Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

DIPLOMOVÁ PRÁCE

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Nekonečné hry a jejich aplikace

Katedra matematické analýzy

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Studijní program: Matematická analýza

Rád bych zde poděkoval vedoucímu své diplomové práce doc. RNDr. Miroslavu Zelenému, Ph.D. za vypsání práce na toto zajímavé téma, za výborné vedení a za mnoho užitečných rad a námětů k přemýšlení.

Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne 7. srpna 2009 Martin Doležal

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Název práce: Nekonečné hry a jejich aplikace
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Abstrakt: Charakterizujeme σ-P-pórovité množiny v úplném metrickém prostoru pomocí nekonečné hry, kde P je libovolná relace pórovitosti. Tato charakterizace může být použita zejména pro případ obyčejné pórovitosti, ale také pro mnoho jiných variant pórovitosti. Modifikaci popsané hry pak použijeme k důkazu existence kompaktní podmnožiny, která není σ-pórovitá, dané borelovské množiny, která není σ-pórovitá, v libovolném lokálně kompaktním metrickém prostoru. To pak dokážeme i pro případ silné pórovitosti, což je nový výsledek. S použitím naší hry také ukážeme existenci uzavřené množiny, která je σ-$(1-\varepsilon)$-symetricky pórovitá pro každé $0 < \varepsilon < 1$, ale není σ-1-symetricky pórovitá.
Klíčová slova: determinovanost, nekonečné hry, pórovitost

Title: Infinite games and their applications
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Abstract: We characterize σ-P-porous sets in a complete metric space via an infinite game where P is an arbitrary porosity-like relation. This can be applied to ordinary porosity above all but also to many other variants of porosity. We use a modification of the game above to prove that there exists a compact and non-σ-porous subset of a given Borel and non-σ-porous set in any locally compact metric space. We also prove the same result for strong porosity (this is a new result). Further, we show that there exists a closed set which is σ-$(1-\varepsilon)$-symmetrically porous for every $0 < \varepsilon < 1$ but which is not σ-1-symmetrically porous.
Keywords: determinacy, infinite games, porosity

1. Introduction.

The theory of porous and σ-porous sets forms an important part of real analysis and Banach space theory for more than forty years. It originated in 1967 when E. P. Dolženko used for the first time the nomenclature 'porous set' and proved that some sets of his interest are σ-porous (see [1]). Since then the porosity has been used many times especially in the differentiation theory (see [6] for an example). A very useful fact is that every σ-porous set (in $\mathbb{R}^n$) is of the first category and has Lebesgue measure zero. In many cases, it is much more comfortable to prove that a given set is σ-porous than proving that the set is small both in the sense of category and in the sense of measure. On the other hand, not every set of the first category and measure zero is also σ-porous which was first proved by L. Zajíček in [9] (although E. P. Dolženko stated this assertion without proof earlier).

The main question I will consider in this work is the following one:

Question 1.1. *Let A be an analytic subset of a metric space X. Suppose that A is not σ-porous. Does there exist a closed set $F \subseteq A$ which is not σ-porous?*

This question was posed by L. Zajíček in [10] (for a Borel set A) and can be also easily reformulated for various other types of porosity. An affirmative answer was given independently by J. Pelant (for any topologically complete metric space X) and M. Zelený (for any compact metric space X). Their results are demonstrated in a joint paper (see [13]) which combines the original idea of J. Pelant (giving an explicit construction of the set F) and techniques developed by M. Zelený. The case of some other types of porosity (including the ordinary one in a locally compact metric space X but also $\langle g \rangle$-porosity in a locally compact metric space X and symmetrical porosity in $\mathbb{R}$) was solved (also affirmatively) by M. Zelený and L. Zajíček in [14]. They offer a less complicated method using so called 'porosity-like' relations and giving a non-constructive proof based on an earlier idea of M. Zelený. However, the authors admitted that their method cannot be applied to strong porosity and so Question 1.1 for strong porosity still remained open (even in a compact metric space X). Meanwhile, J. Zapletal introduced a new powerful tool to describe σ-porous sets. This was an infinite game which can be used to characterize σ-porous sets in the topological space $2^{\mathbb{N}}$ (which are defined in a very natural way). This can be found as an example in a joint paper of J. Zapletal and I. Farah (see [4]). This game is used to show that every analytic subset of $2^{\mathbb{N}}$ which is not σ-porous has a compact subset which is not σ-porous which answers another variant of Question 1.1. The only attempt to answer Question 1.1 for strong porosity (and ordinary porosity once again) was made by D. Rojas-Rebolledo, who generalized the ideas from [4] (see [8]). He managed to give an affirmative answer to Question 1.1 in any zero-dimensional compact metric space X.

For my work, the most inspirational source was [4]. It is the first one which answers a variant of Question 1.1 by using an infinite game (although some connection between σ-porosity and infinite games was already shown by M. Zelený in [12]) as well as I will do. My main aim was finding a similar infinite game which could be used to characterize σ-porous sets in as much general metric space as possible and using this game (or more precisely its modification) to answer some variants of Question 1.1 (hopefully also some unanswered so far). The characterization should be (as it is also in [4]) similar to the very well known characterization of meager sets using so called Banach-Mazur game. This means, I would like to find an infinite game (which is played with a set A) such that A is σ-porous if and only if the second player has a winning strategy in this game.

Let us look at the contents of this work a little closer. In Chapter 2, there are some definitions and well known results which will be necessary for my work. Chapter 3 introduces an infinite game which can be used to characterize σ-P-porous sets in a complete metric space X where P is an arbitrary porosity-like relation on X. This is the first main result of my work. In Chapter 4, we prove that every Borel and non-σ-P-porous set in any locally compact metric space has a compact and non-σ-P-porous subset if the porosity-like relation P satisfies some additional conditions. This is obtained using Martin's determinacy theorem for Borel infinite games. In a few words, if we know that the game above is determined then the first player has to have a winning strategy in the game played with the given Borel and non-σ-P-porous set. I find a compact subset of this given set such that the first player still has a winning strategy in the game played with this subset. This means that the second player does not have a winning strategy and so the subset is not σ-P-porous. In Chapter 5, we apply the last result to concrete porosities and obtain an (affirmative) answer to two different variants of Question 1.1. The first one refers to an ordinary porosity and the second one to strong porosity. As it is described earlier, the former result was already known but the method used in my work (based on a modification of the infinite game described in Chapter 3) aspires to be more elegant and easier than the known proofs. The latter result is new since any of the methods used in previous works (except the one from [8] which concerns only a very special case) cannot be applied to strong porosity. Finally, we show that there exists a closed set in $\mathbb{R}$ which is σ-$(1-\varepsilon)$-symmetrically porous for every $0 < \varepsilon < 1$ but which is not σ-1-symmetrically porous. This answers a question posed by M. J. Evans and P. D. Humke in [3].

2. Preliminaries.

Let (X, d) be a metric space. An open ball with center $x \in X$ and radius $r > 0$ is denoted by $B(x, r)$. Since an open ball (considered as a set) does not uniquely determine its center and radius, we will identify every open ball with the pair (center, radius) throughout this work. Therefore two different open balls (i.e. two different pairs (center, radius)) can still determine the same subset of X. Now, for $p > 0$ and an open ball B with center $x \in X$ and radius $r > 0$, we can define $p \star B$ as an open ball with center x and radius pr. The closed ball with center $x \in X$ and radius $r > 0$ is denoted by $\overline{B}(x, r)$. If $A \subseteq X$ is nonempty and $r > 0$ then $B(A, r) = \{x \in X : \operatorname{dist}(x, A) < r\}$ where $\operatorname{dist}(x, A) = \inf\{d(x, a) : a \in A\}$. We also set $B(\emptyset, r) = \emptyset$. If $A \subseteq X$ is nonempty then $\operatorname{diam} A = \sup\{d(a, b) : a \in A, b \in A\}$.

Let us begin with the definition of porosity and σ-porosity (and some of its variants). From various equivalent definitions, the following ones are probably the most convenient for our purpose.

Definition 2.1. Let (X, d) be a metric space, $A \subseteq X$, $x \in X$ and $q \in (0, 1]$. We say that

- A is *q-(ordinary) porous at x* if there exist sequences $\{B(x_n, r_n)\}_{n=1}^{\infty}$ of open balls in X and $\{q_n\}_{n=1}^{\infty}$ of real numbers from $(0, 1)$ such that
  - $\lim\limits_{n\to\infty} x_n = x$,
  - $\lim\limits_{n\to\infty} q_n = q$,
  - $B(x_n, r_n) \cap A = \emptyset$ for every $n \in \mathbb{N}$,
  - $x \in B\left(x_n, \frac{r_n}{q_n}\right)$ for every $n \in \mathbb{N}$,
- A is *q-(ordinary) porous* if it is q-porous at every its point,
- A is *σ-q-(ordinary) porous* if it is a countable union of q-porous sets,
- A is *(ordinary) porous at x* if it is q-porous at x for some $q \in (0, 1]$,
- A is *(ordinary) porous* if it is porous at every its point,
- A is *σ-(ordinary) porous* if it is a countable union of porous sets,
- A is *strongly porous at x* (resp. *strongly porous* or *σ-strongly porous*) if it is 1-porous at x (resp. 1-porous or σ-1-porous).

If moreover $X = \mathbb{R}$ (with the Euclidean metric), then we say that

- A is *q-symmetrically porous at x* if there exist sequences $\{B(x_n, r_n)\}_{n=1}^{\infty}$ of open balls in X and $\{q_n\}_{n=1}^{\infty}$ of real numbers from $(0, 1)$ such that
  - $\lim\limits_{n\to\infty} x_n = x$,
  - $\lim\limits_{n\to\infty} q_n = q$,
  - $(B(x_n, r_n) \cup B(2x - x_n, r_n)) \cap A = \emptyset$ for every $n \in \mathbb{N}$,
  - $x \in B\left(x_n, \frac{r_n}{q_n}\right)$ for every $n \in \mathbb{N}$,

- A is *q-symmetrically porous* if it is q-symmetrically porous at every its point,
- A is *σ-q-symmetrically porous* if it is a countable union of q-symmetrically porous sets,
- A is *symmetrically porous at x* if it is q-symmetrically porous at x for some $q \in (0, 1]$,
- A is *symmetrically porous* if it is symmetrically porous at every its point,
- A is *σ-symmetrically porous* if it is a countable union of symmetrically porous sets.

We will need the next theorem which is a particular case of [9, Proposition 4.4] (the notation used in [9] differs from the one used in this work).

Theorem 2.2 ([9, Proposition 4.4])**.** *Let X be a metric space, $A \subseteq X$ be a σ-porous set and $q \in (0, 1)$. Then A is σ-q-porous.*

The immediate consequence of this theorem is that the σ-ideal of all σ-porous sets coincides with the σ-ideal of all σ-q-porous sets for every $q \in (0, 1)$.

