Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

BAKALÁŘSKÁ PRÁCE

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Algebraická geometrie v souvislosti s počítačovým modelováním

Katedra didaktiky matematiky

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Děkuji RNDr. Zbyněku Šírovi, Ph.D. za hodnotné rady a odborné vedení během mé práce.

Prohlašuji, že jsem svou bakalářskou práci napsala samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

V Praze dne 28.5.2009 Eva Černohorská

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Abstrakt: Cílem této práce je zkoumat hermitovskou interpolaci polynomiálními kubikami s pythagorejským hodografem (tzv. PH kubiky). Všechny polynomiální PH kubiky jsou podobné části Tschirnhausenovy kubiky. Na základě parametrizace normálou a podrobnější analýzy této kubiky pak rozhoduji o řešitelnosti a počtu řešení interpolační úlohy pro různá vstupní data. Uvádím jak explicitní podmínky pro řešitelnost, tak explicitní formuli interpolační křivky. Dále využívám těchto teoretických znalostí a odvozuji algoritmus interpolace a demonstruji ho na různých příkladech.

Klíčová slova: Tschirnhausenova kubika, hermitovská interpolace, PH křivky

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Abstract: In this work we study the Hermite interpolation with polynomial Pythagorean-Hodograph (PH) cubic. Every polynomial PH cubic is similar to some segment of the Tschirnhausen cubic. A necessary condition of solvability of the Hermite interpolation problem is given, the number of solution is given, too. The solution of this problem is based on parametrizing the Tschirnhausen cubic using support function and on its analysis. Using this theory we construct an algorithm for interpolating with PH cubic. We apply this algorithm to various data.

Keywords: Tschirnhausen cubic, Hermite interpolation, PH curves

Chapter 1

Introduction

Polynomial Pythagorean-Hodograph (PH) curves were introduced by Farouki in 1990 [2]. There are not as many polynomial PH curves as polynomial curves but they have some properties which are very useful in practical use in geometry. For example the arc length of these curves can be expressed as a polynomial function the of parameter, and their offsets are rational curves. From these properties follow others interesting properties like that curvature can be expressed as rational function of parameter.

The most simple non trivial polynomial PH curves are polynomial PH cubics. They are also simple to describe, they have only one shape freedom, they are similar to a segment of the Tschirnhausen cubic as proved in [2].

The Hermite interpolation is a problem with practical use. The solution of this problem allows to construct spline or approximate smooth curve. Problem of Hermite interpolation with Tschirnhausen cubic was introduced in 1997 by Meek in [4]. In this paper he answers questions what Hermite data could be interpolated by Tschirnhausen cubic. He searches for the control polygon of Tschirnhausen cubic using following necessary and sufficient condition for polygon of PH curve from [2].

Theorem. *For a plane cubic $r(t)$ with Bezier control points $p_i, i = 0, 1, 2, 3$ let L_1, L_2, L_3 be lengths of the control polygon legs, and let θ_1, θ_2 be the control polygon angles at the interior vertices p_1, p_2. Then the conditions $L_2 = \sqrt{L_1 L_3}$ and $\theta_1 = \theta_2$ are sufficient and necessary to ensure that $r(t)$ has a Pythagorean hodograph.*

This solution is very technical. We give more elegant solution which is based on taking all possible pairs of points on Tschirnhausen cubic and searching what data could be obtained in this way.

First we parametrize Tschirnhausen cubic by its normal using the support function. Than we fix the angle between tangent vectors and we rotate Tschirnhausen cubic what is easy because we have the parametrization by normal. In this way we obtain a different solution of this problem than Meek in [4]. This work yields interpolants for certain data not included in [4], on the other hand we show some data mentioned in [4] can not be interpolated.

This thesis is organized as follows. In section 2 we recall some basic claims about PH curves and support function. We give support function of Tschirnhausen cubic. Third section show the structure of the Tschirnhausen cubic in the context of Hermite interpolation problem. We deduce the necessary and sufficient condition of solvability of Hermite interpolation problem and number of solutions. The remainder of section 3 is devoted to algorithm of solving this problem and examples. Finally, we conclude this thesis.

Chapter 2

Preliminaries

2.1 Pythagorean-Hodograph curves

Definition. A polynomial planar curve $C = [x(t), y(t)]$ is called *Pythagorean-Hodograph (PH) curve* if it satisfies

$$x'^2(t) + y'^2(t) = \sigma^2(t) \tag{2.1}$$

for some polynomial $\sigma(t)$.

We can construct polynomial PH curves using following lemma (proved by Kubota in [3]).

Lemma 1. *The condition (2.1) is satisfied if and only if there exist polynomials $u(t), v(t), h(t)$ such that*

$$\begin{aligned} x'(t) &= 2u(t)v(t)h(t) \text{ and} \\ y'(t) &= [u^2(t) - v^2(t)]h(t). \end{aligned} \tag{2.2}$$

Example. (**Tschirnhausen cubic**) Let $h(t) = 1, u(t) = t, v(t) = -1$ are polynomials from the previous lemma. We obtain $x'(t) = -2t, y'(t) = t^2 - 1$, by integration we get PH curve $T = \left[-t^2, \frac{t^3}{3} - t\right]$.

This cubic is very special PH cubic as show lemma. The proof of this lemma is in [2].

Lemma 2. *Any PH cubic can be obtained as scaled, rotated, shifted and reparametrized Tschirnhausen cubic.*

In this work we will consider (2.2) as a default position of Tschirnhausen cubic.

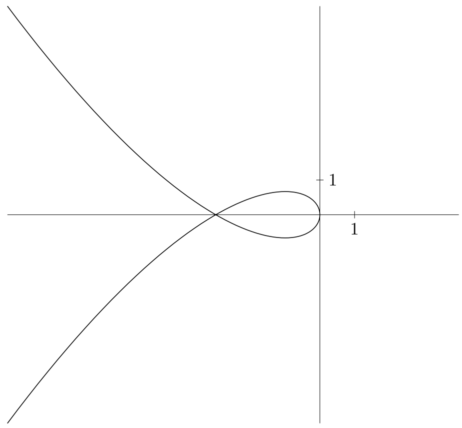

Figure 2.1: Tschirnhausen cubic with singularity in [-3,0]

2.2 Support function

Definition. *Gauss map* G_C of an oriented planar smooth curve C is the mapping $G_C : C \to S^1$ which assigns to a point the unit normal of this point, i. e. $G_C : \mathbf{p} \to \vec{\mathbf{n}}_{\mathbf{p}}$.

