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Moduli spaces of Lie algebroid connections

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Contents

Introduction

The topic of this thesis lies on the crossroad of mathematics (geometry) and theoretical physics (quantum field theory, string theory). Theories arising on the interface of these two sciences always contribute significantly to development of both fields. As an example, we can mention mirror symmetry or geometric Langlands program. Both themes are at present very active research areas, which may bring interesting and surprising results.

The main theme is a study of Lie algebroid connections on fiber bundles, in particular, vector bundles and principal fiber bundles, and a description of the moduli space of gauge equivalence classes of flat linear Lie algebroid connections on a real or complex vector bundle over a connected compact manifold for a wide class of Lie algebroids. The special case of this moduli space is the moduli space of flat linear connections on a vector bundle over a connected compact manifold and the moduli space of holomorphic structures on a complex vector bundle over a connected compact complex manifold. These two examples play a very important role in geometry and quantum field theory, therefore we describe them later in detail.

The concept of a Lie algebroid was first introduced by Jean Pradines in 1966–68 who, in series of notes [1], [2], [3], [4], developed a Lie theory for Lie groupoids. The theory of Lie algebroids got back into the center of interest in the late 1980s with the work of Almeida and Molino [5] and the work of Mackenzie on theory of connections [6]. These works were devoted almost exclusively to transitive Lie algebroids, and it was Weinstein [7] and Karasëv [8], who studied non-transitive Lie algebroids. The theory of connections was a strong motivation for the Mackenzie's approach to Lie groupoid and alegebroid theory. A geometric approach to the theory of connections on Lie algebroids was worked out by Fernandes in [9], [10]. Representations of Lie algebroids were introduced first for transitive Lie algebroids by Mackenzie [6], and they appear in a study of cohomological invariants attached to Lie algebroids. More details on relations between Lie algebroids and Cartan's equivalence method can be found in [11], [12].

Moduli spaces arise naturally in classification problems in geometry. Typically, one has a set whose elements represent algebro-geometric objects of some fixed kind and an equivalence relation on this set saying when two such objects are identical a suitable sense. The problem then is to describe the set of equivalence classes. One would like to give the set of equivalence classes some structure of a geometric space (usually of a smooth manifold, a scheme or an algebraic stack). If it can be done, then one can parametrize such objects by introducing coordinates on the resulting space.

The word *moduli* is due to Bernhard Riemann who used it as a synonym for parameters, when he showed that the space of equivalence classes of Riemann surfaces of a given genus g (for $g > 1$) depends on $3g - 3$ complex numbers. This is the reason why the moduli spaces were first understood as spaces of parameters rather than as spaces of objects.

We have many basic but important examples of moduli spaces, e.g. the moduli space of algebraic curves, moduli space of vector bundles, moduli space of algebraic varieties and many others. We proceed by describing two cases of moduli spaces in detail as mentioned above.

Given a connected compact manifold X and a compact Lie group G, the moduli space of principal G-connections on a principal G-bundle $P \to X$ is the space $\mathcal{M}(P,G) = \mathcal{H}(P,G)/\operatorname{Gau}(P)$, where $\mathcal{H}(P,G)$ is the space of flat principal G-connections and $\operatorname{Gau}(P)$ is the group of gauge transformations. The disjoint union of these moduli spaces over representatives for the isomorphism classes of principal G-bundles gives the moduli space $\mathcal{M}(X,G)$ of all flat principal G-connections over X. Holonomy provides a mapping $\mathcal{H}(P,G) \to \mathrm{Hom}(\pi_1(X,x_0),G)$ which, by Uhlenbeck compactness, induces a homeomorphism

$$\mathrm{Hom}(\pi_1(X,x_0),G)/G \simeq \mathcal{M}(X,G),$$

called the *Riemann–Hilbert correspondence.*

This moduli space has a very close relationship to the *Chern–Simons theory* which is a 3-dimensional topological field theory. The Chern–Simons theory leads to new topological invariants of 3-manifolds, as was proposed by Edward Witten [13] in the late 1980s. The quantum Chern–Simons invariants are closely related to the Jones invariants [14] of links which have many applications in knot theory. These invariants can be approached by defining a vector space $\mathcal{H}_\Sigma$ canonically associated to a closed (compact and without boundary) surface Σ. The underlying idea behind the vector space $\mathcal{H}_\Sigma$ is that of *geometric quantization* of a symplectic manifold $\mathcal{M}(\Sigma,G)$.

Consider a complex vector bundle E over a connected compact complex manifold M and denote by $\mathcal{H}(M,E)$ the space of all holomorphic structures on E. Let $\mathrm{Gau}(E)$ be the group of automorphism of E covering the identity on M. Then $\mathrm{Gau}(E)$ acts on $\mathcal{H}(M,E)$ and we define the moduli space $\mathcal{M}(M,E) = \mathcal{H}(M,E)/\mathrm{Gau}(E)$ as the space of equivalence classes of holomorphic structures on E.

The moduli space of holomorphic vector bundles over a connected compact complex manifold has a very long history. Even the simplest possible case, when the manifold M is a Riemann surface, has been studied intensively for a long time. After the classification of holomorphic vector bundles for genus 0 by Alexander Gronthendieck [15] and genus 1 by Michal Atiyah [16], vector bundles on higher genus Riemann surfaces have been studied extensively with the fundamental work of David Mumford [17] and of Narasimham and Seshadri [18], who introduced the concept of stable vector bundles and constructed the moduli spaces which classify these bundles. In their theorem Narasimhan and Sashedri identified the moduli space of stable vector bundles over a compact Riemann surface with the moduli space of irreducible projective unitary representations of the fundamental group of the surface. More details about the moduli space of holomorphic structures can be found in [19].

These last two examples of the moduli spaces of flat Lie algebroid connections on a vector bundle or on a principal fiber bundle over a connected compact manifold show that they have a fundamental importance both for geometry and for quantum field theory. In fact, there is one more example of the moduli space of this type which was the motivation for a study of the moduli space of Lie algebroid connections. It is the moduli space of topological A-branes and B-branes, see [20].

During last decades, a lot of attention was concentrated to the problem of a unified description of different geometries. In 2002, Nigel Hichtin [21] introduced a concept of generalized complex geometry, which was further developed by his students Marco Gualtieri [22] and Gil Cavalcanti [23]. It contains complex and symplectic geometry as its extremal special cases. It seems that this unifying concept of these two geometries will play a central role in the understanding of mirror symmetry [24] and geometric Langlands program [25].

Mirror symmetry is an example of a general phenomenon known as duality, which occurs when two seemingly different physical systems are isomorphic in a non-trivial way. The non-triviality of this isomorphism involves the fact that quantum corrections must be taken into account. There are many forms of mirror symmetry and they are all closely related.

A mathematical explanation for this phenomenon is the homological mirror symmetry. It is a mathematical conjecture formulated by Maxim Kontsevich at the International Congress of Mathematicians in Zurich in 1994, see [26]. He considered mirror symmetry for a pair of Calabi–Yau manifolds X and Y as an equivalence of the triangulated category $\mathcal{D}^b(\mathrm{Coh}(X))$ constructed from the complex geometry of X and the other triangulated category $\mathrm{Fuk}(Y)$ constructed from the symplectic geometry of Y and vice versa. The triangulated category $\mathcal{D}^b(\mathrm{Coh}(X))$ is a bounded derived category of coherent sheaves on X and $\mathrm{Fuk}(Y)$ is the Fukaya category. Therefore the homological mirror symmetry conjecture can be formulated as

$$\mathcal{D}^b(\mathrm{Coh}(X)) \simeq \mathrm{Fuk}(Y),$$
$$\mathrm{Fuk}(X) \simeq \mathcal{D}^b(\mathrm{Coh}(Y)),$$

where X and Y is a pair of mirror Calabi–Yau manifolds. In fact, this formulation could be understood as a mathematical definition of a mirror pair of Calabi–Yau manifolds.

Another formulation relates two different two-dimensional topological field theories called A-model and B-model. The topological A-model and B-model were originally introduced by Edward Witten [27] in 1988 as the topological twisting of the $\mathcal{N} = (2,2)$ supersymmetric two-dimensional conformal field theory. These models involve maps from a worldsheet Σ (Riemann surface) into a target space M (usually a Calabi–Yau manifold). There are more general cases of a target space than Calabi-Yau manifolds for which the $\mathcal{N} = (2,2)$ supersymmetric two-dimensional conformal field theory exists. Such examples can be described in a very elegant way using generalized complex structures as manifolds involving a generalized Kähler structure or bi-Hermitian structure (first discovered by physicists investigating supersymmetric nonlinear sigma models, see [28]). Riemann surfaces without boundary represent the worldsheet of closed strings, while in the case of Riemann surfaces with boundary describe the worldsheet of open strings. In the second case, we must introduce boundary conditions to preserve the supersymmery. These boundary conditions correspond to objects called topological A-branes and B-branes. These topological branes in a Calabi–Yau manifold M can be described through the generalized complex structure as a complex vector bundle supported on some submanifold of M with a flat linear Lie algebroid connection on this vector bundle. This concept was introduced by Marco Gualtieri in [20].

Moduli spaces of topological A-branes and B-branes play a crucial role in the so called SYZ conjecture formulated by Andrew Strominger, Shing-Tung Yau and Eric Zaslow in [29]. This picture relates the homological mirror symmetry of two Calabi–Yau manifolds X and Y to the T-duality of dual special Lagrangian fibrations in X and Y. A special case of this fibration is the Hitchin fibration in geometric Langlands program.

Our main results about Lie algebroid connections and moduli spaces of Lie algebroid connections are contained in the second and third chapter of this thesis.

In the first chapter some important definitions and notions are reviewed, for example the basic definition of a real and complex Lie algebroid is given and also many examples of Lie algebroids are mentioned, among others an example of the Atiyah algebroid, which is crucial for the definition of Lie algebroid connections on principal fiber bundles, is described. Further, the notion of an L-path is given. This is important for the concept of the parallel transport and for introducing the holomomy group of a Lie algebroid connection. Because Lie algebroids can be understood as generalized tangent bundles, the notions like forms, vector fields, de Rham differential are generalized in a natural way. At the end a wide class of complex Lie algebroids coming from generalized complex structures is presented together with the explanation of generalized complex geometry and necessary tools.

The second chapter is devoted to the study of Lie algebroid connections on vector bundles or linear Lie algebroid connections. After the definition is given, we prove some basic results generalizing well-know facts about linear connections related with the curvature, covariant exterior derivative, flat connections, Bianchi identity and others. We continue by recalling the definition of the group of gauge transformations of a vector bundle. We define an action of this group on the space of Lie algebroid connections and introduce the notion of moduli spaces for Lie algebroid connections. Some basic results about Lebesgue and Sobolev spaces are mentioned. We also recall some well-know facts for elliptic complexes on compact manifolds. Then we define Sobolev completions of these moduli spaces which allow us to give the moduli space the structure of a geometric space. We prove that the irreducible linear Lie algebroid connection together with the action of the reduced group of gauge transformations form (possibly non-Hausdorff) principal fiber bundle. The last section is devoted to the study of the moduli space of smooth irreducible flat Lie algebroid connections. It is proved that this moduli space has the structure of a smooth finite dimensional manifold near a smooth point and its dimension is the dimension of the first Lie algebroid cohomology group. These results were partially published in [30].

In the third chapter we describe the general concept of Lie algebroid connections on a fiber bundle through the horizontal lift and we concentrate more on principal Lie algebroid connections on principal fiber bundles. We generalize some results from the previous chapter which in fact correspond to the special case (general linear group) in the choice of the structure group of a principal fiber bundle. We define the concept of the covariant exterior derivative, the induced linear Lie algebroid connection on an associated vector bundle and the parallel transport along an L-path. The natural action of the group of gauge transformations of a principal fiber bundle on the space of principal Lie algebroid connections is studied. The main result is the proof of the isomorphism between the isotropy group of a principal Lie algebroid connection and the holonomy group of a principal Lie algebroid connection.

The conclusion focuses at the further study of Lie algebroid connections. One possibility is a generalization of the Riemann–Hilbert correspondence.

Chapter 1

Lie and Courant algebroids

1.1 Lie algebroids

Lie algebroids were first introduced and studied by J. Pradines [2], following the work by C. Ehresmann and P. Libermann on *differential groupoids* (later called *Lie groupoids*), as infinitesimal objects for differential groupoids. Just as Lie algebras are the infinitesimal objects of Lie groups, Lie algebroids are the infinitesimal objects of Lie groupoids. They are generalizations of both Lie algebras and tangent vector bundles.

Definition 1. A real (complex) Lie algebroid $(L \xrightarrow{\pi} M, [\cdot\,,\cdot], a)$ is a real (complex) vector bundle $\pi\colon L \to M$ together with a real (complex) Lie algebra bracket $[\cdot\,,\cdot]$ on the space of sections $\Gamma(M, L)$ and a homomorphism of vector bundles $a\colon L \to TM$ $(a\colon L \to TM_{\mathbb{C}})$, called the *anchor map*, covering the identity on M, i.e., the following diagram

$$\begin{array}{ccc} L & \xrightarrow{a} & TM \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_M} \\ M & \xrightarrow[\mathrm{id_M}]{} & M \end{array} \qquad \text{resp.} \qquad \begin{array}{ccc} L & \xrightarrow{a} & TM_{\mathbb{C}} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_M} \\ M & \xrightarrow[\mathrm{id}_M]{} & M \end{array}$$

commutes. Moreover, the anchor map fulfills

i) $a([\xi_1, \xi_2]) = [a(\xi_1), a(\xi_2)]$ resp. $a([\xi_1, \xi_2]) = [a(\xi_1), a(\xi_2)]_{\mathbb{C}}$

ii) $[\xi_1, f\xi_2] = f[\xi_1, \xi_2] + (a(\xi_1)f)\xi_2$, (the Leibniz rule)

for all $\xi_1, \xi_2 \in \Gamma(M, L)$ and $f \in C^\infty(M, \mathbb{R})$ resp. $f \in C^\infty(M, \mathbb{C})$.

Definition 2. If $(L_1 \to M, [\cdot\,,\cdot]_{L_1}, a_{L_1})$ and $(L_2 \to M, [\cdot\,,\cdot]_{L_2}, a_{L_2})$ are Lie algebroids, then a vector bundle homomorphism $\varphi\colon L_1 \to L_2$ covering the identity on M is a *Lie algebroid morphism* if $a_{L_2} \circ \varphi = a_{L_1}$ and φ induces a Lie algebra homomorphism form $\mathfrak{X}_{L_1}(M)$ to $\mathfrak{X}_{L_2}(M)$.

Before the continuing with the study of Lie algebroids, we would like to show that Lie algebroids are interesting themselves. We look at equivalence problems in geometry. Élie Cartan observed that many equivalence problems in geometry can be best formulated in terms of coframe fields. He was able to come up with a method, now called Cartan's equivalence method, to deal with such problems.

A local version of Cartan's formulation of equivalence problems can be described as follows. Consider a family of functions f_i^a and $c_{j,k}^i = -c_{k,j}^i$ defined on some nonempty open set $X \subset \mathbb{R}^n$, where $1 \le i, j, k \le r$, $1 \le a \le n$ (n, r are positive integers).

Cartan's problem: find a manifold N, a coframe field $\{\eta^i\}_{i=1}^r$ on N, and a smooth mapping $h\colon N \to X$ satisfying

$$d\eta^k = \frac{1}{2}\, c_{i,j}^k(h)\, \eta^i \wedge \eta^j, \qquad dh^a = f_i^a(h)\, \eta^i. \tag{1.1}$$

Necessary conditions on the map $h\colon N \to X$ to solve Cartan's problem can be obtained as immediate consequences of the fact that $d^2 = 0$ and that $\{\eta^i\}$ is a coframe field. An easy computation gives

$$f_i^b(h)\,\frac{\partial f_j^a}{\partial x^b}(h) - f_j^b(h)\,\frac{\partial f_i^a}{\partial x^b}(h) = -c_{i,j}^k(h)\,f_k^a(h) \tag{1.2}$$

and

$$f_j^a(h)\,\frac{\partial c_{k,\ell}^i}{\partial x^a}(h) + f_k^a(h)\,\frac{\partial c_{\ell,j}^i}{\partial x^a}(h) + f_\ell^a(h)\,\frac{\partial c_{j,k}^i}{\partial x^a}(h) = -(c_{m,j}^i(h)\,c_{k,\ell}^m(h) + c_{m,k}^i(h)\,c_{\ell,j}^m(h) + c_{m,\ell}^i(h)\,c_{j,k}^m(h)). \tag{1.3}$$

Unless these equations are identities, they place restrictions on the range of h.

On the other hand, if the above equations are identities on the functions f_i^a and $c_{j,k}^i$, then one might hope to find realizations of (1.1) without placing any further restrictions on the range of h.

Cartan's conditions can be reformulated into a more geometric form as follows. Consider a trivializable vector bundle $L \to X$ of $\operatorname{rk} L = r$ over X and any local frame field $\{e_i\}_{i=1}^r$ for L over X. If we define a vector bundle homomorphism $a\colon L \to TX$ by

$$a(g^i e_i) = g^i f_i^a \frac{\partial}{\partial x^a} \tag{1.4}$$

and a bilinear mapping $[\cdot,\cdot]\colon \Gamma(X,L) \times \Gamma(X,L) \to \Gamma(X,L)$ by

$$[g^i e_i, h^j e_j] = -g^i h^j c_{i,j}^k\, e_k + g^i f_i^a \frac{\partial h^j}{\partial x^a}\, e_j - h^j f_j^a \frac{\partial g^i}{\partial x^a}\, e_i, \tag{1.5}$$

where $g^i, h^j \in C^\infty(X,\mathbb{R})$, then the necessary conditions (1.2) and (1.3) are equivalent to the fact that $(L \to X, [\cdot,\cdot], a)$ is a Lie algebroid. More about the reformulation of Cartan's equivalence problems through Lie algebroids can be found in [11] and [12].

Now we express a Lie algebroid structure on a vector bundle $\pi\colon L \to M$ in local coordinates. For any $x \in M$ there exists an open neighborhood $U \subset M$, a local chart (U,u) on M and a vector bundle chart (U,ψ) on L. Then $\left\{\frac{\partial}{\partial u^a}\right\}_{a=1}^n$ is a local frame field for TM over U and moreover there exists a local frame field $\{e_i\}_{i=1}^r$ for L over U. We define local structure functions f_i^a and $c_{j,k}^i$ on U, where $1 \leq i,j,k \leq r$, $1 \leq a \leq n$, $\dim M = n$, $\operatorname{rk} L = r$, by

$$[e_i, e_j] = c_{i,j}^k\, e_k, \qquad a(e_i) = f_i^a \frac{\partial}{\partial u^a}. \tag{1.6}$$

The requirement, that a is a Lie algebra homomorphism, is equivalent to the condition

$$f_i^b \frac{\partial f_j^a}{\partial u^b} - f_j^b \frac{\partial f_i^a}{\partial u^b} = c_{i,j}^k\, f_k^a, \tag{1.7}$$

while the Jacobi identity is equivalent to

$$c_{i,j}^\ell\, c_{k,\ell}^m + c_{k,i}^\ell\, c_{j,\ell}^m + c_{j,k}^\ell\, c_{i,\ell}^m + f_i^a \frac{\partial c_{j,k}^m}{\partial u^a} + f_k^a \frac{\partial c_{i,j}^m}{\partial u^a} + f_j^a \frac{\partial c_{k,i}^m}{\partial u^a} = 0. \tag{1.8}$$

These equations are called the *local structure equations.*

Remark. Let A be a commutative $\mathbb{K}$-algebra[1] with unit. We denote by $\operatorname{Der}_{\mathbb{K}}(A)$ the A-module of $\mathbb{K}$-linear derivations of A. Recall that $\operatorname{Der}_{\mathbb{K}}(A)$ is naturally a Lie algebra over $\mathbb{K}$ with respect to the usual commutator.

A Lie–Rinehart A-algebra is an A-module L endowed with a structure of a Lie algebra over $\mathbb{K}$ and with a morphism $a\colon L \to \operatorname{Der}_{\mathbb{K}}(A)$ of A-modules, called the *anchor map*, satisfying the following axioms:

[1]The letter $\mathbb{K}$ stands for the field $\mathbb{R}$ or $\mathbb{C}$.

i) $a([x,y]_L) = [a(x), a(y)]$ for $x, y \in L$, i.e., a is a morphism of Lie algebras over $\mathbb{K}$,
ii) $[x, fy]_L = f[x,y]_L + (a(x)f)y$ for $x, y \in L$ and $f \in A$.

Consider the commutative $\mathbb{R}$-algebra $A = C^\infty(M, \mathbb{R})$, then $\mathrm{Der}_{\mathbb{R}}(A)$ is the Lie algebra of vector fields on M. Afterwards the space of sections $\Gamma(M, L)$ of a real Lie algebroid $(L \to M, [\cdot\,,\cdot]_L, a)$ is a Lie–Rinehart A-algebra.

In fact, Lie–Rinehart algebras are the algebraic counterparts of Lie algebroids, just as modules over a ring are the algebraic counterpart of vector bundles.

Definition 3. Given a Lie algebroid $(L \xrightarrow{\pi} M, [\cdot\,,\cdot], a)$ over M, a smooth path $\alpha\colon [0,1] \to L$ is an *L-path*, if

$$a(\alpha(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\, \pi(\alpha(t)) \tag{1.9}$$

for all $t \in [0,1]$. The smooth path $\gamma\colon [0,1] \to M$ given by $\gamma = \pi \circ \alpha$ will be called the *base path* of the L-path α. We denote by $\mathcal{P}(L)$ the set of all L-paths.

If $\tau\colon [0,1] \to [0,1]$ is a smooth change of parameter, i.e., a diffeomorphism, and $\alpha\colon [0,1] \to L$ is an L-path, then its reparametrization $\alpha^\tau\colon [0,1] \to L$ given by $\alpha^\tau(t) = \tau'(t)\alpha(\tau(t))$ is an L-path and for τ satisfying $\tau(0) = 0$ and $\tau(1) = 1$ is L-homotopic to the L-path α.

We say that two L-paths α_0 and α_1 are composable, if $\pi(\alpha_0(1)) = \pi(\alpha_1(0))$. In this case we define the concatenation of paths α_0 and α_1 by

$$(\alpha_1 \odot \alpha_0)(t) = \begin{cases} 2\alpha_0(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ 2\alpha_1(2t-1) & \text{for } \frac{1}{2} < t \le 1. \end{cases} \tag{1.10}$$

This is essentially the multiplication of L-paths. However it is not associative and $\alpha_1 \odot \alpha_0$ is only piecewise smooth. One possibility around this difficulty is allowing for piecewise smooth L-paths. Instead we choose a cutoff function $\tau \in C^\infty(\mathbb{R})$ with the following properties:
i) $\tau(t) = 0$ for $t \le 0$ and $\tau(t) = 1$ for $t \ge 1$,
ii) $\tau'(t) > 0$ for $t \in (0,1)$.
We now define the multiplication of composable L-paths by

$$\alpha_1 \cdot \alpha_0 = \alpha_1^\tau \odot \alpha_0^\tau, \tag{1.11}$$

where α_0^τ and α_1^τ are reparametrizations of α_0 and α_1.

Now we can define an equivalence relation $\sim_L$ on a manifold M as follows. We say that $x \sim_L y$ for $x, y \in M$ if there exists an L-path α, with the base path γ, such that $\gamma(0) = x$ and $\gamma(1) = y$. An equivalence class of this relation will be called an *orbit of L*. In the case, when a is surjective, i.e., L is a transitive Lie algebroid, each connected component of M is an orbit of L.

1.2 Examples of Lie algebroids

Let us present now a few basic examples of Lie algebroids.

Example. (tangent bundles) One of the trivial examples of a Lie algebroid over M is the tangent bundle $L = TM$ of M, with the identity mapping as the anchor map and the Lie bracket of vector fields as the Lie bracket.

Example. (Lie algebras) Any real (complex) Lie algebra $\mathfrak{g}$ is a real (complex) Lie algebroid over a one-point manifold, with zero anchor map.

Example. (foliations) Let $L \subset TM$ be an involutive regular distribution on a manifold M. Then L has a Lie algebroid structure with the inclusion as the anchor map and the Lie bracket is the usual Lie bracket of vector fields. By the Frobenius theorem the distribution L gives a regular foliation on M. On the other hand to any regular foliation on M is associated an involutive regular distribution and therefore a Lie algebroid over M.

Example. (bundles of Lie algebras) A bundle of Lie algebras is a vector bundle $L \to M$ with a skew-symmetric $C^\infty(M,\mathbb{R})$-bilinear mapping $[\cdot\,,\cdot]\colon \Gamma(M,L)\times\Gamma(M,L)\to\Gamma(M,L)$, i.e., $[\cdot\,,\cdot]\in\Gamma(M,\Lambda^2L^*\otimes L)$, satisfying the Jacobi identity. If we define the anchor map by $a(\xi)=0$ for $\xi\in\Gamma(M,L)$, then $(L\to M,[\cdot\,,\cdot],a)$ is a Lie algebroid. On the other hand, any Lie algebroid with zero anchor map is a bundle of Lie algebras. Because $[\xi_1,f\xi_2]=f[\xi_1,\xi_2]+(a(\xi_1)f)\xi_2=f[\xi_1,\xi_2]$, we obtain $[\cdot\,,\cdot]\in\Gamma(M,\Lambda^2L^*\otimes L)$.

Note that the notion of a bundle of Lie algebras is weaker than of a Lie algebra bundle, when one requires that L is locally trivial bundle of Lie algebras (in particular, all Lie algebras L_x are isomorphic).

Example. (vector fields) Lie algebroid structures on the trivial real line bundle over M are in a one-to-one correspondence with vector fields on M. Given a vector field $X\in\mathfrak{X}(M)$, we denote by L_X the induced Lie algebroid. As a vector bundle $L_X=M\times\mathbb{R}$. Because $\Gamma(M,L_X)\simeq C^\infty(M,\mathbb{R})$, the anchor map is given by the multiplication by X, i.e., $a(f)=fX$, and the Lie bracket of two sections $f,g\in\Gamma(M,L_X)$ is defined by

$$[f,g]=f\mathcal{L}_X(g)-g\mathcal{L}_X(f). \tag{1.12}$$

Example. (action Lie algebroids) Consider an infinitesimal right action of a real Lie algebra $\mathfrak{g}$ on a manifold M, i.e., a Lie algebra homomorphism $\zeta\colon\mathfrak{g}\to\mathfrak{X}(M)$. The usual situation is when we have a right action $r\colon M\times G\to M$ of a Lie group G with the Lie algebra $\mathfrak{g}$. Then

$$\zeta_X(x)=T_er_x.X=\frac{\mathrm{d}}{\mathrm{d}t}_{|0}\,x.\exp(tX), \tag{1.13}$$

where $X\in\mathfrak{g}$ and $x\in M$, defines an infinitesimal right action of $\mathfrak{g}$ on M. We define a Lie algebroid $\mathfrak{g}\ltimes M$, called the action Lie algebroid or the transformation Lie algebroid, by the following way. As a vector bundle $\mathfrak{g}\ltimes M=M\times\mathfrak{g}$, it is a trivial vector bundle over M. Seeing that $\Gamma(M,\mathfrak{g}\ltimes M)\simeq C^\infty(M,\mathfrak{g})$, the anchor map is given by

$$a(f)(x)=\zeta_{f(x)}(x), \tag{1.14}$$

while the Lie bracket on sections is defined by

$$[f,g](x)=[f(x),g(x)]_{\mathfrak{g}}+(\zeta_{f(x)}g)(x)-(\zeta_{g(x)}f)(x). \tag{1.15}$$

The Lie bracket is uniquely determined by the Leibniz rule and the condition that

$$[c_X,c_Y]=c_{[X,Y]} \tag{1.16}$$

for all $X,Y\in\mathfrak{g}$, where c_X denotes the constant section of $\mathfrak{g}$.

Example. (two forms) For any closed 2-form $\omega\in\Omega^2(M,\mathbb{R})$, we define a Lie algebroid L_ω as follows. As a vector bundle $L_\omega=TM\oplus(M\times\mathbb{R})$, the anchor map is the projection on the first component, while the Lie bracket on sections $\Gamma(M,L_\omega)\simeq\mathfrak{X}(M)\oplus C^\infty(M,\mathbb{R})$ is given by

$$[(X,f),(Y,g)]=([X,Y],\mathcal{L}_X(g)-\mathcal{L}_Y(f)+\omega(X,Y)). \tag{1.17}$$

Example. (Atiyah sequences) In 1957, Atiyah [16] constructed in the setting of vector bundles the following key example of a Lie algebroid. Let (P,p,M,G) be a principal fiber bundle, then there is an associated transitive Lie algebroid $\mathcal{A}(P)$ over M, called the *Atiyah algebroid.*

Theorem 1. Let (P,p,M,G) be a principal fiber bundle. If $r\colon P\times G\to P$ is the principal right action then $\hat{r}\colon TP\times G\to TP$ denotes the right action given by $\hat{r}^g=Tr^g$.

i) The space TP/G of orbits of the right action $\hat{r}$ carries a unique smooth manifold structure such that the quotient mapping $q\colon TP\to TP/G$ is a surjective submersion.

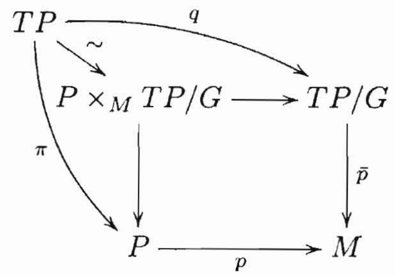
ii) $\bar{p}\colon TP/G \to M$ is a vector bundle in a canonical way, where $\bar{p}$ is given by

$$\begin{CD} TP @>q>> TP/G \\ @V{\pi}VV @VV{\bar{p}}V \\ P @>>p> M \end{CD}$$

and $q_u\colon T_uP \to (TP/G)_{p(u)}$ is a linear diffeomorphism for each $u \in P$, moreover q is a homomorphism of vector bundles.

iii) $q\colon TP \to TP/G$ is a principal G-bundle with the principal right action $\hat{r}$.

iv) The following diagram



commutes, i.e., TP is a topological pullback.

Notation. We will denote TP/G by $\mathcal{A}(P)$. We also define the smooth mapping $\tau\colon P \times_M \mathcal{A}(P) \to TP$ by $\tau(u_x, v_x) = q_{u_x}^{-1}(v_x)$. It satisfies $\tau(u, q(\xi_u)) = \xi_u$, $q(\tau(u_x, v_x)) = v_x$ and $\tau(u_x.g, v_x) = \tau(u_x, v_x).g$. The vector bundle $\mathcal{A}(P) \to M$ is called the *Atiyah bundle.*

Proof. First of all we verify that the right action $\hat{r}\colon TP \times G \to TP$ is free and proper. Suppose that $\xi_u.g_1 = \xi_u.g_2$, then $u.g_1 = \pi(\xi_u.g_1) = \pi(\xi_u.g_2) = u.g_2$. Because the principal right action $r\colon P \times G \to P$ is free, the right action $\hat{r}$ is also free. Now let $\xi_n.g_n \to \xi'$ and $\xi_n \to \xi$ in TP for some $\xi_n, \xi, \xi' \in TP$ and $g_n \in G$. If we denote $u_n = \pi(\xi_n)$, $u = \pi(\xi)$ and $u' = \pi(\xi')$ then $u_n.g_n = \pi(\xi_n.g_n) \to \pi(\xi') = u'$ and $u_n = \pi(\xi_n) \to \pi(\xi) = u$, because π is continuous. But G acts properly on P, hence g_n has a convergent subsequence in G and thus $\hat{r}$ is proper. Immediately, from the characterization of principal fiber bundles it follows that the orbit space TP/G is a smooth manifold, the quotient mapping $q\colon TP \to TP/G$ is a surjective submersion and $q\colon TP \to TP/G$ is a principal G-bundle.

In the setting of the diagram in (ii) the mapping $p \circ \pi$ is constant on orbits of the action $\hat{r}$, so $\bar{p}$ exists as a mapping. Because $q\colon TP \to TP/G$ is a fibered manifold and $\bar{p} \circ q$ is smooth, we obtain that $\bar{p}$ is also smooth.

Let $(U_\alpha, \varphi_\alpha)$ be a principal bundle atlas for P with transition functions $\varphi_{\alpha\beta}\colon U_{\alpha\beta} \to G$ and let (U_α, u_α) be an atlas for M. We define $\chi_\alpha\colon TP_{|p^{-1}(U_\alpha)} \to TU_\alpha \times TG \to U_\alpha \times \mathbb{R}^n \times \mathfrak{g} \times G$ by

$$\chi_\alpha = (Tu_\alpha \times (T\rho)^{-1}) \circ T\varphi_\alpha\colon TP_{|p^{-1}(U_\alpha)} \simeq T(P_{|U_\alpha}) \to TU_\alpha \times TG \to U_\alpha \times \mathbb{R}^n \times \mathfrak{g} \times G,$$

where $T\rho\colon \mathfrak{g} \times G \to TG$ is the right trivialization of TG given by $T\rho.(X, g) = T_e\rho_g.X$. Then χ_α is a diffeomorphism and the diagram

$$\begin{CD} TP_{|p^{-1}(U_\alpha)} @>{\chi_\alpha}>> U_\alpha \times \mathbb{R}^n \times \mathfrak{g} \times G \\ @V{\pi}VV @VV{\mathrm{pr}}V \\ p^{-1}(U_\alpha) @>>{\varphi_\alpha}> U_\alpha \times G \end{CD}$$

commutes. For $\chi_\alpha \circ \chi_\beta^{-1}\colon U_{\alpha\beta} \times \mathbb{R}^n \times \mathfrak{g} \times G \to U_{\alpha\beta} \times \mathbb{R}^n \times \mathfrak{g} \times G$ we obtain

$$(\chi_\alpha \circ \chi_\beta^{-1})(x, v, X, g) = (x, d(u_\alpha \circ u_\beta^{-1})(x, v), \delta\varphi_{\alpha\beta}.((Tu_\beta)^{-1}(x, v)) + \mathrm{Ad}(\varphi_{\alpha\beta}(x))X, \varphi_{\alpha\beta}(x).g),$$

where $\delta\varphi_{\alpha\beta} \in \Omega^1(U_{\alpha\beta}, \mathfrak{g})$ is the right logarithmic derivative of $\varphi_{\alpha\beta}$.

Now we define $\psi_\alpha^{-1}\colon U_\alpha \times \mathbb{R}^n \times \mathfrak{g} \to \bar{p}^{-1}(U_\alpha) \subset TP/G$ by $\psi_\alpha^{-1}(x,v,X) = q(\chi_\alpha^{-1}(x,v,X,e))$, which is a fiber respecting mapping, i.e., the following diagram

$$\begin{array}{ccc} U_\alpha \times \mathbb{R}^n \times \mathfrak{g} & \xrightarrow{\psi_\alpha^{-1}} & \bar{p}^{-1}(U_\alpha) \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \swarrow{\scriptstyle \bar{p}} & \\ U_\alpha & & \end{array}$$

commutes. For each point $q(\xi_u)$ in $\bar{p}^{-1}(x)$ there is exactly one $X \in \mathfrak{g}$ and one $v \in \mathbb{R}^n$ such that the orbit corresponding to this point passes through $\chi_\alpha^{-1}(x,v,X,e)$, i.e., $q(\xi_u) = q(\chi_\alpha^{-1}(x,v,X,e))$. Because χ_α is a diffeomorphism, we can write $\xi_u = \chi_\alpha^{-1}(x,v,X,g)$ for a uniquely determined $v \in \mathbb{R}^n$ and $X \in \mathfrak{g}$, where $\varphi_\alpha(u) = (x,g)$. Then

$$\begin{aligned} T_u r^{g^{-1}}.\chi_\alpha^{-1}(x,v,X,g) &= T_{\varphi_\alpha^{-1}(x,g)} r^{g^{-1}} \circ T_{(x,g)}\varphi_\alpha^{-1} \circ ((Tu_\alpha)^{-1} \times T\rho)(x,v,X,g) \\ &= T_{(x,g)}(r^{g^{-1}} \circ \varphi_\alpha^{-1})((Tu_\alpha)^{-1}(x,v), T_e\rho_g.X) \\ &= T_{(x,g)}(\varphi_\alpha^{-1} \circ \tilde{r}^{g^{-1}})((Tu_\alpha)^{-1}(x,v), T_e\rho_g.X) \\ &= T_{(x,e)}\varphi_\alpha^{-1} \circ T_{(x,g)}(\mathrm{id}_{U_\alpha} \times \rho_{g^{-1}})((Tu_\alpha)^{-1}(x,v), T_e\rho_g.X) \\ &= T_{(x,e)}\varphi_\alpha^{-1}((Tu_\alpha)^{-1}(x,v), T_g\rho_{g^{-1}}.T_e\rho_g.X) \\ &= \chi_\alpha^{-1}(x,v,X,e), \end{aligned}$$

where $\tilde{r}\colon (U_\alpha \times G) \times G \to U_\alpha \times G$ is a right action given by $\tilde{r}((x,g'),g) = (x, g'.g)$. Therefore $\psi_\alpha^{-1}(x,\cdot,\cdot)\colon \mathbb{R}^n \times \mathfrak{g} \to \bar{p}^{-1}(x)$ is bijective, since the principal right action is free. Moreover ψ_α^{-1} is smooth with the invertible tangent mapping, so its inverse $\psi_\alpha\colon \bar{p}^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n \times \mathfrak{g}$ is a fiber respecting diffeomorphism. Furthermore

$$\begin{aligned} \psi_\beta^{-1}(x,v,X) &= q(\chi_\beta^{-1}(x,v,X,e)) \\ &= q(\chi_\alpha^{-1}(x, d(u_\alpha \circ u_\beta^{-1})(x,v), \delta\varphi_{\alpha\beta}.((Tu_\beta)^{-1}(x,v)) + \mathrm{Ad}(\varphi_{\alpha\beta}(x))X, \varphi_{\alpha\beta}(x).e)) \\ &= q(\chi_\alpha^{-1}(x, d(u_\alpha \circ u_\beta^{-1})(x,v), \delta\varphi_{\alpha\beta}.((Tu_\beta)^{-1}(x,v)) + \mathrm{Ad}(\varphi_{\alpha\beta}(x))X, e)) \\ &= \psi_\alpha^{-1}(x, d(u_\alpha \circ u_\beta^{-1})(x,v), \delta\varphi_{\alpha\beta}.((Tu_\beta)^{-1}(x,v)) + \mathrm{Ad}(\varphi_{\alpha\beta}(x))X), \end{aligned}$$

thus $(\psi_\alpha \circ \psi_\beta^{-1})(x,v,X) = (x, d(u_\alpha \circ u_\beta^{-1})(x,v), \delta\varphi_{\alpha\beta}.((Tu_\beta)^{-1}(x,v)) + \mathrm{Ad}(\varphi_{\alpha\beta}(x))X)$, therefore (U_α, ψ_α) is a vector bundle atlas for $\bar{p}\colon TP/G \to M$.

By definition of ψ_α the diagram

$$\begin{array}{ccc} TP_{|p^{-1}(U_\alpha)} & \xrightarrow{\chi_\alpha} & U_\alpha \times \mathbb{R}^n \times \mathfrak{g} \times G \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ \bar{p}^{-1}(U_\alpha) & \xrightarrow[\psi_\alpha]{} & U_\alpha \times \mathbb{R}^n \times \mathfrak{g} \end{array}$$

commutes, if we restrict χ_α on T_uP then we obtain the diagram

$$\begin{array}{ccc} T_uP & \xrightarrow{\chi_\alpha} & \{p(u)\} \times \mathbb{R}^n \times \mathfrak{g} \times \{g\} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ \bar{p}^{-1}(p(u)) & \xrightarrow[\psi_\alpha]{} & \{p(u)\} \times \mathbb{R}^n \times \mathfrak{g} \end{array}$$

in which its lines are linear diffeomorphism, hence we conclude that $q_u\colon T_uP \to \bar{p}^{-1}(p(u)) = (TP/G)_{p(u)}$ is a linear diffeomorphism.

