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DIPLOMOVÁ PRÁCE

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Symbolické reprezentace kompaktních prostorů

Katedra algebry

Vedoucí diplomové práce: prof. RNDr. Petr Kůrka, CSc.
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Abstrakt: Práce se zabývá reprezentací čísel pomocí möbiovských číselných systémů. Tyto systémy reprezentují body pomocí posloupností Möbiových transformací. V práci se věnujeme převážně reprezentacím jednotkové kružnice (které jsou ekvivalentní reprezentacím množiny $\mathbb{R} \cup \{\infty\}$).

Zaměřujeme se především na vylepšování již známých nástrojů pro dokazovaní, že daný posun je möbiovským číselným systémem pro daný möbiovský iterativní systém. Dále studujeme otázku, jak charakterizovat iterativní systémy, pro které existuje posun tvořící möbiovský číselný systém, a naopak, jak popsat posuny, pro které lze najít iterativní systém, že výsledná dvojice je möbiovský číselný systém. Úplnou charakterizaci se nám nepodařilo najít, avšak nabízíme několik pozitivních i negativních částečných výsledků. Krátce se také věnujeme otázce, kdy je daný möbiovský číselný systém sofickým posunem.

Klíčová slova: Möbiova transformace, číselný systém, posun

Title: Symbolic representations of compact spaces

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Abstract: The thesis concerns itself with Möbius number systems. These systems represent points using sequences of Möbius transformations. We are mainly interested in representing the unit circle (which is equivalent to representing $\mathbb{R} \cup \{\infty\}$).

The main aim of the thesis is to improve already known tools for proving that a given subshift–iterative system pair is in fact a Möbius number system. We also study the existence problem: How to describe iterative systems resp. subshifts for which there exists a subshift resp. iterative system such that the resulting pair forms a Möbius number system. While we were unable to provide a complete answer to this question, we present both positive and negative partial results.

As Möbius number systems are also subshifts, we can ask when a given Möbius number system is sofic. We give this problem a short treatment at the end of our thesis.

Keywords: Möbius transformation, numeral system, subshift

Introduction

Numeral systems are recipes for expressing numbers in symbols. The most common are positional systems (usually with base ten, two, eight or sixteen). However, these systems are by no means the only possibility. Various historical systems used different approaches (consider for example the Roman numerals). The reason for positional systems' eventual dominance is the ease with which we can perform basic arithmetic operations on numbers by manipulating their symbolic representations in a positional system (compared to the tedious task that is arithmetic done in, say, Roman numerals).

Modern numeration theorists typically study positional systems with real base (such as the golden mean or -2) or various modifications of continued fractions. Contemporary numeration theory has connections to various other fields, namely the study of fractals and tilings, symbolic dynamics, ergodic theory, computability theory and even cryptography.

In this thesis, we study Möbius number systems as introduced in [6]. A Möbius number system represents numbers as sequences of Möbius transformations obtained by composing a finite starting set of Möbius transformations.

Möbius number systems display complicated dynamical properties and have connections to other kinds of numeral systems. In particular, Möbius number systems can generalize continued fractions (see [9]).

The thesis is organized as follows: First, Chapter 1 prepares the ground for our study, introducing known results on disc preserving Möbius transformations.

In Chapter 2, we present conditions for deciding whether a given sequence of Möbius transformations represents a number. Originally, representation was defined by convergence of measures, but it turns out that there are several other definitions. We collect these definitions in Section 2.2.

Chapter 3 introduces Möbius number systems and contains most of this thesis' results: In Section 3.2 we offer tools to prove (or disprove) that a given iterative system with a given subshift is a Möbius number system. While these tools are far from universal, they are sufficient for practical purposes, as we show on three example number systems.

In Section 3.4, we look at one particular tool, the numbers $Q_n(\mathcal{W}, \Sigma)$, and ask what can these numbers tell us about Möbius number systems. It turns out that if $Q_n(\mathcal{W}, \Sigma)$ are small enough then the corresponding shift $\Sigma_{\mathcal{W}}$ either is not a Möbius number system or is rather badly behaved.

An interesting question is whether a given iterative system admits any Möbius number system at all. It would be quite useful to have a complete characterization of such systems, unfortunately there is still a gap between the sufficient and the necessary conditions that are available. In Section 3.5 we offer a slight improvement of an already known sufficient condition and observe a new necessary condition.

In Section 3.6, we present the findings of a computer experiment based on the theory of Section 3.5. The results of this experiment suggest that the set of Möbius iterative systems admitting a Möbius number system is, up to a small error, equal to the complement of the set of iterative systems having nontrivial inward set. We conjecture that this is true in general, but can not offer a proof.

In Section 3.7 we consider the question "Which subshifts can be Möbius number systems for a suitable Möbius iterative system?" While we don't know the full answer, we show that there is a nontrivial class of subshifts that can not be Möbius number systems.

Proposition 5 in [9] offers a sufficient condition for a number system to be a subshift of finite type. We take off in a similar direction in Section 3.8 and conclude Chapter 3 by giving conditions for a Möbius number system to be sofic. In particular, under some reasonable assumptions on the system $\Sigma_{\mathcal{W}}$, we have a sufficient and necessary condition for $\Sigma_{\mathcal{W}}$ to be sofic.

Finally, in the Appendix one can find various proofs of results belonging in the folklore of the theory of Möbius transformations, symbolic dynamics or (in one case) the theory of measure.

Chapter 1

Preliminaries

1.1 Metric spaces and words

Denote by $\mathbb{T}$ the unit circle and by $\mathbb{D}$ the closed unit disc in the complex plane. For $x, y \in \mathbb{T}$, $x \neq y$ denote by (x, y) resp. $[x, y]$ the open resp. closed interval obtained by going from x to y along $\mathbb{T}$ in the positive (counterclockwise) direction. To make notation more convenient, we define the sum $x + l$ for $x \in \mathbb{T}$ and $l \in \mathbb{R}$ as the point on $\mathbb{T}$ whose argument is equal to $l + \arg x$ modulo 2π.

Let A be a finite alphabet. Any sequence of elements of A is a *word* over A. Let λ be the empty word. Denote by $A^\star$ the monoid of all finite words over A, by A^+ the set $A^\star \setminus \{\lambda\}$ and by A^ω the set of all one-sided infinite words over A. Let $|w|$ denote the length of the word w. If n is finite, let A^n be the set of all words over A of length precisely n. We use the notation $w = w_0 w_1 w_2 \cdots$ and $w_{[i,j]} = w_i w_{i+1} \cdots w_j$. When u is a finite word and v any word we can define the *concatenation* of u and v as $uv = u_0 u_1 \ldots u_{|u|-1} v_0 v_1 \ldots$

Let $v \in A^\star$ be a word of length n. Then we write $[v] = \{w \in A^\omega : w_{[0,n-1]} = v\}$ and call the resulting subset of A^ω the *cylinder* of v. A word u is a *factor* of a word v if there exist i, j such that $u = v_{[i,j]}$.

Let X be a metric space. We denote by ρ the metric function of X, by $\mathrm{Int}(V)$ the interior of the set V and by $B_r(x)$ the open ball of radius r centered at x. If I is an interval, denote by $|I|$ the length of I.

We equip $\mathbb{C}$ with the metric $\rho(x, y) = |x-y|$ and $\mathbb{T}$ with the circle distance metric (i.e. metric measuring distances along the circle). The shift space A^ω of one-sided infinite words comes equipped with the metric $\rho(u, v) = \max\left(\{2^{-k} : u_k \neq v_k\} \cup \{0\}\right)$. It is easy to see that the topology of A^ω is the product topology. A *subshift* $\Sigma \subseteq A^\omega$ is a set that is both topologically closed and invariant under the *shift map* $\sigma(w)_i = w_{i+1}$ (i.e. $\sigma(\Sigma) \subseteq \Sigma$).

As shown in [11, pages 5 and 179], subshifts of A^ω are precisely the subsets of A^ω that can be defined by some set of forbidden factors. More precisely, Σ is a subshift iff there exists $F \subseteq A^+$ such that

$$\Sigma = \{w \in A^\omega : \forall v \in F,\ v \text{ is not a factor of } w\}.$$

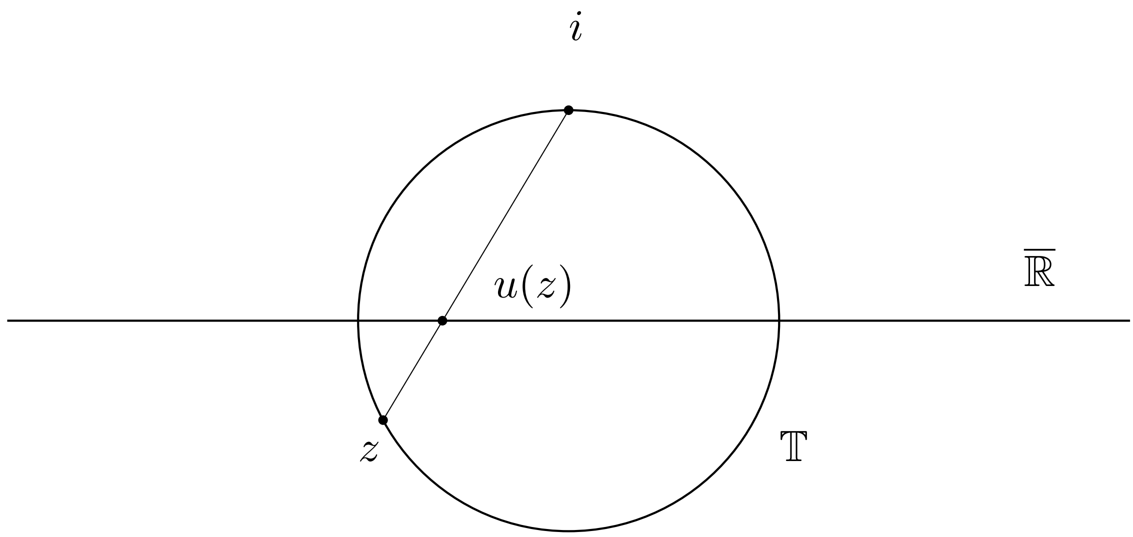

Figure 1.1: The stereographic projection of $\mathbb{T}$ onto $\overline{\mathbb{R}}$

We are going to occasionally define shifts using some such set of forbidden factors.

The *language of a subshift* $\mathcal{L}(\Sigma)$ is the set of all the words $v \in A^\star$ for which there exists $w \in \Sigma$ such that v is a factor of w. See [11] for a more detailed treatment of this topic.

We will mainly consider symbolic representations of $\mathbb{T}$, although representations of the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ will make an appearance as well. Note that $\overline{\mathbb{R}}$ is homeomorphic to $\mathbb{T}$ via the stereographic projection (see Figure 1.1):

$$u : \mathbb{T} \to \overline{\mathbb{R}}, \quad u : z \mapsto \frac{-iz+1}{z-i}.$$

Therefore, as long as we are not interested in arithmetics, representing $\mathbb{T}$ is equivalent to representing the extended real line.

1.2 Möbius transformations

A Möbius transformation (MT for short) of the complex sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is any map of the form

$$F : z \mapsto \frac{az+b}{cz+d}$$

where $(a,b),(c,d)$ are linearly independent vectors from $\mathbb{C}^2$.

Note that the stereographic projection u as defined above is actually a Möbius transformation. Therefore, if we represent $\mathbb{T}$ using the system $\{F_a : a \in A\}$ of MTs, we can represent $\overline{\mathbb{R}}$ in the same way with the system $\{u \circ F_a \circ u^{-1} : a \in A\}$.

To every regular 2×2 complex matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we can associate the MT defined by $F_A(z) = \frac{az+b}{cz+d}$. While the map $A \mapsto F_A$ is surjective, every MT has many preimages: if A is a matrix for F then so is cA for any $c \in \mathbb{C}$, $c \neq 0$. Even normalizing the matrices by demanding $\det A = 1$ is not enough, as it leaves two preimages A and $-A$ for each F.

This ambiguity is, however, a small price to pay: An easy calculation shows that composition of MTs corresponds to multiplying their respective matrices: $F_A \circ F_B = F_{A \cdot B}$. This is why we will often think of MTs as of matrices. It follows that the set of all MTs together with the operation of composition is a group (isomorphic to $SL(2, \mathbb{C})/\{E, -E\}$). In particular, MTs are bijective on $\mathbb{C} \cup \{\infty\}$.

Usually, we will consider *disc preserving* Möbius transformations, i.e. transformations that map $\mathbb{D}$ onto itself. Obviously, disc preserving transformations form a subgroup of the group of all MTs. It turns out that F is disc preserving iff it has the form

$$F = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$$

with the normalizing condition $|\alpha|^2 - |\beta|^2 = 1$. We will provide the proof in the next chapter as Lemma 3.

The geometrical theory of MTs is quite rich and has a strong link to hyperbolic geometry (see [4]). In this thesis, we will need only a handful of basic fragments of this theory. We will use the fact that MTs take circles and lines to circles and lines (possibly turning a circle into a line or vice versa) and the observation that disc preserving transformations also preserve orientation of intervals on the circle (clockwise versus counterclockwise), so the image of the interval $[x, y]$ is the interval $[F(x), F(y)]$ (as compared to $[F(y), F(x)]$).

We can establish a taxonomy of disc preserving MTs by considering the trace of the normalized matrix representing F. While $\operatorname{Tr} F$ does not have a well defined sign, the number $(\operatorname{Tr} F)^2$ is unique and real for each disc preserving F.

Definition 1. Let $F \neq \mathrm{id}$ be a disc preserving MT. We call F:

1. *elliptic* if $(\operatorname{Tr} F)^2 < 4$,
2. *parabolic* if $(\operatorname{Tr} F)^2 = 4$,
3. *hyperbolic* if $(\operatorname{Tr} F)^2 > 4$.

To better understand this classification, consider the fixed points of F. We claim (Lemma 39 in the Appendix) that:

1. F is elliptic iff it has one fixed point inside and one fixed point outside of $\mathbb{T}$ (the outside point might be ∞),
2. F is parabolic iff it has a single fixed point which lies on $\mathbb{T}$,
3. F is hyperbolic iff it has two distinct fixed points, both lying on $\mathbb{T}$.

Remark. Let F be a hyperbolic transformation with fixed points x_1, x_2. Then one of these points (say, x_1) is *stable* and the other is *unstable.* It is $F'(x_1) < 1 < F'(x_2)$ and for every $z \in \overline{\mathbb{C}}, z \neq x_2$, we have $\lim_{n\to\infty} F^n(z) = x_1$.

Similarly, when F is parabolic with the fixed point x, we have $F^n(z) \to x$ for all $z \in \overline{\mathbb{C}}$ and $F'(x) = 1$. See Lemmas 40 and 41 in the Appendix for proofs of these facts.

We will show the significance of this classification in the following chapters.

1.3 Number representation

Möbius number systems assign numbers to sequences of mappings. This principle is actually less exotic than it appears to be. Consider the usual binary representation of the interval $[0, 1]$. Let $A = \{0, 1\}$ be our alphabet. We want to assign to each word $w \in A^\omega$ the number $\Phi(w) = 0.w$ and so obtain the map $\Phi : A^\omega \to [0, 1]$. We need to use some sort of limit process: Taking longer and longer prefixes of w, we obtain better and better approximations, ending with the unique number $0.w$.

The usual construction of the binary system involves letting $\Phi(w)$ to be equal to the limit of the sequence $\{0.w_{[0,k)}\}_{k=1}^{\infty}$. However, we can also define binary numbers in the language of mappings.

Consider the two maps

$$\begin{aligned} F_0 : x &\mapsto x/2 \\ F_1 : x &\mapsto (x+1)/2. \end{aligned}$$

For $v \in A^n$ let $F_v = F_{v_0} \circ F_{v_1} \circ \cdots \circ F_{v_{n-1}}$. Both maps F_0, F_1 are continuous and, more importantly, contractions on the interval $[0, 1]$: For each $x, y \in [0, 1]$ and each $i = 0, 1$ we have $|F_i(x) - F_i(y)| = \frac{1}{2}|F_i(x) - F_i(y)|$. Therefore, for any $w \in A^\omega$, the set $\bigcap_{k=1}^{\infty} F_{w_{[0,k)}}[0, 1]$ is a singleton. What is more, a proof by induction reveals that $F_{w_{[0,k)}}[0, 1]$ is actually precisely the set of all the real numbers whose binary expansion begins with $0.w_{[0,k)}$. We have obtained that $\bigcap_{k=1}^{\infty} F_{w_{[0,k)}}[0, 1] = \{\Phi(w)\} = \{0.w\}$. If we wished, we could go on to prove that Φ is continuous and surjective, both very desirable properties for a number system.

We would like to do the same for Möbius transformations in place of F_0, F_1 and call the result a Möbius number system. However, as MTs are bijective on the complex sphere, we cannot use the contraction property like we did above. To fix this, [6] defined Φ using convergence of measures. We will see that there are other (equivalent) definitions in Theorem 8 but let us give the original definition first.

Denote $m(\mathbb{T})$ the set of all Borel probability measures on $\mathbb{T}$. If ν is a Borel measure on $\mathbb{T}$ and $F : \mathbb{T} \to \mathbb{T}$ an MT, we define the measure $F\nu$ by $F\nu(E) = \nu(F^{-1}(E))$ for all measurable sets E on $\mathbb{T}$. The Dirac measure centered at point x is the measure δ_x such that

$$\delta_x(E) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

for any E measurable subset of $\mathbb{T}$. It is a quite straightforward idea to identify δ_x with the point x itself.

Before we define what does it mean for a sequence of MTs to represent a point, let us give some brief background. Denote by $C(\mathbb{T}, \mathbb{R})$ the vector space of all continuous functions from $\mathbb{T}$ to $\mathbb{R}$ (with the supremum norm). Finite Borel measures act on $C(\mathbb{T}, \mathbb{R})$ as continuous linear functionals: Measure ν assigns to $f \in C(\mathbb{T}, \mathbb{R})$ the number $\int f \mathrm{d}\nu$ and if $\nu \neq \nu'$ then the two measures define different functionals by the Riesz representation theorem (see [1, page 184]).

We have the embedding $m(\mathbb{T}) \subseteq C(\mathbb{T}, \mathbb{R})^*$, where $C(\mathbb{T}, \mathbb{R})^*$ is the dual space to $C(\mathbb{T}, \mathbb{R})$. There are three usual topologies on $C(\mathbb{T}, \mathbb{R})^*$ (listed in the order of strength): The norm topology, the weak topology and the weak* topology.

Definition 2. Denote by μ the uniform probability measure on $\mathbb{T}$. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of Möbius transformations. We say that the sequence $\{F_n\}_{n=1}^{\infty}$ represents the point $x \in \mathbb{T}$ if and only if $\lim_{n\to\infty} F_n\mu = \delta_x$. Here μ is the uniform probability measure on $\mathbb{T}$ and the convergence of measures is taken in the weak* topology, i.e. $\nu_n \to \nu$ if and only if for all $f : \mathbb{T} \to \mathbb{R}$ continuous we have $\int f \mathrm{d}\nu_n \to \int f \mathrm{d}\nu$.

Remark. The reader might wonder why did we choose weak* topology here instead of any the two other common topologies.

One answer is that this is the usual way to define convergence of measures in fields such as ergodic theory. Another answer is that even weak topology is too strong to provide any representation of points at all: Consider any sequence $\{F_n\}_{n=1}^{\infty}$ of MTs. To obtain $\lim_{n\to\infty} F_n\mu = \delta_x$ in the weak topology, we would have to satisfy $\alpha(F_n\mu) = \alpha(\delta_x)$ for any continuous linear functional $\alpha \in C(\mathbb{T}, \mathbb{R})^{**}$.

By the Riesz representation theorem, the space $C(\mathbb{T}, \mathbb{R})^*$ can be identified with the space of all Radon signed measures on $\mathbb{T}$. For λ Radon signed measure on $\mathbb{T}$, define $\alpha(\lambda) = \lambda(\{x\})$. This is a continuous linear functional on $C(\mathbb{T}, \mathbb{R})^*$ (see Lemma 42 in the Appendix). Obviously, $\alpha(\delta_x) = 1$, while $\alpha(F_n\mu) = \mu(\{F_n^{-1}(x)\}) = 0$, so the sequence $\{F_n\}_{n=1}^{\infty}$ does not represent x. As this is true for all sequences and all values of x, Definition 2 would be meaningless in the weak topology and the same is true in the norm topology (which is even stronger that the weak topology).

Chapter 2

Representing points using Möbius transformations

2.1 General properties of Möbius transformations

In this section, we point out several useful properties of disc preserving MTs as well as various equivalent descriptions of what does it mean for a sequence of MTs to represent a point on $\mathbb{T}$.

We begin by fulfilling a promise from Preliminaries:

Lemma 3. *A Möbius transformation F is disc preserving (i.e. $F(\mathbb{D}) = \mathbb{D}$) iff it is of the form*

$$F = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix},$$

where $|\alpha|^2 - |\beta|^2 = 1$.