Example. Image of the Gauss map G_T of the Tschirnhausen cubic is $S^1 \backslash \{[-1, 0]\}$.

Definition. Let C be a smooth curve with (locally) injective Gauss map G_C. Define the *support function* $h : S \supset \mathrm{Im}(G_C) \to \mathbb{R}$ as the function, which to a normal $\vec{\mathbf{n}}$ associates the distance to the origin $[0, 0]$ of the tangent line to C at the point $G_C^{-1}(\vec{\mathbf{n}})$.

In practice we will parametrize S_1 as $[\cos\theta, \sin\theta]$ and identify a normal with the corresponding θ.

The applicability of support function we can see in this lemma. It is proved in [5].

Lemma 3. *Let h be a support function of C, then*

$$Y(\theta) : \begin{cases} x &= h(\theta)\cos\theta - h'(\theta)\sin\theta \\ y &= h(\theta)\sin\theta + h'(\theta)\cos\theta \end{cases} \tag{2.3}$$

is a parametrization of C.

Moreover $Y(\theta)$ is the inverse map to G_C, i.e. unit normal vector at $Y(\theta)$ is $[\cos\theta, \sin\theta]$.

Lemma 4. *The support function of Tschirnhausen cubic is*

$$h(\theta) = \frac{\mathrm{tg}^2\frac{\theta}{2}(3 - \mathrm{tg}^2\frac{\theta}{2})}{3(1 + \mathrm{tg}^2\frac{\theta}{2})}. \tag{2.4}$$

Proof. Let T be Tschirnhausen cubic, $O = [0, 0]$ be the origin, $\mathbf{p} \in T$ and $\vec{\mathbf{n}}_{\mathbf{p}}$ associated unit normal. Let θ be the angle between the normal vector and the positive half of the x-axis and $\mathbf{t}_{\mathbf{p}}$ be the tangent line of T in $\mathbf{p}$. Let N be the point of intersection of $\mathbf{t}_{\mathbf{p}}$ and line in direction $\vec{\mathbf{n}}_{\mathbf{p}}$ and passing trough O, see fig. 2.2.

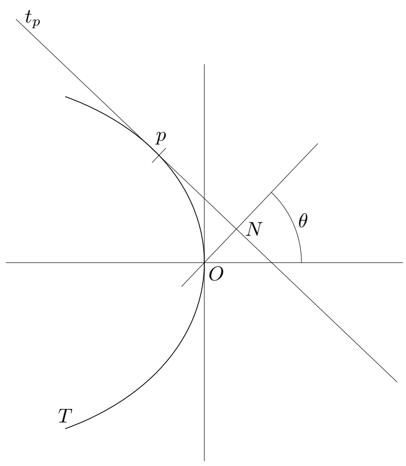

Figure 2.2: Tschirnhausen cubic: searching for its support function

Now, we are looking for the distance $|ON|$. The coordinates of N are the solution of the equation $\mathbf{p} + \alpha\vec{\mathbf{t}}_{\mathbf{p}} = O + \beta\vec{\mathbf{n}}_{\mathbf{p}}$, which we can use for finding the distance $|ON|$.

$$N = \left(\frac{t^2(1-t^2)(3+t^2)}{3(1+t^2)^2}, \frac{2t^3(3-t^2)}{3(1+t^2)^2} \right) \tag{2.5}$$

$$|ON| = \sqrt{\frac{2t^4(3+t^2)^2}{(1+t^2)^2}} \tag{2.6}$$

But the support function is the function of θ, the angle between normal vector $\vec{\mathbf{n}}_{\mathbf{p}}$ and positive x-axis $\vec{e_x}$. Because $\vec{\mathbf{n}}_{\mathbf{p}} = T' = (1-t^2, 2t)$ and $\vec{e_x} = (1,0)$, using formula we obtain $\cos\theta = \frac{1-t^2}{1+t^2}$, which is equivalent with $t = \frac{1-\cos\theta}{1+\cos\theta} = |\operatorname{tg}\frac{\theta}{2}|$. By substitution in (2.6), we verify (2.4). □

Corollary 1. *The parametrization of Tschirnhausen cubicby its normal is*

$$T = \left[-\frac{6-6\cos 2\theta}{12(1+\cos\theta)^2}, \frac{8\sin\theta + 8\sin 2\theta}{12(1+\cos\theta)^2} \right].$$

Chapter 3

Hermite interpolation

3.1 Statement of the problem

We have the situation as in the figure 3.1, i.e. we are given two points and unit tangent vectors and we would like to find a curve (of a given type) which interpolates these data.

We want to fully solve this problem for PH cubics, so we want find the necessary and sufficient condition of existence of interpolants and if given data could be interpolated we want to find the number of different solutions.

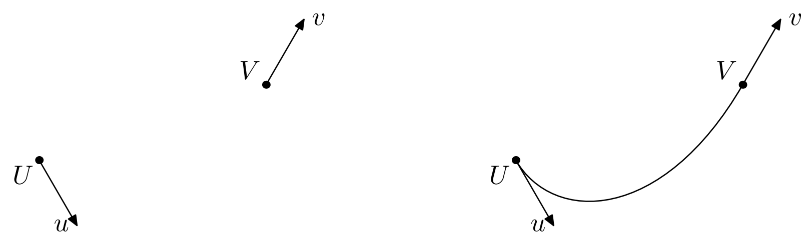

Figure 3.1: Given situation and a solution

The given data are (up to similarity) fully described by pair of angles (β, ω), where β is the angle between given tangent vectors and ω is the angle between bisector of β and the distance vector of the two given points, see figure 3.2.

