asserting that wDNO and wRNA are not sufficient weakenings of DNO and RNA and that we should also think of "more constructive" axiomatization of SUCC.

6

Decidability

The last chapter that looks into the properties of the theories in scope investigates their decidability. The major results are taken from [Smo73b], but we want to supplement them with more details and other results. Sections 6.1 and 6.2 prove the undecidability of theories and are more significant. Section 6.3 shows some consequences of the previous chapters and classical decidability results.

6.1 Undecidability of E and $\mathrm{SUCC}^{\rightarrow}$

The reason why we put the undecidability proofs for E and $\mathrm{SUCC}^{\rightarrow}$ in the same section is that the method used in the proofs is very similar. Initially, we demonstrate that E is undecidable, and subsequently, we show that an easy modification of the proof yields the undecidability result for $\mathrm{SUCC}^{\rightarrow}$.

An essential assumption that is used in the undecidability proofs is the following theorem.

Theorem 6.1 (Maslov, Mints, Orevkov)**.** *Let* M_1 *be an intuitionistic theory with one monadic predicate,* P*, and no non-logical axioms (i.e.,* M_1 *is the intuitionistic monadic predicate calculus). Then,* M_1 *is undecidable.*

The proof of the theorem can be found in [Smo73b, pp. 116–117] and we take it for granted. The core of the following proofs is to show that $\mathrm{Thm}(\mathrm{M}_1)$ is m-reducible to $\mathrm{Thm}(\mathrm{E})$ and to $\mathrm{Thm}(\mathrm{SUCC}^{\rightarrow})$. Indeed, it suffices for claiming that E and $\mathrm{SUCC}^{\rightarrow}$ are undecidable.

First, we need to define a computable function f that will be proved to satisfy the following equivalences:

$$\begin{aligned} \varphi \in \mathrm{Thm}(\mathrm{M}_1) \quad &\Leftrightarrow \quad f(\varphi) \in \mathrm{Thm}(\mathrm{E}), \\ \varphi \in \mathrm{Thm}(\mathrm{M}_1) \quad &\Leftrightarrow \quad f(\varphi) \in \mathrm{Thm}(\mathrm{SUCC}^{\rightarrow}), \end{aligned}$$

where φ is an arbitrary formula in the language of M_1.

Definition 6.2. *Let* φ *be a formula in the language of* M_1*. Then,* $f(\varphi)$ *is a formula in the language* $\{=\}$ *obtained by replacing every occurence of* $P(x)$ *in* φ *by* $\forall y(x = y \vee$

$\neg(x = y))$, *for any variable* x *and certain* $y \neq x$*. We designate* $\forall y(x = y \vee \neg(x = y))$ *by* $Q(x)$*.*

Now, we are prepared for proving the following theorem.

Theorem 6.3 ([Smo73b, pp. 118–120])**.** *For every formula* η *in the language of* M_1*,* $\mathrm{M}_1 \vdash \eta$ *iff* $\mathrm{E} \vdash f(\eta)$*. Hence,* E *is undecidable.*

Proof. The implication from left to right is easy. If $\mathrm{M}_1 \vdash \eta$, then η is a logically valid formula. Consequently, $f(\eta)$ is also logically valid because it was obtained by a mere replacement of $P(x)$ with $Q(x)$. Thus, $\mathrm{E} \vdash f(\eta)$.

The converse direction is not so immediate. Suppose that $\mathrm{M}_1 \nvdash \eta$. Then, there exists a tree Kripke model[1] of M_1, $\mathcal{K} = \langle K, \leq, l, \Vdash\rangle$, with no terminal nodes (all the branches are infinite)[2] such that $\mathcal{K} \nVdash \eta$. We gradually define new models $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$. $\mathcal{K}_1 = \langle K, \leq, l_1, \Vdash_1\rangle$ is defined as follows:

$$\begin{aligned} l_1(\alpha) \quad &= \quad l(\alpha) \cup \{\, q_a^\beta \,;\, \beta \leq \alpha \text{ and } a \in l(\beta) \text{ and } \beta \nVdash P(a) \,\}, \\ \alpha \Vdash_1 P(a) \quad &\text{iff} \quad \alpha \Vdash P(a), \quad \text{for } a \in l(\alpha), \\ \alpha \Vdash_1 P(q_a^\beta) \quad &\text{iff} \quad \alpha \Vdash P(a), \quad \text{for } a \in l(\alpha),\ \beta \leq \alpha, \end{aligned} \tag{6.1}$$

for every node $\alpha \in K$.

The informal explanation of the construction of $\mathcal{K}_1$ is the following: The tree structure of $\mathcal{K}_1$ is the same as in $\mathcal{K}$, i.e., neither K nor $\leq$ has changed. Universes $l(\alpha)$ are enriched with new elements q_a^β. We could say that in $l_1(\alpha)$, there are original elements a_i and for every original element that do not satisfy formula $P(x)$ in some $\beta \leq \alpha$, there is its copy[3] $q_{a_i}^\beta$. Let us say that some original element a does not satisfy formula $P(x)$ in exactly three nodes, $\beta_1, \beta_2, \beta_3 \leq \alpha$. Then, $l_1(\alpha)$ will contain three copies of a, namely $q_a^{\beta_1}, q_a^{\beta_2}, q_a^{\beta_3}$. The realization of predicate symbol P on the original elements is unchanged. Moreover, $a \in P^{l(\alpha)}$ iff for all the copies of a, if there are any, $q_a^{\beta_i} \in P^{l_1(\alpha)}$. It means that the copies of a cannot be distinguished from a by the use of atomic formulas.

By induction on φ, we can easily prove that for any φ in the language of M_1, $\alpha \in K$ and $e : \mathrm{Var} \to l(\alpha)$,

$$\alpha \Vdash \varphi[e] \quad \text{iff} \quad \alpha \Vdash_1 \varphi[e]. \tag{6.2}$$

Now, we define a model $\mathcal{K}_2 = \langle K, \leq, l_1, \Vdash_2\rangle$ by adding the realization of equality to $\mathcal{K}_1$. Forcing of formulas in the language of M_1 is not changed at all. It means that for any $\alpha \in K$, $e : \mathrm{Var} \to l_1(\alpha)$ and any φ in the language of M_1,

$$\alpha \Vdash_2 P(x)[e] \quad \text{iff} \quad \alpha \Vdash_1 P(x)[e].$$

[1] The fact that every Kripke model is equivalent to a tree Kripke model is proved in [Kri65, s. 1.2].

[2] Every tree model with some finite branches can be extended to the model with no terminal nodes by adding infinitely many copies of terminal nodes.

[3] We also call such elements *non-original*.

At the origin, α_0, let every element be equal to itself and to nothing else. If the realization of $=$ is already defined in some α, in any successor β of α, the equivalence relation is extended by putting

$$\beta \quad \Vdash_2 \quad a = q_a^\alpha$$

whenever $q_a^\alpha \in l_1(\beta)$ and closing the relation under reflexivity, tranzitivity and symmetry.

Informally, $\mathcal{K}_2$ has the same tree structure and the same universes as $\mathcal{K}_1$. The realization of predicate P is also unchanged, but additionally, $=$ is realized in $\mathcal{K}_2$ so that all the elements of the form q_a^α which already occured in the preceding node are equal to a in the actual node.