Proof. Let F have the given form. We prove that then $F(\mathbb{D}) = \mathbb{D}$. First, consider $z = e^{i\phi}$. We have:

$$\left|\overline{\beta}e^{i\phi} + \overline{\alpha}\right| = \left|(\overline{\alpha e^{i\phi} + \beta})e^{i\phi}\right| = \left|\alpha e^{i\phi} + \beta\right|.$$

And so

$$|F(e^{i\phi})| = \frac{\left|\alpha e^{i\phi} + \beta\right|}{\left|\overline{\beta}e^{i\phi} + \overline{\alpha}\right|} = 1.$$

Therefore $F(\mathbb{T}) \subseteq \mathbb{T}$. Because F is an MT, the image of $\mathbb{T}$ must be a circle, so $F(\mathbb{T}) = \mathbb{T}$. The unit circle divides $\overline{\mathbb{C}}$ into two components: The inside (containing zero) and the outside (containing ∞). As F is a bijection on $\overline{\mathbb{C}}$, all we have to do to obtain $F(\mathbb{D}) = \mathbb{D}$ is prove that $F(0)$ lies inside $\mathbb{D}$. But this is simple: $F(0) = \frac{\beta}{\overline{\alpha}}$ and $|\beta| < |\alpha|$, so $|F(0)| < 1$.

On the other hand, consider any disc preserving MT $F = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $\det F = 1$. Because F is continuous, it must be $F(\mathbb{T}) = \mathbb{T}$. Therefore, for every ϕ, we must have $\left|ae^{i\phi} + b\right| = \left|ce^{i\phi} + d\right|$. A little thought gives us that if $a = 0$ then $d = 0$ and similarly $b = 0$ implies $c = 0$; in both cases we are done. Assume $a, b, c, d \neq 0$ and continue.

Choose ϕ so that the quantity $\left|ae^{i\phi} + b\right| = \left|ce^{i\phi} + d\right|$ is maximal. The maximal value of the function on the left side is $|a| + |b|$, on the right side $|c| + |d|$, thus $|a| + |b| = |c| + |d|$. Similarly, by choosing the minimal quantity, we obtain that $||a| - |b|| = ||c| - |d||$. Moreover, as ϕ is the same on the right and left, we also have $\arg a - \arg b = \arg c - \arg d$. These three equalities will be enough to complete the proof.

Assume for a moment that the equality $||a| - |b|| = ||c| - |d||$ actually means $|a| - |b| = |c| - |d|$. Then, together with $|a| + |b| = |c| + |d|$, we have $|a| = |c|$ and $|b| = |d|$, obtaining a matrix of the form $F = \left(\begin{smallmatrix} a & b \\ ae^{i\psi} & be^{i\psi} \end{smallmatrix}\right)$. But this matrix is singular, a contradiction.

Therefore, we must have $|a| - |b| = |d| - |c|$, which implies $|a| = |d|, |b| = |c|$ and, after a brief calculation,

$$F = \begin{pmatrix} a & b \\ \overline{b}e^{i\psi} & \overline{a}e^{i\psi} \end{pmatrix}$$

for a suitable ψ.

Now it remains to use the normalization formula $\det F = 1$ to see that ψ is either 0 or π. If $\psi = 0$, we are done. Otherwise, we would have $\det F = |b|^2 - |a|^2 = 1$, so $|b| > |a|$. But then $|F(0)| = \frac{|b|}{|a|} > 1$, so F would turn the disc inside out, a contradiction. $\square$

Definition 4. Let $\alpha \in [0, 2\pi)$, $r \in [1, \infty)$. Call the transformation $R_\alpha(z) = e^{i\alpha}z$ a *rotation* and the transformation

$$C_r = \frac{1}{2}\begin{pmatrix} r + \frac{1}{r} & r - \frac{1}{r} \\ r - \frac{1}{r} & r + \frac{1}{r} \end{pmatrix},$$

a *contraction to 1.*

Obviously, the identity map is both a contraction to 1 and a rotation. Moreover, rotations are precisely those disc preserving MTs whose matrices are diagonal.

Remark. Observe that any contraction to 1 fixes the points ± 1. The name "contraction to 1" comes from the fact that for $r > 1$, the map C_r is a contraction in a suitable neighborhood of 1 as can be seen by computing the derivative $C_r'(1) = \frac{1}{r^2}$. Similarly, C_r expands some neighborhood of -1 as $C_r'(-1) = r^2$. Such C_r is hyperbolic and acts on $\mathbb{T}$ by making all points (with the exception of -1) "flow" towards 1. As we show in the next section, the sequence $\{C_n\}_{n=1}^{\infty}$ represents the point 1.

Lemma 5. *Let F be a Möbius transformation. If F is disc preserving then there exist ϕ_1, ϕ_2 and r such that $F = R_{\phi_1} \circ C_r \circ R_{\phi_2}$. Moreover, if F is not a rotation then $R_{\phi_1}, R_{\phi_2}, C_r$ are uniquely determined by F.*

Proof. We want to satisfy the equation

$$\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} = F = R_{\phi_1} \circ C_r \circ R_{\phi_2} = \begin{pmatrix} \frac{1}{2}\left(r + \frac{1}{r}\right) e^{i\frac{\phi_1 + \phi_2}{2}} & \frac{1}{2}\left(r - \frac{1}{r}\right) e^{i\frac{\phi_1 - \phi_2}{2}} \\ \frac{1}{2}\left(r - \frac{1}{r}\right) e^{-i\frac{\phi_1 - \phi_2}{2}} & \frac{1}{2}\left(r + \frac{1}{r}\right) e^{-i\frac{\phi_1 + \phi_2}{2}} \end{pmatrix}.$$

If $\beta = 0$ then F is a rotation and there are many solutions to the above equation; for example $r = 1$, $\phi_1 = 2\arg\alpha$, $\phi_2 = 0$. Let us now assume $\beta \neq 0$.

Choose r so that $\frac{1}{2}\left(r + \frac{1}{r}\right) = |\alpha|$. It is easy to see that we will then have $\frac{1}{2}\left(r - \frac{1}{r}\right) = |\beta|$ so it remains to get the arguments of α and β right.

Obviously, we need to choose the parameters of the rotations ϕ_1 and ϕ_2 so that we satisfy the conditions $\phi_1 + \phi_2 = 2\arg\alpha$ and $\phi_1 - \phi_2 = 2\arg\beta$. But this is a linear system with a single solution, therefore ϕ_1, ϕ_2 are unique (modulo 2π, of course). $\square$

Denote by $F^\bullet(x)$ the modulus of the derivative of F at x. Direct calculation gives us that when $F = \left(\begin{smallmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{smallmatrix}\right)$, $\det F = 1$ then

$$F^\bullet(x) = |F'(x)| = \frac{1}{\left|\overline{\beta}x + \overline{\alpha}\right|^2}.$$

This number measures whether and how much F expands or contracts the neighborhood of x.

Definition 6. Let F be a Möbius transformation. Then, inspired by [6] and [4], we define the four sets

$$\begin{aligned} U &= \{x \in \mathbb{T} : F^\bullet(x) < 1\} \\ V &= \{x \in \mathbb{T} : (F^{-1})^\bullet(x) > 1\} \\ C &= \{x \in \overline{\mathbb{C}} : F^\bullet(x) \geq 1\} \\ D &= \{x \in \overline{\mathbb{C}} : (F^{-1})^\bullet(x) \geq 1\}. \end{aligned}$$

Call U the *contraction interval of* F, V the *expansion interval of* F^{-1} and C resp. D the *expansion sets* of F resp. F^{-1}.

By Lemma 5 we have that for every F there exist ϕ_1, ϕ_2 and r such that $F = R_{\phi_1} \circ C_r \circ R_{\phi_2}$. As $R_\phi^\bullet = 1$, we have $F^\bullet(x) = C_r{}^\bullet(R_{\phi_2}(x))$ and $(F^{-1})^\bullet(x) = (C_r^{-1})^\bullet(R_{-\phi_1}(x))$. Because the sets U and C are defined using $F^\bullet(x) = C_r{}^\bullet(R_{\phi_2}(x))$, the value of r determines the shapes and sizes of U and C while ϕ_2 rotates U and C clockwise around the point 0. Similarly, the shapes of V and D depend on r while ϕ_1 determines positions of V and D, rotating them (counterclockwise) around 0.

Lemma 7. *Let $F = \left(\begin{smallmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{smallmatrix}\right)$ be a Möbius transformation that is not a rotation. Then the following holds:*

1. *C and D are circles with the same radius $|\beta|^{-1}$ and centers c, d such that $c = -\frac{\overline{\alpha}}{\overline{\beta}}$, $d = \frac{\alpha}{\overline{\beta}}$. Moreover, $|c| = |d| = \sqrt{|\beta|^{-2} + 1}$.*

2. *$U = \mathbb{T} \setminus C$ and $V = \mathbb{T} \cap \operatorname{Int}(D)$*

3. *$F(\overline{\mathbb{C}} \setminus C) = \operatorname{Int} D$*

4. $F(U) = V$

5. $|V| < \pi$

6. $|U| + |V| = 2\pi$

7. *If $x \neq y$ are points in V then I, the shorter of the two intervals joining x, y, lies in V.*

Proof. We prove (1) by direct calculation. We have $F^\bullet(x) = \frac{1}{\left|\overline{\beta}x+\overline{\alpha}\right|^2}$ and therefore $x \in C$ if and only if

$$\left|x + \frac{\overline{\alpha}}{\overline{\beta}}\right| \leq |\beta|^{-1}.$$

This is the equation of a disc with the center $c = -\frac{\overline{\alpha}}{\overline{\beta}}$ and radius $|\beta|^{-1}$. Also, it is

$$|c| = \frac{|\alpha|}{|\beta|} = \frac{\sqrt{1+|\beta|^2}}{|\beta|} = \sqrt{|\beta|^{-2}+1}.$$

The case of F^{-1} is similar.

Observe that $(F^{-1})^\bullet(z) = 1$ precisely on the boundary of D. This (together with $\mathbb{T} = (\mathbb{T} \cap C) \cup U$) gives us (2).

Parts (3) and (4) follow from the formula for the derivative of a composite function. We can write $F^{-1\bullet}(F(z)) = \frac{1}{F^\bullet(z)}$ and so $F^\bullet(z) < 1$ if and only if $F^{-1\bullet}(F(z)) > 1$.

Elementary geometrical analysis of the situation yields (5) and (6) (see Figure 2.1). Finally, (7) is a direct consequence of (5). $\square$

Remark. Observe that the triangles $0de^+$ and $0de^-$ in Figure 2.1 are right by Pythagoras' theorem. Also, we can compute that the length of V is equal to $2\arccos\left(\frac{|\beta|}{\sqrt{1+|\beta|^2}}\right)$ and the distance of d from V is $\sqrt{1+|\beta|^{-2}} - 1$. Therefore, the size of D, length of V and the distance of d and $\mathbb{T}$ are all decreasing functions of $|\beta|$. This will be important in Theorem 8.

2.2 Representing $\mathbb{T}$ and $\overline{\mathbb{R}}$

Recall that a sequence of Möbius transformations $\{F_n\}_{n=1}^\infty$ represents the point $x \in \mathbb{T}$ if and only if $\lim_{n\to\infty} F_n\mu = \delta_x$ in the weak* topology. We will now list several equivalent definitions of what does it mean to represent a point. Some of these results were already known (see Proposition 3 in [9]).

Theorem 8. *Let $\{F_n\}_{n=1}^\infty$ be a sequence of MTs only finitely many of which are rotations. Denote by V_n the expansion interval of F_n^{-1}, by D_n the expansion set of F_n^{-1} and by d_n the center of D_n. Then the following statements are equivalent:*

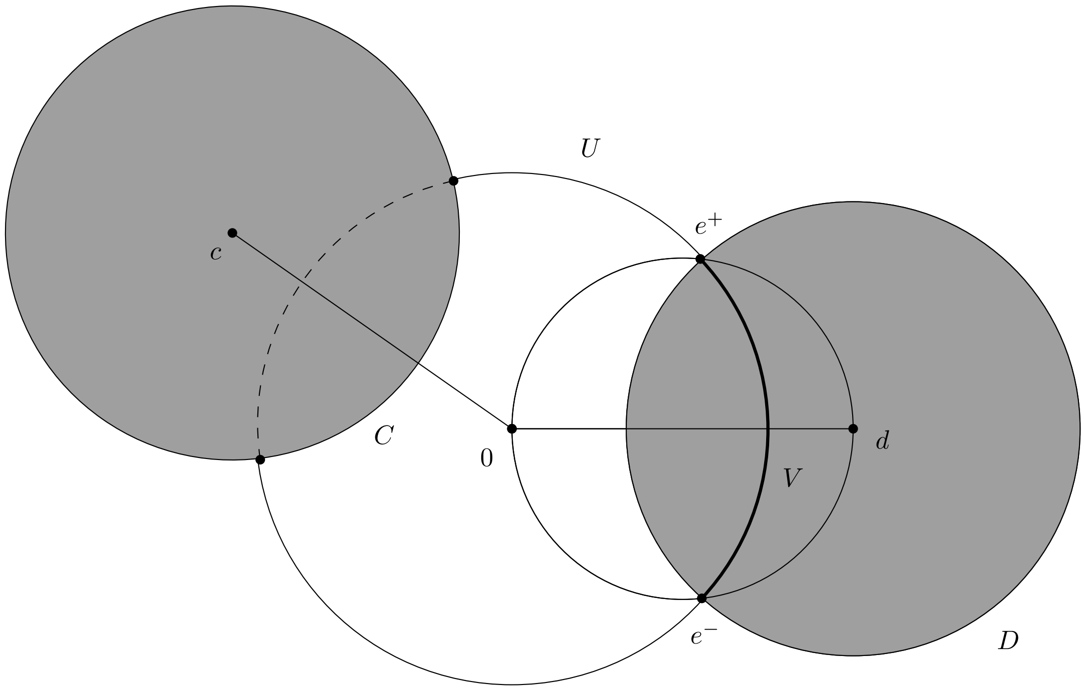

Figure 2.1: The geometry of C and D

1. *The sequence* $\{F_n\}_{n=1}^{\infty}$ *represents* $x \in \mathbb{T}$.

2. *For every open interval* I *on* $\mathbb{T}$ *containing* x *we have* $\lim_{n\to\infty}(F_n\mu)(I) = 1$.

3. *There exists a number* $c > 0$ *such that for every open interval* I *on* $\mathbb{T}$ *containing* x *it is true that* $\liminf_{n\to\infty}(F_n\mu)(I) > c$.

4. $\lim\limits_{n\to\infty} d_n = x$

5. $\lim\limits_{n\to\infty} D_n = \{x\}$

6. $\lim\limits_{n\to\infty} \overline{V}_n = \{x\}$

7. *For all* $K \subseteq \mathrm{Int}(\mathbb{D})$ *compact we have* $\lim\limits_{n\to\infty} F_n(K) = \{x\}$.

8. *For all* $z \in \mathrm{Int}(\mathbb{D})$ *we have* $\lim\limits_{n\to\infty} F_n(z) = x$.

9. *There exists* $z \in \mathrm{Int}(\mathbb{D})$ *such that* $\lim\limits_{n\to\infty} F_n(z) = x$.

10. *The sequence* $\{F_n\}_{n=1}^{\infty}$ *converges to the constant map* $c_x : z \mapsto x$ *in measure, that is*
$$\forall \varepsilon > 0,\ \lim_{n\to\infty} \mu(\{z : \rho(F_n(z), x) > \varepsilon\}) = 0.$$

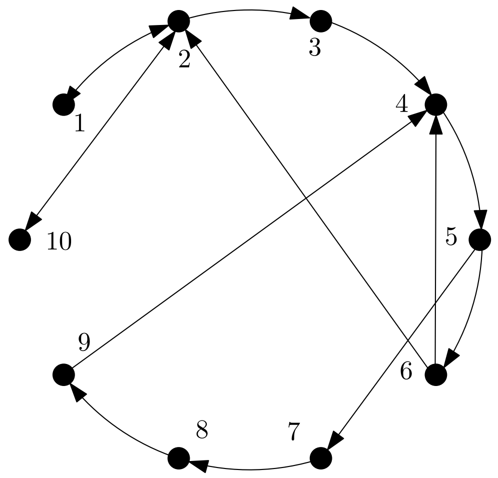

Figure 2.2: The sequence of implications used to prove Theorem 8

Here, μ is the uniform probability measure on $\mathbb{T}$ and ρ the metric on $\mathbb{T}$. In (5), (6) and (7), we take convergence in the Hausdorff metric on the space of nonempty compact subsets of $\mathbb{C}$, $\mathbb{T}$ and $\mathbb{D}$ respectively. In particular, $E_n \to \{x\}$ if and only if for every $\varepsilon > 0$ there exists n_0 such that $\forall n > n_0$ it is $E_n \subseteq B_\varepsilon(x)$.

Proof. We prove a sequence of implications. Unfortunately, the easiest to understand sequence of implications that we found is a bit more complicated than the usual "wheel" used to prove theorems of this type. See Figure 2.2 for our global plan.

Assume (1). Let $I = (a, b)$ be an open interval containing x. Consider the function f defined by:

$$f(z) = \begin{cases} 1 & \text{if } z \in [b, a] \\ \frac{\rho(x,z)}{\rho(x,a)} & \text{if } z \in [a, x] \\ \frac{\rho(x,z)}{\rho(x,b)} & \text{if } z \in [x, b] \end{cases}$$

Obviously, f is continuous on $\mathbb{T}$ and $f(x) = 0$ (see Figure 2.3 for the situation in the case $|I| < \pi$). By definition, it is $\int f \mathrm{d}F_n\mu \to f(x) = 0$. Now consider that $f(I^c) = 1$ and so $\int f \mathrm{d}F_n\mu \geq F_n\mu(I^c)$. This means $F_n\mu(I^c) \to 0$ and so $F_n\mu(I) \to 1$, proving (2).

To prove (2) $\Rightarrow$ (1), consider any continuous function $f : \mathbb{T} \to \mathbb{R}$. As $\mathbb{T}$ is compact, f is bounded by some M. For every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $y \in I = (x - \delta, x + \delta)$, it is $|f(x) - f(y)| < \varepsilon$. By (2), there exists n_0 such that whenever $n > n_0$, we have $F_n\mu(I) > 1 - \varepsilon$. Therefore (assuming without loss of generality $\varepsilon < 1$):

$$\left| f(x) - \int f \mathrm{d}F_n\mu \right| \leq \varepsilon(1 - \varepsilon) + 2M\varepsilon < (2M + 1)\varepsilon.$$

This proves $\lim_{n\to\infty} \int f \mathrm{d}F_n\mu = f(x) = \int f \mathrm{d}\delta_x$, verifying weak* convergence.

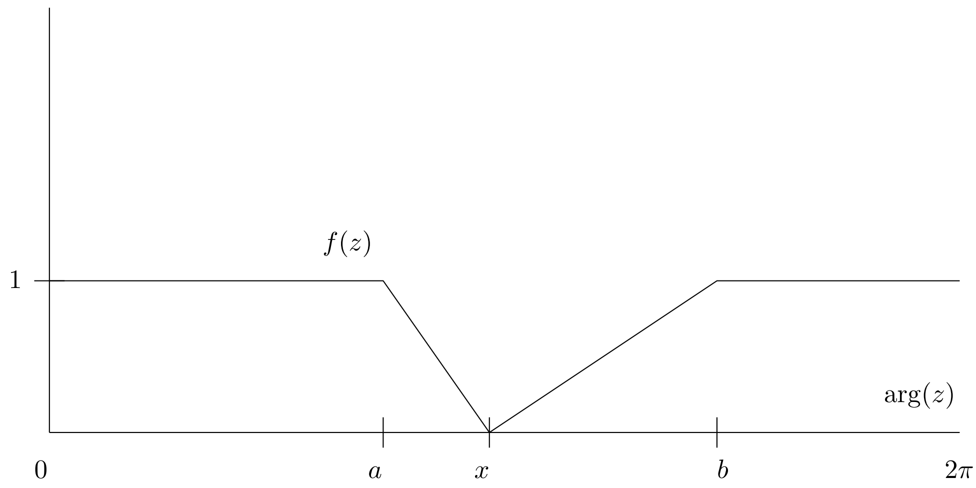

Figure 2.3: The graph of f used to prove (1) $\Rightarrow$ (2)

Claim (3) easily follows from (2) by setting $c = \frac{1}{2}$.

Assume that (3) is true. Denote $F_n = \begin{pmatrix} \alpha_n & \beta_n \\ \overline{\beta}_n & \overline{\alpha}_n \end{pmatrix}$.

Let $\varepsilon > 0$ and take the interval $I = (x - c\varepsilon, x + c\varepsilon)$. There exists n_0 such that for all $n > n_0$ we have $|F_n^{-1}(I)| > \frac{c}{2}$. Therefore, for some $z \in I$ the inequality $(F_n^{-1})^\bullet(z) > \frac{c}{4c\varepsilon} = \frac{1}{4\varepsilon}$ holds.

Recall that $(F_n^{-1})^\bullet(z) = \frac{1}{|-\overline{\beta}_n z + \alpha_n|^2}$. Moreover, for $|z| = 1$ we have $|-\overline{\beta}_n z + \alpha_n| \geq |\alpha_n| - |\beta_n|$, so $(F_n^{-1})^\bullet(z) \leq (|\alpha_n| - |\beta_n|)^{-2}$. Using $|\alpha_n|^2 - |\beta_n|^2 = 1$, we obtain

$$\frac{1}{4\varepsilon} < (F_n^{-1})^\bullet(z) \leq (|\alpha_n| + |\beta_n|)^2,$$

so that for small enough ε, we have $|\beta_n| > 1$ for all $n > n_0$.

