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BAKALÁŘSKÁ PRÁCE

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Lorentzovy prostory

Katedra matematické analýzy

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Prohlašuji, že jsem svou bakalářskou práci napsala samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne 20.5.2009 Kristýna Kuncová

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Abstrakt: Moderní teorie reálné interpolace si vyžádala zavedení mnoha nových pojmů, které dnes mají svůj nezastupitelný význam. My se zaměříme zejména na Lorentzovy prostory, které vznikly jakožto zobecnění slabých Lebesgueových prostorů, a na prostory Marcinkiewiczovy, jež představují příklady Banachových prostorů funkcí. Naším cílem bude nejprve popsat vztahy mezi zobecněními normy a metriky a získané poznatky následně aplikovat na Lorentzovy prostory, které nejsou a obecně ani nemohou být opatřeny vhodnou normou. Využijeme vlastností nerostoucího přerovnání a na speciálních případech Lorentzových prostorů $L^{1,q}$ pro $q \in (1, \infty]$ zavedeme vhodnou α-normu, kvasinormu a jejich metrické ekvivalenty, co nejpodobnější původnímu funkcionálu. Poté popíšeme nutnou a postačující podmínku pro spojité vnoření Marcinkiewiczových prostorů a následně i podmínky pro skoro kompaktní vnoření mezi nimi.
Klíčová slova: Lorentzovy prostory, Marcinkiewiczovy prostory, norma

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Abstract: The modern theory of real interpolation has forced an introduction of many new notions with superordinary importance. We will concentrate on Lorentz spaces, which originated as a generalization of weak Lebesgue spaces, and on Marcinkiewicz spaces, named after the Polish mathematician J. Marcinkiewicz, which represent an example of the so-called Banach function spaces. Our goal is, at the beginning, to describe some relationship between certain generalizations of a norm and a metric, and, afterwards, to apply the knowledge obtained to Lorentz spaces, which, in general, are not, and neither can be, equipped with a suitable norm. At first, we will endow them with an α-norm. Using properties of the nonincreasing rearrangement of a function, we will equip the special cases of Lorentz spaces, more precisely the spaces $L^{1,q}$, where $q \in (1, \infty]$, with a suitable α-norm, a quasinorm and their metric equivalents, as similar to original functional as possible. Hereafter, we induct Marcinkiewicz spaces based on quasiconcave functions and describe a necessary and sufficient condition for continuous and almost-compact embeddings between Marcinkiewicz spaces.
Keywords: Lorentz spaces, Marcinkiewicz spaces, norm

Introduction

The thesis is divided into three parts.

In the first part we treat a norm in a normed linear space and its generalizations, namely, a quasinorm and an α-norm. We describe relationship between these functionals. More precisely, we will concentrate on an α-norm, its dependence on various values of the positive parameter α and its relationship to a quasinorm. The knowledge obtained thereby is then applied to a metric and, analogously to a quasimetric and an α-metric.

In the second part, we introduce the nonincreasing rearrangement of a function and Lorentz spaces. Next, we describe Lorentz spaces with a particular emphasis to their norms. Because the usual functional used in Lorentz spaces does not always satisfy all the norm axioms, we would like to find an α-norm, as similar to original functional as possible. We will use results presented in [1]. We examine separately the cases $L^{1,q}$ for $1 < q < \infty$ and $L^{1,\infty}$, in which case we also prove the optimality of our results. In both cases we are able to find an α-norm and, with the help of the results presented in the first part, we can equip them also by a quasinorm and a quasimetric.

In the final part, we introduce the notion of a quasiconcave function, which leads to Marcinkiewicz spaces, and we explore necessary and sufficient conditions for continuous and almost-compact embeddings between them, using basic properties of quasiconcave functions.

1 Norms and metrics

Our purpose will be to study the question, whether it is possible to equip Lorentz spaces with a suitable generalization of a norm and establish its optimality.

Definition 1.1. Let X be a vector space over the field $\mathbb{C}$ of complex numbers. A mapping $\|\cdot\| : X \longrightarrow [0,\infty)$ is called *norm* on X, if for all $a \in \mathbb{C}$ and $\boldsymbol{x}, \boldsymbol{y} \in$ X

$$\|\boldsymbol{x}\| = 0 \Leftrightarrow \boldsymbol{x} = 0, \tag{1}$$

$$\|a\boldsymbol{x}\| = |a|\|\boldsymbol{x}\|, \tag{2}$$

$$\|\boldsymbol{x}+\boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|. \tag{3}$$

Definition 1.2. Let X be a vector space over the field $\mathbb{C}$ of complex numbers. A mapping $\|\cdot\| : X \longrightarrow [0,\infty)$ is called *quasinorm* on X, if for all $a \in \mathbb{C}$ and $\boldsymbol{x}, \boldsymbol{y} \in$ X

$$\|\boldsymbol{x}\| = 0 \Leftrightarrow \boldsymbol{x} = 0,$$

$$\|a\boldsymbol{x}\| = |a|\|\boldsymbol{x}\|,$$

$$\|\boldsymbol{x}+\boldsymbol{y}\| \leq K(\|\boldsymbol{x}\| + \|\boldsymbol{y}\|) \quad \text{for some } K \geq 1.$$

Definition 1.3. Let X be a vector space over the field $\mathbb{C}$ of complex numbers. Let $\alpha \in (0,1]$, then a mapping $\|\cdot\| : X \longrightarrow [0,\infty)$ is called an *α-norm* on X for some $\alpha \in (0,1]$, if for all $a \in \mathbb{C}$ and $\boldsymbol{x}, \boldsymbol{y} \in$ X

$$\|\boldsymbol{x}\| = 0 \Leftrightarrow \boldsymbol{x} = 0, \tag{4}$$

$$\|a\boldsymbol{x}\| = |a|\|\boldsymbol{x}\|, \tag{5}$$

$$\|\boldsymbol{x}+\boldsymbol{y}\|^\alpha \leq \|\boldsymbol{x}\|^\alpha + \|\boldsymbol{y}\|^\alpha.$$

It is obvious that every norm is a quasinorm with constant $K = 1$ and also an α-norm with $\alpha = 1$.