Consider a homomorphism $(\pi, q)\colon TP \to P \times_M TP/G = p^*(TP/G)$ of vector bundles over P covering the identity on P. Because (π, q) is a linear isomorphism on fibers with the invertible tangent mapping, so (π, q) is an isomorphism of vector bundles. The inverse is denoted by $\tau\colon P \times_M TP/G \to TP$ and given by $\tau(u_x, v_x) = q_{u_x}^{-1}(v_x)$. ♠

Theorem 2. The sections of the Atiyah bundle $\mathcal{A}(P) \to M$ associated to a principal fiber bundle (P, p, M, G) correspond to the G-invariant vector fields on P, moreover we have an isomorphism $\Phi\colon \Gamma(M, \mathcal{A}(P)) \xrightarrow{\sim} \mathfrak{X}(P)^G$ of $C^\infty(M, \mathbb{R})$-modules, where $f\xi = (f \circ p)\xi$ for $f \in C^\infty(M, \mathbb{R})$ and $\xi \in \mathfrak{X}(P)^G$.

Proof. If $\xi \in \mathfrak{X}(P)^G$ then we construct $s_\xi \in \Gamma(M, \mathcal{A}(P))$ in the following way. Because $\xi\colon P \to TP$ is a G-equivariant mapping, the diagram

$$\begin{array}{ccc} P & \xrightarrow{\ \xi\ } & TP \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ M & \xrightarrow[s_\xi]{} & \mathcal{A}(P) \end{array}$$

commutes for a uniquely determined mapping $s_\xi\colon M \to \mathcal{A}(P)$. Further $s_\xi \circ p = q \circ \xi$ is a smooth mapping and $p\colon P \to M$ is a fibered manifold hence s_ξ is a smooth section.

If conversely $s \in \Gamma(M, \mathcal{A}(P))$ we define $\xi_s \in \mathfrak{X}(P)^G$ by $\xi_s = \tau \circ (\mathrm{id}_P \times_M s)\colon P \to P \times_M M \to P \times_M \mathcal{A}(P) \to TP$, i.e., $\xi_s(u) = \tau(u, s(p(u)))$ for $u \in P$. This is a G-invariant vector field since $\xi_s(u.g) = \tau(u.g, s(p(u))) = \tau(u, s(p(u))).g = \xi_s(u).g$ by the G-equivariance of τ.

These two constructions are inverse to each other since we have $\xi_{s(\xi)}(u) = \tau(u, s_\xi(p(u))) = \tau(u, q(\xi(u))) = \xi(u)$ and $s_{\xi(s)}(p(u)) = q(\xi_s(u)) = q(\tau(u, s(p(u)))) = s(p(u))$. ♠

Remark. The space of sections of $\mathcal{A}(P)$ is isomorphic with the space of G-invariant vector fields on P, which is a Lie algebra, hence on sections $\Gamma(M, \mathcal{A}(P))$ there is a natural Lie algebra structure given by $\xi_{[s_1, s_2]} = [\xi_{s_1}, \xi_{s_2}]$.

Because Tp is constant on orbits of the right action $\hat{r}$, this follows from the fact that $Tp \circ \hat{r}^g = T(p \circ r^g) = Tp$, the diagram

$$\begin{array}{ccc} TP & \xrightarrow{\ Tp\ } & TM \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{TM}} \\ \mathcal{A}(P) & \xrightarrow[p_*]{} & TM \end{array}$$

commutes for a uniquely determined smooth mapping $p_*\colon \mathcal{A}(P) \to TM$. Furthermore Tp is a surjective mapping thus p_* is also surjective. Besides it is easy to see that $p_*\colon \mathcal{A}(P) \to TM$ is a homomorphism of vector bundles over M covering the identity on M, because $p_{*|\mathcal{A}(P)_x}\colon \mathcal{A}(P)_x \to T_xM$ is given by $p_{*|\mathcal{A}(P)_x} = T_{u_x}p \circ q_{u_x}^{-1}$ for some $u_x \in p^{-1}(x)$ which is linear.

Now it remains to verify that $(\mathcal{A}(P) \to M, [\cdot,\cdot], p_*)$ is a Lie algebroid. Using the following commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{\ \xi_s\ } & TP & \xrightarrow{\ Tp\ } & TM \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \mathrm{id}_{TM}} \\ M & \xrightarrow[s]{} & \mathcal{A}(P) & \xrightarrow[p_*]{} & TM \end{array}$$

we get $p_*([s_1,s_2]) \circ p = Tp \circ \xi_{[s_1,s_2]} = Tp \circ [\xi_{s_1}, \xi_{s_2}] = [p_*(s_1), p_*(s_2)] \circ p$, where we used the fact that ξ_s and $p_*(s)$ are p-related vector fields, hence $[\xi_{s_1}, \xi_{s_2}]$ and $[p_*(s_1), p_*(s_2)]$ are also p-related vector fields. Because p is surjective, we obtain $p_*([s_1,s_2]) = [p_*(s_1), p_*(s_2)]$. Next we have

$$\begin{aligned}
[s_1, fs_2] \circ p = q \circ \xi_{[s_1,fs_2]} &= q \circ [\xi_{s_1}, \xi_{fs_2}] = q \circ [\xi_{s_1}, \tilde{f}\xi_{s_2}] \\
&= q \circ (\tilde{f}[\xi_{s_1}, \xi_{s_2}] + (\xi_{s_1}(\tilde{f}))\,\xi_{s_2}) \\
&= q \circ \tilde{f}\xi_{[s_1,s_2]} + q \circ (\xi_{s_1}(f \circ p)\,\xi_{s_2}) \\
&= q \circ \xi_{f[s_1,s_2]} + q \circ (p_*(s_1)f \circ p)\,\xi_{s_2} \\
&= f[s_1,s_2] \circ p + q \circ \xi_{(p_*(s_1)f)s_2} \\
&= f[s_1,s_2] \circ p + (p_*(s_1)f)s_2 \circ p,
\end{aligned}$$

where we used that for p-related vector fields ξ_s and $p_*(s)$ is satisfied that $\xi_s(f \circ p) = (p_*(s)f) \circ p$ for any $f \in C^\infty(M,\mathbb{R})$. Again, because p is surjective, we get $[s_1, fs_2] = f[s_1,s_2] + (p_*(s_1)f)s_2$. Because p_* is surjective, we have proved that $(\mathcal{A}(P) \to M, [\cdot\,,\cdot], p_*)$ is a transitive Lie algebroid.

Immediately from the definition of the vertical bundle $VP = \ker Tp$, we obtain the short exact sequence

$$0 \longrightarrow VP \longrightarrow TP \xrightarrow{Tp} TM \longrightarrow 0 \tag{1.18}$$

of vector bundles. Since the vertical bundle VP is isomorphic to the trivial vector bundle $P \times \mathfrak{g}$, we get the short exact sequence

$$0 \longrightarrow P \times \mathfrak{g} \xrightarrow{i} TP \xrightarrow{Tp} TM \longrightarrow 0 \tag{1.19}$$

of vector bundles, where $i\colon P \times \mathfrak{g} \to VP \hookrightarrow TP$ is given by $i(u,X) = T_e r_u.X$. If we define the right action $\hat{r}\colon (P \times \mathfrak{g}) \times G \to P \times \mathfrak{g}$ through $\hat{r}((u,X),g) = (u.g, g^{-1}.X)$, then $i\colon P \times \mathfrak{g} \to TP$ is a G-equivariant mapping. Therefore the following diagram

$$\begin{array}{ccc}
P \times \mathfrak{g} & \xrightarrow{\;i\;} & TP \\
\Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle q} \\
\mathrm{ad}(P) & \xrightarrow[\;i_*\;]{} & \mathcal{A}(P)
\end{array}$$

commutes for a uniquely determined smooth mapping $i_*\colon \mathrm{ad}(P) \to \mathcal{A}(P)$. Hence we get the short exact sequence

$$0 \longrightarrow \mathrm{ad}(P) \xrightarrow{i_*} \mathcal{A}(P) \xrightarrow{p_*} TM \longrightarrow 0 \tag{1.20}$$

of Lie algebroids over M known as the Atiyah sequence associated to a principal G-bundle P, where the Lie bracket on $\Gamma(M, \mathrm{ad}(P))$ is induced from the given one on $\Gamma(M, \mathcal{A}(P))$. The smooth sections of these bundles give rise to the short exact sequence

$$0 \longrightarrow \Gamma(M, \mathrm{ad}(P)) \xrightarrow{i_*} \Gamma(M, \mathcal{A}(P)) \xrightarrow{p_*} \Gamma(M, TM) \longrightarrow 0 \tag{1.21}$$

of Lie algebras. It can be rewritten as

$$0 \longrightarrow \mathfrak{X}_{\mathsf{vert}}(P)^G \longrightarrow \mathfrak{X}(P)^G \xrightarrow{p_*} \mathfrak{X}(M) \longrightarrow 0, \tag{1.22}$$

where $\mathfrak{X}_{\mathsf{vert}}(P)^G$ is the Lie algebra of vertical G-invariant vector fields (the Lie algebra of infinitesimal gauge transformations) and $\mathfrak{X}(P)^G$ is the Lie algebra of G-invariant vector fields. The exactness of the sequence (1.21) follows from the fact that the Atiyah sequence is closely related to principal connections on a principal fiber bundle.

Later we show that a principal connection can be described as a right splitting of the Atiyah sequence, i.e., as a homomorphism $\sigma\colon TM \to \mathcal{A}(P)$ of vector bundles satisfying $p_* \circ \sigma = \mathrm{id}_{TM}$. The curvature of the connection $\sigma \in \Omega^1(M, \mathcal{A}(P))$ is given by

$$\Omega_\sigma(\xi_1, \xi_2) = [\sigma(\xi_1), \sigma(\xi_2)] - \sigma([\xi_1, \xi_2])$$

for $\xi_1, \xi_2 \in \mathfrak{X}(M)$. Furthermore one can verify that $\Omega_\sigma \in \Omega^2(M, \mathcal{A}(P))$. Because the sequence (1.21) is exact and $p_*(\Omega_\sigma(\xi_1, \xi_2)) = 0$, we obtain that there exists a uniquely determined $R_\sigma \in \Omega^2(M, \mathrm{ad}(P))$ such that $i_*(R_\sigma(\xi_1, \xi_2)) = \Omega_\sigma(\xi_1, \xi_2)$ for all $\xi_1, \xi_2 \in \mathfrak{X}(M)$.

If L is a transitive Lie algebroid over M, then the associated short exact sequence

$$0 \to \ker a \xrightarrow{i} L \xrightarrow{a} TM \to 0 \tag{1.23}$$

of Lie algebroids is called the *abstract Atiyah sequence*. Note that not all abstract Atiyah sequence come from sequences associated to a principal fiber bundle. Then we can define a connection on L to be a right splitting of the above exact sequence (1.23), i.e., a homomorphism $\sigma\colon TM \to L$ of vector bundles satisfying $a \circ \sigma = \mathrm{id}_{TM}$. More about connections on transitive Lie algebroids can be found in [6] and [31].

Example. (Poisson manifolds) Any Poisson structure on a manifold M induces, in a natural way, a Lie algebroid structure on the cotangent bundle T^*M of M. Let $\pi \in \Gamma(M, \Lambda^2 TM)$ be a Poisson bivector on M, which is related to the Poisson bracket by $\{f, g\} = \pi(df, dg)$. If we use the notation

$$\pi^\sharp\colon T^*M \to TM \tag{1.24}$$

for the mapping defined by $\beta(\pi^\sharp(\alpha) = \pi(\alpha, \beta)$ for $\alpha, \beta \in \Omega^1(M, \mathbb{R})$, then the Hamiltonian vector field X_f associated to a smooth function f on M is defined by $X_f = \pi^\sharp(df)$. The anchor map is $\pi^\sharp$ and the Lie bracket is given by

$$[\alpha, \beta] = \mathcal{L}_{\pi^\sharp(\alpha)}(\beta) - \mathcal{L}_{\pi^\sharp(\beta)}(\alpha) - d\pi(\alpha, \beta). \tag{1.25}$$

This Lie algebroid structure on T^*M is the unique one with the property that $a(df) = X_f$ and $[df, dg] = d\{f, g\}$ for all $f, g \in C^\infty(M, \mathbb{R})$. When π is nondegenerate, M is a symplectic manifold and this Lie algebra structure of $\Gamma(M, T^*M)$ is isomorphic to that of $\Gamma(M, TM)$.

Example. (Nijenhius manifolds) Let M be a manifold with a Nijenhuis structure, i.e., a vector valued 1-form $\mathcal{N} \in \Omega^1(M, TM)$ with the vanishing Nijenhuis torsion. Recall that the Nijenhuis torsion $T_\mathcal{N} \in \Omega^2(M, TM)$ is defined by

$$T_\mathcal{N}(X, Y) = [\mathcal{N}X, \mathcal{N}Y] - \mathcal{N}[\mathcal{N}X, Y] - \mathcal{N}[X, \mathcal{N}Y] + \mathcal{N}^2[X, Y] \tag{1.26}$$

for $X, Y \in \mathfrak{X}(M)$, note that $T_\mathcal{N} = \frac{1}{2}[\mathcal{N}, \mathcal{N}]$ for the Frölicher-Nijenhuis bracket. A vector valued 1-form $\mathcal{N}$ is called a *Nijenhuis tensor* if its Nijehnius torsion vanishes. To any Nijenhuis structure $\mathcal{N}$, there is associated a new Lie algebroid structure on TM. The anchor map is given by $a(X) = \mathcal{N}(X)$, while the Lie bracket is defined by

$$[X, Y]_\mathcal{N} = [\mathcal{N}X, Y] + [X, \mathcal{N}Y] - \mathcal{N}[X, Y]. \tag{1.27}$$

It is well known that powers of Nijenhuis tensors, considered as endomorphisms of the tangent bundle, are Nijenhuis tensors. Also any complex structure $\mathcal{J}$ on M is a Nijenhuis tensor.

Example. (generalized Nijenhuis manifolds) Let $(L \to M, [\cdot\,,\cdot], a)$ be a Lie algebroid and let $\mathcal{N}\colon L \to L$ be a homomorphism of vector bundles covering the identity on M, such that its Nijenhuis torsion vanishes, i.e.,

$$[\mathcal{N}X, \mathcal{N}Y] - \mathcal{N}[\mathcal{N}X, Y] - \mathcal{N}[X, \mathcal{N}Y] + \mathcal{N}^2[X, Y] = 0 \tag{1.28}$$

for all $X, Y \in \Gamma(M, L)$. If we define the anchor map by $a_{\mathcal{N}}(X) = (a \circ \mathcal{N})(X)$ and the Lie bracket by

$$[X, Y]_{\mathcal{N}} = [\mathcal{N}X, Y] + [X, \mathcal{N}Y] - \mathcal{N}[X, Y]. \tag{1.29}$$

then this gives a new Lie algebroid structure on L.

Example. (trivial Lie algebroids) For any real Lie algebra $\mathfrak{g}$, we define a Lie algebroid $L_{\mathfrak{g}}$ over a manifold M by the following way. As a vector bundle $L_{\mathfrak{g}} = TM \oplus (M \times \mathfrak{g})$, the anchor map is the projection on the first component and the Lie bracket on sections $\Gamma(M, L_{\mathfrak{g}}) \simeq \mathfrak{X}(M) \oplus C^\infty(M, \mathfrak{g})$ is defined by

$$[(X, f), (Y, g)] = ([X, Y], [f, g]), \tag{1.30}$$

where the bracket on sections $\Gamma(M, M \times \mathfrak{g}) \simeq C^\infty(M, \mathfrak{g})$ is given by

$$[f, g](x) = [f(x), g(x)]_{\mathfrak{g}}. \tag{1.31}$$

Example. (jet prolongation of Lie algebroids) Let $(L \xrightarrow{p} M, [\cdot\,,\cdot], a)$ be a Lie algebroid, then the r-th jet prolongations J^rL of L for $r \in \mathbb{N}_0$ has a unique Lie algebroid structure. The anchor map is given by $a_{J^rL} = \pi_0^r \circ a$, where $\pi_0^r \colon J^rL \to L$ is the canonical projection, while the Lie bracket is uniquely determined by requiring that the r-th jet prolongation

$$j^r \colon \Gamma(M, L) \to \Gamma(M, J^rL) \tag{1.32}$$

be a homomorphism of Lie algebroids. More about the relation of jet prolongation Lie algebroids to Cartan's method of equivalence one can find in [12].

1.3 Differential geometry of Lie algebroids

Because we can think of a Lie algebroid as a generalized tangent bundle, we may use a similar construction for it.

Consider a real (complex) Lie algebroid $(L \xrightarrow{\pi} M, [\cdot\,,\cdot], a)$. A section of the vector bundle $\Lambda^k L^*$ for $k \in \mathbb{N}_0$ is called a k-form of L and the space of all k-forms will be denoted by $\Omega_L^k(M)$. Similarly a section of the vector bundle $\Lambda^k L$ for $k \in \mathbb{N}_0$ is called a k-vector field of L and the space of all k-vector fields will be denoted by $\mathfrak{X}_L^k(M)$. Let $\Omega_L^k(M) = \{0\}$ and $\mathfrak{X}_L^k(M) = \{0\}$ for $k < 0$, then we denote by

$$\Omega_L^\bullet(M) = \bigoplus_{k \in \mathbb{Z}} \Omega_L^k(M) \quad \text{resp.} \quad \mathfrak{X}_L^\bullet(M) = \bigoplus_{k \in \mathbb{Z}} \mathfrak{X}_L^k(M) \tag{1.33}$$

the graded vector space of all forms of L resp. of all multivector fields of L. For a real (complex) vector bundle $E \to M$ a section of the vector bundle $\Lambda^k L^* \otimes E$ is called E-valued k-form of L. The space of sections will be denoted by $\Omega_L^k(M, E)$.

The graded vector space $\Omega_L^\bullet(M)$ has a natural structure of a graded commutative algebra via the wedge product

$$(\omega \wedge \tau)(\xi_1, \dots, \xi_{p+q}) = \frac{1}{p!\,q!} \sum_\sigma \operatorname{sign}(\sigma) \cdot \omega(\xi_{\sigma(1)}, \dots, \xi_{\sigma(p)})\, \tau(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)}), \tag{1.34}$$

where $\omega \in \Omega_L^p(M)$, $\tau \in \Omega_L^q(M)$ and $\xi_1, \dots, \xi_{p+q} \in \mathfrak{X}_L(M)$.

Further there is a differential operator $d_L \colon \Omega_L^\bullet(M) \to \Omega_L^{\bullet+1}(M)$ on the graded commutative algebra $\Omega_L^\bullet(M)$ defined by

$$\begin{aligned}(d_L\omega)(\xi_0, \dots, \xi_k) = &\sum_{i=0}^{k} (-1)^i a(\xi_i)\omega(\xi_0, \dots, \widehat{\xi_i}, \dots, \xi_k) \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([\xi_i, \xi_j], \xi_0, \dots, \widehat{\xi_i}, \dots, \widehat{\xi_j}, \dots, \xi_k)\end{aligned} \tag{1.35}$$

for $\omega \in \Omega^k_L(M)$ and $\xi_0, \dots, \xi_k \in \mathfrak{X}_L(M)$. The differential operator d_L is called the *Lie algebroid differential* of L or simply the *de Rham differential* of L. Besides for any $\xi \in \mathfrak{X}_L(M)$ we define the *insertion operator* $i^L_\xi \colon \Omega^\bullet_L(M) \to \Omega^{\bullet-1}_L(M)$ by

$$(i^L_\xi \omega)(\xi_1, \dots, \xi_k) = \omega(\xi, \xi_1, \dots, \xi_k) \tag{1.36}$$

and the *Lie derivative* $\mathcal{L}^L_\xi \colon \Omega^\bullet_L(M) \to \Omega^\bullet_L(M)$ through

$$(\mathcal{L}^L_\xi \omega)(\xi_1, \dots, \xi_k) = a(\xi)\omega(\xi_1, \dots, \xi_k) - \sum_{i=1}^k \omega(\xi_1, \dots, [\xi, \xi_i], \dots, \xi_k) \tag{1.37}$$

for $\omega \in \Omega^k_L(M)$ and $\xi, \xi_1, \dots, \xi_k \in \mathfrak{X}_L(M)$.

Remark. As $\Omega^\bullet_L(M)$ is a graded commutative algebra, the space of all graded derivations

$$\mathrm{Der}\, \Omega^\bullet_L(M) = \bigoplus_{k \in \mathbb{Z}} \mathrm{Der}_k \Omega^\bullet_L(M), \tag{1.38}$$

where $\mathrm{Der}_k \Omega^\bullet_L(M)$ is the space of graded derivations of degree k, has a structure of a graded Lie algebra with the Lie bracket defined by

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{k_1 k_2} D_2 \circ D_1 \tag{1.39}$$

for $D_1 \in \mathrm{Der}_{k_1} \Omega^\bullet_L(M)$ and $D_2 \in \mathrm{Der}_{k_2} \Omega^\bullet_L(M)$.

Lemma 1. The insertion operator $i^L_\xi \colon \Omega^\bullet_L(M) \to \Omega^{\bullet-1}_L(M)$ and the Lie derivative $\mathcal{L}^L_\xi \colon \Omega^\bullet_L(M) \to \Omega^\bullet_L(M)$ have the following properties:

i) $i^L_\xi(\omega \wedge \tau) = i^L_\xi \omega \wedge \tau + (-1)^{\deg(\omega)} \omega \wedge i^L_\xi \tau$, i.e., i^L_ξ is a graded derivation od degree -1,
ii) $\mathcal{L}^L_\xi(\omega \wedge \tau) = \mathcal{L}^L_\xi \omega \wedge \tau + \omega \wedge \mathcal{L}^L_\xi \tau$, i.e., $\mathcal{L}^L_\xi$ is a graded derivation od degree 0,
iii) $[\mathcal{L}^L_\xi, i^L_\eta] = i^L_{[\xi,\eta]}$,
iv) $[\mathcal{L}^L_\xi, \mathcal{L}^L_\eta] = \mathcal{L}^L_{[\xi,\eta]}$,
v) $[i^L_\xi, i^L_\eta] = 0$.

Proof. The proof goes along the same line as the proof of this lemma for a linear connection, see [32]. ♠

Lemma 2. The Lie algebroid differential $d_L \colon \Omega^\bullet_L(M) \to \Omega^\bullet_L(M)$ has the following properties:

i) $d_L(\omega \wedge \tau) = d_L \omega \wedge \tau + (-1)^{\deg(\omega)} \omega \wedge d_L \tau$, i.e., d_L is a graded derivation od degree 1,
ii) $d_L \circ d_L = \frac{1}{2}[d_L, d_L] = 0$, i.e., d_L is a differential,
iii) $[\mathcal{L}^L_\xi, d] = 0$,
iv) $[i^L_\xi, d] = \mathcal{L}^L_\xi$ (Cartan's formula).

Proof. The proof goes along the same line as the proof of this lemma for a linear connection, see [32]. ♠

Because d_L is a graded derivation of degree 1 and a differential, i.e., $d^2_L = 0$, the graded commutative algebra $\Omega^\bullet_L(M)$ is a differential graded commutative algebra. The cohomology of the complex

$$0 \longrightarrow \Omega^0_L(M) \xrightarrow{d_L} \Omega^1_L(M) \xrightarrow{d_L} \dots \xrightarrow{d_L} \Omega^r_L(M) \longrightarrow 0, \tag{1.40}$$

where $r = \mathrm{rk}\, L$, called the *Lie algebroid cohomology* of L, we will denote by $H^\bullet_L(M)$. It unifies de Rham and Chevalley–Eilenberg cohomologies. When $L = TM$, we obtain $H^\bullet_{TM}(M) = H^\bullet_{\mathrm{dR}}(M)$, on the other hand when $L = \mathfrak{g}$, i.e., L is a Lie algebroid over a one-point manifold, we receive $H^\bullet_{\mathfrak{g}}(M) = H^\bullet(\mathfrak{g}, \mathfrak{g})$. Furthermore because d_L is a graded derivation of degree 1, the Lie algebroid cohomology $H^\bullet_L(M)$ of L is a graded commutative algebra.

Furthermore we can ask when is this complex an elliptic complex? For any $f \in C^\infty(M,\mathbb{R})$ and $\omega \in \Omega^k_L(M)$ we have

$$(\mathrm{ad}(f)d_L)\,\omega = d_L(f\omega) - fd_L\omega = d_Lf \wedge \omega - fd_L\omega + fd_L\omega = a^*(df) \wedge \omega,$$

hence for the principal symbol $\sigma_1(d_L)$ we get

$$\sigma_1(d_L)(\xi_x) = a^*(\xi_x)\wedge\colon (\Lambda^k L^*)_x \to (\Lambda^{k+1}L^*)_x$$

for every $x \in M$ and $\xi_x \in T^*_xM$, i.e., the symbol is the exterior multiplication by $a^*(df)$. Therefore we obtain the *Koszul complex*

$$0 \longrightarrow (\Lambda^0 L^*)_x \xrightarrow{a^*(\xi_x)\wedge} (\Lambda^1 L^*)_x \xrightarrow{a^*(\xi_x)\wedge} \dots \xrightarrow{a^*(\xi_x)\wedge} (\Lambda^n L^*)_x \longrightarrow 0, \tag{1.41}$$

where $r = \mathrm{rk}\, L$, which is an exact sequence if and only if $a^*(\xi_x) \neq 0$. Thus, the differential complex is elliptic if and only if the corresponding Koszul complex is an exact sequence for any $x \in M$ and $0 \neq \xi_x \in T^*_xM$, in other words if and only if $a^*(\xi_x) \neq 0$ for any $x \in M$ and $0 \neq \xi_x \in T^*_xM$.

If $L \xrightarrow{a} TM$ is a real Lie algebroid, then the elipticity is equivalent to the requirement that $a^*\colon T^*M \to L^*$ is injective or that $a\colon L \to TM$ is surjective. For a complex Lie algebroid $L \xrightarrow{a} TM_{\mathbb{C}}$ it corresponds to the requirement that $a^*{}_{|T^*M}\colon T^*M \hookrightarrow (TM_{\mathbb{C}})^* \to L^*$ is injective.

Lemma 3. For a Lie algebroid $(L \xrightarrow{\pi} M, [\cdot\,,\cdot], a)$, the graded commutative algebra $\mathfrak{X}^\bullet_L(M)$ of multivector fields of L carries a structure of a Gerstenhaber algebra. The bracket $[\cdot\,,\cdot]$ of the Gerstenhaber alegebra, called an *odd Poisson bracket* or a *Schouten bracket*, generalizes the Schouten-Nijenhius bracket of multivector fields on a manifold. The Schouten bracket is defined as the unique extension of the Lie bracket $[\cdot\,,\cdot]$ on $\mathfrak{X}_L(M)$ on $\mathfrak{X}^\bullet_L(M)$ satisfying

i) $[f,g] = 0$ for $f,g \in C^\infty(M,\mathbb{K}) = \mathfrak{X}^0_L(M)$,
ii) $[\xi, f] = -[f,\xi] = a(\xi)f$ for $f \in C^\infty(M,\mathbb{K})$, $\xi \in \mathfrak{X}_L(M)$,
iii) $[\pi,\sigma] = -(-1)^{(p-1)(q-1)}[\sigma,\pi]$ for $\pi \in \mathfrak{X}^p_L(M)$, $\sigma \in \mathfrak{X}^q_L(M)$,
iv) $[\pi,\sigma\wedge\rho] = [\pi,\sigma]\wedge\rho + (-1)^{(p-1)q}\sigma\wedge[\pi,\rho]$ for $\pi \in \mathfrak{X}^p_L(M)$, $\sigma \in \mathfrak{X}^q_L(M)$ and $\rho \in \mathfrak{X}^\bullet_L(M)$, i.e., $[\pi,\cdot\,]$ is a graded derivation of degree $p-1$ on $\mathfrak{X}^\bullet_L(M)$.

Explicitly, for decomposable multivector fields $\pi = \xi_1 \wedge \xi_2 \wedge \dots \wedge \xi_k$, $\sigma = \eta_1 \wedge \eta_2 \wedge \dots \wedge \eta_\ell$ with $\xi_i, \eta_j \in \mathfrak{X}_L(M)$ and $f \in C^\infty(M,\mathbb{K})$ we obtain

$$[\pi,\sigma] = \sum_{i=1}^{k}\sum_{j=1}^{\ell}(-1)^{i+j}[\xi_i,\eta_j]\wedge\xi_1\wedge\cdots\widehat{\xi_i}\cdots\wedge\xi_k\wedge\eta_1\wedge\cdots\widehat{\eta_j}\cdots\wedge\eta_\ell \tag{1.42}$$

and

$$[f,\pi] = -i^L_{df}\pi = \sum_{i=1}^{k}(-1)^i(a(\xi_i)f)\xi_1\wedge\cdots\widehat{\xi_i}\cdots\wedge\xi_k, \tag{1.43}$$

where $i^L_{df}\colon \mathfrak{X}^\bullet_L(M) \to \mathfrak{X}^{\bullet-1}_L(M)$ is the insertion operator, the adjoint of $df\wedge\colon \Omega^\bullet_L(M) \to \Omega^{\bullet+1}_L(M)$.

Proof. See [33]. ♠

Remark. There are different equivalent ways to define a Lie algebroid structure on a vector bundle $\pi\colon L \to M$, either by a Gerstenhaber algebra structure on $\mathfrak{X}^\bullet_L(M)$ or by a graded derivation of degree 1 on $\Omega^\bullet_L(M)$ that is a differential. Even one can define a Lie algebroid structure on a vector bundle $\pi\colon L \to M$ as the supermanifold ΠL together with a homological vector field d_L of degree 1. It is important that d_L is of degree 1 with respect to the natural $\mathbb{Z}$-grading on functions on ΠL, in order to define a Lie algebroid structure on L.

Definition 4. A pair $(L \to M, [\cdot\,,\cdot]_L, a_L; L^* \to M, [\cdot\,,\cdot]_{L^*}, a_{L^*})$ of Lie algebroids in duality is called a *Lie bialgebroid* if d_L is a derivation of the Schouten bracket $[\cdot,\cdot]_{L^*}$ on $\mathfrak{X}^\bullet_{L^*}(M)$, in the sense that

$$d_L[\xi,\eta]_{L^*} = [d_L\xi,\eta]_{L^*} + [\xi, d_L\eta]_{L^*} \tag{1.44}$$

for all $\xi, \eta \in \Gamma(M, \Lambda^\bullet L^*)$. This condition is satisfied if and only if d_{L^*} is a derivation of $[\cdot\,,\cdot]_L$. Therefore the notion of Lie bialgebroids is self-dual, i.e., (L, L^*) is a Lie bialgebroid if and only if (L^*, L) is a Lie bialgebroid.

1.4 Courant algebroids

The Courant bracket is a generalization of the Lie bracket on sections of the tangent bundle to the bracket on sections of the direct sum of the tangent bundle and the vector bundle of p-forms.

The case $p = 1$ was first introduced in its present form by Thomas Courant in his dissertation thesis based on his work with Alan Weinstein. They used it to define a new geometrical structure called the *Dirac structure*, which unifies the Poisson geometry and the presymplectic geometry (the geometry defined by real closed 2-form) by expressing each structure as a maximal isotropic subbundle of $TM \oplus T^*M$. The integrability condition, namely that the subbundle be closed under the Courant bracket, specializes to the usual integrability conditions in the Poisson and presymplectic cases. The twisted version of the Courant bracket was introduced by Pavol Ševera.

Complex version of the $p = 1$ Courant bracket plays an important role in the generalized complex geometry introduced by Nigel Hitchin. This, like the previous example, unifies the complex geometry on one side and the symplectic geometry on the other hand. Closure under the Courant bracket is the integrability condition of a generalized almost complex structure.

Definition 5. A *Courant algebroid* $(E \xrightarrow{\pi} M, \langle\cdot\,,\cdot\rangle, [\cdot\,,\cdot], a)$ is a real vector bundle $\pi\colon E \to M$ together with a non-degenerate symmetric $C^\infty(M,\mathbb{R})$-bilinear form $\langle\cdot\,,\cdot\rangle\colon \Gamma(M,E) \times \Gamma(M,E) \to C^\infty(M,\mathbb{R})$, a bilinear mapping $[\cdot\,,\cdot]\colon \Gamma(M,E) \times \Gamma(M,E) \to \Gamma(M,E)$, called the *Courant bracket*, and a homomorphism of vector bundles $a\colon E \to TM$, called the *anchor map*, over M covering the identity on M, i.e., the following diagram

$$\begin{array}{ccc} E & \xrightarrow{\;a\;} & TM \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_M} \\ M & \xrightarrow[\mathrm{id_M}]{} & M \end{array}$$

commutes. Moreover they fulfills

i) $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$

ii) $a([e_1, e_2]) = [a(e_1), a(e_2)]$

iii) $[e_1, fe_2] = f[e_1, e_2] + (a(e_1)f)e_2$

iv) $a(e_1)\langle e_2, e_3\rangle = \langle [e_1, e_2], e_3\rangle + \langle e_2, [e_1, e_3]\rangle$

v) $[e_1, e_1] = \frac{1}{2}a^*(d\langle e_1, e_1\rangle)$

for all $e_1, e_2, e_3 \in \Gamma(M, E)$ and $f \in C^\infty(M,\mathbb{R})$.

Remark. Note that the homomorphism $a^*\colon T^*M \to E$ of vector bundles if defined by the formula

$$\langle a^*(\xi), e\rangle = \xi(a(e)), \tag{1.45}$$

where $\xi \in \Omega^1(M,\mathbb{R})$ and $e \in \Gamma(M, E)$.

If the bracket $[\cdot\,,\cdot]$ was skew-symmetric, then $(L \xrightarrow{\pi} M, [\cdot\,,\cdot], a)$ has a structure of a real Lie algebroid; axiom v) indicates that the failure to be a Lie algebroid is measured by the inner product, which itself is invariant under the adjoint action by axiom iv).

Lemma 4. Let $(E \xrightarrow{\pi} M, \langle\cdot\,,\cdot\rangle, [\cdot\,,\cdot], a)$ be a Courant algebroid, then we have $a \circ a^* = 0$.

Proof. From property v) we get $[e_1, e_2] + [e_2, e_1] = a^*(d\langle e_1, e_2\rangle)$ for all $e_1, e_2 \in \Gamma(M, E)$. Further together with property ii) we have $[a(e_1), a(e_2)] + [a(e_2), a(e_1)] = (a \circ a^*)(d\langle e_1, e_2\rangle)$ which implies that $(a \circ a^*)(d\langle e_1, e_2\rangle) = 0$. The last equation is equivalent to the relation $(a \circ a^*)(df) = 0$ for all

$f \in C^\infty(M,\mathbb{R})$. This is because of the nondegeneration of the bilinear pairing. Hence it follows that $a \circ a^* = 0$. ♠

Definition 6. A Courant algebroid is called *exact* when the following sequence

$$0 \to T^*M \xrightarrow{a^*} E \xrightarrow{a} TM \to 0 \tag{1.46}$$

of vector bundles is an exact sequence.

Example. (standard Courant algebroid) A basic example is the so called standard Courant algebroid. As a vector bundle $E = TM \oplus T^*M$, the anchor map is the projection on the first component, the bilinear pairing is given by

$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\xi(Y)+\eta(X)), \tag{1.47}$$

while the Courant bracket is defined via

$$[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X\eta - i_Y d\xi, \tag{1.48}$$

where $X, Y \in \mathfrak{X}(M)$ and $\xi,\eta \in \Omega^1(M,\mathbb{R})$. Moreover because $a^*(\xi) = 2\xi$ for $\xi \in \Omega^1(M,\mathbb{R})$, we obtain that E is an exact Courant algebroid.

Example. (twisted standard Courant algebroids) For any closed 3-form $H \in \Omega^3(M,\mathbb{R})$, we define a Courant algebroid E_H as follows. As a vector bundle $E_H = TM \oplus T^*M$, the anchor map is the projection on the first component, the bilinear pairing is given by

$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\xi(Y)+\eta(X)), \tag{1.49}$$

while the Courant bracket is defined via

$$[X+\xi, Y+\eta]_H = [X,Y] + \mathcal{L}_X\eta - i_Y d\xi + i_X i_Y H, \tag{1.50}$$

where $X, Y \in \mathfrak{X}(M)$ and $\xi,\eta \in \Omega^1(M,\mathbb{R})$. Anyway as in the previous case we have $a^*(\xi) = 2\xi$ for $\xi \in \Omega^1(M,\mathbb{R})$, therefore obtain that E_H is an exact Courant algebroid.

In fact, it was proved by P. Ševera that each exact Courant algebroid is isomorphic to above example for any given closed 3-form $H \in \Omega^3(M,\mathbb{R})$. Explicitly, the theorem says that the exact Courant algebroids are classified by de Rham cohomology $H^3_{\mathrm{dR}}(M,\mathbb{R})$.

Remark. So given Courant bracket is part of a hierarchy of brackets on sections of vector bundles $TM \oplus \Lambda^p T^*M$ for $p \in \mathbb{N}_0$, defined by the similar formula as for $p=1$

$$[X+\sigma, Y+\tau] = [X,Y] + \mathcal{L}_X\tau - i_Y d\sigma + i_X i_Y F, \tag{1.51}$$

where $X, Y \in \mathfrak{X}(M)$, $\sigma,\tau \in \Omega^p(M,\mathbb{R})$ and $F \in \Omega^{p+2}(M,\mathbb{R})$ is a closed $(p+2)$-form.

Example. (Lie bialgebroids) Let $(L \to M, [\cdot,\cdot]_L, a_L; L^* \to M, [\cdot,\cdot]_{L^*}, a_{L^*})$ be a Lie bialgebroid. We define a Courant algebroid E by the following way. As a vector bundle $E = L \oplus L^*$, the anchor map is given by $a = a_L + a_{L^*}$, the bilinear pairing is defined through

$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\xi(Y)+\eta(X)) \tag{1.52}$$

and the Courant bracket via

$$[X+\xi, Y+\eta] = [X,Y]_L + \mathcal{L}^{L^*}_\xi Y - i^{L^*}_\eta d_{L^*}X + [\xi,\eta]_{L^*} + \mathcal{L}^L_X\eta - i^L_Y d_L\xi, \tag{1.53}$$

where $X, Y \in \Gamma(M,L)$ and $\xi,\eta \in \Gamma(M,L^*)$.

In a special case when the Lie algebroid L is $(TM \to M, [\cdot,\cdot], \mathrm{id}_{TM})$ and the Lie algebroid L^* is $(T^*M \to M, [\cdot,\cdot]_{T^*M}, a_{T^*M})$, where $a_{T^*M} = 0$ and the Lie bracket is zero. Then this construction gives on $TM \oplus T^*M$ a structure of the standard Courant algebroid.

Example. (Lie algebras) Let $(E \to M, \langle\cdot,\cdot\rangle_0, [\cdot,\cdot]_0, a_0)$ be a Courant algebroid and let $\mathfrak{g}$ be a Lie algebra with an ad-invariant non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ and with the Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$. Then we define a structure of a Courant algebroid on the vector bundle $E_{\mathfrak{g}} = E_0 \oplus (M \times \mathfrak{g})$ as follows. The anchor map is given by $a = a_0 \circ \mathrm{pr}_{E_0}$. Because $\Gamma(M, E_{\mathfrak{g}}) \simeq \Gamma(M, E_0) \oplus C^\infty(M, \mathfrak{g})$ the Courant bracket is defined through

$$[e_1 + f_1, e_2 + f_2] = [e_1, e_2]_0 + \mathcal{L}_{a_0(e_1)} f_2 - \mathcal{L}_{a_0(e_2)} f_1 + [f_1, f_2]_{\mathfrak{g}} + a_0^* \langle df_1, f_2\rangle_{\mathfrak{g}}, \tag{1.54}$$

and the bilinear pairing by

$$\langle e_1 + f_1, e_2 + f_2\rangle = \langle e_1, e_2\rangle_0 + \langle f_1, f_2\rangle_{\mathfrak{g}}, \tag{1.55}$$

where $e_1, e_2 \in \Gamma(M, E_0)$ and $f_1, f_2 \in C^\infty(M, \mathfrak{g})$. The the bracket on sections $\Gamma(M, M \times \mathfrak{g}) \simeq C^\infty(M, \mathfrak{g})$ is given by

$$[f_1, f_2]_{\mathfrak{g}}(x) = [f_1(x), f_2(x)]_{\mathfrak{g}} \tag{1.56}$$

and the bilinear paring by

$$\langle f_1, f_2\rangle_{\mathfrak{g}}(x) = \langle f_1(x), f_2(x)\rangle_{\mathfrak{g}}, \tag{1.57}$$

where $x \in M$.

