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Flows of fluids with pressure and temperature dependent viscosity in the channel

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BAKALÁŘSKÁ PRÁCE

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Proudění tekutiny s viskozitou závislou na tlaku a teplotě v rovinném kanále

Matematický ústav Univerzity Karlovy

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Prohlašuji, že jsem svou bakalářskou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

V Praze dne 27. května 2010 Adam Janečka

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Abstrakt: V předložené práci studujeme ustálené proudění tepelně vodivé, homogenní, isotropní, nestlačitelné tekutiny s viskozitou závislou na tlaku, teplotě a symetrickém gradientu rychlosti. Uvažujeme proudění v nekonečném rovinném kanále, které je buzeno buď tlakovým gradientem (Poiseuilleovo proudění) či pohybem jedné z desek kanálu (Couetteovo proudění), přičemž na stěnách kanálu jsou předepsány různé typy okrajových podmínek. Řídící rovnice převedeme do bezrozměrné podoby a řešení hledáme ve formě paralelního proudění. Výsledné rovnice řešíme numericky a v několika případech analyticky.

Klíčová slova: tekutina, tok, viskozita, teplota, tlak

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Abstract: In the present work we study steady flows of heat-conducting, homogeneous, isotropic, incompressible fluid with viscosity depending on the pressure, the temperature and the symmetric part of the velocity gradient. We study the flow in infinite plane channel. The flow is driven either by the the pressure gradient (Poiseuille flow) or by the motion of one of the channel plates (Couette flow). Different boundary conditions are prescribed at the channel plates. We non-dimensionalize the governing equations and seek the solution in the form of parallel flow. We solve the final equations numerically and, in some cases, analytically.

Keywords: fluid, flow, viscosity, temperature, pressure

Chapter 1

Flows of heat-conducting incompressible fluids

We will study flows of heat-conducting, homogeneous, isotropic, incompressible fluids with viscosity depending on the pressure and temperature. Mathematical description of the problem requires knowledge of the balance equations (for mass, linear momentum and energy), constitutive equations (for the Cauchy stress and the heat flux) and boundary conditions. Let us briefly comment all these three key components of the mathematical approach.

1.1 Balance equations

We shall assume that the fluid occupies a open connected set $\Omega \subset \mathbb{R}^3$ with the boundary $\partial\Omega$. Motion of such a fluid is well described through the velocity field $\mathbf{v} = (v_1, v_2, v_3) : (0,\infty) \times \Omega \to \mathbb{R}^3$, the density $\rho : (0,\infty) \times \Omega \to \mathbb{R}^+$, the pressure $p : (0,\infty) \times \Omega \to \mathbb{R}^+$ and the internal energy $e : (0,\infty) \times \Omega \to \mathbb{R}^+$.

Let us recall the standard notation. For any scalar field φ, any vector field (e.g. velocity) $\mathbf{v}$ and any tensor field $\mathbf{S}$ we have

$$\begin{gathered} \varphi_{,t} := \frac{\partial \varphi}{\partial t}, \qquad (\nabla\varphi)_i = \frac{\partial \varphi}{\partial x_i}, \qquad (\nabla\mathbf{v})_{ij} = \frac{\partial v_i}{\partial x_j}, \\ \operatorname{div}\mathbf{v} = \sum_i \frac{\partial v_i}{\partial x_i}, \qquad (\operatorname{div}\mathbf{S})_i = \sum_j \frac{\partial S_{ij}}{\partial x_j}. \end{gathered} \tag{1.1}$$

We shall describe the behavior of the fluid through the balance equations expressed in their Eulerian form. The balance of mass is in the form

$$\rho_{,t} + \operatorname{div}(\rho\mathbf{v}) = 0. \tag{1.2}$$

Balance of linear momentum is a generalization of Newton's second law in classical mechanics

$$\rho\left(\mathbf{v}_{,t} + [\nabla\mathbf{v}]\mathbf{v}\right) = \operatorname{div}\mathbf{T}^{\mathrm{T}} + \rho\mathbf{b}, \tag{1.3}$$

where $\mathbf{T}$ denotes the Cauchy stress tensor and $\mathbf{b}$ denotes the specific body force.

In the absence of internal couples (moments per unit volume), the balance of angular momentum implies that the Cauchy stress is symmetric, i.e.,

$$\mathbf{T} = \mathbf{T}^{\mathrm{T}}. \tag{1.4}$$

The balance of energy is given by

$$\rho\,(E_{,t} + \nabla E \cdot \mathbf{v}) = \operatorname{div}(\mathbf{T}\mathbf{v} - \mathbf{q}) + \rho\mathbf{b}\cdot\mathbf{v}, \tag{1.5}$$

where $E = \frac{|\mathbf{v}|^2}{2} + e$ is the sum of kinetic energy and internal energy and $\mathbf{q}$ is the heat flux. By subtracting the scalar product of (1.3) and $\mathbf{v}$ from (1.5) we obtain the the balance of internal energy in the form

$$\rho\,(e_{,t} + \nabla e \cdot \mathbf{v}) = \mathbf{T}\cdot\nabla\mathbf{v} - \operatorname{div}\mathbf{q}. \tag{1.6}$$

The constraint of incompressibility in fluid mechanics is usually in the form

$$\operatorname{div}\mathbf{v} = \operatorname{tr}\mathbf{D} = 0, \tag{1.7}$$

where $\mathbf{D}$ is the symmetric part of the velocity gradient, i.e., $\mathbf{D} = \frac{1}{2}(\nabla\mathbf{v}+\nabla\mathbf{v}^{\mathrm{T}})$ [5]. We will suppose even stronger condition, namely $\rho = \rho^*$, where $\rho^* \in (0, +\infty)$, which ensures the validity of (1.2).

If we identify $\mathbf{T} := \frac{\mathbf{T}}{\rho^*}$ and $\mathbf{q} := \frac{\mathbf{q}}{\rho^*}$, we can re-write the system of balance equations as:

$$\operatorname{div}\mathbf{v} = 0, \tag{1.8}$$
$$\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) = \operatorname{div}\mathbf{T} + \mathbf{b}, \tag{1.9}$$
$$e_{,t} + \operatorname{div}(e\mathbf{v}) = \mathbf{T}\cdot\mathbf{D} - \operatorname{div}\mathbf{q}. \tag{1.10}$$

More details concerning balance equations can be found, for example, in [1].

1.2 Constitutive equations

The constitutive equation for internal energy is specified by simple temperature dependent model

$$e(\theta) = c_V\theta, \tag{1.11}$$

where θ is temperature and c_V is the heat capacity at a constant volume [6].