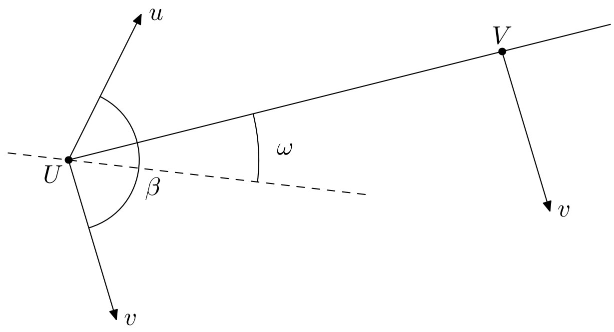

Figure 3.2: Notation

3.2 Analysing Tschirnhausen cubic

Let T be the Tschirnhausen cubic, which is (up to scaling, rotation and translation) the only PH cubic. For this reason it is sufficient to analyse its segments and in this way to get a full understanding of the solutions of the Hermite interpolation problem. We will consider all pairs of points $U, V \in T$ and see which data β, ω are interpolated by the segment U, V. In particular we will consider the inverse problem and decide how many different pairs U, V produce given β, ω.

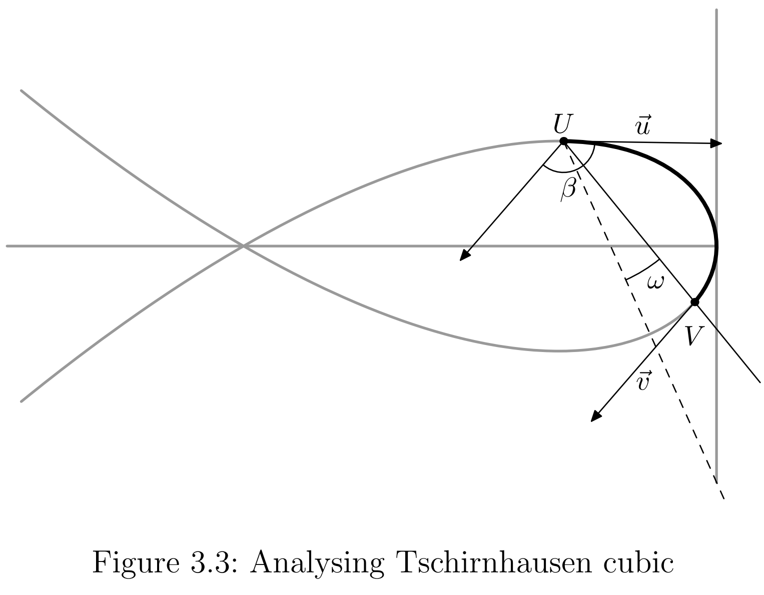

Figure 3.3: Analysing Tschirnhausen cubic

Theorem 1. *Let two angles $\beta \in (-\pi, \pi)$ and $\omega \in (-\pi, \pi)$ be given. Then the number of different pairs $U, V \in T$ producing hermite data described by β, ω is following*

a) There exist two different pairs for

$$\begin{array}{rcl} \beta & \in & \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right) \text{ and} \\ \omega & \in & \left(-\operatorname{arctg}\left(\frac{2\sin\frac{\beta}{2}}{\sqrt{1+2\cos\beta}}\right), \operatorname{arctg}\left(\frac{2\sin\frac{\beta}{2}}{\sqrt{1+2\cos\beta}}\right)\right) \end{array} \tag{3.1}$$

b) There exists one pair for

$$\begin{array}{rcl} \beta & \in & \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right) \text{ and} \\ \omega & = & \pm\operatorname{arctg}\left(\frac{2\sin\frac{\beta}{2}}{\sqrt{1+2\cos\beta}}\right) \end{array} \tag{3.2}$$

c) There exist two pairs for all ω and for

$$\beta \in \left(-\pi, -\frac{2\pi}{3}\right) \cup \left(\frac{2\pi}{3}, \pi\right) \tag{3.3}$$

d) If $\beta = \pm\frac{2\pi}{3}$ then there exists one pair for all ω.

Proof. We know from the corollary 1 that Tschirnhausen cubic T could be parametrized

$$T = T_0 = \left[-\frac{6 - 6\cos 2\theta}{12(1+\cos\theta)^2}, \frac{8\sin\theta + 8\sin 2\theta}{12(1+\cos\theta)^2}\right]. \tag{3.4}$$

Let T_α be Tschirnhausen cubic rotated through α around the origin. If we want to rotate Tschirnhausen cubic through α around the origin, we just substitute $\theta' = \theta + \alpha$ to the formula (3.4) and we get

$$\begin{aligned} T_\alpha \;=\; & \left[\frac{-6\cos\alpha + 4\cos\theta - 4\cos(2\alpha+\theta) + 7\cos(\alpha+2\theta) - \cos(3\alpha+2\theta)}{12(1+\cos(\alpha+\theta))^2},\right. \\ & \left.\frac{6\sin\alpha + 4\sin\theta + 4\sin(2\alpha+\theta) + 7\sin(\alpha+2\theta) + \sin(3\alpha+2\theta)}{12(1+\cos(\alpha+\theta))^2}\right] \end{aligned}$$

We define $U = T_\alpha(\frac{\beta}{2})$ and $V = T_\alpha(-\frac{\beta}{2})$. It is obvious that β is the angle between tangent vectors and that by changing α we obtain all such pairs

U, V. It is easy to see that bisector of the angle of normal vectors is x axis and that bisector of the angle of tangent vectors is y axis.

In this situation, we need to know how the vector $\overrightarrow{UV}$ depends on α, more precisely, what is its angle with the axis y. After short computation we get

$$\overrightarrow{UV} = (r_1, r_2) = \left(-\frac{8 \sin\alpha \sin^3 \frac{\beta}{2}}{3(\cos\alpha + \cos\frac{\beta}{2})^3}, \frac{4(3\cos\frac{\beta}{2} + (2+\cos\beta)\cos\alpha)\sin\frac{\beta}{2}}{3(\cos\alpha + \cos\frac{\beta}{2})^3} \right)$$

Its angle from y-axis must be ω, so we get the condition

$$\operatorname{tg}\omega = \frac{r_1}{r_2} = -\frac{2\sin\alpha \sin^2\frac{\beta}{2}}{3\cos\frac{\beta}{2} + (2+\cos\beta)\cos\alpha} =: F(\alpha, \beta) \qquad (3.5)$$

Our problem is now reduced to the question how many α exist for given (β, ω) so that (3.5) is satisfied. For this purpose, we will analyse function F.