We will prove our results for a general porosity-like relation (satisfying some additional assumptions) and then apply it to concrete cases. To do this, we need the following definition.

Definition 2.3. Let X be a metric space and let $P \subseteq X \times 2^X$ be a relation between points of X and subsets of X. Then P is called a *point-set relation on X*. The symbol $P(x, A)$ where $x \in X$ and $A \subseteq X$ means that $(x, A) \in P$. For $A \subseteq X$ and $B \subseteq X$, we also use the symbol $P(A, B)$ which is equivalent to [$P(a, B)$ for every $a \in A$].

The point-set relation P on X is called a *porosity-like relation* if the following conditions hold for every $A \subseteq X$ and $x \in X$:

(P1) if $B \subseteq A$ and $P(x, A)$ then $P(x, B)$,
(P2) we have $P(x, A)$ if and only if there exists $r > 0$ such that $P(x, A \cap B(x, r))$,
(P3) we have $P(x, A)$ if and only if $P\left(x, \overline{A}\right)$.

If P is a porosity-like relation on X, $A \subseteq X$ and $x \in X$, we say that

- A is *P-porous at x* if $P(x, A)$,
- A is *P-porous* if it is P-porous at every its point,
- A is *σ-P-porous* if it is a countable union of P-porous sets.

Another theorem we will need is the following one which can be found in [11, Lemma 3].

Theorem 2.4 ([11, Lemma 3])**.** *Let X be a metric space, P be a porosity-like relation on X and $A \subseteq X$. Then A is σ-P-porous if and only if for every $x \in A$ there exists $r > 0$ such that $B(x, r) \cap A$ is σ-P-porous.*

It is also necessary to remind some basic definitions which concern infinite games. Let A be a nonempty set and $n \in \mathbb{N}$. We denote by A^n the set of all sequences $s = (s_0, s_1, \ldots, s_{n-1})$ of length n from A. We also set $A^0 = \{\emptyset\}$ where $\emptyset$ is the *empty sequence* (of length 0). We denote by $A^{<\mathbb{N}}$ (resp. $A^{\mathbb{N}\cup\{0\}}$) the set of all finite (resp. infinite) sequences from A. This means that

$$A^{<\mathbb{N}} = \bigcup_{n=0}^{\infty} A^n.$$

The length of a finite sequence s is denoted by $\operatorname{length}(s)$. If $s \in A^{<\mathbb{N}}$ and $n \in \mathbb{N} \cup \{0\}$ such that $n \leq \operatorname{length}(s)$ then $s|n = (s_0, s_1, \ldots, s_{n-1}) \in A^n$. If $s, t \in A^{<\mathbb{N}}$ then we say that s is an *initial segment* of t and t is an *extension* of s if there exists $n \in \mathbb{N} \cup \{0\}$ such that $n \leq \operatorname{length}(t)$ and $s = t|n$. If $s = (s_0, s_1, \ldots, s_{n-1}) \in A^n$ and $t = (t_0, t_1, \ldots, t_{m-1}) \in A^m$, then the *concatenation* of s and t is the sequence $s^\wedge t = (s_0, s_1, \ldots, s_{n-1}, t_0, t_1, \ldots, t_{m-1}) \in A^{n+m}$. If $x = (x_j)_{j=0}^{\infty} \in A^{\mathbb{N}\cup\{0\}}$ and $n \in \mathbb{N} \cup \{0\}$ then $x|n = (x_0, x_1, \ldots, x_{n-1}) \in A^n$. If $s \in A^{<\mathbb{N}}$ and $x \in A^{\mathbb{N}\cup\{0\}}$ then we say that s is an *initial segment* of x and x is an *extension* of s if $s = x|n$ for some $n \in \mathbb{N} \cup \{0\}$.

A subset $T \subseteq A^{<\mathbb{N}}$ is called a *tree* on A if for every $t \in T$ and every initial segment s of t, we have $s \in T$. A sequence $x \in A^{\mathbb{N}\cup\{0\}}$ is called an *infinite branch* of T if $x|n \in T$ for every $n \in \mathbb{N} \cup \{0\}$. The *body* of T is the set of all infinite branches of T and is denoted by $[T]$. This means that

$$[T] = \{x \in A^{\mathbb{N}\cup\{0\}} \colon x|n \in T \text{ for every } n \in \mathbb{N} \cup \{0\}\}.$$

A tree T is called *pruned* if every $s \in T$ has a proper extension in T, i.e. for every $s \in T$ there exists $t \in T$ such that t is an extension of s and $t \neq s$.

Let A be a nonempty set and $X \subseteq A^{\mathbb{N}\cup\{0\}}$. We associate X (which is called a *payoff set* then) with the following game:

$$\begin{array}{lcccccccc} \text{I} & a_0 & & a_2 & & a_4 & & \\ & & & & & & & \ldots \\ \text{II} & & a_1 & & a_3 & & a_5 & \end{array}$$

Player I plays $a_0 \in A$, then player II plays $a_1 \in A$, I plays $a_2 \in A$, etc. Player I wins if $(a_n)_{n=0}^{\infty} \in X$, II wins in the opposite case. We denote this game by $G(A, X)$.

A *strategy* for player I in the game $G(A, X)$ is a tree $\sigma \subseteq A^{<\mathbb{N}}$ on A such that

- σ is nonempty,
- if $i \in \mathbb{N} \cup \{0\}$ and $(a_0, a_1, \ldots, a_{2i}) \in \sigma$ then $(a_0, a_1, \ldots, a_{2i}, a_{2i+1}) \in \sigma$ for every $a_{2i+1} \in A$,
- if $i \in \mathbb{N} \cup \{0\}$ and $(a_0, a_1, \ldots, a_{2i-1}) \in \sigma$ then there exists a unique $a_{2i} \in A$ such that $(a_0, a_1, \ldots, a_{2i-1}, a_{2i}) \in \sigma$.

If we say that player I follows the strategy σ, we mean the following. Player I starts with the unique $a_0 \in A$ such that $(a_0) \in \sigma$. If II replies by $a_1 \in A$ then $(a_0, a_1) \in \sigma$ and I plays the unique $a_2 \in A$ such that $(a_0, a_1, a_2) \in \sigma$, etc.

A strategy for player I is *winning* in the game $G(A, X)$ if for every run $(a_n)_{n=0}^{\infty} \in A^{\mathbb{N}\cup\{0\}}$ of the game, in which I follows the strategy, we have $(a_n)_{n=0}^{\infty} \in X$ (and so I wins the run).

A (winning) strategy for II is defined in an analogous way.

The game $G(A, X)$ is *determined* if one of the players has a winning strategy.

In the game $G(A, X)$, both players play arbitrary elements of a given nonempty set A. In many cases, it is more convenient to let them obey some rules which are represented by a nonempty pruned tree $T \subseteq A^{<\mathbb{N}}$ (which determines so called *legal positions*). Let $X \subseteq [T]$ (X is called a *payoff set* again), then we define the game $G(T, X)$ as follows:

$$\begin{array}{lccccccc} \mathrm{I} & a_0 & & a_2 & & a_4 & & \\ & & & & & & & \cdots \\ \mathrm{II} & & a_1 & & a_3 & & a_5 & \end{array}$$

Again, I plays $a_0 \in A$, II plays $a_1 \in A$, etc. But both players have now to choose their moves such that $(a_0, a_1, \ldots, a_n) \in T$ for every $n \in \mathbb{N}\cup\{0\}$. Player I wins if $(a_n)_{n=0}^{\infty} \in X$, II wins in the opposite case. The notions of (winning) strategy and determinacy are defined analogously as before. However, the game $G(T, X)$ is only a special case of the previous game. Indeed, it is easy to see that if we denote

$$\begin{aligned} X' = \{x \in A^{\mathbb{N}\cup\{0\}} \colon \big(&\text{there exists } n \in \mathbb{N} \text{ such that } x|n \notin T \\ &\text{and the smallest such } n \text{ is even}\big) \text{ or } \ (x \in X)\}, \end{aligned}$$

then I (resp. II) has a winning strategy in the game $G(T, X)$ if and only if I (resp. II) has a winning strategy in the game $G(A, X')$.

Now, we can formulate the well known (and very deep) Martin's theorem. Its proof can be found in [5, Theorem 20.5]. In this Theorem, we consider the discrete topology on a nonempty set A, the product topology on $A^{\mathbb{N}\cup\{0\}}$ and the derived topology on $[T] \subseteq A^{\mathbb{N}\cup\{0\}}$ where T is a nonempty pruned tree on A.

Theorem 2.5 ([7])**.** *Let T be a nonempty pruned tree on a nonempty set A and let $X \subseteq [T]$ be a Borel set. Then the game $G(T, X)$ is determined.*

We will also need the definition of a σ-discrete system of sets.

Definition 2.6. Let X be a topological space. A system $\mathcal{V}$ of subsets of X is said to be

- *discrete* if for every $x \in X$ there exists a neighborhood of x which intersects at most one set from the system $\mathcal{V}$,

- σ-*discrete* if it is a countable union of discrete systems.

We will use the existence of a σ-discrete basis of open sets in a metric space. This is guaranteed by the following theorem (proof can be found in [2, Theorem 4.4.3]).

Theorem 2.7 ([2, Theorem 4.4.3])**.** *Let X be a metrizable topological space. Then X has an open basis which is σ-discrete.*

3. Characterization of σ-P-porous sets in a complete metric space.

Let (X,d) be a nonempty complete metric space and $A \subseteq X$. Let P be a porosity-like relation on X. We define a game $G(A)$ between Boulder and Sisyfos as follows:

Boulder	B_1	B_2	B_3	$\dots$
Sisyfos	(S_1^1)	(S_2^1, S_2^2)	(S_3^1, S_3^2, S_3^3)	

(By using the names Boulder and Sisyfos, we follow the original terminology of J. Zapletal.) On the first move, Boulder plays an open ball $B_1 \subseteq X$ and Sisyfos plays an open set $S_1^1 \subseteq B_1$. On the second move, Boulder plays an open ball B_2 such that $\overline{B_2} \subseteq B_1$ and diam $B_2 \leq \frac{1}{2}$diam B_1 and Sisyfos plays open sets $S_2^1 \subseteq B_2$ and $S_2^2 \subseteq B_2$. On the nth move, $n > 1$, Boulder plays an open ball B_n such that $\overline{B_n} \subseteq B_{n-1}$ and diam $B_n \leq \frac{1}{2}$diam B_{n-1} and Sisyfos plays open sets $S_n^1 \subseteq B_n, S_n^2 \subseteq B_n, \dots, S_n^n \subseteq B_n$. After a run of the game $G(A)$, we get a unique point x lying in the intersection of the balls B_n, $n \in \mathbb{N}$ (its existence and uniqueness follows from the completeness of X). We call this point an *outcome* of the run. Sisyfos wins the run if at least one of the following conditions is satisfied:

(i) $x \notin A$,

(ii) there exists $m \in \mathbb{N}$ such that $x \in X \setminus \bigcup_{n=m}^{\infty} S_n^m$ and $P\left(x, X \setminus \bigcup_{n=m}^{\infty} S_n^m\right)$.