The heat flux $\mathbf{q}$ is related to the variation of temperature through the fluid [4]. The relevant constitutive equation takes the form

$$\mathbf{q} = \mathbf{q}^*(p, \theta, \mathbf{D}(\mathbf{v}), \nabla\theta) = -k(p, \theta, |\mathbf{D}(\mathbf{v})|^2)\nabla\theta, \tag{1.12}$$

where k denotes the thermal conductivity and we will assume it to be constant. Relation (1.12) is called the Fourier's law.

It has been know since the time of Newton, that $\mathbf{T}$ and $\nabla\mathbf{v}$ are related [3]. This relationship changes with the variation of θ. A general algebraic relation describing this fact can be written as

$$\mathbf{g}(\theta, \mathbf{T}, \nabla\mathbf{v}) = 0, \tag{1.13}$$

where $\mathbf{g}$ is isotropic tensor function of the second order. From the principle of material frame-indifference follows that (1.13) must satisfy, for all orthogonal tensors $\mathbf{Q}$,

$$\mathbf{g}(\theta, \mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\nabla\mathbf{v}\mathbf{Q}^{\mathrm{T}}) = \mathbf{Q}\mathbf{g}(\theta, \mathbf{T}, \nabla\mathbf{v})\mathbf{Q}^{\mathrm{T}}. \tag{1.14}$$

From (1.14) follows that we may use only the symmetric part of the velocity gradient. Using the representation theorems for isotropic functions of the form (1.14) we obtain

$$\alpha_0\mathbf{I} + \alpha_1\mathbf{T} + \alpha_2\mathbf{D} + \alpha_3\mathbf{T}^2 + \alpha_4\mathbf{D}^2 + \alpha_5(\mathbf{TD} + \mathbf{DT}) \\ + \alpha_6(\mathbf{T}^2\mathbf{D} + \mathbf{DT}^2) + \alpha_7(\mathbf{TD}^2 + \mathbf{D}^2\mathbf{T}) + \alpha_8(\mathbf{T}^2\mathbf{D}^2 + \mathbf{D}^2\mathbf{T}^2) = 0 \quad (1.15)$$

where the material moduli α_i, $i = 0, \dots, 8$, depend on

$$\theta, \quad \operatorname{tr}\mathbf{T}, \quad \operatorname{tr}\mathbf{D}, \quad \operatorname{tr}\mathbf{T}^2, \quad \operatorname{tr}\mathbf{D}^2, \quad \operatorname{tr}\mathbf{T}^3, \quad \operatorname{tr}\mathbf{D}^3, \\ \operatorname{tr}(\mathbf{TD}), \quad \operatorname{tr}(\mathbf{T}^2\mathbf{D}), \quad \operatorname{tr}(\mathbf{D}^2\mathbf{T}), \quad \operatorname{tr}(\mathbf{T}^2\mathbf{D}^2).$$

To obtain a more specific subclass of fluids we shall first consider α_i, $i = 3 \dots 8$, to be zero. Thus (1.15) simplifies to

$$\alpha_0\mathbf{I} + \alpha_1\mathbf{T} + \alpha_2\mathbf{D} = 0. \quad (1.16)$$

If we take the trace of the previous equation and satisfy the incompressibility constraint, we conclude that

$$3\alpha_0 + \alpha_1 \operatorname{tr}\mathbf{T} = 0 \quad (1.17)$$

or

$$\alpha_0 = -\frac{\operatorname{tr}\mathbf{T}}{3}\alpha_1 \quad (1.18)$$

and we define the pressure (the mean normal stress) as $p := -\frac{1}{3}\operatorname{tr}\mathbf{T}$. By substituting the relation for p into (1.16) we obtain

$$\mathbf{T} = -p\mathbf{I} + \frac{\alpha_2}{\alpha_1}\mathbf{D}. \quad (1.19)$$

By defining viscosity as $\mu(\theta, p, |\mathbf{D}|^2) := \frac{1}{2}\frac{\alpha_2}{\alpha_1}$ we finally get the Cauchy stress tensor of the form

$$\mathbf{T} = -p\mathbf{I} + 2\mu(\theta, p, |\mathbf{D}|^2)\mathbf{D}, \qquad \operatorname{tr}\mathbf{D} = 0, \quad (1.20)$$

where we define the deviatoric (or viscous) part of the stress tensor

$$\mathbf{S} := 2\mu(\theta, p, |\mathbf{D}|^2)\mathbf{D}. \quad (1.21)$$

Our subclass subsumes several fluids including the following fluids we shall study later:

1. Navier-Stokes fluid and its generalizations

$$\mathbf{S} = 2\mu(\theta, p)\mathbf{D}(\mathbf{v}) \quad (1.22)$$

   Navier-Stokes fluid is the one with constant viscosity.

2. Power-law fluid

$$\mathbf{S} = 2\mu(\theta, p)|\mathbf{D}(\mathbf{v})|^{r-2}\mathbf{D}(\mathbf{v}), \qquad 1 < r < +\infty \quad (1.23)$$

3. Generalized power-law fluid

$$\mathbf{S} = 2\mu(\theta, p)\left(\kappa + |\mathbf{D}(\mathbf{v})|^2\right)^{\frac{r-2}{2}} \mathbf{D}(\mathbf{v}), \qquad 1 < r < +\infty,\ \kappa \in \mathbb{R}. \tag{1.24}$$

All the above mentioned models can be generalized as

$$\mathbf{S} = 2\mu(\theta, p)\beta(|\mathbf{D}|^2)\mathbf{D}. \tag{1.25}$$

Studying the power-law fluid or the generalized power-law fluid, we will focus on two important values of r, $r = \left\{\frac{3}{2}, \frac{4}{3}\right\}$. The value $r = \frac{4}{3}$ is used in glacier dynamics for models based on Glen's flow law [6].

We shall consider the viscosity to be dependent either on pressure or temperature. We will study two pressure dependent viscosity models

$$\mu(p) = \alpha p^\gamma, \tag{1.26}$$

$$\mu(p) = \mu_0 \mathrm{e}^{\alpha p}, \tag{1.27}$$

where α is a constant, and the following three temperature dependent viscosity models:

1. Constant viscosity model

$$\mu(\theta) = \mu_0, \tag{1.28}$$

2. Reynolds' model

$$\mu(\theta) = \mu_0 \exp(-m\theta), \tag{1.29}$$

3. Vogel's model

$$\mu(\theta) = \mu_0 \exp\left(\frac{a}{b+\theta}\right), \tag{1.30}$$

where μ_0, m, a and b are constants.