Now from $\frac{1}{|-\overline{\beta}_n z + \alpha_n|^2} > \frac{1}{4\varepsilon}$ we obtain $|\frac{\alpha_n}{\overline{\beta}_n} - z| < \frac{2\sqrt{\varepsilon}}{\overline{\beta}_n} < 2\sqrt{\varepsilon}$. Recall that $\frac{\alpha_n}{\overline{\beta}_n}$ is precisely the point d_n. We have just shown that for any small enough $\varepsilon > 0$ there is n_0 such that for all $n > n_0$ we can find z such that:

$$|x - d_n| < |x - z| + |z - d_n| < c\varepsilon + 2\sqrt{\varepsilon}$$

implying $d_n \to x$.

An elementary examination of the geometry of V_n and D_n shows that (4), (5) and (6) are all equivalent. In particular, if $d_n \to x$ then the diameter of D_n tends to zero and so $D_n \to \{x\}$. Moreover, $\overline{V}_n = \mathbb{T} \cap D_n$, so if $D_n \to \{x\}$ then $\overline{V}_n \to \{x\}$ as well. It remains to see that if $\overline{V}_n \to \{x\}$ then $|V_n| \to 0$ which can only happen when $|\beta_n| \to \infty$. Therefore $\rho(V_n, d_n) = \rho(\mathbb{T}, d_n)$ tends to zero, meaning that $d_n \to x$.

To prove (6) $\Rightarrow$ (2), consider any open interval $I \subseteq \mathbb{T}$ containing x. We know that $\overline{V}_n \to \{x\}$, so there exists n_0 such that $n > n_0 \Rightarrow \overline{V}_n \subseteq I$. Furthermore, for all $\varepsilon > 0$, we can find $n_\varepsilon > n_0$ such that $|V_n| < \varepsilon$ whenever $n > n_\varepsilon$. For $n > n_\varepsilon$, we now have the chain of inequalities:

$$|F_n^{-1}(I)| \geq |F_n^{-1}(V_n)| = 2\pi - |V_n| > 2\pi - \varepsilon,$$

where the middle equality comes from Lemma 7. We have proved $\mu(F_n^{-1}(I)) \to 1$.

Denote now by C_n the expansion set of F_n^{-1} and assume (5). Observe that the diameter of C_n is equal to the diameter of D_n and so the diameter of C_n tends to 0. As C_n is a circle with center outside $\mathbb{D}$, for any $K \subseteq \mathrm{Int}(\mathbb{D})$ compact there exists n_0 such that $K \cap C_n = \emptyset$ whenever $n > n_0$. Then $F_n(K) \subseteq D_n$ and so $F_n(K) \to \{x\}$, proving (7).

As $\{z\}$ is a compact set, (8) easily follows from (7). Also, statement (9) is an obvious consequence of (8).

We now prove (9) $\Rightarrow$ (4). First consider the case $z = 0$. Then $F_n(0) \to x$ iff $\frac{\beta_n}{\alpha_n} \to x$. However, $d_n = \frac{\alpha_n}{\beta_n}$, therefore $d_n \to \frac{1}{\bar{x}}$. As $x \in \mathbb{T}$ and the map $z \mapsto \frac{1}{\bar{z}}$ is the circle inversion with respect to $\mathbb{T}$, we have $\frac{1}{\bar{x}} = x$, so $d_n \to x$.

In the case $z \neq 0$, we will make use of the already known equality (2) $\Leftrightarrow$ (4).

Let $F_n(z) \to x \in \mathbb{T}$ with $z \neq 0$. Let M be any disc preserving MT M that sends 0 to z (it is easy to find such an MT, as $M(0) = z$ iff $\frac{\beta}{\bar{\alpha}} = z$).

Let $G_n = F_n \circ M$ and observe that $G_n(0) = F_n(z) \to x$. Therefore, as we have just shown, the sequence $\{G_n\}_{n=1}^{\infty}$ satisfies (4) and (by (4) $\Rightarrow$ (2)) we have $\mu(M^{-1}(F_n^{-1}(I))) \to 1$ whenever I is an open interval containing x.

But $M(\mu)$ is absolutely continuous with respect to μ and so

$$\mu(M^{-1}(F_n^{-1}(I^c))) \to 0 \Rightarrow \mu(F_n^{-1}(I^c)) \to 0.$$

We then have $\mu(F_n^{-1}(I)) \to 1$, therefore the sequence $\{F_n\}_{n=1}^{\infty}$ satisfies (2). Using (2) $\Rightarrow$ (4), we finally obtain (9) $\Rightarrow$ (4).

It remains to show (2) $\Leftrightarrow$ (10), which turns out to be a simple exercise: Assume (2). Then $\forall\varepsilon > 0$ we have

$$\mu(F_n^{-1}(x - \varepsilon, x + \varepsilon)) \to 1 \Rightarrow \mu(F_n^{-1}(\{z : \rho(z, x) > \varepsilon\})) \to 0.$$

But $F_n^{-1}(\{z : \rho(z, x) > \varepsilon\} = \{z : \rho(F_n(z), x) > \varepsilon\}$, proving convergence in measure.

Similarly, if $\{F_n\}_{n=1}^{\infty}$ converges in measure, we obtain that $\mu(F_n^{-1}(x - \varepsilon, x + \varepsilon))$ tends to 1 for each $\varepsilon > 0$. Obviously, every open I such that $x \in I$ contains an interval of the form $(x - \varepsilon, x + \varepsilon)$, proving (2). $\square$

Remark. Note that Theorem 8 is mostly true even if there are infinitely many rotations in the sequence $\{F_n\}_{n=1}^{\infty}$. In this case, all the statements are trivially false, with the exception of (4) and (6) that are not well defined (though (4) can be easily fixed by letting $d_n = \infty$ for F_n rotation).

As an easy corollary of Theorem 8, we can prove that two intuitive ideas are true.

Corollary 9. *Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of MTs representing the point x. Let M be a disc preserving MT. Then*

1. The sequence $\{F_n \circ M\}_{n=1}^{\infty}$ represents x.

2. The sequence $\{M \circ F_n\}_{n=1}^{\infty}$ represents $M(x)$.

Proof. In both cases, we use the fact that if $G_n(z) \to x$ for some $z \in \mathrm{Int}(\mathbb{D})$ then the sequence $\{G_n\}_{n=1}^{\infty}$ represents x.

1. As $M(0)$ lies inside $\mathbb{D}$, we have $F_n(M(0)) \to x$, therefore $(F_n \circ M)(0) \to x$.

2. As $F_n(0) \to x$ and M is continuous, we have $M(F_n(0)) \to M(x)$. $\square$

We now have enough tools to show, like in [6], how do the three classes of MTs behave with respect to point representation:

1. Let F be an elliptic disc preserving transformation. Then the sequence $\{F^n\}_{n=1}^{\infty}$ does not represent any point.

2. Let F be a parabolic disc preserving transformation. Then the sequence $\{F^n\}_{n=1}^{\infty}$ represents the fixed point of F.

3. Let F be a hyperbolic disc preserving transformation. Then the sequence $\{F^n\}_{n=1}^{\infty}$ represents the stable fixed point of F.

For all three claims, we will need part (8) of Theorem 8.

To prove (1), recall that if F is elliptic, there exists a fixed point of F inside $\mathbb{T}$. Denote this point by x. Then for all n, $F^n(x) = x \notin \mathbb{T}$, so $\{F^n\}_{n=1}^{\infty}$ can not represent anything.

In the parabolic and hyperbolic case, denote by x the (stable) fixed point of F. We now use Lemma 41 in the Appendix to obtain that for all $z \in \mathrm{Int}(\mathbb{D})$ we have $F^n(z) \to x$. Therefore, $\{F^n\}_{n=1}^{\infty}$ represents x, proving (2) and (3).

Going in a different direction, we obtain a useful sufficient condition for representing a point.

Corollary 10. *Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of Möbius transformations such that for some $x_0 \in \mathbb{T}$ we have $\lim_{n\to\infty} (F_n^{-1})^{\bullet}(x_0) = \infty$. Then $\{F_n\}_{n=1}^{\infty}$ represents x_0.*

Proof. Let $F_n = \begin{pmatrix} \alpha_n & \beta_n \\ \overline{\beta}_n & \overline{\alpha}_n \end{pmatrix}$, $|\alpha_n|^2 - |\beta_n|^2 = 1$. Obviously, we have $|-\overline{\beta}_n x_0 + \alpha_n| \to 0$. Because $|-\overline{\beta}_n x_0 + \alpha_n| \geq |\alpha_n| - |\beta_n|$, we obtain that $|\beta_n| \geq 1$ for all n large enough. Then from

$$(F_n^{-1})^{\bullet}(x_0) = \frac{1}{|-\overline{\beta}_n x_0 + \alpha_n|^2}$$

we have $\frac{\alpha_n}{\overline{\beta}_n} \to x_0$. But $\frac{\alpha_n}{\overline{\beta}_n} = d_n$, so $d_n \to x_0$. Now $\{F_n\}_{n=1}^{\infty}$ must represent x_0 by part (4) of Theorem 8. $\square$

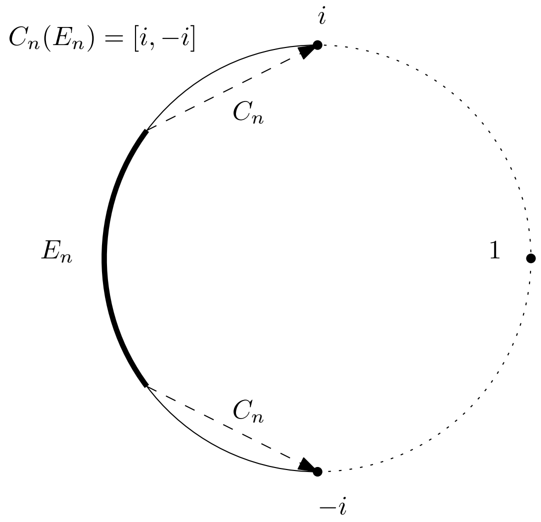

Figure 2.4: The set E_n.

Note that the sequence $\{F^n\}_{n=1}^{\infty}$ with F parabolic is a counterexample to the converse of Corollary 10. This sequence represents the fixed point x of F, yet $(F^{-n})^{\bullet}(x) = 1$ for all n.

Remark. To show that the part (8) of Theorem 8 can not be improved to include points on $\mathbb{T}$, we give an example of a sequence $\{F_n\}_{n=1}^{\infty}$ of MTs such that $\{F_n\}_{n=1}^{\infty}$ represents the point 1 while the $\{F_n(z)\}_{n=1}^{\infty}$ does not converge to 1.

Given the contraction C_n, denote $E_n = C_n^{-1}([i, -i])$. The set E_n is an interval such that $z \in E_n \Rightarrow \rho(C_n(z), 1) > \frac{\pi}{2}$ (see Figure 2.4). For $n \in \mathbb{N}$, let $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,m_n}$ be angles of rotation such that $\bigcup_{i=1}^{m_n} R_{\alpha_{n,i}}^{-1}(E_n) = \mathbb{T}$. It remains to consider the sequence of transformations

$$C_1 \circ R_{\alpha_{1,1}}, C_1 \circ R_{\alpha_{1,2}}, \dots, C_1 \circ R_{\alpha_{1,m_1}}, C_2 \circ R_{\alpha_{2,1}}, \dots, C_2 \circ R_{\alpha_{2,m_2}}, \dots$$

This sequence represents the point 1 by Corollary 10 (although the speed of convergence is quite low). Moreover, for all $z \in \mathbb{T}$ and all n there exists i such that $z \in R_{\alpha_{n,i}}^{-1}(E_n) = (C_n \circ R_{\alpha_{n,i}})^{-1}([i, -i])$, so images of z do not converge to 1.

In contrast to the above construction, we can always achieve pointwise convergence almost everywhere by taking subsequences:

Corollary 11. *If $\{F_n\}_{n=1}^{\infty}$ represents x then there exists a subsequence $\{F_{n_k}\}_{k=1}^{\infty}$ such that $F_{n_k}(z) \to x$ almost everywhere for $k \to \infty$.*

Proof. As the sequence $\{F_n\}_{n=1}^{\infty}$ represents x, it converges in measure to the constant function $c_x(z) = x$. Applying the Riesz theorem (different from the Riesz representation theorem) from [10, page 47], we obtain that there exists a subsequence $\{F_{n_k}\}_{k=1}^{\infty}$ such that $F_{n_k}(z) \to c_x(z) = x$ for almost all z. $\square$

Remark. When representing the real line, analogous results hold. Instead of the interior of $\mathbb{D}$ we have the upper half plane $\{z \in \mathbb{C} : \Im(z) > 0\}$ and instead of μ we can take either the image of μ under the stereographic projection, or any other Borel probabilistic measure that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

Chapter 3

Möbius number systems

3.1 Basic definitions and examples

Let A be an alphabet. Assume we assign to every $a \in A$ a Möbius transformation F_a. The set $\{F_a : a \in A\}$ is then called a *Möbius iterative system*. Given an iterative system, we assign to each word $v \in A^n$ the mapping $F_v = F_{v_0} \circ F_{v_1} \circ \cdots \circ F_{v_{n-1}}$.

Definition 12. Given $w \in A^\omega$, we define $\Phi(w)$ as the point $x \in \mathbb{T}$ such that the sequence $\{F_{w_{[0,n)}}\}_{n=1}^\infty$ represents x. If $\{F_{w_{[0,n)}}\}_{n=1}^\infty$ does not represent any point in $\mathbb{T}$, let $\Phi(w)$ be undefined. Denote the domain of the resulting map Φ by $\mathbb{X}_F$.

Definition 13. The subshift $\Sigma \subseteq A^\omega$ is a *Möbius number system* for a given Möbius iterative system if $\Sigma \subseteq \mathbb{X}_F$, $\Phi(\Sigma) = \mathbb{T}$ and $\Phi_{|\Sigma}$ is continuous.

Using Corollary 9, we observe that if $\Phi(w) = x$ then $\Phi(\sigma(w)) = F_{w_0}^{-1}(x)$. We will use this simple property later.

We now give three examples of Möbius number systems, although the proof that they indeed are Möbius number systems will have to wait until Section 3.3 when we have suitable tools.

Example 14. Let A, B, C be three vertices of an equilateral triangle inscribed in $\mathbb{T}$. Take F_a, F_b, F_c the three parabolic transformations satisfying.

$$\begin{aligned} F_a(A) = A, \quad F_a(C) = B \\ F_b(B) = B, \quad F_b(A) = C \\ F_c(C) = C, \quad F_c(B) = A. \end{aligned}$$

See Figure 3.1. A quick calculation reveals that F_a, F_b, F_c are in fact uniquely determined by the triangle ABC.

Let us define the shift Σ by the three forbidden factors ac, ba, cb. We claim that Σ is a Möbius number system for the iterative system $\{F_a, F_b, F_c\}$.

The following two examples are due to Petr Kůrka, see [9]:

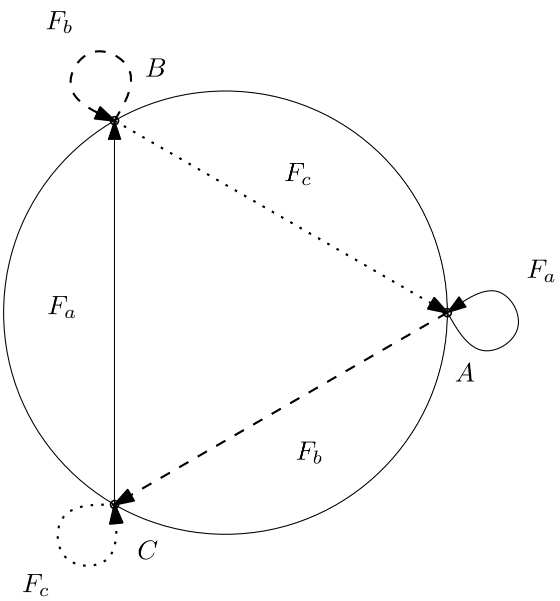

Figure 3.1: The three parabolic maps system

Example 15. The connection between MTs and continued fraction systems is well known. We show how to implement continued fractions as a Möbius number system. Let us take the following three transformations:

$$\begin{aligned}\hat{F}_{\overline{1}}(x) &= x-1\\ \hat{F}_0(x) &= -\frac{1}{x}\\ \hat{F}_1(x) &= x+1.\end{aligned}$$

These transformations are not disc preserving; instead, they preserve the upper half plane (representing $\mathbb{R}\cup\{\infty\}$ instead of $\mathbb{T}$). Conjugating $\hat{F}_{\overline{1}}, \hat{F}_0, \hat{F}_1$ with the stereographic projection, we obtain the following three disc preserving transformations:

$$\begin{aligned}F_{\overline{1}} &= \frac{1}{2}\begin{pmatrix}2-i & -1\\ -1 & 2+i\end{pmatrix}\\ F_0 &= \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}\\ F_1 &= \frac{1}{2}\begin{pmatrix}2+i & 1\\ 1 & 2-i\end{pmatrix}\end{aligned}$$

Words $00, 1\overline{1}$ and $\overline{1}1$ correspond to the identity maps while $\Phi((01)^\infty)$ and $\Phi((0\overline{1})^\infty)$ are not defined (as F_{01} and $F_{0\overline{1}}$ are parabolic). This is why we define the shift Σ by the set of forbidden words $00, 1\overline{1}, \overline{1}1, 101, \overline{1}0\overline{1}$.

It turns out that Σ is the regular continued fraction system as depicted in Figure 3.2. In this picture, the labelled points represent the images of the point 0

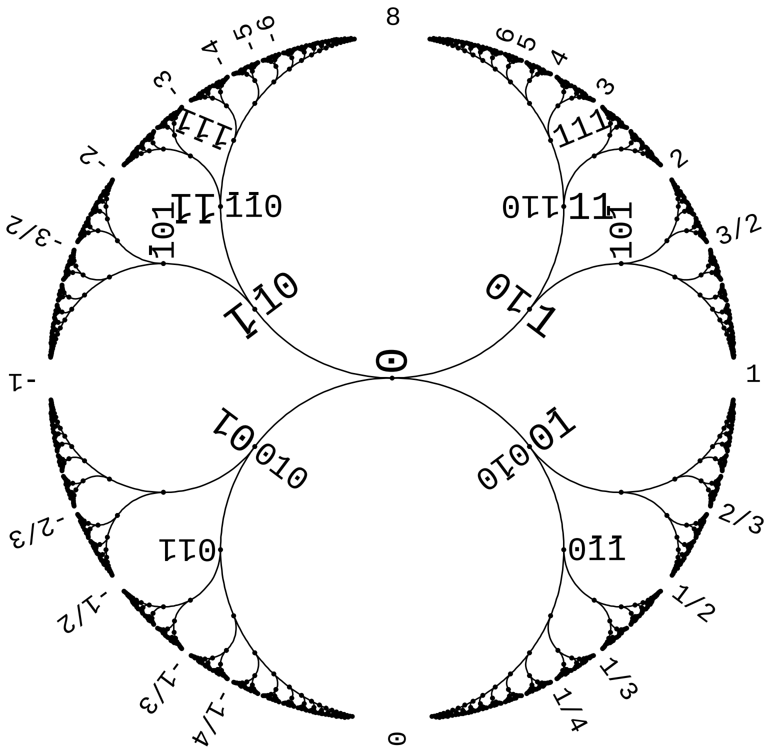

Figure 3.2: The regular continued fractions (as appear in Figure 3 in [9])

under the corresponding sequence of transformations, while curves connect images of 0 that are next to each other in a given sequence. Observe that the images of 0 converge to the boundary of the disc, ensuring convergence.

A slight complication not present in the usual continued fraction system is that we need to juggle with signs, using the transformation $-1/x$ instead of $1/x$ because the latter does not preserve the unit disc (the map $x \mapsto 1/x$ preserves the unit circle but turns the disc inside out). Otherwise, the function $\Phi_{|\Sigma} : \Sigma \to \mathbb{T}$ mirrors the usual continued fraction numeration process.

Remark. Even a quick glance on Figure 3.2 reveals that parts of the circle seem to be missing. While Φ is indeed surjective, the convergence of the images of 0 is sometimes quite slow in this system and so the depth used in the computer graphics was not enough to get near certain points. We can improve the speed of convergence by adding more transformations like in [9].

Example 16. As a last example, we obtain a circle variant of the signed binary number system. Take the following four upper half plane preserving transformations:

$$\begin{aligned}\hat{F}_{\bar{1}}(x) &= (x-1)/2\\ \hat{F}_0(x) &= x/2\\ \hat{F}_1(x) &= (x+1)/2\\ \hat{F}_2(x) &= 2x.\end{aligned}$$

Again, we conjugate these MTs with the stereographic projection to be disc preserving:

$$\begin{aligned}F_{\bar{1}}(x) &= \frac{1}{2\sqrt{2}}\begin{pmatrix}3-i & -1-i\\ -1+i & 3+i\end{pmatrix}\\ F_0(x) &= \frac{1}{2\sqrt{2}}\begin{pmatrix}3 & -i\\ i & 3\end{pmatrix}\\ F_1(x) &= \frac{1}{2\sqrt{2}}\begin{pmatrix}3+i & 1-i\\ 1+i & 3-i\end{pmatrix}\\ F_2(x) &= \frac{1}{2\sqrt{2}}\begin{pmatrix}3 & i\\ -i & 3\end{pmatrix}\end{aligned}$$

We take these transformations as our iterative system and then define the shift $\Sigma \subseteq \{0, 1, \bar{1}, 2\}^\omega$ by forbidding the words $20, 02, 12, \bar{1}2, 1\bar{1}$ and $\bar{1}1$.