It can be easily seen that, for any formula in the language of M_1, any $\alpha \in K$ and $e : \mathrm{Var} \to l_1(\alpha)$,

$$\alpha \Vdash_1 \varphi[e] \quad \text{iff} \quad \alpha \Vdash_2 \varphi[e]. \tag{6.3}$$

Now, the significant point is to demonstrate that, for any $\alpha \in K$ and $e : \mathrm{Var} \to l_1(\alpha)$,

$$\alpha \Vdash_2 P(x)[e] \quad \text{iff} \quad \alpha \Vdash_2 Q(x)[e]. \tag{6.4}$$

(i) Let $e(x) = a$, i.e., $e(x)$ is an original element. Assume first that $\alpha \Vdash_2 P(a)$. By (6.3) and (6.2), $\alpha \Vdash P(a)$ which implies that there is no element of the form q_a^α in $l_1(\alpha)$. Consequently, all the non-original elements from $l_1(\alpha)$ have the form either q_a^β, where $\beta < \alpha$, or q_b^γ, where $a \neq b$ and $\gamma \leq \alpha$. The former implies $\alpha \Vdash_2 a = q_a^\beta$, the latter $\alpha \Vdash_2 \neg(a = q_b^\gamma)$. Also every original element equals to a or never equals to a. Moreover, for all $\delta > \alpha$ and $d \in l_1(\delta) \setminus l_1(\alpha)$, d does not equal to a. Hence, $\alpha \Vdash_2 \forall y(a = y \lor \neg(a = y))$, i.e., $\alpha \Vdash_2 Q(a)$.

Assume now that $\alpha \nVdash_2 P(a)$. By (6.3) and (6.2), $\alpha \nVdash P(a)$, whence $q_a^\alpha \in l_1(\alpha)$. In α, q_a^α does not equal to a, but in the successor[4] of α, q_a^α equals to a. Thus, $\alpha \nVdash_2 Q(a)$.

(ii) Let $e(x) = q_a^\beta$, i.e., $e(x)$ is a non-original element. Assume that $\alpha \Vdash_2 P(q_a^\beta)$ (it means that $\beta < \alpha$). By (6.3) and (6.1), $\alpha \Vdash_2 P(a)$ and by (i), $\alpha \Vdash_2 Q(a)$. We know that $\alpha \Vdash_2 q_a^\beta = a$ (because $\beta < \alpha$) and therefore, $\alpha \Vdash_2 Q(q_a^\beta)$.

Conversely, if $\alpha \nVdash_2 P(q_a^\beta)$, then, by (6.3) and (6.1), $\alpha \nVdash_2 P(a)$. Now by (i), $\alpha \nVdash_2 Q(a)$. If $\beta < \alpha$, then $\alpha \Vdash_2 q_a^\beta = a$ and immediately, $\alpha \nVdash_2 Q(q_a^\beta)$. If $\beta = \alpha$, then $\alpha \nVdash_2 q_a^\beta = a$, but in the successor γ of α, $\gamma \Vdash_2 q_a^\beta = a$. It implies that $\alpha \nVdash_2 q_a^\beta = a \lor \neg(q_a^\beta = a)$, whence $\alpha \nVdash_2 Q(q_a^\beta)$.

By an easy induction, we obtain a stronger formulation of (6.4): For any $\alpha \in K$, $e : \mathrm{Var} \to l_1(\alpha)$ and φ in the language of M_1,

$$\alpha \Vdash_2 \varphi[e] \quad \text{iff} \quad \alpha \Vdash_2 f(\varphi)[e]. \tag{6.5}$$

[4]We make use of the fact that there are no terminal nodes in K_2.

Finally, we define a model $\mathcal{K}_3 = \langle K, \leq, l_1, \Vdash_3 \rangle$ where $\Vdash_3$ is a mere restriction of $\Vdash_2$ to formulas in the language of E. In other words, predicate P is not realized in $\mathcal{K}_3$ and we have obtained a pure model of E. For any formula ψ in the language of E and any $e : \mathrm{Var} \to l_1(\alpha)$, the following equivalence holds:

$$\alpha \Vdash_2 \psi[e] \quad \text{iff} \quad \alpha \Vdash_3 \psi[e]. \tag{6.6}$$

At the beginning of the proof, we assumed a model $\mathcal{K}$ such that $\mathcal{K} \not\Vdash \eta$. To be more specific, there is a node $\alpha \in K$ and an evaluation $e : \mathrm{Var} \to l(\alpha)$ such that $\alpha \not\Vdash \eta[e]$. From $\mathcal{K}$ we have passed to $\mathcal{K}_3$, a model of E. Equivalences (6.2), (6.3), (6.5) and (6.6) show that $\alpha \not\Vdash_3 f(\eta)[e]$ which proves that $E \nvdash f(\eta)$. □

The next theorem shows that $\mathrm{SUCC}^{\to}$ is undecidable. The method of the proof is similar to the previous one—the essential part is the construction of models $\mathcal{K}_1$–$\mathcal{K}_3$ which do not differ much from the models used in the proof of Theorem 6.3. We shall demonstrate that $\mathrm{Thm}(\mathrm{M}_1)$ is m-reducible to $\mathrm{Thm}(\mathrm{SUCC}^{\to})$ via the familiar fuction f.

Theorem 6.4 ([Smo73b, p. 125]). *For every formula η in the language of M_1, $\mathrm{M}_1 \vdash \eta$ iff $\mathrm{SUCC}^{\to} \vdash f(\eta)$. Hence, $\mathrm{SUCC}^{\to}$ is undecidable.*

Proof. Suppose that $\mathrm{M}_1 \vdash \eta$. From Theorem 6.3, it follows that $\mathrm{E} \vdash f(\eta)$. $\mathrm{Thm}(\mathrm{E}) \subseteq \mathrm{Thm}(\mathrm{SUCC}^{\to})$, thus $\mathrm{SUCC}^{\to} \vdash f(\eta)$.

Suppose conversely that $\mathrm{M}_1 \nvdash \eta$. Then, there exists a tree Kripke model, $\mathcal{K} = \langle K, \leq, l, \Vdash \rangle$, with no terminal nodes such that $\mathcal{K} \not\Vdash \eta$. Model $\mathcal{K}_1 = \langle K, \leq, l_1, \Vdash_1 \rangle$ is defined in exactly the same manner as in the previous proof. It means, besides other things, that we have the original elements (a) and the copies of original elements (q_a^α) in universes and that (6.2) holds. Let α_0 be the origin of K. We arbitrarily choose one original element $o \in l_1(\alpha_0)$ to be the realization of 0 (we write $0^{\alpha_0} = o$).

Now, we define a model $\mathcal{K}_{1'} = \langle K, \leq, l_{1'}, \Vdash_{1'} \rangle$ as follows:

$$\begin{aligned} l_{1'}(\alpha) \quad &= \quad \{\, w_n \,;\, w \in l_1(\alpha),\, 0^\alpha = w \text{ or } (0^\alpha = a \text{ and } w = q_a^\beta),\, n \in \mathsf{N} \,\} \\ &\quad\ \cup \{\, w_n \,;\, w \in l_1(\alpha),\, 0^\alpha \neq w \text{ and if } 0^\alpha = a, \text{ then } w \neq q_a^\beta,\, n \in \mathsf{Z} \,\} \\ \alpha \Vdash_{1'} P(w_n) \quad &\text{iff} \quad \alpha \Vdash_1 P(w), \quad \text{for any } w_n \in l_{1'}(\alpha), \end{aligned} \tag{6.7}$$

for every node $\alpha \in K$. An idea of this construction is that every element w of a universe $l_1(\alpha)$ is replaced by denumerably many elements w_n.[5] If w is the realization of 0 or a copy of an element which is the realization of 0, then w is replaced by $\{\, w_n \,;\, n \in \mathsf{N} \,\}$. In all other cases, w is replaced by $\{ w_n \,; n \in \mathsf{Z} \}$. Furthermore, forcing of $P(w_n)$ depends merely on forcing of $P(w)$ in $\mathcal{K}_1$. It means that all elements w_n derived from the same element w behave alike in respect of having property P. It is clear that the following equivalence holds:

$$\alpha \Vdash_1 \varphi[e] \quad \text{iff} \quad \alpha \Vdash_{1'} \varphi[e], \tag{6.8}$$

for any $\alpha \in K$, $e : \mathrm{Var} \to l_1(\alpha)$ and φ in the language of M_1.