Using the mentioned constitutive equations (1.11), (1.12) and (1.20), the system of balance equations can be written as:

$$\operatorname{div} \mathbf{v} = 0, \tag{1.31}$$

$$\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) = -\nabla p + \operatorname{div} \mathbf{S} + \mathbf{b}, \tag{1.32}$$

$$c_V \theta_{,t} + c_V \operatorname{div}(\theta \mathbf{v}) = \mathbf{S} \cdot \mathbf{D} + k\Delta\theta, \tag{1.33}$$

where Δ denotes the Laplace operator.

1.3 Boundary conditions

We assume that the boundary $\partial\Omega$ is not permeable

$$\mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } [0, \infty) \times \partial\Omega, \tag{1.34}$$

where $\mathbf{n}$ is the unit outer normal to the boundary. For the velocity field we will consider the Navier's slip boundary condition

$$\lambda \mathbf{v} \cdot \mathbf{t} + (1 - \lambda)\mathbf{S}\mathbf{n} \cdot \mathbf{t} = 0 \quad \text{on } [0, \infty) \times \partial\Omega, \tag{1.35}$$

where $\mathbf{t}$ is any tangent vector at the boundary, i.e., $\mathbf{t} \cdot \mathbf{n} = 0$ and the parameter λ meets $\lambda \in [0, 1]$ including two limiting cases. If $\lambda = 0$, (1.35) reduces to slip boundary condition. On the contrary, $\lambda = 1$ means no slip boundary condition [2].

Concerning the temperature, we prescribe the temperature values on the boundary

$$\theta|_{\partial\Omega} = \theta_0 \quad \text{on } [0, \infty) \times \partial\Omega. \tag{1.36}$$

For fluids with pressure-dependent viscosity, we need to prescribe the pressure in some point

$$p(\mathbf{x}) = p_0 \quad \text{at any } \mathbf{x} \in \Omega. \tag{1.37}$$

In the specific case of plane Poiseuille flow, when the flow is driven by a pressure potential we need to specify the driving force. This can be done by fixing the pressure gradient along the main flow direction (e.g., x-direction) at some point $\mathbf{x} \in \Omega$

$$\frac{\partial p}{\partial x} = C_0, \tag{1.38}$$

where $C_0 < 0$ is a constant (a datum of the problem).

Chapter 2

Problem geometry

2.1 Plane flow

We will study steady fully developed flow of an incompressible fluid between two infinite parallel plates located in $y = \pm h$ of an orthogonal Cartesian coordinate system (see figure 2.1). We will seek the velocity field of the form

$$\mathbf{v} = u(y)\mathbf{e}_x, \tag{2.1}$$

which means that (1.7) is automatically satisfied. It follows from (2.1) that

$$\nabla\mathbf{v} = \begin{pmatrix} 0 & u' & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{D}(\mathbf{v}) = \frac{1}{2}\begin{pmatrix} 0 & u' & 0 \\ u' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad |\mathbf{D}(\mathbf{v})| = \frac{1}{\sqrt{2}}|u'|, \tag{2.2}$$

where the prime symbol denotes the coordinate derivative, i.e., $u' := \frac{\mathrm{d}u}{\mathrm{d}y}$. In order to eliminate (from aesthetic reasons) the factor $\frac{1}{\sqrt{2}}$ in case of the power-law fluid and the generalized power-law fluid, we redefine the generalized viscosity as $\mu := (\sqrt{2})^{r-2}\mu$. From now on, we will use these rescaled viscosities. From our assumptions, it also immediately follows that

$$\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) = 0, \tag{2.3}$$

so that (1.32) reduces to

$$\nabla p = \operatorname{div}\mathbf{S} + \mathbf{b}. \tag{2.4}$$

In most cases, we will also neglect external body forces, i.e., $\mathbf{b} = \mathbf{0}$, so that we can assume

$$\frac{\partial p}{\partial z} = 0, \tag{2.5}$$

which implies that p is independent of z, i.e., $p = p(x, y)$.

Furthermore, we shall assume the temperature to satisfy

$$\theta = \theta(y), \tag{2.6}$$

meaning that

$$e_{,t} + \operatorname{div}(e\mathbf{v}) = 0. \tag{2.7}$$

Thus (1.33) simplifies to

$$S_{12}u' + k\theta'' = 0, \tag{2.8}$$

or, using (1.25), to

$$\mu(\theta, p)\beta(|u'|^2)[u']^2 + k\theta'' = 0, \tag{2.9}$$

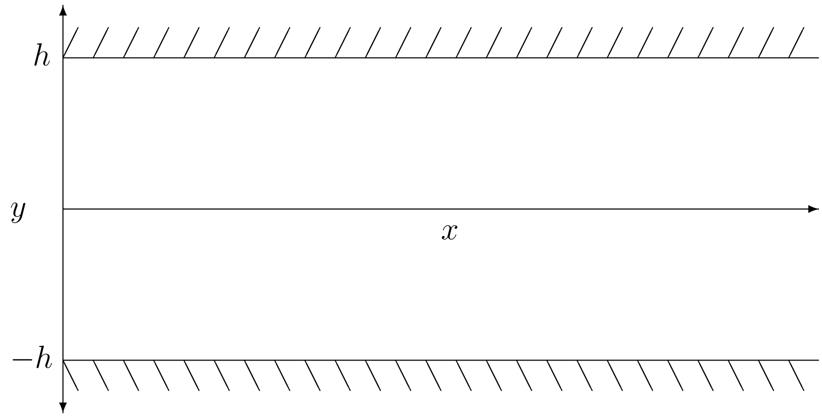

Figure 2.1: Flow between two plates

2.2 Boundary conditions

In our geometry, the tangent vector and the outer normal on the boundary take the form

$$\mathbf{t} = (1, 0, 0), \qquad \mathbf{n} = (0, \pm 1, 0), \tag{2.10}$$

where $+1$ holds at the upper plate and -1 at the lower one. The choice of velocity field of the form (2.1) automatically satisfies the permeability constaint (1.34).

We will deal with the two following types of flow with boundary conditions derived from (1.35):

1. *Plane Poiseuille flow* with boundary conditions

$$\lambda_1 u(-h) - (1-\lambda_1)\mu(\theta,p)\beta(|u'|)u'(-h) = 0 \quad \text{at the lower plate} \tag{2.11}$$

$$\lambda_2 u(h) + (1-\lambda_2)\mu(\theta,p)\beta(|u'|)u'(h) = 0 \quad \text{at the upper plate,} \tag{2.12}$$

   where $0 \le \lambda_1 \le \lambda_2 \le 1$.

