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BAKALÁŘSKÁ PRÁCE

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Vztahy mezi prostory funkcí

Katedra matematické analýzy

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Prohlašuji, že jsem svou bakalářskou práci napsala samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

V Praze dne Lenka Slavíková

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Abstrakt: V předložené práci studujeme skorokompaktní vnoření mezi různými prostory funkcí. Skorokompaktní vnoření představuje vztah mezi dvěma Banachovými prostory funkcí, který je silnější než obyčejné vnoření, ale obecně slabší než vnoření kompaktní. Jde o velice užitečný nástroj například při dokazování kompaktnosti Sobolevových vnoření. Uvádíme několik charakterizací skorokompaktního vnoření, mimo jiné pomocí bodové konvergence posloupnosti funkcí či vlastností jistého operátoru součinu. Podáváme úplnou charakterizaci všech párů koncových prostorů Lorentzova a Marcinkiewiczova typu, pro které platí skorokompaktní vnoření. Uvádíme podmínky pro inkluzi do podprostoru funkcí s absolutně spojitou normou.
Klíčová slova: skorokompaktní vnoření, Banachův prostor funkcí, operátor součinu, fundamentální funkce, absolutně spojitá norma

Title: Relations between function spaces
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Abstract: In the present work we study almost-compact embeddings between Banach function spaces. An almost-compact embedding is a relation which is weaker than a compact embedding but at the same time stronger than an ordinary inclusion. It is very useful for example when compact Sobolev embeddings are established. We present several characterizations of such embeddings between various Banach function spaces. Our criteria are expressed, among others, in terms of a pointwise convergence or in terms of certain product operator. We characterize all possible pairs of Lorentz and Marcinkiewicz endpoint spaces for which such an embedding holds. We point out certain conditions for inclusions into subspaces of functions with absolute continuous norms.
Keywords: almost-compact embedding, Banach function space, product operator, fundamental function, absolutely-continuous norm

1. Introduction

Suppose that X and Y are normed linear spaces. We say that X is *embedded into* Y, denoted $X \hookrightarrow Y$, if $X \subseteq Y$ and the identity operator from X to Y is continuous, i.e.

$$\|f\|_Y \leq C\|f\|_X, \ f \in X,$$

for some constant C independent of f.

Furthermore, X is *compactly embedded into* Y, denoted $X \hookrightarrow\hookrightarrow Y$, if $X \subseteq Y$ and the identity operator from X to Y is compact, equivalently, if for every sequence $(f_n)_{n=1}^{\infty}$ bounded in X, we can find its subsequence convergent in Y.

Compact embeddings play an important role when functional-analytic methods are applied to finding solutions of partial differential equations. However, it is often quite complicated to establish the compactness of an embedding. In this text, we define some other type of embedding, called an almost-compact embedding, which is generally weaker then a compact embedding but in some cases it could be useful for establishing compact embeddings.

Suppose that X and Y are Banach function spaces (in the sense described in the following section) over a measure space (R, μ). We say that X is *almost-compactly embedded into* Y and write $X \overset{*}{\hookrightarrow} Y$ if for every sequence $(E_n)_{n=1}^{\infty}$ of μ-measurable subsets of R satisfying $E_n \to \emptyset$ μ-a.e., we have

$$\lim_{n\to\infty} \sup_{\|f\|_X \leq 1} \|f\chi_{E_n}\|_Y = 0.$$

We first prove an equivalence between an almost-compact embedding to certain type of almost-everywhere convergence. An important corollary of this result shows that an almost-compact embedding combined with a bounded Sobolev embedding leads immediately to a compact Sobolev embedding. This result in some sense justifies the label "almost compact embedding". This is done in Section 3. In Section 4 we study the product operator and find its intimate relation to the almost-compact embedding. An important and useful necessary condition for almost-compact embeddings expressed in terms of fundamental functions is established in Section 5. In Section 6, we study almost-compact embeddings between certain special function spaces, called Lorentz and Marcinkiewicz endpoint spaces. We present a complete characterization of all possible mutual embeddings for such spaces. Compactness of an embedding between function spaces is intimately related to the subspace of functions having absolutely continuous norms of a given Banach function space. In the final section, we study inclusions of endpoint spaces into such subspaces.

2. Preliminaries

In this chapter we shall fix the notation and recall some basic facts from the theory of Banach function spaces and rearrangement-invariant spaces. We shall not prove the well-known results; our standard general reference is [2].

Let (R, μ) be a totally σ-finite measure space. Denote by $\mathcal{M}^+$ the set of all μ-measurable functions on R with values in $[0, \infty]$. A mapping $\rho : \mathcal{M}^+ \to [0, \infty]$ is called a *Banach function norm* if, for all f, g, f_n, $(n = 1, 2, \dots)$, in $\mathcal{M}^+$, for all constants $a \geq 0$, and for all μ-measurable subsets E of R, the following properties hold:

(P1) $\rho(f) = 0 \Leftrightarrow f = 0 \ \mu - a.e., \ \rho(af) = a\rho(f), \ \rho(f+g) \leq \rho(f) + \rho(g),$

(P2) $0 \leq g \leq f \ \mu - a.e. \Rightarrow \rho(g) \leq \rho(f),$

(P3) $0 \leq f_n \uparrow f \ \mu - a.e. \Rightarrow \rho(f_n) \uparrow \rho(f),$

(P4) $\mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty,$

(P5) $\mu(E) < \infty \Rightarrow \int_E f \, d\mu \leq C_E \rho(f)$

for some constant C_E, $0 < C_E < \infty$, depending on E and ρ but independent of f.

Denote by $\mathcal{M}$ the set of all μ-measurable real-valued functions on R. The collection $X = X(\rho)$ of all functions $f \in \mathcal{M}$ for which $\rho(|f|) < \infty$ is called a *Banach function space.* For every $f \in \mathcal{M}$, we define

$$\|f\|_X = \rho(|f|).$$

Let X be a Banach function space. The Fatou lemma says that whenever $(f_n)_{n=1}^{\infty}$ is a sequence in X such that $f_n \to f$ μ-a.e. and $\liminf_{n\to\infty} \|f_n\|_X < \infty$, then $f \in X$ and

$$\|f\|_X \leq \liminf_{n\to\infty} \|f_n\|_X.$$

Given a Banach function space X, the *associate space* X' is a Banach function space consisting of all functions $g \in \mathcal{M}$ such that fg is integrable for every $f \in X$. The norm on X' is given by

$$\|g\|_{X'} = \sup\left\{\int_R |fg| \, d\mu : f \in X, \|f\|_X \leq 1\right\}.$$

Then $X'' = (X')' = X$. Moreover, for every f, g in $\mathcal{M}$, we have the *Hölder inequality*

$$\int_R |fg| \, d\mu \leq \|f\|_X \|g\|_{X'}.$$

If X and Y are Banach function spaces over the same measure space, then $X \hookrightarrow Y$ is equivalent to $Y' \hookrightarrow X'$. Furthermore, $X \hookrightarrow Y$ holds if and only if $X \subseteq Y$ (see [2, Chapter 1, Theorem 1.8]).

Let $(E_n)_{n=1}^{\infty}$ be a sequence of μ-measurable subsets of R. We write $E_n \to \emptyset$ μ-a.e. if the characteristic functions χ_{E_n} converge to 0 pointwise μ-a.e. Moreover, if the sequence $(E_n)_{n=1}^{\infty}$ is nonincreasing, we write $E_n \downarrow \emptyset$ μ-a.e.

A function f in a Banach function function space X is said to have an *absolutely continuous norm* in X if $\|f\chi_{E_n}\| \to 0$ for every sequence $(E_n)_{n=1}^{\infty}$ satisfying $E_n \to \emptyset$ μ-a.e. The set of all functions in X of absolutely continuous norm is denoted by X_a.

Suppose that $f \in \mathcal{M}$. The *nonincreasing rearrangement* of f is the function f^* defined on $[0, \infty)$ by

$$f^*(t) = \inf\{\lambda : \mu\{x \in R : |f(x)| > \lambda\} \leq t\}, \ t \geq 0.$$

Furthermore, f^{**} denotes the *maximal function* of f^*, defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s) \, ds, \ t > 0.$$

A Banach function space X is said to be a *rearrangement-invariant space* if $\|f\|_X = \|g\|_X$ holds whenever f, g belong to X and $f^* = g^*$.

Now suppose that (R, μ) is a nonatomic σ-finite measure space and that X is a rearrangement-invariant Banach function space over (R, μ). Then there is a (not necessarily unique) rearrangement-invariant Banach function space $\bar{X}$ over $[0, \mu(R))$ such that

$$\|f\|_X = \|f^*\|_{\bar{X}}, \ f \in X.$$

The space $\bar{X}$ is called the *representation space* of X.

Because (R,μ) is nonatomic, the range of μ consists of the interval $[0,\mu(R)]$. Thus, for every $t\in[0,\mu(R)]$ (if $\mu(R)<\infty$), or $t\in[0,\infty)$ (if $\mu(R)=\infty$), we can find a set E_t with $\mu(E_t)=t$. Let

$$\varphi_X(t)=\|\chi_{E_t}\|_X.$$

The function φ_X so defined is called the *fundamental function* of X. Then φ_X is nonnegative and nondecreasing, $\varphi_X(t)=0$ if and only if $t=0$, $\varphi_X(t)/t$ is nonincreasing. A function satisfying these properties is said to be *quasiconcave.*

Let X' be the associate space of X. Then

$$\varphi_X(t)\varphi_{X'}(t)=t \tag{2.1}$$

holds for each finite value of t in the range of μ.

Let φ be a quasiconcave function on $(0,\mu(R))$. The *Marcienkiewicz endpoint space* $M_\varphi=M_\varphi(R,\mu)$ consists of all functions f in $\mathcal{M}$ for which the functional

$$\|f\|_{M_\varphi}=\sup_{t\in(0,\mu(R))}\{f^{**}(t)\varphi(t)\}$$

is finite.

For every quasiconcave function φ, we define its *least nondecreasing concave majorant* φ_0 as a pointwise infimum of all nondecreasing concave majorants of φ. Then

$$\frac{1}{2}\varphi_0\le\varphi\le\varphi_0.$$

As a consequence of this, we get that every rearrangement-invariant space X over (R,μ) can be equivalently renormed with a rearrangement-invariant norm in such a way that the resulting fundamental function is concave.

Denote $a=\mu(R)$. Let ψ be a positive nondecreasing concave function on $(0,a)$. The *Lorentz endpoint space* $\Lambda_\psi=\Lambda_\psi(R,\mu)$ consists of all $f\in\mathcal{M}$ for which

$$\|f\|_{\Lambda\psi}=\int_0^a f^*(s)\,d\psi(s)=\|f\|_{L^\infty}\psi(0_+)+\int_0^a f^*(s)\psi'(s)\,ds$$

is finite.

It is not hard to show that both the spaces Λ_φ and M_φ have fundamental function φ.