Why are we forbidding these words? The reason for disallowing 2 and 0 next to each other is that these transformations are inverse to each other. The first four forbidden pairs ensure that twos are going to appear only at the beginning of any word, making the system easier to study, while the last two forbidden words ensure continuity of the function Φ at 2^∞ and, as a side-effect, make the representation nicer (in cryptography, for example, we often wish to only deal with redundant representations of integers without 1 and $\bar{1}$ next to each other).

The result is the Möbius number system depicted in Figure 3.3. On $[-1, 1]$, this is essentially the redundant binary system with $\bar{1}$ playing the role of the digit -1. To represent numbers outside of $[-1, 1]$, we use F_2.

3.2 Systems defined by intervals

This is the main section of our thesis. Our goal here is to obtain sufficient conditions guaranteeing that an iterative Möbius system together with a subshift form a Möbius number system. All the subshifts we consider here are defined using the notion of interval almost cover.

Definition 17. Let $\{F_a : a \in A\}$ be a Möbius iterative system. *Interval almost cover* of $\mathbb{T}$ is any family $\mathcal{W} = \{W_a : a \in A\}$ of sets such that $\bigcup\{\overline{W}_a : a \in A\} = \mathbb{T}$ and each W_a is an union of finitely many open intervals on $\mathbb{T}$.

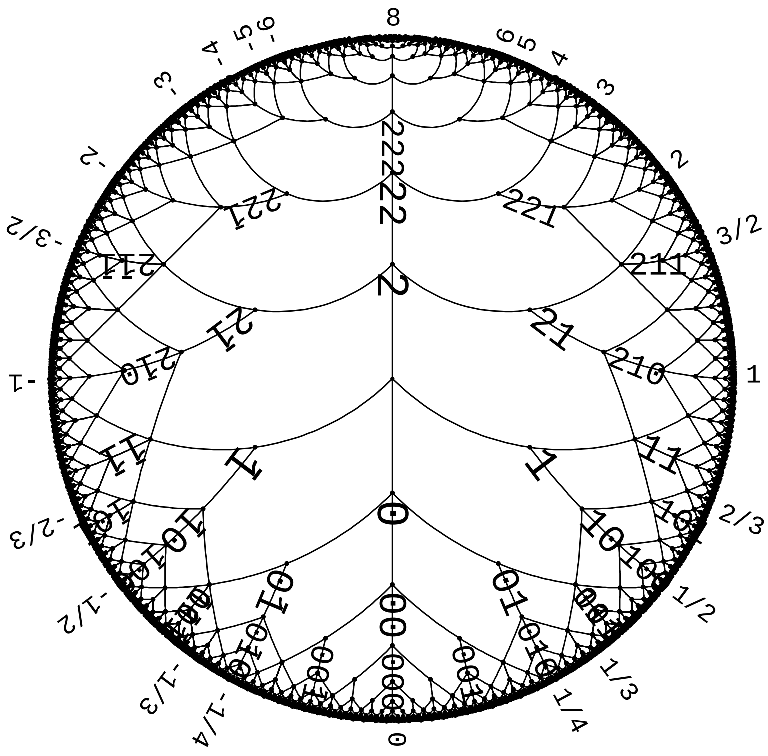

Figure 3.3: The Möbius signed binary system (as appears in Figure 1 in [9])

Given an interval almost cover $\mathcal{W}$ and $u \in A^+$, we define the *refined set* W_u as

$$W_u = W_{u_0} \cap F_{u_0}(W_{u_1}) \cap \cdots \cap F_{u_{[0,|u|-1)}}(W_{|u|-1}).$$

Note that each refined set is again a finite (possibly empty) union of open intervals. For formal reasons, let $W_\lambda = \mathbb{T}$. It is easy to prove by induction that for all $u, v \in A^\star$, we have $W_{uv} = W_u \cap F_u(W_v)$.

Lemma 18. *Let $W_a \subseteq V_a$ for every $a \in A$. Then $W_u \subseteq V_u$ for any $u \in A^\star$.*

Proof. We proceed by induction on the length of u.

For $|u| = 1$, the claim is obvious. Let $u = va$ with $W_v \subseteq V_v$. Then $W_{va} = W_v \cap F_v(W_a)$ and $(F_{va}^{-1})^\bullet(x) = (F_a^{-1})^\bullet(F_v^{-1}(x)) \cdot (F_v^{-1})^\bullet(x)$. When $x \in W_{va}$, we have $x \in W_v \subseteq V_v$ and $F_v^{-1}(x) \in W_a \subseteq V_v$, therefore $(F_a^{-1})^\bullet(F_v^{-1}(x)), (F_v^{-1})^\bullet(x) > 1$, proving the lemma. □

Definition 19. Let $\mathcal{W}$ be an interval almost cover. A subshift Σ is *compatible* with $\mathcal{W}$ if for every $v \in \mathcal{L}(\Sigma)$ it is true that

$$\overline{W}_v = \bigcup_{a,\, va \in \mathcal{L}(\Sigma)} \overline{W}_{va}.$$

Note that the "$\supseteq$" inclusion in the last equality is trivial as $\overline{W}_{va} \subseteq \overline{W}_v$ for all a.

The meaning of the compatibility condition is, roughly speaking, that we can safely extend words of $\mathcal{L}(\Sigma)$. Note in particular that A^ω is compatible with all the interval almost covers on A:

$$\bigcup_{va \in \mathcal{L}(\Sigma)} \overline{W}_{va} = \bigcup_{va \in \mathcal{L}(\Sigma)} \left(\overline{W}_v \cap F_v(\overline{W}_a)\right) = \overline{W}_v \cap F_v\left(\bigcup_{a \in A} \overline{W}_a\right) = \overline{W}_v \cap F_v(\mathbb{T}) = \overline{W}_v$$

Let $\mathcal{W}$ be an interval almost cover and Σ a subshift compatible with $\mathcal{W}$. We then define several other entities:

$$\begin{aligned} \Sigma_{\mathcal{W}} &= \{w \in \Sigma : \forall n,\, W_{w_{[0,n)}} \neq \emptyset\} \\ q(u) &= \min\{(F_u^{-1})^\bullet(x) : x \in \overline{W}_u\} \\ Q_n(\mathcal{W}, \Sigma) &= \min\{q(u) : |u| = n, u \in \mathcal{L}(\Sigma_{\mathcal{W}})\} \end{aligned}$$

We will call $\Sigma_{\mathcal{W}}$ the *interval shift* corresponding to Σ and $\mathcal{W}$ (note that $\Sigma_{\mathcal{W}}$ depends on both). Obviously, $\Sigma_{\mathcal{W}} \subseteq \Sigma$. We show that $\Sigma_{\mathcal{W}}$ is a subshift:

1. Let $w \in \Sigma_{\mathcal{W}}$. Then $\sigma(w) \in \Sigma$ and for all n we have $\emptyset \neq W_{w_{[0,n)}} \subseteq F_{w_0}(W_{w_{[1,n)}})$, implying $W_{w_{[1,n)}} \neq \emptyset$. Therefore, $\sigma(w) \in \Sigma_{\mathcal{W}}$.

2. Let $\{w^{(n)}\}_{n=1}^\infty$ be a sequence of words in $\Sigma_{\mathcal{W}}$ with the limit w. As Σ is closed, $w \in \Sigma$. Moreover, for every k there exists an n such that $w_{[0,k)} = w^{(n)}_{[0,k)}$ and so $W_{w_{[0,k)}} = W_{w^{(n)}_{[0,k)}} \neq \emptyset$. Therefore $w \in \Sigma_{\mathcal{W}}$ and so $\Sigma_{\mathcal{W}}$ is closed.

Remark. We now make an observation that will be useful later. Consider two interval almost covers $\mathcal{W} = \{W_a : a \in A\}$ and $\mathcal{W}' = \{W'_a : a \in A\}$ and two subshifts $\Sigma' \subseteq \Sigma$ such that $\mathcal{W}$ is compatible with Σ and $\mathcal{W}'$ is compatible with Σ'. If for each $a \in A$ we have $W'_a \subseteq W_a$ then easily $\Sigma'_{\mathcal{W}'} \subseteq \Sigma_{\mathcal{W}}$.

Our main goal in this section will be to prove the following theorem:

Theorem 20. *Let $\{F_a : a \in A\}$ be a Möbius iterative system. Assume that $\mathcal{W}$ is such an interval almost cover that $W_a \subseteq V_a$ for all $a \in A$ and Σ is a subshift compatible with $\mathcal{W}$. Then $\Sigma_{\mathcal{W}}$ is a Möbius number system for the iterative system $\{F_a : a \in A\}$.*

Moreover, for every $v \in A^\star$, $\Phi([v] \cap \Sigma_{\mathcal{W}}) = \overline{W}_v$.

Our plan is to first prove several auxiliary claims, then solve the case when $W_a = V_a$ and finally use this special case to prove Theorem 20.

The following lemma is stated (in a different form) in [6] as Lemma 2.

Lemma 21. *Let $\{F_a : a \in A\}$ be a Möbius iterative system and let L be the length of the longest of intervals in $\{\overline{V}_a : a \in A\}$. Then there exists an increasing continuous function $\psi : [0, L] \to \mathbb{R}$ such that:*

1. *$\psi(0) = 0$*
2. *For every $l \in (0, L]$ we have $\psi(l) > l$.*
3. *If I is an interval and $a \in A$ a letter such that $I \subseteq \overline{V}_a$ then $|F_a^{-1}(I)| \geq \psi(|I|)$.*

Proof. Thanks to Lemma 5 we can without loss of generality assume that each F_a is a contraction to 1 with parameter $r_a > 1$ (if $r_a = 1$ then $\overline{V}_a = \emptyset$ so a can be safely omitted from the alphabet).

Choose $a \in A$ so that r_a is minimal and let

$$\psi(l) = \inf\{|F_a^{-1}(I)| \,:\, I \subseteq \overline{V}_a \text{ and } |I| = l\}.$$

By analyzing contractions, it is easy to see that ψ is increasing, continuous and $\psi(l) > l$ for $l > 0$. □

Lemma 22. *Let $\{F_a : a \in A\}$ be a Möbius iterative system. Let x, y be points and $w \in A^\omega$ a word such that for all n we have $F^{-1}_{w_{[0,n)}}(x), F^{-1}_{w_{[0,n)}}(y) \in \overline{V}_{w_n}$. Then $x = y$ or there exists an n_0 such that $F_{w_{[n,n+1]}}$ is a rotation for all $n > n_0$.*

Proof. Denote $x_n = F^{-1}_{w_{[0,n)}}(x)$ and $y_n = F^{-1}_{w_{[0,n)}}(y)$. Assume that $x \neq y$. For each n denote by I_n the closed interval with endpoints x_n, y_n such that $I_n \subseteq \overline{V}_{w_n}$. Möbius transformations are bijective, so $x_n \neq y_n$, implying $|I_n| > 0$ for all n.

Observe first that the sequence $\{|I_n|\}_{n=0}^{\infty}$ is a nondecreasing one. For any particular n we have two possibilities: Either $F^{-1}_{w_n}(I_n) = I_{n+1}$ and therefore $|I_{n+1}| \geq \psi(|I_n|) > |I_n|$ (by Lemma 21), or $I_{n+1} = \overline{\mathbb{T} \setminus F^{-1}_{w_n}(I_n)}$; see Figure 3.4. In the second

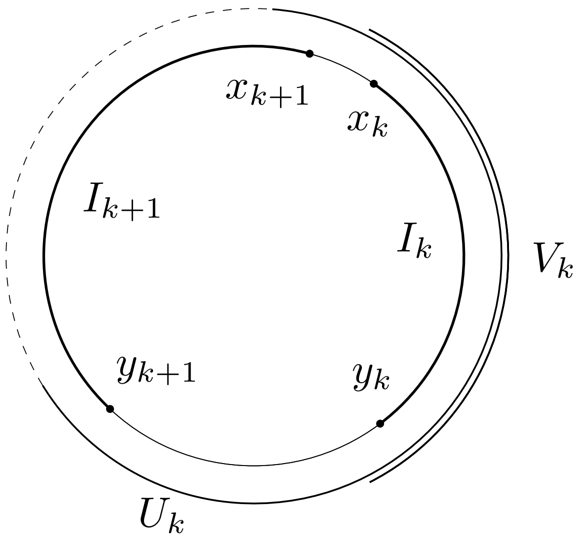

Figure 3.4: A twist.

case, observe that $\mathbb{T} \setminus U_{w_n} \subseteq I_{n+1}$ and recall that $|U_{w_n}| + |V_{w_n}| = 2\pi$. It follows $|I_{n+1}| \geq |V_{w_n}| \geq |I_n|$. Call the second case a *twist*.

Assume first that the number of twists is infinite. We claim that then there must exist an n_0 such that $|I_n|$ is constant for all $n > n_0$: Whenever a twist happens for some n, we have $|V_{w_{n+1}}| \geq |I_{n+1}| \geq |V_{w_n}|$. If any inequality in the previous formula is sharp then the letter w_n does not appear anywhere in $w_{[n+1,\infty)}$ anymore, so if there were infinitely many such sharp inequalities, the alphabet A would have to be infinite. Therefore, $|V_{w_{n+1}}| = |I_{n+1}|$ for all but finitely many twists and the finiteness of A gives us that that there is an n_0 such that $|I_n| = |I_{n_0}|$ whenever $n > n_0$.

Assume now $n > n_0$. Simple case analysis shows that $|I_n| = |I_{n+1}|$ can only happen when x_n, y_n are the endpoints of $\overline{V}_{w_n}$ and the transition is a twist. Similarly, x_{n+2}, y_{n+2} are also endpoints of $\overline{V}_{w_{n+1}}$. Let R be the rotation that sends I_{n+2} to I_n. The map $R \circ F^{-1}_{w_{[n,n+1]}}$ has two fixed points x_n, y_n and

$$(R \circ F^{-1}_{w_{[n,n+1]}})^{\bullet}(x_n) = (R \circ F^{-1}_{w_{[n,n+1]}})^{\bullet}(y_n) = 1.$$

This can happen only when $R \circ F^{-1}_{w_{[n,n+1]}} = \mathrm{id}$ (see the Remark after Definition 1). Therefore, $F^{-1}_{w_{[n,n+1]}}$ is a rotation for all $n > n_0$ and we are done.

It remains to investigate the case when the number of twists is finite, i.e. there exists an n_0 such that for all $n \geq n_0$ we have $I_{n+1} = F^{-1}_{w_n}(I_n)$. By Lemma 21 we obtain $|I_{n+1}| \geq \psi(|I_n|)$ for all $n \geq n_0$ and therefore $|I_n| \geq \psi^{n-n_0}(|I_{n_0}|)$ for all $n \geq n_0$. Denote $l = |I_{n_0}| > 0$.

Consider now the sequence $\{\psi^n(l)\}_{n=0}^{\infty}$. Assume that $\psi^n(l) \leq L$ for all n. Then the sequence is increasing and bounded and therefore it has a limit $\xi \in (0, L]$. As ψ is continuous, $\psi(\xi) = \xi$. But the only fixed point of ψ is 0, a contradiction.

Therefore, there always exists an n such that $\psi^n(l) > L$. But then I_{n+n_0} cannot possibly fit into any of the intervals $\overline{V}_a$, which is a contradiction with the assumption $x_{n+n_0}, y_{n+n_0} \in \overline{V}_{w_{n+n_0}}$. Therefore, $x = y$. $\square$

Let I be an interval on $\mathbb{T}$. Recall that if $I = (a, b)$ then a is the clockwise and b the counterclockwise endpoint of I. The following two easy observations on the geometry of intervals are going to be useful when examining our number system.

Lemma 23. *Let $I_1, \dots, I_k$ be open intervals on $\mathbb{T}$. Then $x \in \overline{\bigcap_{i=1}^k I_i}$ if and only if both of the following conditions hold:*

1. $x \in \bigcap_{i=1}^k \overline{I_i}$
2. *If x is an endpoint of both I_i and I_j then x may not be the counterclockwise endpoint of one interval and clockwise endpoint of the other.*

Proof. Obviously, if $x \in \overline{\bigcap_{i=1}^k I_i}$ then $x \in \bigcap_{i=1}^k \overline{I_i}$. Were x the clockwise endpoint of I_i and counterclockwise endpoint of I_j then there would exist a neighborhood E of x such that $E \cap I_i \cap I_j = \emptyset$ and so x would not belong to $\overline{\bigcap_{i=1}^k I_i}$.

In the other direction, assume that x has both the required properties. Then a simple case analysis shows that for any neighborhood E of x we have $E \cap \bigcap_{i=1}^k I_i \neq \emptyset$ and so $x \in \overline{\bigcap_{i=1}^k I_i}$. $\square$

Lemma 24. *Let $J = [x, y]$ be a nondegenerate interval on $\mathbb{T}$. Let $I_1, \dots, I_k$ be closed intervals such that $J \subseteq \bigcup_{i=1}^k I_i$. Then there exists i such that x is not the counterclockwise endpoint of I_i, that is $[x, x+\varepsilon] \subseteq I_i$ for some $\varepsilon > 0$.*

Proof. Let $I_i = [a_i, b_i]$ for all i.

If for some i, $x \in [a_i, b_i)$, we are done. We have $x \notin I_i$ or $x = b_i$ for each i. But then there exists ε such that $|J| > \varepsilon > 0$ and $(x, x+\varepsilon) \cap I_i = \emptyset$ for all i, a contradiction. $\square$

We are going to construct a candidate for the graph $X = \{(\Phi(w), w) : w \in \Omega\}$ of $\Phi_{|\Omega}$. During the proof, the set Ω turns out to be a subshift.

Given $(x, w) \in \mathbb{T} \times A^\omega$, we use the shorthand notation $x_i = F^{-1}_{w_{[0,i)}}(x)$. For $a \in A$, label e_a^+ the counterclockwise and e_a^- the clockwise endpoint of the interval V_a.

Define $X \subseteq \mathbb{T} \times A^\omega$ to be the set of all pairs (x, w) such that:

(1) For all $i = 1, 2, \dots$, $x_i \in \overline{V}_{w_i}$.

(2) For no i and j is it true that $x_i = e^+_{w_i}$ and $x_j = e^-_{w_j}$.

Note that the second condition says that endpoints cannot "alternate": If x_i, x_j are endpoints of V_{w_i}, V_{w_j} then x_i, x_j are both endpoints of the same type (clockwise or counterclockwise). For an example of a forbidden situation, see Figure 3.5.

Preparing for the future, we investigate the set X. Using Lemma 23, it is easy to see that $(x, w) \in X$ if and only if $x \in \bigcap_{k=0}^\infty \overline{\bigcap_{i=0}^k F_{w_{[0,i)}}(V_{w_i})}$. Remembering the definition of refined sets, we let $W_u = \bigcap_{i=0}^{k-1} F_{u_{[0,i)}}(V_{u_i})$ for $u \in A^k$.

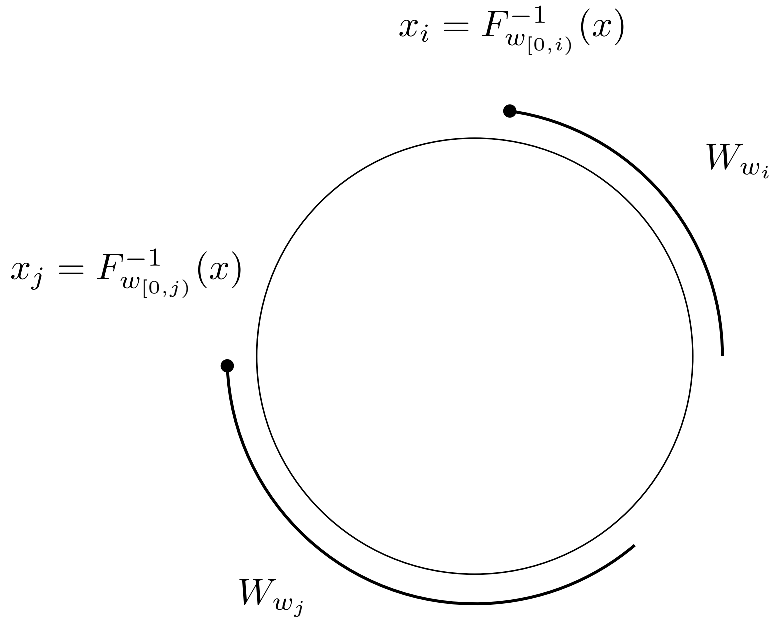

Figure 3.5: A situation contradicting part (2) of the definition of X

We now have:

$$\bigcap_{k=0}^{\infty} \overline{\bigcap_{i=0}^{k} F_{w_{[0,i)}}(V_{w_i})} = \bigcap_{k=1}^{\infty} \overline{W_{w_{[0,k)}}}$$

Note that there is in general a difference between $W_{w_{[0,k)}}$ and $V_{w_{[0,k)}}$. The former is defined as an intersection of preimages of intervals V_{w_i}, while the latter is the expanding interval of $F^{-1}_{w_{[0,k)}}$. These two sets are different in general, but Lemma 18 gives us the inclusion $W_{w_{[0,k)}} \subseteq V_{w_{[0,k)}}$.

Let $P(v) = \overline{W_v}$. We have $(x, w) \in X$ iff $x \in \bigcap_{k=1}^{\infty} P(w_{[0,k)})$.