Boulder wins in the opposite case. If condition (ii) is satisfied for some $m \in \mathbb{N}$, then every such m is called a *witness of Sisyfos' victory*.

We say that a finite (also empty) sequence of open balls $(B_1, B_2, \dots, B_i)$ is *good* if the rules of the game $G(A)$ allow Boulder to play the ball B_n on his nth move, $n = 1, 2, \dots, i$. (In the game $G(A)$, this is independent of Sisyfos' moves.) If $T = (B_1, B_2, \dots, B_i)$ is a good sequence of open balls, we say that a run of the game $G(A)$ satisfies condition $(\star^T)$ if Boulder played the balls $B_1, B_2, \dots, B_i$ in sequence on his first i moves.

Let σ be a strategy for Sisyfos in the game $G(A)$. For $m \in \mathbb{N} \cup \{0\}$ and a good sequence $T = (B_1, B_2, \dots, B_i)$, we denote by M_m^T the set of all

$$x \in \begin{cases} A & \text{if } i = 0 \\ A \cap B_i & \text{if } i > 0 \end{cases}$$

such that in every run V of the game $G(A)$ such that

- the outcome of V is x,
- V satisfies condition $(\star^T)$,
- Sisyfos followed the strategy σ,

all the witnesses of Sisyfos' victory (if there exist any) are greater than m. (The set M_m^T depends on the set A and on the strategy σ. This will not cause any difficulties since if we talk about this set later, both A and σ are always fixed.)

Let Boulder and Sisyfos play a run of the game $G(A)$. Let

$$V = (B_1, \mathcal{S}_1, B_2, \mathcal{S}_2, \dots),$$

$$\mathcal{S}_n = \left(S_n^1, S_n^2, \dots, S_n^n\right), \ n \in \mathbb{N},$$

where Boulder played the ball B_n and Sisyfos played the sets $S_n^1, S_n^2, \dots, S_n^n$ on the nth move of the run, $n \in \mathbb{N}$. Then we will refer to the run itself by V and if we talk about the ball B_n or about the set S_n^m, $m \in \{1, 2, \dots, n\}$, $n \in \mathbb{N}$, we just use the symbols $B_n(V)$ and $S_n^m(V)$, respectively.

First of all, we prove the following lemma which is well known at least for ordinary porosity.

Lemma 3.1. *Let $\mathcal{V}$ be a σ-discrete system of σ-P-porous sets in X. Then $\bigcup \mathcal{V}$ is also σ-P-porous.*

Proof. Let $\mathcal{V} = \bigcup\limits_{n=1}^{\infty} \mathcal{V}_n$ where $\mathcal{V}_n$ is a discrete system for every $n \in \mathbb{N}$. Let us take $n \in \mathbb{N}$ and $x \in X$. There exists $r > 0$ such that $B(x, r)$ intersects at most one set from the system $\mathcal{V}_n$. Therefore $B(x, r) \cap \bigcup \mathcal{V}_n$ is a σ-P-porous set. By Theorem 2.4, the set $\bigcup \mathcal{V}_n$ is σ-P-porous. Finally,

$$\bigcup \mathcal{V} = \bigcup_{n=1}^{\infty} \bigcup \mathcal{V}_n$$

is σ-P-porous as well. □

The next technical lemma will be used to prove Theorem 3.3 which characterizes σ-P-porous sets via the infinite game described earlier.

Lemma 3.2. *Let σ be a strategy for Sisyfos in the game $G(A)$. Let*

$$T_0 = (B_1, B_2, \dots, B_i)$$

be a good sequence of open balls and let $m \in \mathbb{N} \cup \{0\}$. Then there exist a P-porous set $N_m^{T_0}$ and a σ-discrete system $\mathcal{E}$ of sets such that

$$M_m^{T_0} = N_m^{T_0} \cup \bigcup \mathcal{E}$$

and, for every $E \in \mathcal{E}$, there exists a finite sequence T of open balls such that $T_0{}^\wedge T$ is good and $E \subseteq M_{m+1}^{T_0{}^\wedge T}$.

Proof. Whenever we talk about a run of the game $G(A)$ in this proof, we suppose that Sisyfos followed the strategy σ in the run. Let us denote

$$Z = \bigcup \left\{ S_n^{m+1}(V) : n \geq m+1, V \text{ is a run of the game } G(A) \text{ satisfying } (\star^{T_0}) \right\}.$$

For every $x \in Z$, we can find $n(x) \geq m+1$ and a run $V(x)$ of the game $G(A)$ satisfying $\left(\star^{T_0}\right)$ such that x lies in the open set $S_{n(x)}^{m+1}(V(x))$. For $x \in Z$, let us denote

$$T(x) = \left(B_{i+1}(V(x)), B_{i+2}(V(x)), \ldots, B_{n(x)}(V(x))\right).$$

Now, whenever $y \in S_{n(x)}^{m+1}(V(x))$ for some $x \in Z$ and V' is a run giving y as its outcome and satisfying $(\star^{T_0 \wedge T(x)})$ then V' coincides with $V(x)$ in its first $n(x)$ moves, in particular $S_{n(x)}^{m+1}(V') = S_{n(x)}^{m+1}(V(x))$, and so $y \notin X \setminus \bigcup\limits_{n=m+1}^{\infty} S_n^{m+1}(V')$ and $m+1$ is not a witness of Sisyfos' victory in the run V'. Thus, if $y \in S_{n(x)}^{m+1}(V(x)) \cap M_m^{T_0}$ then also $y \in M_{m+1}^{T_0 \wedge T(x)}$, or equivalently

$$S_{n(x)}^{m+1}(V(x)) \cap M_m^{T_0} \subseteq M_{m+1}^{T_0 \wedge T(x)}.$$

Now, if $\mathcal{B}$ is a σ-discrete basis of open sets in X (whose existence is guaranteed by Theorem 2.7) then the system

$$\mathcal{E}' = \left\{ B \in \mathcal{B} \colon B \subseteq S_{n(x)}^{m+1}(V(x)) \text{ for some } x \in Z \right\}$$

is a σ-discrete covering of Z. We can define

$$\mathcal{E} = \left\{ M_{m+1}^{T_0} \right\} \cup \left\{ E' \cap M_m^{T_0} : E' \in \mathcal{E}' \right\}$$

and

$$N_m^{T_0} = M_m^{T_0} \setminus \left(Z \cup M_{m+1}^{T_0}\right).$$

The system $\mathcal{E}$ is obviously σ-discrete and $M_m^{T_0} = N_m^{T_0} \cup \bigcup \mathcal{E}$. Moreover, if $E \in \mathcal{E}$ then either $E = M_{m+1}^{T_0} = M_{m+1}^{T_0 \wedge \emptyset}$ or $E = E' \cap M_m^{T_0}$ for some $E' \in \mathcal{E}'$ and then there exists $x \in Z$ such that

$$E \subseteq S_{n(x)}^{m+1}(V(x)) \cap M_m^{T_0} \subseteq M_{m+1}^{T_0 \wedge T(x)}.$$

It only remains to show that the set $N_m^{T_0}$ is P-porous. Let us choose $x \in N_m^{T_0}$ arbitrarily. Then $x \in M_m^{T_0} \setminus M_{m+1}^{T_0}$ and so there exists a run V of the game $G(A)$ giving x as its outcome and satisfying $(\star^{T_0})$ such that $m+1$ is a witness of Sisyfos' victory in the run V, in particular

$$P\left(x, X \setminus \bigcup_{n=m+1}^{\infty} S_n^{m+1}(V)\right).$$

But

$$N_m^{T_0} \subseteq X \backslash Z \subseteq X \backslash \bigcup_{n=m+1}^{\infty} S_n^{m+1}(V),$$

and by condition (P1) (see p. 8) we have $P\left(x, N_m^{T_0}\right)$. $\square$

Theorem 3.3. *Sisyfos has a winning strategy in the game $G(A)$ if and only if A is a σ-P-porous set.*

Proof. First, let us assume that $A = \bigcup_{n=1}^{\infty} A_n$ where A_n is a P-porous set for every $n \in \mathbb{N}$. On his nth move, let Sisyfos play $S_n^j = \emptyset$ for $j < n$ and $S_n^n = B_n \backslash \overline{A_n}$. Let Boulder and Sisyfos play a run of the game $G(A)$ such that Sisyfos follows the described strategy. Let $x \in X$ be an outcome of this run. We may assume that $x \in A$ because otherwise Sisyfos wins by condition (i) (see p. 12). Then there exists $m \in \mathbb{N}$ such that $x \in A_m$. We have

$$X \backslash \bigcup_{n=m}^{\infty} S_n^m = \overline{A_m} \cup (X \backslash B_m).$$

Therefore

$$x \in A_m \subseteq X \backslash \bigcup_{n=m}^{\infty} S_n^m.$$

Further, P-porosity of A_m implies that $P(x, A_m)$. But this is equivalent to $P\left(x, \overline{A_m}\right)$ by condition (P3) (see p. 8) and this is equivalent to $P\left(x, \overline{A_m} \cup (X \backslash B_m)\right)$ by condition (P2) (see p. 8) since $x \in B_m$. So we have $P\left(x, X \backslash \bigcup_{n=m}^{\infty} S_n^m\right)$. Therefore, Sisyfos wins by condition (ii) (see p. 12) with m as a witness and the described strategy is winning.