1.5 Generalized complex structures

A generalized complex geometry was introduced by Nigel Hitchin [21] and further developed by his students Marco Gualtieri [22], [20] and Gil Cavalcanti [23]. It contains complex and symplectic geometry as its extremal special cases. Generalized complex structures give a wide class of complex Lie algebroids.

Definition 7. Consider a Courant algebroid $(E \to M, \langle\cdot,\cdot\rangle, [\cdot,\cdot], a)$ then a maximal isotropic subbundle L of E is called an *almost Dirac structure.* If L is involutive, i.e., sections of L are closed under the Courant bracket, then an almost Dirac structure is said to be integrable or simply a *Dirac structure.*

Example. The contangent bundle $T^*M \subset TM \oplus T^*M$ is a Dirac structure for any H-twisted standard Courant algebroid with $H \in \Omega^3_{\mathrm{cl}}(M, \mathbb{R})$.

Example. The tangent bundle $TM \subset TM \oplus T^*M$ is an almost Dirac structure for any H-twisted standard Courant algebroid and a Dirac structure only for standard Courant algebroid.

Remark. If L is a Dirac structure, then the restriction of the Courant bracket on sections of L gives a structure of a Lie algebroid on the vector bundle L. This follows from the fact that L is a maximal isotropic subbundle.

Definition 8. A *generalized almost complex structure* on a Courant algebroid E is a vector bundle automorphism $\mathcal{J}\colon E \to E$ covering the identity on M such that $\mathcal{J}^2 = -\mathrm{id}_E$ and which is orthogonal with respect to the inner product (pseudo-Euclidean structure).

Lemma 5. Let E be a Courant algebroid and $\mathcal{J}\colon E \to E$ a vector bundle automorphism covering the identity on M then the following conditions are equivalent:

i) $\mathcal{J}^2 = -\mathrm{id}_E$ and $\mathcal{J}^*\mathcal{J} = \mathrm{id}_E$, i.e., $\langle \mathcal{J}(e_1), \mathcal{J}(e_2)\rangle = \langle e_1, e_2\rangle$,

ii) $\mathcal{J}^2 = -\mathrm{id}_E$ and $\mathcal{J}^* = -\mathcal{J}$, i.e., $\langle \mathcal{J}(e_1), e_2\rangle + \langle e_1, \mathcal{J}(e_2)\rangle = 0$,

where $e_1, e_2 \in \Gamma(M, E)$.

Proof. It follows immediately from the definition of $\mathcal{J}^*$. ♠

As long as $\mathcal{J}$ is a generalized almost complex structure then we can extend $\mathcal{J}$ by linearity on the complexification $E_{\mathbb{C}}$ of vector bundle E. Using the following isomorphism $\Gamma(M, E_{\mathbb{C}}) \simeq$

$\Gamma(M,E)\otimes\mathbb{C}$ we can write $\mathcal{J}_{\mathbb{C}}(e_1+i\,e_2)=\mathcal{J}_{\mathbb{C}}(e_1)+i\,\mathcal{J}_{\mathbb{C}}(e_2)$ for $e_1,e_2\in\Gamma(M,E)$, moreover $\mathcal{J}_{\mathbb{C}}$ is an automorphism of the complex vector bundle $E_{\mathbb{C}}$. Further on the complexification $E_{\mathbb{C}}$ is given a vector bundle morphism $^{-}\colon E_{\mathbb{C}}\to E_{\mathbb{C}}$ by the relation $\overline{e_1+i\,e_2}=e_1-i\,e_2$ for $e_1,e_2\in\Gamma(M,E)$. Note that this is an automorphism of the real vector bundle $E_{\mathbb{C}}$ not the complex vector bundle. Immediately it follows that $(^{-})^2=\mathrm{id}_{E_{\mathbb{C}}}$.

Because $\mathcal{J}_{\mathbb{C}}$ is an automorphism of the complex vector bundle $E_{\mathbb{C}}$, therefore there exists the complex $+i$-eigenbundle $L=\ker(\mathcal{J}_{\mathbb{C}}-i\,\mathrm{id}_{E_{\mathbb{C}}})$ of the automorphism $\mathcal{J}_{\mathbb{C}}$. On the other hand it is quite easy to verify that $\overline{L}=\ker(\mathcal{J}_{\mathbb{C}}+i\,\mathrm{id}_{E_{\mathbb{C}}})$. Further because L is the $+i$-eigenbundle and $\overline{L}$ is the $-i$-eigenbundle, hence $L\cap\overline{L}=0$. Now if e_1,e_2 are two sections of L, then $\langle e_1,e_2\rangle_{\mathbb{C}}=\langle\mathcal{J}_{\mathbb{C}}(e_1),\mathcal{J}_{\mathbb{C}}(e_2)\rangle_{\mathbb{C}}=\langle ie_1,ie_2\rangle_{\mathbb{C}}=-\langle e_1,e_2\rangle_{\mathbb{C}}$, therefore L is a complex almost Dirac structure of the complex Courant algebroid $E_{\mathbb{C}}$. Moreover we know that $L\oplus\overline{L}=E_{\mathbb{C}}$ and furthermore from the fact that $L,\overline{L}$ are isotropic complex subbundles and from whence that $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ is a nondegenerate bilinear form it follows that $L^*\simeq\overline{L}$.

In fact, we have proved the following lemma which provides an equivalent definition of a generalized almost complex structure on a Courant algebroid.

Lemma 6. A generalized almost complex structure on a Courant algebroid E is equivalently given by a complex almost Dirac structure $L\subset E_{\mathbb{C}}$ such that $L\cap\overline{L}=0$ and $L\oplus\overline{L}=E_{\mathbb{C}}$.

Remark. Similarly as in the complex geometry, a complex structure is an almost complex structure such that it satisfies some integrability condition. Therefore we define a generalized complex structure as a generalized almost complex structure with some integrability condition.

Definition 9. A *generalized complex structure* on a Courant algebroid E is a generalized almost complex structure $\mathcal{J}$ for which the complex $+i$-eigenbundle $L\subset E_{\mathbb{C}}$ is a complex Dirac structure.

Accordingly as for a generalized almost complex structure there is an alternative definition of a generalized complex structure expressed through $+i$-eigenbundle.

Lemma 7. A generalized complex structure on a Courant algebroid E is equivalently given by a complex Dirac structure $L\subset E_{\mathbb{C}}$ such that $L\cap\overline{L}=0$ and $L\oplus\overline{L}\simeq E_{\mathbb{C}}$.

The previous definitions are illustrated most clearly with two extremal cases of generalized complex structures on H-twisted standard Courant algebroid $TM\oplus T^*M$.

Example. (complex structures) Consider the automorphism of $TM\oplus T^*M$ defined by

$$\mathcal{J}_J=\begin{pmatrix}-J & 0\\ 0 & J^*\end{pmatrix},$$

where $J\colon TM\to TM$ is a complex structure on M. Then we get $\mathcal{J}_J^2=-\mathrm{id}_{TM\oplus T^*M}$ and $\mathcal{J}_J^*=-\mathcal{J}_J$. The $+i$-eigenbundle $L=T^{(1,0)}M\oplus T^{*(0,1)}M$ is integrable if and only if J is integrable and $H^{(3,0)}=0$.

Example. (symplectic structures) Consider the automorphism of $TM\oplus T^*M$ given via

$$\mathcal{J}_\omega=\begin{pmatrix}0 & -\omega^{-1}\\ \omega & 0\end{pmatrix},$$

where $\omega\in\Omega^2(M,\mathbb{R})$ ($\omega\colon TM\to T^*M$) is a symplectic structure on M. Again, we have $\mathcal{J}_\omega^2=-\mathrm{id}_{TM\oplus T^*M}$ and the $+i$-eigenbundle $L=\{X-i\omega(X);\ X\in\Gamma(M,TM_{\mathbb{C}}\}$ is integrable if and only if $H=0$ and $d\omega=0$.

Chapter 2

Linear Lie algebroid connections

2.1 Linear Lie algebroid connections

In this section we introduce the notion of linear Lie algebroid connections, i.e., Lie algebroid connections on real (complex) vector bundles. The more general definition of Lie algebroid connections on fiber bundles will be presented in Chapter 3. It is a natural generalization of a linear connection on vector bundles, since Lie algebroids can be understood as generalized tangent bundles. Therefore it is possible to use similar constructions for linear Lie algebroid connections as for linear connections.

Remark. We will use notation $\mathbb{K}$ for the field $\mathbb{R}$ of real or for the field $\mathbb{C}$ of complex numbers.

Definition 10. Let $(L \longrightarrow M, [\cdot\,,\cdot], a)$ be a real (complex) Lie algebroid and let $E \longrightarrow M$ be a real (complex) vector bundle. We denote the space of sections of the vector bundle $\Lambda^k L^* \otimes E$ for $k \in \mathbb{N}_0$ by $\Omega_L^k(M, E)$ and sections will be called E-valued k-forms of L or k-forms of L with values in E. A *linear Lie algebroid connection* or an *L-connection* on a vector bundle E is a $\mathbb{K}$-linear mapping

$$\nabla \colon \Omega_L^0(M, E) \to \Omega_L^1(M, E) \tag{2.1}$$

satisfying Leibniz rule $\nabla(fs) = d_L f \otimes s + f\nabla s$ for any $f \in C^\infty(M, \mathbb{K})$ and $s \in \Omega_L^0(M, E)$.

Remark. For any $\xi \in \mathfrak{X}_L(M)$ we have a $\mathbb{K}$-linear mapping $\nabla_\xi \colon \Omega_L^0(M, E) \to \Omega_L^0(M, E)$ given by

$$\nabla_\xi s = i_\xi^L(\nabla s) \tag{2.2}$$

for $s \in \Omega_L^0(M, E)$, called the *covariant derivative* along ξ. Moreover it satisfies

$$\nabla_\xi(fs) = (\mathcal{L}_\xi^L f)s + f\nabla_\xi s \tag{2.3}$$

and

$$\nabla_{\xi_1+\xi_2} s = \nabla_{\xi_1} s + \nabla_{\xi_2} s, \qquad \nabla_{f\xi} s = f\nabla_\xi s \tag{2.4}$$

for all $f \in C^\infty(M, \mathbb{K})$, $\xi, \xi_1, \xi_2 \in \mathfrak{X}_L(M)$ and $s \in \Omega_L^0(M, E)$. Therefore a linear Lie algebroid connection on a vector bundle E can be equivalently defined as a $\mathbb{K}$-bilinear mapping

$$\begin{aligned} \nabla \colon \mathfrak{X}_L(M) \times \Omega_L^0(M, E) &\to \Omega_L^0(M, E), \\ (\xi, s) &\mapsto \nabla_\xi s \end{aligned} \tag{2.5}$$

satisfying (2.3) and (2.4) for all $\xi \in \mathfrak{X}_L(M)$, $f \in C^\infty(M, \mathbb{K})$ and $s \in \Omega_L^0(M, E)$.

Tensorial operations on vector bundles may be extended naturally to vector bundles with L-connections. More precisely, if E_1 and E_2 are two vector bundles with L-connections ∇^{E_1} and ∇^{E_2}, then $E_1 \otimes E_2$ has naturally induced L-connection $\nabla^{E_1 \otimes E_2}$ uniquely determined by the formula

$$\nabla^{E_1 \otimes E_2}_\xi (s_1 \otimes s_2) = \nabla^{E_1}_\xi s_1 \otimes s_2 + s_1 \otimes \nabla^{E_2}_\xi s_2 \tag{2.6}$$

for all $\xi \in \mathfrak{X}_L(M)$, $s_1 \in \Omega^0_L(M, E_1)$ and $s_2 \in \Omega^0_L(M, E_2)$. If we are given a vector bundle E with an L-connection ∇^E then the dual vector bundle E^* has a natural L-connection ∇^{E^*} defined by the identity

$$\mathcal{L}^L_\xi \langle t, s\rangle = \langle \nabla^{E^*}_\xi t, s\rangle + \langle t, \nabla^E_\xi s\rangle \tag{2.7}$$

for all $\xi \in \mathfrak{X}_L(M)$, $s \in \Omega^0_L(M, E)$ and $t \in \Omega^0_L(M, E^*)$, where $\langle \cdot\,, \cdot \rangle \colon \Omega^0_L(M, E^*) \times \Omega^0_L(M, E) \to C^\infty(M, \mathbb{K})$ is the natural pairing. In particular, any L-connection ∇^E on a vector bundle E induces an L-connection $\nabla^{\mathrm{End}(E)}$ on $\mathrm{End}(E) \simeq E^* \otimes E$ by the rule

$$(\nabla^{\mathrm{End}(E)}_\xi T)s = \nabla^E_\xi (Ts) - T(\nabla^E_\xi s) = [\nabla^E_\xi, T]s \tag{2.8}$$

for all $\xi \in \mathfrak{X}_L(M)$, $T \in \Omega^0_L(M, \mathrm{End}(E))$ and $s \in \Omega^0_L(M, E)$.

For any vector bundle E the graded vector space $\Omega^\bullet_L(M, E)$ is a graded $\Omega^\bullet_L(M)$-module through

$$(\alpha \wedge \omega)(\xi_1, \dots, \xi_{p+q}) = \frac{1}{p!\,q!} \sum_\sigma \mathrm{sign}(\sigma) \cdot \alpha(\xi_{\sigma(1)}, \dots, \xi_{\sigma(p)})\, \omega(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)}), \tag{2.9}$$

where $\alpha \in \Omega^p_L(M)$, $\omega \in \Omega^q_L(M, E)$ and $\xi_1, \dots, \xi_{p+q} \in \mathfrak{X}_L(M)$. The graded module homomorphisms $\Phi \colon \Omega^\bullet_L(M, E) \to \Omega^\bullet_L(M, E)$ (so that $\Phi(\alpha \wedge \omega) = \alpha \wedge (-1)^{\deg(\Phi)\cdot \deg(\omega)} \Phi(\omega))$ coincide with the mappings $\mu(A)$ for $A \in \Omega^p_L(M, \mathrm{End}(E))$, which are given by

$$(\mu(A)\,\omega)(\xi_1, \dots, \xi_{p+q}) = \frac{1}{p!\,q!} \sum_\sigma \mathrm{sign}(\sigma) \cdot A(\xi_{\sigma(1)}, \dots, \xi_{\sigma(p)})\, \omega(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)}), \tag{2.10}$$

where $\xi_1, \dots, \xi_{p+q} \in \mathfrak{X}_L(M)$. Moreover, the graded vector space $\Omega^\bullet_L(M, \mathrm{End}(E))$ has a natural structure of a graded associative algebra via

$$(\omega \wedge \tau)(\xi_1, \dots, \xi_{p+q}) = \frac{1}{p!\,q!} \sum_\sigma \mathrm{sign}(\sigma) \cdot (\omega(\xi_{\sigma(1)}, \dots, \xi_{\sigma(p)}) \circ \tau(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)})) \tag{2.11}$$

and a natural structure of a graded Lie algebra through

$$[\omega, \tau](\xi_1, \dots, \xi_{p+q}) = \frac{1}{p!\,q!} \sum_\sigma \mathrm{sign}(\sigma) \cdot [\omega(\xi_{\sigma(1)}, \dots, \xi_{\sigma(p)}), \tau(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)})], \tag{2.12}$$

where $\omega \in \Omega^p_L(M, \mathrm{End}(E))$, $\tau \in \Omega^q_L(M, \mathrm{End}(E))$ and $\xi_1, \dots, \xi_{p+q} \in \mathfrak{X}_L(M)$. Comparing these two definitions we may write

$$[\omega, \tau] = \omega \wedge \tau - (-1)^{\deg(\omega)\deg(\tau)}\, \tau \wedge \omega. \tag{2.13}$$

for $\omega, \tau \in \Omega^\bullet_L(M, \mathrm{End}(E))$.

Let ∇ be an L-connection on a vector bundle E then the *covariant exterior derivative*

$$d^\nabla \colon \Omega^\bullet_L(M, E) \to \Omega^{\bullet+1}_L(M, E) \tag{2.14}$$

is defined by

$$(d^\nabla \omega)(\xi_0, \xi_1, \dots, \xi_k) = \sum_{i=0}^k (-1)^i \nabla_{\xi_i} \omega(\xi_0, \dots, \hat{\xi}_i, \dots, \xi_k) + \sum^k_{0 \le i < j \le k} (-1)^{i+j} \omega([\xi_i, \xi_j], \xi_0, \dots, \hat{\xi}_i, \dots, \hat{\xi}_j, \dots, \xi_k), \tag{2.15}$$

where $\omega \in \Omega^k_L(M,E)$ and $\xi_0,\dots,\xi_k \in \mathfrak{X}_L(M)$.

Lemma 8. The covariant exterior derivative $d^\nabla\colon \Omega^\bullet_L(M,E) \to \Omega^{\bullet+1}_L(M,E)$ has the following properties:

i) $d^\nabla(\Omega^k_L(M,E)) \subset \Omega^{k+1}_L(M,E)$,

ii) $d^\nabla{}_{|\Omega^0_L(M,E)} = \nabla$,

iii) $d^\nabla(\alpha\wedge\omega) = d_L\alpha\wedge\omega + (-1)^{\deg(\alpha)}\alpha\wedge d^\nabla\omega$ for $\alpha\in\Omega^\bullet_L(M)$ and $\omega\in\Omega^\bullet_L(M,E)$ (the graded Leibniz rule),

iv) $d^{\nabla^{\mathrm{End}(E)}}[\omega,\tau] = [d^{\nabla^{\mathrm{End}(E)}}\omega,\tau] + (-1)^{\deg(\omega)}[\omega, d^{\nabla^{\mathrm{End}(E)}}\tau]$ for $\omega,\tau\in\Omega^\bullet_L(M,\mathrm{End}(E))$.

Proof. Properties i) and ii) follows immediately from the definition.
iii) It suffices to investigate decomposable forms $\omega = \beta\otimes s$ for $\beta\in\Omega^q_L(M)$ and $s\in\Omega^0_L(M,E)$. From the definition we obtain $d^\nabla(\beta\otimes s) = d_L\beta\otimes s + (-1)^q\beta\wedge d^\nabla s$. Afterwards for $\alpha\in\Omega^p_L(M)$ we have

$$\begin{aligned} d^\nabla(\alpha\wedge(\beta\otimes s)) &= d^\nabla((\alpha\wedge\beta)\otimes s) = d_L(\alpha\wedge\beta)\otimes s + (-1)^{p+q}(\alpha\wedge\beta)\wedge d^\nabla s\\ &= (d_L\alpha\wedge\beta)\otimes s + (-1)^p(\alpha\wedge d_L\beta)\otimes s + (-1)^{p+q}(\alpha\wedge\beta)\wedge d^\nabla s\\ &= d_L\alpha\wedge(\beta\otimes s) + (-1)^p\alpha\wedge d^\nabla(\beta\otimes s). \end{aligned}$$

iv) For decomposable forms $\omega = \alpha\otimes s$, $\tau = \beta\otimes t$, where $s,t\in\Omega^0_L(M,\mathrm{End}(E))$, $\alpha\in\Omega^p_L(M)$ and $\beta\in\Omega^q_L(M)$, we have $[\alpha\otimes s,\beta\otimes t] = (\alpha\wedge\beta)\otimes[s,t]$. Hence we can write

$$\begin{aligned} d^{\nabla^{\mathrm{End}(E)}}[\alpha\otimes s,\beta\otimes t] &= d^{\nabla^{\mathrm{End}(E)}}((\alpha\wedge\beta)\otimes[s,t])\\ &= d_L(\alpha\wedge\beta)\otimes[s,t] + (-1)^{p+q}(\alpha\wedge\beta)\wedge d^{\nabla^{\mathrm{End}(E)}}[s,t]\\ &= (d_L\alpha\wedge\beta)\otimes[s,t] + (-1)^p(\alpha\wedge d_L\beta)\otimes[s,t]\\ &\quad + (-1)^{p+q}(\alpha\wedge\beta)\wedge[d^{\nabla^{\mathrm{End}(E)}}s,t] + (-1)^{p+q}(\alpha\wedge\beta)\wedge[s,d^{\nabla^{\mathrm{End}(E)}}t]\\ &= [d_L\alpha\otimes s,\beta\otimes t] + (-1)^p[\alpha\otimes s, d_L\beta\otimes t] + (-1)^p[\alpha\wedge d^{\nabla^{\mathrm{End}(E)}}s,\beta\otimes t]\\ &\quad + (-1)^{p+q}[\alpha\otimes s,\beta\wedge d^{\nabla^{\mathrm{End}(E)}}t]\\ &= [d^{\nabla^{\mathrm{End}(E)}}(\alpha\otimes s),\beta\otimes t] + (-1)^p[(\alpha\otimes s), d^{\nabla^{\mathrm{End}(E)}}(\beta\otimes t)], \end{aligned}$$

where we used that $d^{\nabla^{\mathrm{End}(E)}}[s,t] = [d^{\nabla^{\mathrm{End}(E)}}s,t] + [s,d^{\nabla^{\mathrm{End}(E)}}t]$ which follows from the classical Jacobi identity for $\mathbb{K}$-linear mappings on $\Omega^0_L(M,E)$, thus we are done. ♠

Lemma 9. Denote by $\mathcal{A}(E,L)$ the set of all L-connections on a vector bundle E. Then $\mathcal{A}(E,L)$ is an affine space modeled on the vector space $\Omega^1_L(M,\mathrm{End}(E))$.

Proof. We first prove that $\mathcal{A}(E,L)$ is non-empty. Because on any vector bundle E there exists a connection $\tilde\nabla\colon\Omega^0(M,E)\to\Omega^1(M,E)$, we may define an L-connection $\nabla\colon\Omega^0_L(M,E)\to\Omega^1_L(M,E)$ by

$$\nabla_\xi s = \tilde\nabla_{a(\xi)}s$$

for $\xi\in\mathfrak{X}_L(M)$ and $s\in\Omega^0_L(M,E)$. The rest of the proof is very simple. We need to verify that, if ∇ and ∇' are two L-connections, then $(\nabla'-\nabla)\colon\Omega^0_L(M,E)\to\Omega^1_L(M,E)$ is a $C^\infty(M,\mathbb{K})$-linear mapping. But we have $(\nabla'-\nabla)(fs) = d_Lf\otimes s + f\nabla's - d_Lf\otimes s - f\nabla s = f(\nabla'-\nabla)s$ hence there exists a uniquely determined $\alpha\in\Omega^1_L(M,\mathrm{End}(E))$ such that $\nabla'-\nabla = \mu(\alpha)$. ♠

Remark. Thus, if we fix some ∇_0 in $\mathcal{A}(E,L)$, we may write

$$\mathcal{A}(E,L) = \{\nabla_0 + \mu(\alpha);\ \alpha\in\Omega^1_L(M,\mathrm{End}(E))\}. \tag{2.16}$$

This description will permit us to define Sobolev completions of $\mathcal{A}(E,L)$.

Definition 11. If we are given an L-connection ∇ on a vector bundle E, then the *curvature* $R^\nabla \in \Omega^2_L(M, \mathrm{End}(E))$ of the L-connection ∇ is defined by the formula

$$R^\nabla(\xi,\eta)s = \nabla_\xi \nabla_\eta s - \nabla_\eta \nabla_\xi s - \nabla_{[\xi,\eta]} s = [\nabla_\xi, \nabla_\eta] s - \nabla_{[\xi,\eta]} s, \tag{2.17}$$

where $\xi, \eta \in \mathfrak{X}_L(M)$ and $s \in \Omega^0_L(M,E)$.

Remark. An L-connection with zero curvature is called the *flat L-connection.* We will denote the set of all flat L-connections on a vector bundle E by $\mathcal{H}(E,L)$.

Lemma 10. Let ∇ be an L-connection on a vector bundle E, then

$$(d^\nabla \circ d^\nabla)\,\omega = \mu(R^\nabla)\,\omega \tag{2.18}$$

for all $\omega \in \Omega^\bullet_L(M,E)$.

Proof. First we verify that $R^\nabla(\xi,\eta)s = (d^\nabla(d^\nabla s))(\xi,\eta)$. This is a consequence upon the following computation

$$\begin{aligned}(d^\nabla(d^\nabla s))(\xi,\eta) &= \nabla_\xi((d^\nabla s)(\eta)) - \nabla_\eta((d^\nabla s)(\xi)) - (d^\nabla s)([\xi,\eta]) \\ &= \nabla_\xi\nabla_\eta s - \nabla_\eta\nabla_\xi s - \nabla_{[\xi,\eta]} s \\ &= R^\nabla(\xi,\eta)s\end{aligned}$$

for all $\xi,\eta \in \mathfrak{X}_L(M)$ and $s \in \Omega^0_L(M,E)$. Further it suffices to investigate only decomposable forms $\omega = \alpha \otimes s$ for $\alpha \in \Omega^k_L(M)$ and $s \in \Omega^0_L(M,E)$. Afterwards, we can write

$$\begin{aligned}(d^\nabla \circ d^\nabla)(\alpha\otimes s) &= d^\nabla(d_L\alpha \otimes s + (-1)^k \alpha \wedge d^\nabla s) \\ &= 0 + (-1)^{k+1} d_L\alpha \wedge d^\nabla s + (-1)^k d_L \alpha \wedge d^\nabla s + (-1)^{2k}\alpha \wedge (d^\nabla \circ d^\nabla)s \\ &= \alpha \wedge \mu(R^\nabla)\,s \\ &= \mu(R^\nabla)\,(\alpha\otimes s)\end{aligned}$$

hence we have got $d^\nabla \circ d^\nabla = \mu(R^\nabla)$ and this finishes the proof. ♠

Given an L-connection on a vector bundle E, the mapping $\nabla\colon \Omega^0_L(M,E) \to \Omega^1_L(M,E)$ can be extended to the following sequence of first order differential operators

$$0 \longrightarrow \Omega^0_L(M,E) \xrightarrow{d^\nabla} \Omega^1_L(M,E) \xrightarrow{d^\nabla} \dots \xrightarrow{d^\nabla} \Omega^r_L(M,E) \longrightarrow 0, \tag{2.19}$$

where $r = \operatorname{rk} L$. It is a differential complex if and only if the curvature R^∇ of the L-connection ∇ is zero (∇ is a flat L-connection).

A natural question is when is this differential complex an elliptic complex? Let $f \in C^\infty(M,\mathbb{R})$ then we may write

$$(\mathrm{ad}(f)d^\nabla)\,\omega = d^\nabla(f\omega) - f d^\nabla\omega = d_L f \wedge \omega + f d^\nabla \omega - f d^\nabla\omega = a^*(df)\wedge\omega$$

for any $\omega \in \Omega^k_L(M,E)$ hence for the principal symbol $\sigma_1(d^\nabla)$ we obtain

$$\sigma_1(d^\nabla)(\xi_x) = a^*(\xi_x)\wedge\colon (\Lambda^k L^* \otimes E)_x \to (\Lambda^{k+1}L^*\otimes E)_x$$

for every $x \in M$ and $\xi_x \in T^*_x M$, i.e., the symbol is the exterior multiplication by $a^*(\xi_x)$. Therefore we have the *twisted Koszul complex*

$$0 \longrightarrow (\Lambda^0 L^* \otimes E)_x \xrightarrow{a^*(\xi_x)\wedge} \dots \xrightarrow{a^*(\xi_x)\wedge} (\Lambda^r L^*\otimes E)_x \longrightarrow 0, \tag{2.20}$$

where $r = \operatorname{rk} L$, which is an exact sequence, if and only if $a^*(\xi_x) \neq 0$. Thus, the differential complex is elliptic if and only if the corresponding twisted Koszul complex is an exact sequence for any $x \in M$ and $0 \neq \xi_x \in T_x^*M$, in other words if and only if $a^*(\xi_x) \neq 0$ for any $x \in M$ and $0 \neq \xi_x \in T_x^*M$.

If $L \xrightarrow{a} TM$ is a real Lie algebroid, then the elipticity is equivalent to the requirement that $a^*: T^*M \to L^*$ is injective or that $a: L \to TM$ is surjective. For a complex Lie algebroid $L \xrightarrow{a} TM_{\mathbb{C}}$ it corresponds to the requirement that $a^*{}_{|T^*M}: T^*M \hookrightarrow (TM_{\mathbb{C}})^* \to L^*$ is injective. These are the same conditions as for the ellipticity of the complex (1.40). We will call this condition the *ellipticity condition* for a Lie algebroid.

Lemma 11. If ∇ is an L-connection on a vector bundle E then we have

$$d^{\nabla^{\mathrm{End}(E)}} R^{\nabla} = 0. \tag{2.21}$$

This is called the *Bianchi identity* for R^{∇}.

Proof. For any $\xi_1, \xi_2, \xi_3 \in \mathfrak{X}_L(M)$ we may write

$$\begin{aligned}
(d^{\nabla^{\mathrm{End}(E)}} R^{\nabla})(\xi_1, \xi_2, \xi_3) &= [\nabla_{\xi_1}, R^{\nabla}(\xi_2, \xi_3)] - [\nabla_{\xi_2}, R^{\nabla}(\xi_1, \xi_3)] + [\nabla_{\xi_3}, R^{\nabla}(\xi_1, \xi_2)] \\
&\quad - R^{\nabla}([\xi_1, \xi_2], \xi_3) + R^{\nabla}([\xi_1, \xi_3], \xi_2) - R^{\nabla}([\xi_2, \xi_3], \xi_1) \\
&= \sum_{cykl} ([\nabla_{\xi_1}, [\nabla_{\xi_2}, \nabla_{\xi_3}]] - [\nabla_{\xi_1}, \nabla_{[\xi_2, \xi_3]}]) - \sum_{cykl} ([\nabla_{[\xi_1, \xi_2]}, \nabla_{\xi_3}] - \nabla_{[[\xi_1, \xi_2], \xi_3]}) \\
&= -\sum_{cykl} [\nabla_{\xi_1}, \nabla_{[\xi_2, \xi_3]}] - \sum_{cykl} [\nabla_{[\xi_1, \xi_2]}, \nabla_{\xi_3}] \\
&= 0,
\end{aligned}$$

where we used the classical Jacobi identity for commutators of $\mathbb{K}$-linear mappings. ♠

Lemma 12. Consider two L-connections ∇, ∇' on a vector bundle E. There is a uniquely determined $\alpha \in \Omega_L^1(M, \mathrm{End}(E))$ such that $\nabla' - \nabla = \mu(\alpha)$. Then

$$\begin{aligned}
R^{\nabla'} &= R^{\nabla} + d^{\nabla^{\mathrm{End}(E)}} \alpha + \alpha \wedge \alpha && (2.22) \\
&= R^{\nabla} + d^{\nabla^{\mathrm{End}(E)}} \alpha + \frac{1}{2} [\alpha, \alpha]. && (2.23)
\end{aligned}$$

Proof. The proof is a straightforward computation only. We have

$$\begin{aligned}
R^{\nabla'}(\xi, \eta) &= [\nabla'_{\xi}, \nabla'_{\eta}] - \nabla'_{[\xi, \eta]} \\
&= [\nabla_{\xi} + \alpha(\xi), \nabla_{\eta} + \alpha(\eta)] - (\nabla_{[\xi, \eta]} + \alpha([\xi, \eta])) \\
&= [\nabla_{\xi}, \nabla_{\eta}] - \nabla_{[\xi, \eta]} + [\nabla_{\xi}, \alpha(\eta)] - [\nabla_{\eta}, \alpha(\xi)] - \alpha([\xi, \eta]) + [\alpha(\xi), \alpha(\eta)] \\
&= R^{\nabla}(\xi, \eta) + \nabla_{\xi}^{\mathrm{End}(E)} \alpha(\eta) - \nabla_{\eta}^{\mathrm{End}(E)} \alpha(\xi) - \alpha([\xi, \eta]) + [\alpha(\xi), \alpha(\eta)] \\
&= R^{\nabla}(\xi, \eta) + (d^{\nabla^{\mathrm{End}(E)}} \alpha)(\xi, \eta) + (\alpha \wedge \alpha)(\xi, \eta) \\
&= R^{\nabla}(\xi, \eta) + (d^{\nabla^{\mathrm{End}(E)}} \alpha)(\xi, \eta) + \tfrac{1}{2} [\alpha, \alpha](\xi, \eta)
\end{aligned}$$

for all $\xi, \eta \in \mathfrak{X}_L(M)$, so we are done. ♠

Therefore, if we fix some flat L-connection $\nabla_0 \in \mathcal{H}(E, L)$, then, using the result of Lemma 12, we may write

$$\mathcal{H}(E, L) = \{\nabla_0 + \mu(\alpha);\ \alpha \in \Omega_L^1(M, \mathrm{End}(E)),\ d^{\nabla_0^{\mathrm{End}(E)}} \alpha + \alpha \wedge \alpha = 0\}. \tag{2.24}$$

This description, similarly like in the case of $\mathcal{A}(E, L)$, will allow us to define Sobolev completions of $\mathcal{H}(E, L)$.

2.2 Group of gauge transformations

Let $E \xrightarrow{\pi} M$ be a real (complex) vector bundle, then a vector bundle homomorphism is a smooth mapping $\varphi\colon E \to E$ such that there exists mapping $\underline{\varphi}\colon M \to M$, the diagram

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & E \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow[\underline{\varphi}]{} & M \end{array}$$

commutes and and for each $x \in M$ the mapping $\varphi_x = \varphi_{|E_x}\colon E_x \to E_{\underline{\varphi}(x)}$ is $\mathbb{K}$-linear. Because $\pi\colon E \to M$ is a fibered manifold and $\underline{\varphi} \circ \pi$ is smooth, we get that $\underline{\varphi}$ is smooth. If we denote by $\mathrm{Aut}(E)$ the group of vector bundle automorphism $\varphi\colon E \to E$ then the previous diagram commutes for a uniquely determined diffeomorphism $\underline{\varphi}\colon M \to M$. Therefore we have a group homomorphism from $\mathrm{Aut}(E)$ into the group $\mathrm{Diff}(M)$ of all diffeomorphism of M. The kernel $\mathrm{Gau}(E)$ of this homomorphism is called the *group of gauge transformations* and its elements are called *gauge transformations.* Thus $\mathrm{Gau}(E)$ is the group of all vector bundle automorphisms $\varphi\colon E \to E$ satisfying $\pi \circ \varphi = \pi$. Hence we have the following exact sequence

$$\{e\} \to \mathrm{Gau}(E) \to \mathrm{Aut}(E) \to \mathrm{Diff}(M) \tag{2.25}$$

of groups.

Furthermore we define the *Lie algebra of gauge transformations* $\mathfrak{gau}(E)$. As a vector space it is $\Omega^0_L(M, \mathrm{End}(E))$, while the Lie bracket is given by

$$[\gamma_1, \gamma_2] = \gamma_1 \circ \gamma_2 - \gamma_2 \circ \gamma_1 \tag{2.26}$$

for $\gamma_1, \gamma_2 \in \Omega^0_L(M, \mathrm{End}(E))$.

The group of gauge transformations $\mathrm{Gau}(E)$ has a left action on the space $\Omega^k_L(M, \mathrm{End}(E))$ given by

$$(\mathrm{Ad}_\varphi(\omega))(\xi_1, \ldots, \xi_k) = \varphi \circ \omega(\xi_1, \ldots, \xi_k) \circ \varphi^{-1}, \tag{2.27}$$

where $\varphi \in \mathrm{Gau}(E)$, $\omega \in \Omega^k_L(M, \mathrm{End}(E))$ and $\xi_1, \ldots, \xi_k \in \mathfrak{X}_L(M)$. Further this gives a left action of the Lie algebra of gauge transformations $\mathfrak{gau}(E)$ on $\Omega^k_L(M, \mathrm{End}(E))$ via

$$\mathrm{ad}_\gamma(\omega) = [\gamma, \omega] \tag{2.28}$$

for $\gamma \in \mathfrak{gau}(E)$ and $\omega \in \Omega^k_L(M, \mathrm{End}(E))$. So we have got representations of $\mathrm{Gau}(E)$ and $\mathfrak{gau}(E)$ on the graded vector space $\Omega^\bullet_L(M, \mathrm{End}(E))$.

Remark. Furthermore there is a left action of the group $\mathrm{Aut}(E)$ on the space of sections $\Gamma(M, E)$ defined by

$$\varphi \cdot s = \varphi \circ s \circ \underline{\varphi}^{-1}, \tag{2.29}$$

where $\varphi \in \mathrm{Aut}(E)$ and $s \in \Gamma(M, E)$.

2.3 Change of connections

Let $(L \to M, [\cdot, \cdot], a)$ be a real (complex) Lie algebroid and $E \to M$ be a real (complex) vector bundle. Further consider a gauge transformation φ and an L-connection ∇ on E. We define a $\mathbb{K}$-bilinear mapping $\nabla^\varphi\colon \mathfrak{X}_L(M) \times \Omega^0_L(M, E) \to \Omega^0_L(M, E)$ by

$$\nabla^\varphi_\xi s = \varphi(\nabla_\xi(\varphi^{-1}(s))) \tag{2.30}$$

for any $\xi \in \mathfrak{X}_L(M)$ and $s \in \Omega^0_L(M,E)$. Since we may write

$$\begin{aligned}\nabla^\varphi_\xi(fs) &= \varphi(\nabla_\xi(\varphi^{-1}(fs))) = \varphi(\nabla_\xi(f\varphi^{-1}(s)))\\ &= \varphi((\mathcal{L}^L_\xi f)\varphi^{-1}(s) + f\nabla_\xi(\varphi^{-1}(s)))\\ &= (\mathcal{L}^L_\xi f)s + f\varphi(\nabla_\xi(\varphi^{-1}(s)))\\ &= (\mathcal{L}^L_\xi f)s + f\nabla^\varphi_\xi s\end{aligned}$$

and moreover we have

$$\nabla^\varphi_{f\xi}s = \varphi(\nabla_{f\xi}(\varphi^{-1}(s))) = \varphi(f\nabla_\xi(\varphi^{-1}(s))) = f\varphi(\nabla_\xi(\varphi^{-1}(s))) = f\nabla^\varphi_\xi s$$

for all $\xi \in \mathfrak{X}_L(M)$, $f \in C^\infty(M,\mathbb{K})$ and $s \in \Omega^0_L(M,E)$, therefore ∇^φ is an L-connection on E.

As ∇^φ is an L-connection, we can define a natural left action of $\mathrm{Gau}(E)$ on the space $\mathcal{A}(E,L)$ of L-connections by

$$(\varphi,\nabla) \mapsto \varphi\cdot\nabla = \nabla^\varphi. \tag{2.31}$$

It is easy to see that this really defines a left action.

Remark. It would be possible to define a right action instead of a left action by

$$(\nabla,\varphi) \mapsto \nabla\cdot\varphi = \nabla^{\varphi^{-1}}. \tag{2.32}$$

This reverse the role of φ and φ^{-1} in (2.31), but makes no difference in the end.

Lemma 13. Let ∇ be an L-connection on E. Then we have

$$R^{\nabla^\varphi} = \mathrm{Ad}_\varphi(R^\nabla) \tag{2.33}$$

for any gauge transformation $\varphi \in \mathrm{Gau}(E)$.

Proof. It follows immediately that

$$\begin{aligned}R^{\nabla^\varphi}(\xi,\eta) &= [\nabla^\varphi_\xi,\nabla^\varphi_\eta] - \nabla^\varphi_{[\xi,\eta]}\\ &= \varphi\circ[\nabla_\xi,\nabla_\eta]\circ\varphi^{-1} - \varphi\circ\nabla_{[\xi,\eta]}\circ\varphi^{-1}\\ &= \varphi\circ R^\nabla(\xi,\eta)\circ\varphi^{-1}\end{aligned}$$

for all $\xi,\eta \in \mathfrak{X}_L(M)$. ♠

Because $\mathcal{H}(E,L)$ is invariant under the action of $\mathrm{Gau}(E)$, as it follows from Lemma 13, we have the action of $\mathrm{Gau}(E)$ on the space of flat L-connections $\mathcal{H}(E,L)$. Therefore we define the *moduli space*

$$\mathcal{B}(E,L) = \mathcal{A}(E,L)/\mathrm{Gau}(E) \tag{2.34}$$

of gauge equivalence classes of L-connections and the *moduli space*

$$\mathcal{M}(E,L) = \mathcal{H}(E,L)/\mathrm{Gau}(E) \tag{2.35}$$

of gauge equivalence classes of flat L-connections.