Remark 1.4. In all text we will use known the trivial inequalities

$$(A+B)^\alpha \leq A^\alpha + B^\alpha \quad \alpha \leq 1;$$

$$(A+B)^\alpha \geq A^\alpha + B^\alpha \quad \alpha \geq 1;$$

for $A, B \geq 0$.

Now we will prove that if a mapping is an α_0-norm for some α_0, then it is also a β-norm for each $\beta < \alpha_0$.

Proposition 1.5. Let X be a vector space and the mapping $\|\cdot\|$ is an α_0-norm on X for some α_0, that is,

$$\|f+g\|^{\alpha_0} \leq \|f\|^{\alpha_0} + \|g\|^{\alpha_0} \tag{6}$$

for each $f, g \in X$. Let $\beta \in (0, \alpha_0)$. Then for each $f, g \in X$,

$$\|f+g\|^{\beta} \leq \|f\|^{\beta} + \|g\|^{\beta}.$$

Proof. From (6) we have:

$$\|f+g\|^{\beta} = (\|f+g\|^{\alpha_0})^{\beta/\alpha_0} \leq (\|f\|^{\alpha_0} + \|g\|^{\alpha_0})^{\beta/\alpha_0}.$$

Because $\beta/\alpha_0 < 1$, we have

$$(\|f\|^{\alpha_0} + \|g\|^{\alpha_0})^{\beta/\alpha_0} \leq (\|f\|^{\alpha_0})^{\beta/\alpha_0} + (\|g\|^{\alpha_0})^{\beta/\alpha_0} = \|f\|^{\beta} + \|g\|^{\beta}.$$

□

Remark 1.6. We say that an α_0-norm is optimal on a vector space X, if α_0 is the largest value, which satisfies all α-norm axioms, in other words, it is an α-norm if and only if $\alpha \in (0, \alpha_0]$.

Hereafter we will describe the relationship between quasinorms and α-norms and we will apply it to Lorentz spaces.

At the beginning we will prove that a Banach space equipped with an α-norm can be also endowed with a quasinorm.

Proposition 1.7. Let $0 < \alpha < 1$ and assume that for every f and g in a Banach space X one has

$$\|f+g\|^{\alpha} \leq \|f\|^{\alpha} + \|g\|^{\alpha}.$$

Then

$$\|f+g\| \leq 2^{\frac{1}{\alpha}-1}(\|f\| + \|g\|).$$

Proof. Denote $c := \|f+g\|$, $a := \|f\|$ and $b := \|g\|$. Then

$$c^{\alpha} \leq a^{\alpha} + b^{\alpha}$$

and

$$c \leq (a^{\alpha} + b^{\alpha})^{\frac{1}{\alpha}},$$

which we require to be less than or equal to

$$2^{\frac{1}{\alpha}-1}(a+b).$$

If $a = 0$ or $b = 0$, then the inequality holds. So, we find the maximum of

$$\left(\frac{a^\alpha + b^\alpha}{(a+b)^\alpha}\right)^{\frac{1}{\alpha}},$$

where $a > 0$ and $b > 0$, and we hope that this maximum is less than or equal to $2^{\frac{1}{\alpha}-1}$.

Let $b = \lambda a$ and let us find the maximum of

$$\frac{a^\alpha + (\lambda a)^\alpha}{((1+\lambda)a)^\alpha}.$$

Since

$$\frac{a^\alpha + (\lambda a)^\alpha}{((1+\lambda)a)^\alpha} = \frac{(1+\lambda^\alpha)a^\alpha}{(1+\lambda)^\alpha a^\alpha},$$

it is sufficient to find the maximum of

$$\frac{1+\lambda^\alpha}{(1+\lambda)^\alpha}.$$

Next,

$$\left(\frac{1+\lambda^\alpha}{(1+\lambda)^\alpha}\right)' = \frac{(\alpha\lambda^{\alpha-1})(1+\lambda)^\alpha - (1+\lambda^\alpha)\alpha(1+\lambda)^{\alpha-1}}{(1+\lambda)^{2\alpha}},$$

$$\alpha(1+\lambda)^{\alpha-1}[\lambda^{\alpha-1}(1+\lambda) - (1+\lambda^\alpha)] = 0$$

and

$$\lambda^{\alpha-1} + \lambda^\alpha - 1 - \lambda^\alpha = 0.$$

Thus, $\left(\frac{1+\lambda^\alpha}{(1+\lambda)^\alpha}\right)' = 0$ only for $\lambda = 1$, $\left(\frac{1+\lambda^\alpha}{(1+\lambda)^\alpha}\right)' > 0$ for $\lambda < 1$ and $\left(\frac{a+\lambda^\alpha}{(1+\lambda)^\alpha}\right)' < 0$ for $\lambda < 1$ (for $\alpha < 1$). Therefore, the maximum of our expression is attained at $\lambda = 1$ and equals to $\left(\frac{1+1^\alpha}{(1+1)^\alpha}\right)^{\frac{1}{\alpha}} = 2^{\frac{1}{\alpha}-1}$. Consequently, we found a quasinorm on the space X with the constant $K = 2^{\frac{1}{\alpha}-1}$. □

On the other hand, we would like to be able to make an α-norm from a quasinorm. Unfortunately, there exists no general dependence.

Proposition 1.8. There exists a quasinorm $\|\cdot\|$ which is not an α-norm for any $\alpha \in (0, 1)$.