Now, let us assume that Sisyfos has a winning strategy σ in the game $G(A)$. Let us denote $E_0 = A$. By Lemma 3.2, we have

$$A = E_0 = M_0^{\emptyset} = N_0^{\emptyset} \cup \bigcup \mathcal{E} \tag{1}$$

where $N_0^{\emptyset}$ is P-porous and $\mathcal{E}$ is a σ-discrete system of sets such that for every $E_1 \in \mathcal{E}$, there exists a good sequence $T(E_1)$ such that $E_1 \subseteq M_1^{T(E_1)}$. Now, for every $E_1 \in \mathcal{E}$ we have

$$E_1 \subseteq M_1^{T(E_1)} = N_1^{T(E_1)} \cup \bigcup \mathcal{F}^{E_1}$$

where $N_1^{T(E_1)}$ is P-porous and $\mathcal{F}^{E_1}$ is a σ-discrete system of sets such that for every $E_2 \in \mathcal{F}^{E_1}$, there exists a finite sequence $T(E_1, E_2)$ of open balls such that $T(E_1)^\wedge T(E_1, E_2)$ is good and $E_2 \subseteq M_2^{T(E_1)^\wedge T(E_1,E_2)}$. If we denote

$$\mathcal{E}^{E_1} = \left\{E_1 \cap E_2 \colon E_2 \in \mathcal{F}^{E_1}\right\}$$

then we have

$$E_1=\left(E_1\cap N_1^{T(E_1)}\right)\cup\bigcup\mathcal{E}^{E_1}. \tag{2}$$

In the third step, for every $E_1\in\mathcal{E}$ and $E_2\in\mathcal{E}^{E_1}$ we have

$$E_2\subseteq M_2^{T(E_1)^\wedge T(E_1,E_2)}=N_2^{T(E_1)^\wedge T(E_1,E_2)}\cup\bigcup\mathcal{F}^{E_1,E_2}$$

where $N_2^{T(E_1)^\wedge T(E_1,E_2)}$ is P-porous and $\mathcal{F}^{E_1,E_2}$ is a σ-discrete system of sets such that for every $E_3\in\mathcal{F}^{E_1,E_2}$, there exists a finite sequence $T(E_1,E_2,E_3)$ of open balls such that $T(E_1)^\wedge T(E_1,E_2)^\wedge T(E_1,E_2,E_3)$ is good and $E_3\subseteq M_3^{T(E_1)^\wedge T(E_1,E_2)^\wedge T(E_1,E_2,E_3)}$. If we denote

$$\mathcal{E}^{E_1,E_2}=\left\{E_2\cap E_3\colon E_3\in\mathcal{F}^{E_1,E_2}\right\}$$

then we have

$$E_2=\left(E_2\cap N_2^{T(E_1)^\wedge T(E_1,E_2)}\right)\cup\bigcup\mathcal{E}^{E_1,E_2}. \tag{3}$$

By iterating this process, we get a system of P-porous sets

$$\mathcal{U}=\Big\{E_k\cap N_k^{T(E_1)^\wedge T(E_1,E_2)^\wedge\ldots^\wedge T(E_1,E_2,\ldots,E_k)}\colon$$
$$k\in\mathbb{N}\cup\{0\},E_1\in\mathcal{E},E_2\in\mathcal{E}^{E_1},\ldots,E_k\in\mathcal{E}^{E_1,E_2,\ldots,E_{k-1}}\Big\}$$

such that for every $k\in\mathbb{N}\cup\{0\}$ and for every $E_1\in\mathcal{E},E_2\in\mathcal{E}^{E_1},\ldots,E_k\in\mathcal{E}^{E_1,E_2,\ldots,E_{k-1}}$, the sequence $T(E_1)^\wedge T(E_1,E_2)^\wedge\ldots^\wedge T(E_1,E_2,\ldots,E_k)$ is good.

We show that $A\subseteq\bigcup\mathcal{U}$. Suppose that this is not true and so there exist $x\in A\setminus\bigcup\mathcal{U}$. By (1), there exists $E_1\in\mathcal{E}$ such that $x\in E_1$. By (2), there exists $E_2\in\mathcal{E}^{E_1}$ such that $x\in E_2$. By this way (continuing by (3)), we get that there exists a sequence $(E_k)_{k=1}^\infty$ where $E_1\in\mathcal{E}$ and $E_k\in\mathcal{E}^{E_1,E_2,\ldots,E_{k-1}}$ for $k>1$ such that

$$x\in E_k\subseteq M_k^{T(E_1)^\wedge T(E_1,E_2)^\wedge\ldots^\wedge T(E_1,E_2,\ldots,E_k)}$$

for every $k\in\mathbb{N}$. Therefore Boulder can play a run of the game $G(A)$ in the following way. He plays all the balls from $T(E_1)$ in sequence on his first moves, then all the balls from $T(E_1,E_2)$ and so on. (If there exists $k_0\in\mathbb{N}\cup\{0\}$ such that all the sequences $T(E_1,E_2,\ldots,E_k)$, $k>k_0$, are empty then the sequence

$$T(E_1)^\wedge T(E_1,E_2)^\wedge\ldots=T(E_1)^\wedge T(E_1,E_2)^\wedge\ldots^\wedge T(E_1,E_2,\ldots,E_{k_0})$$

is finite. Then Boulder can finish the run arbitrarily such that the outcome of the run is x.) After such a run, x is its outcome and any $m\in\mathbb{N}$ is not a witness of Sisyfos' victory since $x\in M_m^{T(E_1)^\wedge T(E_1,E_2)^\wedge\ldots^\wedge T(E_1,E_2,\ldots,E_m)}$ for every $m\in\mathbb{N}$. This is a contradiction with the assumption that the strategy σ is winning for Sisyfos.

By (P1) (see p. 8), it suffices to show that $\bigcup \mathcal{U}$ is a σ-P-porous set. We have

$$\bigcup \mathcal{U} = \bigcup_{k=0}^{\infty} \bigcup \mathcal{U}_k$$

where

$$\mathcal{U}_k = \big\{ E_k \cap N_k^{T(E_1)^\wedge T(E_1,E_2)^\wedge \ldots ^\wedge T(E_1,E_2,\ldots,E_k)} :$$

$$E_1 \in \mathcal{E}, E_2 \in \mathcal{E}^{E_1}, \ldots, E_k \in \mathcal{E}^{E_1,E_2,\ldots,E_{k-1}} \big\}.$$

We will prove that $\bigcup \mathcal{U}_k$ is a σ-P-porous set for every $k \in \mathbb{N} \cup \{0\}$ which is obviously sufficient. For $k = 0$ we know that $\bigcup \mathcal{U}_0 = N_0^{\emptyset}$ which is a P-porous set. Suppose that $k > 0$ and $E_1 \in \mathcal{E}, E_2 \in \mathcal{E}^{E_1}, \ldots, E_{k-1} \in \mathcal{E}^{E_1,E_2,\ldots,E_{k-2}}$ are fixed. Then

$$C(E_1, E_2, \ldots, E_{k-1}) :=$$

$$\bigcup \left\{ E_k \cap N_k^{T(E_1)^\wedge T(E_1,E_2)^\wedge \ldots ^\wedge T(E_1,E_2,\ldots,E_k)} : E_k \in \mathcal{E}^{E_1,E_2,\ldots,E_{k-1}} \right\}$$

is a union of a σ-discrete system (since $\mathcal{E}^{E_1,E_2,\ldots,E_{k-1}}$ is σ-discrete) of P-porous sets and by Lemma 3.1 it is a σ-P-porous set. Next, if only $E_1 \in \mathcal{E}, E_2 \in \mathcal{E}^{E_1}, \ldots, E_{k-2} \in \mathcal{E}^{E_1,E_2,\ldots,E_{k-3}}$ are fixed, then

$$C(E_1, E_2, \ldots, E_{k-2}) :=$$

$$\bigcup \big\{ E_k \cap N_k^{T(E_1)^\wedge T(E_1,E_2)^\wedge \ldots ^\wedge T(E_1,E_2,\ldots,E_k)} : E_{k-1} \in \mathcal{E}^{E_1,E_2,\ldots,E_{k-2}}, E_k \in \mathcal{E}^{E_1,E_2,\ldots,E_{k-1}} \big\}$$

$$= \bigcup \big\{ C(E_1, E_2, \ldots, E_{k-1}) : E_{k-1} \in \mathcal{E}^{E_1,E_2,\ldots,E_{k-2}} \big\}$$

is a union of a σ-discrete system (indeed, $C(E_1, E_2, \ldots, E_{k-1}) \subseteq E_{k-1}$ and $\mathcal{E}^{E_1,E_2,\ldots,E_{k-2}}$ is σ-discrete) of σ-P-porous sets and by Lemma 3.1 it is σ-P-porous again. Repeating this consideration, we get that only for $E_1 \in \mathcal{E}$ fixed,

$$C(E_1) :=$$

$$\bigcup \big\{ E_k \cap N_k^{T(E_1)^\wedge T(E_1,E_2)^\wedge \ldots ^\wedge T(E_1,E_2,\ldots,E_k)} : E_2 \in \mathcal{E}^{E_1}, E_3 \in \mathcal{E}^{E_1,E_2}, \ldots, E_k \in \mathcal{E}^{E_1,E_2,\ldots,E_{k-1}} \big\}$$

$$= \bigcup \big\{ C(E_1, E_2) : E_2 \in \mathcal{E}^{E_1} \big\}$$

is σ-P-porous as a union of a σ-discrete system of σ-P-porous sets. Finally,

$$\bigcup \mathcal{U}_k = \bigcup \{ C(E_1) : E_1 \in \mathcal{E} \}$$

is σ-P-porous, too. $\square$

This finishes the first main result of my work. The game $G(A)$ is quite simple in comparison with its modification described in the next chapter and should be used instead of this modification whenever possible. Also, the game $G(A)$ can be applied to more porosities than the upcoming modification since there are no restrictions on the porosity-like relation P. Indeed, all commonly used types of porosity (understood as point-set relations in a natural way) are porosity-like relations. Therefore, it is possible to apply Theorem 3.3 to all of them, namely to ordinary porosity, strong porosity, symmetrical porosity, but also right and left porosity (see [10, p. 316]), g-porosity (see [14, p. 35]), etc. However, the upcoming game will be used for inscribing compact and non-σ-P-porous sets in Theorem 4.7.

4. Inscribing compact sets.

Let (K,d) be a nonempty compact metric space. Let R_r^q and R^q, $r>0$, $q\in(0,1]$, be point-set relations on K such that for every $r>0$, $q\in(0,1]$, $A\subseteq K$ and $x\in K$ the following conditions hold:

(R1) $R^q = \bigcap_{0<\tilde{q}<q} \bigcap_{R>0} \bigcup_{0<\tilde{r}\le R} R_{\tilde{r}}^{\tilde{q}}$,
(R2) if $R_r^q(x,A)$ and $0<w<\frac{q}{2}$ then $R_r^{q-2w}(x,B(A,rw))$,
(R3) if $B\subseteq A$ and $R_r^q(x,A)$ then $R_r^q(x,B)$,
(R4) we have $R_r^q(x,A)$ if and only if $R_r^q(x,A\cap B(x,2r))$,
(R5) the set $\{(y,\tilde{r})\in K\times(0,\infty): R_{\tilde{r}}^q(y,A)\}$ is open in $K\times(0,\infty)$.

Claim 4.1. Conditions (R1)-(R5) imply the following:

(S1) if $B\subseteq A$ and $R^q(x,A)$ then $R^q(x,B)$,
(S2) if $0<\tilde{q}<q$ and $R_r^q(x,A)$ then $R_r^{\tilde{q}}(x,A)$.

Proof. Condition (S1) is an immediate consequence of (R1) and (R3).

To verify (S2), let us choose $\tilde{q}\in(0,q)$. By (R2) applied to $w=\frac{q-\tilde{q}}{2}$, we have $R_r^{\tilde{q}}\left(x,B\left(A,\frac{r(q-\tilde{q})}{2}\right)\right)$. By (R3), we have $R_r^{\tilde{q}}(x,A)$. $\square$

Claim 4.2. For every $q\in(0,1]$, the point-set relation R^q is a porosity-like relation.