It is easy to check that the function $F(\alpha, \beta)$ is odd in α and even in β. First we check if function $F(\alpha, \beta)$ is defined on the whole interval, i.e. in this situation we determine if the denominator could be zero and find that it is zero if and only if:

$$\alpha = \pm\alpha_n = \pm \arccos\left(-\tfrac{3\cos\frac{\beta}{2}}{2+\cos\beta}\right) \qquad (3.6)$$

α_n exist if and only if $-\frac{3\cos\frac{\beta}{2}}{2+\cos\beta} \in [-1, 1]$, which is equivalent to $\beta \in (-\pi, -\frac{2\pi}{3}]$ or $\beta \in [\frac{2\pi}{3}, \pi)$. So for all $\beta \in (-\pi, -\frac{2\pi}{3}] \cup [\frac{2\pi}{3}, \pi)$ there exists some α_n for which $F(\alpha, \beta)$ does not exist. Because function is odd, it is not defined in $-\alpha_n$ either. In the rest (i. e. $\beta \in \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right)$) is function $F(\alpha, \beta)$ defined on whole interval $[-\pi, \pi]$. We will study 3 cases.

CASE 1: $\beta \in (-\frac{2\pi}{3}, \frac{2\pi}{3})$

For example $F(\frac{\pi}{2}, \alpha)$ is in figure 3.4. For others $\beta \in (-\frac{2\pi}{3}, \frac{2\pi}{3})$ is the function quite similar.

Let β be fixed. We know that $F_\beta(\alpha)$ is defined on the whole interval $[-\pi, \pi]$. Now we are looking for range of $F_\beta(\alpha)$. We can see that the function is

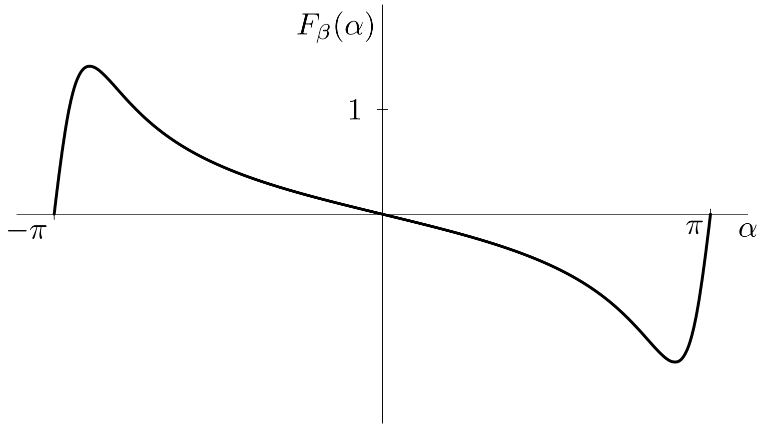

Figure 3.4: $F(\alpha, \beta)$ for $\beta \in (-\frac{2\pi}{3}, \frac{2\pi}{3})$, concretely $\beta = \frac{\pi}{2}$

continuous on whole interval $[-\pi, \pi]$. Let us try to find extremes, so first derivative will be zero.

$$\frac{\partial F(\alpha, \beta)}{\partial \alpha} = \frac{2(2 + 3\cos\alpha\cos\frac{\beta}{2} + \cos\beta)\sin^2\frac{\beta}{2}}{(3\cos\frac{\beta}{2} + (2 + \cos\beta)\cos\alpha)^2} = 0 \tag{3.7}$$

The solution of this equation is following

$$\alpha = \pm\alpha_e = \pm \arccos\left(-\tfrac{1}{3\cos\frac{\beta}{2}}(2 + \cos\beta)\right) \tag{3.8}$$

The first derivative is positive in intervals $[-\pi, -\alpha_e), (-\alpha_e, -\pi]$ and negative in interval $(-\alpha_e, \alpha_e)$. So function $F_\beta(\alpha)$ is strictly increasing in intervals $[-\pi, -\alpha_e), (-\alpha_e, -\pi]$ and strictly decreasing in interval $(-\alpha_e, \alpha_e)$. So in α_e is local minimum and in $-\alpha_e$ is local maximum, but $F(\beta, -\pi) = F(\beta, \pi) = 0$ so local extremes are global on $[-\pi, \pi]$. Coordinates of global maximum and global minimum on $[-\pi, \pi]$ are $\left[-\alpha_e, \frac{2|\sin\frac{\beta}{2}|}{\sqrt{1+2\cos\beta}}\right]$, $\left[-\alpha_e, \frac{2|\sin\frac{\beta}{2}|}{\sqrt{1+2\cos\beta}}\right]$ respectively.

The continuity of F implicate that its range is:

$$\operatorname{tg}\omega = F(\alpha, \beta) \in \left[-\tfrac{2\sin\frac{\beta}{2}}{\sqrt{1+2\cos\beta}}, \tfrac{2\sin\frac{\beta}{2}}{\sqrt{1+2\cos\beta}}\right] \tag{3.9}$$

$$\text{hence } \omega \in \left[-\operatorname{arctg}\left(\frac{2\sin\frac{\beta}{2}}{\sqrt{1+2\cos\beta}}\right), \operatorname{arctg}\left(\frac{2\sin\frac{\beta}{2}}{\sqrt{1+2\cos\beta}}\right)\right] \tag{3.10}$$

From the previous reasoning about monotonicity it is clear that for $\omega \in \left(-\operatorname{arctg}\left(\frac{2\sin\frac{\beta}{2}}{\sqrt{1+2\cos\beta}}\right), \operatorname{arctg}\left(\frac{2\sin\frac{\beta}{2}}{\sqrt{1+2\cos\beta}}\right)\right)$ there exist two solutions and for $\omega = \pm\operatorname{arctg}\left(\frac{2\sin\frac{\beta}{2}}{\sqrt{1+2\cos\beta}}\right)$ there exists one solution as claimed in a) and b).

CASE 2 $\beta \in (-\pi, -\frac{2\pi}{3}) \cup (\frac{2\pi}{3}, \pi)$

Function $F_\beta(\alpha)$ is similar to one shown in figure 3.5.