If X is a rearrangement-invariant Banach function space with a concave fundamental function φ, then

$$\Lambda_\varphi\hookrightarrow X\hookrightarrow M_\varphi.$$

An important example of rearrangement-invariant spaces are the *Lebesgue spaces* $L^p=L^p(R,\mu)$, $(1\le p\le\infty)$, consisting of all $f\in\mathcal{M}$, for which

$$\|f\|_{L^p}=\begin{cases}\left(\int_R|f|^p\,d\mu\right)^{1/p}, & 1\le p<\infty;\\ \operatorname{ess\,sup}_R|f|, & p=\infty;\end{cases}$$

is finite.

Let $\phi:[0,\infty)\to[0,\infty)$ be a nondecreasing and left-continuous function with $\phi(0)=0$, $\phi(s)>0$ for $s>0$. Then the function Φ defined by

$$\Phi(t)=\int_0^t\phi(s)\,ds,\ t\ge0,$$

is said to be a *Young's function.* In particular, every Young's function is convex.

Let Φ be a Young's function. The *Orlicz space* $L^\Phi=L^\Phi(R,\mu)$ is the rearrangement-invariant Banach function space consisting of all $f\in\mathcal{M}$, for which

$$\|f\|_{L^\Phi}=\inf\{k^{-1}:\int_R\Phi(k|f(x)|)\,dx\le1\}$$

is finite.

Suppose that $d \in \mathbb{N}$ and Ω is a nonempty open subset of $\mathbb{R}^d$. Let λ denote the Lebesgue measure on Ω. For a Banach function space X over (Ω, λ), the *Sobolev space* W^1X consists of all real-valued weakly-differentiable functions f in X such that $|\nabla f| \in X$.

3. Almost-compact embeddings and convergence

We first observe that, in the definition of an almost-compact embedding, the sequence (E_n) can be taken nonincreasing.

Theorem 3.1. *Let X and Y be Banach function spaces over a totally σ-finite measure space (R, μ). Then $X \overset{*}{\hookrightarrow} Y$ if and only if*

$$\lim_{n\to\infty} \sup_{\|f\|_X \le 1} \|f\chi_{E_n}\|_Y = 0$$

holds for every sequence $(E_n)_{n=1}^{\infty}$ satisfying $E_n \downarrow \emptyset$ μ-a.e.

This is a well-known fact which follows by replacing (E_n) by $\left(\bigcup_{k\ge n} E_k\right)$. We omit the proof.

We start with an easy observation about almost-compact embedings for associate spaces.

Theorem 3.2. *Let X and Y be Banach function spaces over a totally σ-finite measure space (R, μ). Then $X \overset{*}{\hookrightarrow} Y$ if and only if $Y' \overset{*}{\hookrightarrow} X'$.*

Proof. Suppose that $X \overset{*}{\hookrightarrow} Y$. Let $(E_n)_{n=1}^{\infty}$ be an arbitrary sequence of sets in R satisfying $E_n \downarrow \emptyset$ μ-a.e. Using the definition of the associate norm and the fact that $Y'' = Y$, we get

$$\begin{aligned}
\lim_{n\to\infty} \sup_{\|g\|_{Y'} \le 1} \|g\chi_{E_n}\|_{X'} &= \lim_{n\to\infty} \sup_{\|g\|_{Y'} \le 1} \left(\sup_{\|f\|_X \le 1} \int_R |fg\chi_{E_n}| d\mu \right) \\
&= \lim_{n\to\infty} \sup_{\|f\|_X \le 1} \left(\sup_{\|g\|_{Y'} \le 1} \int_R |fg\chi_{E_n}| d\mu \right) \\
&= \lim_{n\to\infty} \sup_{\|f\|_X \le 1} \|f\chi_{E_n}\|_{Y''} = \lim_{n\to\infty} \sup_{\|f\|_X \le 1} \|f\chi_{E_n}\|_Y = 0,
\end{aligned}$$

i.e. $Y' \overset{*}{\hookrightarrow} X'$, as required.

It remains to show that $Y' \overset{*}{\hookrightarrow} X'$ implies $X \overset{*}{\hookrightarrow} Y$. From the first part of the proof we get $Y' \overset{*}{\hookrightarrow} X'$ implies $X'' \overset{*}{\hookrightarrow} Y''$. Because every Banach function space coincides with its second associate space, we get the result. □

The following theorem provides a characterization of $X \overset{*}{\hookrightarrow} Y$ in terms of convergence μ-a.e.

Theorem 3.3. *Let X and Y be Banach function spaces over a totally σ-finite measure space (R, μ). Then $X \overset{*}{\hookrightarrow} Y$ if and only if for every sequence $(f_n)_{n=1}^{\infty}$ of μ-measurable functions on R satisfying $\|f_n\|_X \le 1$ and $f_n \to 0$ μ-a.e., it holds $\|f_n\|_Y \to 0$.*

Proof. Suppose that $X \overset{*}{\hookrightarrow} Y$. First, we will construct a μ-measurable function g such that $g > 0$ on R and $\|g\|_Y < \infty$. Let $(R_n)_{n=1}^{\infty}$ be a sequence of sets of finite measure satisfying $R_n \uparrow R$. For every positive integer n, consider a function g_n given by

$$g_n = \frac{1}{2^n} \cdot \frac{1}{1 + \|\chi_{R_n}\|_Y} \cdot \chi_{R_n}.$$

Let us also define a function g by $g = \sum_{n=1}^{\infty} g_n$. We have

$$\|g_n\|_Y = \frac{1}{2^n} \cdot \frac{1}{1 + \|\chi_{R_n}\|_Y} \cdot \|\chi_{R_n}\|_Y < \frac{1}{2^n}.$$

Thus

$$\|g\|_Y = \lim_{n\to\infty} \left\| \sum_{k=1}^{n} g_k \right\|_Y \le \lim_{n\to\infty} \sum_{k=1}^{n} \|g_k\|_Y \le \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

Because, obviously, $g > 0$ on R, g has the required properties.

Let $(f_n)_{n=1}^{\infty}$ be a sequence of μ-measurable functions on R satisfying $\|f_n\|_X \le 1$ and $f_n \to 0$ μ-a.e. Choose $\varepsilon > 0$ arbitrarily. Let $E_n = \{x \in R : |f_n(x)| \ge \varepsilon g(x)\}$. Because $f_n \to 0$ μ-a.e. and $\varepsilon g > 0$ on R, for μ-a.e. $x \in R$ we have that $x \in E_n$ holds only for finitely many positive integers n. This implies $E_n \to \emptyset$ μ-a.e.

Observe that

$$\|f_n\|_Y = \|f_n \chi_{E_n} + f_n \chi_{E_n^c}\|_Y \le \|f_n \chi_{E_n}\|_Y + \|f_n \chi_{E_n^c}\|_Y.$$

The assumptions $X \overset{*}{\hookrightarrow} Y$ and $\|f_n\|_X \le 1$ give

$$\lim_{n\to\infty} \|f_n \chi_{E_n}\|_Y \le \lim_{n\to\infty} \sup_{\|h\|_X \le 1} \|h \chi_{E_n}\|_Y = 0.$$

Moreover,

$$\|f_n \chi_{E_n^c}\|_Y \le \|\varepsilon g\|_Y = \varepsilon \|g\|_Y \le \varepsilon.$$

Altogether, we have

$$\limsup_{n\to\infty} \|f_n\|_Y \le \varepsilon,$$

which holds for every $\varepsilon > 0$. So, $\lim_{n\to\infty} \|f_n\|_Y = 0$.

Conversely, suppose that for every sequence $(f_n)_{n=1}^{\infty}$ of μ-measurable functions on R satisfying $\|f_n\|_X \le 1$ and $f_n \to 0$ μ-a.e., it holds $\|f_n\|_Y \to 0$. Let $(E_n)_{n=1}^{\infty}$ be a sequence of subsets of R satisfying $E_n \to \emptyset$ μ-a.e. Then we can find a sequence of functions $(f_n)_{n=1}^{\infty}$ such that $\|f_n\|_X \le 1$ and

$$\|f_n \chi_{E_n}\|_Y + \frac{1}{n} > \sup_{\|f\|_X \le 1} \|f \chi_{E_n}\|_Y.$$

Because $E_n \to \emptyset$ μ-a.e., we have $f_n \chi_{E_n} \to 0$ μ-a.e. Due to the assumption, $\|f_n \chi_{E_n}\|_Y \to 0$. Thus

$$\lim_{n\to\infty} \sup_{\|f\|_X \le 1} \|f \chi_{E_n}\|_Y \le \lim_{n\to\infty} \left(\|f_n \chi_{E_n}\|_Y + \frac{1}{n} \right) = 0.$$

□

In the following two theorems we will show that an almost compact embedding is in general stronger then a regular embedding but weaker then a compact one.

Theorem 3.4. *Suppose that (R, μ) is a totally σ-finite measure space and X and Y are Banach function spaces over (R, μ) satisfying $X \overset{*}{\hookrightarrow} Y$. Then $X \hookrightarrow Y$.*

Proof. Let $(f_n)_{n=1}^{\infty}$ be a sequence in X such that $\|f_n - f\|_X \to 0$ for some $f \in X$. To get a contradiction, assume that $\|f_n - f\|_Y \not\to 0$. Then we can find $\varepsilon > 0$ and a subsequence $(g_k)_{k=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$ satisfying $\|g_k - f\|_Y \ge \varepsilon$ for every $k \in \mathbb{N}$. Because $g_k \to f$ in X, there is a subsequence $(h_l)_{l=1}^{\infty}$ of $(g_k)_{k=1}^{\infty}$ such that $h_l \to f$ μ-a.e. Using that $X \overset{*}{\hookrightarrow} Y$, by Theorem 3.3 we obtain $\|h_l - f\|_Y \to 0$, which gives a contradiction. So, $X \hookrightarrow Y$. □

Theorem 3.5. *Suppose that (R, μ) is a totally σ-finite measure space and X and Y are Banach function spaces over (R, μ) satisfying $X \hookrightarrow\hookrightarrow Y$. Then $X \overset{*}{\hookrightarrow} Y$.*

Proof. Let $(f_n)_{n=1}^{\infty}$ be a sequence in X such that $\|f_n\|_X \le 1$ for every $n \in \mathbb{N}$ and $f_n \to 0$ μ-a.e. To get a contradiction, assume that $\|f_n\|_Y \not\to 0$. Then there is $\varepsilon > 0$ and a subsequence $(g_k)_{k=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$ satisfying $\|g_k\|_Y \ge \varepsilon$ for every $k \in \mathbb{N}$. Because $(g_k)_{k=1}^{\infty}$ is bounded in X and $X \hookrightarrow\hookrightarrow Y$, we can find a subsequence $(h_l)_{l=1}^{\infty}$ of $(g_k)_{k=1}^{\infty}$ such that $(h_l)_{l=1}^{\infty}$ is convergent in Y.

But $h_l \to 0$ μ-a.e., so the limit must be 0. So, $\|h_l\|_Y \to 0$, which contradicts the assumption. Thus, $X \overset{*}{\hookrightarrow} Y$. ☐

The following theorem shows, in fact, that in the cases that might be of a possible interest, a Banach function space cannot be almost-compactly embedded into itself.