Proof. Let us fix $q\in(0,1]$. We need to show that R^q satisfies conditions (P1)-(P3) from Definition 2.3.

Condition (P1) is the same as (S1).

Let us prove condition (P2). Suppose that $R^q(x,A\cap B(x,r_0))$ for some $r_0>0$. By (R1), there exist sequences $(q_k)_{k=1}^{\infty}$ of real numbers from $(0,q)$ and $(r_k)_{k=1}^{\infty}$ of real numbers from $(0,\infty)$ such that $\lim_{k\to\infty} q_k = q$, $\lim_{k\to\infty} r_k = 0$ and $R_{r_k}^{q_k}(x,A\cap B(x,r_0))$ for every $k\in\mathbb{N}$. There exists $k_0\in\mathbb{N}$ such that $2r_k\le r_0$ for every $k\ge k_0$. Then $R_{r_k}^{q_k}(x,A\cap B(x,2r_k))$ for $k\ge k_0$ by (R3) and so $R_{r_k}^{q_k}(x,A)$ for $k\ge k_0$ by (R4). Using (R1) and (S2), we get $R^q(x,A)$. The opposite implication follows by (P1).

It remains to verify condition (P3). Let us suppose that $R^q(x,A)$ and choose $\delta>0$ and $0<\varepsilon<q$. By (R1), there exists $0<\tilde{r}<\delta$ such that $R_{\tilde{r}}^{q-\frac{\varepsilon}{2}}(x,A)$. By (R2) applied to $w=\frac{\varepsilon}{4}$, we have $R_{\tilde{r}}^{q-\varepsilon}\left(x,B\left(A,\frac{\tilde{r}\varepsilon}{4}\right)\right)$ and by (R3), it follows that $R_{\tilde{r}}^{q-\varepsilon}\left(x,\overline{A}\right)$. Since δ and ε were chosen arbitrarily, we have (using (R1)) that $R^q\left(x,\overline{A}\right)$. The opposite implication follows by (P1). $\square$

For the rest of this chapter, let us fix $q\in(0,1]$ and sequences $(R_n)_{n=1}^{\infty}$ and $(a_n)_{n=1}^{\infty}$ of real numbers from $(0,\infty)$ such that

$$R_{n+1}\le\frac{1}{2^{n+2}}R_n \quad \text{for every } n\in\mathbb{N} \tag{4}$$

and

$$\lim_{n\to\infty} \frac{a_n}{R_{n+2}} = 0. \tag{5}$$

For $n \in \mathbb{N}$, let D_n be a finite a_n-net in K (i.e. a finite subset of K such that $K = \bigcup \{B(y, a_n) : y \in D_n\}$) and let

$$M_n = \{B(y, a_n) : y \in D_n\}.$$

Let A be an arbitrary subset of K. We define a game $H(A)$ between Boulder and Sisyfos as follows:

Boulder	B_1		B_2		B_3		$\ldots$
Sisyfos		(S_1^1)		(S_2^1, S_2^2)		(S_3^1, S_3^2, S_3^3)	

On the first move, Boulder plays an open ball $B_1 \subseteq K$ with radius R_1 and Sisyfos plays an open set $S_1^1 \subseteq B_1$ where S_1^1 is a union (possibly empty) of some balls from M_1. On the second move, Boulder plays an open ball B_2 with center in $\frac{1}{2} \star B_1$ and radius R_2 and Sisyfos plays two open sets S_2^1 and S_2^2 such that $S_2^1 \cup S_2^2 \subseteq B_2$ where S_2^j is a union of some balls from M_2, $j = 1, 2$. On the nth move, $n > 1$, Boulder plays an open ball B_n with center in $\left(1 - \frac{1}{2^{n-1}}\right) \star B_{n-1}$ and radius R_n and Sisyfos replies by playing open sets $S_n^1, S_n^2, \ldots, S_n^n$ such that $\bigcup_{j=1}^{n} S_n^j \subseteq B_n$ where S_n^j is a union of some balls from M_n, $j = 1, 2, \ldots, n$.

By (4), we have $\lim_{n\to\infty} \operatorname{diam} B_n = 0$. By this fact and the compactness of K, when a run of the game is over, we get a unique point x lying in the intersection of the balls B_n, $n \in \mathbb{N}$, played by Boulder. We call this point an *outcome* of the run. Sisyfos wins if at least one of the following conditions is satisfied:

(a) $x \notin A$,

(b) there exist $m \in \mathbb{N}$ and sequences $(n_k)_{k=1}^{\infty}$ of integers from $\{m, m+1, \ldots\}$, $(q_k)_{k=1}^{\infty}$ of real numbers from $(0, 1)$ and $(r_k)_{k=1}^{\infty}$ of real numbers from $(0, \infty)$ such that

- $x \in K \setminus \bigcup_{n=m}^{\infty} S_n^m$,
- $\lim_{k\to\infty} n_k = \infty$,
- $\lim_{k\to\infty} q_k = q$,
- $r_k \le \frac{R_{n_k}}{2^{n_k+3}}$, $k \in \mathbb{N}$,
- $R_{r_k}^{q_k}\left(x, K \setminus S_{n_k}^m\right)$, $k \in \mathbb{N}$.

Boulder wins in the opposite case. If condition (b) is satisfied for some $m \in \mathbb{N}$, then every such m is called a *witness of Sisyfos' victory*.

On the first view, condition (b) looks very complicated. For a better understanding, we can observe that it is stronger than the assertion that $R^q\left(x, K\setminus\bigcup\limits_{n=m}^{\infty} S_n^m\right)$ by (R1), (R3) and (S2).

Claim 4.3. For every $n\in\mathbb{N}$, we have $B_{n+1}\subseteq\left(1-\frac{1}{2^{n+1}}\right)\star B_n$.

Proof. Suppose that x_n is the center of B_n, x_{n+1} is the center of B_{n+1} and $z\in B_{n+1}$. Then we have

$$\begin{aligned} d\left(z,x_n\right) &\leq d\left(z,x_{n+1}\right)+d\left(x_{n+1},x_n\right) < R_{n+1}+\left(1-\frac{1}{2^n}\right)R_n \\ &\leq \left(\frac{1}{2^{n+2}}+1-\frac{1}{2^n}\right)R_n=\left(1-\frac{3}{2^{n+2}}\right)R_n<\left(1-\frac{1}{2^{n+1}}\right)R_n. \end{aligned}$$

$\square$

We say that a finite (also empty) sequence of open balls $(B_1,B_2,\ldots,B_i)$ is *good* if the rules of the game $H(A)$ allow Boulder to play the ball B_n on his nth move, $n=1,2,\ldots,i$. (This is independent of Sisyfos' moves.)

For $n\in\mathbb{N}$ and $m\in\mathbb{N}$, let us define

$$d_n^m=\begin{cases}1-\frac{1}{2^{n-m+1}} & \text{if } m\leq n,\\ \frac{1}{4} & \text{if } m>n.\end{cases}$$

Let σ be a strategy for Sisyfos in the game $H(A)$. If $k\in\mathbb{N}\cup\{0\}$ and $l\in\mathbb{N}$ then we say that a good sequence of open balls $(B_1,B_2,\ldots,B_i)$ is (k,l)-*good* (with respect to the strategy σ) if there exists a run of the game $H(A)$ such that the following conditions hold:

- Sisyfos followed the strategy σ,
- Boulder played the ball B_n on his nth move, $n=1,2,\ldots,i$,
- the following conditions are satisfied for every positive $n\in\{k,k+1,\ldots,i-1\}$:
  - (H1) if $\left[l>n \text{ or } \left(l\leq n \text{ and } S_n^l\cap\left(d_n^l\star B_n\right)=\emptyset\right)\right]$ then the center of B_{n+1} lies in $d_n^{l+1}\star B_n$,
  - (H2) if $\left[l\leq n \text{ and } S_n^l\cap\left(d_n^l\star B_n\right)\neq\emptyset\right]$ then the center of B_{n+1} lies in $d_n^l\star B_n$.

Let Boulder and Sisyfos play a run of the game $H(A)$. Let

$$V=\left(B_1,\mathcal{S}_1,B_2,\mathcal{S}_2,\ldots\right),$$
$$\mathcal{S}_n=\left(S_n^1,S_n^2,\ldots,S_n^n\right),\ n\in\mathbb{N},$$

where Boulder played the ball B_n and Sisyfos played the sets $S_n^1,S_n^2,\ldots,S_n^n$ on the nth move of the run, $n\in\mathbb{N}$. Then we will refer to the run itself by V and if we talk about the ball B_n or about the set S_n^m, $m\in\{1,2,\ldots,n\}$, $n\in\mathbb{N}$, we just use the symbols $B_n(V)$ and $S_n^m(V)$, respectively.

We say that a run V of the game $H(A)$ is (k,l)-*good* if Sisyfos followed the strategy σ and the sequence $(B_1(V), B_2(V), \dots, B_j(V))$ is (k,l)-good for every $j \in \mathbb{N}$.

It is easy to see that if $l_1 > l_2$ and a finite sequence of open balls (resp. a run of the game $H(A)$) is (k,l_1)-good then it is also (k,l_2)-good.

If $T = (B_1, B_2, \dots, B_i)$ is a good sequence of open balls, we say that a run V of the game $H(A)$ satisfies condition $(\star^T)$ if $B_n(V) = B_n$ for every $n \in \{1, 2, \dots, i\}$.

For $m \in \mathbb{N} \cup \{0\}$ and a good sequence of open balls $T = (B_1, B_2, \dots, B_i)$, we denote by M_m^T the set of all

$$x \in \begin{cases} A & \text{if } i = 0 \\ A \cap \left(\frac{1}{4} \star B_i\right) & \text{if } i > 0 \end{cases}$$

such that in every $(i, m+1)$-good run of the game $H(A)$ giving x as its outcome and satisfying condition $(\star^T)$, all the witnesses of Sisyfos' victory (if there exist any) are greater than m. (As in Chapter 3, the set M_m^T depends on the set A and on the strategy σ but these will be always fixed.)

Lemma 4.4. *Let σ be a strategy for Sisyfos in the game $H(A)$. Let*

$$T_0 = (B_1, B_2, \dots, B_i)$$

be a good sequence of open balls and let $m \in \mathbb{N} \cup \{0\}$. Then there exist an R^q-porous set $N_m^{T_0}$ and an at most countable collection $\mathcal{T}$ of finite sequences of open balls such that $T_0{}^\wedge T$ is $(i, m+1)$-good for every $T \in \mathcal{T}$ and

$$M_m^{T_0} \subseteq N_m^{T_0} \cup \bigcup \left\{ M_{m+1}^{T_0{}^\wedge T} : T \in \mathcal{T} \right\}.$$

Proof. Define $N_m^{T_0}$ as the set of all $x \in M_m^{T_0}$ such that

(I) there exists an $(i, m+2)$-good run of the game $H(A)$ giving x as its outcome and satisfying $(\star^{T_0})$ such that $m+1$ is a witness of Sisyfos' victory,

(II) for every $(i, m+2)$-good run V of the game $H(A)$ satisfying $(\star^{T_0})$ and for every $n \geq \max\{i, m+1\}$, we have $x \notin S_n^{m+1}(V) \cap (d_n^{m+1} \star B_n(V))$.