Proof. Let X be a Banach space equipped with the norm $\|\cdot\|_X$, and let Y be its closed nontrivial subspace. Let us define

$$\|\cdot\|_{\tilde{X}} = \begin{cases} 2\|y\|_X, & y \in Y; \\ \|x\|_X, & x \in X \backslash Y. \end{cases}$$

At first, we have to confirm that $\|\cdot\|_{\tilde{X}}$ satisfies the axioms of a quasinorm. It is easy to observe that

$$\|x\|_{\tilde{X}} = 0 \Leftrightarrow x = 0$$

and that

$$\|kx\|_{\tilde{X}} = |k|\|x\|_{\tilde{X}} \quad \text{whenever } k \text{ is a scalar.}$$

It remains to prove that $\|\cdot\|_{\tilde{X}}$ is a quasinorm with constant $K = 2$.
For $x + y \in Y$, we have

$$\|x + y\|_{\tilde{X}} = 2\|x + y\|_X \leq 2(\|x\|_X + \|y\|_X).$$

For $x + y \in X \backslash Y$,

$$\|x + y\|_{\tilde{X}} = \|x + y\|_X \leq (\|x\|_X + \|y\|_X) \leq 2(\|x\|_X + \|y\|_X).$$

We shall now prove that, given any $\alpha \in (0, 1)$, $\|\cdot\|_{\tilde{X}}$ is not an α-norm. Let $x, y \in Y$, thus $x + y \in Y$. If $\|\cdot\|_{\tilde{X}}$ was an α-norm, then it would have to obey

$$\|x + y\|^{\alpha}_{\tilde{X}} = 2^{\alpha}\|x + y\|^{\alpha}_X \leq \|x\|^{\alpha}_X + \|y\|^{\alpha}_X.$$

Let $y = mx$ for some scalar m. Then it is required, that for all such m,

$$2^{\alpha}\|x(1 + m)\|^{\alpha}_X \leq \|x\|^{\alpha}_X + \|mx\|^{\alpha}_X.$$

Hence,

$$2^{\alpha}(1 + m)^{\alpha}\|x\|^{\alpha}_X \leq (1 + m^{\alpha})\|x\|^{\alpha}_X$$

and

$$2^{\alpha} \leq \frac{1 + m^{\alpha}}{(1 + m)^{\alpha}}.$$

Because

$$\lim_{m\to\infty} \frac{(1 + m)^{\alpha}}{1 + m^{\alpha}} = 1,$$

it holds that $\frac{(1+m)^{\alpha}}{1+m^{\alpha}} < 2^{\alpha}$ for large enough m. So, for any given α, we found x and y such that $\|\cdot\|^{\alpha}_{\tilde{X}}$ does not satisfy the triangle inequality. Consequently, $\|\cdot\|_{\tilde{X}}$ is not an α-norm, for any $\alpha > 0$. □

Now we define some other important terms; in particular, a metric, which is also called a distance function, and a quasimetric, which is an analogue of a quasinorm in normed spaces.

Definition 1.9. Let X be a (non empty) set. The function $\rho : X \times X \to [0, \infty)$ is called a *metric* if, for all $x, y, z \in X$,

$$\rho(x, y) = 0 \Leftrightarrow x = y;$$

$$\rho(x, y) = \rho(y, x);$$

and

$$\rho(x, y) \leq \rho(x, z) + \rho(z, y).$$

Definition 1.10. Let X be a (non empty) set. The function $\rho : X \times X \to [0, \infty)$ is called a *quasimetric* if, for all $x, y, z \in X$,

$$\rho(x, y) = 0 \Leftrightarrow x = y;$$

$$\rho(x, y) = \rho(y, x);$$

and

$$\rho(x, y) \leq K(\rho(x, z) + \rho(z, y)) \quad \text{for some} \quad K \geq 1.$$

It is obvious that every metric is also a quasimetric with the constant $K = 1$.

Remarks 1.11. (i) Let $(X, \| \cdot \|)$ be a normed vector space. Then we can introduce a metric ρ on X by setting $\rho(x, y) = \|x - y\|$ for all x, y in X.

(ii) Analogically, we can generate a quasimetric by a quasinorm.

2 Lorentz spaces

In this section we shall develop a theory of the nonincreasing rearrangement of a given function and, in particular, of Lorentz and Marcinkiewicz spaces.

Most of the material and further details can be found in [1].

Prior to the definition of Lorentz spaces we have to define some auxiliary notions and their properties.

Let (R, μ) denote a totally σ-finite measure space.

Let $\mathcal{M}$ denote the collection of all extended scalar-valued (real or complex) μ-measurable functions on R, and let $\mathcal{M}_0$ denote the class of functions in $\mathcal{M}$ that are finite μ-a.e.

Definition 2.1. The *distribution function* μ_f of a function f in $\mathcal{M}_0 = \mathcal{M}_0(R, \mu)$ is given by

$$\mu_f(\lambda) = \mu\{x \in R : |f(x)| > \lambda\}, \quad \lambda \geq 0.$$

Observe that the distribution function is a nonnegative, nonincreasing and right-continuous function on $[0, \infty)$.

Definition 2.2. Two functions $f \in \mathcal{M}_0(R, \mu)$ and $g \in \mathcal{M}_0(S, \nu)$ are said to be *equimeasurable* if they have the same distribution function, that is, if $\mu_f(\lambda) = \nu_g(\lambda)$ for all $\lambda \geq 0$.

Definition 2.3. Suppose f belongs to $\mathcal{M}_0(R, \mu)$. The *nonincreasing rearrangement* of f is the function f^*, defined on $[0, \infty)$ by

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leq t\}, \quad t \geq 0.$$

We will presume that $\inf \emptyset = \infty$. Hence, if for some t, we have $\mu_f(\lambda) > t$ for all λ, then $f^*(t) = \infty$. Because μ_f is nonincreasing, we can express f^* also as

$$f^*(t) = \sup\{\lambda : \mu_f(\lambda) > t\}.$$

We can think of the nonincreasing rearrangement as a 'generalized inversion' of μ_f. In the cases, when μ_f is strictly decreasing and continuous, we have $f^* = (\mu_f)^{-1}$.

Now we shall introduce some properties of the nonincreasing rearrangement, which we will need in this section. For the proof see [1, Chapter 2; Proposition 1.7].

Proposition 2.4. Suppose f, g belong to $\mathcal{M}_0(R, \mu)$ and let $a \in \mathbb{R}$ be any scalar. The nonincreasing rearrangement f^* is a nonnegative, nonincreasing, right-continuous function on $[0, \infty)$, and

$$(af)^* = |a| f^*, \tag{7}$$

$$(|f|^p)^* = (f^*)^p, \quad 0 < p < \infty,$$

and

$$(f+g)^*(t_1+t_2) \leq f^*(t_1) + g^*(t_2), \quad t_1, t_2 \geq 0. \tag{8}$$

Observe that for a nonincreasing function f, $f^* = f$.