Observation 25. *If $(x, w) \in X$, where X is defined as above, then $F_{w_{[k,k+1]}}$ is not a rotation for any k.*

Proof. From the condition $(x, w) \in X$ iff $x \in \bigcap_{k=1}^{\infty} P(w_{[0,k)})$ we obtain that the set $P(w_{[0,k+1]}) = \overline{W_{w_{[0,k+1]}}}$ is nonempty. Therefore, $W_{w_{[k,k+1]}} \neq \emptyset$. By Lemma 18, $V_{w_{[k,k+1]}} \neq \emptyset$ and therefore $F_{w_{[k,k+1]}}$ cannot be a rotation. □

We are now ready to prove a special case of Theorem 20.

Theorem 26. *Let $\{F_a : a \in A\}$ be a Möbius iterative system. For $a \in A$ let V_a be the expansive interval of F_a^{-1}. Assume that $\mathcal{V} = \{V_a : a \in A\}$ is an interval almost cover of $\mathbb{T}$ and let $\Omega = (A^\omega)_{\mathcal{V}}$ be the corresponding interval shift. Then Ω is a Möbius number system for the iterative system $\{F_a : a \in A\}$.*

Proof. If $V_a = \emptyset$ for some $a \in A$ then the letter a does not appear in the shift Ω at all. Therefore, without loss of generality $V_a \neq \emptyset$ for all $a \in A$.

Consider the set $X \subseteq \mathbb{T} \times A^\omega$ as introduced above. Taking projections π_1, π_2 of X to the first and second element, we obtain the set of points $\pi_1(X) \subseteq \mathbb{T}$ and the set of words $\pi_2(X) \subseteq A^\omega$. To conclude our proof, we verify that:

1. $\pi_1(X) = \mathbb{T}$
2. $\pi_2(X) = \Omega$
3. For $(x, w) \in X$ we have $\Phi(w) = x$.
4. $\Phi_{|\Omega}$ is continuous.

1. This follows from the fact that $\{\overline{V}_a : a \in A\}$ covers $\mathbb{T}$. Given $x \in \mathbb{T}$ we can construct $w \in A^\omega$ satisfying $(x, w) \in X$ by induction using Lemma 24: Let $x_0 = x$ and choose $w_0 \in A$ such that $x_0 \in \overline{V}_{w_0}$ and x_0 is not the counterclockwise endpoint of $\overline{V}_{w_0}$. Then take $x_1 = F^{-1}_{w_0}(x_0)$, choose w_1 so that $x_1 \in \overline{V}_{w_1}$ and x_1 is not the counterclockwise endpoint of $\overline{V}_{w_1}$, let $x_2 = F^{-1}_{w_2}(x_1)$ and repeat the procedure.

2. Notice that $W_v \neq \emptyset$ iff $P(v) \neq \emptyset$. and $P(w_{[0,k+1)}) \subseteq P(w_{[0,k)})$. By compactness of $\mathbb{T}$, $w \in \Omega$ if and only if there exists $x \in \bigcap_{k=1}^\infty P(w_{[0,k)})$. But $x \in \bigcap_{k=1}^\infty P(w_{[0,k)})$ iff $(x, w) \in X$. Therefore $w \in \Omega$ iff $w \in \pi_2(X)$.

3. Let $l = \min\{|\overline{V}_a| : a \in A\}$ and assume $(x, w) \in X$. Recall the notation $x_i = F^{-1}_{w_{[0,i)}}(x)$. We will divide the proof into several cases.

   If $\lim_{i\to\infty} (F^{-1}_{w_{[0,i)}})^\bullet(x) = \infty$ then we simply use Lemma 10 and are done.

   Therefore, assume that $\lim_{i\to\infty} (F^{-1}_{w_{[0,i)}})^\bullet(x)$ is finite or does not exist and examine the consequences.

   We have

$$(F^{-1}_{w_{[0,i)}})^\bullet(x) = \prod_{k=0}^{i-1} (F^{-1}_{w_k})^\bullet(F^{-1}_{w_{[0,k)}}(x)) = \prod_{k=0}^{i-1} (F^{-1}_{w_k})^\bullet(x_k).$$

   As $x_k \in \overline{V}_{w_k}$, we have that $(F^{-1}_{w_k})^\bullet(x_k) \geq 1$ for each k. From this we deduce that $\lim_{i\to\infty} (F^{-1}_{w_{[0,i)}})^\bullet(x) = \prod_{k=0}^\infty (F^{-1}_{w_k})^\bullet(x_k)$ exists.

   Recall the notation $\overline{V}_a = [e^-, e^+]$. Examining the function $(F_a^{-1})^\bullet$ as in the proof of Lemma 7, we see that for every ξ such that $2l > \xi > 0$ there exists a $\delta > 0$ such that for each $a \in A$ and each $y \in [e_a^- + \xi, e_a^+ - \xi]$ we have $(F_a^{-1})^\bullet(y) > 1 + \delta$ (see Figure 3.6).

   As $\lim_{k\to\infty}(1 + \delta)^k = \infty$ for any $\delta > 0$, the only way that the product $\prod_{k=0}^\infty (F^{-1}_{w_k})^\bullet(x_k)$ can be finite is when for each ξ such that $2l > \xi > 0$ there exists k_0 such that

$$k \geq k_0 \Rightarrow x_k \in [e^-_{w_k}, e^-_{w_k} + \xi) \cup (e^+_{w_k} - \xi, e^+_{w_k}].$$

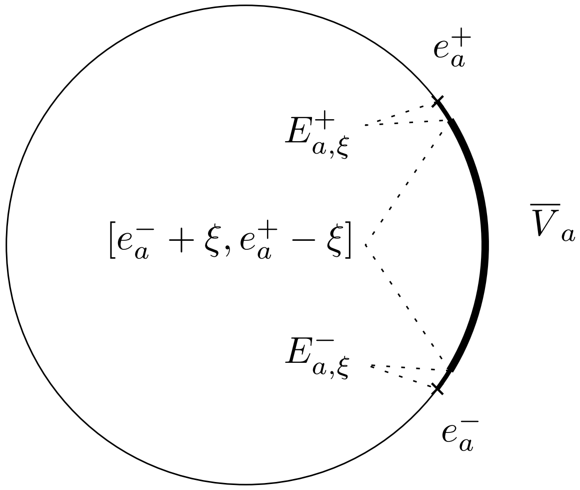

Figure 3.6: The interval $[e_a^- + \xi, e_a^+ - \xi]$ and friends.

Let $E_{a,\xi}^- = [e_a^-, e_a^- + \xi)$ and $E_{a,\xi}^+ = (e_a^+ - \xi, e_a^+]$. As the set $\{e_a^-, e_a^+ : a \in A\}$ is finite, if we choose ξ small enough then for any $a, b \in A$ and any choice of $\triangle, \square$ in $\{+, -\}$, we obtain:

$$F_a^{-1}(E_{a,\xi}^{\triangle}) \cap E_{b,\xi}^{\square} \neq \emptyset \Rightarrow F_a^{-1}(e_a^{\triangle}) = e_b^{\square}.$$

Moreover, if $\triangle$ and $\square$ are different then

$$F_a^{-1}(E_{a,\xi}^{\triangle}) \cap E_{b,\xi}^{\square} = \{e_b^{\square}\}.$$

Take such a small ξ. If $\Phi(w_{[k_0,\infty)}) = x_{k_0}$ then $\Phi(w) = F_{w_{[0,k_0)}}(\Phi(w_{[k_0,\infty)}) = x$. Therefore, we can without loss of generality assume that $k_0 = 0$, i.e.:

$$\forall k \geq 0,\, x_k \in E_{w_k,\xi}^- \cup E_{w_k,\xi}^+$$

Assume that $x_k \in E_{w_k,\xi}^+$ and $x_{k+1} \in E_{w_{k+1},\xi}^-$. By the choice of ξ, this would only be possible when $x_k = e_{w_k}^+$ and $x_{k+1} = e_{w_{k+1}}^-$, a contradiction with $(x, w) \in X$.

Therefore, without loss of generality, $x_k \in E_{w_k,\xi}^-$ for each $k \geq 0$. Let $e_k = F_{[0,k)}^{-1}(e_{w_0}^-)$ and observe that in this case $e_k = e_{w_k}^-$ for each $k \geq 0$.

Now x_k and e_k both belong to $\overline{V}_{w_k}$ for all $k \geq 0$ and, using Lemma 22 and Observation 25, we obtain $x_k = e_k = e_{w_k}^-$ for all $k \geq 0$.

We will conclude the proof using part (3) of Theorem 8. Let J be an open interval containing x. Then there exists $\varepsilon > 0$ such that $I = [x, x + \varepsilon] \subseteq J$. We want to show that

$$\liminf_{k \to \infty} |F_{w_{[0,k)}}^{-1}(I)| \geq l,$$

where again $l = \min\{|\overline{V}_a| : a \in A\}$. Observe that each $F_{w_k}^{-1}$ expands the interval $[x_k, x_k + l]$, therefore it is enough to find a single k such that $|F_{w_{[0,k)}}^{-1}(I)| \geq l$.

But this again follows from Lemma 22 and Observation 25; all we have to do is choose the point y equal to $x + \varepsilon$. Obviously, $y \neq x$, so there must exist k such that $F_{w_{[0,k)}}^{-1}(y) \notin \overline{V}_{w_k}$ which is only possible when $|F_{w_{[0,k)}}^{-1}(I)| > l$.

Therefore, by Theorem 8, we have $\Phi(w) = x$, concluding the proof.

4. We begin by proving that $\bigcap_{i=1}^{\infty} P(w_{[0,i)})$ is a singleton for every $w \in \Omega$. We know that when $(x, w) \in X$ then $x \in P(w)$. Let $y \in P(w)$. Then for all k we have $F_{w_{[0,k)}}^{-1}(\{x, y\}) \in \overline{V}_{w_k}$ and, applying Lemma 22 together with Observation 25, we obtain $x = y$, which is what we need.

   Now given an open set $U \subseteq \mathbb{T}$ we want to show that $\Phi^{-1}(U)$ is open in Ω. Let $w \in \Phi^{-1}(U)$ and assume that for all k there exists $w' \in \Omega$ such that $w_{[0,k]} = w'_{[0,k]}$ and $\Phi(w') \notin U$. Then we must have $\bigcap_{i=1}^{k} P(w_{[0,i]}) \cap U^c \neq \emptyset$ and from compactness of X we obtain that $\cap_{i=1}^{\infty} P(w_{[0,i]}) \cap U^c \neq \emptyset$. But then $\cap_{i=1}^{\infty} P(w_{[0,i]})$ cannot be a singleton as $\Phi(w) \notin U^c$, a contradiction. $\square$

Note that the theorem is also true for a slightly larger subshift than Ω. For details, see [5], Theorem 21.

Finally, we have prepared the groundwork for the proof of Theorem 20.

Theorem 20. *Let $\{F_a : a \in A\}$ be a Möbius iterative system. Assume that $\mathcal{W}$ is such an interval almost cover that $W_a \subseteq V_a$ for all $a \in A$ and Σ is a subshift compatible with $\mathcal{W}$. Then $\Sigma_{\mathcal{W}}$ is a Möbius number system for the iterative system $\{F_a : a \in A\}$.*

Moreover, for every $v \in A^\star$, $\Phi([v] \cap \Sigma_{\mathcal{W}}) = \overline{W}_v$.

Proof. As $W_a \subseteq V_a$ for each $a \in A$ and $\Sigma \subseteq A^\omega$, we have $\Sigma_{\mathcal{W}} \subseteq \Omega$ where Ω is the subshift from Theorem 26. Therefore, using Theorem 26, we obtain that Φ is defined and continuous on $\Sigma_{\mathcal{W}}$.

It remains to prove that $\overline{W}_v = \Phi([v] \cap \Sigma_{\mathcal{W}})$ for each $v \in A^\star$ (this will also show the surjectivity of $\Phi_{|\Sigma_{\mathcal{W}}}$, as $\overline{W}_\lambda = \mathbb{T}$). As usual, we prove two inclusions.

First, $\overline{W}_v \subseteq \Phi([v] \cap \Sigma_{\mathcal{W}})$. Let $x \in \overline{W}_v$ be any point. We want to find an infinite word w such that $vw \in \Sigma$ and we have $x \in \overline{W}_{v \cdot w_{[0,k)}}$ for all k. This will be enough, as for such a w we will have $vw \in \Sigma_{\mathcal{W}}$ as well as $(x, vw) \in X$. Then by the proof of Theorem 26, $x = \Phi(vw)$ and so $x \in \Phi([v] \cap \Sigma_{\mathcal{W}})$.

Without loss of generality assume that there exists $\varepsilon_0 > 0$ such that $[x, x + \varepsilon_0] \subseteq \overline{W}_v$ (if this is not true then there exists $\varepsilon_0 > 0$ such that $[x - \varepsilon_0, x] \subseteq \overline{W}_v$ and the proof is similar).

We now inductively construct w: Assume that $[x, x + \varepsilon_k] \subseteq \overline{W}_{v \cdot w_{[0,k)}}$ for $\varepsilon_k > 0$ (this is true for $k = 0$). Then from the compatibility condition, we obtain that

$$\overline{W}_{v \cdot w_{[0,k)}} = \bigcup_{v \cdot w_{[0,k)} a \in \mathcal{L}(\Sigma)} \overline{W}_{v \cdot w_{[0,k)} a}.$$

Applying Lemma 24 with $J = [x, x + \varepsilon_k]$, we find $w_k = a$ and $\varepsilon_{k+1} > 0$ such that $v \cdot w_{[0,k)} w_k \in \mathcal{L}(\Sigma)$ and $[x, x + \varepsilon_{k+1}] \subseteq W_{v \cdot w_{[0,k)} w_k}$, completing the induction step.

It remains to show that $\overline{W}_v \supseteq \Phi([v] \cap \Sigma_{\mathcal{W}})$. Let $\Phi(vw) = x$ for $vw \in \Sigma_{\mathcal{W}}$. We claim that then $x \in \overline{W}_v$.

By Theorem 8, we have that $\overline{V}_{v \cdot w_{[0,k)}} \to \{x\}$ as k tends to ∞. From the inclusion $\overline{W}_{v \cdot w_{[0,k)}} \subseteq \overline{V}_{v \cdot w_{[0,k)}}$, we obtain $\overline{W}_{v \cdot w_{[0,k)}} \to \{x\}$. To complete the proof, notice that for all k we have $\overline{W}_{v \cdot w_{[0,k)}} \subseteq \overline{W}_v$, which is only possible when $x \in \overline{W}_v$. □

Corollary 27. *Let $\{F_a : a \in A\}$ be a Möbius iterative system and $B \subseteq B^+$ a finite set of words. Assume that $\mathcal{W}$ is such an interval almost cover and Σ such a subshift compatible with $\mathcal{W}$ that $W_b \subseteq V_b$ for every $b \in B$ and each $w \in \Sigma_{\mathcal{W}}$ contains as a prefix some $b \in B$.*

Then $\Sigma_{\mathcal{W}}$ is a Möbius number system for the iterative system $\{F_a : a \in A\}$. Moreover, $\Phi([v] \cap \Sigma_{\mathcal{W}}) = \overline{W}_v$ for all $v \in A^\star$.

Proof. We will take the set B as our new alphabet, solve the problem in B^ω and then return back to A^ω. Denote $\psi : B^\star \to A^\star$ the map that "breaks down" each $b \in B$ into its letters. We can extend ψ in an obvious way to obtain a map $B^\omega \to A^\omega$. Denote this map also by ψ.

Because every $w \in A^\omega$ has a prefix in B, the map $\psi : B^\omega \to A^\omega$ is surjective. A little thought gives us that ψ is continuous in the product topology on B^ω and A^ω.

Let $\Theta = \psi^{-1}(\Sigma)$. We claim that Θ is a subshift of B^ω. As ψ is continuous, Θ must be closed and from the shift-invariance of Σ easily follows shift-invariance of Θ.

Consider the Möbius iterative system $\{F_b : b \in B\}$ and let $\mathcal{W}^B = \{W_b : b \in B\}$. We claim that $\mathcal{W}^B$ is an interval almost cover compatible with the subshift Θ.

Before we proceed, notice that we have now two meanings for the refined set W_v depending on whether $v \in A^\star$ or $v \in B^\star$. However, a quick proof by induction yields that for any $v \in B^\star$ we have $W_v = W_{\psi(v)}$. This is why we will identify v and $\psi(v)$ when talking about refined sets.

We show that for every $v \in B^\star$ we have

$$\overline{W}_v = \bigcup_{b \in B,\, vb \in \mathcal{L}(\Theta)} \overline{W}_{vb}.$$

Not only does this prove compatibility, it also shows that $\mathcal{W}^B$ is an interval almost cover (because $\overline{W}_\lambda = \mathbb{T}$).

Let $v \in B^\star$. Let $n = \max\{|b| : b \in B\}$ and choose any $x \in \overline{W}_v$. From the compatibility of $\mathcal{W}$ and Σ we obtain by induction that there exists a word w of length n such that $x \in \overline{W}_{vw}$ and $vw \in \mathcal{L}(\Sigma)$. But then there exists a $b \in B$ that is a prefix of w. Therefore $\overline{W}_{vb} \subseteq \overline{W}_{vw}$ and so $x \in \overline{W}_{vb}$ and $vb \in \mathcal{L}(\Theta)$. We have shown that

$$\overline{W}_v = \bigcup_{b \in B, vb \in \mathcal{L}(\Theta)} \overline{W}_{vb}.$$

As $W_b \subseteq V_b$ for each $b \in B$, Theorem 20 now yields that $\Theta_{\mathcal{W}^B}$ is a Möbius number system for $\{F_b : b \in B\}$ and that for every $v \in B^\star$, $\Phi([v] \cap \Theta_{\mathcal{W}^B}) = \overline{W}_v$.

As $\psi(\Theta) = \Sigma$ and for each $v \in B^+$, $W_v = \emptyset$ iff $W_{\psi(v)} = \emptyset$, we obtain that $\psi(\Theta_{\mathcal{W}^B}) = \Sigma_{\mathcal{W}}$.

To prove that $\Sigma_{\mathcal{W}}$ is a Möbius number system, we still need to verify that if $w = \psi(u)$ for $u \in \Theta_{\mathcal{W}^B}$ then the sequence $\{F_{w_{[0,n)}}\}_{n=1}^{\infty}$ actually represents $\Phi(u)$. Fortunately, this is not difficult: Let

$$E = \{F_v(0) :\ v \text{ is a prefix of some } b \in B\}.$$

The set E is finite (and therefore compact) and lies inside the unit circle. By Theorem 8, $F_{u_{[0,k)}}(E) \to \{x\}$ for $k \to \infty$. Observe that for every n there exist k and v such that $w_{[0,n)} = \psi(u_{[0,k)})v$ and v is a prefix of some $b \in B$. Therefore, $F_{w_{[0,n)}}(0) \in F_{u_{[0,k)}}(E)$ and so $F_{w_{[0,n)}}(0) \to x$. It follows that the sequence $\{F_{w_{[0,n)}}\}_{n=1}^{\infty}$ represents $\Phi(u)$.

It remains to show that $\Phi([v] \cap \Sigma_{\mathcal{W}}) = \overline{W}_v$ for every $v \in A^\star$. Observe that we already have this result for $v = \psi(u)$ where $u \in B^\star$. Again, we prove two inclusions.

To prove $\overline{W}_v \subseteq \Phi([v] \cap \Sigma_{\mathcal{W}})$, consider any $x \in \overline{W}_v$. Then we can find $z \in \mathcal{L}(\Theta)$ such that v is a prefix of $\phi(z)$ and $x \in \overline{W}_z$. Therefore:

$$x \in \Phi([z] \cap \Theta_{\mathcal{W}^B}) \subseteq \Phi([v] \cap \Sigma_{\mathcal{W}}).$$

Let $w = \psi(u)$ with $u \in \Theta_{\mathcal{W}^B}$. Observe that $\Phi(w) = x$ iff $\bigcap_{k=1}^{\infty} \overline{W}_{\psi(u_{[0,k)})} = \{x\}$. Because the sets $\overline{W}_{w_{[0,k)}}$ form a chain, we can actually rewrite this condition as $\bigcap_{n=1}^{\infty} \overline{W}_{w_{[0,n)}} = \{x\}$. Therefore, $\Phi(w) \in \overline{W}_{w_{[0,n)}}$ for all n. Letting $w_{[0,n)} = v$, we have $\Phi([v] \cap \Sigma_{\mathcal{W}}) \subseteq \overline{W}_v$. □

As the set $B = \{v : v \in \mathcal{L}(\Sigma_{\mathcal{W}}), |v| = n\}$ contains a prefix of every $w \in \Sigma_{\mathcal{W}}$, we obtain an improvement of parts (1)–(3) of Theorem 10 in [7].

Corollary 28. *Let $\{F_a : a \in A\}$ be a Möbius iterative system. Assume that $\mathcal{W}$ is such an interval almost cover and Σ such a subshift compatible with $\mathcal{W}$ that either $Q_n(\mathcal{W}, \Sigma) > 1$ or $Q_n(\mathcal{W}, \Sigma) = 1$ and no F_v, $v \in \mathcal{L}(\Sigma_{\mathcal{W}}) \cap A^n$ is a rotation. Then $\Sigma_{\mathcal{W}}$ is a Möbius number system for the iterative system $\{F_a : a \in A\}$.*

Moreover, $\Phi([v] \cap \Sigma_{\mathcal{W}}) = \overline{W}_v$ for all $v \in A^\star$.