Suppose that $x \in M_m^{T_0} \setminus \left(M_{m+1}^{T_0} \cup N_m^{T_0}\right)$. Then condition (I) holds for x by the definitions of $M_m^{T_0}$ and $M_{m+1}^{T_0}$. Therefore condition (II) cannot be true by the definition of $N_m^{T_0}$, and so there exist an $(i, m+2)$-good run $V(x)$ of the game $H(A)$ satisfying $(\star^{T_0})$ and $n(x) \geq \max\{i, m+1\}$ such that $x \in S_{n(x)}^{m+1}(V(x)) \cap \left(d_{n(x)}^{m+1} \star B_{n(x)}(V(x))\right)$. Denote $B_j(x) = B(x, R_j)$ for $j > n(x)$. Find $N(x) > n(x)$ such that $B_{N(x)}(x) \subseteq S_{n(x)}^{m+1}(V(x))$ and denote

$$T(x) = \left(B_{i+1}(V(x)), \dots, B_{n(x)}(V(x))\right)^\wedge \left(B_{n(x)+1}(x), \dots, B_{N(x)}(x)\right).$$

Then the sequence $T_0{}^\wedge T(x)$ is $(i, m+1)$-good. (Indeed, the sequence

$$T_0{}^\wedge \left(B_{i+1}(V(x)), \dots, B_{n(x)}(V(x))\right)$$

is even $(i, m+2)$-good and the fact that

$$S_{n(x)}^{m+1}(V(x)) \cap \left(d_{n(x)}^{m+1} \star B_{n(x)}(V(x))\right) \neq \emptyset$$

allows Boulder to use condition (H2) (see p. 21) and play the ball with center $x \in d_{n(x)}^{m+1} \star B_{n(x)}(V(x))$ on his $(n(x)+1)$st move.) Since $B_{N(x)}(x) \subseteq S_{n(x)}^{m+1}(V(x))$, we know that $m+1$ cannot become a witness of Sisyfos' victory in any run of the game $H(A)$ satisfying $(\star^{T_0{}^\wedge T(x)})$. Therefore we have

$$M_m^{T_0} \cap \left(\frac{1}{4} \star B_{N(x)}(x)\right) \subseteq M_{m+1}^{T_0{}^\wedge T(x)},$$

and so $x \in M_{m+1}^{T_0{}^\wedge T(x)}$. By Lindelöf's property, there exists an at most countable set

$$\{x_j \colon j \in \mathbb{N}\} \subseteq M_m^{T_0} \setminus \left(M_{m+1}^{T_0} \cup N_m^{T_0}\right)$$

such that $M_m^{T_0} \setminus \left(M_{m+1}^{T_0} \cup N_m^{T_0}\right)$ is covered by the system

$$\left\{\frac{1}{4} \star B_{N(x_j)}(x_j) \colon j \in \mathbb{N}\right\}$$

of open sets and so it is also covered by the countable system

$$\left\{M_{m+1}^{T_0{}^\wedge T(x_j)} \colon j \in \mathbb{N}\right\}.$$

Now, we can define

$$\mathcal{T} = \{\emptyset\} \cup \{T(x_j) \colon j \in \mathbb{N}\}.$$

Then we obviously have

$$M_m^{T_0} \subseteq N_m^{T_0} \cup \bigcup \left\{M_{m+1}^{T_0{}^\wedge T} \colon T \in \mathcal{T}\right\}.$$

It remains to show that $N_m^{T_0}$ is R^q-porous. Suppose that $x \in N_m^{T_0}$ and V is an $(i, m+2)$-good run of the game $H(A)$ satisfying $(\star^{T_0})$ such that x is its outcome and $m+1$ is a witness of Sisyfos' victory. We know that there exist sequences $(n_k)_{k=1}^{\infty}$ of integers from $\{m+1, m+2, \ldots\}$, $(q_k)_{k=1}^{\infty}$ of real numbers from $(0,1)$ and $(r_k)_{k=1}^{\infty}$ of real numbers from $(0, \infty)$ such that

- $x \in K \setminus \bigcup\limits_{n=m+1}^{\infty} S_n^{m+1}(V)$,
- $\lim\limits_{k\to\infty} n_k = \infty$,
- $\lim\limits_{k\to\infty} q_k = q$,
- $r_k \leq \frac{R_{n_k}}{2^{n_k+3}}$, $k \in \mathbb{N}$,
- $R_{r_k}^{q_k}\left(x, K \setminus S_{n_k}^{m+1}(V)\right)$, $k \in \mathbb{N}$.

We may assume that $n_k \geq \max\{i, m+2\}$ for every $k \in \mathbb{N}$. We know that the center of $B_{n+1}(V)$ lies in $d_n^{m+2} \star B_n(V)$ for every $n \geq i$ by conditions (H1) and (H2) (see p. 21). Let us fix $k \in \mathbb{N}$. By condition (R4), we have

$$R_{r_k}^{q_k}\left(x, K \backslash \left(S_{n_k}^{m+1}(V) \cap B(x, 2r_k)\right)\right). \tag{6}$$

By condition (II), we have

$$N_m^{T_0} \subseteq K \backslash \left(S_{n_k}^{m+1}(V) \cap \left(d_{n_k}^{m+1} \star B_{n_k}(V)\right)\right). \tag{7}$$

Now, let x_{n_k} be the center of $B_{n_k}(V)$, x_{n_k+1} be the center of $B_{n_k+1}(V)$ and let us take $z \in B(x, 2r_k)$. Then we have

$$\begin{aligned} d(z, x_{n_k}) &\leq d(z, x) + d(x, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) < 2r_k + R_{n_k+1} + d_{n_k}^{m+2} R_{n_k} \\ &\leq \frac{1}{2^{n_k+2}} R_{n_k} + \frac{1}{2^{n_k+2}} R_{n_k} + d_{n_k}^{m+2} R_{n_k} = \left(\frac{1}{2^{n_k+1}} + 1 - \frac{1}{2^{n_k-m-1}}\right) R_{n_k} \\ &= \left(1 - \frac{2^{m+2}-1}{2^{n_k+1}}\right) R_{n_k} \leq \left(1 - \frac{1}{2^{n_k-m}}\right) R_{n_k} = d_{n_k}^{m+1} R_{n_k}. \end{aligned}$$

Therefore we have $B(x, 2r_k) \subseteq d_{n_k}^{m+1} \star B_{n_k}(V)$, and so

$$K \backslash \left(S_{n_k}^{m+1}(V) \cap \left(d_{n_k}^{m+1} \star B_{n_k}(V)\right)\right) \subseteq K \backslash \left(S_{n_k}^{m+1}(V) \cap B(x, 2r_k)\right). \tag{8}$$

Finally, we have $R_{r_k}^{q_k}\left(x, N_m^{T_0}\right)$ by (6), (7), (8) and (R3). Therefore also $R^q\left(x, N_m^{T_0}\right)$ by (R1) and (S2). $\square$

Theorem 4.5. *Sisyfos has a winning strategy in the game $H(A)$ if and only if the set A is σ-R^q-porous.*

Proof. Suppose first that $A = \bigcup_{n=1}^{\infty} A_n$ where $R^q(A_n, A_n)$, $n \in \mathbb{N}$. For $n \in \mathbb{N}$ and $m \in \{1, 2, \ldots, n\}$, let Sisyfos play S_n^m as the union of all balls $B \in M_n$ for which $B \subseteq B_n \backslash A_m$. Let Boulder and Sisyfos play a run of the game $H(A)$ such that Sisyfos follows this strategy. Let x be an outcome of this run. If $x \notin A$ then Sisyfos satisfies condition (a) (see p. 20) and wins. If $x \in A$ then there exists $m \in \mathbb{N}$ such that $x \in A_m$. Then we have

$$x \notin \bigcup_{n=m}^{\infty} S_n^m.$$

Further, since $R^q(x, A_m)$, we know by condition (R1) that there exist sequences $(q_k)_{k=1}^{\infty}$ of real numbers from $(0,1)$ and $(r_k)_{k=1}^{\infty}$ of real numbers from $(0, \infty)$ such that

- $\lim_{k \to \infty} q_k = q$,
- $\lim_{k \to \infty} r_k = 0$,
- $R_{r_k}^{q_k}(x, A_m)$, $k \in \mathbb{N}$.

There also exists $n_0 \geq m$ such that

$$\frac{2^{n+6}a_n}{R_{n+1}} \leq \inf\{q_k\colon k \in \mathbb{N}\} \tag{9}$$

for $n \geq n_0$ since the expression on the right side is strictly positive and the expression on the left side tends to zero which follows from (5) and the estimate (derived from (4))

$$0 < \frac{2^{n+6}a_n}{R_{n+1}} \leq \frac{8a_n}{R_{n+2}}. \tag{10}$$

We may assume that

$$r_k \leq \frac{R_{n_0}}{2^{n_0+3}}$$

for every $k \in \mathbb{N}$. Let us choose $k \in \mathbb{N}$ and define n_k $(\geq n_0)$ as the greatest integer such that

$$r_k \leq \frac{R_{n_k}}{2^{n_k+3}}. \tag{11}$$

(Obviously, $\lim\limits_{k\to\infty} n_k = \infty$.) Since the previous inequality does not hold for n_k+1 instead of n_k, we get

$$r_k > \frac{R_{n_k+1}}{2^{n_k+4}} \geq \frac{4a_{n_k}}{q_k} \tag{12}$$

(we used estimate (9) for $n = n_k$ in the second inequality). It follows that

$$\frac{q_k}{2} > \frac{2a_{n_k}}{r_k} > 0.$$

By condition (R2) applied to $w = \frac{2a_{n_k}}{r_k}$, we have

$$R_{r_k}^{q_k - \frac{4a_{n_k}}{r_k}}\left(x, B\left(A_m, 2a_{n_k}\right)\right). \tag{13}$$

Let us denote $\tilde{q}_k = q_k - \frac{4a_{n_k}}{r_k}$. Using the first inequality from estimate (12), we get

$$0 \leq \frac{4a_{n_k}}{r_k} \leq \frac{2^{n_k+6}a_{n_k}}{R_{n_k+1}}. \tag{14}$$

By (5), (10) and (14), we have

$$\lim_{n\to\infty} \frac{4a_{n_k}}{r_k} = 0$$

and so

$$\lim_{k\to\infty} \tilde{q}_k = \lim_{k\to\infty} q_k - \lim_{k\to\infty} \frac{4a_{n_k}}{r_k} = q.$$