Now we take up the question of reducible connections. Given an L-connection $\nabla \in \mathcal{A}(E,L)$ then the *isotropy subgroup* or the *stabilizer* of ∇ is the subgroup $\mathrm{Gau}(E)_\nabla$ of $\mathrm{Gau}(E)$ that leaves ∇ fixed, i.e.,

$$\mathrm{Gau}(E)_\nabla = \{\varphi \in \mathrm{Gau}(E);\ \varphi\cdot\nabla = \nabla\}. \tag{2.36}$$

Every such group contains the subgroup $\mathbb{K}^*\cdot \mathrm{id}_E$.

Definition 12. An L-connection ∇ on a vector bundle E is called *irreducible* or *simple*, if $\mathrm{Gau}(E)_\nabla = \mathbb{K}^*\cdot\mathrm{id}_E$, otherwise ∇ is called *reducible*. We denote the set of irreducible L-connections by $\mathcal{A}^*(E,L)$ and the set of irreducible flat L-connections by $\mathcal{H}^*(E,L)$.

Lemma 14. Let ∇ be an L-connection on a vector bundle E over a compact manifold M. Then the following are equivalent:

i) $\mathrm{Gau}(E)_\nabla = \mathbb{K}^* \cdot \mathrm{id}_E$,
ii) $\ker \nabla^{\mathrm{End}(E)} = \mathbb{K} \cdot \mathrm{id}_E$,
iii) $\ker \nabla^{\mathrm{End}(E)}{}_{|\Omega^0_L(M,\mathrm{End}(E))^0} = \{0\}$.

Proof. Consider a gauge transformation $\varphi \in \mathrm{Gau}(E)$. Then the requirement $\varphi \cdot \nabla = \nabla$ means that for any $\xi \in \mathfrak{X}_L(M)$ we have $\varphi \circ \nabla_\xi \circ \varphi^{-1} = \nabla_\xi$ and this is equivalent to $[\nabla_\xi, \varphi] = 0$. Therefore we have got that $\varphi \in \mathrm{Gau}(E)_\nabla$ if and only if $\nabla^{\mathrm{End}(E)}\varphi = 0$ and $\varphi \in \mathrm{Gau}(E)$.

Suppose that $\varphi \in \mathrm{Gau}(E)_\nabla$ then $\nabla^{\mathrm{End}(E)}\varphi = 0$ and, provided that $\ker \nabla^{\mathrm{End}(E)} = \mathbb{K} \cdot \mathrm{id}_E$, we obtain $\varphi = c \cdot \mathrm{id}_E$ for some $c \in \mathbb{K}^*$. Hence we get $\mathrm{Gau}(E)_\nabla \subset \mathbb{K}^* \cdot \mathrm{id}_E$ and because the converse inclusion is trivial, we have proved ii) $\Rightarrow$ i).

To prove the opposite implication, we use the compactness of the manifold M. Assume that $\varphi \in \ker \nabla^{\mathrm{End}(E)}$. Because M is compact, there exists $c \in \mathbb{K}$ (with $|c|$ sufficiently large) so that $c \cdot \mathrm{id}_E + \varphi \in \mathrm{Gau}(E)$. Moreover, $\nabla^{\mathrm{End}(E)}(c \cdot \mathrm{id}_E + \varphi) = 0$ and from the previous consideration, it follows $c \cdot \mathrm{id}_E + \varphi \in \mathrm{Gau}(E)_\nabla$. Besides, if we suppose $\mathrm{Gau}(E)_\nabla = \mathbb{K}^* \cdot \mathrm{id}_E$, we obtain $\ker \nabla^{\mathrm{End}(E)} \subset \mathbb{K} \cdot \mathrm{id}_E$. The converse inclusion is trivial.

The equivalence of ii) and iii) immediately follows form the definition of $\Omega^0_L(M, \mathrm{End}(E))^0$, so this finishes the proof. ♠

From the fact that $\mathrm{Gau}(E)_{\nabla^\varphi} = \varphi \cdot \mathrm{Gau}(E)_\nabla \cdot \varphi^{-1}$ for all $\varphi \in \mathrm{Gau}(E)$ and $\nabla \in \mathcal{A}(E, L)$, we obtain that $\mathcal{A}^*(E, L)$ is invariant under the action of $\mathrm{Gau}(E)$ and the same for $\mathcal{H}^*(E, L)$. Thus we can define, similarly as in (2.34) and (2.35), the moduli space

$$\mathcal{B}^*(E, L) = \mathcal{A}^*(E, L) / \mathrm{Gau}(E) \tag{2.37}$$

of gauge equivalence classes of irreducible L-connections and the moduli space

$$\mathcal{M}^*(E, L) = \mathcal{H}^*(E, L) / \mathrm{Gau}(E) \tag{2.38}$$

of gauge equivalence classes of irreducible flat L-connections.

Because $\mathbb{K}^* \cdot \mathrm{id}_E$ is a normal subgroup of $\mathrm{Gau}(E)$, we define the *reduced group of gauge transformations* $\mathrm{Gau}(E)^{\mathrm{r}}$ by

$$\mathrm{Gau}(E)^{\mathrm{r}} = \mathrm{Gau}(E) / \mathbb{K}^* \cdot \mathrm{id}_E. \tag{2.39}$$

Then the left action of $\mathrm{Gau}(E)$ on $\mathcal{A}(E, L)$ factors trough an action of the reduced group of gauge transformations $\mathrm{Gau}(E)^{\mathrm{r}}$ since the group $\mathbb{K}^* \cdot \mathrm{id}_E$ acts trivially on $\mathcal{A}(E, L)$, similarly for $\mathcal{H}(E, L)$. Therefore for the moduli spaces (2.34), (2.35) of L-connections we may write

$$\mathcal{B}(E, L) = \mathcal{A}(E, L) / \mathrm{Gau}(E)^{\mathrm{r}} \quad \text{and} \quad \mathcal{M}(E, L) = \mathcal{H}(E, L) / \mathrm{Gau}(E)^{\mathrm{r}} \tag{2.40}$$

and similarly for the moduli spaces (2.37), (2.38) of irreducible L-connections we have

$$\mathcal{B}^*(E, L) = \mathcal{A}^*(E, L) / \mathrm{Gau}(E)^{\mathrm{r}} \quad \text{and} \quad \mathcal{M}^*(E, L) = \mathcal{H}^*(E, L) / \mathrm{Gau}(E)^{\mathrm{r}}. \tag{2.41}$$

The set $\mathcal{A}^*(E, L)$ of all irreducible L-connections is the maximal subset of $\mathcal{A}(E, L)$ on which the reduced group of gauge transformations $\mathrm{Gau}(E)^{\mathrm{r}}$ acts freely, likewise for $\mathcal{H}^*(E, L)$.

If we are given a gauge transformation $\varphi \in \mathrm{Gau}(E)$ and an L-connection ∇ on a vector bundle E, then for the changed L-connection ∇^φ we have

$$\nabla^\varphi_\xi = \nabla_\xi + \varphi \circ \nabla^{\mathrm{End}(E)}_\xi \varphi^{-1} = \nabla_\xi - \nabla^{\mathrm{End}(E)}_\xi \varphi \circ \varphi^{-1}, \tag{2.42}$$

where $\xi \in \mathfrak{X}_L(M)$. The last equality follows by differentiating the identity $\varphi \circ \varphi^{-1} = \mathrm{id}_E$. More generally, if we fix some L-connection ∇ and express another L-connection ∇' as $\nabla' = \nabla + \mu(\alpha)$, then

$$\nabla'^\varphi_\xi = \nabla_\xi + \varphi \circ \nabla^{\mathrm{End}(E)}_\xi \varphi^{-1} + \varphi \circ \alpha(\xi) \circ \varphi^{-1}, \tag{2.43}$$

hence, writing $\nabla'^{\varphi} = \nabla + \mu(\alpha^{\varphi})$, we obtain

$$\alpha^{\varphi}(\xi) = \varphi \circ \nabla_{\xi}^{\mathrm{End}(E)}\varphi^{-1} + \varphi \circ \alpha(\xi) \circ \varphi^{-1} \tag{2.44}$$

for $\xi \in \mathfrak{X}_L(M)$. This can be rewritten as

$$\begin{aligned} \alpha^{\varphi} &= \varphi \wedge \nabla^{\mathrm{End}(E)}\varphi^{-1} + \mathrm{Ad}_{\varphi}(\alpha) &(2.45)\\ &= -\nabla^{\mathrm{End}(E)}\varphi \wedge \varphi^{-1} + \mathrm{Ad}_{\varphi}(\alpha) &(2.46)\end{aligned}$$

for $\varphi \in \mathrm{Gau}(E)$.

2.4 Sobolev spaces and elliptic operators

In this section we introduce Lebesgue and Sobolev spaces on manifolds which are an important framework for the construction of moduli spaces of Lie algebroid connections on fiber bundles (in particular vector bundles and principal fiber bundles). More details can be found in [34] and [35].

Let (M, g) be a Riemannian manifold and $\pi\colon E \to M$ be a real (complex) vector bundle endowed with an Euclidean (Hermitian) metric h. The metric g determines the density $\mathrm{vol}(g)$ of the Riemannian metric g, even $\mathrm{vol}(g)$ induces a (regular) Borel measure μ_g on M.

Definition 13. Let $p \in (1, +\infty)$, then an L^p-section of $E \xrightarrow{\pi} M$ is a Borel measurable mapping $\psi\colon M \to E$, i.e., $\psi^{-1}(U)$ is Borel measurable for any open subset $U \subset E$, such that

i) $\pi \circ \psi = \mathrm{id}_M$,

ii) the function $x \mapsto |\psi(x)|_h^p = |h(\psi(x), \psi(x))|^p$ is integrable with respect to the Borel measure μ_g, i.e., belongs to $L^p(M, \mathbb{R})$.

We denote by $L^p(M, E)$ the vector space of equivalence classes of L^p-sections with respect to the equality almost everywhere. With regard to the norm defined by

$$||\psi||_p = \left(\int_M |\psi(x)|_h^p \,\mathrm{d}\mu_g\right)^{\frac{1}{p}} \tag{2.47}$$

is $L^p(M, E)$ a Banach space for any $p \in (1, +\infty)$.

Denote now by ∇^g the Levi-Civita connection of g and by ∇^h a connection compatible with h. Further for each $j \in \mathbb{N}$ we define ∇^j as the composition

$$\Gamma(M, E) \xrightarrow{\nabla^E} \Gamma(M, T^*M \otimes E) \xrightarrow{\nabla^{T^*M\otimes E}} \dots \xrightarrow{\nabla^{T^*M^{\otimes(j-1)}\otimes E}} \Gamma(M, T^*M^{\otimes j} \otimes E), \tag{2.48}$$

where $\nabla^{T^*M^{\otimes k}\otimes E}$ for $k \in \mathbb{N}_0$ denotes the connection on $T^*M^{\otimes k} \otimes E$ induced by ∇^g and ∇^h.

The metrics g and h induce metrics on each of the vector bundle $T^*M^{\otimes j} \otimes E$, hence we can define the spaces $L^p(M, T^*M^{\otimes j} \otimes E)$.

Definition 14. Let $u \in L^p(M, E)$ and $v \in L^p(M, T^*M^{\otimes j} \otimes E)$, then we say that $\nabla^j u = v$ *weakly* if

$$\int_M \langle v, \varphi\rangle \,\mathrm{d}\mu_g = \int_M \langle u, (\nabla^j)^*\varphi\rangle \,\mathrm{d}\mu_g \tag{2.49}$$

for all $\varphi \in \Gamma_0(M, T^*M^{\otimes j} \otimes E)$, i.e., sections with compact support, where $(\nabla^j)^*$ is the formal adjoint of ∇^j.

For $p \in (1, +\infty)$ and $k \in \mathbb{N}_0$ we define the Sobolev space $L^{k,p}(M, E)$ as the space of sections $\psi \in L^p(M, E)$ such that there exists $\psi_j \in L^p(M, T^*M^{\otimes j} \otimes E)$ satisfying $\nabla^j\psi = \psi_j$ weakly for all $j = 1, 2, \dots, k$. This is a Banach space with respect to the norm

$$||\psi||_{k,p} = \left(\sum_{j=0}^{k} ||\nabla^j\psi||_p^p\right)^{\frac{1}{p}}, \tag{2.50}$$

where $\nabla^0\psi = \psi$. The Banach spaces $L^{k,p}(M,E)$ are called the *Sobolev spaces of sections.*

The spaces $L^{k,p}(M,E)$ are *separable*, and for $p > 1$ they are *reflexive*. For $p = 2$ the spaces $L^{k,2}(M,E)$ are Hilbert spaces with the following scalar product

$$\langle \psi, \varphi \rangle_k = \sum_{j=0}^{k} \int_M \langle \nabla^j \psi, \nabla^j \varphi \rangle \, d\mu_g. \tag{2.51}$$

In the special case $p = 2$ we will write $\Gamma(M,E)_k$ instead of $L^{k,2}(M,E)$.

Denote by $C^r(M,E)$ for $r \in \mathbb{N}_0$ the vector space of C^r-sections of a vector bundle $E \to M$. If M is a compact manifold then $C^r(M,E)$ with the norm defined by

$$||\psi||_r = \sum_{j=0}^{r} \max_M |\nabla^j \psi| \tag{2.52}$$

is a Banach space.

Remark. The Sobolev spaces $L^{k,p}(M,E)$ depend on several choices: the metrics on TM and E and the connection on E. When M is non-compact this dependence is very dramatic and has to be seriously taken into consideration.

Theorem 3. Let (M,g) be a compact Riemannian manifold of dimension n and $E \to M$ be a real (complex) vector bundle over M equipped with an Euclidean (Hermitian) metric h and a compatible connection ∇^h on E.

(i) The Sobolev space $L^{k,p}(M,E)$ does not depend on the metrics g, h and on the connection ∇^h. More precisely, if g' is a different Riemannian metric on M and $\nabla^{h'}$ is another connection on E compatible with some metric h' then

$$L^{k,p}(M,E;g,h,\nabla^h) = L^{k,p}(M,E;g',h',\nabla^{h'}) \tag{2.53}$$

as sets of equivalence classes of sections and the identity mapping between these two Banach spaces is continuous.

(ii) If $1 \leq p < +\infty$, then $\Gamma(M,E)$ is dense in $L^{k,p}(M,E)$.

(iii) (Sobolev embedding theorem) If $k_0 - \frac{n}{p_0} \geq k_1 - \frac{n}{p_1}$ and $k_0 \geq k_1$ then

$$L^{k_0,p_0}(M,E) \subset L^{k_1,p_1}(M,E) \tag{2.54}$$

and the embedding is continuous. Moreover if $k_0 - \frac{n}{p_0} > k_1 - \frac{n}{p_1}$ and $k_0 > k_1$ then the embedding $L^{k_0,p_0}(M,E) \hookrightarrow L^{k_1,p_1}(M,E)$ is compact.

(iv) (Lemma of Rellich) If $k - \frac{n}{p} \geq r$ then

$$L^{k,p}(M,E) \subset C^r(M,E) \tag{2.55}$$

and the embedding is continuous. In case we have strict inequality then the embedding is compact. In particular, if one has $\varphi \in L^{k,p}(M,E)$ for some fixed p and all $k \geq k_0$, then $\varphi \in \Gamma(M,E)$.

Remark. Therefore we have the following sequence of compact embeddings

$$\Gamma(M,E) \subset \ldots \hookrightarrow L^{k,2}(M,E) \hookrightarrow \ldots \hookrightarrow L^{1,2}(M,E) \hookrightarrow L^{0,2}(M,E) = L^2(M,E) \tag{2.56}$$

and moreover from Rellich's lemma it follows that

$$\Gamma(M,E) = \bigcap_{k=0}^{\infty} L^{k,p}(M,E) \tag{2.57}$$

for all $p \in \langle 1, +\infty)$.

Theorem 4. (Sobolev multiplication theorem) Let E_1, E_2, F be $\mathbb{K}$-vector bundles over a compact manifold M of dimension n and

$$m\colon \Gamma(M, E_1) \times \Gamma(M, E_2) \to \Gamma(M, F) \tag{2.58}$$

be a $C^\infty(M, \mathbb{K})$-bilinear mapping then m extends to a continuous mapping

(i)

$$m\colon L^{k_1,p_1}(M, E_1) \otimes L^{k_2,p_2}(M, E_2) \to L^{k,p}(M, F) \tag{2.59}$$

provided that $p_1, p_2 \neq 1$, $k_1, k_2 \geq k$, $p_1 \cdot k_1, p_2 \cdot k_2 < n$ and $k_1 - \frac{n}{p_1} + k_2 - \frac{n}{p_2} \geq k - \frac{n}{p}$,

(ii)

$$m\colon L^{k,p}(M, E_1) \otimes L^{k',p'}(M, E_2) \to L^{k',p'}(M, F) \tag{2.60}$$

if $p' \cdot k' > n$, $k > k'$ and $k - \frac{n}{p} \geq k' - \frac{n}{p'}$ for $p \neq 1$ (or $k - n > k' - \frac{n}{p'}$ in the case $p = 1$),

(iii)

$$m\colon L^{k,p}(M, E_1) \otimes L^{k,p}(M, E_2) \to L^{k,p}(M, F) \tag{2.61}$$

if $p \cdot k > n$.

Theorem 5. (Left composition lemma) Let E, F_1, F_2 be $\mathbb{K}$-vector bundles over a compact manifold M of dimension n and $f\colon F_1 \to F_2$ a homomorphism of $\mathbb{K}$-vector bundles covering the identity on M, i.e., $f \in \Gamma(M, \mathrm{Hom}(F_1, F_2))$. Then f defines a mapping

$$f_*\colon \Gamma(M, \mathrm{Hom}(E, F_1)) \to \Gamma(M, \mathrm{Hom}(E, F_2)) \tag{2.62}$$

given by

$$f_*(\varphi) = f \circ \varphi \tag{2.63}$$

which extends to a differentiable mapping of Banach spaces

$$f_*\colon L^{k,p}(M, \mathrm{Hom}(E, F_1)) \to L^{k,p}(M, \mathrm{Hom}(E, F_2)) \tag{2.64}$$

provided that $p \cdot k > n$.

Theorem 6. Let E, F be $\mathbb{K}$-vector bundles over a compact manifold M and $P\colon \Gamma(M, E) \to \Gamma(M, F)$ be a $\mathbb{K}$-linear differential operator of order ℓ. Then P extends to a continuous $\mathbb{K}$-linear mapping

$$P_k\colon L^{k,p}(M, E) \to L^{k-\ell,p}(M, E) \tag{2.65}$$

for $k \geq \ell$.

Theorem 7. (Elliptic regularity) Consider a $\mathbb{K}$-vector bundles E, F over a compact manifold M. Let $P\colon \Gamma(M, E) \to \Gamma(M, F)$ be an elliptic $\mathbb{K}$-linear differential operator of degree ℓ. If for $\psi \in \Gamma(M, E)_k$ one has $P_k\psi \in \Gamma(M, E)_{k-\ell+1}$ then $\psi \in \Gamma(M, E)_{k+1}$. Therefore $P_k\psi \in \Gamma(M, E)$ implies $\psi \in \Gamma(M, E)$ by the Lemma of Rellich, and in particular we have $\ker L_k = \ker L$.

Next we consider a sequence of differential operators

$$0 \longrightarrow \Gamma(M, E_0) \xrightarrow{D_0} \Gamma(M, E_1) \xrightarrow{D_1} \dots \xrightarrow{D_{\ell-1}} \Gamma(M, E_\ell) \longrightarrow 0, \tag{2.66}$$

where E_i are $\mathbb{K}$-vector bundles over a compact manifold M, and D_i are $\mathbb{K}$-linear differential operators of degree r_i. Let us assume that this sequence is an *elliptic complex*, i.e., $D_i \circ D_{i-1} = 0$

for $i = 1, 2, \dots, \ell - 1$ and for all $x \in M$ and $0 \neq \xi_x \in T_x^*M$ the associated sequence of principal symbols

$$0 \longrightarrow (E_0)_x \xrightarrow{\sigma(D_0)(\xi_x)} (E_1)_x \xrightarrow{\sigma(D_1)(\xi_x)} \dots \xrightarrow{\sigma(D_{\ell-1})(\xi_x)} (E_\ell)_x \longrightarrow 0, \tag{2.67}$$

is an exact sequence.

Denote the cohomology of this elliptic compex by $H^i(E_\bullet, D_\bullet)$ for $i = 0, 1, \dots, \ell$. Endow each E_i with an Euclidean (Hermitian) metric h_i and a compatible connection ∇^{h_i}. Furthermore let g be a Riemannian metric on M. Then we define the formal selfadjoint elliptic operators

$$\Delta_i = D_i^* \circ D_i + D_{i-1} \circ D_{i-1}^* \colon \Gamma(M, E_i) \to \Gamma(M, E_i) \tag{2.68}$$

of degree $\max\{2r_{i-1}, 2r_i\}$ for $i = 0, 1, \dots, \ell$, where D_i^* is a formal adjoint of D_i and D_{-1}, D_ℓ are zero operators. Because Δ_i is an elliptic operator, the *i-th vector space of harmonic sections*

$$\mathcal{H}^i(E_\bullet, D_\bullet) = \{\psi \in \Gamma(M, E_i);\ \Delta_i\psi = 0\} = \ker D_i \cap \ker D_{i-1}^* \tag{2.69}$$

of the elliptic complex (2.66) is finite dimensional for $i = 0, 1, \dots, \ell$.

Theorem 8. Let $H_i\colon \Gamma(M, E_i) \to \mathcal{H}^i(E_\bullet, D_\bullet)$ for $i = 0, 1, \dots, \ell$ be L^2-orthogonal projections.

i) There exist unique continuous linear operators $G_i\colon \Gamma(M, E_i) \to \Gamma(M, E_i)$ for $i = 0, 1, \dots, \ell$ satisfying

$$\mathrm{id}_{\Gamma(M,E_i)} = H_i + \Delta_i \circ G_i = H_i + G_i \circ \Delta_i \tag{2.70}$$

and the following commutation relation

$$H_i \circ G_i = G_i \circ H_i, \quad D_i \circ G_i = G_{i+1} \circ D_i, \quad D_i^* \circ G_{i+1} = G_i \circ D_i^*. \tag{2.71}$$

Moreover G_i is a pseudo-differential operator of degree $\min\{-2r_{i-1}, -2r_i\}$, called the *Green operator* associated to Δ_i.

ii) There are L^2-orthogonal decompositions

$$\begin{aligned}
\Gamma(M, E_i) &= \mathcal{H}^i(E_\bullet, D_\bullet) \oplus \mathrm{im}(D_{i-1} \circ D_{i-1}^* \circ G_i) \oplus \mathrm{im}(D_i^* \circ D_i \circ G_i), && (2.72)\\
&= \mathcal{H}^i(E_\bullet, D_\bullet) \oplus \mathrm{im}(G_i \circ D_{i-1} \circ D_{i-1}^*) \oplus \mathrm{im}(G_i \oplus D_i^* \circ D_i), && (2.73)\\
&= \mathcal{H}^i(E_\bullet, D_\bullet) \oplus \mathrm{im}\, D_{i-1} \oplus \mathrm{im}\, D_i^*, && (2.74)\\
&= \ker D_i \oplus \mathrm{im}\, D_i^*, && (2.75)\\
&= \mathrm{im}\, D_{i-1} \oplus \ker D_i^*, && (2.76)\\
\ker D_i &= \mathcal{H}^i(E_\bullet, D_\bullet) \oplus \mathrm{im}\, D_{i-1}, && (2.77)\\
\ker D_i^* &= \mathcal{H}^{i+1}(E_\bullet, D_\bullet) \oplus \mathrm{im}\, D_{i+1}^* && (2.78)
\end{aligned}$$

of $\Gamma(M, E_i)$ into the closed subspaces.

iii) There are natural isomorphisms

$$\mathcal{H}^i(E_\bullet, D_\bullet) \simeq H^i(E_\bullet, D_\bullet) \tag{2.79}$$

between the i-th vector space of harmonic sections and the i-th cohomology group for any $i = 0, 1, \dots, \ell$. Furthermore we have $\dim \mathcal{H}^i(E_\bullet, D_\bullet) < \infty$.

iv) We have decompositions

$$\begin{aligned}
\psi &= H_i\psi + (D_{i-1} \circ D_{i-1}^* \circ G_i)\psi + (D_i^* \circ D_i \circ G_i)\psi, && (2.80)\\
&= H_i\psi + (G_i \circ D_{i-1} \circ D_{i-1}^*)\psi + (G_i \circ D_i^* \circ D_i)\psi && (2.81)
\end{aligned}$$

of $\psi \in \Gamma(M, E_i)$ called the *Hodge decompositions* of ψ.

All operators extend to continuous linear mappings between appropriate Sobolev completions $\Gamma(M, E_i)_k$, i.e.,

$$D_{i,k} : \Gamma(M, E_i)_k \to \Gamma(M, E_{i+1})_{k-r_i}, \qquad D^*_{i,k} : \Gamma(M, E_i)_k \to \Gamma(M, E_{i-1})_{k-r_i}, \tag{2.82}$$

$$\Delta_{i,k} : \Gamma(M, E_i)_k \to \Gamma(M, E_i)_{k-s_i}, \qquad G_{i,k} : \Gamma(M, E_i)_k \to \Gamma(M, E_i)_{k+s_i}, \tag{2.83}$$

where s_i is the order of the differential operator Δ_i. Moreover

$$\ker \Delta_{i,k} = \ker \Delta_i = \mathcal{H}^i(E_\bullet, D_\bullet) \tag{2.84}$$

by elliptic regularity. All statements in Theorem 8 remain true in we replace the spaces by the correct Sobolev completions, e.g. there are L^2-orthogonal (not L^2_k-orthogonal) decompositions

$$\Gamma(M, E_i)_k = \mathcal{H}^i(E_\bullet, D_\bullet) \oplus \operatorname{im} D_{i-1,k+r_{i-1}} \oplus \operatorname{im} D^*_{i,k+r_i}, \tag{2.85}$$

$$= \ker D_{i,k} \oplus \operatorname{im} D^*_{i,k+r_i}, \tag{2.86}$$

$$= \operatorname{im} D_{i-1,k+r_{i-1}} \oplus \ker D^*_{i,k} \tag{2.87}$$

of $\Gamma(M, E_i)_k$ into closed subspaces.

2.5 Moduli spaces

Moduli spaces arise naturally in classification problems in geometry. Typically, one has a set whose elements represent algebro-geometric objects of some fixed kind and an equivalence relation on this set saying when two such objects are the same in some sense, and the problem is to describe the set of equivalence classes. One would like to give the set of equivalence classes some structure of a geometric space (usually of a smooth manifold, a scheme or an algebraic stack). If it can be done then one can parametrize such objects by introducing coordinates on the resulting space.

The word *moduli* is due to B. Riemann, who used it as a synonym for parameters when he showed that the space of equivalence classes of Riemann surfaces of a given genus g (for $g > 1$) depends on $3g - 3$ complex numbers. Moduli spaces were first understood as spaces of parameters rather than as spaces of objects.

The moduli spaces (2.34), (2.35), (2.37) and (2.38) introduced in the previous section were only sets of gauge equivalence classes of L-connections. In this part we define a geometric structure on these sets.

From now on we will assume that M is a connected compact manifold. To endow the sets of gauge equivalence classes of L-connections with some geometric structure it is most convenient, and standard practise, to work in the framework of Sobolev spaces.

Let $(L \to M, [\cdot\,,\cdot], a)$ be a real (complex) Lie algebroid satisfying the ellipticity condition and let $E \to M$ be a real (complex) vector bundle. Further consider a Riemannian metric g on M and denote by h_E, h_L an Euclidean (Hermitian) metric on E, L respectively. These metrics induce natural metrics on E^*, $\operatorname{End}(E) \simeq E^* \otimes E$, $\Lambda^k L^* \otimes \operatorname{End}(E)$ and others. The metric g on M defines the density $\operatorname{vol}(g)$ of the Riemannian metric and even induces a (regular) Borel measure μ_g on M. Therefore we can construct appropriate Sobolev completions defined in the previous section. The Hilbert spaces $L^{2,\ell}(M, \Lambda^k L^* \otimes \operatorname{End}(E))$ will be denoted by $\Omega^k_L(M, \operatorname{End}(E))_\ell$.

Furthermore note that the metric on $\operatorname{End}(E) \simeq E^* \otimes E$ induced by the metric h_E on E is given by

$$(f_1, f_2) = \int_M \operatorname{tr}(f_1 \circ f_2^*)\, d\mu_g \tag{2.88}$$

for $f_1, f_2 \in \Omega^0_L(M, \operatorname{End}(E))$, where * denotes the adjoint with respect to h_E. If we define the space $\Omega^0_L(M, \operatorname{End}(E))^0$ of traceless endomorphisms by

$$\Omega^0_L(M, \operatorname{End}(E))^0 = \{f \in \Omega^0_L(M, \operatorname{End}(E));\ \int_M \operatorname{tr}(f)\, d\mu_g = 0\}, \tag{2.89}$$

then obviously we obtain

$$\Omega^0_L(M, \mathrm{End}(E)) = \Omega^0_L(M, \mathrm{End}(E))^0 \oplus \mathbb{K} \cdot \mathrm{id}_E \tag{2.90}$$

and the decomposition is L^2-orthogonal with respect to (2.88). The orthogonal projection p_r of $\Omega^0_L(M, \mathrm{End}(E))$ onto $\Omega^0_L(M, \mathrm{End}(E))^0$ is given by the following formula

$$p_r(f) = f - \frac{1}{n \cdot \mathrm{vol}(M)} \left(\int_M \mathrm{tr}(f)\, d\mu_g \right) \cdot \mathrm{id}_E, \tag{2.91}$$

where $n = \mathrm{rk}\, E$ and $\mathrm{vol}(M)$ is the volume of the manifold M.

For $\ell > \frac{1}{2} \dim M$ and a fixed L-connection ∇_0 in $\mathcal{A}(E, L)$, we define Sobolev completions $\mathcal{A}(E, L)_\ell$ of the space of L-connections, using (2.16), as

$$\mathcal{A}(E, L)_\ell = \{\nabla_0 + \alpha;\ \alpha \in \Omega^1_L(M, \mathrm{End}(E))_\ell\}. \tag{2.92}$$

Further a mapping $\chi\colon \mathcal{A}(E, L)_\ell \to \Omega^1_L(M, \mathrm{End}(E))_\ell$ defined by $\chi(\nabla_0 + \alpha) = \alpha$ is a bijection and therefore gives the set $\mathcal{A}(E, L)_\ell$ a structure of a Hilbert manifold whose tangent space at ∇ is

$$T_\nabla \mathcal{A}(E, L)_\ell = \Omega^1_L(M, \mathrm{End}(E))_\ell. \tag{2.93}$$

Sobolev completions of the group of gauge transformations $\mathrm{Gau}(E)$ take a bit more work since it can not be identified with the space of sections of any vector bundle, nevertheless $\mathrm{Gau}(E) \subset \Omega^0_L(M, \mathrm{End}(E))$. In case $\ell > \frac{1}{2} \dim M$, the Sobolev space $\Omega^0_L(M, \mathrm{End}(E))_{\ell+1}$ consists of continuous sections[1] and, using the Sobolev multiplication theorem, we obtain that the product $\varphi \cdot \psi = \varphi \circ \psi$ in $\Omega^0_L(M, \mathrm{End}(E))$ can be extended to a continuous bilinear mapping

$$\Omega^0_L(M, \mathrm{End}(E))_{\ell+1} \times \Omega^0_L(M, \mathrm{End}(E))_{\ell+1} \to \Omega^0_L(M, \mathrm{End}(E))_{\ell+1}. \tag{2.94}$$

Therefore there exists a positive constat c such that $||\varphi \cdot \psi||_{\ell+1} \leq c||\varphi||_{\ell+1}||\psi||_{\ell+1}$ for all $\varphi, \psi \in \Omega^0_L(M, \mathrm{End}(E))_{\ell+1}$. Now if we take a new equivalent norm given by $||\cdot||'_{\ell+1} = c||\cdot||_{\ell+1}$, then the Banach space $\Omega^0_L(M, \mathrm{End}(E))_{\ell+1}$ is a Banach algebra with unit id_E. Because the set of invertible elements is an open subset in $\Omega^0_L(M, \mathrm{End}(E))_{\ell+1}$ and forms a topological group under multiplication, we define $\mathrm{Gau}(E)_{\ell+1}$ by

$$\mathrm{Gau}(E)_{\ell+1} = \{\varphi \in \Omega^0_L(M, \mathrm{End}(E))_{\ell+1};\ \exists\, \psi \in \Omega^0_L(M, \mathrm{End}(E))_{\ell+1},\ \varphi \cdot \psi = \psi \cdot \varphi = \mathrm{id}_E\}. \tag{2.95}$$

Since $\mathrm{Gau}(E)_{\ell+1}$ is an open subset in the Hilbert space $\Omega^0_L(M, \mathrm{End}(E))_{\ell+1}$, thus $\mathrm{Gau}(E)_{\ell+1}$ is a Hilbert manifold. In fact, one can easy show that $\mathrm{Gau}(E)_{\ell+1}$ is a Hilbert–Lie group with a Lie algebra

$$\mathfrak{gau}(E)_{\ell+1} = \Omega^0_L(M, \mathrm{End}(E))_{\ell+1}, \tag{2.96}$$

where the Lie bracket is given by

$$[\gamma_1, \gamma_2] = \gamma_1 \cdot \gamma_2 - \gamma_2 \cdot \gamma_1 \tag{2.97}$$

for all $\gamma_1, \gamma_2 \in \Omega^0_L(M, \mathrm{End}(E))_{\ell+1}$.

The multiplication on the graded vector space $\Omega^\bullet_L(M, \mathrm{End}(E))$ defined by (2.11) extends, using the Sobolev multiplication theorem, to a continuous bilinear mapping on the graded Hilbert space $\Omega^\bullet_L(M, \mathrm{End}(E))_k$ in the range $k > \frac{1}{2} \dim M$. With this bilinear mapping

$$\begin{aligned} \Omega^p_L(M, \mathrm{End}(E))_k \times \Omega^q_L(M, \mathrm{End}(E))_k &\to \Omega^{p+q}_L(M, \mathrm{End}(E))_k, \\ (\varphi, \psi) &\mapsto \varphi \cdot \psi, \end{aligned} \tag{2.98}$$

[1] Note that this is still true for $\ell + 1 > \frac{1}{2} \dim M$.

$\Omega^\bullet_L(M, \mathrm{End}(E))_k$ is a graded associative algebra.

Using the formula (2.43), we extend the action of $\mathrm{Gau}(E)$ on $\mathcal{A}(E,L)$ to an action of $\mathrm{Gau}(E)_{\ell+1}$ on $\mathcal{A}(E,L)_\ell$ via

$$\varphi \cdot \nabla = \varphi \cdot (\nabla_0 + \alpha) = \nabla_0 + \varphi \cdot \left(d^{\nabla_0}\varphi^{-1}\right) + \varphi \cdot \alpha \cdot \varphi^{-1}, \tag{2.99}$$

where $\alpha \in \Omega^1_L(M, \mathrm{End}(E))_\ell$, $d^{\nabla_0}\colon \Omega^0_L(M, \mathrm{End}(E))_{\ell+1} \to \Omega^1_L(M, \mathrm{End}(E))_\ell$ is a continuous extension of the linear operator d^{∇_0} defined on $\Omega^0_L(M, \mathrm{End}(E))$ and the multiplication $\cdot$ is an extension of (2.11) to a continuous bilinear mapping $\Omega^0_L(M, \mathrm{End}(E))_{\ell+1} \times \Omega^1_L(M, \mathrm{End}(E))_\ell \to \Omega^1_L(M, \mathrm{End}(E))_\ell$ eventually $\Omega^0_L(M, \mathrm{End}(E))_\ell \times \Omega^1_L(M, \mathrm{End}(E))_{\ell+1} \to \Omega^1_L(M, \mathrm{End}(E))_\ell$ in the range $\ell > \frac{1}{2}\dim M$. Moreover in this range $\Omega^1_L(M, \mathrm{End}(E))_\ell$ is a topological $\Omega^0_L(M, \mathrm{End}(E))_{\ell+1}$-bimodule.

It is easy to see that this action is a smooth mapping of Hilbert manifolds and that, if $\nabla = \nabla_0 + \alpha \in \mathcal{A}(E,L)_\ell$ is fixed, the mapping of $\mathrm{Gau}(E)_{\ell+1}$ to $\mathcal{A}(E,L)_\ell$ given by $\varphi \mapsto \varphi \cdot \nabla$ has a tangent mapping at id_E equal to

$$-d^\nabla : \Omega^0_L(M, \mathrm{End}(E))_{\ell+1} \to \Omega^1_L(M, \mathrm{End}(E))_\ell, \tag{2.100}$$

where d^∇ is defined through

$$d^\nabla \gamma = d^{\nabla_0}\gamma + [\alpha, \gamma] \tag{2.101}$$

and $[\cdot,\cdot]\colon \Omega^1_L(M, \mathrm{End}(E))_\ell \times \Omega^0_L(M, \mathrm{End}(E))_{\ell+1} \to \Omega^1_L(M, \mathrm{End}(E))_\ell$ is a continuous extension of (2.12) by Sobolev multiplication theorem in the range $\ell > \frac{1}{2}\dim M$.

Furthermore the curvature of an L-connection $\nabla = \nabla_0 + \alpha \in \mathcal{A}(E,L)_\ell$ is defined, using (2.23), by

$$R^\nabla = R^{\nabla_0+\alpha} = R^{\nabla_0} + d^{\nabla_0}\alpha + \frac{1}{2}[\alpha, \alpha], \tag{2.102}$$

where $\alpha \in \Omega^1_L(M, \mathrm{End}(E))_\ell$, $d^{\nabla_0}\colon \Omega^1_L(M, \mathrm{End}(E))_\ell \to \Omega^2_L(M, \mathrm{End}(E))_{\ell-1}$ is a continuous extension of the linear operator d^{∇_0} defined on $\Omega^1_L(M, \mathrm{End}(E))$ and the bracket $[\cdot,\cdot]$ is an extension of (2.12) to a continuous bilinear mapping $\Omega^1_L(M, \mathrm{End}(E))_\ell \times \Omega^1_L(M, \mathrm{End}(E))_\ell \to \Omega^1_L(M, \mathrm{End}(E))_\ell$ in the range $\ell > \frac{1}{2}\dim M$.

It is easy to see that $F\colon \mathcal{A}(E,L)_\ell \to \Omega^2_L(M, \mathrm{End}(E))_{\ell-1}$, defined by $F(\nabla) = R^\nabla$, is a smooth mapping of Hilbert manifolds, and the tangent mapping

$$T_\nabla F\colon \Omega^1_L(M, \mathrm{End}(E))_\ell \to \Omega^2_L(M, \mathrm{End}(E))_{\ell-1}$$

is given by

$$(T_\nabla F)(\gamma) = d^{\nabla_0}\gamma + [\alpha, \gamma] = d^\nabla \gamma, \tag{2.103}$$

where $\nabla = \nabla_0 + \alpha$ and $\gamma \in \Omega^1_L(M, \mathrm{End}(E))_\ell$.

Remark. For $\ell > \frac{1}{2}\dim M$ we denote by $\mathcal{H}(E,L)_\ell$ the space of flat Sobolev L-connections. Because $F\colon \mathcal{A}(E,L)_\ell \to \Omega^2_L(M, \mathrm{End}(E))_{\ell-1}$ is a continuous mapping, $\mathcal{H}(E,L)_\ell$ is a closed subset in $\mathcal{A}(E,L)_\ell$. Moreover, if we fix some flat L-connection $\nabla_0 \in \mathcal{H}(E,L)$, then

$$\mathcal{H}(E,L)_\ell = \{\nabla_0 + \alpha;\ \alpha \in \Omega^1_L(M, \mathrm{End}(E))_\ell,\ d^{\nabla_0}\alpha + \frac{1}{2}[\alpha,\alpha] = 0\}. \tag{2.104}$$

Furthermore, we need to show that $\mathcal{H}(E,L)_\ell$ is invariant under the action of the group of gauge transformations $\mathrm{Gau}(E)_{\ell+1}$.