Theorem 3.6. *We say that a totally σ-finite measure space (R,μ) has the property $(*)$ if there exists a sequence $(E_n)_{n=1}^{\infty}$ of μ-measurable subsets of R such that $E_n \downarrow \emptyset$ μ-a.e. and $\mu(E_n) > 0$ for every $n \in \mathbb{N}$.*

(i) *Assume that the measure space (R,μ) has the property $(*)$. Let X be a Banach function space over (R,μ). Then $X \overset{*}{\not\hookrightarrow} X$.*

(ii) *Conversely, assume that (R,μ) does not have this property. Then all Banach function spaces over (R,μ) coincide, and, moreover, $X \overset{*}{\hookrightarrow} X$ holds for each Banach function space X.*

Proof. We start with proving the assertion (ii). We first observe that in the definition of an almost-compact embedding, it is enough to consider only those sequences $E_n \downarrow \emptyset$ μ-a.e., for which $\mu(E_n) > 0$ $(n = 1, 2, \dots)$. Indeed, for Banach function spaces X and Y, the condition

$$\lim_{n\to\infty} \sup_{\|f\|_X \le 1} \|f\chi_{E_n}\|_Y = 0$$

trivially holds if $\mu(E_n) = 0$ for some $n \in \mathbb{N}$ (then $\mu(E_m) = 0$ for every $m \ge n$, thus also $f\chi_{E_m} = 0$ μ-a.e. and $\|f\chi_{E_m}\|_Y = 0$ whenever $m \ge n$ and $f \in X$).

Assume that (R,μ) does not have the property $(*)$. Then for every pair of Banach function spaces X and Y, we have $X \overset{*}{\hookrightarrow} Y$ and also $Y \overset{*}{\hookrightarrow} X$. So, by Theorem 3.4, $X \hookrightarrow Y$ and $Y \hookrightarrow X$, i.e. X and Y coincide and $X \overset{*}{\hookrightarrow} X$. This shows (ii).

As for the statement (i), suppose that (R,μ) has the property $(*)$. To get a contradiction, assume that $X \overset{*}{\hookrightarrow} X$ holds for some Banach function space X. Let $(E_n)_{n=1}^{\infty}$ be the sequence of subsets of R satisfying $\mu(E_n) > 0$ for every $n \in \mathbb{N}$ and $E_n \downarrow \emptyset$ μ-a.e. Consider a sequence $(f_n)_{n=1}^{\infty}$ of functions in X defined by $f_n = \frac{1}{\|\chi_{E_n}\|_X}\chi_{E_n}$. Then, for every $n \in \mathbb{N}$, we have

$$\sup_{\|f\|_X \le 1} \|f\chi_{E_n}\|_X \ge \|f_n\|_X = 1,$$

which contradicts the assumption $X \overset{*}{\hookrightarrow} X$. ☐

The condition from Theorem 3.3 is often used as a crucial step in proofs of compact embeddings, for example of Sobolev spaces (cf. [4, Section 9] or [3]). For that matter, so is the almost-compactness, hence their equivalence is very reasonable. Let us now present a result that illustrates the importance of almost-compact embeddings.

Theorem 3.7. *Let $d \in \mathbb{N}$ and let Ω be a nonempty open subset of $\mathbb{R}^d$. Suppose that X, Y, Z are Banach function spaces over (Ω,λ), where λ denotes the n-dimensional Lebesgue measure. Moreover, assume that $W^1X \hookrightarrow Y$ and $Y \overset{*}{\hookrightarrow} Z$. Then $W^1X \hookrightarrow\hookrightarrow Z$.*

Proof. Whenever $x \in \Omega$, we can find a ball B_x centered in x such that $B_x \subseteq \Omega$. For $x \in \Omega$, consider also a ball $\tilde{B}_x$ with center x and with radius equal to one half of the radius of B_x. Then the set $\{\tilde{B}_x : x \in \Omega\}$ forms an open covering of Ω. Because Ω is separable, we can find a sequence $(x_n)_{n=1}^{\infty}$ of points in Ω such that $\{\tilde{B}_n = \tilde{B}_{x_n} : n \in \mathbb{N}\}$ covers Ω. Furthermore, we denote $B_n = B_{x_n}$ $(n = 1, 2, \dots)$.

Let $(g_k)_{k=1}^{\infty}$ be a bounded sequence in $W^1X = W^1X(\Omega)$. By induction, for every $n \in \mathbb{N}$ we will find a subsequence $(g_k^n)_{k=1}^{\infty}$ of the sequence $(g_k^{n-1})_{k=1}^{\infty}$ (here we formally put $g_k^0 = g_k$) converging μ-a.e. on $\tilde{B}_n$. Then, the diagonal sequence $(g_n^n)_{n=1}^{\infty}$ will converge μ-a.e. to some function g on the entire Ω (because $\{\tilde{B}_n : n \in \mathbb{N}\}$ forms a covering of Ω).

Fix $n \in \mathbb{N}$ and suppose that we already know the sequence $(g_k^{n-1})_{k=1}^\infty$. Then $(g_k^{n-1})_{k=1}^\infty$ is bounded in $W^1X(B_n)$ and (by property (P5) of Banach function spaces) also in $W^{1,1}(B_n)$. Consider a function ψ defined on R^d by

$$\psi(x) = \begin{cases} \exp(-\frac{1}{1-|x|^2}), & |x| < 1; \\ 0, & |x| \geq 1. \end{cases}$$

Denote by ψ_n the function satisfying $\psi_n(x) = \psi((x - x_n)/r_n)$ for $x \in \mathbb{R}^d$ (r_n denotes the radius of $\tilde{B}_n$). Then ψ_n is a C^∞-function on $\mathbb{R}^d$ and $\psi_n(x) \neq 0$ if and only if $x \in \tilde{B}_n$. Define a sequence $(u_k^n)_{k=1}^\infty$ by $u_k^n(x) = g_k^{n-1}(x)\psi_n(x)$, $x \in B_n$. The sequence $(u_k^n)_{k=1}^\infty$ is bounded in $W_0^{1,1}(B_n)$, so we can extend it by 0 out of B_n and consider $(u_k^n)_{k=1}^\infty$ to be bounded in $W^{1,1}(\mathbb{R}^d)$. Thus, there is a subsequence $(u_{k_l}^n)_{l=1}^\infty$ which converges μ-a.e. to some function u_n (see [4, Lemma 9.2]). We will denote by $(g_k^n)_{k=1}^\infty$ the sequence $(g_{k_l}^{n-1})_{l=1}^\infty$. Because $\psi_n(x) \neq 0$ for $x \in \tilde{B}_n$, we have $g_k^n \to u_n/\psi_n$ μ-a.e. on $\tilde{B}_n$, as required.

By the assumption, $(g_n^n)_{n=1}^\infty$ is bounded in Y. Hence, by the Fatou lemma,

$$\|g\|_Y \leq \liminf_{n\to\infty} \|g_n^n\|_Y < \infty,$$

so $g \in Y$. By assumption $Y \overset{*}{\hookrightarrow} Z$ and by Theorem 3.3, $\|g_n^n - g\|_Z \to 0$, i.e. $g_n^n \to g$ in Z. Thus, $W^1X \hookrightarrow\hookrightarrow Z$. □

4. The product operator

In the first half of this section we observe that the fact that a rearrangement-invariant Banach function space does not coincide with either of L^1, L^∞, can be characterized by its fundamental function and also by the almost-compact embedding. We shall finish the section with a characterization of an almost-compact embedding by some properties of a certain product operator.

Definition 4.1. Suppose that (R, μ) is a measure space. Then we define the *product operator* $P : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ by

$$P(f, g) = f \cdot g.$$

Theorem 4.2. *Let (R, μ) be a nonatomic measure space satisfying $0 < \mu(R) < \infty$ and let X be a rearrangement-invariant Banach function space over (R, μ). Denote by φ the fundamental function of X. Then the following statements are equivalent:*
(i) $X \neq L^1$;
(ii) $\lim_{t\to 0+} \frac{t}{\varphi(t)} = 0$;
(iii) $X \overset{*}{\hookrightarrow} L^1$.

Proof. (i) ⇒ (ii) Suppose $X \neq L^1$. Then there exists a function $f \in L^1 \setminus X$. We may suppose that f is nonnegative (otherwise we may consider the function $|f|$ which is nonnegative and belongs to $L^1 \setminus X$) and $\|f\|_{L^1} = 1$. (Because f does not belong to X it cannot be equal to 0 μ-a.e., so it has a positive norm in L^1. If this norm is different from 1 we may consider the function $\|f\|_{L^1}^{-1} f$ instead of f.) Let $(u_n)_{n=1}^\infty$ be a sequence of nonnegative nontrivial simple functions satisfying $u_n \uparrow f$. Then $u_n \in X$ for every $n \in \mathbb{N}$, $\|u_n\|_X \uparrow \infty$ and $\|u_n\|_{L^1} \leq \|f\|_{L^1} = 1$. Thus

$$\lim_{n\to\infty} \frac{\|u_n\|_X}{\|u_n\|_{L^1}} \geq \lim_{n\to\infty} \|u_n\|_X = \infty.$$

Choose $K > 0$ arbitrarily. Then we can find $n \in \mathbb{N}$ such that $\|u_n\|_X \geq K\|u_n\|_{L^1}$. Suppose

$$u_n = \sum_{i=1}^k a_i \chi_{A_i},$$

where A_i are pairwise disjoint subsets of R and a_i are different positive constants. By the triangle inequality, we have

$$\|u_n\|_X \leq \sum_{i=1}^{k} a_i \|\chi_{A_i}\|_X,$$

moreover

$$\|u_n\|_{L^1} = \sum_{i=1}^{k} a_i \|\chi_{A_i}\|_{L^1}.$$

Now we use $\|u_n\|_X \geq K\|u_n\|_{L^1}$ to get the following inequality

$$\sum_{i=1}^{k} a_i \|\chi_{A_i}\|_X \geq K \sum_{i=1}^{k} a_i \|\chi_{A_i}\|_{L^1}.$$

There must exist $i \in \{1, 2, \dots, k\}$ such that $a_i\|\chi_{A_i}\|_X \geq K a_i \|\chi_{A_i}\|_{L^1}$, i.e. $\frac{\|\chi_{A_i}\|_X}{\|\chi_{A_i}\|_{L^1}} \geq K$. Denote by t the measure of A_i. It holds $\varphi(t) = \|\chi_{A_i}\|_X$ and $t = \|\chi_{A_i}\|_{L^1}$. So, for an arbitrary $K > 0$ we have found $t > 0$ such that $\frac{\varphi(t)}{t} \geq K$. Together with the fact that $\frac{\varphi(t)}{t}$ is nonincreasing, it implies $\lim_{t \to 0_+} \frac{\varphi(t)}{t} = \infty$, in other words $\lim_{t \to 0_+} \frac{t}{\varphi(t)} = 0$.