To verify condition (b) (see p. 20), it suffices to show that $R_{r_k}^{\tilde{q}_k}\left(x, K \backslash S_{n_k}^m\right)$, $k \in \mathbb{N}$. Let us fix $k \in \mathbb{N}$ and suppose that $z \in B\left(x, 2r_k\right) \backslash B\left(A_m, 2a_{n_k}\right)$. Then

$$B\left(z, 2a_{n_k}\right) \subseteq K \backslash A_m \tag{15}$$

by the definition of $B\left(A_m, 2a_{n_k}\right)$. Let us denote the center of B_{n_k} by x_{n_k}. If we use

- Claim 4.3 and the fact that $x \in B_{n_k+1}$ (in the second inequality of the upcoming estimate),
- an immediate consequence of (12) that $a_{n_k} \leq r_k$ (in the third inequality),
- estimate (11) (in the fourth inequality),

then we have for arbitrary $y \in B\left(z, 2a_{n_k}\right)$ the following:

$$\begin{aligned} d\left(y, x_{n_k}\right) &\leq d\left(y, z\right) + d\left(z, x\right) + d\left(x, x_{n_k}\right) < 2a_{n_k} + 2r_k + \left(1 - \frac{1}{2^{n_k+1}}\right) R_{n_k} \\ &\leq 4r_k + \left(1 - \frac{1}{2^{n_k+1}}\right) R_{n_k} \leq \frac{1}{2^{n_k+1}} R_{n_k} + \left(1 - \frac{1}{2^{n_k+1}}\right) R_{n_k} = R_{n_k}. \end{aligned}$$

This gives us the inclusion

$$B\left(z, 2a_{n_k}\right) \subseteq B_{n_k}. \tag{16}$$

By putting (15) and (16) together, we get

$$B\left(z, 2a_{n_k}\right) \subseteq B_{n_k} \backslash A_m$$

and it easily follows from the definitions of D_{n_k} and M_{n_k} that $z \in S_{n_k}^m$. So we have

$$B\left(x, 2r_k\right) \backslash B\left(A_m, 2a_{n_k}\right) \subseteq S_{n_k}^m$$

and thus

$$B\left(x, 2r_k\right) \backslash S_{n_k}^m \subseteq B\left(A_m, 2a_{n_k}\right). \tag{17}$$

By (13), (17) and (R3), we get

$$R_{r_k}^{\tilde{q}_k}\left(x, B\left(x, 2r_k\right) \backslash S_{n_k}^m\right).$$

By (R4), this is equivalent to

$$R_{r_k}^{\tilde{q}_k}\left(x, K \backslash S_{n_k}^m\right)$$

as we wanted.

Now, let us assume that Sisyfos has a winning strategy σ in the game $H(A)$ and that he follows this strategy in every run of he game $H\left(A\right)$. We have $A = M_0^{\emptyset}$ and by Lemma 4.4, it follows

$$A = M_0^{\emptyset} \subseteq N_0^{\emptyset} \cup \bigcup \left\{M_1^{T_1}; T_1 \in \mathcal{T}\right\} \tag{18}$$

where $N_0^{\emptyset}$ is R^q-porous and $\mathcal{T}$ is an at most countable collection of $(0,1)$-good sequences of open balls. Now, for every $T_1 \in \mathcal{T}$ we have

$$M_1^{T_1} \subseteq N_1^{T_1} \cup \bigcup \left\{ M_2^{T_1{}^\wedge T_2}; T_2 \in \mathcal{T}^{T_1} \right\} \tag{19}$$

where $N_1^{T_1}$ is R^q-porous and $\mathcal{T}^{T_1}$ is an at most countable collection of finite sequences of open balls such that $T_1{}^\wedge T_2$ is $(\operatorname{length}(T_1), 2)$-good for every $T_2 \in \mathcal{T}^{T_1}$. By iterating this process, we get a countable system of R^q-porous sets

$$\mathcal{U} = \left\{ N_k^{T_1{}^\wedge T_2{}^\wedge \ldots {}^\wedge T_k} \colon k \in \mathbb{N} \cup \{0\}, T_1 \in \mathcal{T}, T_2 \in \mathcal{T}^{T_1}, \ldots, T_k \in \mathcal{T}^{T_1, \ldots, T_{k-1}} \right\}$$

such that for every $k \in \mathbb{N} \cup \{0\}$ and for every $T_1 \in \mathcal{T}, T_2 \in \mathcal{T}^{T_1}, \ldots, T_k \in \mathcal{T}^{T_1, T_2, \ldots, T_{k-1}}$, the sequence $T_1{}^\wedge T_2{}^\wedge \ldots {}^\wedge T_k$ is $(\operatorname{length}(T_1{}^\wedge T_2{}^\wedge \ldots {}^\wedge T_{k-1}), k)$-good. By (S1), it suffices to show that $A \subseteq \bigcup \mathcal{U}$. Suppose that this is not true and so there exists $x \in A \setminus \bigcup \mathcal{U}$. By (18), there exists $T_1 \in \mathcal{T}$ such that $x \in M_1^{T_1}$. By (19), there exists $T_2 \in \mathcal{T}^{T_1}$ such that $x \in M_2^{T_1{}^\wedge T_2}$. In this way, we get that there exists a sequence $(T_k)_{k=1}^{\infty}$ where $T_1 \in \mathcal{T}$ and $T_k \in \mathcal{T}^{T_1, T_2, \ldots, T_{k-1}}$ for $k > 1$ such that $x \in M_k^{T_1{}^\wedge T_2{}^\wedge \ldots {}^\wedge T_k}$ for every $k \in \mathbb{N}$. Therefore Boulder can play all the balls from T_1 in sequence on his first moves of the game $H(A)$, then all the balls from T_2 and so on. (If there exists $k_0 \in \mathbb{N} \cup \{0\}$ such that all the sequences T_k, $k > k_0$, are empty then the sequence

$$T_1{}^\wedge T_2{}^\wedge \ldots = T_1{}^\wedge T_2{}^\wedge \ldots {}^\wedge T_{k_0}$$

is finite. Then Boulder can finish the run such that the center of all the remaining balls is x. The outcome of such a run is x. Moreover, since $x \in M_k^{T_1{}^\wedge T_2{}^\wedge \ldots {}^\wedge T_{k_0}}$, we have $x \in \frac{1}{4} \star B_{k_0}$. It follows that the run is $(\operatorname{length}(T_1{}^\wedge T_2{}^\wedge \ldots {}^\wedge T_{k_0}), m+1)$-good for every $m \in \mathbb{N}$.) Then, x is the outcome of the run and any $m \in \mathbb{N}$ cannot be a witness of Sisyfos' victory since $x \in M_m^{T_1{}^\wedge T_2{}^\wedge \ldots {}^\wedge T_m}$ and the run is $(\operatorname{length}(T_1{}^\wedge T_2{}^\wedge \ldots {}^\wedge T_m), m+1)$-good for every $m \in \mathbb{N}$. This is a contradiction since the strategy σ is winning for Sisyfos. $\square$

Theorem 4.6. *Let (K,d) be a nonempty compact metric space and let $A \subseteq K$ be a Borel set. Then the game $H(A)$ is determined.*

Proof. On his nth move, Boulder plays an open ball with radius R_n. Since we identify every open ball with the pair (center, radius) (see p. 7), this is the same as choosing the center of the ball which is an element of K. Thus, we may assume that Boulder plays $x_n \in K$ (which is the center of B_n) on his nth move. Meanwhile, Sisyfos plays an element of $\left(2^K\right)^n$ on his nth move. If we denote the tree of all legal positions of the game $H(A)$ by T then the payoff set P for the game $H(A)$ is the set of all $t \in [T]$ of the form

$$t = \left(x_1, \left(S_1^1\right), x_2, \left(S_2^1, S_2^2\right), \ldots\right) \tag{20}$$

such that neither of the conditions (a) and (b) (see p. 20) is satisfied for t. Let us consider the discrete topology on K and on $\left(2^K\right)^n$ for every $n \in \mathbb{N}$. Then $[T]$ is a subset of

$$K \times 2^K \times K \times \left(2^K\right)^2 \times \dots$$

which will be considered as a topological space with the product topology. By Theorem 2.5, it is sufficient to show that the payoff set P is Borel.

We define mappings $f\colon [T] \to K$ and $h_n^j\colon [T] \to 2^K$, $j \in \{1, 2, \dots, n\}$, $n \in \mathbb{N}$, such that for $t \in [T]$ of the form (20) and $j, n \in \mathbb{N}$, we have:

- $f(t) \in \bigcap\limits_{n=1}^{\infty} B_n$ (i.e. $f(t)$ is the outcome of the appropriate run),
- $h_n^j(t) = S_n^j$.

Let us choose $n \in \mathbb{N}$ arbitrarily. When the beginnings, sufficiently long, of two sequences $t_1 \in [T]$ and $t_2 \in [T]$ coincide then both $f(t_1)$ and $f(t_2)$ lie in the same ball B_n played by Boulder. This means that $d\left(f(t_1), f(t_2)\right) \leq 2R_n$. Since $\lim\limits_{n\to\infty} R_n = 0$, it follows that the mapping f is continuous from $[T]$ to (K, d).

The mapping h_n^j is also continuous since its values depend only on the projection of t to $\left(2^K\right)^n$ (with the discrete topology).

Next, we define

$$T_m = \big\{t \in [T]\colon \ m \text{ is a witness of Sisyfos' victory}$$

$$\text{in the run of the game } H\left(A\right) \text{ which corresponds to } t\big\}.$$

Then we have

$$P = f^{-1}\left(A\right) \setminus \bigcup_{m=1}^{\infty} T_m.$$

The set $f^{-1}\left(A\right)$ is a continuous preimage of a Borel set and so it is Borel.

To finish the proof, it remains to show that T_m is a Borel set for every $m \in \mathbb{N}$. Let us fix $m \in \mathbb{N}$. After taking into consideration (S2) and (R5) (see p. 19), we have $t \in T_m$ if and only if

- $f(t) \in K \setminus \bigcup\limits_{n=m}^{\infty} h_n^m(t)$ and
- for every $k \in \mathbb{N}$ there exist $n_k \geq \max\{m, k\}$, $q_k \in (q - \frac{1}{k}, q + \frac{1}{k}) \cap (0,1) \cap \mathbb{Q}$ and $r_k \in \left(0, \frac{R_{n_k}}{2^{n_k+3}}\right) \cap \mathbb{Q}$ such that $R_{r_k}^{q_k}\left(f(t), K \setminus h_{n_k}^m(t)\right)$.

Further, we have $f(t) \in K \backslash \bigcup_{n=m}^{\infty} h_n^m(t)$ if and only if

$$t \in \bigcap_{n=m}^{\infty} \bigcup_{\substack{G \text{ is a union} \\ \text{of some balls} \\ \text{from } M_n}} \left((h_n^m)^{-1}(G) \cap f^{-1}(K \backslash G) \right)$$

and the set on the right side is G_δ since

- $f^{-1}(K \backslash G)$ is closed as a continuous preimage of a closed set,
- $(h_n^m)^{-1}(G)$ is open as a continuous preimage of an open set,
- the set M_n is finite for every $n \in \mathbb{N}$.