Lemma 15. Let $\nabla = \nabla_0 + \alpha \in \mathcal{A}(E,L)_\ell$ be an L-connection then we have

$$R^{\nabla^\varphi} = \mathrm{Ad}_\varphi(R^\nabla), \tag{2.105}$$

where $\mathrm{Ad}\colon \mathrm{Gau}(E)_{\ell+1} \times \Omega^2_L(M,\mathrm{End}(E))_{\ell-1} \to \Omega^2_L(M,\mathrm{End}(E))_{\ell-1}$ is a continuous extension of (2.27) to the appropriate Sobolev spaces using the Sobolev multiplication theorem.

Proof. If $\nabla = \nabla_0 + \alpha \in \mathcal{A}(E,L)_\ell$ then $R^\nabla = R^{\nabla_0} + d^{\nabla_0}\alpha + \alpha\cdot\alpha$. Consider $\varphi \in \mathrm{Gau}(E)_{\ell+1}$ then we have $\varphi\cdot\nabla = \nabla_0 + \varphi\cdot(d^{\nabla_0}\varphi^{-1}) + \varphi\cdot\alpha\cdot\varphi^{-1}$. Therefore we can write

$$\begin{aligned}
R^{\nabla^\varphi} &= R^{\nabla_0} + d^{\nabla_0}(\varphi\cdot(d^{\nabla_0}\varphi^{-1}) + \varphi\cdot\alpha\cdot\varphi^{-1})\\
&\quad + (\varphi\cdot(d^{\nabla_0}\varphi^{-1}) + \varphi\cdot\alpha\cdot\varphi^{-1})\cdot(\varphi\cdot(d^{\nabla_0}\varphi^{-1}) + \varphi\cdot\alpha\cdot\varphi^{-1})\\
&= R^{\nabla_0} + (d^{\nabla_0}\varphi)\cdot(d^{\nabla_0}\varphi^{-1}) + \varphi\cdot((d^{\nabla_0}\circ d^{\nabla_0})\varphi^{-1}) + (d^{\nabla_0}\varphi)\cdot\alpha\cdot\varphi^{-1}\\
&\quad + \varphi\cdot(d^{\nabla_0}\alpha)\cdot\varphi^{-1} - \varphi\cdot\alpha\cdot(d^{\nabla_0}\varphi^{-1}) + (\varphi\cdot(d^{\nabla_0}\varphi^{-1}))\cdot(\varphi\cdot(d^{\nabla_0}\varphi^{-1}))\\
&\quad + (\varphi\cdot(d^{\nabla_0}\varphi^{-1}))\cdot(\varphi\cdot\alpha\cdot\varphi^{-1}) + (\varphi\cdot\alpha\cdot\varphi^{-1})\cdot(\varphi\cdot(d^{\nabla_0}\varphi^{-1})) + (\varphi\cdot\alpha\cdot\varphi^{-1})\cdot(\varphi\cdot\alpha\cdot\varphi^{-1})\\
&= R^{\nabla_0} + (d^{\nabla_0}\varphi)\cdot(d^{\nabla_0}\varphi^{-1}) + \varphi\cdot[R^{\nabla_0},\varphi^{-1}] + (d^{\nabla_0}\varphi)\cdot\alpha\cdot\varphi^{-1}\\
&\quad + \varphi\cdot(d^{\nabla_0}\alpha)\cdot\varphi^{-1} - \varphi\cdot\alpha\cdot(d^{\nabla_0}\varphi^{-1}) - (d^{\nabla_0}\varphi)\cdot(d^{\nabla_0}\varphi^{-1})\\
&\quad + \varphi\cdot\alpha\cdot(d^{\nabla_0}\varphi^{-1}) - (d^{\nabla_0}\varphi)\cdot\alpha\cdot\varphi^{-1} + \varphi\cdot\alpha\cdot\alpha\cdot\varphi^{-1}\\
&= \varphi\cdot R^{\nabla_0}\cdot\varphi^{-1} + \varphi\cdot(d^{\nabla_0}\alpha)\cdot\varphi^{-1} + \varphi\cdot\alpha\cdot\alpha\cdot\varphi^{-1}\\
&= \varphi\cdot R^\nabla\cdot\varphi,
\end{aligned}$$

where we used the fact that $\varphi\cdot(d^{\nabla_0}\varphi^{-1}) = -(d^{\nabla_0}\varphi)\cdot\varphi^{-1}$ and that $(d^{\nabla_0}\circ d^{\nabla_0})\varphi = [R^{\nabla_0},\varphi]$. ♠

Analogously to the smooth case we define the notion of irreducibility of Sobolev L-connection. A stabilizer $\mathrm{Gau}(E)^\nabla_{\ell+1}$ of any Sobolev L-connection contains the subgroup $\mathbb{K}^*\cdot\mathrm{id}_E$ of $\mathrm{Gau}(E)_{\ell+1}$. In case $\mathrm{Gau}(E)^\nabla_{\ell+1} = \mathbb{K}^*\cdot\mathrm{id}_E$, we will say that the connections ∇ is *irreducible*; otherwise, ∇ is *reducible*. We can prove the following characterization of irreducibility.

Lemma 16. Let $\nabla \in \mathcal{A}(E,L)_\ell$ be a Sobolev L-connection. Then the following are equivalent:

i) $\mathrm{Gau}(E)^\nabla_{\ell+1} = \mathbb{K}^*\cdot\mathrm{id}_E$,

ii) $\ker d^\nabla = \mathbb{K}\cdot\mathrm{id}_E$,

iii) $\ker d^\nabla{}_{|\Omega^0_L(M,\mathrm{End}(E))^0_{\ell+1}} = \{0\}$.

Proof. The proof goes along the similar line as in Lemma 14. Let $\nabla = \nabla_0 + \alpha$ be an L-connection and consider a gauge transformation $\varphi \in \mathrm{Gau}(E)_{\ell+1}$. Note that the condition $\varphi\cdot\nabla = \nabla$ means that $-d^{\nabla_0}\varphi\cdot\varphi^{-1} + \varphi\cdot\alpha\cdot\varphi^{-1} = \alpha$. If we multiply this equation by φ from the right, we obtain $d^{\nabla_0}\varphi + [\alpha,\varphi] = 0$ and using (2.101) we have $d^\nabla\varphi = 0$. Therefore $\varphi \in \mathrm{Gau}(E)^\nabla_{\ell+1}$ if and only if $d^\nabla\varphi = 0$ and $\varphi \in \mathrm{Gau}(E)_{\ell+1}$.

Suppose that $\varphi \in \mathrm{Gau}(E)^\nabla_{\ell+1}$ then $d^\nabla\varphi = 0$ and, provided that $\ker d^\nabla = \mathbb{K}\cdot\mathrm{id}_E$, we obtain $\varphi = c\cdot\mathrm{id}_E$ for some $c \in \mathbb{K}^*$. Thus we get $\mathrm{Gau}(E)^\nabla_{\ell+1} \subset \mathbb{K}^*\cdot\mathrm{id}_E$ and because the converse inclusion is trivial, we have proved ii) ⇒ i).

Now assume that $\varphi \in \ker d^\nabla$. Because $\Omega^0_L(M,\mathrm{End}(E))_{\ell+1}$ with the norm $||\cdot||'_{\ell+1}$ is a Banach algebra with unit id_E, for $c \in \mathbb{K}$ such that $|c| > ||\varphi||'_{\ell+1}$ we obtain $c\cdot\mathrm{id}_E + \varphi \in \mathrm{Gau}(E)_{\ell+1}$. Furthermore $d^\nabla(c\cdot\mathrm{id}_E + \varphi) = 0$ hence, from the previous consideration, we have $c\cdot\mathrm{id}_E + \varphi \in \mathrm{Gau}(E)^\nabla_{\ell+1}$. Moreover if we suppose $\mathrm{Gau}(E)^\nabla_{\ell+1} = \mathbb{K}^*\cdot\mathrm{id}_E$, we obtain $\ker d^\nabla \subset \mathbb{K}\cdot\mathrm{id}_E$. Converse inclusion is trivial, so we have proved the converse inclusion.

The equivalence of ii) and iii) immediately follows form the definition of $\Omega^0_L(M,\mathrm{End}(E))^0_{\ell+1}$, so we are done. ♠

We will denote by $\mathcal{A}^*(E,L)_\ell$ the subset of $\mathcal{A}(E,L)_\ell$ consisting of irreducible L-connections and similarly by $\mathcal{H}^*(E,L)_\ell$ the subset of $\mathcal{H}(E,L)_\ell$ containing irreducible flat L-connections. It follows from the fact $\mathrm{Gau}(E)^{\nabla^\varphi}_{\ell+1} = \varphi\cdot\mathrm{Gau}(E)^\nabla_{\ell+1}\cdot\varphi^{-1}$ that the irreducibility of L-connection is invariant under gauge transformations. In addition to $\mathcal{H}(E,L)_\ell$ is invariant under gauge transformations as well.

In analogy with (2.34), (2.35), (2.37) and (2.38) we define the *moduli space*

$$\mathcal{B}(E,L)_\ell = \mathcal{A}(E,L)_\ell/\mathrm{Gau}(E)_{\ell+1} \quad\text{and}\quad \mathcal{M}(E,L)_\ell = \mathcal{H}(E,L)_\ell/\mathrm{Gau}(E)_{\ell+1} \tag{2.106}$$

of L-connections and flat L-connections on E and similarly the *moduli space*

$$\mathcal{B}^*(E,L)_\ell = \mathcal{A}^*(E,L)_\ell/\mathrm{Gau}(E)_{\ell+1} \qquad \text{and} \qquad \mathcal{M}^*(E,L)_\ell = \mathcal{H}^*(E,L)_\ell/\mathrm{Gau}(E)_{\ell+1} \tag{2.107}$$

of irreducible L-connections and irreducible flat L-connections on E. Each of these is assumed to have the quotient topology and in the next we shall show that $\mathcal{B}^*(E,L)_\ell$ is open in $\mathcal{B}(E,L)_\ell$ and that $\mathcal{M}^*(E,L)_\ell$ is open in $\mathcal{M}(E,L)_\ell$. Furthermore we will denote by

$$p_\ell \colon \mathcal{A}(E,L)_\ell \to \mathcal{B}(E,L)_\ell \tag{2.108}$$

possibly by

$$\hat{p}_\ell \colon \mathcal{A}^*(E,L)_\ell \to \mathcal{B}^*(E,L)_\ell \tag{2.109}$$

the canonical projection.

For $\alpha \in \Omega^1_L(M,\mathrm{End}(E))$ the zero order operator $\mathrm{ad}(\alpha)^* \colon \Omega^1_L(M,\mathrm{End}(E)) \to \Omega^0_L(M,\mathrm{End}(E))$, defined as a formal adjoint of $\mathrm{ad}(\alpha)\colon \Omega^0_L(M,\mathrm{End}(E)) \to \Omega^1_L(M,\mathrm{End}(E))$, $\mathrm{ad}(\alpha)(\gamma) = [\alpha,\gamma]$, with respect to the Hermitian metric on $\mathrm{End}(E)$ given by $(f_1,f_2) \mapsto \mathrm{tr}(f_1 \circ f_2^*)$, yields a mapping

$$\begin{aligned} \Omega^1_L(M,\mathrm{End}(E)) \times \Omega^1_L(M,\mathrm{End}(E)) &\to \Omega^0_L(M,\mathrm{End}(E)), \\ (\alpha,\beta) &\mapsto \mathrm{ad}(\alpha)^*(\beta), \end{aligned} \tag{2.110}$$

which is $C^\infty(M,\mathbb{K})$-sesquilinear in the first component and $C^\infty(M,\mathbb{K})$-linear in the second component. This mapping can be extend by Sobolev multiplication theorem to a continuous sesquilinear-linear mapping

$$\Omega^1_L(M,\mathrm{End}(E))_\ell \times \Omega^1(M,\mathrm{End}(E))_\ell \to \Omega^0_L(M,\mathrm{End}(E))_\ell$$

hence the mapping $\mathrm{ad}(\alpha)^* \colon \Omega^1_L(M,\mathrm{End}(E))_\ell \to \Omega^0_L(M,\mathrm{End}(E))_\ell$ for every $\alpha \in \Omega^1_L(M,\mathrm{End}(E))_\ell$ is continuous. Then for $\nabla = \nabla_0 + \alpha \in \mathcal{A}(E,L)_\ell$ we may write

$$d^\nabla = d^{\nabla_0} + \mathrm{ad}(\alpha) \circ i, \tag{2.111}$$

where $i\colon \Omega^0_L(M,\mathrm{End}(E))_{\ell+1} \to \Omega^0(M,\mathrm{End}(E))_\ell$ is a compact embedding. Furthermore, we define

$$\delta^\nabla \colon \Omega^1_L(M,\mathrm{End}(E))_\ell \to \Omega^0_L(M,\mathrm{End}(E))_{\ell-1} \tag{2.112}$$

through

$$\delta^\nabla = \delta^{\nabla_0} + i \circ \mathrm{ad}(\alpha)^*, \tag{2.113}$$

where $i\colon \Omega^0_L(M,\mathrm{End}(E))_\ell \to \Omega^0(M,\mathrm{End}(E))_{\ell-1}$ is a compact embedding and δ^{∇_0} is a continuous extension of formal adjoint of d^{∇_0} with respect to the Hermitian metric on $\mathrm{End}(E)$.

Lemma 17. For $\ell > \max\{\frac{1}{2}\dim M, 1\}$ the natural mapping

$$j_\ell \colon \mathcal{B}(E,L) \to \mathcal{B}(E,L)_\ell \tag{2.114}$$

is injective.

Proof. Let $\nabla = \nabla_0 + \alpha$ and $\nabla' = \nabla_0 + \alpha'$ be smooth L-connections, and suppose we have a gauge transformation $\varphi \in \mathrm{Gau}(E)_{\ell+1}$ satisfying $\varphi \cdot \nabla = \nabla'$, then for the injectivity of j_ℓ it suffices to show that φ is smooth. If we denote $\beta = \alpha - \alpha'$, then the requirement $\varphi \cdot \nabla = \nabla'$ is equivalent to $d^\nabla \varphi = \beta \cdot \varphi$ and we have

$$\Delta(\varphi) = (\delta^\nabla \circ d^\nabla)(\varphi) = \delta^\nabla(\beta \cdot \varphi).$$

If $k > \max\{\frac{1}{2}\dim M, 1\}$, then $\varphi \in \Omega^0_L(M,\mathrm{End}(E))_k$ implies, by the Sobolev multiplication theorem, that $\beta \cdot \varphi \in \Omega^1_L(M,\mathrm{End}(E))_k$, because β is smooth. Since ∇ is a smooth L-connection, the term on the right hand side in the equation above belongs to $\Omega^0_L(M, \mathrm{End}(E))_{k-1}$, and the Elliptic Regularity (Lemma 7), applied to the elliptic operator Δ, gives $\varphi \in \Omega^0_L(M, \mathrm{End}(E))_{k+1}$. Using the induction on k we get $\varphi \in \Omega^0_L(M, \mathrm{End}(E))_k$ for all $k \geq \ell$. From the Lemma of Rellich (Theorem 3) it follows that φ is smooth. ♠

Lemma 18. Let $\nabla \in \mathcal{A}(E, L)_\ell$ be an L-connection then the operator

$$\delta^\nabla \circ d^\nabla \colon \Omega^0_L(M, \mathrm{End}(E))_{\ell+1} \to \Omega^0_L(M, \mathrm{End}(E))_{\ell-1} \tag{2.115}$$

is a Fredholm operator for $\ell > \frac{1}{2}\dim M$.

Proof. For $\nabla = \nabla_0 + \alpha$, we may write $\Delta_\alpha = \delta^\nabla \circ d^\nabla = (\delta^{\nabla_0} + i \circ \mathrm{ad}(\alpha)^*) \circ (d^{\nabla_0} + \mathrm{ad}(\alpha) \circ i)$. Because $\mathrm{ad}(\alpha) \circ i$ and $i \circ \mathrm{ad}(\alpha)^*$ are compact operators,

$$i \circ \mathrm{ad}(\alpha)^* \circ d^{\nabla_0} + \delta^{\nabla_0} \circ \mathrm{ad}(\alpha) \circ i + i \circ \mathrm{ad}(\alpha)^* \circ \mathrm{ad}(\alpha) \circ i$$

is also compact operator. The rest of the proof is to show that $\delta^{\nabla_0} \circ d^{\nabla_0}$ is a Fredholm operator. It is enough to show that $\delta^{\nabla_0} \circ d^{\nabla_0} \colon \Omega^0_L(M, \mathrm{End}(E)) \to \Omega^0_L(M, \mathrm{End}(E))$ is an elliptic operator, i.e., that the principal symbol $\sigma_2(\delta^{\nabla_0} \circ d^{\nabla_0})(\xi_x) \colon \mathrm{End}(E)_x \to \mathrm{End}(E)_x$ is an isomorphism for all $x \in M$ and $0 \neq \xi_x \in T^*_x M$. Obviously,

$$\sigma_2(\delta^{\nabla_0} \circ d^{\nabla_0})(\xi_x) = \sigma_1(\delta^{\nabla_0})(\xi_x) \circ \sigma_1(d^{\nabla_0})(\xi_x) = -\big(\sigma_1(d^{\nabla_0})(\xi_x)\big)^* \circ \sigma_1(d^{\nabla_0})(\xi_x)$$

and this is an isomorphism if and only if $\sigma_1(d^{\nabla_0})(\xi_x)$ is an isomorphism. But $\sigma_1(d^{\nabla_0})(\xi_x) = a^*(\xi_x) \otimes$, i.e., the symbol is the tensor multiplication by $a^*(\xi_x)$, hence it is an isomorphism if $a^*(\xi_x) \neq 0$. Thus, $\sigma_2(\delta^{\nabla_0} \circ d^{\nabla_0})$ is an isomorphism for all $x \in M$ and $0 \neq \xi_x \in T^*_x M$ if and only if a^* is injective or equivalently if and only if a is surjective. This is true because L satisfies the ellipticity condition. ♠

Lemma 19. For any $\nabla \in \mathcal{A}(E, L)_\ell$ we have an L^2-orthogonal decomposition

$$\Omega^1_L(M, \mathrm{End}(E))_\ell = \mathrm{im}\, d^\nabla \oplus \ker \delta^\nabla \tag{2.116}$$

for $\ell > \frac{1}{2}\dim M$.

Proof. Let $\nabla = \nabla_0 + \alpha$ be an L-connection and denote $\Delta_\alpha = \delta^\nabla \circ d^\nabla$. From the previous lemma we know that Δ_α is a Fredholm operator, thus $\dim\ker\Delta_\alpha < \infty$ and $\mathrm{im}\,\Delta_\alpha$ is a closed subspace. Therefore $\Omega^0_L(M, \mathrm{End}(E))_{\ell+1} = \ker\Delta_\alpha \oplus (\ker\Delta_\alpha)^\perp$ is an L^2-orthogonal (not $L^2_{\ell+1}$) decomposition into closed subspaces in $\Omega^0_L(M, \mathrm{End}(E))_{\ell+1}$.

Furthermore, $\mathrm{im}\,\Delta_\alpha$ is a closed subspace, thus $\Delta_{\alpha|(\ker\Delta_\alpha)^\perp} \colon (\ker\Delta_\alpha)^\perp \to \mathrm{im}\,\Delta_\alpha$ is a bijective continuous linear operator between Banach spaces, therefore, using the Banach's Open Mapping Theorem, $G_\alpha = (\Delta_{\alpha|(\ker\Delta_\alpha)^\perp})^{-1}$ is a continuous linear operator. If $X \subset \Omega^1_L(M, \mathrm{End}(E))_\ell$ denotes the closed subspace given by $X = (\delta^\nabla)^{-1}(\mathrm{im}\,\Delta_\alpha)$, then $\mathrm{id}_{|X} - d^\nabla \circ G_\alpha \circ \delta^\nabla{}_{|X}$ is a continuous linear operator. Because $\ker d^\nabla = \ker\Delta_\alpha$, we get $\mathrm{im}\, d^\nabla = \ker(\mathrm{id}_{|X} - d^\nabla \circ G_\alpha \circ \delta^\nabla{}_{|X})$, therefore $\mathrm{im}\, d^\nabla$ is a closed subspace in $\Omega^1_L(M, \mathrm{End}(E))_\ell$.

Thus we get an L^2-orthogonal decomposition $\Omega^1_L(M, \mathrm{End}(E))_\ell = \mathrm{im}\, d^\nabla \oplus (\mathrm{im}\, d^\nabla)^\perp$ into closed subspaces. On the other hand for $\varphi \in \Omega^0_L(M, \mathrm{End}(E))_{\ell+1}$ and $\psi \in \Omega^1_L(M, \mathrm{End}(E))_\ell$ we have $(d^\nabla\varphi, \psi) = (\varphi, \delta^\nabla\psi)$, hence we obtain that $(\mathrm{im}\, d^\nabla)^\perp = \ker\delta^\nabla$. ♠

Lemma 20. The set of irreducible Sobolev L-connections $\mathcal{A}^*(E, L)_\ell$ is an open subset in $\mathcal{A}(E, L)_\ell$ for $\ell > \frac{1}{2}\dim M$.

Proof. Let $\nabla = \nabla_0 + \alpha$ be an L-connection. From Lemma 18 it follows that $\Delta_\alpha = \delta^\nabla \circ d^\nabla$ is a Fredholm operator. Moreover, the mapping

$$\mathcal{A}(E, L)_\ell \to \mathcal{L}(\Omega^0_L(M, \mathrm{End}(E))_{\ell+1}, \Omega^0_L(M, \mathrm{End}(E))_{\ell-1})$$

given by $\nabla_0 + \alpha \mapsto \Delta_\alpha$ is a continuous family of Fredholm operators, hence

$$\nabla_0 + \alpha \mapsto \dim\ker\Delta_\alpha$$

is an upper semicontinuous mapping from $\mathcal{A}(E,L)_\ell$ to $\mathbb{R}$, see [36]. Because we have $\ker d^\nabla = \ker \Delta_\alpha$ and $\dim \ker d^\nabla \geq 1$, hence the upper semicontinuity implies that $\mathcal{A}^*(E,L)_\ell$ is an open subset. ♠

Remark. We have just proved that $\mathcal{A}^*(E,L)_\ell$ is an open subset in $\mathcal{A}(E,L)_\ell$. Because $\mathcal{B}(E,L)_\ell$ is assumed to have the quotient topology and $p_\ell^{-1}(\mathcal{B}^*(E,L)_\ell) = \mathcal{A}^*(E,L)_\ell$, we get that $\mathcal{B}^*(E,L)_\ell$ is open in $\mathcal{B}(E,L)_\ell$.

Now, for $\nabla = \nabla_0 + \alpha \in \mathcal{A}(E,L)_\ell$ and $\varepsilon > 0$ we consider the Hilbert submanifold

$$\mathcal{O}_{\alpha,\varepsilon} = \{\nabla_0 + \alpha + \beta;\ \beta \in \Omega_L^1(M, \mathrm{End}(E))_\ell,\ \delta^\nabla \beta = 0,\ ||\beta||_\ell < \varepsilon\} \tag{2.117}$$

of the Hilbert manifold $\mathcal{A}(E,L)_\ell$. Because $\mathcal{O}_{\alpha,\varepsilon}$ is a Hilbert manifold modeled on $\ker \delta^\nabla$, thus we have

$$T_\nabla(\mathcal{O}_{\alpha,\varepsilon}) = \ker \delta^\nabla. \tag{2.118}$$

First note that if $\nabla \in \mathcal{A}^*(E,L)_\ell$, then we may take ε small enough to ensure $\mathcal{O}_{\alpha,\varepsilon} \subset \mathcal{A}^*(E,L)_\ell$, since $\mathcal{A}^*(E,L)_\ell$ is open in $\mathcal{A}(E,L)_\ell$. Next, we define the *reduced group of gauge transformations* $\mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1}$ by

$$\mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1} = {\mathrm{Gau}(E)_{\ell+1}}/{\mathbb{K}^* \cdot \mathrm{id}_E}. \tag{2.119}$$

Because $\mathbb{K}^* \cdot \mathrm{id}_E$ is a normal Hilbert–Lie subgroup of $\mathrm{Gau}(E)_{\ell+1}$, Theorem 9 bellow implies that the reduced group of gauge transformations is a Hilbert–Lie group with the Lie algebra

$$\mathfrak{gau}(E)^{\mathrm{r}}_{\ell+1} = \Omega_L^0(M, \mathrm{End}(E))^0_{\ell+1}, \tag{2.120}$$

where the Lie bracket descents from the one on $\mathfrak{gau}(E)_{\ell+1}$. Moreover, if

$$q\colon \mathrm{Gau}(E)_{\ell+1} \to \mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1} = {\mathrm{Gau}(E)_{\ell+1}}/{\mathbb{K}^* \cdot \mathrm{id}_E} \tag{2.121}$$

denotes the canonical projection, then q is a smooth mapping and any mapping $f\colon \mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1} \to X$, where X is a smooth Banach manifold, is smooth if and only if $f \circ q\colon \mathrm{Gau}(E)_{\ell+1} \to X$ is smooth.

Theorem 9. Let G be a Banach–Lie group over $\mathbb{K}$ with Lie algebra $\mathfrak{g}$ and suppose that N is a normal Banach–Lie subgroup over $\mathbb{K}$ of G with Lie algebra $\mathfrak{n}$. Then G/N is a Banach–Lie group over $\mathbb{K}$ with Lie algebra $\mathfrak{g}/\mathfrak{n}$ in a unique way such that the quotient mapping $q\colon G \to G/N$ is smooth. Moreover, for any Banach manifold X a mapping $f\colon G/N \to X$ is smooth if and only if $f \circ q$ is smooth.

Proof. See [37], [38] and [39]. ♠

Theorem 10. $\mathcal{B}^*(E,L)_\ell$ is a locally Hausdorff Hilbert manifold and $\hat{p}_\ell\colon \mathcal{A}^*(E,L)_\ell \to \mathcal{B}^*(E,L)_\ell$ is a principal $\mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1}$-bundle.

Proof. Let $\nabla = \nabla_0 + \alpha$ be an irreducible L-connection. Consider the smooth mapping of Hilbert manifolds

$$\begin{aligned} &\Psi_\nabla\colon \mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1} \times \mathcal{O}_{\alpha,\varepsilon} \to \mathcal{A}^*(E,L)_\ell, \\ &\Psi_\nabla(\varphi, \nabla_0 + \alpha + \beta) = \varphi \cdot (\nabla_0 + \alpha + \beta), \end{aligned} \tag{2.122}$$

then the tangent mapping at (id_E, ∇) equals to

$$\begin{aligned} &T_{(\mathrm{id}_E,\nabla)}\Psi_\nabla\colon \Omega_L^0(M, \mathrm{End}(E))^0_{\ell+1} \oplus \ker \delta^\nabla \to \Omega_L^1(M, \mathrm{End}(E))_\ell, \\ &(T_{(\mathrm{id}_E,\nabla)}\Psi_\nabla)(\gamma, \beta) = -d^\nabla \gamma + \beta. \end{aligned} \tag{2.123}$$

From Lemma 19 it follows that $T_{(\mathrm{id}_E,\nabla)}\Psi_\nabla$ is surjective. Moreover, because ∇ is assumed to be an irreducible L-connection, we obtain, using Lemma 16, that $T_{(\mathrm{id}_E,\nabla)}\Psi_\nabla$ is injective. Hence by the Banach's open mapping theorem $T_{(\mathrm{id}_E,\nabla)}\Psi_\nabla$ is an isomorphism. Therefore the inverse function theorem for Banach manifolds implies that Ψ_∇ is a local diffeomorphism near (id_E,∇). Consequently, there is an open neighborhood $\mathcal{U}_\alpha$ of ∇ in $\mathcal{A}^*(E,L)_\ell$ and an open neighborhood $\mathcal{N}_{\mathrm{id}_E}$ of id_E in $\mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1}$ such that

$$\Psi_\nabla\colon \mathcal{N}_{\mathrm{id}_E}\times\mathcal{O}_{\alpha,\varepsilon}\to\mathcal{U}_\alpha \tag{2.124}$$

is a diffeomorphism sufficiently small $\varepsilon>0$.

Next we show that, for ε small enough, the mapping $p_{\alpha,\varepsilon}=\hat{p}_{|\mathcal{O}_{\alpha,\varepsilon}}\colon \mathcal{O}_{\alpha,\varepsilon}\to\mathcal{B}^*(E,L)_\ell$ is injective. We have to show that if for two elements $\nabla_0+\alpha+\beta_1,\nabla_0+\alpha+\beta_2\in\mathcal{O}_{\alpha,\varepsilon}$ there exists a gauge transformation $\varphi\in\mathrm{Gau}(E)_{\ell+1}$ satisfying

$$\varphi\cdot(\nabla_0+\alpha+\beta_1)=\nabla_0+\alpha+\beta_2, \tag{2.125}$$

then $\beta_1=\beta_2$. First observe that (2.125) is equivalent to

$$d^\nabla\varphi=\varphi\cdot\beta_1-\beta_2\cdot\varphi. \tag{2.126}$$

Further, because $\Omega^0_L(M,\mathrm{End}(E))_{\ell+1}=\ker d^\nabla\oplus(\ker d^\nabla)^\perp$ is an L^2-orthogonal decomposition into closed subspaces, we can write $\varphi=c\cdot\mathrm{id}_E+\varphi_0$, where $c\in\mathbb{K}$ and $\varphi_0\in(\ker d^\nabla)^\perp$. Moreover $\mathrm{im}\, d^\nabla$ is a closed subspace in $\Omega^1_L(M,\mathrm{End}(E))_\ell$, hence we obtain by the Banach's open mapping theorem that

$$d^\nabla\colon(\ker d^\nabla)^\perp\to\mathrm{im}\, d^\nabla \tag{2.127}$$

is an isomorphism of Hilbert spaces. Therefore it is lower bounded operator, i.e., there exists a positive constant c_1 such that

$$||d^\nabla\psi||_\ell\geq c_1||\psi||_{\ell+1} \tag{2.128}$$

for all $\psi\in(\ker d^\nabla)^\perp$. Thus we may write

$$c_1||\varphi_0||_{\ell+1}\leq||d^\nabla\varphi_0||_\ell=||d^\nabla\varphi||_\ell=||\varphi\cdot\beta_1-\beta_2\cdot\varphi||_\ell\leq 2\,c_0\cdot\varepsilon\cdot(|c|\cdot||\mathrm{id}_E||_{\ell+1}+||\varphi_0||_{\ell+1}), \tag{2.129}$$

where we used the fact that $||\psi\cdot\alpha||_\ell\leq c_0\cdot||\psi||_{\ell+1}||\alpha||_\ell$ and $||\alpha\cdot\psi||_\ell\leq c_0\cdot||\alpha||_\ell||\psi||_{\ell+1}$ for all $\psi\in\Omega^0_L(M,\mathrm{End}(E))_{\ell+1}$ and $\alpha\in\Omega^1_L(M,\mathrm{End}(E))_\ell$. As a consequence we have

$$||\varphi_0||_{\ell+1}\leq\frac{2\,c_0\cdot|c|\cdot\varepsilon}{c_1-2\,c_0\cdot\varepsilon}||\mathrm{id}_E||_{\ell+1} \tag{2.130}$$

for $\varepsilon<\frac{c_1}{2c_0}$. If $c=0$, then we obtain immediately $||\varphi_0||_{\ell+1}=0$, thus $\varphi=c\cdot\mathrm{id}_E+\varphi_0=0$ and this is a contradiction. Because $c\neq0$, we get

$$||c^{-1}\cdot\varphi-\mathrm{id}_E||_{\ell+1}=\frac{1}{|c|}||\varphi_0||_{\ell+1}\leq\frac{2\,c_0\cdot\varepsilon}{c_1-2\,c_0\cdot\varepsilon}||\mathrm{id}_E||_{\ell+1}. \tag{2.131}$$

Since $q^{-1}(\mathcal{N}_{\mathrm{id}_E})$ is open set in $\mathrm{Gau}(E)_{\ell+1}$ and $\mathrm{id}_E\in q^{-1}(\mathcal{N}_{\mathrm{id}_E})$, therefore for ε small enough is φ near id_E in $\mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1}$, i.e., $\varphi\in\mathcal{N}_{\mathrm{id}_E}$. And if we use that Ψ_∇ is injective, we obtain $\beta_1=\beta_2$.

Let $\mathcal{U}_{\alpha,\varepsilon}=\hat{p}\,(\mathcal{O}_{\alpha,\varepsilon})$, then we have $\hat{p}^{-1}(\mathcal{U}_{\alpha,\varepsilon})=\lambda(\mathrm{Gau}(E)_{\ell+1}\times\mathcal{O}_{\alpha,\varepsilon})$, where $\lambda\colon\mathrm{Gau}(E)_{\ell+1}\times\mathcal{A}^*(E,L)_\ell\to\mathcal{A}^*(E,L)_\ell$ is the left action. From the previous considerations it follows that $\hat{p}^{-1}(\mathcal{U}_{\alpha,\varepsilon})$ is open in $\mathcal{A}^*(E,L)_\ell$, thus $\mathcal{U}_{\alpha,\varepsilon}$ is open in $\mathcal{B}^*(E,L)_\ell$. Moreover $p_{\alpha,\varepsilon}\colon\mathcal{O}_{\alpha,\varepsilon}\to\mathcal{U}_{\alpha,\varepsilon}$ is a homeomorphism. The mapping

$$\begin{aligned}&\Psi_\nabla\colon\ \mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1}\times\mathcal{O}_{\alpha,\varepsilon}\to\hat{p}^{-1}(\mathcal{U}_{\alpha,\varepsilon}),\\&\Psi_\nabla(\varphi,\nabla_0+\alpha+\beta)=\varphi\cdot(\nabla_0+\alpha+\beta)\end{aligned} \tag{2.132}$$

is surjective because $p^{-1}(\mathcal{U}_{\alpha,\varepsilon}) = \lambda(\mathrm{Gau}(E)_{\ell+1} \times \mathcal{O}_{\nabla,\varepsilon})$, the injectivity follows from the previous consideration and from the fact that the action of $\mathrm{Gau}(E)^{\mathfrak{r}}_{\ell+1}$ on $\mathcal{A}^*(E,L)_\ell$ is free. We will show that it is in fact diffeomorphism of Hilbert manifolds.

For an arbitrary $\varphi \in \mathrm{Gau}(E)^{\mathfrak{r}}_{\ell+1}$ we find an open neighborhood $\mathcal{W}_\varphi$ of φ such that the mapping $\Psi_{\nabla|L_{\varphi^{-1}}(\mathcal{W}_\varphi)\times\mathcal{O}_{\nabla,\varepsilon}}$ is a diffeomorphism, where $L_{\varphi^{-1}}$ is the left translation by φ^{-1} in $\mathrm{Gau}(E)^{\mathfrak{r}}_{\ell+1}$. In particular, we can take $\mathcal{W}_\varphi = L_\varphi(\mathcal{N}_{\mathrm{id}_E})$. Therefore we have

$$\Psi_{\nabla|\mathcal{W}\times\mathcal{O}_{\alpha,\varepsilon}} = L_\varphi \circ \Psi_\nabla \circ (L_{\varphi^{-1}} \times \mathrm{id}_{\hat{\mathcal{A}}(E,L)_\ell})_{|\mathcal{W}_\varphi\times\mathcal{O}_{\alpha,\varepsilon}}, \tag{2.133}$$

which is a diffeomorphism.

Now to show that $\hat{p}_\ell \colon \mathcal{A}^*(E,L)_\ell \to \mathcal{B}^*(E,L)_\ell$ is a principal $\mathrm{Gau}(E)^{\mathfrak{r}}_{\ell+1}$-bundle over a Hilbert manifold, we only need to glue together the local charts $\sigma_\alpha \colon \mathcal{U}_{\alpha,\varepsilon} \to \mathcal{O}_{\alpha,\varepsilon}$, $\sigma_\alpha = p^{-1}_{\alpha,\varepsilon}$. Consider the smooth mapping

$$g_\nabla = \mathrm{pr} \circ \Psi_\nabla^{-1} \colon \hat{p}^{-1}(\mathcal{U}_{\alpha,\varepsilon}) \to \mathrm{Gau}(E)^{\mathfrak{r}}_{\ell+1}, \tag{2.134}$$

where $\mathrm{pr}\colon \mathrm{Gau}(E)^{\mathfrak{r}}_{\ell+1} \times \mathcal{O}_{\alpha,\varepsilon} \to \mathrm{Gau}(E)^{\mathfrak{r}}_{\ell+1}$ is the projection. Then for any $\nabla' = \nabla_0 + \alpha' \in \hat{\mathcal{A}}(E,L)_\ell$ with $\hat{p}(\nabla_0 + \alpha') \in \mathcal{U}_{\alpha,\varepsilon}$ we have

$$\sigma_\alpha(\hat{p}(\nabla_0 + \alpha')) = (g_\nabla(\nabla_0 + \alpha'))^{-1} \cdot (\nabla_0 + \alpha'). \tag{2.135}$$

Hence it is easy to see that over $\sigma_{\alpha'}(\mathcal{U}_{\alpha',\varepsilon'} \cap \mathcal{U}_{\alpha,\varepsilon})$ we have

$$(\sigma_\alpha \circ \sigma_{\alpha'}^{-1})(\nabla_0 + \alpha' + \beta) = \sigma_\alpha(\hat{p}(\nabla_0 + \alpha' + \beta)) = (g_\nabla(\nabla_0 + \alpha' + \beta))^{-1} \cdot (\nabla_0 + \alpha' + \beta), \tag{2.136}$$

and this is clearly smooth in β. ♠

2.6 Moduli spaces – local model

In this section we give a local description of the moduli space $\mathcal{M}(E,L)$ of flat L-connections and the moduli space $\mathcal{M}^*(E,L)$ of irreducible flat L-connections around a given point. We will adopt to this situation the Kuranishi argument for describing the moduli space of complex structures near a given one on a compact manifold and the moduli space of anti-self-dual connections on a compact 4-manifold given by Atiyah, Hitchin and Singer, see [40].

The *Kuranishi description* provides local models of the moduli space, i.e., it gives an explicit description of the germ of the moduli space in a given point. This makes it possible to estimate the dimension of the moduli space in a given point, and provides a simple smoothness criteria.

Let $(L \to M, [\cdot\,,\cdot], a)$ be a real (complex) Lie algebroid satisfying the *ellipticity condition* and $E \to M$ be a real (complex) vector bundle. Further assume that M is a connected compact manifold. Then to any flat L-connection ∇ on E is associated a *fundamental elliptic complex* $\mathcal{E}(\nabla)$ playing a cental role in the subsequent discussion.

Consider a sequence of linear differential operators

$$0 \longrightarrow \Omega^0_L(M,\mathrm{End}(E)) \xrightarrow{d^\nabla} \Omega^1_L(M,\mathrm{End}(E)) \xrightarrow{d^\nabla} \dots \xrightarrow{d^\nabla} \Omega^r_L(M,\mathrm{End}(E)) \longrightarrow 0, \tag{2.137}$$

where $r = \mathrm{rk}\, L$. Because $R^\nabla = 0$ and

$$R^{\nabla^{\mathrm{End}(E)}}(\xi,\eta)\,\gamma = [R^\nabla(\xi,\eta),\gamma] = [R^\nabla,\gamma](\xi,\eta), \tag{2.138}$$

where $\xi,\eta \in \mathfrak{X}_L(M)$ and $\gamma \in \Omega^0_L(M,\mathrm{End}(E))$, we obtain $R^{\nabla^{\mathrm{End}(E)}} = 0$. Further, using Lemma 10 and the fact that the Lie algebroid satisfies the condition of ellipticity, we get that the sequence (2.137) of differential operators is an elliptic complex, called the *deformation complex*.