2. *Plane Couette flow* when the upper plate moves with a prescribed velocity $(V, 0, 0)$. The velocity has to satisfy (2.11) at the lower plate and

$$\lambda_2(u(h) - V) + (1-\lambda_2)\mu(\theta,p)\beta(|u'|)u'(h) = 0 \quad \text{at the upper plate.} \tag{2.13}$$

The prescribed temperatures at the lower and the upper plate are

$$\theta_1 = \theta(-h), \qquad \theta_2 = \theta(h). \tag{2.14}$$

2.3 Non-dimensional equations

We shall introduce the following variables and parameter

$$\bar{y} = \frac{y}{h}, \qquad \bar{u} = \frac{u}{V}, \qquad \bar{\mu} = \frac{\mu}{\mu_0}, \qquad T = \frac{\theta - \theta_1}{\theta_2 - \theta_1}, \qquad \Gamma = \frac{\mu_0 V^2}{k(\theta_2 - \theta_1)}, \tag{2.15}$$

where V is the characteristic velocity. In case of plane Couette flow, V corresponds to the velocity of the upper plate. Concerning the plane Poiseuille flow, V is given by

$$V = -\frac{h^2}{2\mu_0}\frac{\mathrm{d}p}{\mathrm{d}x}, \tag{2.16}$$

which is the velocity for plane Poiseuille flow for Navier-Stokes fluid. In case of the power-law fluid and the generalized power-law fluid, we redefine the parameter Γ as $\Gamma := [\frac{h}{V}]^{r-2}\Gamma$.

Then, (2.9) in its the non-dimensional form is

$$T'' + \Gamma\bar{\mu}(T,\bar{p})\beta(|\bar{u}'|^2)[\bar{u}']^2 = 0 \tag{2.17}$$

subject to the boundary conditions

$$T(-1) = 0, \qquad T(1) = 1. \tag{2.18}$$

We can also re-write the temperature dependent viscosity models:

1. Constant viscosity model
$$\mu(T) = 1, \tag{2.19}$$

2. Reynolds' model
$$\mu(T) = \exp(-MT), \tag{2.20}$$
where $M = m(\theta_2 - \theta_1)$,

3. Vogel's model
$$\mu(T) = \exp[\theta(T) - \theta(0)], \tag{2.21}$$
where $\theta = \frac{A}{B+T}$, $A = \frac{a}{\theta_2-\theta_1}$ and $B = \frac{(b+\theta_1)}{(\theta_2-\theta_1)}$ [7].

We shall also introduce the non-dimensional form of (2.4). The equation is already divided by the pressure ρ^* as mentioned in section 1.1. Thus we have

$$\nabla\left(\frac{p}{\rho^*}\right) = \text{div}\left(\frac{\mu}{\rho^*}\beta(|\mathbf{D}|^2)\mathbf{D}\right) + \mathbf{b}. \tag{2.22}$$

If we assume $\mathbf{b}$ of the form $\mathbf{b} = (0, -g, 0)$ and use (2.15) we obtain

$$\bar{\nabla}\left(\frac{p}{\rho^* V^2}\right) = \overline{\text{div}}\left(\frac{\mu_0}{\rho^* V L}\bar{\mu}(T,\bar{p})\beta(|\bar{\mathbf{D}}|^2)\bar{\mathbf{D}}\right) + \frac{Lg}{V^2}\bar{\mathbf{b}}, \tag{2.23}$$

where L is the characteristic linear dimension. By defining an analogue of the Reynolds number

$$\text{Re} = \frac{\rho^* V L}{\mu_0}, \tag{2.24}$$

and the Froude number

$$\text{Fr} = \frac{V}{\sqrt{Lg}}, \tag{2.25}$$

we finally obtain the sought non-dimensional form

$$\bar{\nabla}\bar{p} = \overline{\text{div}}\left(\frac{1}{\text{Re}}\bar{\mu}\beta(|\bar{\mathbf{D}}|^2)\bar{\mathbf{D}}\right) + \frac{1}{\text{Fr}^2}\bar{\mathbf{b}}. \tag{2.26}$$

In the following, we will omit the bars above the non-dimensional variables.

Chapter 3

Solutions of selected problems

3.1 Pressure-dependent viscosities

We will consider the flow of a fluid modelled by (1.22) and seek the pressure field in the form $p(x, y) = F(x)G(y)$. Considering the plane Poiseuille flow and on neglecting the body forces, only trivial solutions exist for viscosity of the form (1.26) and for $\gamma \neq 1$. For viscosity of the form (1.27), solution is not possible [3].

In the following, some of the results and methods are adpoted from [3].

(i) $\mu(p) = \alpha p^\gamma$

If $\gamma = 1$ and we assume $u'(y) > 0$ on $(-1, 0)$, from (2.4) follows

$$\frac{\partial p}{\partial x} = \alpha \frac{\partial p}{\partial y} u' + \alpha p u'', \tag{3.1a}$$

$$\frac{\partial p}{\partial y} = \alpha \frac{\partial p}{\partial x} u', \tag{3.1b}$$

which is equivalent to

$$\frac{\partial p}{\partial x}(1 - \alpha^2 [u']^2) = \alpha p u'', \tag{3.2a}$$

$$\frac{\partial p}{\partial y}(1 - \alpha^2 [u']^2) = \alpha^2 p u' u''. \tag{3.2b}$$

The first equation can be written as

$$\frac{\partial}{\partial x} \ln|p(x, y)| = \frac{\partial}{\partial y} \left(\ln \left| \frac{1 + \alpha u'}{1 - \alpha u'} \right|^{1/2} \right) =: C_1(y), \tag{3.3}$$

which implies that

$$p(x, y) = C_2(y) \mathrm{e}^{C_1(y)x}, \tag{3.4}$$

where C_2 is either non-negative or non-positive. Substituting (3.4) into (3.2b) leads to

$$C_2'(y) + C_2(y) C_1'(y) x = \alpha u'(y) C_2(y) C_1(y), \tag{3.5}$$

which implies that $C_1(y) \equiv C_0 = const.$, $\forall y \in (-1, 0)$. Denoting $C_3(y)$ by $L(y)$ we then have

$$p(x, y) = \pm L(y) \mathrm{e}^{C_0 x}, \qquad L \geq 0, \tag{3.6}$$

and

$$\frac{1 + \alpha u'}{1 - \alpha u'} = \pm M e^{2C_0 y}, \qquad M > 0. \tag{3.7}$$

Thus

$$u'(y) = \frac{1}{\alpha} \frac{M e^{2C_0 y} - 1}{M e^{2C_0 y} + 1}, \qquad M \in \mathbb{R}. \tag{3.8}$$

We assume that $\lambda_1 = \lambda_2$, so we consider the problem to be symmetric. Hence, we require that $u'(0) = 0$ which leads to $M = 1$. So that

$$u'(y) = \frac{1}{\alpha} \frac{e^{2C_0 y} - 1}{e^{2C_0 y} + 1} = \frac{1}{\alpha} \frac{\sinh C_0 y}{\cosh C_0 y}, \qquad C_0 < 0, \tag{3.9}$$

and the constant C_0 is related to the pressure gradient along the x-direction. By integrating and satisfying the boundary condition for $\lambda_1 = \lambda_2 = 1$ we can find the explicit formula

$$u(y) = \frac{1}{\alpha C_0} \ln\left(\frac{\cosh C_0 y}{\cosh C_0}\right). \tag{3.10}$$

Different boundary conditions (λ_1, $\lambda_2 \neq 1$) were also tried but led to a sulution which did not fulfilled assumption (2.1) (due to the form of p).