Definition 2.5. Let f belong to $\mathcal{M}_0(R,\mu)$. Then f^{**} will denote the *maximal function* of f^*, defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\mathrm{d}s, \quad t > 0.$$

We will now summarize some properties of the maximal function. For the proof see [1, Chapter 2; Proposition 3.2] and [1, Chapter 2; Theorem 3.4].

Proposition 2.6. Suppose f, g belong to $\mathcal{M}_0$ and let $a \in \mathbb{R}$ be any scalar. Then f^{**} is nonnegative, nonincreasing, and continuous on $(0,\infty)$. Furthermore, the following properties hold:

$$f^{**} \equiv 0 \Leftrightarrow f = 0 \quad \mu\text{-a.e};$$

$$(af)^{**} = |a| f^{**};$$

$$(f+g)^{**}(t) \leq f^{**}(t) + g^{**}(t).$$

Definition 2.7. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed linear spaces such that $X \subseteq Y$. We say that X is *continuously embedded* into Y, denoted $X \hookrightarrow Y$, if the identity function $\mathrm{Id} : X \to Y$ is continuous, in other words, if there exists a constant $c \geq 0$ such that $\|x\|_Y \leq c\|x\|_X$ for every $x \in X$.

Definition 2.8. Let (R,μ) be a totally σ-finite measure space and suppose $0 < p$, $q \leq \infty$. The *Lorentz space* $L^{p,q} = L^{p,q}(R,\mu)$ consists of all f in $\mathcal{M}_0(R,\mu)$ for which the quantity

$$\|f\|_{p,q} = \begin{cases} \left\{\int_0^\infty [t^{1/p} f^*(t)]^q \frac{\mathrm{d}t}{t}\right\}^{1/q} & \text{if } 0 < q < \infty \\ \sup\limits_{0<t<\infty} \{t^{1/p} f^*(t)\} & \text{if } q = \infty. \end{cases}$$

is finite.

Now we recall some properties of Lorentz spaces. If $p = q \in (0,\infty)$, then $L^{p,p} = L^p$ and $\|f\|_{p,p} = \|f\|_p$, which results from (7) and the fact, that f and f^* are equimeasurable. If $q = \infty$, then $L^{p,\infty}$ is the so-called *weak Lebesgue space* and $L^{p,\infty} \supsetneq L^p$ for $p < \infty$. For $p = \infty$ and $q < \infty$, then $L^{\infty,q} = \{0\}$.

The next proposition describes an embedding of Lorentz spaces for a fixed p. For the proof see [1, Chapter 4; Proposition 4.2].

Proposition 2.9. Let $0 < p \leq \infty$ and $0 < q \leq r \leq \infty$. Then

$$L^{p,q} \hookrightarrow L^{p,r}.$$

(There exists a positive c such that for every f it holds that $\|f\|_{p,r} \leq c\|f\|_{p,q}$.)

The inclusions between $L^{p,q}$ spaces with different p is similar to those between Lebesgue spaces L^p, independent of q [1, Chapter 4; page 217].

Despite our notation, the functional $\|\cdot\|_{p,q}$ is not always a norm. More precisely, it is a norm if and only if $1 \leq q \leq p$ and $1 < p < \infty$. In cases when $1 < p < \infty$ and $q \in [1, \infty]$, it is at least equivalent to the norm $\|f^{**}\|_{p,q}$.

However, the Lorentz space $L^{1,q}$, where $q \in (1, \infty]$, can not be equipped with any norm equivalent to $\|\cdot\|_{1,q}$.

Henceforward we will concentrate on the most interesting case $p = 1$ and our goal will be to equip Lorentz spaces with a suitable α-norm or to show that such an α-norm does not exist.

Now let us demonstrate the necessity of this search and prove that, for $q = \infty$, the functional $\|f\|_{1,\infty}$ is not a norm.

Proposition 2.10. Let $(L^{1,\infty}, \|\cdot\|_{1,\infty})$ be as in Definition 2.8. Then, the mapping $\|\cdot\|_{1,\infty}$ does not satisfy (3) – the triangle inequality.

Proof. It is sufficient to find a counterexample. Let $f := x$ and $g := 1 - x$, then $f + g \equiv 1$. We will compute their norms using 2.8. We have:

$$\|f\|_{1,\infty} = \sup_{0<t<1} \{tf^*(t)\} = \max_{0<t<1} t(1-t) = \frac{1}{4},$$

$$\|g\|_{1,\infty} = \sup_{0<t<1} \{tg^*(t)\} = \max_{0<t<1} t(1-t) = \frac{1}{4}$$

and

$$\|f+g\|_{1,\infty} = \sup_{0<t<1} \{t(f+g)^*(t)\} = \sup_{0<t<1} t \cdot 1 = 1,$$

but $1 \not\leq \frac{1}{4} + \frac{1}{4}$, hence $\|f+g\| \not\leq \|f\| + \|g\|$. ☐

Our purpose now is to equip $L^{1,\infty}$ with an optimal α-norm. Because the functional $\|\cdot\|_{1,\infty}$ obviously satisfies the norm axioms (1) and (2) , we have to prove only the triangle-inequality. At first, we will prove it for $\alpha = 1/2$.

Theorem 2.11. Let $(L^{1,\infty}, \|\cdot\|_{1,\infty})$ be as in Definition 2.8. Then

$$\|f+g\|_{1,\infty}^{1/2} \leq \|f\|_{1,\infty}^{1/2} + \|g\|_{1,\infty}^{1/2}.$$

Proof. We require:

$$(\sup_{0<t<\infty} t(f+g)^*(t))^{1/2} \leq (\sup_{0<t<\infty} tf^*(t))^{1/2} + (\sup_{0<t<\infty} tg^*(t))^{1/2},$$

that is, by the properties of the supremum,

$$\sup_{0<t<\infty} \sqrt{t}\sqrt{(f+g)^*(t)} \leq \sup_{0<t<\infty} \sqrt{t}\sqrt{f^*(t)} + \sup_{0<t<\infty} \sqrt{t}\sqrt{g^*(t)}.$$