We will denote the cohomology of this elliptic compex by $H^i(E,\nabla)$ for $i = 0,1,\dots,r$. Endow E, L with an Euclidean (Hermitian) metric h_E, h_L respectively. This gives an Euclidean (Hermitian) metric on each vector bundle $\Lambda^k L^* \otimes \mathrm{End}(E)$. Furthermore, let g be a Riemannian metric on M. Then we have the formal selfadjoint elliptic operators of second order

$$\Delta_i = \delta_i^\nabla \circ d_i^\nabla + d_{i-1}^\nabla \circ \delta_{i-1}^\nabla \colon \Omega_L^i(M, \mathrm{End}(E)) \to \Omega_L^i(M, \mathrm{End}(E)) \tag{2.139}$$

where δ_i^∇ is a formal adjoint of d_i^∇ and $d_{-1}^\nabla, d_r^\nabla$ are zero operators. Besides the kernel of Δ_i

$$\mathcal{H}^i(E, \nabla) = \{\alpha \in \Omega_L^k(M, \mathrm{End}(E));\ \Delta_i \alpha = 0\} = \ker d_i^\nabla \cap \ker \delta_{i-1}^\nabla \tag{2.140}$$

is a finite dimensional vector space for $i = 0, 1, \dots, r$ and moreover there exists a natural isomorphism $\mathcal{H}^i(E, \nabla) \simeq H^i(E, \nabla)$. Because all cohomology groups are finite dimensional vector spaces, we may define the *index* of $\mathcal{E}(\nabla)$ by

$$\operatorname{Ind} \mathcal{E}(\nabla) = \sum_{i=0}^{r} (-1)^i \dim H^i(E, \nabla) = \sum_{i=0}^{r} (-1)^i \dim \ker \Delta_i. \tag{2.141}$$

A fundamental result of the Hodge theory for the elliptic complex (2.137) is the Hodge decomposition theorem, which states that there is an L^2-orthogonal decomposition

$$\Omega_L^i(M, \mathrm{End}(E)) = \mathcal{H}^i(E, \nabla) \oplus \operatorname{im} d_{i-1}^\nabla \oplus \operatorname{im} \delta_i^\nabla. \tag{2.142}$$

Furthermore there exists a unique linear operator

$$G_i \colon \Omega_L^i(M, \mathrm{End}(E)) \to \Omega_L^i(M, \mathrm{End}(E)), \tag{2.143}$$

called the Green operator associated to Δ_i, satisfying

$$\mathrm{id}_{\Omega_L^i(M, \mathrm{End}(E))} = \mathrm{pr}_{\mathcal{H}^i(E,\nabla)} + \Delta_i \circ G_i = \mathrm{pr}_{\mathcal{H}^i(E,\nabla)} + G_i \circ \Delta_i. \tag{2.144}$$

and the following commutation relations

$$H_i \circ G_i = G_i \circ H_i, \quad d_i^\nabla \circ G_i = G_{i+1} \circ d_i^\nabla, \quad \delta_i^\nabla \circ G_{i+1} = G_i \circ \delta_i^\nabla, \tag{2.145}$$

where $H_i \colon \Omega_L^i(M, \mathrm{End}(E)) \to \mathcal{H}^i(E, \nabla)$ for $i = 0, 1, \dots, r$ are L^2-orthogonal projections. Moreover G_i is a pseudo-differential operator of degree -2. Further all associated operators d_i^∇, δ_i^∇, Δ_i, G_i can be extended to continuous linear operators between appropriate Sobolev completions, e.g.

$$\begin{aligned}
d_{i,k}^\nabla &\colon \Omega_L^i(M, \mathrm{End}(E))_k \to \Omega_L^{i+1}(M, \mathrm{End}(E))_{k-1}, && (2.146)\\
\delta_{i,k}^\nabla &\colon \Omega_L^i(M, \mathrm{End}(E))_k \to \Omega_L^{i-1}(M, \mathrm{End}(E))_{k-1}, && (2.147)\\
\Delta_{i,k} &\colon \Omega_L^i(M, \mathrm{End}(E))_k \to \Omega_L^i(M, \mathrm{End}(E))_{k-2}, && (2.148)\\
G_{i,k} &\colon \Omega_L^i(M, \mathrm{End}(E))_k \to \Omega_L^i(M, \mathrm{End}(E))_{k+2}, && (2.149)
\end{aligned}$$

and note that

$$\ker \Delta_{i,k} = \ker \Delta_i = \mathcal{H}^i(E, \nabla). \tag{2.150}$$

All statements in Theorem 8 remain true in we replace the spaces by the correct Sobolev completions, e.g. there are L^2-orthogonal (not L_k^2-orthogonal) decompositions

$$\begin{aligned}
\Omega_L^i(M, \mathrm{End}(E))_k &= \mathcal{H}^i(E, \nabla) \oplus \operatorname{im} d_{i-1,k+1}^\nabla \oplus \operatorname{im} \delta_{i,k+1}^\nabla, && (2.151)\\
&= \ker d_{i,k}^\nabla \oplus \operatorname{im} \delta_{i,k+1}^\nabla, && (2.152)\\
&= \operatorname{im} d_{i-1,k+1}^\nabla \oplus \ker \delta_{i,k}^\nabla && (2.153)
\end{aligned}$$

of $\Omega_L^i(M, \mathrm{End}(E))_k$ into closed subspaces.

Remark. Note that $H^0(E, \nabla) = \ker d_0^\nabla = \ker \Delta_0$, thus $\dim H^0(E, \nabla) = 1$, if ∇ is an irreducible L-connection and $\dim H^0(E, \nabla) > 1$ otherwise.

Recall that if we fix some flat L-connection $\nabla_0 \in \mathcal{H}(E, L)$ then the Sobolev completions is defined by

$$\mathcal{H}(E, L)_\ell = \{\nabla_0 + \alpha;\ \alpha \in \Omega^1_L(M, \mathrm{End}(E))_\ell,\ d^{\nabla_0}\alpha + \frac{1}{2}[\alpha, \alpha] = 0\} \tag{2.154}$$

for $\ell > \frac{1}{2}\dim M$. Furthermore from the previous we know that the curvature

$$F\colon \mathcal{A}(E, L)_\ell \to \Omega^2_L(M, \mathrm{End}(E))_{\ell-1} \tag{2.155}$$

defined by $F(\nabla_0 + \alpha) = d^{\nabla_0}\alpha + \frac{1}{2}[\alpha, \alpha]$ is a smooth mapping of Hilbert manifolds for $\ell > \frac{1}{2}\dim M$ and

$$\mathcal{H}(E, L)_\ell = F^{-1}(0). \tag{2.156}$$

Consider a smooth irreducible flat L-connection $\nabla = \nabla_0 + \alpha \in \mathcal{H}^*(E, L)$. Then from Theorem 10 we have that there exists a Hilbert submanifold $\mathcal{O}_{\alpha,\varepsilon}$ of $\mathcal{A}^*(E, L)_\ell$ for $\varepsilon > 0$ small enough such that $p_{\alpha,\varepsilon} = \hat{p}_{\ell\,|\mathcal{O}_{\alpha,\varepsilon}}\colon \mathcal{O}_{\alpha,\varepsilon} \to \mathcal{U}_{\alpha,\varepsilon} \subset \mathcal{B}^*(E, L)_\ell$, where $\mathcal{U}_{\alpha,\varepsilon} = \hat{p}_\ell(\mathcal{O}_{\alpha,\varepsilon})$ in open in $\mathcal{B}^*(E, L)_\ell$, is a homeomorphism, ($\mathcal{O}_{\alpha,\varepsilon}$ is a slice to the $\mathrm{Gau}(E)_{\ell+1}$-orbits of the action of the group of gauge transformations $\mathrm{Gau}(E)_{\ell+1}$ on $\mathcal{A}^*(E, L)_\ell$). Furthermore consider a closed subset

$$\mathcal{S}_{\alpha.\varepsilon} = \{\nabla_0 + \alpha + \beta;\ \beta \in \Omega^1_L(M, \mathrm{End}(E))_\ell,\ \delta^\nabla\beta = 0,\ d^\nabla\beta + \frac{1}{2}[\beta, \beta] = 0,\ ||\beta||_\ell < \varepsilon\} \tag{2.157}$$

of $\mathcal{O}_{\alpha,\varepsilon}$. Because $\mathcal{S}_{\alpha,\varepsilon} \subset \mathcal{H}^*(E, L)_\ell$, we obtain that $p_{\alpha,\varepsilon}\colon \mathcal{S}_{\alpha,\varepsilon} \to \mathcal{V}_{\alpha,\varepsilon} = \mathcal{U}_{\alpha,\varepsilon} \cap \mathcal{M}^*(E, L)_\ell$ is a homeomorphism on open subset in $\mathcal{M}^*(E, L)_\ell$ for $\ell > \frac{1}{2}\dim M + 1$.

Now if we apply the Hodge decomposition (2.144) to the element $d^\nabla_1\beta + \frac{1}{2}[\beta, \beta]$ for $\beta \in \Omega^1_L(M, \mathrm{End}(E)_\ell$, we obtain

$$\begin{aligned} d^\nabla_1\beta + \tfrac{1}{2}[\beta, \beta] &= \mathrm{pr}_{\mathcal{H}^2(E,\nabla)}(d^\nabla_1\beta + \tfrac{1}{2}[\beta, \beta]) + (\delta^\nabla_2 \circ d^\nabla_2 \circ G_2)(d^\nabla_1\beta + \tfrac{1}{2}[\beta, \beta]) \\ &\quad + (d^\nabla_1 \circ \delta^\nabla_1 \circ G_2)(d^\nabla_1\beta + \tfrac{1}{2}[\beta, \beta]) \\ &= \tfrac{1}{2}\,\mathrm{pr}_{\mathcal{H}^2(E,\nabla)}([\beta, \beta]) + \tfrac{1}{2}(\delta^\nabla_2 \circ d^\nabla_2 \circ G_2)([\beta, \beta]) \\ &\quad + d^\nabla_1((\delta^\nabla_1 \circ G_2 \circ d^\nabla_1)\beta + \tfrac{1}{2}(\delta^\nabla_1 \circ G_2)([\beta, \beta])), \end{aligned}$$

where we used that $G_2 \circ d^\nabla_1 = d^\nabla_1 \circ G_1$. Besides we have

$$\begin{aligned} \delta^\nabla_1 \circ G_2 \circ d^\nabla_1 &= \delta^\nabla_1 \circ d^\nabla_1 \circ G_1 = \Delta_1 \circ G_1 - d^\nabla_0 \circ \delta^\nabla_0 \circ G_1 \\ &= \mathrm{id}_{\Omega^1_L(M,\mathrm{End}(E))_\ell} - \mathrm{pr}_{\mathcal{H}^1(E,\nabla)} - d^\nabla_0 \circ \delta^\nabla_0 \circ G_1, \end{aligned}$$

therefore substituting this into the equation above, we get

$$\begin{aligned} d^\nabla_1\beta + \tfrac{1}{2}[\beta, \beta] &= \tfrac{1}{2}\,\mathrm{pr}_{\mathcal{H}^2(E,\nabla)}([\beta, \beta]) + \tfrac{1}{2}\,(\delta^\nabla_2 \circ d^\nabla_2 \circ G_2)([\beta, \beta]) \\ &\quad + d^\nabla_1(\beta + \tfrac{1}{2}(\delta^\nabla_1 \circ G_2)([\beta, \beta])). \end{aligned}$$

From this L^2-orthogonal decomposition we have

$$d^\nabla_1\beta + \frac{1}{2}[\beta, \beta] = 0 \Longleftrightarrow \begin{cases} d^\nabla_1(\beta + \frac{1}{2}(\delta^\nabla_1 \circ G_2)([\beta, \beta])) = 0, \\ (\delta^\nabla_2 \circ d^\nabla_2 \circ G_2)([\beta, \beta]) = 0, \\ \mathrm{pr}_{\mathcal{H}^2(E,\nabla)}([\beta, \beta]) = 0. \end{cases} \tag{2.158}$$

Furthermore for the irreducible flat L-connection ∇ we define the *Kuranishi mapping*

$$K_\nabla\colon \Omega^1_L(M, \mathrm{End}(E))_\ell \to \Omega^1_L(M, \mathrm{End}(E))_\ell$$

by the formula

$$K_\nabla(\beta) = \beta + \frac{1}{2}(\delta_1^\nabla \circ G_2)([\beta,\beta]) \tag{2.159}$$

for $\beta \in \Omega_L^1(M, \mathrm{End}(E))_\ell$. It is a smooth mapping of Hilbert manifolds with the tangent mapping $T_\beta K_\nabla \colon \Omega_L^1(M, \mathrm{End}(E))_\ell \to \Omega_L^1(M, \mathrm{End}(E))_\ell$ at β equals to

$$T_\beta K_\nabla \gamma = \gamma + (\delta_1^\nabla \circ G_2)([\beta,\gamma]), \tag{2.160}$$

where $\gamma \in \Omega_L^1(M, \mathrm{End}(E))_\ell$. Since $T_0 K_\nabla = \mathrm{id}_{\Omega_L^1(M,\mathrm{End}(E))_\ell}$, using the inverse function theorem for Banach manifolds, we immediately obtain that K_∇ is a local diffeomorphism at 0. Further we define a subset

$$\mathcal{S}_\varepsilon = \{\beta \in \Omega_L^1(M, \mathrm{End}(E))_\ell,\ \delta_0^\nabla \beta = 0,\ d_1^\nabla \beta + \frac{1}{2}[\beta,\beta] = 0,\ ||\beta||_\ell < \varepsilon\} \tag{2.161}$$

of $\Omega_L^1(M, \mathrm{End}(E))_\ell$ for $\varepsilon > 0$.

Lemma 21. Let $\ell > \max\{\frac{1}{2}\dim M, 1\}$ then $K_\nabla(\mathcal{S}_\varepsilon) \subset \mathcal{H}^1(E,\nabla)$ and $\mathcal{S}_\varepsilon \subset \Omega_L^1(M, \mathrm{End}(E))$.

Proof. The first observation is trivial, it is enough to show that $d_1^\nabla(K_\nabla(\beta)) = 0$ and $\delta_0^\nabla(K_\nabla(\beta)) = 0$ for $\alpha \in \mathcal{S}_\varepsilon$, since $\mathcal{H}^1(E,\nabla) = \ker d_1^\nabla \cap \ker \delta_0^\nabla$. We have $\delta_0^\nabla(K_\nabla(\beta)) = \delta_0^\nabla \beta = 0$ furthermore, using (2.158), we obtain $d_1^\nabla(K_\nabla(\beta)) = d_1^\nabla(\beta + \frac{1}{2}(\delta_1^\nabla \circ G_2)([\beta,\beta])) = 0$.

Consider $\beta \in \mathcal{S}_\varepsilon$ and assume that $\beta \in \Omega_L^1(M, \mathrm{End}(E))_k$ for $k > \max\{\frac{1}{2}\dim M, 1\}$. Because $\Delta_1(K_\nabla(\beta)) = 0$, we get

$$\Delta_1 \beta = -\frac{1}{2}(\Delta_1 \circ \delta_1^\nabla \circ G_2)([\beta,\beta]).$$

The term on the right hand side in the equation above belongs to $\Omega_L^1(M, \mathrm{End}(E))_{k-1}$, and the Elliptic Regularity (Lemma 7), applied to the elliptic operator Δ_1, gives $\beta \in \Omega_L^1(M, \mathrm{End}(E))_{k+1}$. Using the induction on k we get $\beta \in \Omega_L^1(M, \mathrm{End}(E))_k$ for all $k \geq \ell$. From the Rellich's lemma (Theorem 3) it follows that β is smooth, so we are done. ♠

Lemma 22. For $\ell > \frac{1}{2}\dim M + 1$ the mapping $j_\ell \colon \mathcal{M}^*(E,L) \to \mathcal{M}^*(E,L)_\ell$ is injective and has an open image.

Proof. The injectivity of j_ℓ follows from Lemma 17 and the fact that $j_\ell(\mathcal{M}^*(E,L)) \subset \mathcal{M}^*(E,L)_\ell$. Further let $\nabla = \nabla_0 + \alpha$ be a smooth irreducible flat L-connection then from the previous consideration there exists $\mathcal{S}_{\alpha,\varepsilon} \subset \mathcal{H}^*(E,L)_\ell$ such that $\hat{p}_\ell(\mathcal{S}_{\alpha,\varepsilon})$ is an open neighbourhood of $j_\ell([\nabla])$ in $\mathcal{M}^*(E,L)_\ell$. But from Lemma 21 we get $\mathcal{S}_{\alpha,\varepsilon} \subset \mathcal{H}^*(E,L)$ therefore we have $\hat{p}_\ell(\mathcal{S}_{\alpha,\varepsilon}) \subset j_\ell(\mathcal{M}^*(E,L))$, so we are done. ♠

Theorem 11. The moduli space $\mathcal{M}^*(E,L)$ of gauge equivalence classes of irreducible flat L-connections on E has a structure of a topological space such that for each $[\nabla] \in \mathcal{M}^*(E,L)$ represented by $\nabla = \nabla_0 + \alpha \in \mathcal{H}^*(E,L)$ there exist an open neighbourhood $\mathcal{U}_\alpha$ of $[\nabla]$ in $\mathcal{M}^*(E,L)$, an open neighborhood $\mathcal{O}_\alpha$ of 0 in $\mathcal{H}^1(E,\nabla)$ and a smooth mapping

$$\Phi \colon \mathcal{O}_\alpha \to \mathcal{H}^2(E,\nabla), \tag{2.162}$$

called the *obstruction mapping*, satisfying $\Phi(0) = 0$ and

$$\mathcal{U}_\alpha \simeq \Phi^{-1}(0). \tag{2.163}$$

Thus $\mathcal{U}_\alpha$ is homeomorphic to a closed subset in an open subset in a finite dimensional vector space.

Proof. Because the Kuranishi mapping $K_\nabla \colon \Omega_L^1(M, \mathrm{End}(E))_\ell \to \Omega_L^1(M, \mathrm{End}(E))_\ell$ is a local diffeomorphism at 0, there exist open neighborhoods $\mathcal{U}, \mathcal{V}$ of 0 in $\Omega_L^1(M, \mathrm{End}(E))_\ell$ such that $K_{\nabla|\mathcal{U}} \colon \mathcal{U} \to \mathcal{V}$ is a diffeomorphism of Hilbert manifolds. We can take $\mathcal{U} = \{\beta \in \Omega_L^1(M, \mathrm{End}(E))_\ell;\ ||\beta||_\ell < \varepsilon\}$ for $\varepsilon > 0$ small enough, therefore $\mathcal{S}_\varepsilon \subset \mathcal{U}$. Denote $F = (K_{\nabla|\mathcal{U}})^{-1} \colon \mathcal{V} \to \mathcal{U}$. Because $\mathcal{H}^1(E,\nabla)$ is a closed subspace in $\Omega^1_L(M,\mathrm{End}(E))_\ell$ and $\mathcal{O}=\mathcal{V}\cap\mathcal{H}^1(E,\nabla)$ is an open set in $\mathcal{H}^1(E,\nabla)$, therefore $\mathcal{O}$ is a Hilbert submanifold of $\Omega^1_L(M,\mathrm{End}(E))_\ell$. If we define the obstruction map $\Phi\colon\mathcal{O}\to\mathcal{H}^2(E,\nabla)$ by

$$\Phi(\gamma)=\mathrm{pr}_{\mathcal{H}^2(E,\nabla)}\big([F(\gamma),F(\gamma)]\big),$$

then Φ is a smooth mapping of Hilbert manifolds.

From the previous we have $K_\nabla(\mathcal{S}_\varepsilon)\subset\mathcal{V}\cap\mathcal{H}^1(E,\nabla)=\mathcal{O}$. It remains to show that $K_\nabla(\mathcal{S}_\varepsilon)=\Phi^{-1}(0)$. In case $\beta\in\mathcal{S}_\varepsilon$, then we obtain $(\Phi\circ K_\nabla)(\beta)=\mathrm{pr}_{\mathcal{H}^2(E,\nabla)}([\beta,\beta])$, using (2.158), we get $(\Phi\circ K_\nabla)(\beta)=0$. On the other hand if $\gamma\in\Phi^{-1}(0)$, then there exists a unique $\beta\in\mathcal{U}$ satisfying $K_\nabla(\beta)=\gamma$. Hence $0=\Phi(\gamma)=(\Phi\circ K_\nabla)(\beta)=\mathrm{pr}_{\mathcal{H}^2(E,\nabla)}([\beta,\beta])$. Since $\gamma\in\mathcal{H}^1(E,\nabla)$, we get

$$\begin{aligned}0&=d_1^\nabla\gamma=d_1^\nabla\Big(\beta+\frac{1}{2}(\delta_1^\nabla\circ G_2)([\beta,\beta])\Big),\\ 0&=\delta_0^\nabla\gamma=\delta_0^\nabla\beta.\end{aligned}$$

Applying the Hodge decomposition (2.144) to the element $\frac{1}{2}[\beta,\beta]$ and using the above equations, we obtain

$$\begin{aligned}d_1^\nabla\beta+\frac{1}{2}[\beta,\beta]&=d_1^\nabla\beta+\frac{1}{2}(\delta_2^\nabla\circ d_2^\nabla\circ G_2)([\beta,\beta])+\frac{1}{2}(d_1^\nabla\circ\delta_1^\nabla\circ G_2)([\beta,\beta])+\frac{1}{2}\mathrm{pr}_{\mathcal{H}^2(E,\nabla)}([\beta,\beta])\\ &=\frac{1}{2}(\delta_2^\nabla\circ d_2^\nabla\circ G_2)([\beta,\beta])=\frac{1}{2}(\delta_2^\nabla\circ d_2^\nabla\circ G_2)([\beta,\beta]).\end{aligned}$$

Denoting the left hand side of the equation above by ψ, we have

$$\begin{aligned}\psi=d_1^\nabla\beta+\frac{1}{2}[\beta,\beta]&=\frac{1}{2}(\delta_2^\nabla\circ d_2^\nabla\circ G_2)([\beta,\beta])\\ &=\frac{1}{2}(\delta_2^\nabla\circ G_3\circ d_2^\nabla)([\beta,\beta])=\frac{1}{2}(\delta_2^\nabla\circ G_3)([d_1^\nabla\beta,\beta]-[\beta,d_1^\nabla\beta])\\ &=\frac{1}{2}(\delta_2^\nabla\circ G_3)([\psi,\beta]-[\beta,\psi])=(\delta_2^\nabla\circ G_3)([\psi,\beta]),\end{aligned}$$

where we used that $[[\beta,\beta],\beta]=0$. Using the fact that there exists a positive constant c such that

$$||(\delta_2^\nabla\circ G_3)\,\varphi||_\ell\le c\,||\varphi||_{\ell-1},$$

for all $\varphi\in\Omega^3_L(M,\mathrm{End}(E))_{\ell-1}$, we make the following estimate

$$||\psi||_{\ell-1}\le||\psi||_\ell=||(\delta_2^\nabla\circ G_3)([\gamma,\beta])||_\ell\le c||[\psi,\beta]||_{\ell-1}\le c'||\psi||_{\ell-1}||\beta||_\ell<c'\cdot\varepsilon\,||\psi||_{\ell-1},$$

where c' is another positive constant and the last inequality is provided that $||\psi||_{\ell-1}>0$. If we take $\varepsilon<\frac{1}{c'}$, then we have $\psi=0$. Thus, together with $\delta_0^\nabla\beta=0$, we obtain that $\beta\in\mathcal{S}_\varepsilon$.

Further because $j_\ell\colon\mathcal{M}^*(E,L)\to\mathcal{M}^*(E,L)_\ell$ is injective for all $\ell>\frac{1}{2}\dim M+1$, so the mapping

$$j_{k\ell|j_\ell(\mathcal{M}^*(E,L))}\colon j_\ell(\mathcal{M}^*(E,L))\to j_k(\mathcal{M}^*(E,L))$$

is bijective for $\ell\ge k>\frac{1}{2}\dim M+1$ since $j_{k\ell}\circ j_\ell=j_k$. Moreover form Lemma 22 we know that j_ℓ has an open image, therefore for each $\nabla_0+\alpha\in\mathcal{H}^*(E,L)$ there exists $\varepsilon>0$ satisfying that $\hat{p}_\ell(\mathcal{S}^\ell_{\alpha,\varepsilon})$ is an open neighbourhood of $j_\ell([\nabla_0+\alpha])$ in $j_\ell(\mathcal{M}^*(E,L))$. Furthermore from the previous we have that the following mapping

$$\hat{p}_\ell(\mathcal{S}^\ell_{\alpha,\varepsilon})\xrightarrow{(p^\ell_{\alpha,\varepsilon})^{-1}}\mathcal{S}^\ell_{\alpha,\varepsilon}\xrightarrow{\chi^\ell_\alpha}\mathcal{S}^\ell_\varepsilon\xrightarrow{K_\nabla}K_\nabla(\mathcal{S}^\ell_\varepsilon)\subset\mathcal{O}^\ell_\varepsilon=\mathcal{V}^\ell_\varepsilon\cap\mathcal{H}^1(E,\nabla),$$

where $\chi^\ell_\alpha\colon\mathcal{S}^\ell_{\alpha,\varepsilon}\to\mathcal{S}^\ell_\varepsilon$ is given via $\chi^\ell_\alpha(\nabla_0+\alpha+\beta)=\beta$, is a homeomorphism. Since $K_\nabla(\mathcal{S}^\ell_\varepsilon)\subset K_\nabla(\mathcal{S}^k_\varepsilon)$, for ε small enough we have the following commutative diagram

$$\begin{array}{ccc}\hat{p}_\ell(\mathcal{S}^\ell_{\alpha,\varepsilon}) & \longrightarrow & K_\nabla(\mathcal{S}^\ell_\varepsilon)\\ {\scriptstyle j_{k\ell}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_{\mathcal{H}^1(E,\nabla)}}\\ \hat{p}_k(\mathcal{S}^k_{\alpha,\varepsilon}) & \longrightarrow & K_\nabla(\mathcal{S}^k_\varepsilon)\end{array}$$

in which $\mathrm{id}_{\mathcal{H}^1(E,\nabla)}$ is a continuous mapping with respect to the norms $||\cdot||_k$ and $||\cdot||_\ell$ on $\mathcal{H}^1(E,\nabla)$ because all norms on a finite dimensional vector space are equivalent. On the other hand because we can find $\varepsilon' \leq \varepsilon$ such that $K_\nabla(\mathcal{S}^k_{\varepsilon'}) \subset K_\nabla(\mathcal{S}^\ell_\varepsilon)$, we obtain the following commutative diagram

$$\begin{array}{ccc} \hat{p}_\ell(\mathcal{S}^\ell_{\alpha,\varepsilon}) & \longrightarrow & K_\nabla(\mathcal{S}^\ell_\varepsilon) \\ {\scriptstyle (j_{k\ell})^{-1}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}_{\mathcal{H}^1(E,\nabla)}} \\ \hat{p}_k(\mathcal{S}^k_{\alpha,\varepsilon'}) & \longrightarrow & K_\nabla(\mathcal{S}^k_{\varepsilon'}) \end{array}$$

which gives that $j_{k\ell|j_\ell(\mathcal{M}^*(E,L))}\colon j_\ell(\mathcal{M}^*(E,L)) \to j_k(\mathcal{M}^*(E,L))$ is a homeomorphism.

Therefore we have proved that $j_{k\ell|j_\ell(\mathcal{M}^*(E,L))}\colon j_\ell(\mathcal{M}^*(E,L)) \to j_k(\mathcal{M}^*(E,L))$ is a homeomorphism. Thus j_ℓ gives a topology on $\mathcal{M}^*(E,L)$ which is independent on the Sobolev index ℓ for $\ell > \frac{1}{2}\dim M+1$ and for each $\nabla = \nabla_0+\alpha$ there exists an open neighbourhood $\mathcal{U}_\alpha = (j_\ell)^{-1}(\hat{p}_\ell(\mathcal{S}^\ell_{\alpha,\varepsilon}))$ of $[\nabla]$ homeomorphic to $\Phi^{-1}(0)$. ♠

Remark. Note that if $\dim\mathcal{H}^2(E,\nabla) = 0$ then $\Phi^{-1}(0) = \mathcal{O}_\alpha$. Therefore $\mathcal{M}^*(E,L)$ is at $[\nabla]$ locally homeomorphic to an open subset in $\mathcal{H}^1(E,\nabla)$. Thus $\mathcal{M}^*(E,L)$ has near this point a structure of a manifold of dimension $\dim\mathcal{H}^1(E,L)$.

Chapter 3

Principal Lie algebroid connections

3.1 Lie algebroid connections

The theory of connections is a classical topic in differential geometry. They provide an extremely important tool to the study of geometric structures on manifolds.

Lie algebroid connections based on the notion of a horizontal lift were introduced by R. L. Fernandes in [10] for the special case of Poisson manifolds and in [9] for general Lie algebroids. It is defined by analogy with an Ehresmann connection on an arbitrary fiber bundle. There are two distinguished cases, linear connections on vector bundles and principal connections on principal fiber bundles.

Definition 15. Let $(L \xrightarrow{\pi} M, [\cdot,\cdot], a)$ be a Lie algebroid. A *Lie algebroid connection* on a fiber bundle (E, p, M, S) with the standard fiber S is a homomorphism $\eta\colon p^*L \to TE$ of vector bundles over E covering the identity on E, which is horizontal, i.e., the following diagram

$$\begin{array}{ccc} p^*L & \xrightarrow{\eta} & TE \\ {\scriptstyle \hat{p}}\downarrow & & \downarrow{\scriptstyle Tp} \\ L & \xrightarrow[a]{} & TM \end{array}$$

commutes, where p^*L is the pullback

$$\begin{array}{ccc} p^*L & \xrightarrow{\hat{p}} & L \\ \downarrow & & \downarrow \\ E & \xrightarrow[p]{} & M \end{array}$$

of the vector bundle L by p. The vector bundle homomorphism η is called the *horizontal lift*.

Depending on a structure of the fiber bundle E, we may require some additional conditions on the *horizontal lift* η.

The subspace $\operatorname{im}\eta_u$ of T_uE formed by all horizontal lifts is denoted by H_uE, furthermore HE is a smooth distribution on E called the *horizontal distribution* of the connection η. Note that HE is not a regular distribution (a smooth distribution of constant rank) more and that this distribution does not define the Lie algebroid connection uniquely.

In general, we have neither $H_uE \cap V_uE = \{0\}$ nor $T_uE = H_uE + V_uE$. As usual, a vector $\xi_u \in T_uE$ will be called *vertical* resp. *horizontal*, if it belongs to V_uE resp. H_uE.

Consider a fiber bundle (E, p, M, S). Then there are two equivalent descriptions of a connection on the fiber bundle E either via a horizontal bundle or through a connection form.

i) A connection on the fiber bundle (E, p, M, S) is a vector valued 1-form $\Phi \in \Omega^1(E, TE)$ such that $\Phi \circ \Phi = \Phi$ and $\operatorname{im}\Phi = VE$, i.e., Φ is a projection on the vertical bundle VE.
ii) A connection on the fiber bundle (E, p, M, S) is a vector subbundle HE of the tangent bundle TE, called the horizontal bundle, such that $TE = HE \oplus VE$.

How these definitions of a connection on a fiber bundle are related to the definition of a Lie algebroid connection on a fiber bundle?

Let (E, p, M, S) be a fiber bundle and consider a Lie algebroid connection $\eta\colon p^*TM \to TE$ for the Lie algebroid $(TM \to M, [\cdot\,,\cdot], \mathrm{id}_{TM})$. Then $HE = \operatorname{im}\eta$ is the horizontal distribution of the connection η. If $\xi_u \in H_uE \cap V_uE$ then there exists $v_x \in T_xM$ for $x = p(u)$ satisfying $\eta_u(u, v_x) = \xi_u$. From the commutative diagram

$$\begin{array}{ccc} p^*TM & \xrightarrow{\ \eta\ } & TE \\ \Big\downarrow{\scriptstyle \hat{p}} & & \Big\downarrow{\scriptstyle Tp} \\ TM & \xrightarrow[\mathrm{id}_{TM}]{} & TM \end{array}$$

and from the fact that $\xi_u \in V_uE$ we get $0 = T_up.\xi_u = T_up.\eta_u(u, v_x) = \hat{p}_u(u, v_x) = v_x$. Therefore $\xi_u = 0$ and $H_uE \cap V_uE = \{0\}$. Let $\xi_u \in T_uE$ and take the decomposition

$$\xi_u = \eta_u(u, T_up.\xi_u) + (\xi_u - \eta_u(u, T_up.\xi_u))$$

then $T_up.(\xi_u - \eta_u(u, T_up.\xi_u)) = 0$. Because $\eta_u(u, T_up.\xi_u) \in H_uE$ and $(\xi_u - \eta_u(u, T_up.\xi_u)) \in V_uE$, we have proved that $T_uE = H_uE \oplus V_uE$. Hence HE is a vector subbundle of TE such that $TE = HE \oplus VE$ and for that reason η defines a connection on the fiber bundle (E, p, M, S) in the sense of (ii).

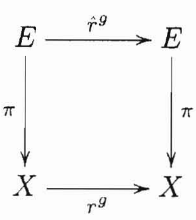
On the other hand if we are given a connection on the fiber bundle (E, p, M, S) in the sense of (ii) then there exists a unique Lie algebroid connection $\eta\colon p^*TM \to TE$ such that $\operatorname{im}\eta = HE$. Consider the homomorphism $(\pi_E, Tp)\colon TE \to E \times_M TM = p^*TM$ of vector bundles over E covering the identity on E. By definition we have $\ker(\pi_E, Tp) = VE$, hence $(\pi_E, Tp)_{|HE}\colon HE \to p^*TM$ is injective on fibers and by reason of dimensions it is a linear isomorphism on fibres. Because $(\pi_E, Tp)_{|HE}$ is a smooth bijection with the invertible tangent mapping, so its inverse is a homomorphism of vector bundles. If we denote

$$\eta = ((\pi_E, Tp)_{|HE})^{-1}\colon p^*TM \to HE \hookrightarrow TE$$

then η satisfies $Tp \circ \eta = \hat{p}$ and $\operatorname{im}\eta = HE$. Thus η is a right inverse for (π_E, Tp). The uniqueness follows from the following fact. If η_1 and η_2 are Lie algebroid connections on (E, p, M, S) such that $\operatorname{im}\eta_1 = HE$ and $\operatorname{im}\eta_2 = HE$ then $\operatorname{im}(\eta_1 - \eta_2) \subset HE$. Because $Tp \circ (\eta_1 - \eta_2) = 0$, we obtain $\operatorname{im}(\eta_1 - \eta_2) \subset HE \cap VE$, therefore we have $\eta_1 = \eta_2$.

These two constructions are inverse to each other therefore Lie algebroid connections on the fiber bundle (E, p, M, S) for the Lie algebroid $(TM \to M, [\cdot\,,\cdot], \mathrm{id}_{TM})$ are in a one-to-one correspondence with connections on the fiber bundle (E, p, M, S).

Definition 16. Let X be a manifold with a right action $r\colon X \times G \to X$ of a Lie group G on X and let $\pi\colon E \to X$ be a vector bundle over X. We say that E is a G-equivariant vector bundle if we are given a right action $\hat{r}\colon E \times G \to E$ of the group G on E satisfying that



is an isomorphism of vector bundles for all $g \in G$.

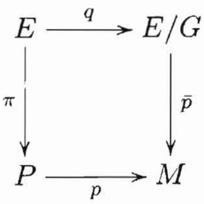
Definition 17. Consider a principal fiber bundle (P, p, M, G) with the principal right action $r\colon P \times G \to P$ and a G-equivariant vector bundle $\pi\colon E \to P$ over P. We say that a vector bundle atlas (U_α, ψ_α) for E is G-equivariant if U_α is a p-saturated set, i.e., $U_\alpha = p^{-1}(V_\alpha)$ for an open set V_α in M, and

$$\psi_\alpha^{-1}(u.g, v) = \psi_\alpha^{-1}(u, v).g \tag{3.1}$$

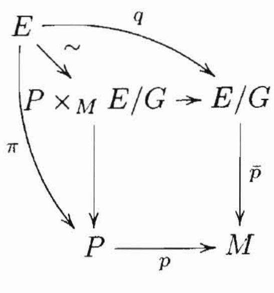
for all $u \in U_\alpha$, $v \in V$ and $g \in G$. It is easy to see that for transition functions $\psi_{\alpha\beta}\colon U_{\alpha\beta} \to \mathrm{GL}(V)$ we get $\psi_{\alpha\beta}(u.g) = \psi_{\alpha\beta}(u)$, where V is the standard fiber of E.

Theorem 12. Let (P, p, M, G) be a principal fiber bundle and let $\pi\colon E \to P$ be a G-equivariant vector bundle with a G-equivariant vector bundle atlas. Denote by $\hat{r}\colon E \times G \to E$ the right action on E.

i) The space E/G of orbits of the right action $\hat{r}$ carries a unique smooth manifold structure such that the quotient map $q\colon E \to E/G$ is a surjective submersion.
ii) $\bar{p}\colon E/G \to M$ is a vector bundle in a canonical way, where $\bar{p}$ is given by



and $q_u\colon E_u \to (E/G)_{p(u)}$ is a linear diffeomorphism for each $u \in P$, moreover q is a homomorphism of vector bundles.
iii) $q\colon E \to E/G$ is a principal G-bundle with the principal right action $\hat{r}$.
iv) The following diagram



commutes, i.e., E is a topological pullback.

Notation. We will denote E/G by E_G. We also define the smooth mapping $\tau\colon P \times_M E_G \to E$ by $\tau(u_x, v_x) = q_{u_x}^{-1}(v_x)$. It satisfies $\tau(u, q(\xi_u)) = \xi_u$, $q(\tau(u_x, v_x)) = v_x$ and $\tau(u_x.g, v_x) = \tau(u_x, v_x).g$.

Proof. First of all we verify that the right action $\hat{r}\colon E \times G \to E$ is free and proper. Suppose that $\xi_u.g_1 = \xi_u.g_2$, then $u.g_1 = \pi(\xi_u.g_1) = \pi(\xi_u.g_2) = u.g_2$. Because the principal right action $r\colon P \times G \to P$ is free, the right action $\hat{r}$ is also free. Now let $\xi_n.g_n \to \xi'$ and $\xi_n \to \xi$ in E for some $\xi_n, \xi, \xi' \in E$ and $g_n \in G$. If we denote $u_n = \pi(\xi_n)$, $u = \pi(\xi)$ and $u' = \pi(\xi')$, then

$u_n.g_n = \pi(\xi_n.g_n) \to \pi(\xi') = u'$ and $u_n = \pi(\xi_n) \to \pi(\xi) = u$, because π is continuous. But G acts properly on P, hence g_n has a convergent subsequence in G and thus $\hat{r}$ is proper. Immediately, from the characterization of principal fiber bundles it follows that the orbit space E/G is a smooth manifold, the quotient mapping $q\colon E \to E/G$ is a surjective submersion and $q\colon E \to E/G$ is a principal G-bundle.

In the setting of the diagram in (ii) the mapping $p \circ \pi$ is constant on orbits of the action $\hat{r}$, so $\bar{p}$ exists as a mapping. Because $q\colon E \to E/G$ is a fibered manifold and $\bar{p} \circ q$ is smooth, we obtain that $\bar{p}$ is also smooth.