In figure 3.1, there are shown the velocity profiles for certain values C_0. The profiles are scaled so that

$$\int_{-1}^{1} u(y)\, dy = 1. \tag{3.11}$$

Pressure decline in the middle of the channel is depicted in figure 3.2.

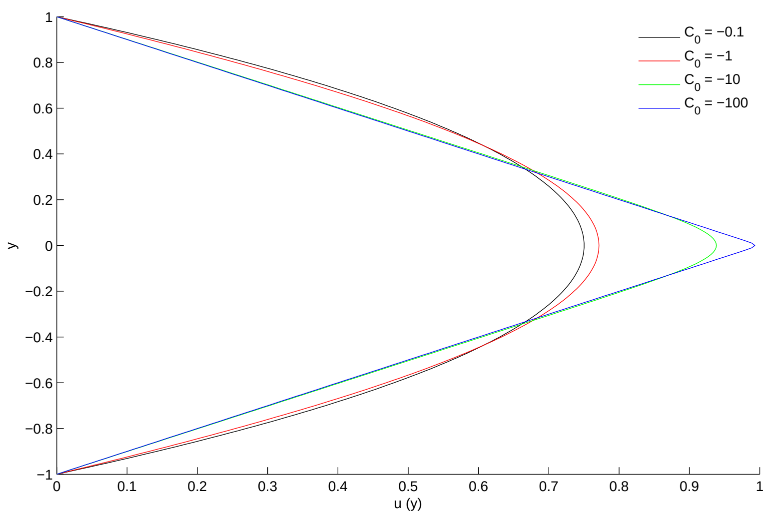

Figure 3.1: Poiseuille flow - velocity profiles for different values of C_0 ($\alpha = 1$)

(ii) $\mu(p) = e^{\alpha p}$

Since the solution for plane Poiseuille flow does not exist, we will study the plane Couette flow with an external body force, e.g., gravity, in the form of

$$\mathbf{b} = (0, -g, 0). \tag{3.12}$$

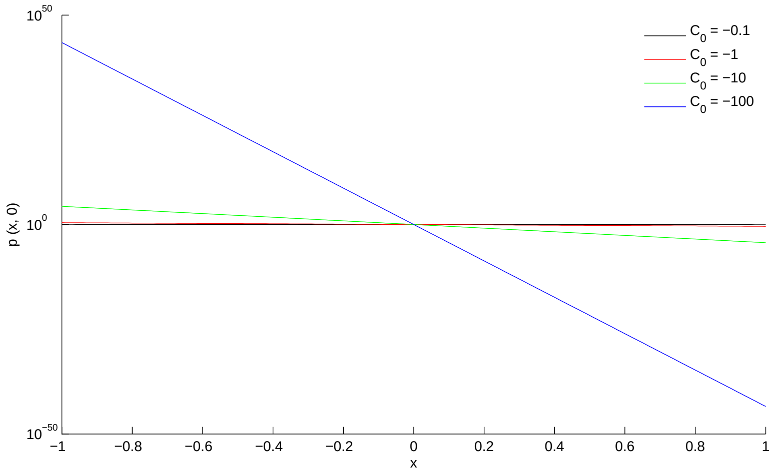

Figure 3.2: Poiseuille flow - pressure along the channel for different values of C_0 $(L = 1)$

In [3] is shown that p has to be in the form of

$$p(x, y) = R(x) + S(y) \tag{3.13}$$

and, in addition, $R(x)$ has to be a constant function. Thus viscosity is in the form of

$$\mu(p) = \mu(S(y)) = C\mathrm{e}^{\alpha S(y)}, \tag{3.14}$$

where $C = const$. Then, it immediately follows from (2.4) that

$$0 = \alpha\mu(S)S'u' + \mu(S)u'', \tag{3.15a}$$
$$S' = -g. \tag{3.15b}$$

From the second equation, we obtain the form of pressure in the channel

$$p = -gy + D, \tag{3.16}$$

D being a constant. By substituting the second equation into the first, we obtain a ordinary differential equation of the second order

$$\mu(S)\,(u'' - \alpha g u') = 0. \tag{3.17}$$

By solving it and satisfying the boundary conditions (2.11) and (2.13) we get the solution

$$u(y) = C_1 \left(\frac{\mathrm{e}^{\alpha g y} - \mathrm{e}^{-\alpha g}}{\alpha g} + \frac{1 - \lambda_1}{\lambda_1} \right), \tag{3.18}$$

where

$$C_1 = V \Big/ \left(\frac{2 \sinh \alpha g}{\alpha g} + \frac{1 - \lambda_1}{\lambda_1} + \frac{1 - \lambda_2}{\lambda_2} \right), \tag{3.19}$$

and λ_1, $\lambda_2 \neq 0$. The velocity profiles for certain values of λ_1, λ_2, V and g (with $\alpha = 1$) are depicted in figures 3.3, 3.4 and 3.5.

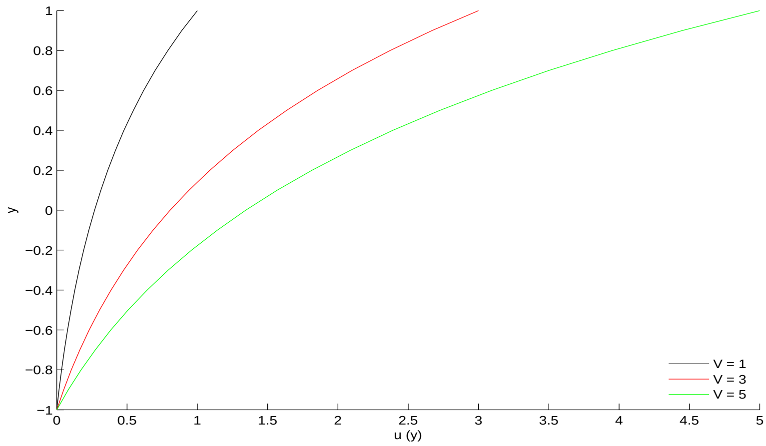

Figure 3.3: Couette flow - velocity profiles for different values of V ($g = 1$, $\lambda_1 = \lambda_2 = 1$)