Proof. Choose $B = \{v : v \in \mathcal{L}(\Sigma_{\mathcal{W}}), |v| = n\}$. We need to verify that $W_b \subseteq V_b$ for each $b \in B$. If $Q_n(\mathcal{W}, \Sigma) > 1$ then even $\overline{W}_b \subseteq V_b$, while if $Q_n(\mathcal{W}, \Sigma) = 1$ and no F_b is a rotation then for all $x \in W_b$ we have $(F_n^{-1})^\bullet(x) > 1$ (the inequality is sharp), therefore $W_b \subseteq V_b$ and we can apply Corollary 27. □

3.3 Examples revisited

We now return to the three number systems presented at the end of Section 3.1 and prove that they indeed are Möbius number systems. The main practical advantage of using Theorem 20 and Corollary 27 is that they turn verifying convergence, continuity and surjectivity of $\Phi_{|\Sigma}$ into a set of combinatorial problems (finding $\mathcal{W}$ and Σ, describing $\Sigma_{\mathcal{W}}$ and finding B so that $W_b \subseteq V_b$).

First, let us revisit **Example 14**. We have the three parabolic transformations F_a, F_b and F_c. Observe that $V_a = (A, B)$, $V_b = (B, C)$ and $V_c = (C, A)$. What is more, the interval shift $\Omega = (A^\omega)_{\mathcal{V}}$ defined using the interval almost cover $\mathcal{V} = \{V_a, V_b, V_c\}$ is precisely the shift Σ obtained by forbidding the words ac, ba, cb. One way to show this is to first show that $W_{ac} = W_{ba} = W_{cb} = \emptyset$ and then verify that whenever $u \in A^n$ does not contain any forbidden factor then $W_u = F_{u_{[0,n-1)}}(W_{w_{n-1}})$.

We can prove the last equality by induction on n: For $n = 1$ the claim is trivial, while for $n = 2$ we can examine all the (finitely many) cases. Assume that the claim is true for some n and let $u \in A^{n+1}$. Then:

$$\begin{aligned} W_u &= W_{u_{[0,n)}} \cap F_{u_{[0,n)}}(W_{u_n}) = F_{u_{[0,n-1)}}(W_{u_{n-1}}) \cap F_{u_{[0,n)}}(W_{u_n}) \\ &= F_{u_{[0,n-1)}}(W_{u_{n-1}} \cap F_{u_{n-1}}(W_{u_n})) = F_{u_{[0,n-1)}}(W_{u_{[n-1,n]}}). \end{aligned}$$

Now $W_{u_{[n-1,n]}} = F_{u_{n-1}}(W_{u_n})$ by the induction hypothesis for $n = 2$ and so:

$$W_u = F_{u_{[0,n-1)}}(F_{u_{n-1}}(W_{u_n})) = F_{u_{[0,n)}}(W_{u_n}).$$

Having obtained $\Sigma = \Omega$, Theorem 26 gives us that Σ is a Möbius number system.

In the case of the continued fraction system from **Example 15**, let

$$W_{\overline{1}} = (i, -1), W_0 = (-1, 1) \text{ and } W_1 = (1, i).$$

It is straightforward to see that then $(A^\omega)_{\mathcal{W}}$ is precisely the subshift Σ defined by forbidding $00, 1\overline{1}, \overline{1}1, 101, 10\overline{1}$. It remains to choose the set $B = \{01, 0\overline{1}, 1, \overline{1}\}$ (which contains a prefix of every word $w \in \Sigma$) and verify that $W_b \subseteq V_b$ for each $b \in B$.

By Corollary 27, we conclude that $\Sigma_{\mathcal{W}}$ is a Möbius number system.

Finally, we analyze the *signed binary system* from **Example 16**. Proving that this system is indeed a Möbius number system requires a reasonable amount of computation which we have decided to skip here and present only the main points of the proof.

Define the shift Σ_0 by forbidding the words $02, 20, 12$ and $\overline{1}2$. Then let $v : \overline{\mathbb{R}} \to \mathbb{T}$ denote the inverse of the stereographic projection and consider the interval almost cover $\mathcal{W}$:

$$W_{\overline{1}} = (-1, q^-), W_0 = (h^-, h^+), W_1 = (q^+, 1), W_2 = (h^-, h^+),$$

where

$$\begin{aligned} q^- &= v(-1/4) = \frac{-8 - 15i}{17} \\ q^+ &= v(1/4) = \frac{8 - 15i}{17} \\ h^- &= v(-1/2) = \frac{-4 - 3i}{5} \\ h^+ &= v(1/2) = \frac{4 - 3i}{5}. \end{aligned}$$

Next, we should show that $\mathcal{W}$ is compatible with Σ_0 and that $(\Sigma_0)_{\mathcal{W}} = \Sigma$ is the subshift defined by the forbidden words $20, 02, 12, \overline{1}2, 1\overline{1}, \overline{1}1$. This follows from the set of identities:

$$
\begin{aligned}
F_2(\overline{W}_2) \cup F_2(\overline{W}_1) \cup F_2(\overline{W}_{\overline{1}}) &= \overline{W}_2 \\
F_0(\overline{W}_0) \cup F_0(\overline{W}_1) \cup F_0(\overline{W}_{\overline{1}}) &= \overline{W}_0 \\
F_1(\overline{W}_0) \cup F_1(\overline{W}_1) &= \overline{W}_1 \\
F_{\overline{1}}(\overline{W}_0) \cup F_{\overline{1}}(\overline{W}_{\overline{1}}) &= \overline{W}_{\overline{1}}
\end{aligned}
$$

together with

$$F_1(W_{\overline{1}}) \cap W_1 = F_{\overline{1}}(W_1) \cap W_{\overline{1}} = \emptyset.$$

It remains to take $B = \{0, 1, \overline{1}, 21, 2\overline{1}, 22\}$ and check the requirements of Corollary 27. It is easy to see that each $w \in \Sigma$ contains as a prefix a member of B. Finally, a detailed calculation (which we omit here) will verify that for every $b \in B$ we have $W_b \subseteq V_b$, therefore the signed binary system is a Möbius number system by Corollary 27.

3.4 The numbers $Q_n(\mathcal{W}, \Sigma), Q(\mathcal{W}, \Sigma)$ and $Q(\Sigma)$

In the previous section, we have shown that if $Q_n(\mathcal{W}, \Sigma) > 1$ then the interval shift $\Sigma_{\mathcal{W}}$ is a Möbius number system. We offer a partial converse to this statement.

Definition 29. For an interval almost cover $\mathcal{W}$, let $Q(\mathcal{W}, \Sigma) = \lim_{n\to\infty} \sqrt[n]{Q_n(\mathcal{W}, \Sigma)}$.

Remark. Observe that $Q_{n+m}(\mathcal{W}, \Sigma) \geq Q_n(\mathcal{W}, \Sigma) \cdot Q_m(\mathcal{W}, \Sigma)$. Therefore

$$\log Q_{n+m}(\mathcal{W}, \Sigma) \geq \log Q_n(\mathcal{W}, \Sigma) + \log Q_m(\mathcal{W}, \Sigma)$$

and so, by Fekete's lemma (see Lemma 43 in the Appendix), we have that $Q(\mathcal{W}, \Sigma)$ always exists and is equal to $\sup_{n\in\mathbb{N}} \sqrt[n]{Q_n(\mathcal{W}, \Sigma)}$. In particular, we see the inequality $Q(\mathcal{W}, \Sigma)^n \geq Q_n(\mathcal{W}, \Sigma)$ for all n.

Obviously, if $Q(\mathcal{W}, \Sigma) > 1$ then there exists n such that $Q_n(\mathcal{W}, \Sigma) > 1$ and, by Corollary 28, $\Sigma_{\mathcal{W}}$ is a Möbius number system. Let us now examine what happens when $Q(\mathcal{W}, \Sigma) < 1$.

Theorem 30. *Let $\{F_a : a \in A\}$ be a Möbius iterative system. Then there is no interval almost cover $\mathcal{W}$ and no subshift Σ compatible with $\mathcal{W}$ simultaneously satisfying:*

1. *$Q(\mathcal{W}, \Sigma) < 1$*
2. *$\Sigma_{\mathcal{W}}$ is a Möbius number system for $\{F_a : a \in A\}$.*
3. *For every $w \in \Sigma_{\mathcal{W}}$, $\bigcap_{k=1}^{\infty} \overline{W}_{w_{[0,k)}} = \{\Phi(w)\}$.*

Remark. The condition (3) is a quite reasonable demand. In particular, it is satisfied by all systems such that $\Phi([u] \cap \Sigma_{\mathcal{W}}) = \overline{W}_u$ for every $u \in A^+$.

Proof. Assume such $\mathcal{W}$ and Σ exist. Choose any number R satisfying the inequalities $Q(\mathcal{W}, \Sigma) < R < 1$. We first find $w \in \Sigma_{\mathcal{W}}$ and $x = \Phi(w)$ such that for all k it is true that $(F^{-1}_{w_{[0,k)}})^\bullet(x) \leq R^k$, then we prove that such a pair may not exist in any number system. The method used for the first step is a modification of the approach from the proof of Lemma 1.1 in [3].

For $v \in \mathcal{L}(\Sigma_{\mathcal{W}})$ denote

$$\alpha(v) = \frac{\min\{(F_v^{-1})^\bullet(x) : x \in \overline{W}_v\}}{R^{|v|}}.$$

Denote by $v^{(n)}$ the word of length at most n such that $v^{(n)} \in \mathcal{L}(\Sigma_{\mathcal{W}})$ and $\alpha(v^{(n)})$ is minimal. Label $x^{(n)}$ the minimum point corresponding to $\alpha(v^{(n)})$. Observe that

$$\alpha(v^{(n)}) = \frac{Q_{|v^{(n)}|}(\mathcal{W}, \Sigma)}{R^{|v^{(n)}|}} \leq \left(\frac{Q(\mathcal{W}, \Sigma)}{R}\right)^{|v^{(n)}|}.$$

Elementary calculus shows that for n tending to infinity $|v^{(n)}| \to \infty$ and $\alpha(v^{(n)}) \to 0$.

The crucial observation is that when $v^{(n)} = uv$ for $u, v \in A^\star$, we have the inequality $(F_u^{-1})^\bullet(x^{(n)}) \leq R^{|u|}$. Assume the contrary. Then:

$$\alpha(v^{(n)}) = \frac{(F_{uv}^{-1})^\bullet(x^{(n)})}{R^{|uv|}} = \frac{(F_v^{-1})^\bullet(F_u^{-1}(x^{(n)}))}{R^{|v|}} \cdot \frac{(F_u^{-1})^\bullet(x^{(n)})}{R^{|u|}}$$

and as $(F_u^{-1})^\bullet(x^{(n)}) > R^{|u|}$ we have

$$\alpha(v^{(n)}) > \frac{(F_v^{-1})^\bullet(F_u^{-1}(x^{(n)}))}{R^{|v|}} = \alpha(v),$$

so we should have chosen v instead of $v^{(n)}$.

As the length of $v^{(n)}$ tends to infinity, we can find $w \in A^\omega$ such that each $w_{[0,k)}$ is a prefix of infinitely many members of $\{v^{(n)}\}_{n=1}^\infty$. Take the subsequence $\{v^{(n_k)}\}_{k=1}^\infty$ so that $w_{[0,k)} = v^{(n_k)}_{[0,k)}$ for every k. As $\mathbb{T}$ is compact we can furthermore choose n_k so that $x^{(n_k)}$ converge to some x for $k \to \infty$.

We want to show that $(F^{-1}_{w_{[0,k)}})^\bullet(x) \leq R^k$ and $\Phi(w) = x$.

To prove the first claim, fix k. Then $w_{[0,k)}$ is the prefix of $v^{(n_l)}$ for all $l \geq k$. Therefore, $(F^{-1}_{w_{[0,k)}})^\bullet(x^{(n_l)}) \leq R^k$ whenever $l \geq k$. But the function $(F^{-1}_{w_{[0,k)}})^\bullet$ is continuous and $x^{(n_l)}$ converge to x. This means $(F^{-1}_{w_{[0,k)}})^\bullet(x) \leq R^k$.

To prove $\Phi(w) = x$, fix k again. Then whenever $l \geq k$, $x^{(n_l)} \in \overline{W}_{w_{[0,n_l)}} \subseteq \overline{W}_{w_{[0,k)}}$. Because $\overline{W}_{w_{[0,k)}}$ is closed and x the limit of $x^{(n_l)}$, we have $x \in \overline{W}_{w_{[0,k)}}$. Because k in the above argument was arbitrary, $x \in \overline{W}_{w_{[0,k)}}$ for all k. By the assumption (3) on Σ and $\mathcal{W}$ we obtain $\Phi(w) = x$.

It remains to show that such a pair w, x can not exist in any Möbius number system. We will prove that when $(F^{-1}_{w_{[0,k)}})^\bullet(x) \leq R^k$ for all k then it is not true that $\overline{V}_{w_{[0,k)}} \to \{x\}$.

We will need a little observation: For any $\gamma > 1$ there exists $\delta > 0$ such that whenever $x, y \in \mathbb{T}$ are such that $\rho(x,y) < \delta$ then $\forall a \in A$, $(F_a^{-1})^\bullet(y) \leq \gamma \cdot (F_a^{-1})^\bullet(x)$.

This observation is actually an easy consequence of $F^\bullet(x) > 0$ and $F^\bullet$ being (uniformly) continuous for any MT F: The function $\ln(F^\bullet(x))$ is continuous, therefore for any $\varepsilon > 0$ there exists δ such that $\rho(x,y) < \delta$ implies:

$$\begin{aligned} \ln(F^\bullet(y)) &\leq \ln(F^\bullet(x)) + \varepsilon \\ F^\bullet(y) &\leq e^\varepsilon \cdot F^\bullet(x) \end{aligned}$$

Letting $F = F_a^{-1}$ and $\varepsilon = \ln\gamma$, we obtain some $\delta_a > 0$. Now let $\delta = \min\{\delta_a : a \in A\}$ to prove our observation.

Choose $\gamma > 1$ so that $\gamma R < 1$ and let $\delta > 0$ be such that if $\rho(z,y) < \delta$ then $\forall a \in A$, $(F_a^{-1})^\bullet(y) \geq \gamma \cdot (F_a^{-1})^\bullet(z)$.

It will be enough to prove that whenever $\rho(x,y) < \delta$ and $k \geq 1$, the inequality

$$(F^{-1}_{w_{[0,k)}})^\bullet(y) \leq (\gamma R)^k$$

holds. As $(\gamma R)^k < 1$, we have $(x-\delta, x+\delta) \cap \overline{V}_{w_{[0,k)}} = \emptyset$, a contradiction with $\Phi(w) = x$ by part (6) of Theorem 8.

Denote $x_i = F^{-1}_{w_{[0,i)}}(x)$ and $y_i = F^{-1}_{w_{[0,i)}}(y)$ and let $k \geq 1$.

For $k = 1$ we have from the definition of δ:

$$(F^{-1}_{w_0})^\bullet(y) \leq \gamma (F^{-1}_{w_0})^\bullet(x) \leq \gamma R.$$

For $k > 1$ write:

$$\begin{aligned} (F^{-1}_{w_{[0,k)}})^\bullet(x) &= (F^{-1}_{w_0})^\bullet(x) \cdot (F^{-1}_{w_1})^\bullet(x_1) \cdots (F^{-1}_{w_{k-1}}{}^\bullet)(x_{k-1}) \\ (F^{-1}_{w_{[0,k)}})^\bullet(y) &= (F^{-1}_{w_0})^\bullet(y) \cdot (F^{-1}_{w_1})^\bullet(y_1) \cdots (F^{-1}_{w_{k-1}}{}^\bullet)(y_{k-1}). \end{aligned}$$

We have $\rho(x,y) < \delta$. Let I be the shorter closed interval between x and y. Then for $z \in I$ we have $\rho(x,z) < \delta$ and so $(F^{-1}_{w_0})^\bullet(z) < 1$. Thus $F^{-1}_{w_0}$ contracts I and so

$$\rho(x_1, y_1) = \rho(F^{-1}_{w_0}(x_0), F^{-1}_{w_0}(y_0)) \leq \rho(x_0, y_0) < \delta.$$

By the same argument, $F^{-1}_{w_1}$ contracts the shorter interval between x_1 and y_1 and so $\rho(x_2, y_2) \leq \rho(x_1, y_1) < \delta$. Continuing in this manner, we obtain that for all i we have $\rho(x_i, y_i) < \delta$ and so $(F^{-1}_{w_i})^\bullet(y_i) \leq \gamma F^{-1\bullet}_{w_i}(x_i)$. Calculating the products, we obtain:

$$(F^{-1}_{w_{[0,k)}})^\bullet(y) \leq \gamma^k (F^{-1}_{w_{[0,k)}})^\bullet(x) \leq \gamma^k R^k = (\gamma R)^k < 1,$$

concluding the proof. □

One might ask whether the condition (3) was necessary in Theorem 30. There exist trivial and less trivial examples showing that (1) and (2) can indeed happen at once.

As a trivial example, consider the signed binary system from Example 16 with $W_i = \mathbb{T}$ for $i \in \{0, 1, \overline{1}, 2\}$ and $\Sigma \subseteq \{0, 1, \overline{1}, 2\}^\omega$ defined by forbidding words $20, 02, 12, \overline{1}2, 1\overline{1}, \overline{1}1$. It is easy to see that Σ is compatible with $\mathcal{W}$ and $\Sigma_{\mathcal{W}} = \Sigma$. As we have already shown, Σ is a Möbius number system. Observe that F_0 is hyperbolic with fixed points $\pm i$ and $(F_0^{-1})^\bullet(i) = \frac{1}{2}$. Because $0^\omega \in \Sigma$, we have $Q_n(\mathcal{W}, \Sigma) \leq \frac{1}{2^n}$ for each n and so $Q(\mathcal{W}, \Sigma) \leq \frac{1}{2} < 1$.

One might argue that we have cheated by taking $W_i = \mathbb{T}$, that maybe some sort of size limit on W_i or allowing only shifts of the form $(A^\omega)_{\mathcal{W}}$ might prevent (1) and (2) from happening simultaneously. While we are unable to account for all the possible modifications of Theorem 30, we show a less trivial number system that demonstrates the limitations of Theorem 30.

Example 31. Take the hyperbolic number system for $n = 4$ and $r = \sqrt{2} - 1$ from [6]. This iterative system consists of four hyperbolic transformations as depicted in Figure 3.7. This system consists of four hyperbolic MTs $\{F_0, F_1, F_2, F_3\}$ conjugated by rotation. The transformation F_1 is a contraction to 1, F_0 has the stable fixed point $-i$ and unstable fixed point i, $F_2 = F_0^{-1}$ and $F_3 = F_1^{-1}$. Moreover, the parameter of the contraction is chosen so that

$$V_0 = (e^{-3\pi/4i}, e^{-\pi/4i}),\ V_1 = (e^{-\pi/4i}, e^{\pi/4i}),\ V_2 = (e^{\pi/4i}, e^{3\pi/4i}),\ V_3 = (e^{3\pi/4i}, e^{-3\pi/4i}).$$

As $\{V_i : i = 0, 1, 2, 3\}$ covers $\mathbb{T}$, we obtain that the shift Ω from Theorem 26 is a Möbius number system. An argument similar to the one used when analyzing Example 14 shows that Ω is defined by the forbidden factors $02, 20, 13, 31$.

However, we can obtain Ω as the interval shift of a rather different interval almost cover. Take $W_i = \mathbb{T} \setminus \overline{V}_i$ for $i = 0, 1, 2, 3$. It turns out that $(A^\omega)_{\mathcal{W}} = \Omega$. Moreover, W_{0^n} always contains -1, the unstable fixed point of F_0. Then $(F_0^{-1})^\bullet(-1) = q < 1$ gives us $Q_n(\mathcal{W}, \Sigma) \leq q^n$ and so $Q(\mathcal{W}, \Sigma) \leq q < 1$.

We have shown that if $Q(\mathcal{W}, \Sigma) > 1$ then $\Sigma_{\mathcal{W}}$ is a Möbius number system, while if $Q(\mathcal{W}, \Sigma) < 1$ then $\Sigma_{\mathcal{W}}$ might be a number system only if it does not satisfy the rather reasonable condition $\Phi([u] \cap \Sigma_{\mathcal{W}}) = \overline{W}_u$.

In the remaining case $Q(\mathcal{W}, \Sigma) = 1$, the scales might tilt either way. We know that if $Q_n(\mathcal{W}, \Sigma) = 1$ for some n and no F_v, $v \in A^n$ is a rotation, then $\Sigma_{\mathcal{W}}$ is a Möbius number system by Corollary 28. However, having $Q(\mathcal{W}, \Sigma) = 1$ together with all the transformations F_u different from rotations is not enough to obtain a Möbius number system:

Example 32. Take the three parabolic transformations from Example 14 and let $W_a = (C, A)$, $W_b = (A, B)$, $W_c = (B, C)$ and $\Sigma = A^\omega$. It is easy to see that $W_{e^n} = W_e$ for every $e \in \{a, b, c\}$ while $W_u = \emptyset$ whenever u contains two different letters. Therefore $\Sigma_{\mathcal{W}} = \{a^\omega, b^\omega, c^\omega\}$, so this is not a Möbius number system.