Finally, we have $R_{r_k}^{q_k}\left(f(t), K \backslash h_{n_k}^m(t)\right)$ if and only if

$$t \in \bigcup_{\substack{G \text{ is a union} \\ \text{of some balls} \\ \text{from } M_{n_k}}} \left(\left(h_{n_k}^m\right)^{-1}(G) \cap f^{-1}\left(\{y \in K \colon R_{r_k}^{q_k}(y, K \backslash G)\}\right) \right)$$

and the last set is open by (R5). □

Theorem 4.7. *Let (K, d) be a nonempty compact metric space and let $A \subseteq K$ be a Borel set which is not σ-R^q-porous. Then there exists a compact set $F \subseteq A$ which is not σ-R^q-porous.*

Proof. Sisyfos does not have a winning strategy in the game $H(A)$ by Theorem 4.5. But by Theorem 4.6, the game is determined and so Boulder has a winning strategy μ. The fact that Sisyfos has only finitely many possible choices on each of his moves of the game $H(A)$ easily implies that the body $[\mu]$ is compact in the topology derived from the topological space

$$K \times 2^K \times K \times \left(2^K\right)^2 \times \dots$$

with the topology described in Theorem 4.6. Every $u \in [\mu]$ corresponds to some run V_u of the game $H(A)$ (won by Boulder) in a natural way. We can define a mapping $\varphi \colon [\mu] \to K$ assigning an outcome of the run V_u to $u \in [\mu]$. The mapping φ is continuous since it is a restriction of the continuous mapping $f \colon [T] \to K$ from the proof of Theorem 4.6. Define F as the compact set $\varphi([\mu])$. Then F is a subset of A by condition (a) (see p. 20) because the strategy μ is winning for Boulder.

It remains to show that F is not σ-R^q-porous. Since satisfying of condition (b) (see p. 20) does not depend on the set which the game is played with, it is obvious that μ is a winning strategy for Boulder also in the game $H(F)$. Therefore Sisyfos does not have a winning strategy in the game $H(F)$ and using Theorem 4.5 again, we get the conclusion. □

5. Applications to porosities.

In the following two theorems, we use Theorem 4.7 to prove the existence of non-σ-porous (resp. non-σ-strongly porous) compact subset of a given non-σ-porous (resp. non-σ-strongly porous) Borel set in any locally compact metric space. As it was already described, Theorem 5.1 was already known (it was proved in [13] for the first time) but Theorem 5.2 provides a new result.

Theorem 5.1 ([13, Theorem 3.1])**.** *Let (X,d) be a locally compact metric space. Let $A \subseteq X$ be a non-σ-porous Borel set. Then there exists a non-σ-porous compact set $F \subseteq A$.*

Proof. First, suppose that the space (X,d) is compact. We define point-set relations R_r^q and R^q, $r>0$, $q \in (0,1]$, on X such that for every $r>0$, $q \in (0,1]$, $M \subseteq X$ and $x \in X$ we have

- $R_r^q(x,M)$ if there exists an open ball $B(y,\tilde{r})$ such that $x \in \left(B\left(y,r\right) \setminus \overline{B}(y,\frac{r}{2})\right) \cap B\left(y,\frac{\tilde{r}}{q}\right)$ and $B(y,\tilde{r}) \cap M = \emptyset$,
- $R^q = \bigcap_{0<\tilde{q}<q} \bigcap_{R>0} \bigcup_{0<r\leq R} R_r^{\tilde{q}}$.

The relations R_r^q and R^q, $r>0$, $q \in (0,1]$, satisfy conditions (R1)-(R5) (see p. 19). Let us verify only (R2) and (R4), the other conditions are easy to check.

First, we verify condition (R2). Let us take $r>0$, $q \in (0,1]$, $M \subseteq X$, $x \in X$ and $0<w<\frac{q}{2}$ and suppose that $R_r^q(x,M)$. We want to show that $R_r^{q-2w}\left(x, B\left(M,rw\right)\right)$. We know that there exists an open ball $B(y,\tilde{r})$ such that

$$x \in \left(B\left(y,r\right) \setminus \overline{B}(y,\frac{r}{2})\right) \cap B\left(y,\frac{\tilde{r}}{q}\right)$$

and

$$B(y,\tilde{r}) \cap M = \emptyset.$$

So we have

$$\frac{\tilde{r}}{q} > d(x,y) > \frac{r}{2} \tag{21}$$

and so

$$\tilde{r} - rw > r\left(\frac{q}{2} - w\right) > 0.$$

Since clearly

$$B\left(y,\tilde{r}-rw\right) \cap B\left(M,rw\right) = \emptyset,$$

it suffices to show that $x \in B\left(y, \frac{\tilde{r}-rw}{q-2w}\right)$. But indeed, by (21) we have

$$\frac{\tilde{r}-rw}{q-2w} > \frac{\tilde{r}\left(1-\frac{2w}{q}\right)}{q-2w} = \frac{\tilde{r}}{q} > d(x,y).$$

Now, we verify condition (R4). Let us assume that $r>0$, $q \in (0,1]$, $M \subseteq X$ and $x \in X$ such that $R_r^q\left(x, M \cap B\left(x, 2r\right)\right)$. Then there exists an open ball $B\left(y, \tilde{r}\right)$ such that

$$x \in \left(B\left(y,r\right) \setminus \overline{B}(y, \frac{r}{2})\right) \cap B\left(y, \frac{\tilde{r}}{q}\right)$$

and

$$B(y,\tilde{r}) \cap M \cap B\left(x, 2r\right) = \emptyset.$$

First, let us assume that $\tilde{r} \le r$. If $z \in B\left(y, \tilde{r}\right)$ then

$$d\left(z,x\right) \le d\left(z,y\right) + d\left(y,x\right) < \tilde{r} + r \le 2r.$$

So we have $B\left(y,\tilde{r}\right) \subseteq B\left(x, 2r\right)$ and therefore

$$B(y,\tilde{r}) \cap M = B(y,\tilde{r}) \cap M \cap B\left(x,2r\right) = \emptyset.$$

It follows that $R_r^q\left(x, M\right)$. Now, let us assume that $\tilde{r} > r$. Then we have

$$B\left(y,r\right) \cap M = B\left(y,r\right) \cap M \cap B\left(x,2r\right) \subseteq B(y,\tilde{r}) \cap M \cap B\left(x,2r\right) = \emptyset$$

and the open ball $B\left(y,r\right)$ witnesses that $R_r^q\left(x,M\right)$. The other implication of condition (R4) is obvious.

It is also straightforward to verify that $M \subset X$ is q-porous at $x \in X$ if and only if M is R^q-porous at x, $q \in (0,1]$. Moreover, using Theorem 2.2, we know that $M \subset X$ is σ-porous if and only if M is σ-$R^{\frac{1}{2}}$-porous. Therefore, A is not σ-$R^{\frac{1}{2}}$-porous and by Theorem 4.7, there exists a non-σ-$R^{\frac{1}{2}}$-porous (and thus also non-σ-porous) compact set $F \subseteq A$.

Now, suppose that (X,d) is an arbitrary locally compact metric space. Since A is a non-σ-porous subset of X, there exists $x \in X$ such that $A \cap B(x,r)$ is a non-σ-porous subset of X for every $r>0$ by Theorem 2.4. Let us take $r_0 > 0$ such that $\overline{B\left(x, r_0\right)}$ is compact and denote $A' = A \cap B(x, r_0)$. Since porosity is a local property, every $M \subseteq \overline{B\left(x,r_0\right)}$ is σ-porous in X if and only if M is σ-porous in the compact metric space $\overline{B\left(x,r_0\right)}$. Therefore, A' is non-σ-porous in $\overline{B\left(x,r_0\right)}$. Due to the previous part of the proof, there exists a non-σ-porous (in $\overline{B\left(x,r_0\right)}$ and therefore also in X) compact set $F \subseteq A' \subseteq A$. $\square$

Theorem 5.2. *Let (X,d) be a locally compact metric space. Let $A \subseteq X$ be a non-σ-strongly porous Borel set. Then there exists a non-σ-strongly porous compact set $F \subseteq A$.*

Proof. Let us use the same notation as in the proof of Theorem 5.1. Then $M \subseteq X$ is σ-strongly porous if and only if A is σ-R^1-porous. If the space (X, d) is compact, we can use Theorem 4.7. The general case can be shown in the same way as in Theorem 5.1, we only have to substitute ordinary porosity by strong porosity. □

Finally, we apply Theorem 4.7 to answer a question posed by M. J. Evans and P. D. Humke in [3]. This is the following question.

Question 5.3 ([3, page 178])**.** *Does there exist an F_σ set in $[0, 1]$ which is σ-$(1 - \varepsilon)$-symmetrically porous for every $0 < \varepsilon < 1$ but which is not σ-1-symmetrically porous?*

We answer this question positively by proving the next theorem.

Theorem 5.4. *There exists a closed set $F \subseteq [0, 1]$ which is σ-$(1 - \varepsilon)$-symmetrically porous for every $0 < \varepsilon < 1$ but which is not σ-1-symmetrically porous.*

Proof. We define point-set relations R^q_r and R^q, $r > 0$, $q \in (0, 1]$, on $[0, 1]$ such that for every $r > 0$, $q \in (0, 1]$, $M \subseteq [0, 1]$ and $x \in [0, 1]$ we have

- $R^q_r(x, M)$ if there exists an open ball $B(y, \tilde{r})$ in $[0, 1]$ such that $x \in \left(B\left(y, r\right) \setminus \overline{B}(y, \frac{r}{2})\right) \cap B\left(y, \frac{\tilde{r}}{q}\right)$ and $(B(y, \tilde{r}) \cup B(2x - y, \tilde{r})) \cap M = \emptyset$,
- $R^q = \bigcap_{0<\tilde{q}<q} \bigcap_{R>0} \bigcup_{0<r\leq R} R^{\tilde{q}}_r$.

Similarly as in Theorem 5.1, we can verify that

- relations R^q_r and R^q, $r > 0$, $q \in (0, 1]$, satisfy conditions (R1)-(R5),
- $M \subset (0, 1)$ is σ-1-symmetrically porous (in $\mathbb{R}$) if and only if A is σ-R^1-porous (in $[0, 1]$).

As it is written in [3], it is known that there exists a Borel set $A \subseteq (0, 1)$ which is σ-$(1 - \varepsilon)$-symmetrically porous for every $0 < \varepsilon < 1$ but which is not σ-1-symmetrically porous. By Theorem 4.7, there exists a compact non-σ-1-symmetrically porous set $F \subseteq A$. Since F is a subset of A, it is still a σ-$(1 - \varepsilon)$-symmetrically porous set for every $0 < \varepsilon < 1$. □

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