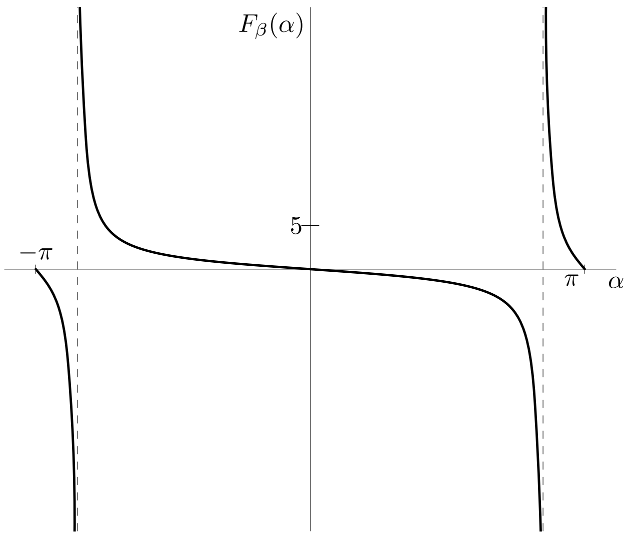

Figure 3.5: $F(\alpha, \beta)$ for $\beta \in (-\pi, -\frac{2\pi}{3}) \cup (\frac{2\pi}{3}, \pi)$, concretely $\beta = \frac{3\pi}{4}$

We fix $\beta \in (-\pi, -\frac{2\pi}{3}) \cup (\frac{2\pi}{3}, \pi)$. We have two points where $F_\beta(\alpha)$ is not continuous: $\pm\alpha_n$. We can compute limits in these points

$$\begin{aligned}
\lim_{\alpha\to\alpha_n^-} -\frac{2\sin\alpha\sin^2\frac{\beta}{2}}{3\cos\frac{\beta}{2} + (2+\cos\beta)\cos\alpha} &= -\infty \\
\lim_{\alpha\to\alpha_n^+} -\frac{2\sin\alpha\sin^2\frac{\beta}{2}}{3\cos\frac{\beta}{2} + (2+\cos\beta)\cos\alpha} &= +\infty
\end{aligned} \tag{3.11}$$

and the limits in $-\alpha_n$ are clear from oddity of F. Because function $F_\beta(\alpha)$ is continuous on $(-\alpha_n, \alpha_n)$, solution exists on whole $[-\pi, \pi]$.

From the first derivative (3.7) we have that $F_\beta(\alpha)$ is strictly decreasing in $[-\pi, -\alpha_n)$, $(-\alpha_n, \alpha_n)$ and $(\alpha_n, \pi]$. So there exist two solutions for all $\omega \in [-\pi, \pi]$.

CASE 3 $\beta = \pm\frac{2\pi}{3}$

In the case $\beta = \pm\frac{2\pi}{3}$ the α_n become π and we have only one solutions for all $\omega \in [-\pi, \pi]$. The function $F_\beta(\alpha)$ is in figure 3.6. ☐

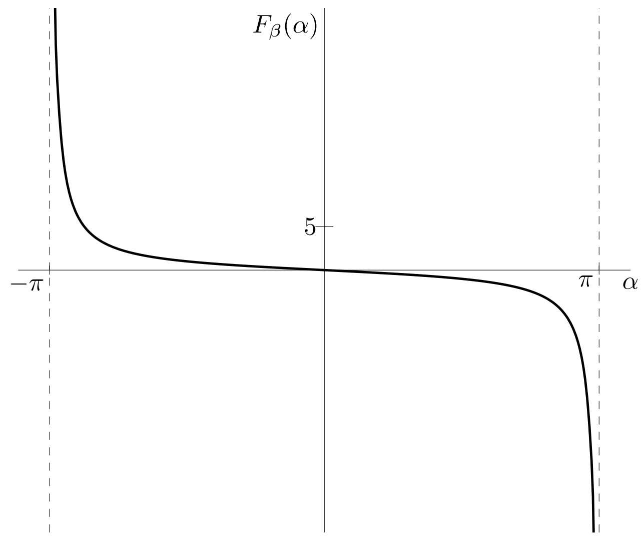

Figure 3.6: $F(\alpha, \beta)$ for $\beta = \pm\frac{2\pi}{3}$

3.3 Interpolation

In this section we construct the Hermite interpolants. We have two given angles ω and β and we are looking for Tschirnhausen cubic. It is sufficient to find suitable angle of rotation α. So we solve the equation:

$$\operatorname{tg}\omega = -\frac{2\sin\alpha\sin^2\frac{\beta}{2}}{3\cos\frac{\beta}{2} + (2+\cos\beta)\cos\alpha} \quad \text{where } \omega, \beta \text{ are fixed.} \tag{3.12}$$

The number of solutions follow from theorem 1. In order to solve (3.12) we make the substitution $k = \operatorname{tg}\omega$, $b = \cos\beta$ and $x = \cos\alpha$, which after squaring and some computation becomes

$$9k^2 - 9k^2b - 2 - 2b^2 - 4b + x(6\sqrt{2(1-b)}k^2b + 12k^2x\sqrt{2(1-b)}) +$$
$$+ x^2(8k^2 + 2b^2k^2 + 8bk^2 + 2 + 2b^2 + 4b) = 0$$

We solve this quadratic equation and we obtain two roots for x. We have four solutions and by testing we find out that two are correct, see below. For the seek of brevity let's define

$$M = \sqrt{-\operatorname{tg}^2\omega(-2 + \operatorname{tg}^2\omega + 2(1 + \operatorname{tg}^2\omega)\cos\beta)(3\sin\frac{\beta}{2} + \sin\frac{3\beta}{2})^2}$$