(ii) $\Rightarrow$ (iii) Fix an arbitrary sequence $(E_n)_{n=1}^{\infty}$ of subsets of R with $E_n \downarrow \emptyset$ μ-a.e. Moreover, suppose that $\mu(E_n) > 0$ for every $n \in \mathbb{N}$ (in the proof of Theorem 3.6, we observed that it is enough to consider only sequences of this type). We will show that for every $n \in \mathbb{N}$

$$\sup_{\|f\|_X \leq 1} \|f\chi_{E_n}\|_{L_1} \leq \frac{\mu(E_n)}{\varphi(\mu(E_n))}. \tag{4.1}$$

Indeed, using the Hölder inequality and (2.1), we get

$$\|f\chi_{E_n}\|_{L_1} = \int_R |f\chi_{E_n}|\, d\mu \leq \|f\|_X \|\chi_{E_n}\|_{X'} = \frac{\mu(E_n)}{\varphi(\mu(E_n))}\|f\|_X$$

for every $n \in \mathbb{N}$ and $f \in X$, which implies (4.1). Because $\mu(R) < \infty$ and $E_1 \supseteq E_2 \supseteq \dots$, it holds that $\mu(E_n) \to \mu(\bigcap_{n=1}^{\infty} E_n) = 0$. Thus, by (ii),

$$\lim_{n\to\infty} \sup_{\|f\|_X \leq 1} \|f\chi_{E_n}\|_{L_1} \leq \lim_{n\to\infty} \frac{\mu(E_n)}{\varphi(\mu(E_n))} = 0.$$

So we have proved that $X \overset{*}{\hookrightarrow} L^1$.

(iii) $\Rightarrow$ (i) It follows from Theorem 3.6 that $L^1 \overset{*}{\hookrightarrow} L^1$ cannot be true. $\square$

Near the other endpoint space, L^∞, we have an analogous result.

Theorem 4.3. *Let (R, μ) be a nonatomic measure space satisfying $0 < \mu(R) < \infty$ and let X be a rearrangement-invariant Banach function space over (R, μ). Denote by φ the fundamental function of X. Then the following statements are equivalent:*
(i) $X \neq L^\infty$*;*
(ii) $\lim_{t\to 0_+} \varphi(t) = 0$*;*
(iii) $L^\infty \overset{*}{\hookrightarrow} X$*.*

Proof. Denote by ψ the fundamental function of the associate space X'. Then $\psi(t) = \frac{t}{\varphi(t)}$ for every $t \in (0, \mu(R)]$. This gives $\lim_{t\to 0_+} \varphi(t) = 0$ if and only if $\lim_{t\to 0_+} \frac{t}{\psi(t)} = 0$. Because $(L^\infty)' = L^1$ we have $X \neq L^\infty$ if and only if $X' \neq L^1$ and (by Theorem 3.2) $L^\infty \overset{*}{\hookrightarrow} X$ if and only if $X' \overset{*}{\hookrightarrow} L^1$. The assertion thus follows from Theorem 4.2. $\square$

Lemma 4.4. *Suppose that (R,μ) is a nonatomic measure space with $0<\mu(R)<\infty$ and X and Y are Banach function spaces over (R,μ). Then the following two statements are equivalent:*

(i) $$X \overset{*}{\hookrightarrow} Y;$$

(ii) $$\lim_{t\to 0+} \sup_{\|f\|_X\le 1} \sup_{\mu(E)\le t} \|f\chi_E\|_Y = 0.$$

Proof. (i) ⇒ (ii) Consider a function H defined by

$$H(t) = \sup_{\|f\|_X\le 1} \sup_{\mu(E)\le t} \|f\chi_E\|_Y,\ t\in(0,\mu(R)].$$

Clearly, H is nondecreasing on $(0,\mu(R)]$. Thus, it will be enough to prove

$$\lim_{n\to\infty} \sup_{\|f\|_X\le 1} \sup_{\mu(E)\le a_n} \|f\chi_E\|_Y = 0 \tag{4.2}$$

for some sequence $(a_n)_{n=1}^\infty$ satisfying $a_n\downarrow 0$. We will choose the sequence $a_n = 1/n^2$. Observe that $\sum_{n=1}^\infty a_n < \infty$. For every $n\in\mathbb{N}$ we can find $f_n\in X$, $E_n\subseteq R$ such that $\|f_n\|_X\le 1$, $\mu(E_n)\le a_n$ and

$$\sup_{\|f\|_X\le 1} \sup_{\mu(E)\le a_n} \|f\chi_E\|_Y < \|f_n\chi_{E_n}\|_Y + \frac{1}{n}. \tag{4.3}$$

Denote $F_n = \bigcup_{k=n}^\infty E_k$. Then $F_1\supseteq F_2\supseteq\dots$ and

$$\mu(\bigcap_{n=1}^\infty F_n) = \lim_{n\to\infty}\mu(F_n) \le \lim_{n\to\infty}\sum_{k=n}^\infty \mu(E_k) = \lim_{n\to\infty}\sum_{k=n}^\infty a_k = 0.$$

This implies $F_n\downarrow\emptyset$ μ-a.e. Because $E_n\subseteq F_n$ for every $n\in\mathbb{N}$ and $X \overset{*}{\hookrightarrow} Y$, we have

$$\lim_{n\to\infty}\|f_n\chi_{E_n}\|_Y \le \lim_{n\to\infty}\sup_{\|f\|_X\le 1}\|f\chi_{E_n}\|_Y \le \lim_{n\to\infty}\sup_{\|f\|_X\le 1}\|f\chi_{F_n}\|_Y = 0.$$

Using the inequality (4.3), we obtain (4.2).

(ii) ⇒ (i) Choose an arbitrary sequence $(E_n)_{n=1}^\infty$ of subsets of R such that $E_n\downarrow\emptyset$ μ-a.e. Because $\mu(R)<\infty$ we have

$$\lim_{n\to\infty}\mu(E_n) = \mu(\bigcap_{n=1}^\infty E_n) = 0.$$

Thus

$$\lim_{n\to\infty}\sup_{\|f\|_X\le 1}\|f\chi_{E_n}\|_Y \le \lim_{n\to\infty}\sup_{\|f\|_X\le 1}\sup_{\mu(E)=\mu(E_n)}\|f\chi_E\|_Y = 0,$$

so $X \overset{*}{\hookrightarrow} Y$. □

The following simple but useful lemma shows that the set inclusion $P(X\times Y)\subseteq Z$ already implies the norm boundedness $P: X\times Y\to Z$. Both the result and the proof are modeled upon [2, Chapter 1, Theorem 1.8].

Lemma 4.5. *Let X, Y, Z be Banach function spaces over a measure space (R,μ). Suppose that $P(X\times Y)\subseteq Z$. Then there exists $K>0$ such that*

$$\|fg\|_Z \le K\|f\|_X\|g\|_Y, \tag{4.4}$$

whenever $f\in X$, $g\in Y$.

Proof. Suppose that (4.4) fails. Then for every $n \in \mathbb{N}$ we can find $\tilde{f_n} \in X$, $\tilde{g_n} \in Y$ such that

$$\|\tilde{f_n}\tilde{g_n}\|_Z > n^5 \|\tilde{f_n}\|_X \|\tilde{g_n}\|_Y.$$

In particular, we have $\|\tilde{f_n}\tilde{g_n}\|_Z > 0$, so $\tilde{f_n}\tilde{g_n} \neq 0$, and thus $\|\tilde{f_n}\|_X > 0$, $\|\tilde{g_n}\|_Y > 0$. Dividing the previous inequality by $\|\tilde{f_n}\|_X \|\tilde{g_n}\|_Y$, we get

$$\left\| \frac{|\tilde{f_n}|}{\|\tilde{f_n}\|_X} \frac{|\tilde{g_n}|}{\|\tilde{g_n}\|_Y} \right\|_Z = \left\| \frac{\tilde{f_n}}{\|\tilde{f_n}\|_X} \frac{\tilde{g_n}}{\|\tilde{g_n}\|_Y} \right\|_Z > n^5.$$

Denote $f_n = \frac{|\tilde{f_n}|}{\|\tilde{f_n}\|_X}$, $g_n = \frac{|\tilde{g_n}|}{\|\tilde{g_n}\|_Y}$. Then $f_n \geq 0$, $g_n \geq 0$, $\|f_n\|_X = \|g_n\|_Y = 1$, $\|f_n g_n\|_Z > n^5$. Consider the functions

$$f = \sum_{n=1}^{\infty} \frac{f_n}{n^2}, \ g = \sum_{n=1}^{\infty} \frac{g_n}{n^2}.$$

We have

$$\|f\|_X = \lim_{n\to\infty} \left\| \sum_{k=1}^{n} \frac{f_k}{k^2} \right\|_X \leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\|f_k\|_X}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

so $f \in X$. Analogously we obtain $g \in Y$. Moreover, $0 \leq \frac{f_n}{n^2} \leq f$ and $0 \leq \frac{g_n}{n^2} \leq g$ for every $n \in \mathbb{N}$. Multiplying these two inequalities, we get

$$0 \leq \frac{f_n g_n}{n^4} \leq fg,$$

which gives

$$\|fg\|_Z \geq \frac{\|f_n g_n\|_Z}{n^4} > n$$

for every $n \in \mathbb{N}$. This means $\|fg\|_Z = \infty$, i.e. $fg \notin Z$, which contradicts the assumption $P(X \times Y) \subseteq Z$. □

The following theorem is the main result of this section.

Theorem 4.6. *Suppose that (R, μ) is a nonatomic measure space with $0 < \mu(R) < \infty$ and X and Y are Banach function spaces over (R, μ). Then the following two conditions are equivalent.*
(i) $X \overset{*}{\hookrightarrow} Y$*;*
(ii) *there exists a rearrangement-invariant Banach function space Z over (R, μ) such that $Z \neq L^1$ and $P(X \times Y') \subseteq Z$.*

Proof. (i) ⇒ (ii) We will find a quasiconcave function φ such that $\lim_{t\to 0+} \frac{t}{\varphi(t)} = 0$ and $\|fg\|_{M_\varphi} \leq \|f\|_X \|g\|_{Y'}$, whenever $f \in X$, $g \in Y'$. Once this is done, M_φ will be a rearrangement-invariant Banach function space different from L^1 (Theorem 4.2) satisfying $P : X \times Y' \to M_\varphi$, so the proof will be complete.

Let $f \in X$, $g \in Y'$. Let us recall that

$$\|fg\|_{M_\varphi} = \sup_{t\in(0,\mu(R))} \frac{\varphi(t)}{t} \int_0^t (fg)^*(s)\, ds.$$

Fix $t \in (0, \mu(R))$. Then (see e.g. [2, Chapter 2, Lemma 2.5]) there exists $E_t \subseteq R$ with $\mu(E_t) = t$ such that

$$\int_0^t (fg)^*(s)\, ds = \int_{E_t} |fg|\, d\mu.$$

Due to Hölder inequality,

$$\int_{E_t} |fg|\, d\mu \leq \|f\chi_{E_t}\|_Y \|g\|_{Y'},$$

and because

$$\left\| \frac{f}{\|f\|_X} \chi_{E_t} \right\|_Y \leq \sup_{\|h\|_X \leq 1} \sup_{\mu(E) \leq t} \|h\chi_E\|_Y,$$

we also get

$$\|f\chi_{E_t}\|_Y \|g\|_{Y'} \leq \|f\|_X \|g\|_{Y'} \sup_{\|h\|_X \leq 1} \sup_{\mu(E) \leq t} \|h\chi_E\|_Y.$$

Define the function H by

$$H(t) = \sup_{\|h\|_X \leq 1} \sup_{\mu(E) \leq t} \|h\chi_E\|_Y,$$

$t \in (0, \mu(R))$. Because $X \overset{*}{\hookrightarrow} Y$, we have in particular $X \hookrightarrow Y$, so there exists a constant $C > 0$ such that $\|h\|_Y \leq C\|h\|_X$ for every $h \in X$. Using that $\|h\chi_E\|_Y \leq \|h\|_Y$ holds for every $h \in Y$ (in particular, for $h \in X$) and $E \subseteq R$, we observe that the function H is bounded by the constant C. Moreover, due to Lemma 4.4, $\lim_{t \to 0_+} H(t) = 0$. Our goal is to find a function ψ such that ψ is a nondecreasing concave majorant of H and $\lim_{t \to 0_+} \psi(t) = 0$.