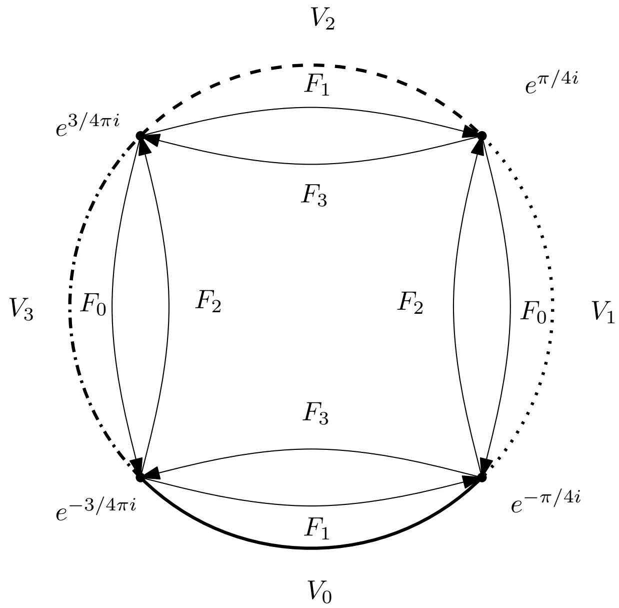

Figure 3.7: The hyperbolic number system for $n = 4$, $r = \sqrt{2} - 1$

Now consider the transformation F_{a^n}. It is parabolic with the fixed point a and $(F_{a^n}^{-1}(x))^{\bullet}$ is minimal for $x = C$. Denote $f(n) = (F_{a^n}^{-1})^{\bullet}(C)$. Easily, $0 < f(n) < 1$. What is more, $f(n) = |g(n)|^{-2}$ where g is a linear polynomial in n because F_a is parabolic (and therefore the matrix of F_a is similar to $\left(\begin{smallmatrix} 1 & \pm 1 \\ 0 & 1 \end{smallmatrix}\right)$).

It follows that $\sqrt[n]{f(n)} \to 1$ and the same argument holds for b and c, so we have $Q(\mathcal{W}, \Sigma) = 1$.

As we have shown, the number $Q(\mathcal{W}, \Sigma)$, while useful, is not a perfect solution for characterizing Möbius number systems. We will shortly mention one direction (due to Petr Kůrka), in which it can be improved.

Let $Q(\Sigma) = \sup\{Q(\mathcal{W}, \Sigma) : \mathcal{W} \text{ is an interval almost cover compatible with } \Sigma\}$.

The number $Q(\Sigma)$ depends only on the iterative system and Σ. Observe that $Q(\Sigma) > 1$ means that there exists an interval almost cover $\mathcal{W}$ such that $\Sigma_{\mathcal{W}}$ is a Möbius number system while $Q(\Sigma) < 1$ signifies that any possible $\Sigma_{\mathcal{W}}$ is either not a Möbius number system at all or it is badly behaved with respect to the condition $\Phi([u] \cap \Sigma_{\mathcal{W}}) = \overline{W}_u$. It remains open if $Q(\Sigma)$ has any other interesting properties. For example, it might be possible that $Q(\Sigma) < 1$ implies that no $\Sigma_{\mathcal{W}}$ is a Möbius system.

3.5 Existence results

The most basic existence question one might ask is whether there exists any Möbius number system at all for a given iterative system. This problem is not solved yet, however, we can offer a partial answer.

Originally, the following theorem comes from [9]. We have slightly modified it so that it refers to $\mathbb{T}$ instead of the extended real line. It gives one sufficient and one necessary condition for the existence of a Möbius iterative system.

Theorem 33 (Theorem 9, [9])**.** *Let $F : A^+ \times \mathbb{T} \to \mathbb{T}$ be a Möbius iterative system.*

1. *If $\overline{\bigcup_{u \in A^+} V_u} \neq \mathbb{T}$ then $\Phi(\mathbb{X}_F) \neq \mathbb{T}$.*
2. *If $\{V_u : u \in A^+\}$ is a cover of $\mathbb{T}$ then $\Phi(\mathbb{X}_F) = \mathbb{T}$ and there exists a subshift $\Sigma \subseteq \mathbb{X}_F$ on which Φ is continuous and $\Phi(\Sigma) = \mathbb{T}$.*

Note that if the condition of (1) is satisfied then there is no Möbius number system for $\{F_a : a \in A\}$. We improve Theorem 33 by weakening the condition in part (2).

Observe that (by compactness of $\mathbb{T}$) if $\{V_u : u \in A^+\}$ cover $\mathbb{T}$ then there exists a finite $B \subseteq A^+$ such that $\{V_b : b \in B\}$ cover $\mathbb{T}$. In the spirit of our previous results, we show that in part (2) of Theorem 33 it suffices to demand that there exists a finite $B \subseteq A^+$ such that the *closed sets* $\{\overline{V}_b : b \in B\}$ cover $\mathbb{T}$.

Corollary 34. *Let $\{F_a : a \in A\}$ be a Möbius iterative system. Assume that there exists a finite subset B of A^+ such that $\{\overline{V}_b : b \in B\}$ is a cover of $\mathbb{T}$. Then there exists a subshift $\Sigma \subseteq A^\omega$ that, together with the iterative system $\{F_a : a \in A\}$, forms a Möbius number system.*

Proof. Take B as our new alphabet. The set $\{V_b : b \in B\}$ is an interval almost cover. Therefore, we can apply Theorem 26 and obtain the Möbius number system $\Omega \subseteq B^+$.

We now proceed similarly to the proof of Corollary 27: Denote by ψ the natural map from B^ω to A^ω. Let $\Sigma = \bigcup_{i=0}^{d} \sigma^d(\psi(\Omega))$ where $d = \max\{|b| : b \in B\}$. It is then straightforward to show that Σ is a Möbius number system for the iterative system $\{F_a : a \in A\}$. □

While Theorem 33 gives a necessary condition for a Möbius number system to exist, this condition is not very comfortable to use. In the spirit of [8], we offer a condition that is easier to check.

Let $\{F_a : a \in A\}$ be a Möbius iterative system. A nonempty closed set $W \subseteq \mathbb{T}$ is *inward* if $\bigcup_{a \in A} F_a(W) \subseteq \mathrm{Int}(W)$. All iterative systems have the trivial inward set $\mathbb{T}$ and some systems have nontrivial inward sets as well.

Theorem 35. *Let $\{F_a : a \in A\}$ be an iterative system with a nontrivial inward set. Then there is no Möbius number system for $\{F_a : a \in A\}$.*

Proof. Assume that W is a nontrivial inward set. Let $z \notin W$. Because W^c is open there exists an open interval I disjoint with W and containing z. Assume that $\Phi(w) = z$. Then $\lim_{k\to\infty} |F^{-1}_{w_{[0,k)}}(I)| = 2\pi$. However, $\mathrm{Int}\, W$ is nonempty so there exists an open interval $J \subseteq \mathrm{Int}\, W$. Now for all k we have $F_{w_{[0,k)}}(J) \subseteq W$ so $J \cap F^{-1}_{w_{[0,k)}}(I) = \emptyset$. But then $|F^{-1}_{w_{[0,k)}}(I)| \leq 2\pi - |J|$, a contradiction. □

3.6 Empirical data

Searching for the solution to the existence problem, we have used a numerical simulation to obtain insight in the behavior of MTs. The results suggest that closing the gap in Theorem 33 is an achievable task.

We have studied the behavior of the iterative system $\{F_a, F_b\}$ consisting of two hyperbolic transformations. The transformation F_a has fixed points 1 (stable) and $-i$ (unstable), while the transformation F_b has fixed points -1 (stable) and i (unstable). We have parameterized F_a, F_b by the values q_a, q_b of $(F_i)^\bullet$ at stable points, so we have:

$$\begin{aligned} F_a &= \frac{1}{2\sqrt{q_a}} \begin{pmatrix} 1+q_a-i(1-q_a) & 1-q_a+i(1-q_a) \\ 1-q_a-i(1-q_a) & 1+q_a+i(1-q_a) \end{pmatrix} \\ F_b &= \frac{1}{2\sqrt{q_b}} \begin{pmatrix} 1+q_b-i(1-q_b) & -1+q_b-i(1-q_b) \\ -1+q_b+i(1-q_b) & 1+q_b+i(1-q_b) \end{pmatrix}. \end{aligned}$$

Denote

$$Y = \{(q_a, q_b) : q_a, q_b \in (0,1), \text{ there exists a Möbius number system for } \{F_a, F_b\}\}.$$

We wrote a C program that tries various pairs (q_a, q_b), constructs F_a, F_b, then computes the intervals $\{\overline{V}_v : |v| \le m\}$ (where m is the number of iterations to consider) and finally checks whether these intervals cover the whole $\mathbb{T}$. If they do, the program puts a white dot on the corresponding place in the graph, otherwise we leave it black.

As we are interested in characterization, we have plotted (in gray) a second set in the graph: The set U of all choices of (q_a, q_b) such that the iterative system $\{F_a, F_b\}$ has a nontrivial inward set. The formula for U, as shown in [8], is $U = \bigcup_{n\in\mathbb{Z}} U_n$, where for $n > 0$ we have:

$$\begin{aligned} U_0 &= U_{ab} \cap \left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right) \\ U_n &= U_{a^n b} \cap U_{a^{n+1} b} \cap \left(\frac{1}{\sqrt[n]{2}}, \frac{1}{\sqrt[n+1]{2}}\right) \times \left(0, \frac{1}{2}\right) \\ U_{-n} &= U_{ab^n} \cap U_{ab^{n+1}} \cap \left(0, \frac{1}{2}\right) \times \left(\frac{1}{\sqrt[n]{2}}, \frac{1}{\sqrt[n+1]{2}}\right) \end{aligned}$$

and $U_v = \{(q_a, q_b) : F_v \text{ is hyperbolic}\}$ for $v \in A^\star$. For practical reasons, we have only drawn the sets U_n with $|n| \le m$ (the same m as the number of iterations in the first part of the program). The result for $m = 10$ and resolution 1000×1000 is shown in Figure 3.8.

By Theorem 35, $U \cap Y = \emptyset$. We are interested in the size of the complement of $U \cup Y$ in $(0,1)^2$. It turns out that $U \cup Y$ covers most of the unit square and U and Y appear to fit rather well together. The area between the two sets is likely to get smaller and smaller as we let m grow, possibly shrinking to zero in the limit.

However, there might still exist points that do not belong either to U or to Y. We (vaguely) conjecture, that $U \cup Y$ is equal to the whole $(0,1)^2$ perhaps up to a small set of exceptional points.

3.7 Subshifts not admitting Möbius number systems

An interesting question one can pose is whether, given a subshift $\Sigma \subseteq A^\omega$, there exists a collection of MTs $\{F_a : a \in A\}$ such that Σ is a Möbius number system.

Trivially, the cardinality of Σ must be precisely continuum, as for smaller Σ there is no projection from Σ onto $\mathbb{T}$. We offer a less trivial necessary condition.

A (non-erasing) *substitution* is any mapping $\psi : B \to A^+$. We can extend ψ to a map from B^ω to A^ω in a natural way. As there is no risk of confusion, we will denote the resulting map by ψ as well and call it a *substitution map*. Observe that $\psi : B^\omega \to A^\omega$ is continuous in the product topology. We have already met substitution maps in the proof of Corollary 27.

Theorem 36. *Let $\Sigma \subseteq A^\omega$ be a Möbius number system for an iterative system $\{F_a : a \in A\}$. Then for all alphabets B and all substitution maps ψ we have $\Sigma \neq \psi(B^\omega)$.*

Proof. Assume that there exist B and ψ for which the theorem is false.

Then B^ω together with the maps $\{G_b : b \in B\}$ such that $G_b = F_{\psi(b)}$ is a Möbius number system. Denote by Φ the resulting projection of B^ω to $\mathbb{T}$ and observe that $\Phi = \Phi_\Sigma \circ \psi$ where $\Phi_\Sigma : \Sigma \to \mathbb{T}$ is the number system on Σ. We see that Φ is surjective and continuous on B^ω.

Fix $x \in \mathbb{T}$ and $u \in B^\omega$. There exists $v \in B^\omega$ with $\Phi(v) = x$. Consider the sequence $\{\Phi(u_{[0,k)}v)\}_{k=1}^\infty$. We have

$$\Phi(u_{[0,k)}v) = G_{u_{[0,k)}}(\Phi(v)) = G_{u_{[0,k)}}(x).$$

However, Φ is continuous, so $\Phi(u_{[0,k)}v)$ tends to $\Phi(u)$ when k tends to infinity. Therefore, $G_{u_{[0,k)}}(x) \to \Phi(u)$.

As x, u were arbitrary, we have shown that for every $u \in B^\omega$ and each point x of $\mathbb{T}$ the sequence$\{G_{u_{[0,k)}}(x)\}_{k=1}^\infty$ converges to $\Phi(u)$.

Let u be periodic with some period $w \in B^+$. Then $\Phi(u)$ is the (stable) fixed point of G_w. Therefore G_w may not be elliptic ($\Phi(u)$ would not be defined) nor hyperbolic (the images of the unstable fixed point of G_w would not converge). This means that for all $w \in B^+$ the transformation G_w must be parabolic.

If all the the transformations G_b, $b \in B$ were parabolic with the same fixed point then all the transformations $G_w, w \in B^+$ would be parabolic with the same fixed point and we would not be able to represent anything except this fixed point. Therefore there exist $a, b \in B$ such that G_a, G_b have different fixed points. The

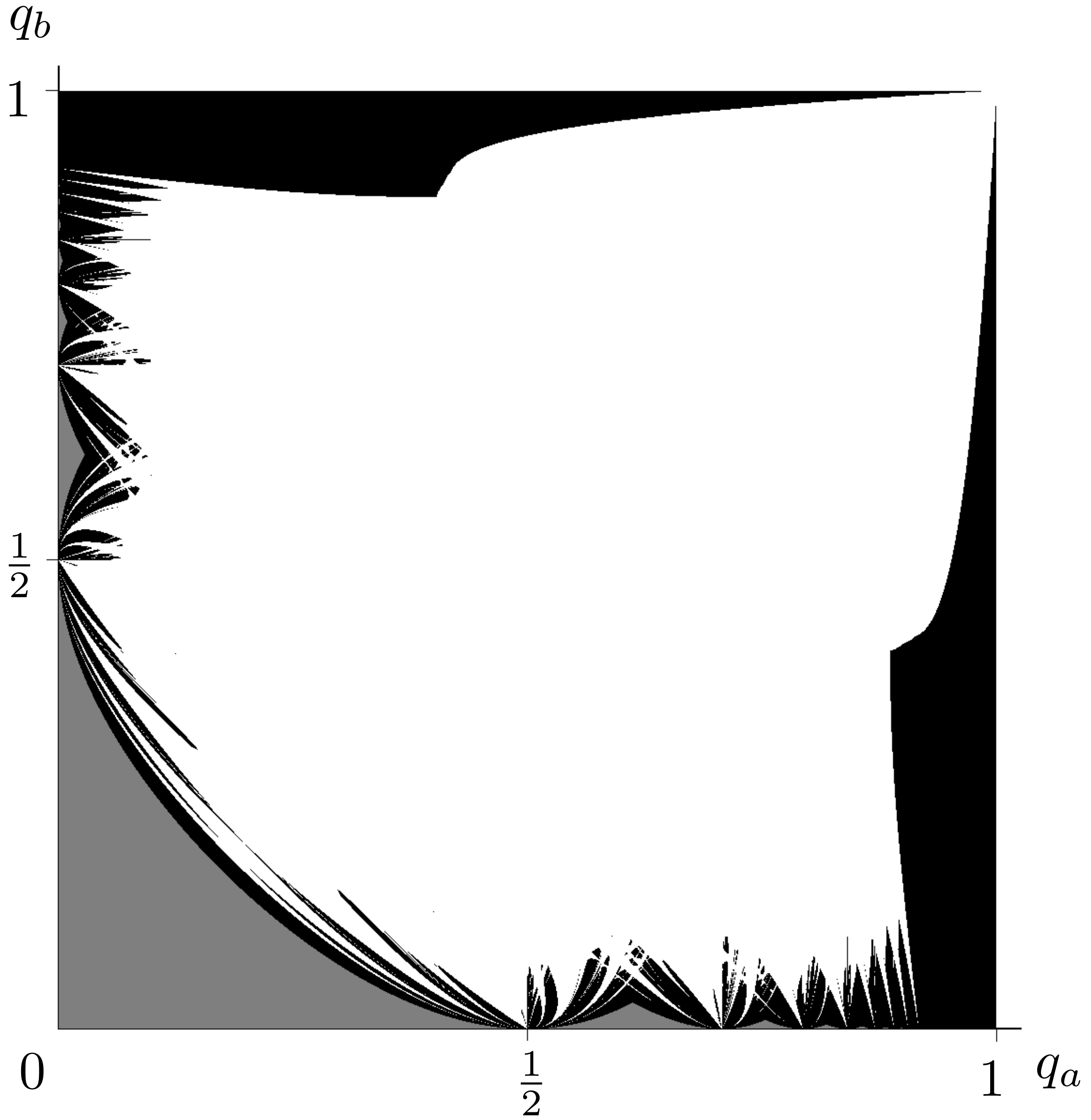

Figure 3.8: The graph of U and Y for the depth $m = 10$ and resolution 1000×1000.

rest of the proof consists of a straightforward (but technical) calculation that such a situation is impossible.

Choose $a, b \in B$ so that G_a, G_b have different fixed points. We know that G_{ab} and G_{aab} must both be parabolic. Without loss of generality assume that G_a is similar to the matrix $J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Write $G_a = MJM^{-1}$ where M is a regular matrix. Recall that a transformation F is parabolic iff $\mathrm{Tr}(F)^2 = 4$ and observe that:

$$\mathrm{Tr}(G_{ab}) = \mathrm{Tr}(MJM^{-1}G_b) = \mathrm{Tr}(JM^{-1}G_bM)$$

and

$$\mathrm{Tr}(G_{aab}) = \mathrm{Tr}(MJ^2M^{-1}G_b) = \mathrm{Tr}(J^2M^{-1}G_bM),$$

where we used the equality $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Let

$$M^{-1}G_bM = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and observe that $(a+d)^2 = \mathrm{Tr}(G_b)^2 = 4$.

We now have

$$JM^{-1}G_bM = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}, \quad J^2M^{-1}G_bM = \begin{pmatrix} a+2c & b+2d \\ c & d \end{pmatrix}.$$

Calculating the traces, we obtain the equalities

$$(a+2c+d)^2 = (a+c+d)^2 = (a+d)^2 = 4$$

that can only be satisfied when $c = 0$. But then matrices J and $M^{-1}G_bM$ share the eigenvector $(1,0)^T$ and so $G_a = MJM^{-1}$ and G_b share the eigenvector $M(1,0)^T$. However, eigenvectors of matrices are in one to one correspondence with fixed points of Möbius transformations (see the proof of Lemma 39 in the Appendix) and so G_a, G_b have the same fixed point, a contradiction. □

Theorem 36 tells us in particular that there are no Möbius number systems on the full shift. We must always take something smaller in order to limit bad concatenations. An important weakness of Theorem 36 is that the set $\psi(B^\omega)$ *need not be a subshift*: while it is always closed, σ-invariance is not guaranteed.

However, there are cases when $\psi(B^\omega)$ is a nontrivial subshift. Consider the Fibonacci shift Σ_F defined on the alphabet $\{0,1\}$ by forbidding the factor 11. It is easy to see that $\Sigma_F = \psi(\{0,1\}^\omega)$ under the substitution $\psi : 0 \mapsto 0, 1 \mapsto 10$. Therefore, Σ_F can never be a Möbius number system.

3.8 Sofic Möbius number systems

In this section, we will explore another facet of Möbius number systems. As every number system is a subshift, we can ask how complicated (in the sense of formal language theory, not information theory) is the language of this subshift.

A subshift Σ is of *finite type* if Σ can be defined using a finite set of forbidden words (note that this was the case in all our example subshifts). A subshift Σ is called *sofic* if and only if the language of Σ is regular (recognizable by a finite automaton). A little thought gives us that subshifts of finite type are always sofic. Sofic subshifts and subshifts of finite type are quite popular in practice, as they are easier to manipulate than general subshifts. There are numerous results and algorithms available for sofic subshifts and subshifts of finite type.

The papers [6] and [9] contain several examples of Möbius number systems that are subshifts of finite type. Furthermore, Proposition 5 in [9] states a sufficient condition for a number system to be of finite type. We now present a similar condition for $\Sigma_{\mathcal{W}}$ to be sofic.

Theorem 37. *Let Σ be a sofic subshift and $\mathcal{W}$ such an interval almost cover that the set $\{F_v^{-1}(W_v) : v \in A^\star\}$ is finite. Then $\Sigma_{\mathcal{W}}$ is sofic.*

Proof. We construct a finite automaton $\mathcal{A}$ that recognizes all the words $v \in A^\star$ such that $W_v \neq \emptyset$. We then intersect the resulting regular language with the language $\mathcal{L}(\Sigma)$ to obtain $\mathcal{L}(\Sigma_{\mathcal{W}})$. Because regular languages are closed under intersection, $\mathcal{L}(\Sigma_{\mathcal{W}})$ is regular.

The states of our automaton will be all the sets $Z_v = F_v^{-1}(W_v)$, $v \in A^\star$. We let Z_λ to be the initial state and all states except $\emptyset$ to be accepting states. A transition labelled by the letter a leads from Z_v to Z_{va} for every $v \in A^\star$ and every $a \in A$.