Let $(p^{-1}(U_\alpha), \chi_\alpha)$ be a G-equivariant vector bundle atlas for E. Assume, by shrinking U_α if necessary, that $(U_\alpha, \varphi_\alpha)$ is a principal bundle atlas for P with transition functions $\varphi_{\alpha\beta}\colon U_{\alpha\beta} \to G$. We define $\psi_\alpha^{-1}\colon U_\alpha \times V \to \bar{p}^{-1}(U_\alpha) \subset E/G$ by $\psi_\alpha^{-1}(x,v) = q(\chi_\alpha^{-1}(\varphi_\alpha^{-1}(x,e),v))$, which is a fiber respecting mapping, i.e., the following diagram

$$\begin{array}{ccc} U_\alpha \times V & \xrightarrow{\psi_\alpha^{-1}} & \bar{p}^{-1}(U_\alpha) \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \swarrow{\scriptstyle \bar{p}} & \\ U_\alpha & & \end{array}$$

commutes. For each point $q(\xi_{u_x})$ in $\bar{p}^{-1}(x)$ there is exactly one $v \in V$ such that the orbit corresponding to this point passes through $\chi_\alpha^{-1}(\varphi_\alpha^{-1}(x,e),v)$, i.e., $q(\xi_{u_x}) = q(\chi_\alpha^{-1}(\varphi_\alpha^{-1}(x,e),v))$. Because χ_α is a diffeomorphism, we can write $\xi_{u_x} = \chi_\alpha^{-1}(\varphi_\alpha^{-1}(x,g),v)$ for a uniquely determined $v \in V$, where $\varphi_\alpha(u_x) = (x,g)$. Then

$$\chi_\alpha^{-1}(\varphi_\alpha^{-1}(x,g),v).g^{-1} = \chi_\alpha^{-1}(\varphi_\alpha^{-1}(x,g).g^{-1},v) = \chi_\alpha^{-1}(\varphi_\alpha^{-1}(x,e),v),$$

where we used the fact that χ_α is a G-equivariant chart. Therefore $\psi_\alpha^{-1}(x,\cdot)\colon V \to \bar{p}^{-1}(x)$ is bijective, since the principal right action is free. Moreover ψ_α^{-1} is smooth with the invertible tangent mapping, so its inverse $\psi_\alpha\colon \bar{p}^{-1}(U_\alpha) \to U_\alpha \times V$ is a fiber respecting diffeomorphism. Furthermore

$$\begin{aligned}
\psi_\beta^{-1}(x,v) &= q(\chi_\beta^{-1}(\varphi_\beta^{-1}(x,e),v)) \\
&= q(\chi_\alpha^{-1}(\varphi_\beta^{-1}(x,e),\chi_{\alpha\beta}(\varphi_\beta^{-1}(x,e)).v)) \\
&= q(\chi_\alpha^{-1}(\varphi_\alpha^{-1}(x,\varphi_{\alpha\beta}(x).e),\chi_{\alpha\beta}(\varphi_\beta^{-1}(x,e)).v)) \\
&= q(\chi_\alpha^{-1}(\varphi_\alpha^{-1}(x,e).\varphi_{\alpha\beta}(x),\chi_{\alpha\beta}(\varphi_\beta^{-1}(x,e)).v)) \\
&= q(\chi_\alpha^{-1}(\varphi_\alpha^{-1}(x,e),\chi_{\alpha\beta}(\varphi_\beta^{-1}(x,e)).v)) \\
&= \psi_\alpha^{-1}(x,\chi_{\alpha\beta}(\varphi_\beta^{-1}(x,e)).v),
\end{aligned}$$

thus $(\psi_\alpha \circ \psi_\beta^{-1})(x,v) = (x,\chi_{\alpha\beta}(\varphi_\beta^{-1}(x,e)).v)$, hence (U_α, ψ_α) is a vector bundle atlas for $\bar{p}\colon E/G \to M$. By definition of ψ_α the diagram

$$\begin{array}{ccc} E_{|p^{-1}(U_\alpha)} & \xrightarrow{(\varphi_\alpha \times \mathrm{id}_V)\circ\chi_\alpha} & U_\alpha \times G \times V \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ \bar{p}^{-1}(U_\alpha) & \xrightarrow[\psi_\alpha]{} & U_\alpha \times V \end{array}$$

commutes, if we restrict χ_α on E_u then we obtain the diagram

$$\begin{array}{ccc} E_u & \xrightarrow{(\varphi_\alpha \times \mathrm{id}_V)\circ \chi_\alpha} & \{p(u)\} \times \{g\} \times V \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ \bar{p}^{-1}(p(u)) & \xrightarrow[\psi_\alpha]{} & \{p(u)\} \times V \end{array}$$

in which its lines are linear diffeomorphism, hence we conclude that $q_u\colon E_u \to \bar{p}^{-1}(p(u)) = (E/G)_{p(u)}$ is a linear diffeomorphism.

Consider a homomorphism $(\pi, q)\colon E \to P \times_M E/G = p^*(E/G)$ of vector bundles over P covering the identity on P. Because (π, q) is a linear isomorphism on fibers with the invertible tangent mapping, so (π, q) is an isomorphism of vector bundles. The inverse is denoted by $\tau\colon P \times_M E/G \to E$ and given by $\tau(u_x, v_x) = q_{u_x}^{-1}(v_x)$. ♠

Theorem 13. The sections of the vector bundle $E_G \to M$ correspond to the G-invariant sections of the G-equivariant vector bundle $E \to P$, moreover we have an isomorphism $\Phi\colon \Gamma(M, E_G) \xrightarrow{\sim} \Gamma(P, E)^G$ of $C^\infty(M, \mathbb{R})$-modules, where $f\xi = (f \circ p)\xi$ for $f \in C^\infty(M, \mathbb{R})$ and $\xi \in \Gamma(P, E)^G$.

Proof. If $\xi \in \Gamma(P, E)^G$ then we construct $s_\xi \in \Gamma(M, E_G)$ in the following way. Because $\xi\colon P \to E$ is a G-equivariant mapping, the diagram

$$\begin{array}{ccc} P & \xrightarrow{\xi} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ M & \xrightarrow[s_\xi]{} & E_G \end{array}$$

commutes for a uniquely determined mapping $s_\xi\colon M \to E_G$. Further $s_\xi \circ p = q \circ \xi$ is a smooth mapping and $p\colon P \to M$ is a fibered manifold hence s_ξ is a smooth section.

If conversely $s \in \Gamma(M, E_G)$ we define $\xi_s \in \Gamma(P, E)^G$ by $\xi_s = \tau \circ (\mathrm{id}_P \times_M s)\colon P \to P \times_M M \to P \times_M E_G \to E$, i.e., $\xi_s(u) = \tau(u, s(p(u)))$ for $u \in P$. This is a G-invariant section since $\xi_s(u.g) = \tau(u.g, s(p(u))) = \tau(u, s(p(u))).g = \xi_s(u).g$ by the G-equivariance of τ.

These two constructions are inverse to each other since we have $\xi_{s(\xi)}(u) = \tau(u, s_\xi(p(u))) = \tau(u, q(\xi(u))) = \xi(u)$ and $s_{\xi(s)}(p(u)) = q(\xi_s(u)) = q(\tau(u, s(p(u)))) = s(p(u))$. ♠

Theorem 14. (i) Let (P, p, M, G) be a principal fiber bundle and $\pi\colon E \to P$ be a G-equivariant vector bundle with a G-equivariant vector bundle atlas. Consider a vector bundle $q\colon F \to N$. If we are given a homomorphism $\varphi\colon E \to F$ of vector bundles covering $f\colon P \to N$ satisfying $\varphi(\xi_u.g) = \varphi(\xi_u)$ and $f(u.g) = f(u)$, i.e., $\varphi \circ \hat{r}^g = \varphi$ and $f \circ r^g = f$, then there exists a unique vector bundle homomorphism

$$\begin{array}{ccc} E_G & \xrightarrow{\varphi^G} & F \\ {\scriptstyle \bar{p}}\downarrow & & \downarrow{\scriptstyle q} \\ M & \xrightarrow[f^G]{} & N \end{array}$$

such that $\varphi = \varphi^G \circ q^E$ and $f = f^G \circ p$.

(ii) Let (P, p, M, G) and (P', p', M', G') be principal fiber bundles. Consider a G-equivariant resp. G'-equivariant vector bundle $\pi\colon E \to P$ resp. $\pi'\colon E' \to P'$ with a G-equivariant resp. G'-equivariant vector bundle atlas. Let $\Phi\colon G \to G'$ be a homomorphism of Lie groups. If we are given a homomorphism $\varphi\colon E \to E'$ of vector bundles covering $f\colon P \to P'$ such that $\varphi(\xi_u.g) = \varphi(\xi_u).\Phi(g)$

and $f(u.g) = f(u).\Phi(g)$, i.e., $\varphi \circ \hat{r}^g = \hat{r}^{\Phi(g)} \circ \varphi$ and $f \circ r^g = r^{\Phi(g)} \circ f$, then there exists a unique vector bundle homomorphism

$$\begin{array}{ccc} E_G & \xrightarrow{\varphi^G} & E'_{G'} \\ {\scriptstyle \bar{p}^E}\downarrow & & \downarrow{\scriptstyle \bar{p}^{E'}} \\ M & \xrightarrow[f^G]{} & M' \end{array}$$

such that $q^{E'} \circ \varphi = \varphi^G \circ q^E$ and $p' \circ f = f^G \circ p$.

Proof. We prove the second part only, because (i) is a special case of (ii). Since φ is G-equivariant and q^E is surjective, so there exists a unique mapping φ^G such that the following diagram

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & E' \\ {\scriptstyle q^E}\downarrow & & \downarrow{\scriptstyle q^{E'}} \\ E_G & \xrightarrow[\varphi^G]{} & E'_{G'} \end{array}$$

commutes. Moreover because $q^E \colon E \to E_G$ is a fibered manifold and $\varphi^G \circ q^E$ is smooth mapping, thus φ^G is also smooth. By the same argument we get there exists a uniquely determined smooth mapping $f^G \colon M \to M'$ satisfying $p' \circ f = f^G \circ p$. In fact $f \colon P \to P'$ is a principal fiber bundle homomorphism. The rest of the proof is to verify that $\varphi^G \colon E_G \to E'_{G'}$ is a homomorphism of vector bundles covering f^G. Because $\varphi^G_x = q^{E'}_{f(u_x)} \circ \varphi_{u_x} \circ (q^E_{u_x})^{-1} \colon (E_G)_x \to (E'_{G'})_{f^G(x)}$ is a linear mapping, hence φ^G is a homomorphism of vector bundles covering f^G. ♠

The previous framework can be used to the construction of an associated vector bundle to a principal fiber bundle.

Let (P, p, M, G) be a principal fiber bundle and $\rho \colon G \to \mathrm{GL}(V)$ be a representation of G on a finite dimensional vector space V. We consider the right action $\hat{r} \colon (P \times V) \times G \to P \times V$ given by $\hat{r}((u, v), g) = (u.g, g^{-1}.v)$. With this right action the trivial vector bundle $\pi \colon P \times V \to P$ is a G-equivariant vector bundle over P. Further let $(U_\alpha, \varphi_\alpha)$ be a principal bundle atlas for P then we define a vector bundle atlas $(p^{-1}(U_\alpha), \psi_\alpha)$ for $P \times V$, where $\psi_\alpha \colon (P \times V)_{|p^{-1}(U_\alpha)} \to p^{-1}(U_\alpha) \times V$, through

$$\psi_\alpha(u, v) = (u, \mathrm{pr}_G(\varphi_\alpha(u)).v).$$

Because

$$\begin{aligned} \psi_\alpha^{-1}(u.g, v) &= (u.g, (\mathrm{pr}_G(\varphi_\alpha(u.g)))^{-1}.v) \\ &= (u.g, g^{-1}.(\mathrm{pr}_G(\varphi_\alpha(u)))^{-1}.v) \\ &= (u, (\mathrm{pr}_G(\varphi_\alpha(u)))^{-1}.v).g \\ &= \psi_\alpha^{-1}(u, v).g, \end{aligned}$$

we get that $(p^{-1}(U_\alpha), \psi_\alpha)$ is a G-equivariant vector bundle atlas for $P \times V$. Using the construction in Theorem 12 we obtain the associated vector bundle $\bar{p} \colon P \times_G V \to M$. Moreover by Theorem 13 we have $\Gamma(M, P \times_G V) \simeq \Gamma(P, P \times V)^G \simeq C^\infty(P, V)^G$.

There is another important example of this construction. Consider a principal fiber bundle (P, p, M, G) and a vector bundle $\pi \colon E \to M$. Then the pullback $p^*E = P \times_M E$ carries a natural right action $\hat{r} \colon p^*E \times G \to p^*E$ of G defined by

$$\hat{r} = r \times_{\mathrm{id}_M} \mathrm{id}_E \colon (P \times_M E) \times G \xrightarrow{\sim} (P \times G) \times_M E \to P \times_M E. \tag{3.2}$$

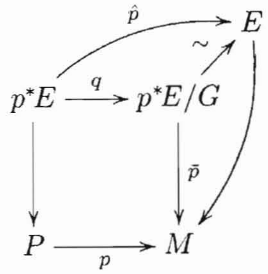
Moreover $\hat{r}^g = r^g \times_{\mathrm{id}_M} \mathrm{id}_E\colon p^*E \to p^*E$ is an isomorphism of vector bundles covering r^g for all $g \in G$, hence with this right action p^*E is a G-equivariant vector bundle over P. Let (U_α, χ_α) be a vector bundle atlas for E, i.e., $\chi_\alpha\colon E_{|U_\alpha} \to U_\alpha \times V$, and let $(U_\alpha, \varphi_\alpha)$ be a principal bundle atlas for P then a vector bundle atlas $(p^{-1}(U_\alpha), \psi_\alpha)$ for p^*E, where $\psi_\alpha\colon p^*E_{|p^{-1}(U_\alpha)} \to p^{-1}(U_\alpha) \times V$, is given by

$$\psi_\alpha(u_x, \xi_x) = (u_x, \mathrm{pr}_V(\chi_\alpha(\xi_x))).$$

Further

$$\begin{aligned}\psi_\alpha^{-1}(u.g, v) &= (u.g, \chi_\alpha^{-1}(p(u.g), v)) \\ &= (u.g, \chi_\alpha^{-1}(p(u), v)) \\ &= (u, \chi_\alpha^{-1}(p(u), v)).g \\ &= \psi_\alpha^{-1}(u, v).g,\end{aligned}$$

hence $(p^{-1}(U_\alpha), \psi_\alpha)$ is a G-equivariant vector bundle atlas for p^*E. From the characterization of principal fiber bundles and using the following commutative diagram

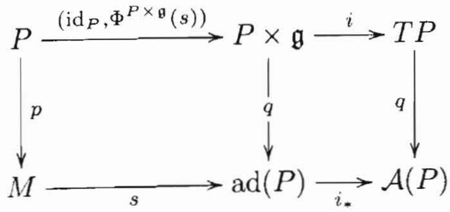


we get that $p^*E/G \to M$ and $E \to M$ are isomorphic vector bundles over M. Furthermore we have $\Gamma(M, E) \simeq \Gamma(M, p^*E/G) \simeq \Gamma(P, p^*E)^G$.

If we define the mapping $j\colon C^\infty(P, \mathfrak{g})^G \to \mathfrak{X}(P)^G$ through

$$j(f)(u) = T_e r_u . f(u), \tag{3.3}$$

where $u \in P$, for $f \in C^\infty(P, \mathfrak{g})^G$ then from the following commutative diagram



we obtain

$$j \circ \Phi^{P\times\mathfrak{g}} = \Phi^{TP} \circ i_*, \tag{3.4}$$

where $\Phi^{TP}\colon \Gamma(M, \mathcal{A}(P)) \to \mathfrak{X}(P)^G$ is a $C^\infty(M, \mathbb{R})$-module isomorphism.

Consider a principal fiber bundle (P, p, M, G) and denote by $r\colon P \times G \to P$ the principal right action of G on P. Let $(L \to M, [\cdot, \cdot], a)$ be a Lie algebroid then the pullback $p^*L = P \times_M L$ carries a natural right action $\hat{r}\colon p^*L \times G \to p^*L$ of G. Moreover $p^*L \to P$ is a G-equivariant vector bundle and the vector bundle $p^*L/G \to M$ is isomorphic to $L \to M$.

Definition 18. Let $(L \to M, [\cdot, \cdot], a)$ be a Lie algebroid. A *principal Lie algebroid* connection on a principal fiber bundle (P, p, M, G) is a homomorphisms $\eta\colon p^*L \to TP$ of vector bundles over P covering the identity on P such that

i) η is horizontal, i.e., the following diagram

$$\begin{array}{ccc} p^*L & \xrightarrow{\ \eta\ } & TP \\ {\scriptstyle \hat{p}}\downarrow & & \downarrow{\scriptstyle Tp} \\ L & \xrightarrow[\ a\ ]{} & TM \end{array}$$

commutes,

ii) η is G-equivariant, i.e., $Tr^g \circ \eta = \eta \circ \hat{r}^g$ for all $g \in G$.

Note that a principal Lie algebroid connection is a Lie algebroid connection which is G-equivariant.

By its G-equivariance, a principal Lie algebroid connection η on P defines a homomorphism $\omega_\eta \colon L \to \mathcal{A}(P)$ of vector bundles over M covering the identity on M, called the *connection form* of η, satisfying $p_* \circ \omega_\eta = a$. On the other hand if $\omega \in \Omega^1_L(M, \mathcal{A}(P))$ is a connection form then there exists a unique principal Lie algebroid connection $\eta \colon p^*L \to TP$ with the given connection form, i.e., $\omega_\eta = \omega$. Using Theorem 14 it is defined by

$$\eta = \tau^{TP} \circ (\mathrm{id}_P \times_{\mathrm{id}_M} \omega_\eta) \colon P \times_M L \to P \times_M \mathcal{A}(P) \to TP. \tag{3.5}$$

Therefore there is a one-to-one correspondence between principal Lie algebroid connections and connection forms hence we will not distinguish between them.

If η is a principal Lie algebroid connection then we define the *horizontal lift* $\eta\xi \in \mathfrak{X}(P)$ of $\xi \in \mathfrak{X}_L(M)$ by

$$\eta\xi = \eta \circ (\mathrm{id}_P \times_{\mathrm{id}_M} \xi) \circ (\mathrm{id}_P, p) \colon P \xrightarrow{\sim} P \times_M M \longrightarrow P \times_M L \longrightarrow TP. \tag{3.6}$$

Because η is G-equivariant, we have

$$(\eta\xi)(u.g) = \eta(u.g, \xi(p(u.g))) = \eta(u.g, \xi(p(u))) = T_u r^g . \eta(u, \xi(p(u))) = T_u r^g . (\eta\xi)(u)$$

hence $\eta\xi \in \mathfrak{X}(P)^G$. Recall that the $C^\infty(M, \mathbb{R})$-module isomorphism $\Phi^{TP} \colon \Gamma(M, \mathcal{A}(P)) \to \mathfrak{X}(P)^G$ is given by

$$\Phi^{TP}(s)(u) = \tau^{TP}(u, (s \circ p)(u)). \tag{3.7}$$

Thus we get, using (3.5) and (3.7),

$$(\eta\xi)(u) = \eta(u, \xi(p(u))) = \tau^{TP}(u, \omega_\eta(\xi)(p(u))) = \Phi^{TP}(\omega_\eta(\xi))(u),$$

thus we have obtained the horizontal lift $\eta\xi$ given by the connection form ω_η, i.e., $\eta\xi = \Phi^{TP}(\omega_\eta(\xi))$. Moreover $\eta\xi$ and $a(\xi)$ are p-related vector fields, since

$$(Tp \circ \eta\xi)(u) = (Tp \circ \eta)(u, \xi(p(u))) = (a \circ \hat{p})(u, \xi(p(u))) = (a(\xi) \circ p)(u).$$

For a principal Lie algebroid connection η with the connection form $\omega_\eta \in \Omega^1_L(M, \mathcal{A}(P))$ we define the *curvature form* $\Omega_\eta \in \Omega^2_L(M, \mathcal{A}(P))$ by

$$\Omega_\eta(\xi_1, \xi_2) = [\omega_\eta(\xi_1), \omega_\eta(\xi_2)] - \omega_\eta([\xi_1, \xi_2]), \tag{3.8}$$

where $\xi_1, \xi_2 \in \mathfrak{X}_L(M)$. We should verify that $\Omega_\eta(\xi_1, f\xi_2) = f\Omega_\eta(\xi_1, \xi_2)$ for $f \in C^\infty(M, \mathbb{R})$, but

$$\begin{aligned}
\Omega_\eta(\xi_1, f\xi_2) &= [\omega_\eta(\xi_1), \omega_\eta(f\xi_2)]_{\mathcal{A}(P)} - \omega_\eta([\xi_1, f\xi_2]_L) \\
&= [\omega_\eta(\xi_1), f\omega_\eta(\xi_2)]_{\mathcal{A}(P)} - \omega_\eta(f[\xi_1, \xi_2]_L + (a_L(\xi_1)f)\,\xi_2) \\
&= f[\omega_\eta(\xi_1), \omega_\eta(\xi_2)]_{\mathcal{A}(P)} + (a_{\mathcal{A}(P)}(\omega_\eta(\xi_1))f)\,\omega_\eta(\xi_2) - f\omega_\eta([\xi_1, \xi_2]_L) - (a_L(\xi_1)f)\,\omega_\eta(\xi_2) \\
&= f\Omega_\eta(\xi_1, \xi_2) + (a_{\mathcal{A}(P)}(\omega_\eta(\xi_1))f)\,\omega_\eta(\xi_2) - (a_L(\xi_1)f)\,\omega_\eta(\xi_2) \\
&= f\Omega_\eta(\xi_1, \xi_2) + ((p_* \circ \omega_\eta)(\xi_1)f)\,\omega_\eta(\xi_2) - (a_L(\xi_1)f)\,\omega_\eta(\xi_2) \\
&= f\Omega_\eta(\xi_1, \xi_2),
\end{aligned}$$

where we used that $p_* \circ \omega_\eta = a_L$. For any $\omega \in \Omega^k_L(M, \mathrm{ad}(P))$ we define $i_*(\omega) \in \Omega^k_L(M, \mathcal{A}(P))$ by

$$i_*(\omega)(\xi_1, \dots, \xi_k) = i_* \circ \omega(\xi_1, \dots, \xi_k), \tag{3.9}$$

where $\xi_1, \dots, \xi_k \in \mathfrak{X}_L(M)$ and similarly for $\omega \in \Omega^k_L(M, \mathcal{A}(P))$ we define $p_*(\omega) \in \Omega^k_L(M, TM)$ through

$$p_*(\omega)(\xi_1, \dots, \xi_k) = p_* \circ \omega(\xi_1, \dots, \xi_k), \tag{3.10}$$

where $\xi_1, \dots, \xi_k \in \mathfrak{X}_L(M)$. Because

$$\begin{aligned} p_* \circ \Omega_\eta(\xi_1, \xi_2) &= p_* \circ [\omega_\eta(\xi_1), \omega_\eta(\xi_2)]_{\mathcal{A}(P)} - p_* \circ \omega_\eta([\xi_1, \xi_2]_L) \\ &= [p_* \circ \omega_\eta(\xi_1), p_* \circ \omega_\eta(\xi_2)] - a_L([\xi_1, \xi_2]_L) \\ &= [a_L(\xi_1), a_L(\xi_2)] - a_L([\xi_1, \xi_2]_L) \\ &= 0, \end{aligned}$$

there exists, using the exactness of the sequence (1.21), a unique $R_\eta \in \Omega^2_L(M, \mathrm{ad}(P))$ such that $\Omega_\eta = i_*(R_\eta)$.

Notation. A principal Lie algebroid connection with zero curvature form is called *flat principal Lie algebroid connection.* We will denote the set of all connection forms by $\mathcal{A}(P, L)$ and the set of all flat connection forms by $\mathcal{H}(P, L)$.

Now we show a similar correspondence between principal Lie algebroid connections and principal connections as for Lie algebroid connections and connections.

Consider a principal fiber bundle (P, p, M, G). Then there are two equivalent descriptions of a principal connection on a principal fiber bundle either via a horizontal bundle or through a connection form.

i) A principal connection on the principal fiber bundle (P, p, M, G) is a vector valued 1-form $\Phi \in \Omega^1(P, TP)$ such that $\Phi \circ \Phi = \Phi$, $\mathrm{im}\,\Phi = VP$ and $Tr^g \circ \Phi = \Phi \circ Tr^g$.
ii) A principal connection on the principal fiber bundle (P, p, M, G) is a vector subbundle HP of the tangent bundle TP such that $TP = HP \oplus VP$ and $H_{u.g}P = T_u r^g(H_u P)$.

Let (P, p, M, G) be a principal fiber bundle and consider a principal Lie algebroid connection $\eta\colon p^*TM \to TP$ for the Lie algebroid $(TM \to M, [\cdot\,,\cdot], \mathrm{id}_{TM})$. Therefore η defines a connection on P given by the horizontal bundle $HP = \mathrm{im}\,\eta$. Because η is G-equivariant, we obtain $H_{u.g}P = \mathrm{im}\,\eta_{u.g} = \mathrm{im}\,(T_u r^g \circ \eta_u) = T_u r^g(\mathrm{im}\,\eta_u) = T_u r^g(H_u P)$. Thus HP is G-invariant subbundle and defines a principal connection on P in the sense of (ii).

On the other hand if we are given a principal connection on the principal bundle (P, p, M, G) in the sense of (ii) then there is a unique Lie algebroid connection $\eta\colon p^*TM \to TP$ given as

$$\eta = ((\pi_P, Tp)_{|HP})^{-1}\colon p^*TM \to HP \hookrightarrow TP,$$

where $(\pi_P, Tp)\colon TP \to p^*TM$. Because $(\pi_P, Tp)\colon TP \to p^*TM$ is G-equivariant, i.e., $(\pi_P, Tp) \circ Tr^g = \hat{r}^g \circ (\pi_P, Tp)$, and HP is G-invariant, thus η is also G-equivariant. These two construction are inverse to each other.

Lemma 23. The set $\mathcal{A}(P, L)$ of connection forms of principal Lie algebroid connections on a principal fiber bundle (P, p, M, G) for the Lie algebroid $(L \to M, [\cdot\,,\cdot], a)$ is an affine space modeled on the vector space $\Omega^1_L(M, \mathrm{ad}(P))$.

Proof. We first prove that $\mathcal{A}(P, L)$ is non-empty. Because any principal fiber bundle admits a principal connection, this gives an existence of a principal Lie algebroid connection η for the Lie algebroid $(TM \to M, [\cdot\,,\cdot], \mathrm{id}_{TM})$ with the connection form $\omega_\eta \in \mathcal{A}(P, TM)$. Now we define 1-form $\omega \in \Omega^1_L(M, \mathcal{A}(P))$ by $\omega = \omega_\eta \circ a$. Since $p_* \circ \omega = p_* \circ \omega_\eta \circ a = \mathrm{id}_{TM} \circ a = a$, we have proved that $\mathcal{A}(P, L)$ is non-empty.

The rest of the proof is very simple. If ω_1 and ω_0 are two connection forms then $p_* \circ (\omega_1 - \omega_0) = a - a = 0$. Because the following sequence

$$0 \longrightarrow \Gamma(M, \mathrm{ad}(P)) \xrightarrow{i_*} \Gamma(M, \mathcal{A}(P)) \xrightarrow{p_*} \Gamma(M, TM) \longrightarrow 0$$

is exact, there is a uniquely determined 1-form $\alpha \in \Omega^1_L(M, \mathrm{ad}(P))$ such that $\omega_1 - \omega_0 = i_*(\alpha)$. Therefore $\mathcal{A}(P, L)$ is an affine space modeled on $\Omega^1_L(M, \mathrm{ad}(P))$. ♠

Remark. Thus, if we fix some ω_0 in $\mathcal{A}(P, L)$, we may write

$$\mathcal{A}(P, L) = \{\omega_0 + i_*(\alpha);\ \alpha \in \Omega^1_L(M, \mathrm{ad}(P))\}. \tag{3.11}$$

This description will permit us to define Sobolev completions of $\mathcal{A}(P, L)$.

We equip the graded vector spaces $\Omega^\bullet_L(M, \mathcal{A}(P))$ in a canonical way with the structure of a graded Lie algebra by

$$[\omega, \tau](\xi_1, \ldots, \xi_{p+q}) = \frac{1}{p!\,q!} \sum_\sigma \mathrm{sign}(\sigma) \cdot [\omega(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(p)}), \tau(\xi_{\sigma(p+1)}, \ldots, \xi_{\sigma(p+q)})], \tag{3.12}$$

where $\omega \in \Omega^p_L(M, \mathcal{A}(P))$, $\tau \in \Omega^q_L(M, \mathcal{A}(P))$ and $\xi_1, \ldots, \xi_{p+q} \in \mathfrak{X}_L(M)$. Furthermore, the graded vector space $\Omega^\bullet_L(M, \mathcal{A}(P))$ is a graded $\Omega^\bullet_L(M)$-module through

$$(\alpha \wedge \omega)(\xi_1, \ldots, \xi_{p+q}) = \frac{1}{p!\,q!} \sum_\sigma \mathrm{sign}(\sigma) \cdot \alpha(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(p)})\, \omega(\xi_{\sigma(p+1)}, \ldots, \xi_{\sigma(p+q)}), \tag{3.13}$$

where $\alpha \in \Omega^p_L(M)$, $\omega \in \Omega^q_L(M, \mathcal{A}(P))$ and $\xi_1, \ldots, \xi_{p+q} \in \mathfrak{X}_L(M)$.

Definition 19. Let (P, p, M, G) be a principal fiber bundle and let $(L \to M, [\cdot,\cdot], a)$ be a Lie algebroid. If $\eta\colon p^*L \to TP$ is a principal Lie algebroid connection with the connection form $\omega_\eta \in \Omega^1_L(M, \mathcal{A}(P))$ then we define the *exterior derivative* $d_{\omega_\eta}\colon \Omega^\bullet_L(M, \mathcal{A}(P)) \to \Omega^{\bullet+1}_L(M, \mathcal{A}(P))$ by

$$\begin{aligned}(d_{\omega_\eta}\omega)(\xi_0, \ldots, \xi_k) &= \sum_{i=0}^k (-1)^i [\omega_\eta(\xi_i), \omega(\xi_0, \ldots, \widehat{\xi_i}, \ldots, \xi_k)] \\ &\quad + \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([\xi_i, \xi_j], \xi_0, \ldots, \widehat{\xi_i}, \ldots, \widehat{\xi_j}, \ldots, \xi_k),\end{aligned} \tag{3.14}$$

where $\omega \in \Omega^k_L(M, \mathcal{A}(P))$ and $\xi_0, \ldots, \xi_k \in \mathfrak{X}_L(M)$.

If we denote by $d\colon \Omega^\bullet_L(M, \mathcal{A}(P)) \to \Omega^{\bullet+1}_L(M, \mathcal{A}(P))$ the usual *Chevalley differential* given via

$$(d\omega)(\xi_0, \ldots, \xi_k) = \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([\xi_i, \xi_j], \xi_0, \ldots, \hat{\xi_i}, \ldots, \hat{\xi_j}, \ldots, \xi_k), \tag{3.15}$$

where $\omega \in \Omega^k_L(M, \mathcal{A}(P))$ and $\xi_0, \ldots, \xi_k \in \mathfrak{X}_L(M)$, then the covariant derivative d_{ω_η} can be written as

$$d_{\omega_\eta} = d + \mathrm{ad}_{\omega_\eta}, \tag{3.16}$$

where $\mathrm{ad}_{\omega_\eta}$ is defined through $\mathrm{ad}_{\omega_\eta}\omega = [\omega_\eta, \omega]$ for $\omega \in \Omega^k_L(M, \mathcal{A}(P))$.

Theorem 15. The covariant derivative d_{ω_η} has the following properties.

i) $d_{\omega_\eta}(\alpha \wedge \omega) = d_L\alpha \wedge \omega + (-1)^{\deg(\alpha)} \alpha \wedge d_{\omega_\eta}\omega$ for $\alpha \in \Omega^\bullet_L(M)$ and $\omega \in \Omega^\bullet_L(M, \mathcal{A}(P))$.
ii) $d_{\omega_\eta}[\omega, \tau] = [d_{\omega_\eta}\omega, \tau] + (-1)^{\deg(\omega)}[\omega, d_{\omega_\eta}\tau]$ for $\omega, \tau \in \Omega^\bullet_L(M, \mathcal{A}(P))$, i.e., d_{ω_η} is a graded derivation of degree 1.
iii) $\Omega_\eta = d\omega_\eta + \frac{1}{2}[\omega_\eta, \omega_\eta]$, the Maurer–Cartan formula for the curvature form.
iv) $\Omega_\eta = d_{\omega_\eta}\omega_\eta - \frac{1}{2}[\omega_\eta, \omega_\eta]$, the curvature form.

v) $d_{\omega_\eta}\Omega_\eta = 0$, the Bianchi identity.
vi) $d_{\omega_\eta} \circ d_{\omega_\eta} = \mathrm{ad}_{\Omega_\eta}$.

Proof. i) It suffices to investigate decomposable forms $\omega = \beta \otimes s$ for $s \in \Omega^0_L(M, \mathcal{A}(P))$ and $\beta \in \Omega^q_L(M)$. From the definition we obtain $d_{\omega_\eta}(\beta \otimes s) = d_L\beta \otimes s + (-1)^q \beta \wedge d_{\omega_\eta} s$. Afterwards for $\alpha \in \Omega^p_L(M)$ we have

$$\begin{aligned} d_{\omega_\eta}(\alpha \wedge (\beta \otimes s)) &= d_{\omega_\eta}((\alpha \wedge \beta) \otimes s) = d_L(\alpha \wedge \beta) \otimes s + (-1)^{p+q}(\alpha \wedge \beta) \wedge d_{\omega_\eta} s \\ &= (d_L\alpha \wedge \beta) \otimes s + (-1)^p(\alpha \wedge d_L\beta) \otimes s + (-1)^{p+q}(\alpha \wedge \beta) \wedge d_{\omega_\eta} s \\ &= d_L\alpha \wedge (\beta \otimes s) + (-1)^p \alpha \wedge d_{\omega_\eta}(\beta \otimes s). \end{aligned}$$

ii) For decomposable forms $\omega = \alpha \otimes s$, $\tau = \beta \otimes t$, where $s, t \in \Omega^0_L(M, \mathcal{A}(P))$, $\alpha \in \Omega^p_L(M)$ and $\beta \in \Omega^q_L(M)$, we have $[\alpha \otimes s, \beta \otimes t] = (\alpha \wedge \beta) \otimes [s,t]$. Hence we can write

$$\begin{aligned} d_{\omega_\eta}[\alpha \otimes s, \beta \otimes t] &= d_{\omega_\eta}((\alpha \wedge \beta) \otimes [s,t]) \\ &= d_L(\alpha \wedge \beta) \otimes [s,t] + (-1)^{p+q}(\alpha \wedge \beta) \wedge d_{\omega_\eta}[s,t] \\ &= (d_L\alpha \wedge \beta) \otimes [s,t] + (-1)^p(\alpha \wedge d_L\beta) \otimes [s,t] \\ &\quad + (-1)^{p+q}(\alpha \wedge \beta) \wedge [d_{\omega_\eta} s, t] + (-1)^{p+q}(\alpha \wedge \beta) \wedge [s, d_{\omega_\eta} t] \\ &= [d_L\alpha \otimes s, \beta \otimes t] + (-1)^p[\alpha \otimes s, d_L\beta \otimes t] + (-1)^p[\alpha \wedge d_{\omega_\eta} s, \beta \otimes t] \\ &\quad + (-1)^{p+q}[\alpha \otimes s, \beta \wedge d_{\omega_\eta} t] \\ &= [d_{\omega_\eta}(\alpha \otimes s), \beta \otimes t] + (-1)^p[(\alpha \otimes s), d_{\omega_\eta}(\beta \otimes t)], \end{aligned}$$

where we used $d_{\omega_\eta}[s,t] = [d_{\omega_\eta} s, t] + [s, d_{\omega_\eta} t]$ which follows from the Jacobi identity, thus we are done.
iii) Immediately from the definition we get

$$\begin{aligned} \Omega_\eta(\xi_1, \xi_2) &= [\omega_\eta(\xi_1), \omega_\eta(\xi_2)] - \omega_\eta([\xi_1, \xi_2]) \\ &= \tfrac{1}{2}[\omega_\eta, \omega_\eta](\xi_1, \xi_2) + (d\omega_\eta)(\xi_1, \xi_2). \end{aligned}$$

iv) We have

$$\begin{aligned} \Omega_\eta(\xi_1, \xi_2) &= [\omega_\eta(\xi_1), \omega_\eta(\xi_2)] - \omega_\eta([\xi_1, \xi_2]) \\ &= [\omega_\eta(\xi_1), \omega_\eta(\xi_2)] - [\omega_\eta(\xi_2), \omega_\eta(\xi_1)] - \omega_\eta([\xi_1, \xi_2]) - [\omega_\eta(\xi_1), \omega_\eta(\xi_2)] \\ &= (d_{\omega_\eta}\omega_\eta)(\xi_1, \xi_2) - \tfrac{1}{2}[\omega_\eta, \omega_\eta](\xi_1, \xi_2). \end{aligned}$$

v) Using (i), (iv) and (vi) we obtain

$$\begin{aligned} d_{\omega_\eta}\Omega_\eta &= d_{\omega_\eta}(d_{\omega_\eta}\omega_\eta - \tfrac{1}{2}[\omega_\eta, \omega_\eta]) \\ &= d_{\omega_\eta}d_{\omega_\eta}\omega_\eta - \tfrac{1}{2}([d_{\omega_\eta}\omega_\eta, \omega_\eta] - [\omega_\eta, d_{\omega_\eta}\omega_\eta]) \\ &= \mathrm{ad}_{\Omega_\eta}\omega_\eta - [d_{\omega_\eta}\omega_\eta, \omega_\eta] \\ &= [d_{\omega_\eta}\omega_\eta, \omega_\eta] - \tfrac{1}{2}[[\omega_\eta, \omega_\eta], \omega_\eta] - [d_{\omega_\eta}\omega_\eta, \omega_\eta] \\ &= 0, \end{aligned}$$

where we used the fact that $[[\omega_\eta, \omega_\eta], \omega_\eta] = 0$.
vi) First we verify that $[\Omega_\eta(\xi_1, \xi_2), s] = (d_{\omega_\eta}d_{\omega_\eta} s)(\xi_1, \xi_2)$. This is a consequence upon the following computation

$$\begin{aligned} (d_{\omega_\eta}(d_{\omega_\eta} s))(\xi_1, \xi_2) &= [\omega_\eta(\xi_1), (d_{\omega_\eta} s)(\xi_2)] - [\omega_\eta(\xi_2), (d_{\omega_\eta} s)(\xi_1)] - (d_{\omega_\eta} s)([\xi_1, \xi_2]) \\ &= [\omega_\eta(\xi_1), [\omega_\eta(\xi_2), s]] - [\omega_\eta(\xi_2), [\omega_\eta(\xi_1), s]] - [\omega_\eta([\xi_1, \xi_2]), s] \\ &= [[\omega_\eta(\xi_1), \omega_\eta(\xi_2)], s] - [\omega_\eta([\xi_1, \xi_2]), s] = [[\omega_\eta(\xi_1), \omega_\eta(\xi_2)] - \omega_\eta([\xi_1, \xi_2]), s] \\ &= [\Omega_\eta(\xi_1, \xi_2), s] \end{aligned}$$

for all $\xi_1, \xi_2 \in \mathfrak{X}_L(M)$ and $s \in \Omega^0_L(M, \mathcal{A}(P))$. Because it suffices to deal with decomposable forms $\omega = \alpha \otimes s$ for $\alpha \in \Omega^k_L(M)$ and $s \in \Omega^0_L(M, \mathcal{A}(P))$, we can write

$$\begin{aligned} d_{\omega_\eta} d_{\omega_\eta}(\alpha \otimes s) &= d_{\omega_\eta}(d_L\alpha \otimes s + (-1)^k \alpha \wedge d_{\omega_\eta} s \\ &= 0 + (-1)^{k+1} d_L\alpha \wedge d_{\omega_\eta} s + (-1)^k d_L \alpha \wedge d_{\omega_\eta} s + (-1)^{2k} \alpha \wedge d_{\omega_\eta} d_{\omega_\eta} s \\ &= \alpha \wedge \mathrm{ad}_{\Omega_\eta} s \\ &= \mathrm{ad}_{\Omega_\eta}(\alpha \otimes s) \end{aligned}$$

hence we have got $d_{\omega_\eta} \circ d_{\omega_\eta} = \mathrm{ad}_{\Omega_\eta}$ and thus we are done. ♠

Consider a flat principal Lie algebroid connection η with the connection form ω_η. From the previous theorem we have $d_{\omega_\eta}[\omega, \tau] = [d_{\omega_\eta}\omega, \tau] + (-1)^{\deg(\omega)}[\omega, d_{\omega_\eta}\tau]$ for $\omega, \tau \in \Omega^\bullet_L(M, \mathcal{A}(P))$, i.e., d_{ω_η} is a graded derivation of degree 1. Moreover because $\Omega_\eta = 0$, we get $d_{\omega_\eta} \circ d_{\omega_\eta} = 0$. Therefore the graded Lie algebra $\Omega^\bullet_L(M, \mathcal{A}(P)$ with the Lie bracket given by (3.12) has a structure of a differential graded Lie algebra.