First, observe that the constant function C is a nondecreasing concave majorant of H. Now consider all nondecreasing concave majorants of H and denote by ψ its pointwise infimum. Then ψ is itself a nondecreasing concave majorant of H, so it remains to prove that $\lim_{t \to 0_+} \psi(t) = 0$.

Choose $\varepsilon \in (0, 2C)$. Because $\lim_{t \to 0_+} H(t) = 0$, there exists $\tilde{\delta} > 0$ such that $H(t) < \frac{\varepsilon}{2}$ for $t \in (0, \tilde{\delta})$. From this it follows that the function $F(t) = \varepsilon/2 + t(C - \varepsilon/2)/\tilde{\delta}$ is a nondecreasing concave majorant of H, so $\psi(t) \leq F(t)$ on $(0, \mu(R))$. But we can find $\delta > 0$ (namely, $\delta = \varepsilon\tilde{\delta}/(2C - \varepsilon)$) such that $F(t) < \varepsilon$ for $t \in (0, \delta)$, therefore $\psi(t) < \varepsilon$ on $(0, \delta)$, which gives the result.

Now we are in a position to define the function φ. So, put $\varphi(t) = \frac{t}{\psi(t)}$, $t \in (0, \mu(R))$. Then φ is quasiconcave and $\lim_{t \to 0_+} \frac{t}{\varphi(t)} = \lim_{t \to 0_+} \psi(t) = 0$. Moreover, for $f \in X$, $g \in Y'$ we have

$$\begin{aligned} \|fg\|_{M_\varphi} &= \sup_{t \in (0, \mu(R))} \frac{\varphi(t)}{t} \int_0^t (fg)^*(s)\, ds \leq \|f\|_X \|g\|_{Y'} \sup_{t \in (0, \mu(R))} \frac{\varphi(t)}{t} H(t) \\ &\leq \|f\|_X \|g\|_{Y'} \sup_{t \in (0, \mu(R))} \frac{\varphi(t)}{t} \psi(t) = \|f\|_X \|g\|_{Y'}, \end{aligned}$$

which completes the proof.

(ii) $\Rightarrow$ (i) Choose an arbitrary sequence $(E_n)_{n=1}^\infty$ of subsets of R satisfying $E_n \downarrow \emptyset$ μ-a.e. We have

$$\sup_{\|f\|_X \leq 1} \|f\chi_{E_n}\|_Y = \sup_{\|f\|_X \leq 1} \sup_{\|g\|_{Y'} \leq 1} \int_R |fg\chi_{E_n}|\, d\mu.$$

Due to Lemma 4.5, there exists a constant $K > 0$ such that $\|fg\|_Z \leq K\|f\|_X \|g\|_{Y'}$ for every $f \in X$, $g \in Y'$. Thus, if $\|f\|_X \leq 1$ and $\|g\|_{Y'} \leq 1$ then $\|fg\|_Z \leq K$, which implies

$$\begin{aligned} \sup_{\|f\|_X \leq 1} \sup_{\|g\|_{Y'} \leq 1} \int_R |fg\chi_{E_n}|\, d\mu &\leq \sup_{\|h\|_Z \leq K} \int_R |h\chi_{E_n}|\, d\mu = \sup_{\|h\|_Z \leq K} K \int_R \left| \frac{h}{K} \chi_{E_n} \right| d\mu \\ &= K \sup_{\|h\|_Z \leq 1} \int_R |h\chi_{E_n}|\, d\mu. \end{aligned}$$

According to Theorem 4.2, $Z \overset{*}{\hookrightarrow} L^1$, so

$$\lim_{n \to \infty} \sup_{\|f\|_X \leq 1} \|f\chi_{E_n}\|_Y \leq K \lim_{n \to \infty} \sup_{\|h\|_Z \leq 1} \int_R |h\chi_{E_n}|\, d\mu = 0,$$

i.e. $X \overset{*}{\hookrightarrow} Y$. ☐

Our next example illustrates that the restriction to rearrangement-invariant spaces in Theorem 4.6 is indispensable.

Example 4.7. Suppose that (R,μ) is a nonatomic measure space with $0<\mu(R)<\infty$ and X and Y are Banach function spaces over (R,μ). We will show that if a Banach function space $Z\neq L^1$ is not rearrangement-invariant, then the condition $P(X\times Y')\subseteq Z$ does not necessarily imply $X\overset{*}{\hookrightarrow}Y$. To do this, it is enough to find a Banach function space $Z\neq L^1$ such that it does not hold $Z\overset{*}{\hookrightarrow}L^1$ (recall that this is impossible if Z is rearrangement-invariant). Having such a space Z, we set $X=Z$, $Y=L^1$, so $Y'=L^\infty$. Suppose that $f\in Z$, $g\in L^\infty$. There must exist a positive constant K such that $|g|\leq K$ a.e. Then $\|fg\|_Z\leq K\|f\|_Z<\infty$, so $fg\in Z$. Thus, we have $P(X\times Y')=P(Z\times L^\infty)\subseteq Z\neq L^1$, but $X\overset{*}{\hookrightarrow}Y$ is not true, as required.

Consider the measure space $((0,1),\lambda)$. For a measurable function f, define

$$\|f\|_Z=\|f\chi_{(0,\frac{1}{2})}\|_{L^\infty}+\|f\chi_{[\frac{1}{2},1)}\|_{L^1}.$$

Using the facts that $L^\infty((0,\frac{1}{2}),\lambda)$ and $L^1([\frac{1}{2},1),\lambda)$ are Banach function spaces, it is easy to see that Z is a Banach function space as well. We have $Z\neq L^1$, because the function $f(t)=1/\sqrt{t}$ belongs to L^1 but not to Z. Set $E_n=(1-1/n,1)$ and $f_n=n\chi_{E_n}$ $(n\in\mathbb{N})$. Then $E_n\downarrow\emptyset$ and, $\|f_n\|_Z=1$ for $n>1$. Thus

$$\sup_{\|f\|_Z\leq 1}\|f\chi_{E_n}\|_{L^1}\geq\|f_n\chi_{E_n}\|_{L^1}=1,\ (n>1)$$

so $Z\overset{*}{\hookrightarrow}L^1$ does not hold.

5. Almost-compact embeddings and the fundamental function

In this section we shall present an important necessary condition for an almost-compact embedding between two rearrangement-invariant spaces in terms of their fundamental functions.

Lemma 5.1. *Suppose that (R,μ) is a nonatomic measure space satisfying $0<\mu(R)<\infty$ and let X and Y be rearrangement-invariant Banach function spaces over (R,μ). Let S denote the set of nonnegative nonzero simple functions on R. Then the following conditions are equivalent.*

(i) $$X\overset{*}{\hookrightarrow}Y;$$

(ii) $$\lim_{t\to 0+}\sup_{\|f\|_X\leq 1}\|f^*\chi_{[0,t)}\|_{\bar{Y}}=0;$$

(iii) $$\lim_{t\to 0+}\sup_{u\in S}\frac{\|u^*\chi_{[0,t)}\|_{\bar{Y}}}{\|u^*\chi_{[0,t)}\|_{\bar{X}}}=0.$$

Proof. (i) $\Leftrightarrow$ (ii) Due to Lemma 4.4, $X\overset{*}{\hookrightarrow}Y$ is equivalent to

$$\lim_{t\to 0+}\sup_{\|f\|_X\leq 1}\sup_{\mu(E)\leq t}\|f\chi_E\|_Y=0.$$

Thus it is sufficient to show that, for every $f\in X$ and for every $t\in(0,\mu(R))$,

$$\sup_{\mu(E)\leq t}\|f\chi_E\|_Y=\|f^*\chi_{[0,t)}\|_{\bar{Y}}.$$

Fix $f\in X$ and $t\in(0,\mu(R))$. Whenever E is a measurable subset of R with $\mu(E)\leq t$, we have

$$\|f\chi_E\|_Y=\|(f\chi_E)^*\|_{\bar{Y}}=\|(f\chi_E)^*\chi_{[0,t)}\|_{\bar{Y}}\leq\|f^*\chi_{[0,t)}\|_{\bar{Y}},$$

and therefore

$$\sup_{\mu(E)\leq t}\|f\chi_E\|_Y\leq\|f^*\chi_{[0,t)}\|_{\bar{Y}}.$$

For $f \in X$ and $t \in (0, \mu(R))$, we can find a measurable set $F \subseteq R$ with $\mu(F) = t$ such that $f^*\chi_{[0,t)} = (f\chi_F)^*$ (this follows from the proof of [2, Chapter 2, Lemma 2.5]). Thus, we can write

$$\sup_{\mu(E)\le t} \|f\chi_E\|_Y = \sup_{\mu(E)\le t} \|(f\chi_E)^*\|_{\bar{Y}} \ge \|(f\chi_F)^*\|_{\bar{Y}} = \|f^*\chi_{[0,t)}\|_{\bar{Y}},$$

which gives the reverse inequality.

(ii) $\Leftrightarrow$ (iii) We will show that for every $t \in (0, \mu(R))$

$$\sup_{\|f\|_X\le 1} \|f^*\chi_{[0,t)}\|_{\bar{Y}} = \sup_{u\in S} \frac{\|u^*\chi_{[0,t)}\|_{\bar{Y}}}{\|u^*\chi_{[0,t)}\|_{\bar{X}}},$$

which is obviously enough for the proof.