Observe that when $Z_v = Z_u$ then $Z_{va} = Z_{ua}$, so the definition of our automaton is correct:

$$Z_{va} = F_a^{-1}F_v^{-1}(W_v \cap F_v(W_a)) = F_a^{-1}(F_v^{-1}(W_v)) \cap F_a^{-1}(W_a) = F_a^{-1}(Z_v) \cap Z_a$$

Similarly, $Z_{ua} = F_a^{-1}(Z_u) \cap Z_a$ and as $Z_u = Z_v$ we obtain $Z_{va} = Z_{ua}$.

To finish the proof, we observe that the automaton $\mathcal{A}$ accepts the word v iff $Z_v \neq \emptyset$. Because $Z_v \neq \emptyset$ iff $W_v \neq \emptyset$, $\mathcal{A}$ recognizes precisely those $v \in A^\star$ with $W_v \neq \emptyset$. $\square$

Under an additional assumption, we can prove the converse of Theorem 37:

Theorem 38. *Assume that $\mathcal{W}$ is an interval almost cover compatible with the subshift Σ. Let the subshift $\Sigma_{\mathcal{W}}$ be a sofic Möbius number system such that $\Phi([v] \cap \Sigma_{\mathcal{W}}) = \overline{W_v}$ for every word v. Then the set $\{F_v^{-1}(W_v) : v \in A^\star\}$ is finite.*

Proof. To prove this theorem we define a chain of several finite sets, each obtained from the previous, with the final set being $\{F_v^{-1}(W_v) : v \in A^\star\}$.

Denote by $\mathcal{F}(v)$ the *follower set of v in $\Sigma_{\mathcal{W}}$*, i.e. the set of all words $u \in A^\star$ such that $vu \in \mathcal{L}(\Sigma_{\mathcal{W}})$. By the Myhill-Nerode theorem, we have that if $\Sigma_{\mathcal{W}}$ is sofic, then $\{\mathcal{F}(v) : v \in A^\star\}$ is a finite set. Let $v \in A^\star$ and denote $\mathcal{F}^\omega(v) = \{w \in A^\omega : \forall k, w_{[0,k)} \in \mathcal{F}(v)\}$. The set $\{\mathcal{F}^\omega(v) : v \in A^\star\}$ is finite as each $\mathcal{F}^\omega(v)$ depends only on $\mathcal{F}(v)$.

A little thought gives us that $\mathcal{F}^\omega(v) = \{w \in A^\omega : vw \in \Sigma_{\mathcal{W}}\}$. Finally, denote $Z_v = \Phi(\mathcal{F}^\omega(v))$ and observe again that the set $\{Z_v : v \in A^\star\}$ is finite. It remains to notice that

$$Z_v = \Phi(\{w \in A^\omega : vw \in \Sigma_{\mathcal{W}}\}) = F_v^{-1}(\Phi([v] \cap \Sigma_{\mathcal{W}})) = F_v^{-1}(W_v)$$

to see that the set $\{F_v^{-1}(W_v) : v \in A^\star\}$ must be finite. $\square$

Note that Theorems 37 and 38 give us that if Σ is sofic then the interval shifts considered in Theorem 20 or Corollary 27 are sofic if and only if $\{F_v^{-1}(W_v) : v \in A^\star\}$ is a finite set.

Conclusions and open problems

In the whole thesis we have explored various topics in the theory of Möbius number systems. We have obtained tools to prove that a subshift is a Möbius number system for a given iterative system as well as various existence results and criteria for sofic number systems. However, we have left quite a few open problems, practical as well as theoretical, in this area.

The first open question is how "nice" the representation of points is for sequences of the type $\{F_{w_{[0,n)}}\}_{n=1}^{\infty}$. In general, we can have a sequence of Möbius transformations $\{F_n\}_{n=1}^{\infty}$ that represents $x \in \mathbb{T}$, yet for all $z \in \mathbb{T}$ the sequence $\{F_n(z)\}_{n=1}^{\infty}$ does not converge to x. We think that such sequences must always exhibit low speed of convergence of $\{F_n(0)\}_{n=1}^{\infty}$ to x. We further conjecture that in the more special case of $w \in A^{\omega}$ and $\{F_{w_{[0,n)}}\}_{n=1}^{\infty}$ representing some x, the set of points $z \in \mathbb{T}$ such that $F_{w_{[0,n)}}(z)$ converges to x is nonempty, perhaps even of measure one.

Interval shifts together with computing $Q(\mathcal{W}, \Sigma)$ and $Q(\Sigma)$ seem to be useful when dealing with concrete examples. Theorem 20 and Corollary 26 offer practical tools to prove that a given subshift is a Möbius number system for a given iterative system. We wonder how the available toolbox for this kind of proof could be further improved. We see numerous areas open to incremental improvements.

For examples and applications, we would like to have a sufficient and necessary condition for the existence of a Möbius number system for a given iterative system. Ideally, this condition should be effectively verifiable (for reasonable iterative systems, say when real and imaginary parts of coefficients are rational). While we doubt that a general effective algorithm exists, improvements in the tools for proving the existence of Möbius iterative systems would be welcome indeed.

Regarding iterative systems, we know that $Q(\mathcal{W}, \Sigma) > 1$ or $Q(\Sigma) > 1$ guarantees the existence of a number system while $Q(\mathcal{W}, \Sigma) < 1$ or $Q(\Sigma) < 1$ means that if a number system $\Sigma_{\mathcal{W}}$ exists at all then it is going to be badly behaved with respect to $\mathcal{W}$. We conjecture that when $Q(\Sigma) < 1$ there actually does not exist any number system on any interval shift $\Sigma_{\mathcal{W}} \subseteq \Sigma$.

Again, we do not know if there exist general algorithms for computing $Q(\mathcal{W}, \Sigma)$ and $Q(\Sigma)$ even when Σ is of finite type or sofic. Perhaps we could gain some inspiration in the symbolic dynamic tools for computing entropy of shifts of finite type. Being able to compute, or at least estimate $Q(\mathcal{W}, \Sigma)$ would make examining examples easier.

Another, perhaps less practical, but combinatorially interesting problem is when a given subshift Σ can be a Möbius number system. So far, we have some sufficient and some necessary conditions and a large gap in between.

To manipulate number systems, it would be nice to have a sofic Möbius number system. Theorems 37 and 38 offer useful checks to perform when verifying if a number system is sofic. What is completely missing is a condition, similar to Theorem 20, for the existence of a sofic number system for a given iterative set. We are hoping that obtaining such a result is possible but the current amount of knowledge on sofic number systems is rather small. For example, we can not even tell whether existence of a Möbius number system implies existence of a sofic system for the same iterative system or not.

A large part the complexity of above problems seems to come not from the number systems themselves but from the fact that we don't properly understand how do large numbers of MTs compose (or, equivalently, how long sequences of matrices multiply). This suggests that maybe the way forward lies in studying the limits of products of matrices. Unfortunately, this area is full of hard questions, see for example [2].

Hard problems notwithstanding , we conclude on a positive note: Although there are numerous open questions about Möbius number systems, and some properties of these systems might turn out to be undecidable, current tools do allow us to deal with systems that are likely to be used elsewhere (for example, the continued fraction number system).

Appendix

The Appendix contains various proofs that we felt should be included in this thesis, yet their length or technical nature would disturb the flow of the rest of the text. Note that these are all well known results; the proofs here are just for the sake of completeness and understanding of the topic.

First, we present a series of three lemmas concerning the classification of disc preserving MTs into elliptic, parabolic and hyperbolic transformations. We will use a bit of linear algebra machinery. We will understand each disc preserving Möbius transformation F both as a map and as the corresponding normalized matrix

$$F = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}, \ |\alpha|^2 - |\beta|^2 = 1.$$

An important role in the following proofs belongs to the eigenvalues of the matrix F. However, these eigenvalues are not uniquely defined: The Möbius transformation F always has two corresponding normalized matrices $F, -F$, therefore it also has two different sets of eigenvalues. We deal with this problem by always fixing one matrix of F for the whole proof.

Lemma 39. *Let F be a disc preserving Möbius transformation. Fix a matrix of F such that* $\det F = 1$*. Then the following holds:*

1. *F is elliptic iff the eigenvalues of F are not real iff F has one fixed point inside and one fixed point outside of $\mathbb{T}$ (the outside point might be ∞),*

2. *F is parabolic iff F has the single eigenvalue equal to* 1 *or* -1 *iff F has a single fixed point and it lies on $\mathbb{T}$,*

3. *F is hyperbolic iff F has two different real eigenvalues iff F has two different fixed points, both lying on $\mathbb{T}$.*

Proof. We begin by providing a connection between $(\operatorname{Tr} F)^2$ and the eigenvalues of F. The characteristic polynomial of F is:

$$(\alpha - \lambda)(\overline{\alpha} - \lambda) - \beta\overline{\beta} = |\alpha|^2 - |\beta|^2 - \operatorname{Tr} F \cdot \lambda + \lambda^2 = 1 - \operatorname{Tr} F \cdot \lambda + \lambda^2.$$

We see that for $(\operatorname{Tr} F)^2 < 4$, F has two distinct complex conjugate eigenvalues, while if $(\operatorname{Tr} F)^2 > 4$, then F has two distinct real eigenvalues. Finally, if $(\operatorname{Tr} F)^2 = 4$, there is only one eigenvalue $\lambda = 1$ or $\lambda = -1$.

Observe that for all $z \in \mathbb{C}$ such that $F(z) \neq \infty$ we have:

$$F \cdot \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha z + \beta \\ \overline{\beta} z + \overline{\alpha} \end{pmatrix} = (\overline{\beta} z + \overline{\alpha}) \cdot \begin{pmatrix} F(z) \\ 1 \end{pmatrix}.$$

For $z \in \mathbb{C}$ we obtain that z is a fixed point of F iff $(z, 1)^T$ is an eigenvector of F. Similarly, the point ∞ is fixed iff $(1, 0)^T$ is an eigenvector of F.

Let λ be an eigenvalue of F. If F is not the identity, then the eigenspace of λ must have dimension 1 (otherwise all the points of $\overline{\mathbb{C}}$ would be fixed points of F). Therefore, we have a one to one correspondence between the eigenvalues and fixed points of F.

Let $v = (v_1, v_2)^T$ be an eigenvector corresponding to the eigenvalue λ. Then:

$$\begin{aligned} F \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} \alpha v_1 + \beta v_2 \\ \overline{\beta} v_1 + \overline{\alpha} v_2 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ F \cdot \begin{pmatrix} \overline{v}_2 \\ \overline{v}_1 \end{pmatrix} &= \begin{pmatrix} \beta \overline{v}_1 + \alpha \overline{v}_2 \\ \overline{\alpha} \overline{v}_1 + \overline{\beta} \overline{v}_2 \end{pmatrix} = \begin{pmatrix} \overline{\overline{\beta} v_1 + \overline{\alpha} v_2} \\ \overline{\alpha v_1 + \beta v_2} \end{pmatrix} = \begin{pmatrix} \overline{\lambda v_2} \\ \overline{\lambda v_1} \end{pmatrix} = \overline{\lambda} \begin{pmatrix} \overline{v}_2 \\ \overline{v}_1 \end{pmatrix}. \end{aligned}$$

If λ is real, then the vectors $(v_1, v_2)^T$ and $(\overline{v}_2, \overline{v}_1)^T$ must be linearly dependent as they both belong to the same eigenspace. In particular, if $v_2 = 0$ we would have $v_1 = 0$, so we can assume that $v_1 = z, v_2 = 1$. But then the linear dependence is equivalent with

$$\det \begin{pmatrix} z & 1 \\ 1 & \overline{z} \end{pmatrix} = 0$$

which is equivalent with $|z| = 1$ and so $z \in \mathbb{T}$.

On the other hand, if λ is not real then $\overline{\lambda} \neq \lambda$ and the vectors $(v_1, v_2)^T$ and $(\overline{v}_2, \overline{v}_1)^T$ must be linearly independent as they are eigenvectors of different eigenvalues. Assume $z \in \mathbb{C}$ is a fixed point of F. Were $|z| = 1$ then the determinant argument above would give us a contradiction with linear independence. Therefore $z \notin \mathbb{T}$. But there is more: If z is a fixed point of F then so is $\frac{1}{\overline{z}}$, the image of z under circle inversion with respect to $\mathbb{T}$:

$$F \cdot \begin{pmatrix} \frac{1}{\overline{z}} \\ 1 \end{pmatrix} = \frac{1}{\overline{z}} F \cdot \begin{pmatrix} 1 \\ \overline{z} \end{pmatrix} = \frac{1}{\overline{z}} \overline{\lambda} \begin{pmatrix} 1 \\ \overline{z} \end{pmatrix} = \overline{\lambda} \begin{pmatrix} \frac{1}{\overline{z}} \\ 1 \end{pmatrix}.$$

Similarly, if ∞ is a fixed point then so is 0. Therefore, if F has an eigenvalue that is not real, then F has one fixed point outside $\mathbb{T}$ and one inside $\mathbb{T}$ (and we can even map one onto another using the circle inversion with respect to $\mathbb{T}$). This is precisely the case of elliptic F.

On the other hand, if F is hyperbolic, then F has two distinct real eigenvalues and, therefore, two distinct fixed points on $\mathbb{T}$.

If F is elliptic then there is a single real eigenvalue of F and so the Möbius transformation F has a single fixed point on $\mathbb{T}$. $\square$

Lemma 40. *Assume F is a hyperbolic transformation, x_1 and x_2 its fixed points. Then $F'(x_1) = \lambda_2/\lambda_1$ and $F'(x_2) = \lambda_1/\lambda_2$ where λ_1, λ_2 are the eigenvalues of F associated to x_1 and x_2.*

Similarly, if F is a parabolic transformation and x its fixed point then $F'(x) = 1$.

Proof. First observe that the ratio of λ_1 and λ_2 does not depend on the choice of the matrix for F so the claim is sensible.

We will fix a normalized matrix corresponding to F. Let J be the Jordan matrix similar to the matrix F, i.e. then exists an MT M such that $F = M \circ J \circ M^{-1}$.

Let x be a fixed point of F. Then we have:

$$F'(x) = (M \circ J \circ M^{-1})'(x) = M'(J \circ M^{-1}(x)) \cdot J'(M^{-1}(x)) \cdot (M^{-1})'(x).$$

Because $F(x) = x$, we must have $J \circ M^{-1}(x) = M^{-1}(x)$, so:

$$F'(x) = M'(M^{-1}(x)) \cdot J'(M^{-1}(x)) \cdot (M^{-1})'(x) = J'(M^{-1}(x)),$$

where we used the formula $M'(M^{-1}(x)) \cdot (M^{-1})'(x) = 1$.

However, $J'(M^{-1}(x))$ is easy to compute because $M^{-1}(x)$ is the fixed point of J corresponding to the same eigenvalue as x. The only problem is that we have to ensure $M^{-1}(x) \neq \infty$ to have the derivative well defined. We deal with this problem by carefully choosing our J.

Let us begin with the hyperbolic case. To calculate, x_1 choose the Jordan matrix $J = \left(\begin{smallmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{smallmatrix}\right)$ (note the switched order of λ_1, λ_2). Now $J(z) = \frac{\lambda_2}{\lambda_1} z$ and $M^{-1}(x_1) = 0$. Thus $F'(x_1) = J'(0) = \frac{\lambda_2}{\lambda_1}$. Similarly, we calculate $F'(x_2) = \frac{\lambda_1}{\lambda_2}$ from the Jordan form $J = \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$.

In the parabolic case, we avoid problems with the point at ∞, by taking a matrix similar to the usual Jordan form but with different interpretation as a Möbius transformation: $J = \left(\begin{smallmatrix} 1 & 0 \\ \mp 1 & 1 \end{smallmatrix}\right)$ (the sign in the lower left corner depends on the Jordan matrix for F; there are two possibilities). Now $J(z) = \frac{z}{\mp z + 1}$ and $M^{-1}(x) = 0$, so $F'(x) = J'(0) = 1$ and we are done. $\square$

Lemma 41. *Let F be a parabolic or a hyperbolic transformation, let x be the (stable) fixed point of F. Let $z \in \overline{\mathbb{C}}$ (assume that z is not the unstable fixed point of F in the hyperbolic case). Then $\lim_{n\to\infty} F^n(z) = x$.*

Proof. Let us again fix a matrix of F such that $\det F = 1$. Denote by J the Jordan matrix similar to F. Then $F = M \cdot J \cdot M^{-1}$ and $F^n = M \cdot J^n \cdot M^{-1}$. We know that J^n has one of the two possible forms:

$$\begin{pmatrix} 1 & \pm n \\ 0 & 1 \end{pmatrix}, \text{ or } \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$$

where without loss of generality $|\lambda_1| > 1 > |\lambda_2|$. Let $z \in \overline{\mathbb{C}}$ (if F is hyperbolic, let z be different from the unstable fixed point of F) and consider the vector

$$F^n \begin{pmatrix} z \\ 1 \end{pmatrix} = M^{-1} \cdot J^n \cdot M \cdot \begin{pmatrix} z \\ 1 \end{pmatrix}.$$

It is easy to see that the first component of $J^n \cdot M \cdot \left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right)$ tends to infinity, while the second is bounded. Therefore (understanding J and M as Möbius transformations), we have $\lim_{n\to\infty} J^n \circ M(z) = \infty$.

However, $M^{-1}(\infty) = x$ as $(x,1)^T$ and $(1,0)^T$ are eigenvectors of F and J belonging to the same eigenvalues. It follows that

$$\lim_{n\to\infty} M^{-1} \circ J^n \circ M(z) = M^{-1}(\infty) = x. \qquad \square$$

We conclude the Appendix with two miscellaneous lemmas: a lemma proving continuity of a certain linear functional and Fekete's lemma about superadditive series.

The following lemma is a consequence of the Riesz representation theorem. See ([1, page 184] for details). In its statement, we are going to identify $C^*(\mathbb{T},\mathbb{R})$ with the space of signed Radon measures on $\mathbb{T}$.

Lemma 42. *Let E be a measurable set on $\mathbb{T}$. Then the map $\alpha : \lambda \mapsto \lambda(E)$ from $C^*(\mathbb{T},\mathbb{R})$ to $\mathbb{R}$ is linear and continuous on $C^*(\mathbb{T},\mathbb{R})$.*

Proof. Linearity of α is obvious. To obtain continuity, it is enough to show that $|\alpha(\lambda)|$ is bounded whenever $|\lambda|$ is bounded. By definition,

$$|\lambda| = \sup\{\lambda(f) : f \in C(\mathbb{T},\mathbb{R}), |f| \le 1\}.$$

We first observe that every λ can be written as $\lambda_1 - \lambda_2$ where λ_1, λ_2 are positive measures. Moreover, as shown in [1], we can choose λ_1, λ_2 so that

$$\begin{aligned} \lambda_1(f) &= \sup\{\lambda(g) : 0 \le g \le f\} \\ \lambda_2(f) &= \sup\{\lambda(g) : -f \le g \le 0\} \end{aligned}$$

for all $f \ge 0$.

Then we have

$$|\lambda| \ge \sup\{\lambda(f) :\ 0 \le f \le 1\} = \lambda_1(1) = \lambda_1(\mathbb{T}) \ge \lambda_1(E) \ge \lambda(E)$$

as well as

$$|\lambda| \ge \sup\{\lambda(f) :\ -1 \le f \le 0\} = \lambda_2(1) = \lambda_2(\mathbb{T}) \ge \lambda_2(E) \ge -\lambda(E).$$

Therefore, $|\lambda| \ge |\lambda(E)| = |\alpha(\lambda)|$ for every λ, proving the continuity of α. $\square$

Lemma 43 (Fekete's lemma)**.** *Let $\{R_n\}_{n=1}^{\infty}$ be a* superadditive sequence *of real numbers, that is, a sequence such that for all $m,n \in \mathbb{N}$ we have $R_{m+n} \ge R_m + R_n$. Then the limit $\lim_{n\to\infty} \frac{1}{n} R_n$ exists and is equal to $\sup_{n\in\mathbb{N}} \frac{1}{n} R_n$.*

Proof. Denote $s = \sup_{n\in\mathbb{N}} \frac{1}{n} R_n$.

Assume first $s < \infty$. Let $\varepsilon > 0$ and suppose that n is such that $\frac{1}{n}R_n > s - \varepsilon$. From the superaditivity condition, we obtain for any $k \in \mathbb{N}$ and any $m < n$ the inequalities:

$$
\begin{aligned}
\frac{1}{kn+m}R_{kn+m} &\geq \frac{1}{kn+m}\left(kR_n + mR_1\right) \\
&\geq \frac{kn(s-\varepsilon) + mR_1}{kn+m} \\
&\geq \left(1 - \frac{1}{k+1}\right)(s-\varepsilon) + \frac{m}{kn+m}R_1
\end{aligned}
$$

In particular, there exists k such that whenever $l > kn$, the value $\frac{1}{l}R_l$ belongs to the interval $[s - 2\varepsilon, s]$. Therefore, the sequence $\{\frac{1}{n}R_n\}_{n=1}^{\infty}$ converges to s.

If $s = \infty$, replacing $s - \varepsilon$ with arbitrarily large $K > 0$ and performing the same argument gives us that there exist n and k such that $l > kn$ implies $\frac{1}{l}R_l \geq K - 1$, proving the lemma. $\square$

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