Denote $a := \sup_{0<t<\infty} \sqrt{t}\sqrt{f^*(t)}$ and $b := \sup_{0<t<\infty} \sqrt{t}\sqrt{g^*(t)}$. We need to show that, for every $t \in (0,\infty)$,

$$t(f+g)^*(t) \leq (a+b)^2,$$

that is

$$(f+g)^*(t) \leq \frac{1}{t}(a+b)^2.$$

From (8), we get

$$(f+g)^*(t) \leq f^*(\lambda t) + g^*((1-\lambda)t), \quad \text{for every } \lambda \in [0,1]. \tag{9}$$

Now, it suffices to prove that

$$f^*(\lambda t) + g^*((1-\lambda)t) \leq \frac{1}{t}(a+b)^2. \tag{10}$$

Recall that

$$a = \sup_{0<t<\infty} \sqrt{tf^*(t)},$$

hence

$$a = \sup_{0<t<\infty} \sqrt{\lambda t f^*(\lambda t)}$$

and

$$a^2 = \sup_{0<t<\infty} \lambda t f^*(\lambda t).$$

Thus,

$$\lambda t f^*(\lambda t) \leq \sup_{0<t<\infty} \lambda t f^*(\lambda t) = a^2,$$

that is

$$f^*(\lambda t) \leq \frac{a^2}{\lambda t}.$$

Similarly, we get

$$g^*((1-\lambda)t) \leq \frac{b^2}{(1-\lambda)t}.$$

Let us insert this into (10), we then obtain

$$\frac{a^2}{\lambda t} + \frac{b^2}{(1-\lambda)t} \leq \frac{1}{t}(a+b)^2$$

and

$$\frac{a^2}{\lambda} + \frac{b^2}{(1-\lambda)} \leq (a+b)^2.$$

Finally, we need to find λ satisfying

$$(1-\lambda)a^2 + \lambda b^2 \leq \lambda(1-\lambda)(a+b)^2.$$

Because λ can depend on a and b, we need to solve the quadratic inequation: let $\lambda := \frac{a}{a+b}$, then $\lambda \in (0,1)$ and

$$(1-\lambda)a^2 + \lambda b^2 \leq \lambda(1-\lambda)(a+b)^2.$$

So, we found λ satisfying (10), and from (9) we obtain the required inequality. □

Combining this result with the Proposition 1.5, we immediately obtain that $\|\cdot\|$ is an α-norm also for all $\alpha \in (0, 1/2]$.

Now we shall demonstrate that $\alpha = 1/2$ is the largest (optimal) value for which the functional $\|\cdot\|_{1,\infty}$ is an α-norm.

Proposition 2.12. Let $\beta > 1/2$, then exist $f, g \in \mathcal{M}_0$ such that

$$\|f+g\|_{1,\infty}^{\beta} > \|f\|_{1,\infty}^{\beta} + \|g\|_{1,\infty}^{\beta}.$$

Proof. We will use the same counterexample as above. Let $f := x$ and $g := 1-x$, then $f + g \equiv 1$. Then

$$\|f\|_{1,\infty} = \frac{1}{4},$$
$$\|g\|_{1,\infty} = \frac{1}{4}$$

and

$$\|f+g\|_{1,\infty} = 1.$$

Let us define $h(\beta) := 2(\frac{1}{4}^{\beta})$, then h is a strictly decreasing function, $h(1/2) = 1$, so, for each $\beta > 1/2$, $h(\beta) < 1$, proving the claim. □

Corollary 2.13. Let $(L^{1,\infty}, \|\cdot\|_{1,\infty})$ be as in Definition 2.8. Then $\|\cdot\|_{1,\infty}$ is the α-norm if and only if $\alpha \in (0, 1/2]$.

Using the relationship between a α-norm and a quasinorm, we can formulate the following corollary.

Corollary 2.14. We can equip the Lorentz space $L^{1,\infty}$ with a quasinorm with constant $K = 2$.

Using Remark 1.11 we can formulate yet another corollary.

Corollary 2.15. We can equip the Lorentz space $L^{1,\infty}$ with a quasimetric with constant $K = 2$.

We shall now leave the case $q = \infty$ and will explore other cases. Before doing that, we will formulate a result by Godfrey Harold Hardy, for its proof see [1, Chapter 2; Proposition 3.6].

Proposition 2.16 (Hardy's lemma)**.** Let f and g be nonnegative measurable functions on $(0, \infty)$ and suppose

$$\int_0^t f(s)\mathrm{d}s \leq \int_0^t g(s)\mathrm{d}s$$

for all $t > 0$. Let h be any nonnegative nonincreasing function on $(0, \infty)$. Then

$$\int_0^\infty f(s)h(s)\mathrm{d}s \leq \int_0^\infty g(s)h(s)\mathrm{d}s.$$

Now we are able to show that $\|f\|_{1,q}$ for $q \in (0, 1)$ is a q-norm. The first two α-norm axioms (4) and (5) are obvious, so we will prove only the triangle-inequality.

Proposition 2.17. Let $0 < q < 1$, then

$$\|f + g\|_{1,q}^q \leq \|f\|_{1,q}^q + \|g\|_{1,q}^q.$$

Proof. We know that

$$\int_0^s (f + g)^*(t)\mathrm{d}t \leq \int_0^s f^*(t)\mathrm{d}t + \int_0^s g^*(t)\mathrm{d}t.$$

Thus, for h nonincreasing, we have:

$$\int_0^\infty (f + g)^*(t)h(t)\mathrm{d}t \leq \int_0^\infty f^*(t)h(t)\mathrm{d}t + \int_0^\infty g^*(t)h(t)\mathrm{d}t.$$

Let $h := t^{q-1}$, then, for $q < 1$, h is decreasing, and

$$\|f + g\|_{1,q}^q \leq \|f\|_{1,q}^q + \|g\|_{1,q}^q.$$

□

Next, we will describe the spaces $L^{1,q}$, where $1 < q < \infty$. The following proposition shows that we are able to equip all such Lorentz spaces by an 1/2-norm. Because of the properties of the nonincreasing rearrangement, it is necessary to prove only the triangle-inequality.