[5]Note that w_n has either the form a_n or the form $(q_a^\beta)_n$.

As in the proof of Theorem 6.3, we proceed with defining a model $\mathcal{K}_2 = \langle K, \leq, l_{1'}, \Vdash_2 \rangle$ which can be regarded as a bridge from a model of M_1 to a model of $\mathrm{SUCC}^{\rightarrow}$. Forcing of formulas in the language of M_1 does not differ from $\Vdash_{1'}$, i.e., for any node $\alpha \in K$ and any element $w_n \in l_{1'}(\alpha)$,

$$\alpha \Vdash_2 P(w_n) \quad \text{iff} \quad \alpha \Vdash_{1'} P(w_n).$$

Moreover, the forcing relation is extended to the language of $\mathrm{SUCC}^{\rightarrow}$ by putting

$$\begin{aligned} \alpha_0 &\Vdash_2 o_0 = 0, && \\ \alpha &\Vdash_2 w_{n+1} = \mathrm{S}(w_n), && \text{for all } w \in l_1(\alpha) \text{ and all } n, \\ \beta &\Vdash_2 (q_a^\alpha)_n = a_n, && \text{for } a \in l(\alpha),\ \beta \text{ a successor of } \alpha, \text{ whenever } q_a^\alpha \in l_1(\beta), \end{aligned} \tag{6.9}$$

for any $\alpha \in K$ and the origin α_0. Indeed, in α_0, every element is defined to be equal to itself and to no other element.

$\mathcal{K}_2$ has the same tree structure, the same universes and the same realization of predicate symbol P as $\mathcal{K}_{1'}$. Moreover, 0, S and = are realized in $\mathcal{K}_2$. 0 is realized as o_0, where o is the arbitrarily chosen element from $l_1(\alpha_0)$. S is realized very intuitively so that the universes look as if they were comprised of several copies of all natural numbers and all integers. All the elements of the form $(q_a^\alpha)_n$ which already occured in the preceding node are equal to a_n in the actual node. It can be easily seen that for any formula φ in the language of M_1, any $\alpha \in K$ and $e : \mathrm{Var} \rightarrow l_{1'}(\alpha)$,

$$\alpha \Vdash_{1'} \varphi[e] \quad \text{iff} \quad \alpha \Vdash_2 \varphi[e]. \tag{6.10}$$

Again, the important point is to demonstrate that for any $\alpha \in K$ and $e : \mathrm{Var} \rightarrow l_{1'}(\alpha)$,

$$\alpha \Vdash_2 P(x)[e] \quad \text{iff} \quad \alpha \Vdash_2 Q(x)[e].$$

(i) Let $e(x) = a_n$. Assume first that $\alpha \Vdash_2 P(a_n)$. By (6.10), (6.7) and (6.2), $\alpha \Vdash P(a)$ which implies that there is no element of the form q_a^α in $l_1(\alpha)$. Consequently, there is no element of the form $(q_a^\alpha)_n$ in $l_{1'}(\alpha)$. For all the elements $(q_a^\beta)_n$, where $\beta < \alpha$, $\alpha \Vdash_2 a_n = (q_a^\beta)_n$. For all the other elements $u \in l_{1'}(\alpha)$, $\alpha \Vdash_2 \neg(a_n = u)$. Moreover, for all $\delta > \alpha$ and $d \in l_{1'}(\delta) \setminus l_{1'}(\alpha)$, d never equals to a_n. Hence, $\alpha \Vdash_2 \forall y(a_n = y \vee \neg(a_n = y))$, i.e., $\alpha \Vdash_2 Q(a_n)$.

   Assume now that $\alpha \nVdash_2 P(a_n)$. By (6.10), (6.7) and (6.2), $\alpha \nVdash P(a)$, whence $q_a^\alpha \in l_1(\alpha)$. Consequently, $(q_a^\alpha)_n \in l_{1'}(\alpha)$. In α, $(q_a^\alpha)_n$ does not equal to a_n, but in the successor of α, $(q_a^\alpha)_n$ equals to a_n. Thus, $\alpha \nVdash_2 Q(a_n)$.

(ii) Let $e(x) = (q_a^\beta)_n$. Assume that $\alpha \Vdash_2 P((q_a^\beta)_n)$. By (6.10), (6.7) and (6.1), we have $\alpha \Vdash P(a)$ and $\beta < \alpha$. By (6.9), we obtain $\alpha \Vdash_2 (q_a^\beta)_n = a_n$. The fact that $\alpha \Vdash P(a)$ ensures us that there is not any element of the form $(q_a^\gamma)_n$, where $\gamma \geq \alpha$, in any universe of model $\mathcal{K}_2$. If $\gamma < \alpha$, then $\alpha \Vdash_2 (q_a^\gamma)_n = (q_a^\beta)_n$. For all the other elements $u \in \bigcup_{\gamma \geq \alpha} l_{1'}(\gamma)$, $\alpha \Vdash_2 \neg(u = (q_a^\beta)_n)$. Hence, $\alpha \Vdash_2 \forall y((q_a^\beta)_n = y \vee \neg((q_a^\beta)_n = y))$, i.e. $\alpha \Vdash_2 Q((q_a^\beta)_n)$.

Conversely, if $\alpha \not\Vdash_2 P((q_a^\beta)_n)$, then, by (6.10), (6.7) and (6.1), we have $\alpha \not\Vdash P(a)$. It means that $\alpha \not\Vdash_2 P(a_n)$ and by (i), $\alpha \not\Vdash_2 Q(a_n)$. If $\beta < \alpha$, then, by (6.9), $\alpha \Vdash_2 (q_a^\beta)_n = a_n$ and immediately, $\alpha \not\Vdash_2 Q((q_a^\beta)_n)$. If $\beta = \alpha$, then $\alpha \not\Vdash_2 (q_a^\beta)_n = a_n$, but in the successor γ of α, $\gamma \Vdash_2 (q_a^\beta)_n = a_n$. It implies that $\alpha \not\Vdash_2 (q_a^\beta)_n = a_n \vee \neg((q_a^\beta)_n = a_n)$, whence $\alpha \not\Vdash_2 Q((q_a^\beta)_n)$.