In the case a) of the theorem 1 we have these solutions

$$\alpha = -\arccos\left(-\frac{\sin^2\frac{\beta}{2}M - 3\operatorname{tg}^3\omega\cos\frac{\beta}{2}(2+\cos\beta)^2}{\operatorname{tg}^3\omega(2+\cos\beta)^3 + 4\operatorname{tg}\omega(2+\cos\beta)\sin^4\frac{\beta}{2}}\right) \tag{3.13}$$

$$\alpha = -\arccos\left(\frac{(\cos\beta - 1)M - 6\operatorname{tg}^3\omega\cos\frac{\beta}{2}(2+\cos\beta)^2}{-2\operatorname{tg}^3\omega(2+\cos\beta)^3 - 8\operatorname{tg}\omega(2+\cos\beta)\sin^4\frac{\beta}{2}}\right) \tag{3.14}$$

In the case b) of the theorem 1 the solutions (3.13) and (3.14) become equal, so we have only one solution.
In the case c) of the theorem 1 we have following solutions:

$$\alpha = \arccos\left(-\frac{\sin^2\frac{\beta}{2}M - 3\operatorname{tg}^3\omega\cos\frac{\beta}{2}(2+\cos\beta)^2}{\operatorname{tg}^3\omega(2+\cos\beta)^3 + 4\operatorname{tg}\omega(2+\cos\beta)\sin^4\frac{\beta}{2}}\right) \tag{3.15}$$

$$\alpha = -\arccos\left(\frac{(\cos\beta - 1)M - 6\operatorname{tg}^3\omega\cos\frac{\beta}{2}(2+\cos\beta)^2}{-2\operatorname{tg}^3\omega(2+\cos\beta)^3 - 8\operatorname{tg}\omega(2+\cos\beta)\sin^4\frac{\beta}{2}}\right) \tag{3.16}$$

In the case d) of the theorem 1 we have one solution

$$\alpha = 2\omega \tag{3.17}$$

3.4 Algorithm

1. given data: points U, V and vectors from these points $\vec{u}, \vec{v}$
2. compute angles β, ω as in the figure 3.2
3. compute α:
   - if $\beta \in \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right)$ and $\omega \in \left(\operatorname{arctg}\left(-\frac{2\sin\frac{\beta}{2}}{\sqrt{1+2\cos\beta}}\right), \operatorname{arctg}\left(\frac{2\sin\frac{\beta}{2}}{\sqrt{1+2\cos\beta}}\right)\right)$ exist 2 solutions described by α_1 see (3.13) and α_2 see (3.14)
   - if $\beta \in \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right)$ and $\omega = \pm\operatorname{arctg}\left(-\frac{2\sin\frac{\beta}{2}}{\sqrt{1+2\cos\beta}}\right)$ exists one solution described by α_1 see (3.13)
   - if $\beta \in \left(-\pi, -\frac{2\pi}{3}\right) \cup \left(\frac{2\pi}{3}, \pi\right)$ exist 2 solutions described by α_1 see (3.15) and α_2 see (3.16)
   - if $\beta = \pm\frac{2\pi}{3}$ exists one solution described by α_1 see (3.17)
   - in the others cases no solution exists
4. for each α_i compute $\theta_i < \theta_i'$:

$$\{\theta_i, \theta_i'\} = \begin{cases} \left\{-\frac{\beta}{2} + \alpha_i, \frac{\beta}{2} + \alpha_i\right\} & \text{if } |\alpha_i| + |\frac{\beta}{2}| \leq \pi \\ \left\{-2\pi + \frac{\beta}{2} + \alpha, -\frac{\beta}{2} + \alpha\right\} & \text{otherwise} \end{cases} \tag{3.18}$$

5. for each α_i compute t_i and t_i' corresponding to θ_i and θ_i' respectively: $t = \frac{-1\pm|\cos\theta|}{\sin\theta}$ so as $[-t^2, \frac{t^3}{3} - t] = T_\alpha(\theta)$
6. compute Bezier control polygon:

$$\begin{aligned} P_0 &= [-t_i^2, \frac{t_i^3}{3} - t_i] \\ P_1 &= [\frac{1}{3}(-2t_it_i' - t_i'^2), \frac{1}{3}(-t_i - 2t_i' + t_it_i'^2)] \\ P_2 &= [\frac{1}{3}(-2t_it_i' - t_i^2), \frac{1}{3}(-t_i' - 2t_i + t_i^2t_i')] \\ P_3 &= [-t_i'^2, \frac{t_i'^3}{3} - t_i'] \end{aligned} \tag{3.19}$$

7. shift, rotate and scale the control polygon so as the $\tilde{P}_0 = U$ and $\tilde{P}_3 = V$, general transformation is of the form:

$$\tilde{B} = \frac{|UV|}{|P_0P_3|}\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} B + U \tag{3.20}$$

where ϕ is the angle between UV and P_3P_0

8. determine the solution as Bezier curve given by transformed control polygon

Remark. (explanation of some steps of algorithm)

ad 3 Follow from the theorem 1.

ad 4 The first case is not possible to consider when this interval contain angle $\pm\pi$, which has the limit tangent, so in this case, we need to consider the complement to $[-\pi, \pi]$ which is equivalent to second second case in 4.

ad 5 The PH cubic is the segment (which should be rotated, shifted and scaled) of Tschirnhausen cubic, which correspond to $\theta \in [\theta_i, \theta_i']$ in the parametrization (3.4). Using formulae $\sin\theta = \frac{2t}{t^2+1}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$ we can find corresponding t_i and t_i' in the parametrization $\left[-t^2, \frac{t^3}{3} - t\right]$ mentioned in 5.

ad 6 We reparametrize cubic again substituting $t = \bar{t}(t_i' - t_i) + t_i$. Now $\bar{t} \in [0, 1]$ and we can express Tschirnhausen cubic in Bernstein-Bezier base and coefficients are control points.

3.5 Examples

In this section we will demonstrate algorithm from the previous section on three representatives examples.

Example 1.