Theorem 2.18. Let $1 < q < \infty$ and $(L^{1,q}, \|\cdot\|_{1,q})$ be as in Definition 2.8. Then

$$\|f+g\|_{1,q}^{1/2} \leq \|f\|_{1,q}^{1/2} + \|g\|_{1,q}^{1/2}. \tag{11}$$

Proof. Let denote $a := \|f\|_{1,q}$ and $b := \|g\|_{1,q}$. We need to show that

$$\left(\int_0^\infty (f+g)^*(t)^q t^{q-1} \mathrm{d}t\right)^{1/q} \leq (\sqrt{a}+\sqrt{b})^2. \tag{12}$$

From (8), we have

$$\left(\int_0^\infty (f+g)^*(t)^q t^{q-1}\right)^{1/q} \leq \left(\int_0^\infty [f^*(\lambda t) + g^*((1-\lambda)t)]^q t^{q-1} \mathrm{d}t\right)^{1/q}.$$

Using the Minkowski inequality, we obtain

$$\begin{aligned}\left(\int_0^\infty [f^*(\lambda t) + g^*((1-\lambda)t)]^q t^{q-1} \mathrm{d}t\right)^{1/q} \leq \\ \left(\int_0^\infty f^*(\lambda t)^q t^{q-1} \mathrm{d}t\right)^{1/q} + \left(\int_0^\infty g^*((1-\lambda)t)^q t^{q-1} \mathrm{d}t\right)^{1/q}.\end{aligned} \tag{13}$$

We set $s = \lambda t$ and get

$$\left(\int_0^\infty f^*(\lambda t)^q t^{q-1} \mathrm{d}t\right)^{1/q} = \left(\int_0^\infty f^*(s)^q \left(\frac{s}{\lambda}\right)^{q-1} \frac{1}{\lambda} \mathrm{d}t\right)^{1/q},$$

and, similarly,

$$\left(\int_0^\infty g^*((1-\lambda)t)^q t^{q-1} \mathrm{d}t\right)^{1/q} = \left(\int_0^\infty g^*(s)^q \left(\frac{s}{1-\lambda}\right)^{q-1} \frac{1}{1-\lambda} \mathrm{d}t\right)^{1/q}.$$

Then we insert this into (13), and we obtain

$$\frac{1}{\lambda}\|f\|_{1,q} + \frac{1}{1-\lambda}\|g\|_{1,q} = \frac{1}{\lambda}a + \frac{1}{1-\lambda}b.$$

From (12) it sufficient to show

$$\frac{1}{\lambda}a + \frac{1}{1-\lambda}b \leq (\sqrt{a}+\sqrt{b})^2.$$

Now, because λ can depend on a and b we can just take $\lambda = \frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$, and the desired inequality (11) follows. □

As in the case $L^{1,\infty}$, we can equip the Lorentz space $L^{1,q}$ with a quasinorm and a quasimetric.

Corollary 2.19. We can equip the Lorentz space $L^{1,\infty}$ with a quasinorm with constant $K = 2$ and with a quasimetric with constant $K = 2$.

3 Marcinkiewicz spaces

Now we would like to define Marcinkiewicz spaces and describe embeddings between them. Prior to the definition of Marcinkiewicz spaces we also have to insert some preliminary material.

Definition 3.1. Let φ be a nonnegative function defined on the interval $\mathbb{R}^+ = [0, \infty)$. If

$$\begin{aligned} \varphi(t) &\quad \text{is nondecreasing on } (0,\infty); \\ \varphi(t) &= 0 \Leftrightarrow t = 0; \\ \varphi(t)/t &\quad \text{is nonincreasing on } (0,\infty), \end{aligned} \tag{14}$$

then φ is said to be *quasiconcave*.

Definition 3.2. Let φ be a quasiconcave function on $\mathbb{R}^+$. The *Marcinkiewicz space* $m_\varphi = m_\varphi(R,\mu)$ consists of all functions f in $\mathcal{M}_0(R,\mu)$ for which the functional

$$\|f\|_{m_\varphi} = \sup_{0<t<\infty} \{f^*(t)\varphi(t)\}$$

is finite.

Definition 3.3. Let X be a normed vector space. The *unit ball* is the set

$$B_X = \{x \in X : \|x\| \le 1\}.$$

The next proposition describes embeddings between Marcinkiewicz spaces.

Proposition 3.4. Let φ and ψ be a quasiconcave functions. Then

$$m_\varphi \hookrightarrow m_\psi \tag{15}$$

if and only if

$$\sup_{0<t<\infty} \frac{\psi(t)}{\varphi(t)} < \infty. \tag{16}$$

Proof. At first, we will prove the sufficiency of the condition.

We would like to prove that, assuming (16), we have

$$\sup_{f \not\equiv 0} \frac{\|f\|_{m_\psi}}{\|f\|_{m_\varphi}} < \infty.$$

Suppose that $\sup\limits_{0<t<\infty} \frac{\psi(t)}{\varphi(t)} = c$ for some $c > 0$. Then, for every $t > 0$,

$$\psi(t) < c\varphi(t).$$

Let f be a fixed but arbitrary function in $\mathcal{M}_0(R,\mu)$. Then, by Definition 3.2 and the properties of the supremum, we have

$$\frac{\sup\limits_{0<t<\infty} f^*(t)\psi(t)}{\sup\limits_{0<s<\infty} f^*(s)\varphi(s)} < \frac{\sup\limits_{0<t<\infty} f^*(t)c\varphi(t)}{\sup\limits_{0<s<\infty} f^*(s)\varphi(s)} = c\frac{\sup\limits_{0<t<\infty} f^*(t)\varphi(t)}{\sup\limits_{0<s<\infty} f^*(s)\varphi(s)} = c.$$

Thus,

$$\sup_{f\in m_\varphi} \frac{\|f\|_{m_\psi}}{\|f\|_{m_\varphi}} \le c.$$

Now we shall prove the necessity of the condition (16).

Suppose, for contradiction, that (15) holds and simultaneously, for each $n\in\mathbb{N}$, there exists a sequence $a_n \in (0,\infty)$, $n\in\mathbb{N}$, such that

$$\frac{\psi(a_n)}{\varphi(a_n)} \ge n. \tag{17}$$

Let f_n be a function defined by the following formula:

$$f_n(t) = \begin{cases} 1, & t\in[0,a_n]; \\ 0, & t\in(a_n,\infty). \end{cases}$$

Since f_n is nonincreasing, we have $f_n = f_n^*$.