By an easy induction, we obtain for any $\alpha \in K$, $e : Var \to l_{1'}(\alpha)$ and φ in the language of M_1,

$$\alpha \Vdash_2 \varphi[e] \quad \text{iff} \quad \alpha \Vdash_2 f(\varphi)[e]. \tag{6.11}$$

Finally, we define a model $\mathcal{K}_3 = \langle K, \leq, l_{1'}, \Vdash_3 \rangle$ where $\Vdash_3$ is a mere restriction of $\Vdash_2$ to formulas in the language of $\mathrm{SUCC}^{\rightarrow}$. It means that for any formula ψ in the language of $\mathrm{SUCC}^{\rightarrow}$ and any $e : \mathrm{Var} \to l_{1'}(\alpha)$, the following equivalence holds:

$$\alpha \Vdash_2 \psi[e] \quad \text{iff} \quad \alpha \Vdash_3 \psi[e]. \tag{6.12}$$

We have to verify that $\mathcal{K}_3$ is a model of $\mathrm{SUCC}^{\rightarrow}$. It is easy to show that all the axioms are forced in every node. We focus on the most interesting axiom which is $\mathrm{Q3}^{\rightarrow}$. The only elements of $l_{1'}(\alpha)$ without a predecessor are o_0 and those of the form $(q_o^\beta)_0$. But all of them gradually merge with 0 in some node of every path in the tree. Thus, the antecedent of the implication in $\mathrm{Q3}^{\rightarrow}$ is not satisfied by them and axiom $\mathrm{Q3}^{\rightarrow}$ holds.

Note that $\mathcal{K}_3$ need not be a model of $\mathrm{SUCC}^{\vee}$ because there can be some element $(q_o^\beta)_0$ in the universes which is not equal to 0 and at the same time has no predecessor. This is why we cannot prove the undecidability of $\mathrm{SUCC}^{\vee}$ by the same construction as in this proof.

Suppose that there is a node $\alpha \in K$ and an evaluation $e : \mathrm{Var} \to l(\alpha)$ such that $\alpha \not\Vdash \eta[e]$. Equivalences (6.2), (6.8), (6.10), (6.11) and (6.12) show that $\alpha \not\Vdash_3 f(\eta)[e]$ which proves that $\mathrm{SUCC}^{\rightarrow} \nvdash f(\eta)$. □

To sum up, we proved that E and $\mathrm{SUCC}^{\rightarrow}$ are undecidable theories. Note that it stands in contrast to the decidability of classical versions of E and $\mathrm{SUCC}^{\rightarrow}$. In the proofs, we made use of the fact that the theory of one monadic predicate is undecidable and we demonstrated that $\mathrm{Thm}(\mathrm{M}_1)$ is m-reducible to $\mathrm{Thm}(\mathrm{E})$ and to $\mathrm{Thm}(\mathrm{SUCC}^{\rightarrow})$. We used the same computable function f for both reductions; f substitutes every occurence of $P(x)$ by $\forall y(x = y \vee \neg(x = y))$. At the end, we found out that the construction cannot be used for proving the undecidability of $\mathrm{SUCC}^{\vee}$. The decidability of $\mathrm{SUCC}^{\vee}$ is an open problem.

6.2 Undecidability of wDNO, wLO and LO

The aim of this section is to prove that wDNO, wLO and LO are undecidable theories. Analogously to the previous section, we demonstrate that $\mathrm{Thm}(\mathrm{M}_1)$ is m-reducible to Thm(wDNO), Thm(wLO) and Thm(LO), but the method is not so straight. The results are proved with a help of new theories $\mathrm{wDNO}^{<}$ and $\mathrm{wLO}^{d<}$.

To begin with, we should perceive that wDNO $\vdash \neg(x < y \vee y < x) \equiv x = y$; it follows from axioms AP and LO2. This fact immediately leads to the idea that we may interpret wDNO within its order fragment. Let wDNO$^<$ be the theory with language $\{<\}$ and axioms LO1, LO2, wLO3, DN1–DN3. We show that all the axioms of wDNO, where $=$ is interpreted as $\neg(x < y \vee y < x)$, are theorems of wDNO$^<$.

(E1) wDNO$^< \vdash \neg(x < x)$, thus wDNO$^< \vdash \neg(x < x \vee x < x)$.

(E2) wDNO$^< \vdash \neg(x < y \vee y < x) \rightarrow \neg(y < x \vee x < y)$.

(E3) wDNO$^< \vdash \neg(x < y \vee y < x) \ \&\ \neg(y < z \vee z < y) \rightarrow \neg(x < z \vee z < x)$, since if x was comparable to z, then, by wLO3, y would be comparable to x or z.

(E5) Suppose that $\neg(x_1 < y_1 \vee y_1 < x_1)$ and $\neg(x_2 < y_2 \vee y_2 < x_2)$ and $x_1 < x_2$. We want to prove that $y_1 < y_2$. Axiom wLO3 implies that $y_1 < x_2$ or $x_1 < y_1$, since $x_1 < x_2$. We assumed that y_1 is not comparable to x_1, which means that $y_1 < x_2$. By the similar argument, $x_1 < y_2$. Again, wLO3 entails that $y_1 < y_2$ or $x_1 < y_1$, since $x_1 < y_2$. But x_1 is incomparable to y_1, thus $y_1 < y_2$.

If we supposed $y_1 < y_2$ instead of $x_1 < x_2$ at the beginning, we would obtain $x_1 < x_2$. Hence, we proved wDNO$^< \vdash \neg(x_1 < y_1 \vee y_1 < x_1) \ \&\ \neg(x_2 < y_2 \vee y_2 < x_2) \rightarrow (x_1 < x_2 \equiv y_1 < y_2)$.

(AP) wDNO$^< \vdash \neg(x < y \vee y < x) \rightarrow \neg(x < y \vee y < x)$.

For convenience, let $x \# y$ denote $x < y \vee y < x$. We read $x \# y$ as x *is comparable to* y which indicates that x and y are comparable by means of $<$. However, such a reading implies that x is not comparable to x.

In the following paragraphs, we formulate and prove a theorem that demonstrates the undecidability of wDNO$^<$. It is clear that the undecidability of wDNO$^<$ immediately implies the undecidability of wDNO, since Thm(wDNO$^<$) can be m-reduced to Thm(wDNO) via the identity function. (Particularly, the preceding paragraphs showed that if any φ in the language $\{<\}$ is a theorem of wDNO, then φ is also a theorem of wDNO$^<$.)

First, we define a computable function g that will be proved to satisfy

$$\varphi \in \mathrm{Thm}(\mathrm{M}_1) \quad \text{iff} \quad g(\varphi) \in \mathrm{Thm}(\mathrm{wDNO}^<),$$

for every formula φ in the language of M_1.

Definition 6.5. *Let φ be a formula in the language of M_1. Then, $g(\varphi)$ is a formula in the language $\{<\}$ obtained by replacing every occurence of $P(x)$ in φ by $\forall y(x \# y \vee \neg(x \# y))$, for any variable x and certain $y \neq x$. We designate $\forall y(x \# y \vee \neg(x \# y))$ by $R(x)$.*

Theorem 6.6 ([Smo73b, pp. 126–128])**.** *For every formula η in the language of M_1, $\mathrm{M}_1 \vdash \eta$ iff* wDNO$^< \vdash g(\eta)$. *Hence,* wDNO$^<$ *is undecidable.*

Proof. The implication from left to right is easy and proved similarly as in Theorem 6.3. Assume that $\mathrm{M}_1 \nVdash \eta$. There exists a tree model $\mathcal{K}$ of M_1 with origin α_0 and infinite domains[6] and an evaluation $e : \mathrm{Var} \to l(\alpha_0)$ such that $\alpha_0 \nVdash \eta[e]$.