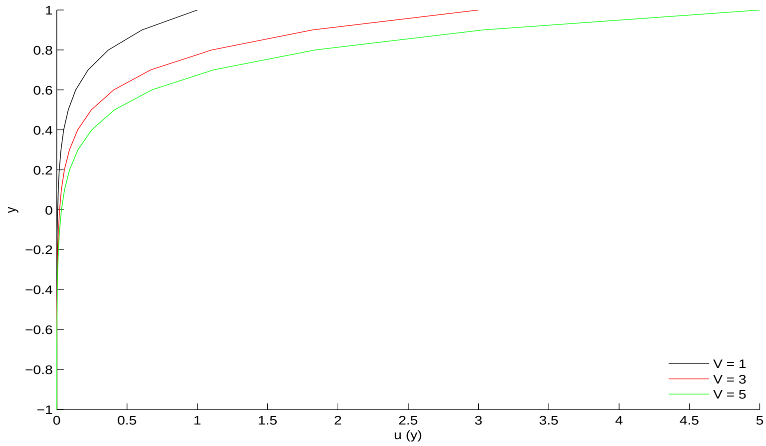

Figure 3.4: Couette flow - velocity profiles for different values of V ($g = 5$, $\lambda_1 = \lambda_2 = 1$)

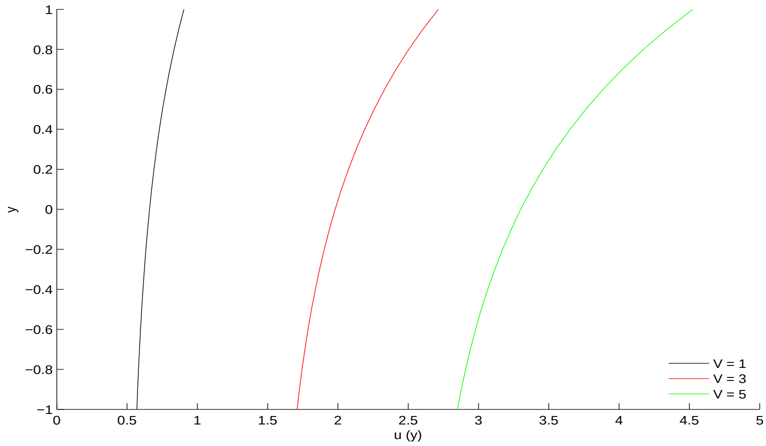

Figure 3.5: Couette flow - velocity profiles for different values of V ($g = 1$, $\lambda_1 = 0.2$, $\lambda_2 = 0.6$)

3.2 Temperature-dependent viscosities

In this section, we shall study the plane Poisseuille flow with no-slip boundary conditions at the plates

$$u(\pm h) = 0, \tag{3.20}$$

i.e., $\lambda_1 = \lambda_2 = 1$. The external body forces will be neglected and all the velocity profiles will be scaled according to (3.11).

Firstly, we focused on the generalized Navier-Stokes fluid (1.22). We were not able to find the analytical solutions, therefore, we had to solve the problems numerically.

Reynolds' viscosity model represents oils well [7]. The velocity profiles in the absence of viscous heating, i.e., $\Gamma = 0$, are shown in figure 3.6 and the temperature distributions with $\Gamma = 10$ are depicted in figure 3.7.

In figures 3.8-3.9 are displayed solutions for the Vogel's model which describes lubricating oils [7]. Again, the velocity profiles are in the absence of viscous heating and the temperature distributions are with $\Gamma = 10$.

The constant viscosity model describing the behaviour of the Navier-Stokes fluid is a special subclass of both Reynolds' and Vogel's models with $M = 0$ and $A = B = 0$ respectively.

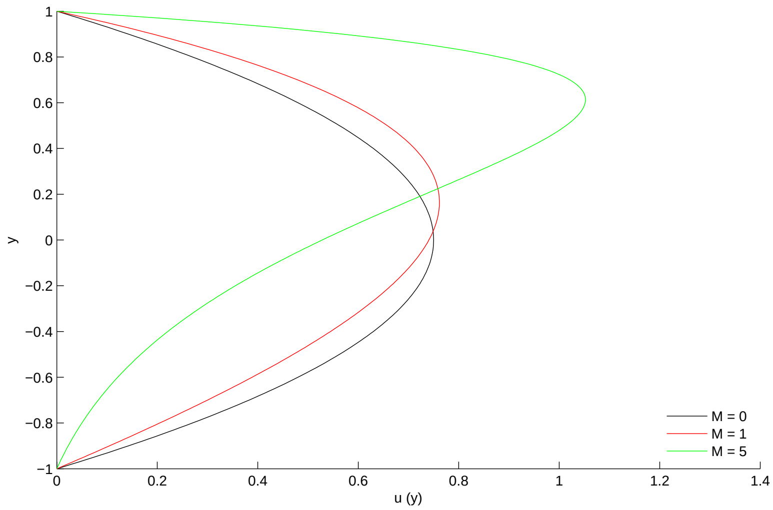

Figure 3.6: Generalized Navier-Stokes fluid with Reynolds' viscosity model - velocity profiles for different values of M ($\Gamma = 0$)

For the power-law fluid model (1.23), we were not able to find neither the analytical nor the numerical solution (using the standard ODE BVP solvers) for any but the constant viscosity model which can be solved analyticaly as

$$u(y) = \begin{cases} -L_r(-y)^{\frac{r}{r-1}} + L_r & \text{on } (-1, 0), \\ -L_r y^{\frac{r}{r-1}} + L_r & \text{on } (0, 1), \end{cases} \tag{3.21}$$

where

$$L_r = \frac{r-1}{r} \left(-\frac{C_0}{\mu} \right)^{\frac{1}{r-1}}. \tag{3.22}$$

The velocity profiles for different values of r are depicted in figure 3.10. More details concerning the power-law fluids with constant viscosity can be found in [2].