1. given data: points $U = [-2, 1]$ and $V = [1, 2]$ and tangent vectors at these points $\vec{u} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \vec{v} = (1, 0)$
2. compute angles $\beta = \frac{3\pi}{4}, \omega \doteq 0.7144$
3. $\beta \in [-\pi, \frac{2\pi}{3}]$ so the interpolation problem has 2 solutions described by $\alpha_1 \doteq 3.0696$, $\alpha_2 \doteq -1.0904$

4. because of $|\alpha_1| + \left|\frac{\beta}{2}\right| \doteq 4.2477 > \pi$ we have $\theta_1 \doteq 1.8915$ and $\theta_1' \doteq -2.0354$ and similarly $|\alpha_2|+\left|\frac{\beta}{2}\right| \doteq 2.2685 < \pi$ we have $\theta_2 \doteq 0.0877$ and $\theta_2' \doteq -2.2685$

5. compute corresponding t: $t_1 \doteq 1.6199, t_1' \doteq -1.3860, t_2 \doteq 2.1433$, $t_2' \doteq -0.0439$

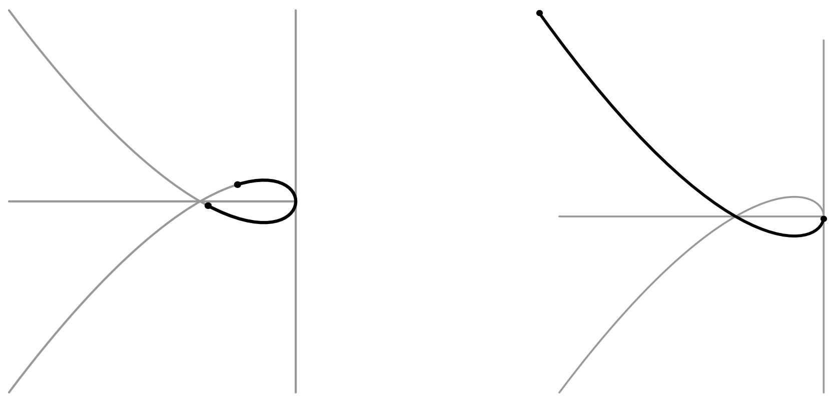

Figure 3.7: interpolants in example 1 are segments of Tschirnhausen cubic

6. find the Bezier control polygon for the first and the second solution:

$$\begin{array}{rclrcl} P_0 &=& [-2.6239. - 0.2031] & P_0' &=& [-4.5938, 1.1387] \\ P_1 &=& [0.8564, 1.4212] & P_1' &=& [0.0620, -0.6838] \\ P_2 &=& [0.6221, -1.8301] & P_2' &=& [-1.4686, -1.4814] \\ P_3 &=& [-1.9209, 0.4985] & P_3' &=& [-0.0019, 0.0438] \end{array} \quad (3.21)$$

7. shift, rotate and scale the control polygon, angles between UV and P_3P_0 and $P_3'P_0'$ are $\phi_1 \doteq -0.4626$ and $\phi_2 \doteq 0.5558$. Transformed polygons are for the first and the second solution:

$$\begin{array}{rclrcl} \tilde{P}_0 &=& [-2, 1] & \tilde{P}_0' &=& [-2, 1] \\ \tilde{P}_1 &=& [10.2241, 0.6823] & \tilde{P}_1' &=& [1.2936, 1.6085] \\ \tilde{P}_2 &=& [4.9362, -8.2483] & \tilde{P}_2' &=& [0.7045, 0.6136] \\ \tilde{P}_3 &=& [1, 2] & \tilde{P}_3' &=& [1, 2] \end{array} \quad (3.22)$$

8. The Bezier curve determined by transformed polygons is interpolation cubic:

$$
\begin{aligned}
C_1 = \; & [-2 + 22.2987s - 11.8196s^2 - 7.4791s^3, \\
& 1 - 22.2987s + 47.5974s^2 - 24.2987s^3] \\
C_2 = \; & [1 - 8.5646s + 7.5584s^2 - 1.9938s^3, \\
& 2s - 2.4292s^2 + 1.4292s^3]
\end{aligned}
\tag{3.23}
$$

9. These cubic are in the figure:

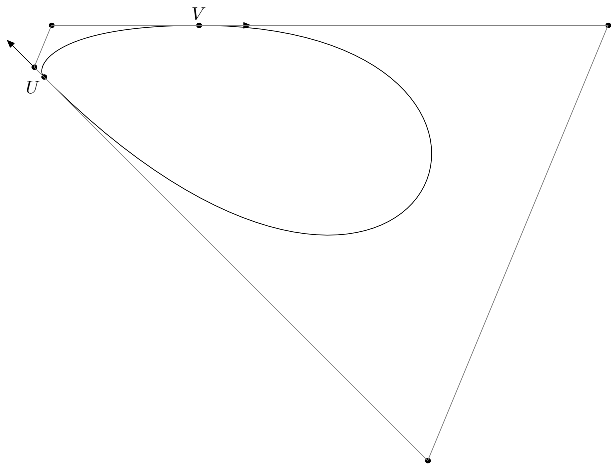

Figure 3.8: solution of example 1

Example 2.

1. given data: points $U = [-2, 1]$ and $V = [1, 2]$ and tangent vectors at these points $\vec{u} = (1, 0)\,, \vec{v} = (0, 1)$

2. compute angles $\beta = \frac{\pi}{2}, \omega \doteq -0.4636$

3. $\beta \in \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right)$ and $\omega \in (-0.9553, 0.9553)$ so the interpolation problem has 2 solutions described by $\alpha_1 \doteq -1.6335$, $\alpha_2 \doteq -3.0789$

4. because of $|\alpha_1| + \left|\frac{\beta}{2}\right| \doteq 2.4189 < \pi$ we have $\theta_1 \doteq -2.4189$ and $\theta_1' \doteq -0.8481$ and similarly $|\alpha_2| + \left|\frac{\beta}{2}\right| \doteq 3.8643 > \pi$ we have $\theta_2 \doteq -3.8643$ and $\theta_2' \doteq -8.5767$