Now, we formally define a model $\mathcal{K}_1 = \langle K, \leq, l_1, \Vdash_1 \rangle$. The following lines may look quite complicated, but we shall accompany them with a more intelligible explanation.

$$
\begin{aligned}
l_1(\alpha_0) &= \{\, q^{\alpha_0}_{a,n} \,;\, (\alpha_0 \Vdash P(a) \text{ and } n = 0) \text{ or } (\alpha_0 \nVdash P(a) \text{ and } n \in \mathsf{Q}) \,\} \\
l_1(\beta) &= \{\, q^{\beta}_{a,\mathbf{n}n} \,;\, q^{\alpha}_{a,\mathbf{n}} \in l_1(\alpha) \text{ and } \beta \nVdash P(a) \text{ and } n \in \mathsf{Q} \,\} && (6.13)\\
&\cup \{\, q^{\beta}_{a,n} \,;\, a \in l(\beta) \text{ and } a \notin l(\alpha) \text{ and } \beta \nVdash P(a) \text{ and } n \in \mathsf{Q} \,\} && (6.14)\\
&\cup \{\, q^{\beta}_{a,\mathbf{n}0} \,;\, q^{\alpha}_{a,\mathbf{n}} \in l_1(\alpha) \text{ and } \beta \Vdash P(a) \,\} && (6.15)\\
&\cup \{\, q^{\beta}_{a,0} \,;\, a \in l(\beta) \text{ and } a \notin l(\alpha) \text{ and } \beta \Vdash P(a) \,\}, && (6.16)
\end{aligned}
$$

where β is a successor of α and $\mathbf{n}$ denotes a sequence of rationals. Furthermore, for all $\alpha \in K$, we define

$$\alpha \Vdash_1 P(q^{\alpha}_{a,\mathbf{n}}) \quad \text{iff} \quad \alpha \Vdash P(a).$$

Here is the explanation of the preceding lines. First, look at the universe $l_1(\alpha_0)$. For every $a \in l(\alpha_0)$ such that $\alpha_0 \Vdash P(a)$, there is one element $q^{\alpha_0}_{a,0}$ in $l_1(\alpha_0)$. For every $a \in l(\alpha_0)$ such that $\alpha_0 \nVdash P(a)$, there are incomparables $\{\, q^{\alpha_0}_{a,n} \,;\, n \in \mathsf{Q} \,\}$ in $l_1(\alpha_0)$, where a is identified with $q^{\alpha_0}_{a,0}$.

Now, suppose that the universe $l_1(\alpha)$ has been already defined and β is a successor of α. The domain of $l_1(\alpha)$ contains four types of elements:

(i) For every $q^{\alpha}_{a,\mathbf{n}} \in l_1(\alpha)$ such that $\beta \nVdash P(a)$, there are elements $\{\, q^{\beta}_{a,\mathbf{n}n} \,;\, n \in \mathsf{Q} \,\}$, where $q^{\alpha}_{a,\mathbf{n}}$ is identified with $q^{\beta}_{a,\mathbf{n}0}$. These elements are defined in (6.13).

(ii) If a is a new element of $l(\beta)$, i.e. $a \notin l(\alpha)$, and $\beta \nVdash P(a)$, then there are incomparables $\{\, q^{\beta}_{a,n} \,;\, n \in \mathsf{Q} \,\}$ in $l_1(\beta)$. These elements are defined in (6.14).

(iii) For every $q^{\alpha}_{a,\mathbf{n}} \in l_1(\alpha)$ such that $\beta \Vdash P(a)$, there is one element $q^{\beta}_{a,\mathbf{n}0}$ in $l_1(\beta)$, as it is defined in (6.15).

(iv) If a is a new element of $l(\beta)$, i.e. $a \notin l(\alpha)$, and $\beta \Vdash P(a)$, then there is one element $q^{\beta}_{a,0}$ in $l_1(\beta)$, as it is defined in (6.16).

Note that each $l_1(\alpha)$ contains only the elements with superscripted α. Every element $q^{\alpha}_{a,\mathbf{n}}$ is forced in node α of $\mathcal{K}_1$ if and only if a is forced in α of $\mathcal{K}$. $\mathbf{n}$ is a sequence of rationals and the length of $\mathbf{n}$ is the same as the length of the path from α_0 to α increased by 1.

For any formula φ in the language of M_1, any $\alpha \in K$ and any $e : \mathrm{Var} \to l(\alpha)$,

$$\alpha \Vdash_1 \varphi[e] \quad \text{iff} \quad \alpha \Vdash \varphi[e].$$

[6] If $l(\alpha_0)$ was finite, then we would choose one element $a \in l(\alpha_0)$ and add infinitely many copies, $\{\, q_n \,;\, n \in \mathsf{N} \,\}$, of a into $l(\alpha_0)$. We would define $\alpha \Vdash P(q_n)$ iff $\alpha \Vdash P(a)$, for every $\alpha \in K$ and $n \in \mathsf{N}$. Elements q_n could not be distinguished from a, thus we would get new $\mathcal{K}$ with infinite domains, but satisfying the same formulas as the original $\mathcal{K}$.

The next point is the definition of model $\mathcal{K}_2 = \langle K, \leq, l_1, \Vdash_2 \rangle$. First, we order every $l(\alpha)$ in a dense linear order ($<$) such that the persistence is not violated. It is possible, since the domains are infinite. Second, we define[7]

$$\begin{aligned} \alpha_0 \Vdash_2 q^{\alpha_0}_{a,n_1} < q^{\alpha_0}_{b,n_2} \quad & \text{iff} \quad a < b, \\ \beta \Vdash_2 q^{\beta}_{a,\mathbf{n_1}n_1} < q^{\beta}_{b,\mathbf{n_2}n_2} \quad & \text{iff} \quad a < b \text{ or } (a = b \text{ and } \mathbf{n_1} \prec \mathbf{n_2}), \end{aligned}$$

for origin α_0 and a successor β of α. Again, there is a short explanation. Elements $q^{\alpha_0}_{a,n_1}$ and $q^{\alpha_0}_{b,n_2}$ are incomparable iff $a = b$. If $q^{\alpha}_{a,\mathbf{n}n_1}$ and $q^{\alpha}_{a,\mathbf{n}n_2}$ are incomparable, then $q^{\beta}_{a,\mathbf{n}n_1n_3}$ and $q^{\beta}_{a,\mathbf{n}n_2n_4}$ are ordered according to numbers n_1 and n_2. Further, if $\beta \nVdash P(a)$, then $\{\, q^{\beta}_{a,\mathbf{n}n_1n} \,;\, n \in \mathsf{Q} \,\}$, $\{\, q^{\beta}_{a,\mathbf{n}n_2n} \,;\, n \in \mathsf{Q} \,\}$, ... are blocks of new incomparables.

Model $\mathcal{K}_3 = \langle K, \leq, l_1, \Vdash_3 \rangle$ is, as usual, obtained by restricting $\Vdash_2$ to formulas in the language $\{<\}$. The following two equivalences can be easily proved:

$$\begin{aligned} \alpha \Vdash_1 \varphi[e] \quad & \text{iff} \quad \alpha \Vdash_2 \varphi[e], \\ \alpha \Vdash_2 \psi[e] \quad & \text{iff} \quad \alpha \Vdash_3 \psi[e], \end{aligned}$$

for any φ in the language of M_1, ψ in the language of $\mathrm{wDNO}^{<}$, $\alpha \in K$ and $e : \mathrm{Var} \to l_1(\alpha)$. The essential point is the proof of

$$\alpha \Vdash_2 \varphi[e] \quad \text{iff} \quad \alpha \Vdash_2 g(\varphi)[e], \tag{6.17}$$