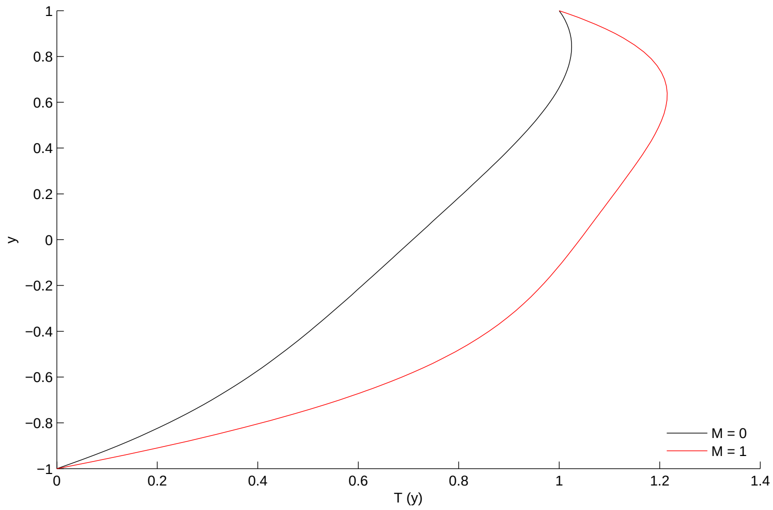

Figure 3.7: Generalized Navier-Stokes fluid with Reynolds' viscosity model - temperature distributions for different values of M $(\Gamma = 10)$

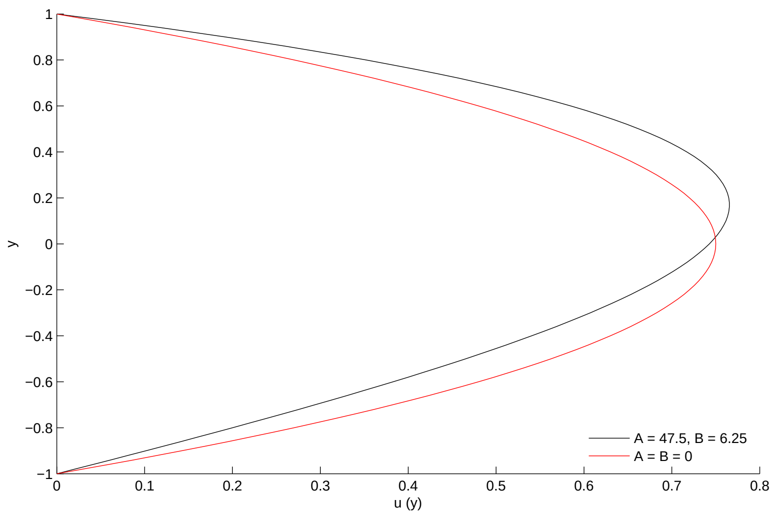

Figure 3.8: Generalized Navier-Stokes fluid with Vogel's viscosity model - velocity profiles for different values of A and B $(\Gamma = 0)$

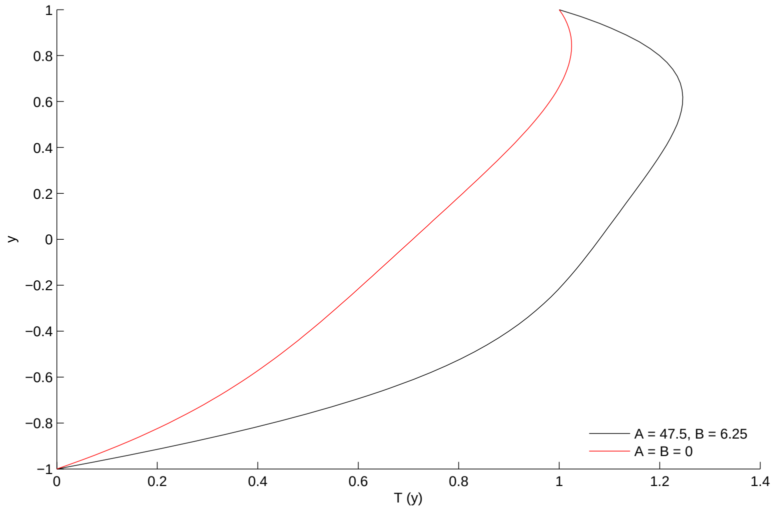

Figure 3.9: Generalized Navier-Stokes fluid with Vogel's viscosity model - temperature distributions for different values of M ($\Gamma = 10$)

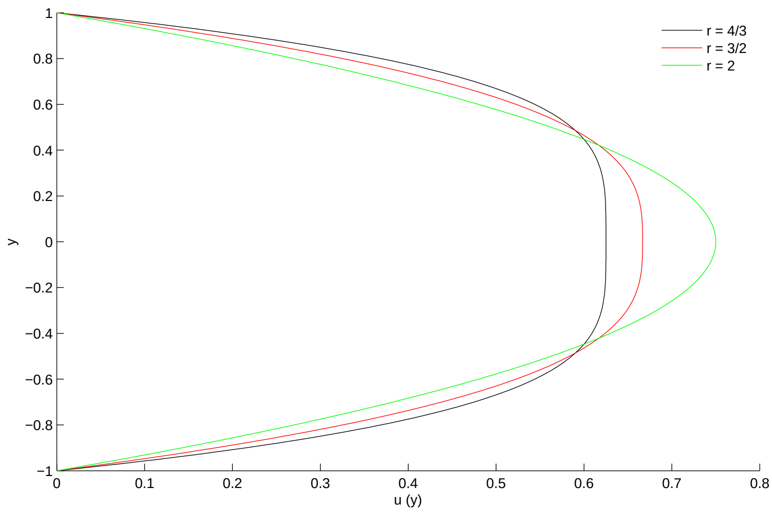

Figure 3.10: Power-law fluid with constant viscosity model - velocity profiles for different values of r

Concerning the generalized power-law fluids (1.24), with some effort, we were able to find the numerical solution but only for some parameters' values. The velocity profiles and the temperature distributions for Reynolds' and Vogel's model are displayed in figures 3.11-3.15. The temperature distributions are not mentioned for Vogel's model because we were not able to obtain the solution with viscous heating, i.e., $\Gamma > 0$.

Once again, the constant viscosity model is a special subclass of, e.g., Reynolds' model with $M = 0$.

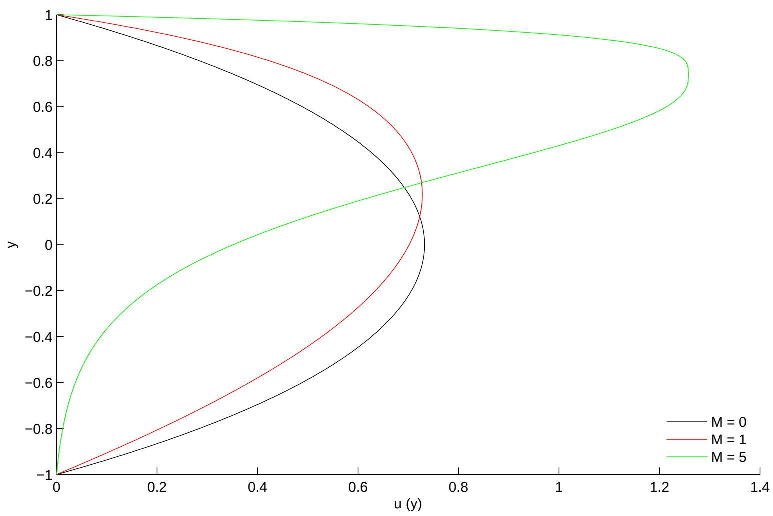

Figure 3.11: Generalized power-law fluid with Reynolds' viscosity model - velocity profiles for different values of M ($r = \frac{3}{2}$, $\Gamma = 0$, $\kappa = 1$)