Now we will compute the norm $\|f_n\|_{m_\varphi}$:

$$\sup_{0<t<\infty} f_n^*(t)\varphi(t) = \max\{\sup_{t\in(0,a_n]} f_n^*(t)\varphi(t), \sup_{t\in(a_n,\infty)} f_n^*(t)\varphi(t)\} =$$
$$\max\{\sup_{t\in(0,a_n]} 1\cdot\varphi(t), \sup_{t\in(a_n,\infty)} 0\cdot\varphi(t)\} = \sup_{t\in(0,a_n]} \varphi(t).$$

Because φ is nondecreasing,

$$\|f_n\|_{m_\varphi} = \varphi(a_n), \tag{18}$$

and $f_n \in m_\varphi$.

Now, let us estimate the value of $\|f_n\|_{m_\psi}$: from the properties of the supremum we have

$$\sup_{0<t<\infty} f_n^*(t)\psi(t) \ge f_n(a_n)\psi(a_n) = 1\psi(a_n),$$

hence, from (17) and (18)

$$\psi(a_n) \ge n\varphi(a_n) = n\|f_n\|_{m_\varphi}.$$

So

$$\frac{\|f_n\|m_\psi}{\|f_n\|m_\varphi} \ge n$$

for each $n \in \mathbb{N}$. Consequently,

$$\sup_{f \in m_\varphi} \frac{\|f\|_{m_\psi}}{\|f\|_{m_\varphi}} = \infty,$$

which is a contradiction with (15). □

In the theory of compact embeddings between function spaces [2], the following notion is of importance.

Definition 3.5. We say that one Marcinkiewicz space m_φ is *almost compactly embedded* into another one, m_ψ, if

$$\lim_{a \to 0+} \sup_{f \in B_{m_\varphi}} \|f^* \chi_{(0,a)}\|_{m_\psi} = 0$$

and

$$\lim_{a \to \infty} \sup_{f \in B_{m_\varphi}} \|f^* \chi_{(a,\infty)}\|_{m_\psi} = 0.$$

The aim of this chapter is to characterize when this happens. We begin with two properties.

Proposition 3.6. Let m_φ, m_ψ be Marcinkiewicz spaces. Then

$$\lim_{a \to 0+} \sup_{f \in B_{m_\varphi}} \|f^* \chi_{(0,a)}\|_{m_\psi} = 0 \tag{19}$$

if and only if

$$\lim_{a \to 0+} \sup_{0<t<a} \frac{\psi(t)}{\varphi(t)} = 0. \tag{20}$$

Proof. At first, we will prove the sufficiency of (20). We begin with writing out the definition of the unit ball. We have

$$f \in B_{m_\varphi}$$

if and only if

$$\|f\|_{m_\varphi} \leq 1.$$

By Definition 3.2, this is the same as

$$\sup_{0<t<\infty} f^*(t)\varphi(t) \leq 1,$$

that is,

$$f^*(t) \leq \frac{1}{\varphi(t)} \quad \text{for every } t \in (0,\infty).$$

Because φ is nondecreasing, then $1/\varphi$ is nonincreasing, and, for every $t \in (0, \infty)$

$$\frac{1}{\varphi}(t) = \left(\frac{1}{\varphi}\right)^* (t),$$

hence

$$f^*(t) \leq \left(\frac{1}{\varphi}\right)^* (t),$$

then

$$f^*(t)\chi_{(0,a)}(t) \leq \left(\frac{1}{\varphi}\right)^* (t)\chi_{(0,a)}(t).$$

It is obvious that

$$(f^*\chi_{(0,a)})^*(t) = f^*(t)\chi_{(0,a)}(t)$$

and

$$\left(\left(\frac{1}{\varphi}\right)^* \chi_{(0,a)}\right)^* (t) = \left(\frac{1}{\varphi}\right)^* (t)\chi_{(0,a)}(t).$$

Hence

$$\|f^*\chi_{(0,a)}\|_{m_\psi} \leq \left\| \left(\frac{1}{\varphi}\right)^* \chi_{(0,a)} \right\|_{m_\psi}.$$

Now we will test the condition (19):

$$\lim_{a\to 0+} \sup_{f\in B_{m_\varphi}} \|f^*\chi_{(0,a)}\|_{m_\psi} \leq \lim_{a\to 0+} \left\| \left(\frac{1}{\varphi}\right)^* \chi_{(0,a)} \right\|_{m_\psi}$$

$$= \lim_{a\to 0+} \sup_{0<t<\infty} \left(\frac{1}{\varphi}\right)^* (t)\chi_{(0,a)}(t)\psi(t) = \lim_{a\to 0+} \sup_{0<t<a} \frac{\psi(t)}{\varphi(t)}$$

and from the condition (20)

$$\lim_{a\to 0+} \sup_{f\in B_{m_\varphi}} \|f^*\chi_{(0,a)}\|_{m_\psi} = 0,$$

which we needed.

Now we shall prove the necessity of (20). Suppose for contradiction, that (19) holds and

$$\lim_{a\to 0+} \sup_{0<t<a} \frac{\psi(t)}{\varphi(t)} = c > 0. \tag{21}$$

Let $f := \frac{1}{\varphi}$. Then $f^* = \left(\frac{1}{\varphi}\right)^* = \frac{1}{\varphi}$ and $f \in B_{m_\varphi}$. Let us compute

$$\lim_{a\to 0+} \|f^*\chi_{(0,a)}\|_{m_\psi} = \lim_{a\to 0+} \left\| \frac{\chi_{(0,a)}}{\varphi} \right\|_{m_\psi},$$

using the computation above, we know that

$$\lim_{a\to 0+}\left\|\frac{\chi_{(0,a)}}{\varphi}\right\|_{m_\psi} = \lim_{a\to 0+}\sup_{0<t<a}\frac{\psi(t)}{\varphi(t)},$$

but from (21),

$$\lim_{a\to 0+}\sup_{0<t\le a}\frac{\psi(t)}{\varphi(t)} = c > 0,$$

so

$$\lim_{a\to 0+}\sup_{f\in B_{m_\varphi}}\|f^*\chi_{(0,a)}\|_{m_\psi} \ge c > 0,$$

which is in a contradiction with (19). □

Proposition 3.7. Let m_φ, m_ψ be Marcinkiewicz spaces. Then

$$\lim_{a\to\infty}\sup_{f\in B_{m_\varphi}}\|f^*\chi_{(a,\infty)}\|_{m_\psi} = 0 \tag{22}$$

if and only if

$$\lim_{a\to\infty}\sup_{a<t<\infty}\frac{\psi(t)}{\varphi(t)} = 0. \tag{23}$$

Proof. Analogously to the last proof, at first we will prove the sufficiency of (23).