for any φ in the language of M_1, $\alpha \in K$ and $e : \mathrm{Var} \to l_1(\alpha)$. It follows from the construction that if $a \neq b$, then $q^{\alpha}_{a,\mathbf{n_1}}$ and $q^{\alpha}_{b,\mathbf{n_2}}$ are always comparable, i.e., $\alpha \Vdash_2 q^{\alpha}_{a,\mathbf{n_1}} \# q^{\alpha}_{b,\mathbf{n_2}}$. In contrast, elements $q^{\alpha}_{a,\mathbf{n_1}}$ and $q^{\alpha}_{a,\mathbf{n_2}}$ need not be comparable, but if $\alpha \Vdash P(a)$ and $\mathbf{n_1} \neq \mathbf{n_2}$, they are. If $\alpha \nVdash P(a)$ and $n_1 \neq n_2$, then for any $q^{\alpha}_{a,\mathbf{n}n_1}$ and $q^{\alpha}_{a,\mathbf{n}n_2}$, $\alpha \nVdash_2 q^{\alpha}_{a,\mathbf{n}n_1} \# q^{\alpha}_{a,\mathbf{n}n_2}$, but in the successor β of α, $\beta \Vdash_2 q^{\beta}_{a,\mathbf{n}n_1 0} \# q^{\beta}_{a,\mathbf{n}n_2 0}$. The last case is that for any $q^{\alpha}_{a,\mathbf{n}}$ and α, $\alpha \nVdash_2 q^{\alpha}_{a,\mathbf{n}} \# q^{\alpha}_{a,\mathbf{n}}$ and thus $\alpha \Vdash_2 \neg(q^{\alpha}_{a,\mathbf{n}} \# q^{\alpha}_{a,\mathbf{n}})$. To sum up, for any $\alpha \in K$ and $e : \mathrm{Var} \to l_1(\alpha)$,

$$\alpha \Vdash_2 P(x)[e] \quad \text{iff} \quad \alpha \Vdash_2 R(x)[e],$$

and consequently, (6.17) holds.

The very last point of the proof is to verify that $\mathcal{K}_3$ is a model of $\mathrm{wDNO}^{<}$. Axioms DN1–DN3 are satisfied, since the elements of $l(\alpha)$ are ordered in a dense linear order (i.e., for $a, b \in l(\alpha)$, there are $c, d, e \in l(\alpha)$ such that $c < a$, $a < d$ and if $a < b$, then $a < e < b$) and consequently, for every $q^{\alpha}_{a,\mathbf{n_1}}, q^{\alpha}_{b,\mathbf{n_2}} \in l_1(\alpha)$, there are $q^{\alpha}_{c,\mathbf{n_3}}, q^{\alpha}_{d,\mathbf{n_4}}, q^{\alpha}_{e,\mathbf{n_5}} \in l_1(\alpha)$ such that $\alpha \Vdash_3 q^{\alpha}_{c,\mathbf{n_3}} < q^{\alpha}_{a,\mathbf{n_1}}$, $\alpha \Vdash_3 q^{\alpha}_{a,\mathbf{n_1}} < q^{\alpha}_{d,\mathbf{n_4}}$ and if $a \neq b$ and $\alpha \Vdash_3 q^{\alpha}_{a,\mathbf{n_1}} < q^{\alpha}_{b,\mathbf{n_2}}$, then $\alpha \Vdash_3 q^{\alpha}_{a,\mathbf{n_1}} < q^{\alpha}_{e,\mathbf{n_5}} < q^{\alpha}_{b,\mathbf{n_2}}$. If $\alpha \Vdash_3 q^{\alpha}_{a,\mathbf{n_1}} < q^{\alpha}_{a,\mathbf{n_2}}$, then there also exists $q^{\alpha}_{a,\mathbf{n}}$ between them: if p is the first place where $\mathbf{n_1}$ and $\mathbf{n_2}$ differ, then $p(\mathbf{n_1}) < p(\mathbf{n_2})$[8] and we put $\mathbf{n}$ to be $\mathbf{n_1}$ with changed number at position p — $p(\mathbf{n_1})$ is replaced by $\frac{p(\mathbf{n_1})+p(\mathbf{n_2})}{2}$.

The verification of the other axioms of $\mathrm{wDNO}^{<}$ is left to the reader. We have shown that $\mathcal{K}_3$ is a model of $\mathrm{DNO}^{<}$ such that $\alpha_0 \nVdash_3 \eta$. Hence, $\mathrm{DNO}^{<} \nvdash \eta$. □

[7] $\mathbf{n_1} \prec \mathbf{n_2}$ means that $\mathbf{n_1}$ is less than $\mathbf{n_2}$ at the first place they differ.

[8] $p(\mathbf{n})$ denotes the rational that is at position p in $\mathbf{n}$.

Corollary 6.7. wDNO *is undecidable.*

Corollary 6.8. wLO *is undecidable.*

Proof. If wLO was decidable, we could decide whether any formula of the form DN1 & DN2 & DN3 $\to \varphi$ is a theorem of wLO. By the deduction theorem, wDNO would be decidable. $\square$

We finish this section by proving the undecidability of LO. It will be done with a help of a new theory $\mathrm{wLO}^{d<}$ which is defined as follows. Theory $\mathrm{wLO}^{d<}$ has the language $\{<\}$ and the axioms LO1, LO2, wLO3 and $\forall x \forall y (x < y \ \vee \ \neg(x < y))$. First, we demonstrate that $\mathrm{wLO}^{d<}$ is undecidable by showing, as usual, that $\mathrm{Thm}(\mathrm{M}_1)$ is m-reducible to $\mathrm{Thm}(\mathrm{wLO}^{d<})$ via h, where h is defined as follows:

Definition 6.9. *Let φ be a formula in the language of M_1. Then $h(\varphi)$ is a formula in the language $\{<\}$ obtained by replacing every occurence of $P(x)$ in φ by $\forall y \exists z (x < y \to x < z \ \& \ z < y)$, for any variable x and certain $y, z \neq x$. We designate $\forall y \exists z (x < y \to x < z \ \& \ z < y)$ by $S(x)$.*

Proposition 6.10 ([Smo73b, pp. 128–129])**.** *For every formula η in the language of M_1, $\mathrm{M}_1 \vdash \eta$ iff $\mathrm{LO}^{d<} \vdash h(\eta)$. Hence, $\mathrm{wLO}^{d<}$ is undecidable.*

Proof. The implication from left to right has a similar expanation as in Theorem 6.3. Conversely, assume that $\mathcal{K} = \langle K, \leq, l, \Vdash \rangle$ is a tree model of M_1 with origin α_0 and at most countable universes[9] such that $\alpha_0 \nVdash \eta[e]$, for some $e : \mathrm{Var} \to l(\alpha_0)$.