Suppose that $f \in X$, $\|f\|_X \le 1$. Then we have

$$f^*\chi_{[0,t)} \le \frac{f^*}{\|f\|_X}\chi_{[0,t)} = \left(\frac{f}{\|f\|_X}\right)^*\chi_{[0,t)},$$

which gives

$$\|f^*\chi_{[0,t)}\|_{\bar{Y}} \le \left\|\left(\frac{f}{\|f\|_X}\right)^*\chi_{[0,t)}\right\|_{\bar{Y}} \le \sup_{\|g\|_X=1} \|g^*\chi_{[0,t)}\|_{\bar{Y}},$$

so

$$\sup_{\|f\|_X\le 1} \|f^*\chi_{[0,t)}\|_{\bar{Y}} \le \sup_{\|f\|_X=1} \|f^*\chi_{[0,t)}\|_{\bar{Y}}.$$

The reverse inequality is obvious, thus

$$\sup_{\|f\|_X\le 1} \|f^*\chi_{[0,t)}\|_{\bar{Y}} = \sup_{\|f\|_X=1} \|f^*\chi_{[0,t)}\|_{\bar{Y}}.$$

Furthermore,

$$\sup_{\|f\|_X=1} \|f^*\chi_{[0,t)}\|_{\bar{Y}} = \sup_{0\ne f\in X} \left\|\left(\frac{f}{\|f\|_X}\right)^*\chi_{[0,t)}\right\|_{\bar{Y}} = \sup_{0\ne f\in X} \frac{\|f^*\chi_{[0,t)}\|_{\bar{Y}}}{\|f\|_X} = \sup_{0\ne f\in X} \frac{\|f^*\chi_{[0,t)}\|_{\bar{Y}}}{\|f^*\|_{\bar{X}}}.$$

We need to show that

$$\sup_{0\ne f\in X} \frac{\|f^*\chi_{[0,t)}\|_{\bar{Y}}}{\|f^*\|_{\bar{X}}} = \sup_{0\ne f\in X} \frac{\|f^*\chi_{[0,t)}\|_{\bar{Y}}}{\|f^*\chi_{[0,t)}\|_{\bar{X}}}.$$

Because $\|f^*\|_{\bar{X}} \ge \|f^*\chi_{[0,t)}\|_{\bar{X}}$, it must be

$$\sup_{0\ne f\in X} \frac{\|f^*\chi_{[0,t)}\|_{\bar{Y}}}{\|f^*\|_{\bar{X}}} \le \sup_{0\ne f\in X} \frac{\|f^*\chi_{[0,t)}\|_{\bar{Y}}}{\|f^*\chi_{[0,t)}\|_{\bar{X}}}.$$

On the other hand, whenever $f \in X$, $f \ne 0$, we can find a measurable set F such that $\mu(F) = t$ and $f^*\chi_{[0,t)} = (f\chi_F)^* = (f\chi_F)^*\chi_{[0,t)}$. Then

$$\frac{\|f^*\chi_{[0,t)}\|_{\bar{Y}}}{\|f^*\chi_{[0,t)}\|_{\bar{X}}} = \frac{\|(f\chi_F)^*\chi_{[0,t)}\|_{\bar{Y}}}{\|(f\chi_F)^*\|_{\bar{X}}} \le \sup_{0\ne g\in X} \frac{\|g^*\chi_{[0,t)}\|_{\bar{Y}}}{\|g^*\|_{\bar{X}}},$$

which gives the reverse inequality.

Finally, we observe that the supremum can be taken over the (smaller) set S instead of $X\setminus\{0\}$. Indeed, for every $f \in X$, $f \ne 0$, we can find a sequence $(u_n)_{n=1}^\infty$ with $u_n \in S$, $(n \in \mathbb{N})$, and $u_n \uparrow |f|$. This implies $u_n^*\chi_{[0,t)} \uparrow f^*\chi_{[0,t)}$, and thus

$$\lim_{n\to\infty} \frac{\|u_n^*\chi_{[0,t)}\|_{\bar{Y}}}{\|u_n^*\chi_{[0,t)}\|_{\bar{X}}} = \frac{\|f^*\chi_{[0,t)}\|_{\bar{Y}}}{\|f^*\chi_{[0,t)}\|_{\bar{X}}},$$

which gives the result. □

Now we are in a position to state and prove our main result of this section.

Theorem 5.2. *If (R,μ) is a nonatomic measure space satisfying $0<\mu(R)<\infty$ and X and Y are rearrangement-invariant Banach function spaces over (R,μ) such that $X \overset{*}{\hookrightarrow} Y$, then*

$$\lim_{t\to 0_+} \frac{\varphi_Y(t)}{\varphi_X(t)} = 0,$$

where φ_X, φ_Y are fundamental functions of X, Y respectively.

Proof. The function $f=1$ on R belongs to S defined in the Lemma 5.1. Thus, for $t\in(0,\mu(R)]$,

$$\frac{\varphi_Y(t)}{\varphi_X(t)} = \frac{\|f^*\chi_{[0,t)}\|_Y}{\|f^*\chi_{[0,t)}\|_X} \le \sup_{u\in S} \frac{\|u^*\chi_{[0,t)}\|_Y}{\|u^*\chi_{[0,t)}\|_X}.$$

According to the lemma, we have

$$\lim_{t\to 0_+} \frac{\varphi_Y(t)}{\varphi_X(t)} = 0.$$

□

Corollary 5.3. *Let (R,μ) be a nonatomic measure space satisfying $0<\mu(R)<\infty$ and let φ be a positive nondecreasing concave function on $(0,\mu(R))$. Then it does not hold that $\Lambda_\varphi \overset{*}{\hookrightarrow} M_\varphi$.*

Proof. Λ_φ and M_φ have the same fundamental function φ so the necessary condition for almost compact embedding from Theorem 5.2 cannot hold. □

Our next example shows that the necessary condition from Theorem 5.2 is not sufficient for an almost-compact embedding.

Example 5.4. Suppose that X, Y are rearrangement-invariant Banach function spaces with fundamental functions φ_X, φ_Y, respectively. We will show that the condition $\lim_{t\to 0_+} \frac{\varphi_Y(t)}{\varphi_X(t)} = 0$ does not imply $X \hookrightarrow Y$. In particular, it does not imply $X \overset{*}{\hookrightarrow} Y$.

Let $p\in(1,\infty)$. Denote by p' the conjugate index satisfying $\frac{1}{p}+\frac{1}{p'}=1$. We will consider for X the Marcinkiewicz space $L_{p,\infty}$ and for Y the *Lorentz-Zygmund space* $L_{p,1;-1}$ over $((0,1),\lambda)$, consisting of all measurable functions f such that

$$\|f\|_{p,\infty} = \sup_{t\in(0,1)} f^*(t) t^{\frac{1}{p}} < \infty$$

and

$$\|f\|_{p,1;-1} = \int_0^1 \frac{f^*(s) s^{\frac{1}{p}-1}}{e-\log s} ds = \int_0^1 \frac{f^*(s)}{s^{\frac{1}{p'}}(e-\log s)} ds < \infty,$$

respectively. We note that the functionals $\|\cdot\|_{p,\infty}$ and $\|\cdot\|_{p,1;-1}$ are equivalent to rearrangement-invariant Banach function norms.

Then it follows from [1, Theorem 9.3] that $L_{p,\infty} \not\hookrightarrow L_{p,1;-1}$. Moreover, if we denote by φ the fundamental function of $L_{p,\infty}$ and by ψ the fundamental function of $L_{p,1;-1}$, then, for every $t\in(0,1)$, we have

$$\varphi(t) = \sup_{s\in(0,1)} \chi_{(0,t)}(s) s^{\frac{1}{p}} = t^{\frac{1}{p}},$$

while

$$\psi(t) = \int_0^t \frac{1}{s^{\frac{1}{p'}}(e-\log s)} ds.$$

Because

$$\lim_{t\to 0_+} \varphi(t) = \lim_{t\to 0_+} \psi(t) = 0,$$

we can use the L'Hospital rule to get

$$\lim_{t\to 0_+}\frac{\psi(t)}{\varphi(t)}=\lim_{t\to 0_+}\frac{\psi'(t)}{\varphi'(t)}=\lim_{t\to 0_+}\frac{p}{e-\log t}=0.$$

Example 5.5. Let (R,μ) be a nonatomic measure space such that $\mu(R)=1$. Recall that for every Young's function Φ and every $f\in L^\Phi=L^\Phi(R,\mu)$, we have

$$\|f\|_{L^\Phi}=\inf\{k^{-1}:\int_R\Phi(k|f(x)|)\,dx\le 1\}=\inf\{k^{-1}:\int_0^1\Phi(kf^*(x))\,dx\le 1\}=\|f^*\|_{\bar L^\Phi}.$$

Moreover, for $f\in L^\Phi$, $t\in(0,1)$ and $\lambda>0$, we have $\|f^*\chi_{[0,t)}\|_{\bar L^\Phi}\le 1/\lambda$ if and only if $\int_0^t\Phi(\lambda f^*(x))\,dx\le 1$.

Suppose that A, B are Young's functions. We will consider Orlicz spaces $L^A=L^A(R,\mu)$, $L^B=L^B(R,\mu)$. Our goal is to show that $L^A\overset{*}{\hookrightarrow}L^B$ if and only if for every $\lambda>0$,

$$\lim_{t\to\infty}\frac{B(\lambda t)}{A(t)}=0. \tag{5.1}$$

Note that a necessary and sufficient condition for $L^A\hookrightarrow L^B$ to be true is that there exists $C>0$ such that $B(t)\le A(Ct)$ for every $t>0$, while $L^A\hookrightarrow\hookrightarrow L^B$ never holds because $L^\infty\not\hookrightarrow\hookrightarrow L^1$.

It follows from the proof of Lemma 5.1 that the condition $L^A\overset{*}{\hookrightarrow}L^B$ is equivalent to

$$\lim_{t\to 0_+}\sup_{0\ne f\in L^A}\frac{\|f^*\chi_{[0,t)}\|_{\bar L^B}}{\|f^*\chi_{[0,t)}\|_{\bar L^A}}=0.$$

Furthermore, for $t\in(0,1)$, we have

$$\begin{aligned}\sup_{0\ne f\in L^A}\frac{\|f^*\chi_{[0,t)}\|_{\bar L^B}}{\|f^*\chi_{[0,t)}\|_{\bar L^A}}&=\sup\{\|f^*\chi_{[0,t)}\|_{\bar L^B}:\ \|f^*\chi_{[0,t)}\|_{\bar L^A}=1\}\\&=\sup\{\|f^*\chi_{[0,t)}\|_{\bar L^B}:\ \|f^*\chi_{[0,t)}\|_{\bar L^A}\le 1\}.\end{aligned}$$

So, $L^A\overset{*}{\hookrightarrow}L^B$ holds if and only if for every $\lambda>0$ there exists $\delta>0$ such that the condition $\int_0^\delta A(f^*(x))\,dx\le 1$ implies $\int_0^\delta B(\lambda f^*(x))\,dx\le 1$ (we are using the fact that the expression $\sup\{\|f^*\chi_{[0,t)}\|_{\bar L^B}:\ \|f^*\chi_{[0,t)}\|_{\bar L^A}\le 1\}$ increases with t).

Assume that $L^A\overset{*}{\hookrightarrow}L^B$. We claim that, for every $\lambda>0$, there is $t_0>0$ such that $B(\lambda t)\le A(t)$ for $t\ge t_0$. Suppose that this is not true. Then we can find $\lambda>0$ and a sequence $t_n\to\infty$ satisfying $B(\lambda t_n)>A(t_n)$ for every $n\in\mathbb{N}$. For this $\lambda>0$, choose $\delta>0$ as above. Because $\lim_{t\to\infty}A(t)=\infty$, there is $m\in\mathbb{N}$ such that $1/A(t_m)<\delta$. Denote $\delta_0=1/A(t_m)$. Since (R,μ) is nonatomic, we can find a set $F\subseteq R$ with $\mu(F)=\delta_0$. The function $f=t_m\chi_F$ satisfies $\int_0^\delta A(f^*(x))\,dx=\delta_0A(t_m)=1$ but $\int_0^\delta B(\lambda f^*(x))\,dx=\delta_0B(\lambda t_m)>\delta_0A(t_m)=1$, which gives a contradiction.