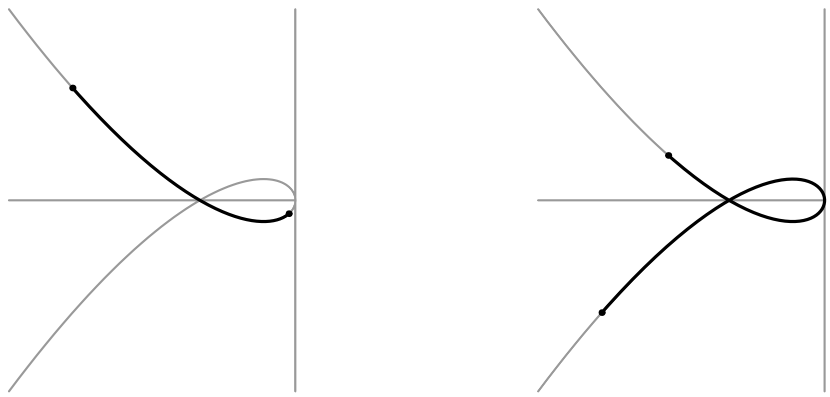

Figure 3.9: interpolants in example 2 are segments of Tschirnhausen cubic

5. compute corresponding t: $t_1 \doteq 2.6458, t'_1 \doteq 0.4514, t_2 \doteq -2.6456$, $t'_2 \doteq 2.2153$

6. find the Bezier control polygon for the first and the second solution:

$$\begin{array}{lll} P_0 = [-7, 3.5277] & P'_0 = [-4.9073, 1.4084] \\ P_1 = [-0.8641, -1.0032] & P'_1 = [1.574, 6.1943] \\ P_2 = [-3.1296, -0.8610] & P'_2 = [2.2716, -4.9228] \\ P_3 = [-0.2038, -0.4208] & P'_3 = [-7, -3.5277] \end{array} \quad (3.24)$$

7. shift, rotate and scale the control polygon, angles between UV and P_3P_0 and $P'_3P'_0$ are $\phi_1 \doteq 0.8481$ and $\phi_2 \doteq 2.2935$, transformed polygons are for the first and the second solution:

$$\begin{array}{ll} \tilde{P}_0 = [-2, 1] & \tilde{P}'_0 = [-2, 1] \\ \tilde{P}_1 = [1, 1.6456] & \tilde{P}'_1 = [5.6458, 1] \\ \tilde{P}_2 = [0.3542, 1] & \tilde{P}'_2 = [1, -3.6458] \\ \tilde{P}_3 = [1, 2] & \tilde{P}'_3 = [1, 2] \end{array} \quad (3.25)$$

8. The Bezier curve determined by transformed polygons is interpolation cubic:

$$\begin{array}{rl} C_1 = & [-2 + 7.0628s - 5.1255s^2 + 1.0628s^3, \\ & s + 1.9373s^2 - 0.9373s^3] \\ C_2 = & [s + 13.9373s^2 - 16.9373s^3, \\ & 2 - 16.9373s + 30.8745s^2 - 14.9373s^3] \end{array} \quad (3.26)$$

9. These cubic are in the figure 3.10

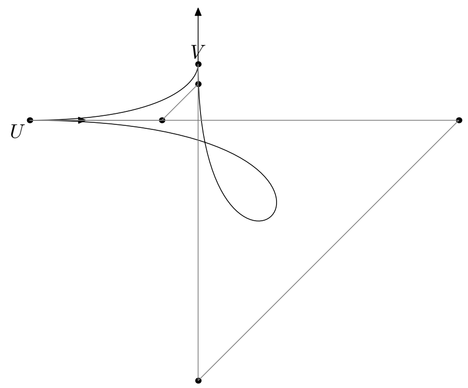

Figure 3.10: solution of example 2

Example 3.

1. given data: points $U = [-2, 1]$ and $V = [1, 2]$ and tangent vectors at these points $\vec{u} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{3}\right), \vec{v} = \left(\frac{1}{2}, \frac{\sqrt{3}}{3}\right)$

2. compute angles $\beta = \frac{2\pi}{3}, \omega \doteq 0.3218$

3. $\beta = \frac{2\pi}{3}$ so the interpolation problem has one solution described by $\alpha_1 \doteq 0.6435$

4. because of $|\alpha_1| + \left|\frac{\beta}{2}\right| \doteq 1.6907 < \pi$ we have $\theta_1 \doteq -0.4037$ and $\theta_1' \doteq 1.6907$

5. compute corresponding t: $t_1 \doteq -1.1277, t_1' \doteq 0.2046$

6. find the Bezier control polygon for the first and the second solution:

$$\begin{aligned} P_0 &= [-0.0419, -0.2018] \\ P_1 &= [-0.2701, 0.7703] \\ P_2 &= [0.1399, 0.2237] \\ P_3 &= [-1.2717, 0.6497] \end{aligned} \tag{3.27}$$

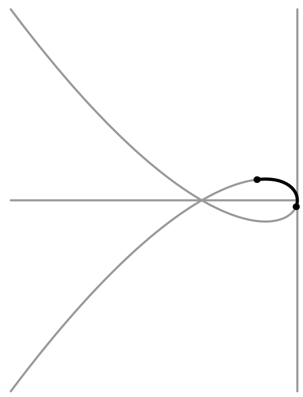

Figure 3.11: interpolant in example 3 is segment of Tschirnhausen cubic

7. shift, rotate and scale the control polygon, angle between UV and P_3P_0 is $\phi_1 \doteq 2.2143$, transformed polygon is:

$$\begin{array}{rcl} \tilde{P_0} & = & [-2, 1] \\ \tilde{P_1} & = & [-0.0665, 0.1528] \\ \tilde{P_2} & = & [-1.5109, 0.1528] \\ \tilde{P_3} & = & [1, 2] \end{array} \tag{3.28}$$

8. The Bezier curve determined by transformed polygons is interpolation cubic:

$$C_1 = [-2 + 1.4673s + 2.866s^2 - 1.3333s^3, 1 - 2.5415s + 2.5415s^2 + 1.s^3] \tag{3.29}$$

9. This cubic are in the figure:

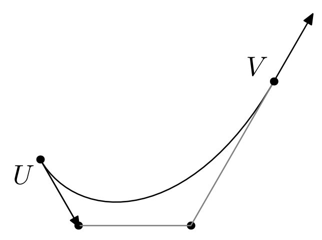

Figure 3.12: solution of example 3

Chapter 4

Conclusion

In this work we have fully analysed and solved the problem of the Hermite interpolation with PH cubics. We determined the sufficient and necessary condition of solvability and we also found number of different solution in cases when given data could be interpolated. We also give an explicit algorithm for the construction of the interpolants.

In the future, we intend to study interpolation with PH curves of higher degree and with other special curves.

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