Model $\mathcal{K}_1 = \langle K, \leq, l_1, \Vdash_1 \rangle$ is defined as follows:

$$
\begin{array}{rcl}
l_1(\alpha_0) & = & l(\alpha_0) \cup \{\, q^{\alpha_0}_{a,n} \,;\, \alpha_0 \Vdash P(a),\ n \in \mathsf{Q} \,\} \\
l_1(\beta) & = & l_1(\alpha) \cup \{\, q^{\beta}_{a,n} \,;\, \beta \Vdash P(a),\ \alpha \nVdash P(a),\ n \in \mathsf{Q} \,\} \\
\alpha \Vdash_1 P(a) & \text{iff} & \alpha \Vdash P(a), \\
\alpha & \Vdash_1 & P(q^{\gamma}_{a,n}),
\end{array}
$$

for any nodes α and a successor β of α, $\gamma \leq \alpha$ and any original element a. Let the elements of the domains of $\mathcal{K}$ be indexed by natural numbers (a_0,$a_1 \ldots \in l(\alpha)$). There is the definition of $\mathcal{K}_2 = \langle K, \leq, l_1, \Vdash_2 \rangle$:

$$
\begin{array}{rcl}
\alpha \Vdash_2 P(a) & \text{iff} & \alpha \Vdash_1 P(a), \\
\alpha \Vdash_2 P(q^{\alpha}_{a,n}) & \text{iff} & \alpha \Vdash_1 P(q^{\alpha}_{a,n}), \\
\alpha \Vdash_2 a_i < a_j & \text{iff} & i < j, \\
\alpha \Vdash_2 q^{\gamma}_{a_i,n} < q^{\delta}_{a_j,m} & \text{iff} & i < j \text{ or } (i = j \text{ and } n < m), \\
\alpha \Vdash_2 a_i < q^{\gamma}_{a_j,n} & \text{iff} & i \leq j, \\
\alpha \Vdash_2 q^{\gamma}_{a_i,n} < a_j & \text{iff} & i < j,
\end{array}
$$

[9] Löwenheim-Skolem theorem holds for intuitionistic logic, thus it is sure that such model exists.

for any nodes α and $\gamma,\delta \leq \alpha$, any original elements a, a_i, a_j, any non-original elements q and any $n, m \in \mathsf{Q}$, $i, j \in \mathsf{N}$. $\mathcal{K}_3 = \langle K, \leq, l_1, \Vdash_3 \rangle$ is obtained, again, by a mere restriction of $\Vdash_2$ to the formulas in the language $\{<\}$.

Informally, we start with model $\mathcal{K}$, order the elements of domains and if for some $a_i \in l(\alpha)$, $\alpha \Vdash P(a_i)$, then we add a block of ordered rationals between a_i and a_{i+1}.

As in the previous proofs, equations similar to (6.2), (6.3), (6.5) and (6.6) can be proved. Particularly,

$$\alpha \Vdash_2 P(x)[e] \quad \text{iff} \quad \alpha \Vdash_2 S(x)[e],$$

since if $\alpha \nVdash_2 P(a_i)$, then there is not any block of rationals between a_i and a_{i+1}, thus there is not any element between a_i and a_{i+1} and $\alpha \nVdash_2 S(a_i)$. Finally, $\mathcal{K}_3 \nVdash h(\eta)$ and $\mathrm{LO}^{d<} \nvdash h(\eta)$. $\square$

Proposition 6.11. LO *is undecidable.*

Proof. We make use of the previous proposition. Note that $\mathrm{LO} \vdash \neg(x < y \lor y < x) \equiv x = y$, because LO3 is intuitionistically stronger than $\neg(x < y \lor y < x) \equiv x = y$. It enables us to interpret LO within $\mathrm{LO}^{d<}$. At the beginning of section 6.2, p. 54, we demonstrated that DNO can be interpreted in $\mathrm{wDNO}^{<}$. We showed that $\mathrm{wDNO}^{<}$ proves axioms E1–E3, E5, AP, where $x = y$ is $\neg(x < y \lor y < x)$, but in the proof, we used only axioms LO1, LO2 and wLO3. Hence, $\mathrm{wLO}^{d<}$ proves E1–E3, E5.

We need to demonstrate that even LO3 is a theorem of $\mathrm{wLO}^{d<}$.

$$\mathrm{wLO}^{d<} \;\vdash\; (y < x \lor \neg(y < x)) \;\&\; (x < y \lor \neg(x < y))$$

and by distributivity,

$$\begin{aligned}\mathrm{wLO}^{d<} \;\vdash\; & (y < x \;\&\; x < y) \lor (y < x \;\&\; \neg(x < y)) \lor \\ & (\neg(y < x) \;\&\; x < y)) \lor (\neg(y < x) \;\&\; \neg(x < y)).\end{aligned}$$

But each disjunct of the previous formula implies

$$x < y \lor y < x \lor (\neg(x < y) \;\&\; \neg(y < x)),$$

thus we have

$$\mathrm{wLO}^{d<} \;\vdash\; x < y \lor y < x \lor \neg(x < y \lor y < x),$$

which is trichotomy translated to the language $\{<\}$.

The fact that LO can be interpreted in $\mathrm{wLO}^{d<}$ implies that for any formula φ in the language $\{<\}$, if $\mathrm{LO} \vdash \varphi$, then $\mathrm{wLO}^{d<} \vdash \varphi$. It is clear that

$$\mathrm{LO3} \;\vdash\; \mathrm{wLO3} \;\&\; \forall x \forall y (x < y \lor \neg(x < y)).$$

Hence, if $\mathrm{wLO}^{d<} \vdash \varphi$, then $\mathrm{LO} \vdash \varphi$. We have that $\mathrm{Thm}(\mathrm{wLO}^{d<})$ is m-reducible to $\mathrm{Thm}(\mathrm{LO})$ and consequently, LO is undecidable. $\square$

6.3 Minor (un)decidability results

In this short section, we mention the (un)decidability results of theories which have not been dealt with in this chapter. The following propositions are easy consequences of the previous parts and classical results.

To begin with, we look into theories DNO and RNA. Corollaries 3.6 and 3.7 showed that both theories are, in fact, classical theories. The classical decidability results for DNO and RNA are well known, thus we have the following proposition.

Proposition 6.12. DNO *and* RNA *are decidable.*

Other theories of our interest are two versions of Robinson arithmetic. Classical Robinson arithmetic is undecidable, but neither $\mathrm{Q}^{\vee}$ nor $\mathrm{Q}^{\rightarrow}$ coincides with it. Anyway, the Gödel translation will help us show that Qs are undecidable.

Proposition 6.13. $\mathrm{Q}^{\vee}$ *and* $\mathrm{Q}^{\rightarrow}$ *are undecidable.*

Proof. Suppose that $\mathrm{Q} \vdash_{\mathrm{c}} \varphi$. Then, by Theorem 2.21, $\mathrm{Q}^{\mathrm{g}} \vdash_{\mathrm{i}} \varphi^{\mathrm{g}}$. But every formula that is translated by the Gödel translation is intuitionistically weaker than the original formula, whence $\mathrm{Q} \vdash_{\mathrm{i}} \gamma$, for every $\gamma \in \mathrm{Q}^{\mathrm{g}}$. Consequently, $\mathrm{Q} \vdash_{\mathrm{i}} \varphi^{\mathrm{g}}$. Conversely, if $\mathrm{Q} \vdash_{\mathrm{i}} \varphi^{\mathrm{g}}$, then $\mathrm{Q} \vdash_{\mathrm{c}} \varphi^{\mathrm{g}}$ and $\mathrm{Q} \vdash_{\mathrm{c}} \varphi$, since φ^{g} and φ are classically equivalent. To sum up, $\mathrm{Thm}^{\mathrm{c}}(\mathrm{Q})$ is m-reducible to $\mathrm{Thm}^{\mathrm{i}}(\mathrm{Q})$. Hence, Q is undecidable. $\square$

An interesting fact is that we cannot prove Proposition 6.13 by mere mimicking the classical proof presented in [Šve02, s. 4.4], since we cannot prove the Σ-completeness of Q. Particularly,

Lemma 6.14. $\mathrm{Q} \nvdash \forall x(x \leq \overline{n} \rightarrow x = 0 \vee \ldots \vee x = \overline{n})$*, for any numeral* $\overline{n}$*.*

Proof. Look at the model of Q that was constructed in the proof of Proposition 3.21. We show that $\alpha \not\Vdash (x \leq \overline{n} \rightarrow x = 0 \vee \ldots \vee x = \overline{n})[e]$. Let $\overline{n}$ be $\mathrm{S}(\mathrm{S}(0))$, i.e. a_2, and $e(x)$ be b_1. It is clear that $\alpha \Vdash b_1 \leq a_2$, since $\alpha \Vdash b_1 + b_1 = a_2$. But $\alpha \not\Vdash b_1 = 0 \vee \ldots \vee b_1 = \overline{n}$, because $\alpha \not\Vdash b_1 = a_1$. $\square$

The last theory that we inspect is wRNA. It is closely related to wDNO, so we try to make use of the undecidability of wDNO for showing that wRNA is undecidable. The following lemma is clear.