Because B is convex and $B(0)=0$, we have for every $k\in\mathbb{N}$ and $t\in[0,\infty)$

$$B(t)=B\left(\frac{1}{k}\cdot kt+\frac{k-1}{k}\cdot 0\right)\le\frac{1}{k}\cdot B(kt)+\frac{k-1}{k}\cdot B(0)=\frac{1}{k}\cdot B(kt).$$

Fix $\lambda>0$. Then for every $k\in\mathbb{N}$ there is $t_k>0$ such that $B(k\lambda t)\le A(t)$, $t\ge t_k$. Thus

$$\frac{B(\lambda t)}{A(t)}\le\frac{1}{k}\cdot\frac{B(k\lambda t)}{A(t)}\le\frac{1}{k},\ (t\ge t_k),$$

which gives (5.1), as required.

Now assume that (5.1) holds for every $\lambda > 0$. Choose $\lambda > 0$ arbitrarily. Then we can find $t_0 > 0$ such that $B(\lambda t) \le 1/2 \cdot A(t)$, whenever $t \ge t_0$. Set $\delta = 1/(2B(\lambda t_0))$. Let f be an arbitrary μ-measurable function on R. Denote $a = \min(\mu\{x \in R : |f(x)| \ge t_0\}, \delta)$. We have

$$\int_0^\delta B(\lambda f^*(x))\,dx = \int_0^a B(\lambda f^*(x))\,dx + \int_a^\delta B(\lambda f^*(x))\,dx \le \frac{1}{2}\int_0^a A(f^*(x))\,dx + \int_a^\delta B(\lambda t_0)\,dx$$
$$\le \frac{1}{2}\int_0^\delta A(f^*(x))\,dx + \int_0^\delta B(\lambda t_0)\,dx \le 1,$$

whenever $\int_0^\delta A(f^*(x))\,dx \le 1$. Thus $L^A \overset{*}{\hookrightarrow} L^B$.

Remark 5.6. Suppose that $p \in [1, \infty)$ and consider the Young function $\Phi(t) = t^p$. Then the Orlicz space L^Φ is exactly the Lebesgue space L^p. The characterization of almost compact embedding between Orlicz spaces from the previous example together with Theorem 4.3 shows that for $1 \le p, q \le \infty$, $L^p \overset{*}{\hookrightarrow} L^q$ holds if and only if $q < p$. Note that $L^p \hookrightarrow L^q$ if and only if $q \le p$, while $L^p \hookrightarrow\hookrightarrow L^q$ is never true.

6. Almost-compact embeddings between spaces of type Λ and M

In this section we present a complete characterization of all possible mutual almost-compact embeddings among the Lorentz and Marcinkiewicz endpoint spaces. We shall work for our typographical convenience on the measure space $((0,1), \lambda)$. This of course can be done with no loss of generality and the results of this section can be easily extended to all non-atomic finite-measure spaces.

Suppose that φ is a quasiconcave function on $(0,1)$. In the following text, $\tilde{\varphi}$ denotes the quasiconcave function satisfying $\tilde{\varphi}(t) = \frac{t}{\varphi(t)}$ for every $t \in (0,1)$.

Lemma 6.1. *Let φ and ψ be quasiconcave functions on $(0,1)$. Suppose that there exist positive constants C_1, C_2 such that*

$$C_1\varphi(t) \le \psi(t) \le C_2\varphi(t)$$

for every $t \in (0,1)$. Then $M_\varphi = M_\psi$.

Proof. Assume that $f \in M_\varphi$. Then $f \in M_\psi$, because

$$\|f\|_{M_\psi} = \sup_{t\in(0,1)} \psi(t) f^{**}(t) \le C_2 \sup_{t\in(0,1)} \varphi(t) f^{**}(t) = C_2\|f\|_{M_\varphi} < \infty.$$

The converse embedding follows from symmetry. □

Lemma 6.2. *Suppose that φ is a quasiconcave function on $(0,1)$. Let α be the least nondecreasing concave majorant of $\tilde{\varphi}$. Then $M'_\varphi = \Lambda_\alpha$.*

Proof. The function α satisfies $\frac{\alpha(t)}{2} \le \frac{t}{\varphi(t)} \le \alpha(t)$ for $t \in (0,1)$. Thus also $\frac{\varphi(t)}{2} \le \frac{t}{\alpha(t)} \le \varphi(t)$ on $(0,1)$. Due to Lemma 6.1, $M_\varphi = M_{\tilde{\alpha}}$.

First, we will show that $M'_\varphi \subseteq \Lambda_\alpha$. This is equivalent to $\Lambda'_\alpha \subseteq M''_\varphi = M_\varphi = M_{\tilde{\alpha}}$. But Λ'_α has the fundamental function $\tilde{\alpha}$ and $M_{\tilde{\alpha}}$ is the largest rearrangement-invariant space with this fundamental function, so $\Lambda'_\alpha \subseteq M_{\tilde{\alpha}}$.

On the other hand, because $M_\varphi = M_{\tilde{\alpha}}$, we have $M'_\varphi = M'_{\tilde{\alpha}}$. Using that $M'_{\tilde{\alpha}}$ has the fundamental function α and Λ_α is the smallest rearrangement-invariant space with fundamental function α, we obtain $\Lambda_\alpha \subseteq M'_{\tilde{\alpha}} = M'_\varphi$. □

Lemma 6.3. *Let φ and ψ be positive nondecreasing concave functions on $(0,1)$. Suppose that there exist positive constants C_1, C_2 such that*

$$C_1\varphi(t) \le \psi(t) \le C_2\varphi(t) \tag{6.1}$$

for every $t \in (0,1)$. Then $\Lambda_\varphi = \Lambda_\psi$.

Proof. According to Lemma 6.2, $\Lambda_\varphi = M'_{\tilde\varphi}$ and $\Lambda_\psi = M'_{\tilde\psi}$. The assumption (6.1) gives

$$C_1\tilde\psi(t) \le \tilde\varphi(t) \le C_2\tilde\psi(t)$$

for every $t \in (0,1)$. So, due to Lemma 6.1, $M_{\tilde\varphi} = M_{\tilde\psi}$, thus also $\Lambda_\varphi = M'_{\tilde\varphi} = M'_{\tilde\psi} = \Lambda_\psi$. □

Theorem 6.4. *Suppose that φ and ψ are positive nondecreasing concave functions on $(0,1)$. Then the following four statements are equivalent.*

(i) $$\Lambda_\varphi \overset{*}{\hookrightarrow} \Lambda_\psi;$$

(ii) $$M_\varphi \overset{*}{\hookrightarrow} M_\psi;$$

(iii) $$\Lambda_\varphi \overset{*}{\hookrightarrow} M_\psi;$$

(iv) $$\lim_{t\to 0_+} \frac{\psi(t)}{\varphi(t)} = 0.$$

Proof. According to Theorem 5.2, each of the conditions (i), (ii), (iii) implies (iv).

(iv) ⇒ (i) Due to Lemma 5.1, we only need to prove that

$$\lim_{t\to 0_+} \sup_{u\in S} \frac{\|u^*\chi_{(0,t)}\|_{\Lambda_\psi}}{\|u^*\chi_{(0,t)}\|_{\Lambda_\varphi}} = 0,$$

where S denotes the set of nonnegative nonzero simple functions on $(0,1)$.

Suppose that $u \in S$. Given $t \in (0,1)$, we have

$$u^*\chi_{(0,t)} = \sum_{i=1}^n c_i \chi_{(0,t_i)},$$

where $c_i > 0$, $i = 1,2,\dots,n$ and $0 < t_1 < \dots < t_n \le t$. Because

$$\frac{\psi(t_i)}{\varphi(t_i)} \le \sup_{0<s\le t} \frac{\psi(s)}{\varphi(s)},$$

we have

$$\frac{\|u^*\chi_{(0,t)}\|_{\Lambda_\psi}}{\|u^*\chi_{(0,t)}\|_{\Lambda_\varphi}} = \frac{\sum_{i=1}^n c_i\psi(t_i)}{\sum_{i=1}^n c_i\varphi(t_i)} \le \frac{\sum_{i=1}^n c_i\varphi(t_i)\sup_{0<s\le t}\frac{\psi(s)}{\varphi(s)}}{\sum_{i=1}^n c_i\varphi(t_i)} = \sup_{0<s\le t} \frac{\psi(s)}{\varphi(s)}.$$

Thus

$$\lim_{t\to 0_+} \sup_{u\in S} \frac{\|u^*\chi_{(0,t)}\|_{\Lambda_\psi}}{\|u^*\chi_{(0,t)}\|_{\Lambda_\varphi}} \le \lim_{t\to 0_+} \sup_{0<s\le t} \frac{\psi(s)}{\varphi(s)} = 0.$$

(iv) ⇒ (ii) Denote by α, β the least nondecreasing concave majorant of $\tilde\varphi$, $\tilde\psi$, respectively. Then

$$\frac{\alpha(t)}{\beta(t)} \le 2\frac{\tilde\varphi(t)}{\tilde\psi(t)} = 2\frac{\psi(t)}{\varphi(t)}$$

for every $t \in (0,1)$. The assumption (iv) gives

$$\lim_{t\to 0_+} \frac{\alpha(t)}{\beta(t)} \le \lim_{t\to 0_+} 2\frac{\psi(t)}{\varphi(t)} = 0.$$

Using the implication (iv) ⇒ (i), which was just proved, for functions α, β, we obtain $\Lambda_\beta \overset{*}{\hookrightarrow} \Lambda_\alpha$. Due to Lemma 6.2, $\Lambda_\alpha = M'_\varphi$ and $\Lambda_\beta = M'_\psi$. Thus we have $M'_\psi \overset{*}{\hookrightarrow} M'_\varphi$, which (by Theorem 3.2) implies $M_\varphi \overset{*}{\hookrightarrow} M_\psi$, as required.

(ii) ⇒ (iii) This is a consequence of the facts that $\Lambda_\varphi \hookrightarrow M_\varphi$ and $M_\varphi \overset{*}{\hookrightarrow} M_\psi$. □

Theorem 6.5. *Suppose that φ and ψ are positive nondecreasing concave functions on $(0,1)$ and α is the least nondecreasing concave majorant of $\tilde{\varphi}$. Assume that $\lim_{t\to 0_+}\psi(t)=0$ and $\lim_{t\to 0_+}\frac{t}{\varphi(t)}=0$. Then the following statements are equivalent:*

(i) $$M_\varphi \hookrightarrow \Lambda_\psi,$$

(ii) $$M_\varphi \overset{*}{\hookrightarrow} \Lambda_\psi,$$

(iii) $$\int_0^1 \alpha'(s)\psi'(s)\,ds<\infty.$$

Moreover, if ψ'' exists on $(0,1)$, then the conditions (i), (ii), (iii) *are equivalent to*

(iv) $$\int_0^1 \frac{s(-\psi''(s))}{\varphi(s)}\,ds<\infty.$$

Proof. (i) $\Rightarrow$ (iii) The function α satisfies $\frac{\alpha(t)}{2}\le\frac{t}{\varphi(t)}\le\alpha(t)$ for every $t\in(0,1)$. We have

$$\lim_{t\to 0_+}\alpha(t)\le\lim_{t\to 0_+}\frac{2t}{\varphi(t)}=0.$$

We also have

$$\|\alpha'\|_{M_\varphi}=\sup_{t\in(0,1)}\frac{\varphi(t)}{t}\int_0^t\alpha'(s)ds=\sup_{t\in(0,1)}\frac{\varphi(t)}{t}\cdot\alpha(t)\le\sup_{t\in(0,1)}\frac{\varphi(t)}{t}\cdot\frac{2t}{\varphi(t)}=2<\infty,$$

so $\alpha'\in M_\varphi$, and because $M_\varphi\hookrightarrow\Lambda_\psi$, we obtain $\alpha'\in\Lambda_\psi$, i.e.

$$\int_0^1\alpha'(s)\psi'(s)\,ds<\infty,$$

as required.