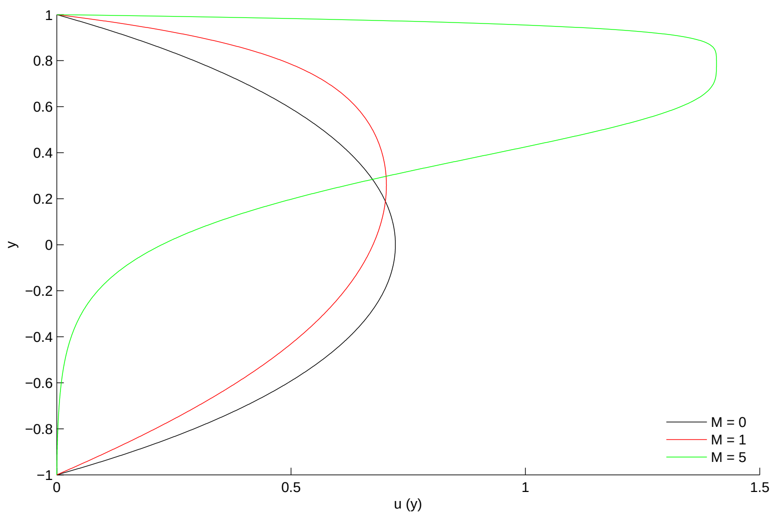

Figure 3.12: Generalized power-law fluid with Reynolds' viscosity model - velocity profiles for different values of M ($r = \frac{4}{3}$, $\Gamma = 0$, $\kappa = 1$)

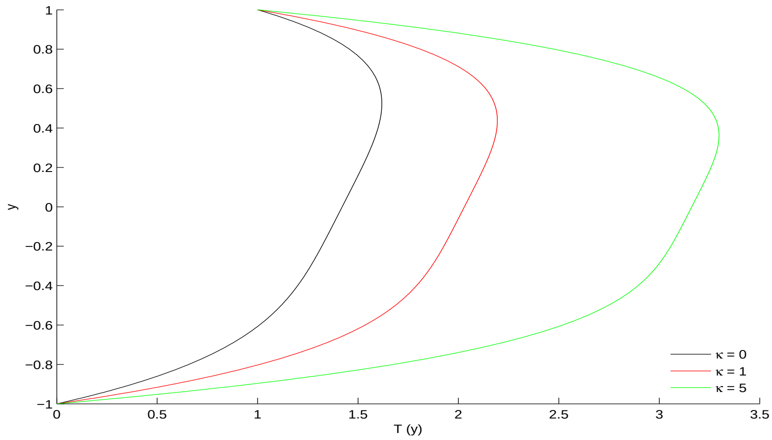

Figure 3.13: Generalized power-law fluid with Reynolds' viscosity model - temperature distributions for different values of κ ($r = \frac{3}{2}$, $M = 0$, $\Gamma = 10$)

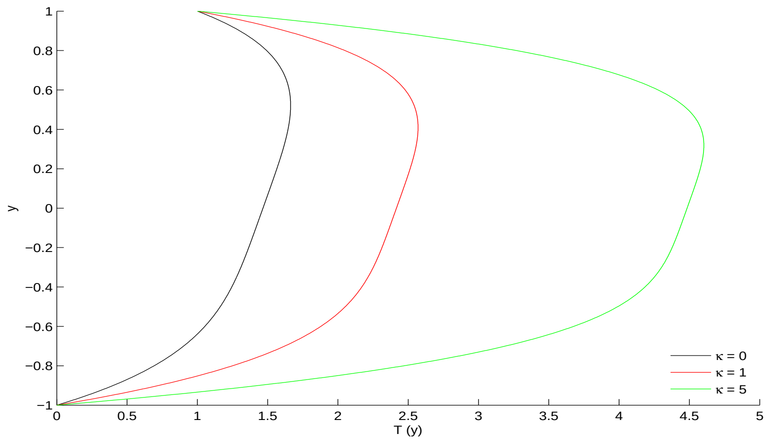

Figure 3.14: Generalized power-law fluid with Reynolds' viscosity model - temperature distributions for different values of κ ($r = \frac{4}{3}$, $M = 0$, $\Gamma = 10$)

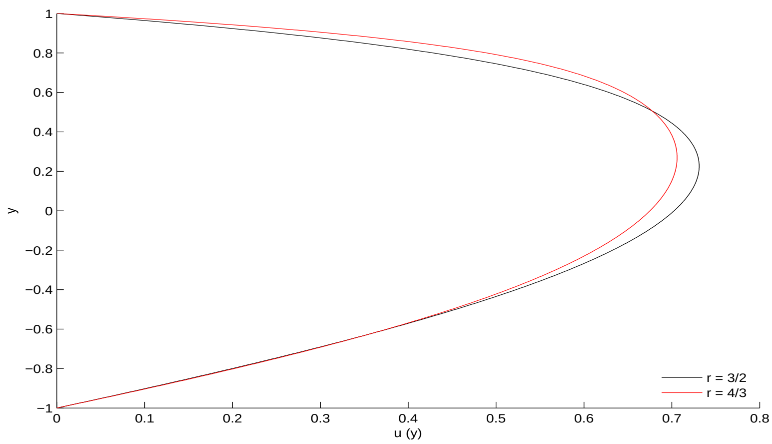

Figure 3.15: Generalized power-law fluid with Vogel's viscosity model - velocity profiles for different values of r ($\kappa = 1$, $\Gamma = 0$, $A = 47.5$, $B = 6.25$)

Chapter 4

Concluding remarks

In chapter 3, we have summed up so far known results. To date, the mechanic (the pressure and the symmetric part of the velocity gradient) and the temperature effects were studied separately and in our work, we try to combine both of these effects even with respect to different boundary conditions.

The flow between two parallel plates was chosen as a simple problem geometry in which we can easily compare the mechanic and the temperature effects.

From the qualitative point of view, it is obvious that the temperature gradient shifts the maximum of the velocity profile towards the warmer plate. This effect cannot be achieved by a pure mechanic model.

We were not able to obtain the numerical solution directly in case of the power-law fluid and the generalized power-law fluid (solution only for some parameters' values). These problems would need a deeper analysis.

In the future, we can further study the fluids with pressure and temperature dependent viscosities. We could also try to optimize the flow by setting up the temperature at the channel plates.

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