Lemma 6.15. *If* wRNA *is a conservative extension of* wDNO*, then* wRNA *is undecidable.*

Proof. Corollary 6.7 states that wDNO is undecidable. If wRNA was decidable, then for any formula φ in the language of wDNO, we could decide whether wRNA proves φ or not. But $\mathrm{wRNA} \vdash \varphi$ iff $\mathrm{wDNO} \vdash \varphi$, thus wDNO would be decidable. $\square$

Proposition 6.16. wRNA *is a conservative extension of* wDNO*. Thus,* wRNA *is undecidable.*

Proof. Suppose that wRNA is not a conservative extension of wDNO. Then, there exists a formula φ in the language of wDNO such that wRNA $\vdash \varphi$, but wDNO $\nvdash \varphi$. It means that φ is valid in every model of wRNA and there exists a model of wDNO in which φ is not valid. Let $\mathcal{K}$ be a one-node model of wRNA where the elements of the universe are all rationals and nothing else. 0, 1, + and $<$ are realized as in the rational numbers. Now, φ is valid in $\mathcal{K}$, but should not be valid in some model of wDNO. Nevertheless, every model of wDNO contains in all its universes a block of rationals with $<$ realized normally, whence φ must not claim anything about the rationals in the block, otherwise φ would be valid in every model of wDNO. But in the language $\{=, <\}$, we are not able to tell the block of rationals from other elements, thus φ always relates to the rationals in the block. φ is valid in every model of wDNO. □

To sum up this chapter, we obtained a few interesting decidability results. All of the theories in our scope, except for DNO, RNA and SUCC$^{\vee}$, were proved to be undecidable. We presented a method useful for demonstrating that an intuitionistic theory is undecidable—we reduced the monadic predicate calculus with one predicate symbol to theories E, SUCC$^{\rightarrow}$, wDNO$^{<}$ and LO$^{d<}$. Theories DNO and RNA are decidable, since they coincide with their classical extensions which are decidable. We did not manage to show any decidability result for SUCC$^{\vee}$.

When we look at the results from the perspective of searching for a "good" intuitionistic theory, we must admit that they are not very positive. The only decidable theories that we have are, in fact, classical theories. The slight weakening of axioms of DNO and RNA swaps decidability into undecidability and in the case of SUCC$^{\rightarrow}$, the underlying logic changes the decidability result. Anyway, there is a chance that it is the unexplored theory SUCC$^{\vee}$ that yields the example of decidable intuitionistic theory in demand.

7

Conclusion

The main aim of this thesis was exploring four properties of the theories in our scope. The following table presents the results. (For each theory T, there are the answers of the questions “Does T coincide with its classical extension?”, “Is T saturated?”, “Does T satisfy De Jongh’s theorem?” and “Is T decidable?”.)

Theory	**Coincides**	**Saturated**	**De Jongh’s**	**Decidable**
E	no	no	yes	no
LO	no	no	yes	no
wLO	no	no	yes	no
DNO	yes	no	no	yes
wDNO	no	no	no	no
$\mathrm{SUCC}^{\vee}$	no	yes	no	?
$\mathrm{SUCC}^{\rightarrow}$	no	yes	no	no
$\mathrm{Q}^{\vee}$	no	yes	yes	no
$\mathrm{Q}^{\rightarrow}$	no	yes	yes	no
RNA	yes	no	no	yes
wRNA	no	no	no	no

Let us interpret the table and add some further information. Theories E, LO and wLO have exactly the same properties, yet, there are differences in the details. Atomic formulas are decidable in LO, whereas they are not decidable either in E or in wLO. Similarly, two versions of SUCC and Q differ by the decidability of formulas $x = \mathrm{S}^n(0)$—in the weaker versions of the theories these formulas are not decidable; in the stronger versions they are decidable. The other pairs of theories with the same properties are DNO – RNA and wDNO – wRNA. Both DNO and RNA coincide with their classical extensions and this fact affects the other properties. Theories wDNO and wRNA avoid the coincidence, but at the same time lose their decidability.

Now, we try to arrive at rather more general conclusions. One of the questions was how the different (but classically equivalent) formulations of an axiom influences the properties of the theories. The answer is not clear, since the effect can be quite dramatic, but also very slight. Theories DNO and RNA are dramatically changed by weakening of axiom LO3, because they stop trivializing intuitionistic logic, lose their classical properties and by this become more constructive. In contrast, LO is affected by the same reformulation of LO3 only slightly. It seems that also the formulation of axiom Q3 as implication instead of disjunction does not have any enormous consequences. Although, we would immediately change our opinion if $\mathrm{SUCC}^{\vee}$ was decidable.

In our considerations, we not only investigated the properties, but we also expressed, implicitly or explicitly, which properties we prefer. We hoped to find a "genuine" intuitionistic theory, i.e., a theory that does not coincide with its classical extension, is saturated, decidable and satisfies De Jongh's theorem. Unfortunately, none of the theories in scope is "genuine". Anyway, there are some remarks that could help in constructing such a theory.

If we have a theory that trivializes intuitionistic logic, we may weaken its axioms so that it proves less classical tautologies. This method was successfully used for theories DNO and RNA, but there is always a danger that by the weakening of axioms, we lose the concept that the theory should describe. If we have a theory that is not saturated and we do not want to change its axioms, we may try adding some constants into the language of the theory (as we suggested for theory RNA). However, such an extension of the language may negatively affect the other properties. To satisfy De Jongh's theorem, a theory should have infinitely many models such that every two of them differ in forcing a sentence and a basal structure that can be put into all the nodes below these models. As for the decidability, we cannot imagine any criteria, since all the theories in scope that do not coincide with their classical versions are undecidable.

In the introduction, we declared that we are not only interested in the properties of the theories, but also in the methods that lead to the results. The main methods presented in this thesis were invented (or at least strongly inspired) by C. Smorynski and D. de Jongh. The methods are the following: To prove that a theory coincides with its classical extension, we showed a criterion (see Theorem 3.5) that made use of the fact that the classical extension is model complete. As for the saturation results, we demonstrated a method based on the Aczel slash (see Corollary 4.10). De Jongh's theorem was investigated by modifying the original proof for Heyting arithmetic (see Proposition 5.3) and by the considerations of the form of models (see Propositions 5.7 and 5.8). Finally, for showing the undecidability of the theories, we have a method based on the reduction of the theory of one monadic predicate symbol (see Theorems 6.3, 6.4 and 6.6).

An unsolved problem is the decidability of $\mathrm{SUCC}^{\vee}$. It is not possible to use the same method that proved the undecidability of $\mathrm{SUCC}^{\rightarrow}$, but at the same time, we did not manage to eliminate the quantifiers and prove that it is decidable. Apart from solving this problem, a future work could investigate some other theories or take a theory and try to change the axioms and the language in order to obtain the theory with "positive" intuitionistic properties.

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