(iii) $\Rightarrow$ (ii) According to Lemma 5.1, it is enough to show that

$$\lim_{t\to 0_+}\sup_{\|f\|_{M_\varphi}\le 1}\|f^*\chi_{(0,t)}\|_{\Lambda_\psi}=0.$$

Using the Hölder inequality and the fact that $M'_\varphi=\Lambda_\alpha$, in particular $\Lambda_\alpha\hookrightarrow M'_\varphi$, we have for $t\in(0,1)$ and $f\in M_\varphi$

$$\begin{aligned}\|f^*\chi_{(0,t)}\|_{\Lambda_\psi}&=\int_0^1 f^*(s)\psi'(s)\chi_{(0,t)}\,ds\le\|f\|_{M_\varphi}\|\psi'\chi_{(0,t)}\|_{M'_\varphi}\\&\le C\|f\|_{M_\varphi}\|\psi'\chi_{(0,t)}\|_{\Lambda_\alpha}=C\|f\|_{M_\varphi}\int_0^t\psi'(s)\alpha'(s)\,ds,\end{aligned}$$

where C is a positive constant independent of f. Thus

$$\lim_{t\to 0_+}\sup_{\|f\|_{M_\varphi}\le 1}\|f^*\chi_{(0,t)}\|_{\Lambda_\psi}\le\lim_{t\to 0_+}C\int_0^t\psi'(s)\alpha'(s)\,ds=0,$$

which completes the proof.

(ii) $\Rightarrow$ (i) $X\overset{*}{\hookrightarrow}Y$ implies $X\hookrightarrow Y$ for every pair of Banach function spaces X and Y.

(iii) $\Leftrightarrow$ (iv) The functions α' and ψ' are measurable and nonnegative a.e., thus $\alpha'\psi'$ is measurable and nonnegative a.e., so $\int_0^1\alpha'(s)\psi'(s)\,ds$ exists. Because ψ'' exists on $(0,1)$, we have

$\psi'(s) - \psi'(1_-) = \int_s^1 -\psi''(r)\,dr$. Using the Fubini theorem, we obtain

$$\begin{aligned}\int_0^1 \alpha'(s)\psi'(s)\,ds &= \int_0^1 \alpha'(s)\int_s^1 -\psi''(r)\,dr\,ds + \int_0^1 \psi'(1_-)\alpha'(s)\,ds\\ &= \int_0^1 -\psi''(r)\int_0^r \alpha'(s)\,ds\,dr + \alpha(1_-)\psi'(1_-)\\ &= \int_0^1 \alpha(r)(-\psi''(r))\,dr + \alpha(1_-)\psi'(1_-).\end{aligned}$$

The product $\alpha(1_-)\psi'(1_-)$ is always finite, so

$$\int_0^1 \alpha'(s)\psi'(s)\,ds < \infty \Leftrightarrow \int_0^1 \alpha(s)(-\psi''(s))\,ds < \infty.$$

For $s \in (0,1)$, we have $\frac{\alpha(s)}{2} \le \frac{s}{\varphi(t)} \le \alpha(s)$ and $-\psi''(s) \ge 0$, which implies

$$\frac{1}{2}\int_0^1 \alpha(s)(-\psi''(s))\,ds \le \int_0^1 \frac{s(-\psi''(s))}{\varphi(s)}\,ds \le \int_0^1 \alpha(s)(-\psi''(s))\,ds.$$

Thus

$$\int_0^1 \alpha(s)(-\psi''(s))\,ds < \infty \Leftrightarrow \int_0^1 \frac{s(-\psi''(s))}{\varphi(s)}\,ds < \infty,$$

and we are done. □

Remark 6.6. The conditions $\lim_{t\to 0_+}\psi(t) = 0$ and $\lim_{t\to 0_+}\frac{t}{\varphi(t)} = 0$ are equivalent to $\Lambda_\psi \neq L^\infty$ and $M_\varphi \neq L^1$.

Example 6.7. We will show that for general concave function ψ on $(0,1)$, the condition

$$\int_0^1 \frac{s(-\psi''(s))}{\varphi(s)}\,ds < \infty$$

does not necessarily imply $M_\varphi \hookrightarrow \Lambda_\psi$.

Suppose that $p \in (0,1)$. Let $\varphi(t) = t^p$. Define a function ψ_0 to be linear on each of the intervals $[\frac{1}{n+1}, \frac{1}{n}]$ $(n \in \mathbb{N})$ in such a way that $\psi_0(\frac{1}{n}) = \frac{1}{n^p}$ holds for every $n \in \mathbb{N}$. Put $\psi = \psi_0 \restriction (0,1)$. It is easy to see that ψ has the required properties. Observe that $\psi'' = 0$ a.e., which gives

$$\int_0^1 \frac{s(-\psi''(s))}{\varphi(s)}\,ds = 0 < \infty.$$

Now we will show that $\Lambda_\psi = \Lambda_\varphi$. Proving this, we will get $M_\varphi \not\hookrightarrow \Lambda_\psi$ because $M_\varphi \not\hookrightarrow \Lambda_\varphi$ (the function $f(t) = \frac{1}{t^p}$ belongs to M_φ but does not belong to Λ_φ).

Fix $n \in \mathbb{N}$ and suppose that t belongs to $[\frac{1}{n+1}, \frac{1}{n})$. Then $\varphi(t) \in [\frac{1}{(n+1)^p}, \frac{1}{n^p})$, $\psi(t) \in [\frac{1}{(n+1)^p}, \frac{1}{n^p})$. Thus

$$\left(\frac{n}{n+1}\right)^p \le \frac{\psi(t)}{\varphi(t)} \le \left(\frac{n+1}{n}\right)^p.$$

But for every $n \in \mathbb{N}$, we have $\frac{1}{2^p} \le (\frac{n}{n+1})^p$ and $\left(\frac{n+1}{n}\right)^p \le 2^p$. This gives

$$\frac{1}{2^p} \le \frac{\psi(t)}{\varphi(t)} \le 2^p$$

for every $t \in (0,1)$. Due to Lemma 6.3, $\Lambda_\psi = \Lambda_\varphi$.

Corollary 6.8. *Let X and Y be Banach function spaces of type M or Λ (not necessarily both of the same type) over $((0,1),\lambda)$. Denote by φ_X, φ_Y the fundamental functions of X and Y, respectively. Then $X \overset{*}{\hookrightarrow} Y$ if and only if $X \hookrightarrow Y$ and $\lim_{t\to 0+}\frac{\varphi_Y(t)}{\varphi_X(t)} = 0$.*

Proof. The conditions $X \hookrightarrow Y$ and $\lim_{t\to 0+} \frac{\varphi_Y(t)}{\varphi_X(t)} = 0$ are necessary for an almost compact embedding between rearrangement-invariant Banach function spaces over a finite nonatomic measure space. Conversely, Theorems 6.4 and 6.5 show that these conditions are also sufficient. □

7. Embeddings into the subspace of functions of absolutely continuous norm

Theorem 7.1. *Let X be a rearrangement-invariant Banach function space over $((0,1),\lambda)$ and let φ be a positive nondecreasing concave function on $(0,1)$ such that $\lim_{t\to 0_+} \varphi(t) = 0$. Suppose that $\Lambda_\varphi \hookrightarrow X$. Then $\Lambda_\varphi \subseteq X_a$.*

Proof. Because $\Lambda_\varphi \hookrightarrow X$, we have

$$\|f\|_X \le C\|f\|_{\Lambda_\varphi}$$

for every $f \in \Lambda_\varphi$ and for a constant $C > 0$ independent of f. Thus,

$$\lim_{t\to 0_+} \|f^*\chi_{(0,t)}\|_X \le \lim_{t\to 0_+} C\|f^*\chi_{(0,t)}\|_{\Lambda_\varphi} = C \lim_{t\to 0_+} \int_0^t f^*(s)\varphi'(s)\, ds = 0,$$

so every function $f \in \Lambda_\varphi$ belongs to X_a. □

Corollary 7.2. *Suppose that φ is a positive nondecreasing concave function on $(0,1)$ satisfying $\lim_{t\to 0_+} \varphi(t) = 0$. Then*

$$\Lambda_\varphi \subseteq (M_\varphi)_a.$$

Proof. Just use the fact that $\Lambda_\varphi \hookrightarrow M_\varphi$ and the previous theorem. □

Remark 7.3. In the case that $\lim_{t\to 0_+} \varphi(t) > 0$, the inclusion $\Lambda_\varphi \subseteq (M_\varphi)_a$ fails because $(M_\varphi)_a = \{0\}$ and $\Lambda_\varphi \neq \{0\}$.

Theorem 7.4. *Let X be a rearrangement-invariant Banach function space over $((0,1),\lambda)$ and let φ be a positive nondecreasing concave function on $(0,1)$ such that $\lim_{t\to 0_+} \frac{t}{\varphi(t)} = 0$. Suppose that $M_\varphi \subseteq X_a$. Then $M_\varphi \overset{*}{\hookrightarrow} X$.*

Proof. Choose a concave function α satisfying $\frac{\alpha(t)}{2} \le \frac{t}{\varphi(t)} \le \alpha(t)$ for every $t \in (0,1)$. To prove that $M_\varphi \overset{*}{\hookrightarrow} X$ it is enough to show that $X' \overset{*}{\hookrightarrow} M'_\varphi = \Lambda_\alpha$, which is equivalent to

$$\lim_{t\to 0_+} \sup_{\|f\|_{X'} \le 1} \|f^*\chi_{(0,t)}\|_{\Lambda_\alpha} = 0.$$

In the proof of Theorem 6.5 it was shown that $\lim_{t\to 0_+} \alpha(t) = 0$ and $\alpha' \in M_\varphi$. The latter condition implies, together with the assumption $M_\varphi \subseteq X_a$, that $\alpha' \in X_a$ (in particular, $\alpha' \in X$). Thus, due to Hölder inequality, for $t \in (0,1)$

$$\|f^*\chi_{(0,t)}\|_{\Lambda_\alpha} = \int_0^1 f^*(s)\alpha'(s)\chi_{(0,t)}\, ds \le \|f\|_{X'}\|\alpha'\chi_{(0,t)}\|_X.$$

So, using that $\alpha' \in X_a$,

$$\lim_{t\to 0_+} \sup_{\|f\|_{X'} \le 1} \|f^*\chi_{(0,t)}\|_{\Lambda_\alpha} \le \lim_{t\to 0_+} \|\alpha'\chi_{(0,t)}\|_X = 0.$$

This completes the proof. □

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