Using the definition of the unit ball and Definition 3.2, we know that for every $f \in B_{m_\varphi}$

$$f^*(t) \le \frac{1}{\varphi(t)} \quad \text{for every } t\in(0,\infty).$$

Since φ is nondecreasing, $1/\varphi$ is nonincreasing and for every $t \in (0,\infty)$

$$\frac{1}{\varphi(t)} = \left(\frac{1}{\varphi}\right)^*(t).$$

Hence

$$f^*(t)\chi_{(a,\infty)}(t) \le \left(\frac{1}{\varphi}\right)^*(t)\chi_{(a,\infty)}(t)$$

and

$$\|f^*(t)\chi_{(a,\infty)}\|_{m_\psi} \le \left\|\left(\frac{1}{\varphi}\right)^*\chi_{(a,\infty)}\right\|_{m_\psi}.$$

It is obvious that

$$\left(\frac{\chi_{(a,\infty)}}{\varphi}\right)^*(t) = \frac{1}{\varphi(a+t)}.$$

Now we will test the inequality (22):

$$\begin{aligned}\lim_{a\to\infty} \sup_{f\in B_{m_\varphi}} \|f^*\chi_{(a,\infty)}\|_{m_\psi} &\le \lim_{a\to\infty} \left\| \left(\frac{1}{\varphi}\right)^* \chi_{(a,\infty)} \right\|_{m_\psi} \\ &= \lim_{a\to\infty} \sup_{0<t<\infty} \left(\frac{\chi_{(a,\infty)}}{\varphi}\right)^*(t)\psi(t) \\ &= \lim_{a\to\infty} \sup_{0<t<\infty} \frac{\psi(t)}{\varphi(t+a)} \\ &= \lim_{a\to\infty} \sup_{0<t<\infty} \frac{\psi(t)}{\psi(t+a)} \frac{\psi(t+a)}{\varphi(t+a)}\end{aligned}$$

Because ψ is nondecreasing, $\psi(t)/\psi(t+a) \le 1$. Therefore

$$\lim_{a\to\infty} \sup_{0<t<\infty} \frac{\psi(t)}{\psi(t+a)} \frac{\psi(t+a)}{\varphi(t+a)} \le \lim_{a\to\infty} \sup_{0<t<\infty} \frac{\psi(t+a)}{\varphi(t+a)} = \lim_{a\to 0+} \sup_{a<t<\infty} \frac{\psi(t)}{\varphi(t)}$$

and from the condition (23)

$$\lim_{a\to\infty} \sup_{f\in B_{m_\varphi}} \|f^*\chi_{(0,a)}\|_{m_\psi} = 0,$$

which we needed.

Now we shall prove the necessity of (23). Suppose for contradiction, that (22) holds and

$$\lim_{a\to\infty} \sup_{a<t<\infty} \frac{\psi(t)}{\varphi(t)} = c > 0. \qquad (24)$$

Let $f := \frac{1}{\varphi}$. Then $f^* = \left(\frac{1}{\varphi}\right)^* = \frac{1}{\varphi}$ and $f \in B_{m_\varphi}$. Let us compute

$$\lim_{a\to\infty} \|f^*\chi_{(a,\infty)}\|_{m_\psi} = \lim_{a\to\infty} \left\| \frac{\chi_{(a,\infty)}}{\varphi} \right\|_{m_\psi},$$

using the computation above, we know that

$$\begin{aligned}\lim_{a\to\infty} \left\| \frac{\chi_{(a,\infty)}}{\varphi} \right\|_{m_\psi} &= \lim_{a\to\infty} \sup_{0<t<\infty} \frac{\psi(t)}{\varphi(t+a)} \\ &= \lim_{a\to\infty} \max\left\{ \sup_{0<t\le a} \frac{\psi(t)}{\varphi(t+a)}, \sup_{a<t<\infty} \frac{\psi(t)}{\varphi(t+a)} \right\}.\end{aligned}$$

It is enough to consider only the case $t > a$. We have

$$\lim_{a\to\infty} \sup_{a<t<\infty} \frac{\psi(t)}{\varphi(t+a)} = \lim_{a\to\infty} \sup_{a<t<\infty} \frac{\psi(t)}{\psi(t+a)} \frac{\psi(t+a)}{\varphi(t+a)}.$$

Because ψ is quasiconcave, from (14)

$$\psi(a+t) \leq \psi(2t) \leq 2\psi(t)$$

and

$$\lim_{a\to\infty} \sup_{a<t<\infty} \frac{\psi(t)}{\psi(t+a)} \frac{\psi(t+a)}{\varphi(t+a)} \geq \lim_{a\to\infty} \sup_{a<t<\infty} \frac{1}{2} \frac{\psi(t+a)}{\varphi(t+a)}$$
$$= \frac{1}{2} \lim_{a\to\infty} \sup_{2a<t<\infty} \frac{\psi(t)}{\varphi(t)}.$$

that is, from (24)

$$\frac{1}{2} \lim_{a\to\infty} \sup_{2a<t<\infty} \frac{\psi(t)}{\varphi(t)} \geq \frac{c}{2},$$

hence

$$\lim_{a\to\infty} \left\| \frac{\chi_{(a,\infty)}}{\varphi} \right\|_{m_\psi} \geq \frac{c}{2},$$

which is in a contradiction with (22). □

The following theorem is a direct consequence of Propositions 3.6 and 3.7.

Theorem 3.8. A Marcinkiewicz space m_φ is almost compactly embedded into another one, m_ψ, if and only if (20) and (23) hold.

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