Lemma 24. Consider two principal Lie algebroid connections η, η' on a principal fiber bundle (P, p, M, G) for a Lie algebroid $(L \to M, [\cdot\,,\cdot], a)$. If we denote $\omega_{\eta'} - \omega_\eta = \alpha \in \Omega^1_L(M, \mathcal{A}(P))$ then

$$\Omega_{\eta'} = \Omega_\eta + d_{\omega_\eta}\alpha + \frac{1}{2}[\alpha, \alpha]. \tag{3.17}$$

Proof. The proof is a straightforward computation only. We have

$$\begin{aligned} \Omega_{\eta'}(\xi_1, \xi_2) &= [\omega_{\eta'}(\xi_1), \omega_{\eta'}(\xi_2)] - \omega_{\eta'}([\xi_1, \xi_2]) \\ &= [\omega_\eta(\xi_1) + \alpha(\xi_1), \omega_\eta(\xi_2) + \alpha(\xi_2)] - \omega_\eta([\xi_1, \xi_2]) - \alpha([\xi_1, \xi_2]) \\ &= [\omega_\eta(\xi_1), \omega_\eta(\xi_2)] - \omega_\eta([\xi_1, \xi_2]) + [\alpha(\xi_1), \omega_\eta(\xi_2)] + [\omega_\eta(\xi_1), \alpha(\xi_2)] \\ &\quad + [\alpha(\xi_1), \alpha(\xi_2)] - \alpha([\xi_1, \xi_2]) \\ &= \Omega_\eta(\xi_1, \xi_2) + [\omega_\eta(\xi_1), \alpha(\xi_2)] - [\omega_\eta(\xi_2), \alpha(\xi_1)] - \alpha([\xi_1, \xi_2]) + [\alpha(\xi_1), \alpha(\xi_2)] \\ &= \Omega_\eta(\xi_1, \xi_2) + d_{\omega_\eta}\alpha + \tfrac{1}{2}[\alpha, \alpha](\xi_1, \xi_2) \end{aligned}$$

for all $\xi_1, \xi_2 \in \mathfrak{X}_L(M)$. ♠

Let $(L \to M, [\cdot\,,\cdot], a)$ be a Lie algebroid and let (P, p, M, G) be a principal fiber bundle. Consider a principal Lie algebroid connection η with the connection form ω_η. If $\rho\colon G \to \mathrm{GL}(\mathbb{E})$ is a representation of the structure group G on a finite dimensional vector space $\mathbb{E}$ then the principal Lie algebroid connection η induces an L-connection $\nabla\colon \Omega^0_L(M, E) \to \Omega^1_L(M, E)$ on the associated vector bundle $E = P \times_G \mathbb{E}$.

We define a bilinear mapping $\nabla\colon \mathfrak{X}_L(M) \times \Omega^0_L(M, E) \to \Omega^0_L(M, E)$ through

$$\nabla_\xi s = \Phi^{-1}\big((\eta\xi)\Phi(s)\big) = \Phi^{-1}\big(\Phi^{TP}(\omega_\eta(\xi))\Phi(s)\big), \tag{3.18}$$

where $\Phi\colon \Gamma(M, E) \xrightarrow{\sim} \Gamma(P, P \times \mathbb{E})^G \xrightarrow{\sim} C^\infty(P, \mathbb{E})^G$ is a $C^\infty(M, \mathbb{R})$-module isomorphism defined in Theorem 14, $\eta\xi \in \mathfrak{X}(P)^G$ is the horizontal lift of $\xi \in \mathfrak{X}_L(M)$ and $s \in \Omega^0_L(M, E)$. Because we have $\nabla_{f\xi} s = f\nabla_\xi s$ and since we may write

$$\begin{aligned} \nabla_\xi(fs) &= \Phi^{-1}\big((\eta\xi)\Phi(fs)\big) = \Phi^{-1}\big((\eta\xi)((f \circ p)\Phi(s))\big) \\ &= \Phi^{-1}\big((\eta\xi)(f \circ p)\Phi(s) + (f \circ p)(\eta\xi)\Phi(s)\big) \\ &= \Phi^{-1}\big((\eta\xi)(f \circ p)\Phi(s)\big) + \Phi^{-1}\big((f \circ p)(\eta\xi)\Phi(s)\big) \\ &= \Phi^{-1}\big(((a(\xi)f) \circ p)\Phi(s)\big) + f\Phi^{-1}\big((\eta\xi)\Phi(s)\big) \\ &= (a(\xi)f)\Phi^{-1}\big(\Phi(s)\big) + f\Phi^{-1}\big((\eta\xi)(\Phi(s)\big) \\ &= (a(\xi)f)s + f\nabla_\xi s, \end{aligned}$$

so $\nabla\colon \Omega^0_L(M,E) \to \Omega^1_L(M,E)$ is a linear Lie algebroid connection on the associated vector bundle E, called the *induced L-connection.*

Lemma 25. Let η be a principal Lie algebroid connection and let $\nabla\colon \Omega^0_L(M,E) \to \Omega^1_L(M,E)$ be the induced connection on the associated vector bundle $E = P \times_G \mathbb{E}$. Then the curvature $R^\nabla \in \Omega^2_L(M, \mathrm{End}(E)$ and the connection form $R_\eta \in \Omega^2_L(M, \mathrm{ad}(P))$, where $\Omega_\eta = i_*(R_\eta)$, are related by

$$R^\nabla(\xi_1,\xi_2)s = -(\rho'_{R_\eta}s)(\xi_1,\xi_2), \tag{3.19}$$

where $\rho'\colon \mathfrak{g} \to \mathrm{End}(\mathbb{E})$ is the derivative of the representation $\rho\colon G \to \mathrm{GL}(\mathbb{E})$.

Proof. From the previous we get

$$\begin{aligned}
R^\nabla(\xi_1,\xi_2)s &= \nabla_{\xi_1}\nabla_{\xi_2}s - \nabla_{\xi_2}\nabla_{\xi_1}s - \nabla_{[\xi_1,\xi_2]}s\\
&= \Phi^{-1}((\eta\xi_1)((\eta\xi_2)\Phi(s))) - \Phi^{-1}((\eta\xi_2)((\eta\xi_1)\Phi(s))) - \Phi^{-1}((\eta[\xi_1,\xi_2])\Phi(s))\\
&= \Phi^{-1}([\eta\xi_1,\eta\xi_2]\Phi(s)) - \Phi^{-1}((\eta[\xi_1,\xi_2])\Phi(s))\\
&= \Phi^{-1}(([\eta\xi_1,\eta\xi_2] - \eta[\xi_1,\xi_2])\Phi(s))\\
&= \Phi^{-1}(([\Phi^{TP}(\omega_\eta(\xi_1)),\Phi^{TP}(\omega_\eta(\xi_2))] - \Phi^{TP}(\omega_\eta([\xi_1,\xi_2])))\Phi(s))\\
&= \Phi^{-1}((\Phi^{TP}([\omega_\eta(\xi_1),\omega_\eta(\xi_2)]) - \Phi^{TP}(\omega_\eta([\xi_1,\xi_2])))\Phi(s))\\
&= \Phi^{-1}(\Phi^{TP}(\Omega_\eta(\xi_1,\xi_2))\Phi(s))\\
&= \Phi^{-1}((\Phi^{TP}\circ i_*)(R_\eta(\xi_1,\xi_2))\Phi(s))\\
&= \Phi^{-1}((j\circ\Phi^{P\times\mathfrak{g}})(R_\eta(\xi_1,\xi_2))\Phi(s))\\
&= -\Phi^{-1}(\rho'(\Phi^{P\times\mathfrak{g}}(R_\eta(\xi_1,\xi_2)))\Phi(s))\\
&= -(\rho'_{R_\eta}s)(\xi_1,\xi_2),
\end{aligned}$$

where we used (3.4) and (3.31). ♠

Lemma 26. Let η, η' be two principal Lie algebroid connections and denote by $\alpha \in \Omega^1_L(M, \mathrm{ad}(P))$ a uniquely determined 1-form satisfying that $\omega_{\eta'} - \omega_\eta = i_*(\alpha)$. Then the corresponding induced L-connection ∇, ∇' on the associated vector bundle $E = P \times_G \mathbb{E}$ are related through

$$\nabla' = \nabla - \rho'_\alpha \tag{3.20}$$

where $\rho'\colon \mathfrak{g} \to \mathrm{End}(\mathbb{E})$ is the derivative of the representation $\rho\colon G \to \mathrm{GL}(\mathbb{E})$.

Proof. Using the definition of the induced L-connection, we obtain

$$\begin{aligned}
\nabla'_\xi s &= \Phi^{-1}(\Phi^{TP}(\omega_{\eta'}(\xi))\Phi(s))\\
&= \Phi^{-1}(\Phi^{TP}(\omega_\eta(\xi) + i_*(\alpha)(\xi))\Phi(s))\\
&= \Phi^{-1}(\Phi^{TP}(\omega_\eta(\xi))\Phi(s)) + \Phi^{-1}(\Phi^{TP}(i_*(\alpha)(\xi))\Phi(s))\\
&= \nabla_\xi s + \Phi^{-1}((\Phi^{TP}\circ i_*)(\alpha(\xi))\Phi(s))\\
&= \nabla_\xi s + \Phi^{-1}((j\circ\Phi^{P\times\mathfrak{g}})(\alpha(\xi))\Phi(s))\\
&= \nabla_\xi s - \Phi^{-1}(\rho'(\Phi^{P\times\mathfrak{g}}(\alpha(\xi)))\Phi(s))\\
&= \nabla_\xi s - (\rho'_\alpha s)(\xi),
\end{aligned}$$

hence we have found how the induced L-connection changes. ♠

Let η be a principal Lie algebroid connection and assume that $\alpha \in \Omega^k_L(M, \mathrm{ad}(P))$. Because $p_*\circ i_* = 0$, after an easy computation we obtain $p_*(d_{\omega_\eta} i_*(\alpha)) = 0$. Therefore there exists a unique $\beta \in \Omega^{k+1}_L(M, \mathrm{ad}(P))$ such that $i_*(\beta) = d_{\omega_\eta} i_*(\alpha)$.

We can write

$$(di_*(\alpha))(\xi_0,\dots,\xi_k) = \sum_{0\leq i<j\leq k} (-1)^{i+j} i_*(\alpha)([\xi_i,\xi_j],\dots,\hat{\xi}_i,\dots,\hat{\xi}_j,\dots,\xi_k)$$
$$= \sum_{0\leq i<j\leq k} (-1)^{i+j} i_*(\alpha([\xi_i,\xi_j],\dots,\hat{\xi}_i,\dots,\hat{\xi}_j,\dots,\xi_k)).$$

Further we have

$$\Phi^{TP}((\mathrm{ad}_{\omega_\eta} i_*(\alpha))(\xi_0,\dots,\xi_k)) = \sum_{i=0}^{k}(-1)^i \Phi^{TP}([\omega_\eta(\xi_i), i_*(\alpha)(\xi_0,\dots,\hat{\xi}_i,\dots,\xi_k)])$$
$$= \sum_{i=0}^{k}(-1)^i [\Phi^{TP}(\omega_\eta(\xi_i)), \Phi^{TP}(i_*(\alpha)(\xi_0,\dots,\hat{\xi}_i,\dots,\xi_k))]$$
$$= \sum_{i=0}^{k}(-1)^i [\eta\xi_i, (\Phi^{TP}\circ i_*)(\alpha(\xi_0,\dots,\hat{\xi}_i,\dots,\xi_k))]$$
$$= \sum_{i=0}^{k}(-1)^i [\eta\xi_i, (j\circ \Phi^{P\times\mathfrak{g}})(\alpha(\xi_0,\dots,\hat{\xi}_i,\dots,\xi_k))]$$
$$= \sum_{i=0}^{k}(-1)^i j((\eta\xi_i)(\Phi^{P\times\mathfrak{g}}(\alpha(\xi_0,\dots,\hat{\xi}_i,\dots,\xi_k)))).$$

This can be rewritten as

$$(\mathrm{ad}_{\omega_\eta} i_*(\alpha))(\xi_0,\dots,\xi_k) = \sum_{i=0}^{k}(-1)^i ((\Phi^{TP})^{-1}\circ j)((\eta\xi_i)(\Phi^{P\times\mathfrak{g}}(\alpha(\xi_0,\dots,\hat{\xi}_i,\dots,\xi_k))))$$
$$= \sum_{i=0}^{k}(-1)^i (i_*\circ (\Phi^{P\times\mathfrak{g}})^{-1})((\eta\xi_i)(\Phi^{P\times\mathfrak{g}}(\alpha(\xi_0,\dots,\hat{\xi}_i,\dots,\xi_k))))$$
$$= \sum_{i=0}^{k}(-1)^i i_*((\Phi^{P\times\mathfrak{g}})^{-1}((\eta\xi_i)(\Phi^{P\times\mathfrak{g}}(\alpha(\xi_0,\dots,\hat{\xi}_i,\dots,\xi_k)))))$$
$$= \sum_{i=0}^{k}(-1)^i i_*(\nabla_{\xi_i}\alpha(\xi_0,\dots,\hat{\xi}_i,\dots,\xi_k)).$$

If we give this together, then we obtain

$$(d_{\omega_\eta} i_*(\alpha))(\xi_0,\dots,\xi_k) = (\mathrm{ad}_{\omega_\eta} i_*(\alpha))(\xi_0,\dots,\xi_k) + (di_*(\alpha))(\xi_0,\dots,\xi_k)$$
$$= \sum_{i=0}^{k}(-1)^i i_*(\nabla_{\xi_i}\alpha(\xi_0,\dots,\hat{\xi}_i,\dots,\xi_k))$$
$$+ \sum_{0\leq i<j\leq k} (-1)^{i+j} i_*(\alpha([\xi_i,\xi_j],\dots,\hat{\xi}_i,\dots,\hat{\xi}_j,\dots,\xi_k))$$
$$= i_*(d^\nabla\alpha)(\xi_0,\dots,\xi_k),$$

therefore we have

$$d_{\omega_\eta} i_*(\alpha) = i_*(d^\nabla\alpha) \tag{3.21}$$

for $\alpha\in\Omega^k_L(M,\mathrm{ad}(P))$.

Let η and η' be principal Lie algebroid connections on a principal fiber bundle (P,p,M,G) for a Lie algebroid $(L\to M,[\cdot,\cdot],a)$. Then there exists a unique $\alpha\in\Omega^1_L(M,\mathrm{ad}(P))$ such that

$\omega' - \omega = i_*(\alpha)$. From Lemma 24 we have

$$\Omega_{\eta'} = \Omega_\eta + d_{\omega_\eta} i_*(\alpha) + \frac{1}{2}\left[i_*(\alpha), i_*(\alpha)\right]$$

but from the previous result we obtain

$$\Omega_{\eta'} = \Omega_\eta + i_*(d^\nabla \alpha) + \frac{1}{2} i_*([\alpha, \alpha]) = \Omega_\eta + i_*(d^\nabla \alpha + \frac{1}{2}\left[\alpha, \alpha\right]).$$

Therefore, if we fix some flat connection form $\omega_0 \in \mathcal{H}(P, L)$, then we may write

$$\mathcal{H}(P, L) = \{\omega_0 + i_*(\alpha);\ \alpha \in \Omega^1_L(M, \mathrm{ad}(P)),\ d^\nabla \alpha + \frac{1}{2}\left[\alpha, \alpha\right] = 0\}. \tag{3.22}$$

This description, similarly as in the case of $\mathcal{A}(P, L)$, will allow us to define Sobolev completions of $\mathcal{H}(P, L)$.

3.2 Group of gauge transformations

Let (P, p, M, G) be a principal fiber bundle with the principal right action $r\colon P \times G \to P$, then a *principal fiber bundle homomorphism* is a smooth G-equivariant mapping $\varphi\colon P \to P$, i.e., $\varphi \circ r^g = r^g \circ \varphi$ for all $g \in G$. Then obviously the diagram

$$\begin{array}{ccc} P & \xrightarrow{\ \varphi\ } & P \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ M & \xrightarrow[\ \underline{\varphi}\ ]{} & M \end{array}$$

commutes for a uniquely determined smooth mapping $\underline{\varphi}\colon M \to M$. For each $x \in M$ the mapping $\varphi_x = \varphi_{|P_x}\colon P_x \to P_{\underline{\varphi}(x)}$ is G-equivariant and therefore a diffeomorphism. If we denote by $\mathrm{Aut}(P)$ the group of all G-equivariant diffeomorphisms $\varphi\colon P \to P$ then the previous diagram commutes for a unique diffeomorphism $\underline{\varphi}\colon M \to M$. Hence we have a group homomorphism from $\mathrm{Aut}(P)$ into the group $\mathrm{Diff}(M)$ of all diffeomorphism of M. The kernel $\mathrm{Gau}(P)$ of this homomorphism is called the *group of gauge transformations*. Thus $\mathrm{Gau}(P)$ is the group of all G-equivariant diffeomorphism $\varphi\colon P \to P$ satisfying $p \circ \varphi = p$. Therefore we get the following exact sequence

$$\{e\} \to \mathrm{Gau}(P) \to \mathrm{Aut}(P) \to \mathrm{Diff}(M) \tag{3.23}$$

of groups.

Furthermore we define the *Lie algebra of infinitesimal gauge transformations* $\mathfrak{gau}(P)$. As a vector space it is the vector spaces of vertical G-invariant vector fields $\mathfrak{X}_{\mathrm{vert}}(P)^G$, while the Lie bracket is the Lie bracket of vector fields.

The group of gauge transformations and the Lie algebra of infinitesimal gauge transformations can be described by another equivalent ways. If we denote by

$$\mathrm{Ad}\, P = P \times_G G \tag{3.24}$$

the associated bundle for the action of G on itself given by the conjugation then sections of this bundle can be identified with the space

$$C^\infty(P, G)^G = \{f \in C^\infty(P, G);\ f(u.g) = \mathrm{conj}_{g^{-1}} f(u)\} \tag{3.25}$$

which is a group under pointwise multiplication. If can be identified with the group $\mathrm{Gau}(P)$. For $\varphi \in \mathrm{Gau}(P)$ we define $f_\varphi \in C^\infty(P, G)^G$ by $f_\varphi = \tau \circ (\mathrm{id}_P, \varphi)$, where $\tau\colon P \times_M P \to G$.

Then $f_\varphi(u.g) = \tau(u.g, \varphi(u.g)) = g^{-1}.\tau(u, \varphi(u)).g = \operatorname{conj}_{g^{-1}} f_\varphi(u)$, thus f_φ is G-equivariant. If conversely $f \in C^\infty(P, G)^G$ is given we define $\varphi_f \in \operatorname{Gau}(P)$ by $\varphi_f(u) = u.f(u)$. Because $\varphi_f(u.g) = u.g.f(u.g) = u.g.g^{-1}.f(u).g = \varphi_f(u).g$, we indeed get $\varphi_f \in \operatorname{Gau}(P)$. These two constructions are inverse to each other since $f_{\varphi_f}(u) = \tau(u, \varphi_f(u)) = \tau(u, u.f(u)) = \tau(u, u).f(u) = f(u)$ and $\varphi_{f_\varphi}(u) = u.f_\varphi(u) = u.\tau(u.\varphi(u)) = \varphi(u)$.

Now let $\xi \in \mathfrak{X}_{\text{vert}}(P) = \Gamma(P, VP)$ be a vertical vector field then there is a uniquely determined mapping $f_\xi \in C^\infty(P, \mathfrak{g})$ via $\xi(u) = T_e r_u.f_\xi(u)$. The mapping f_ξ is G-equivariant if and only if

$$\begin{aligned} T_e r_u.f_\xi(u) = \xi(u) &= ((r^g)^*\xi)(u) = T_{u.g} r^{g^{-1}}.\xi(u.g) \\ &= T_{u.g} r^{g^{-1}}.T_e r_{u.g}.f_\xi(u.g) = T_e(r^{g^{-1}} \circ r_{u.g}).f_\xi(u) \\ &= T_e(r_u \circ \operatorname{conj}_g).f_\xi(u) = T_e r_u.\operatorname{Ad}(g).f_\xi(u), \end{aligned}$$

i.e., if and only if $\xi \in \mathfrak{X}_{\text{vert}}(P)^G$. Therefore we have the following isomorphism

$$\operatorname{Gau}(P) \simeq C^\infty(P, G)^G \simeq \Gamma(M, \operatorname{Ad}(P)) \tag{3.26}$$

of groups and isomorphism

$$\mathfrak{X}_{\text{vert}}(P)^G \simeq C^\infty(P, \mathfrak{g})^G \simeq \Gamma(M, \operatorname{ad}(P)) \tag{3.27}$$

of Lie algebras.

Let $\rho\colon G \to \operatorname{GL}(\mathbb{E})$ be a representation of the structure group G on a finite dimensional vector space $\mathbb{E}$. If E denotes the corresponding associated vector bundle $P \times_G \mathbb{E}$ then there is a natural left action of the group of gauge transformations $\operatorname{Gau}(P)$ on the vector space $\Omega^k_L(M, E)$.

Consider a gauge transformation φ then there exists an isomorphism $\varphi_\mathbb{E}\colon E \to E$ of vector bundles over M covering the identity on M defined by the following diagram

$$\begin{array}{ccc} P \times \mathbb{E} & \xrightarrow{\varphi \times \operatorname{id}_\mathbb{E}} & P \times \mathbb{E} \\ \Big\downarrow q & & \Big\downarrow q \\ E & \xrightarrow[\varphi_\mathbb{E}]{} & E \end{array}$$

which in a unique way determines $\varphi_\mathbb{E}$. This gives a left action of $\operatorname{Gau}(P)$ on $\Omega^k_L(M, E)$ through

$$(\rho_\varphi(\omega))(\xi_1, \dots, \xi_k) = \varphi_\mathbb{E} \circ \omega(\xi_1, \dots, \xi_k), \tag{3.28}$$

where $\xi_1, \dots, \xi_k \in \mathfrak{X}_L(M)$. This action can be described otherwise. If $\Phi\colon \Gamma(M, E) \to C^\infty(P, \mathbb{E})^G$ denotes a $C^\infty(M, \mathbb{R})$-module isomorphism then for any $\varphi \in \operatorname{Gau}(P)$ and $s \in \Gamma(M, E)$ we have $\Phi(s) \circ \varphi^{-1} \in C^\infty(P, \mathbb{E})^G$. Furthermore from the following commutative diagram

$$\begin{array}{ccccccc} P & \xrightarrow{\varphi^{-1}} & P & \xrightarrow{(\operatorname{id}_P, \Phi(s))} & P \times \mathbb{E} & \xrightarrow{\varphi \times \operatorname{id}_\mathbb{E}} & P \times \mathbb{E} \\ \Big\downarrow p & & \Big\downarrow p & & \Big\downarrow q & & \Big\downarrow q \\ M & \xrightarrow[\operatorname{id}_M]{} & M & \xrightarrow[s]{} & E & \xrightarrow[\varphi_\mathbb{E}]{} & E \end{array}$$

we get $\Phi(\varphi_\mathbb{E} \circ s) = \Phi(s) \circ \varphi^{-1}$. Therefore the action (3.28) can be rewritten as

$$(\rho_\varphi(\omega))(\xi_1, \dots, \xi_k) = \Phi^{-1}\big(\Phi(\omega(\xi_1, \dots, \xi_k)) \circ \varphi^{-1}\big) \tag{3.29}$$

$$= \Phi^{-1}\big(\rho(g_\varphi)\Phi(\omega(\xi_1, \dots, \xi_k))\big), \tag{3.30}$$

where in the last equality we used the fact that $\Phi(s) \circ \varphi^{-1} = \rho(g_\varphi)\Phi(s)$ following using the G-equivariance of $\Phi(s)$ and the definition of g_φ.

If $\rho'\colon \mathfrak{g} \to \mathrm{End}(\mathbb{E})$ denotes the corresponding representation of the Lie algebra $\mathfrak{g}$ then for any $\tau \in \Omega^p_L(M, \mathrm{ad}(P))$ we define a graded $\Omega^\bullet_L(M)$-module homomorphism $\rho'_\tau\colon \Omega^\bullet_L(M,E) \to \Omega^\bullet_L(M,E)$ (so that $\rho'_\tau(\alpha \wedge \omega) = \alpha \wedge (-1)^{\deg(\tau)\deg(\omega)}\rho'_\tau(\omega)$ for $\alpha \in \Omega^\bullet_L(M)$ and $\omega \in \Omega^\bullet_L(M,E)$) by

$$\begin{aligned}(\rho'_\tau(\omega))&(\xi_1,\dots,\xi_{p+q})\\ &= \frac{1}{p!\,q!}\sum_\sigma \mathrm{sign}(\sigma)\cdot \Phi^{-1}\big(\rho'(\Phi^{P\times\mathfrak{g}}(\tau(\xi_{\sigma(1)},\dots,\xi_{\sigma(p)})))\Phi(\omega(\xi_{\sigma(p+1)},\dots,\xi_{\sigma(p+q)}))\big),\end{aligned} \tag{3.31}$$

where $\xi_1,\dots,\xi_{p+q} \in \mathfrak{X}_L(M)$. In case $\rho' = \mathrm{ad}$ then this gives the structure of a graded Lie algebra on $\Omega^\bullet_L(M, \mathrm{ad}(P))$. Because the Lie algebra $\Gamma(M, \mathrm{ad}(P)) = \Omega^0_L(M, \mathrm{ad}(P))$ is isomorphic to the Lie algebra of gauge transforamtions $\mathfrak{gau}(P)$ then (3.31) is a representation of $\mathfrak{gau}(P)$ on $\Omega^\bullet_L(M,E)$.

Further we define a left action of the group of gauge transformations $\mathrm{Gau}(P)$ on $\Omega^k_L(M, \mathcal{A}(P))$ via

$$(\mathrm{Ad}_\varphi(\omega))(\xi_1,\dots,\xi_k) = \varphi_* \circ \omega(\xi_1,\dots,\xi_k), \tag{3.32}$$

where $\xi_1,\dots,\xi_k \in \mathfrak{X}_L(M)$.

Lemma 27. For any gauge transformation $\varphi \in \mathrm{Gau}(P)$ we have

$$\mathrm{Ad}_\varphi \circ i_* = i_* \circ \mathrm{Ad}_\varphi, \tag{3.33}$$

where $i_*\colon \Omega^\bullet_L(M, \mathrm{ad}(P)) \to \Omega^\bullet_L(M, \mathcal{A}(P))$.

Proof. For any $\omega \in \Omega^k_L(M, \mathrm{ad}(P))$ we have

$$\begin{aligned}(\mathrm{Ad}_\varphi(i_*(\omega)))(\xi_1,\dots,\xi_k) &= \varphi_* \circ i_*(\omega)(\xi_1,\dots,\xi_k)\\ &= (\Phi^{TP})^{-1}(\Phi^{TP}(\varphi_* \circ (i_*(\omega)(\xi_1,\dots,\xi_k))))\\ &= (\Phi^{TP})^{-1}(\varphi_*^{-1}\Phi^{TP}(i_*(\omega)(\xi_1,\dots,\xi_k)))\\ &= (\Phi^{TP})^{-1}(\varphi_*^{-1}((\Phi^{TP} \circ i_*)(\omega(\xi_1,\dots,\xi_k))))\\ &= (\Phi^{TP})^{-1}(T\varphi \circ ((j \circ \Phi^{P\times\mathfrak{g}})(\omega(\xi_1,\dots,\xi_k))) \circ \varphi^{-1})\\ &= (\Phi^{TP})^{-1}(j((\Phi^{P\times\mathfrak{g}}(\omega(\xi_1,\dots,\xi_k)) \circ \varphi^{-1}))\\ &= (\Phi^{TP})^{-1}(j(\Phi^{P\times\mathfrak{g}}(\varphi_{\mathfrak{g}} \circ \omega(\xi_1,\dots,\xi_k))))\\ &= (\Phi^{TP})^{-1}((\Phi^{TP} \circ i_*)(\varphi_{\mathfrak{g}} \circ \omega(\xi_1,\dots,\xi_k)))\\ &= (i_*(\mathrm{Ad}_\varphi(\omega)))(\xi_1,\dots,\xi_k),\end{aligned}$$

therefore we are done. ♠

3.3 Geometry of principal Lie algebroid connections

Let $(L \to M, [\cdot,\cdot], a)$ be a Lie algebroid and consider a principal Lie algebroid connection $\eta\colon p^*L \to TP$ with the connection form ω_η. For any gauge transformation $\varphi \in \mathrm{Gau}(P)$ we define a homomorphism $\eta^\varphi\colon p^*L \to TP$ of vector bundles over P covering the identity on P by the following commutative diagram

$$\begin{array}{ccccccc} p^*L & \xrightarrow{\hat{\varphi}} & p^*L & \xrightarrow{\eta} & TP & \xrightarrow{T\varphi^{-1}} & TP \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P & \xrightarrow[\varphi]{} & P & \xrightarrow[\mathrm{id}_P]{} & P & \xrightarrow[\varphi^{-1}]{} & P, \end{array}$$

where $\hat{\varphi} = \varphi \times_{\mathrm{id}_M} \mathrm{id}_L \colon P \times_M L \to P \times_M L$. Because

$$Tp \circ \eta^\varphi = Tp \circ T\varphi^{-1} \circ \eta \circ \hat{\varphi} = T(p \circ \varphi^{-1}) \circ \eta \circ \hat{\varphi} = Tp \circ \eta \circ \hat{\varphi} = a \circ \hat{p} \circ \hat{\varphi} = a \circ \hat{p},$$

so η^φ is a Lie algebroid connection. Moreover we have

$$\begin{aligned} Tr^g \circ \eta^\varphi &= Tr^g \circ T\varphi^{-1} \circ \eta \circ \hat{\varphi} = T\varphi^{-1} \circ Tr^g \circ \eta \circ \hat{\varphi} = T\varphi^{-1} \circ \eta \circ \hat{r}^g \circ \hat{\varphi} \\ &= T\varphi^{-1} \circ \eta \circ \hat{\varphi} \circ \hat{r}^g = \eta^\varphi \circ \hat{r}^g \end{aligned}$$

hence η^φ is a principal Lie algebroid connection. It is easy to see that the corresponding connection form is

$$\omega_{\eta^\varphi} = \varphi_*^{-1} \circ \omega_\eta, \tag{3.34}$$

where $\varphi_* = (T\varphi)^G$. Therefore we can consider a natural right action of the group of gauge transformations $\mathrm{Gau}(P)$ on the space $\mathcal{A}(P, L)$ of connection forms given by

$$(\omega, \varphi) \mapsto \omega \cdot \varphi = \varphi_*^{-1} \circ \omega = \mathrm{Ad}_{\varphi^{-1}}(\omega). \tag{3.35}$$

Remark. It would be possible to define a left action instead of a right action by

$$(\varphi, \omega) \mapsto \varphi \cdot \omega = \varphi_* \circ \omega = \mathrm{Ad}_\varphi(\omega) \tag{3.36}$$

but it has no essential meaning.

Let $\Phi \colon \Gamma(M, \mathcal{A}(P)) \to \mathfrak{X}(P)^G$ be a $C^\infty(M, \mathbb{R})$-module isomorphism given by Theorem 2. Further consider $s \in \Gamma(M, \mathcal{A}(P))$ and $\varphi \in \mathrm{Gau}(P)$. Because $(r^g)^*(\varphi^{-1})^*\Phi(s) = (\varphi^{-1})^*\Phi(s)$, i.e., $(\varphi^{-1})^*\Phi(s) \in \mathfrak{X}(P)^G$ and

$$q \circ (\varphi^{-1})^*\Phi(s) = q \circ T\varphi \circ \Phi(s) \circ \varphi^{-1} = \varphi_* \circ q \circ \Phi(s) \circ \varphi^{-1} = \varphi_* \circ s \circ p \circ \varphi^{-1} = \varphi_* \circ s \circ p,$$

thus $\Phi(\varphi_* \circ s) = (\varphi^{-1})^*\Phi(s)$. Now let $s_1, s_2 \in \Gamma(M, \mathcal{A}(P))$ then

$$\begin{aligned} [\varphi_* \circ s_1, \varphi_* \circ s_2] &= \Phi^{-1}([\Phi(\varphi_* \circ s_1), \Phi(\varphi_* \circ s_2)] \\ &= \Phi^{-1}([(\varphi^{-1})^*\Phi(s_1), (\varphi^{-1})^*\Phi(s_2)] \\ &= \Phi^{-1}((\varphi^{-1})^*\Phi([s_1, s_2])) \\ &= \varphi_* \circ [s_1, s_2] \end{aligned}$$

and because $p_* \circ \varphi_* = p_*$, so $\varphi_* \colon \mathcal{A}(P) \to \mathcal{A}(P)$ is an isomorphism of the Atiyah algebroid.

Lemma 28. Let η be a principal Lie algebroid connection on P with the connection form ω_η and the curvature form Ω_η. Then we have

$$\Omega_{\eta^\varphi} = \mathrm{Ad}_{\varphi^{-1}}(\Omega_\eta) \tag{3.37}$$

for any $\varphi \in \mathrm{Gau}(P)$.

Proof. It follows immediately that

$$\begin{aligned} \Omega_{\eta^\varphi}(\xi_1, \xi_2) &= [\omega_{\eta^\varphi}(\xi_1), \omega_{\eta^\varphi}(\xi_2)] - \omega_{\eta^\varphi}([\xi_1, \xi_2]) \\ &= [\varphi_*^{-1} \circ \omega_\eta(\xi_1), \varphi_*^{-1} \circ \omega_\eta(\xi_2)] - \varphi_*^{-1} \circ \omega_\eta([\xi_1, \xi_2]) \\ &= \varphi_*^{-1} \circ [\omega_\eta(\xi_1), \omega_\eta(\xi_2)] - \varphi_*^{-1} \circ \omega_\eta([\xi_1, \xi_2]) \\ &= \varphi_*^{-1} \circ \Omega_\eta(\xi_1, \xi_2) \end{aligned}$$

for all $\xi_1, \xi_2 \in \mathfrak{X}_L(M)$. So we are done. ♠

Because $\mathcal{H}(P,L)$ is invariant under the action of $\mathrm{Gau}(P)$, as it follows from Lemma 28, we have the action of $\mathrm{Gau}(P)$ on the space of flat connection forms $\mathcal{H}(P,L)$. Therefore we define the *moduli space*

$$\mathcal{B}(P,L) = \mathcal{A}(P,L)/\mathrm{Gau}(P) \tag{3.38}$$

of gauge equivalence classes of connection forms and the *moduli space*

$$\mathcal{M}(P,L) = \mathcal{H}(P,L)/\mathrm{Gau}(P) \tag{3.39}$$

of gauge equivalence classes of flat connection forms.

Theorem 16. Let η and η_0 be principal Lie algebroid connections on a principal fiber bundle (P,p,M,G) for a Lie algebroid $(L \to M, [\cdot,\cdot], a)$. Further consider a gauge transformation $\varphi \in \mathrm{Gau}(P)$. Then there exists a uniquely determined $\alpha^\varphi \in \Omega^1_L(M, \mathrm{ad}(P))$ such that $\omega_{\eta^\varphi} - \omega_{\eta_0} = i_*(\alpha^\varphi)$ and is given by

$$\alpha^\varphi(\xi) = -(\Phi^{P\times\mathfrak{g}})^{-1}((g_\varphi^*\theta)(\Phi^{TP}(\omega_{\eta_0}(\xi)))) + \varphi_{\mathfrak{g}} \circ \alpha(\xi), \tag{3.40}$$

where $\theta \in \Omega^1(G,\mathfrak{g})$ is the Maurer–Cartan form of the Lie group G, $\alpha \in \Omega^1_L(M,\mathrm{ad}(P))$ and satisfies $\omega_\eta - \omega_{\eta_0} = i_*(\alpha)$.

Proof. We can write

$$\begin{aligned}\omega_{\eta^\varphi} &= \varphi_*^{-1} \circ (\omega_{\eta_0} + i_*(\alpha)) = \varphi_*^{-1} \circ \omega_{\eta_0} + \varphi_* \circ i_*(\alpha)\\ &= \varphi_*^{-1} \circ \omega_{\eta_0} + i_*(\varphi_{\mathfrak{g}} \circ \alpha).\end{aligned}$$

Further from the previous we know that $\varphi_*^{-1} \circ \omega_{\eta_0}$ can be written as $\omega_{\eta_0} + i_*(\beta)$ for a uniquely determined $\beta \in \Omega^1_L(M,\mathrm{ad}(P))$. We have

$$\begin{aligned}\varphi_*^{-1} \circ \omega_{\eta_0}(\xi) &= (\Phi^{TP})^{-1}(\Phi^{TP}(\varphi_*^{-1} \circ \omega_{\eta_0}(\xi)))\\ &= (\Phi^{TP})^{-1}(\varphi^*(\Phi^{TP}(\omega_{\eta_0}(\xi))))\\ &= (\Phi^{TP})^{-1}(T\varphi^{-1} \circ \Phi^{TP}(\omega_{\eta_0}(\xi)) \circ \varphi).\end{aligned}$$

Furthermore if $\xi \in \mathfrak{X}(P)^G$, then we get

$$(\varphi^*\xi)(u) = T_{\varphi(u)}\varphi^{-1}.\xi(\varphi(u)) = T_{\varphi(u)}\varphi^{-1}.\xi(u.g_\varphi(u)) = T_{\varphi(u)}\varphi^{-1}.T_u r^{g_\varphi(u)}.\xi(u)$$

but because $\varphi = r \circ (\mathrm{id}_P, g_\varphi)$ we obtain

$$T_u\varphi = T_{(u,g_\varphi(u))}r \circ T_u(\mathrm{id}_P, g_\varphi) = T_u r^{g_\varphi(u)} \circ T_u\mathrm{id}_P + T_{g_\varphi(u)}r_u \circ T_u g_\varphi.$$

Therefore we have

$$\begin{aligned}(\varphi^*\xi)(u) &= T_{\varphi(u)}\varphi^{-1}.T_u r^{g_\varphi(u)}.\xi(u)\\ &= T_{\varphi(u)}\varphi^{-1}.(T_u\varphi - T_{g_\varphi(u)}r_u \circ T_u g_\varphi).\xi(u)\\ &= \xi(u) - T_{\varphi(u)}\varphi^{-1}.T_{g_\varphi(u)}r_u.T_u g_\varphi.\xi(u)\\ &= \xi(u) - T_{g_\varphi(u)}(\varphi^{-1} \circ r_u).T_u g_\varphi.\xi(u)\\ &= \xi(u) - T_{g_\varphi(u)}(r_u \circ \lambda_{g_\varphi^{-1}(u)}).T_u g_\varphi.\xi(u)\\ &= \xi(u) - T_e r_u.T_{g_\varphi(u)}\lambda_{g_\varphi^{-1}(u)}.T_u g_\varphi.\xi(u)\\ &= \xi(u) - T_e r_u.\delta^{\mathrm{left}} g_\varphi.\xi(u).\end{aligned}$$

Denote by $\theta \in \Omega^1(G,\mathfrak{g})$ the Maurer–Cartan form of the Lie group G, then for $\xi \in \mathfrak{X}(P)^G$ we get

$$(\varphi^*\xi) = \xi - \jmath((g_\varphi^*\theta)(\xi)).$$

I we get this together we obtain

$$\begin{aligned}\varphi_*^{-1} \circ \omega_{\eta_0}(\xi) &= (\Phi^{TP})^{-1}(\varphi^*(\Phi^{TP}(\omega_{\eta_0}(\xi))))\\ &= (\Phi^{TP})^{-1}(\Phi^{TP}(\omega_{\eta_0}(\xi)) - j((g_\varphi^*\theta)(\Phi^{TP}(\omega_{\eta_0}(\xi)))))\\ &= \omega_{\eta_0}(\xi) - (\Phi^{TP})^{-1}(j((g_\varphi^*\theta)(\Phi^{TP}(\omega_{\eta_0}(\xi)))))\\ &= \omega_{\eta_0}(\xi) - (i_* \circ (\Phi^{P\times\mathfrak{g}})^{-1})((g_\varphi^*\theta)(\Phi^{TP}(\omega_{\eta_0}(\xi)))),\end{aligned}$$

thus we are done. ♠

Lemma 29. Let η be a principal Lie algebroid connection with the connection form ω_η and let $\varphi \in \mathrm{Gau}(P)$ be a gauge transformation. Then for η and η^φ we get induced L-connections ∇ and ∇^φ on the associated vector bundle $E = P \times_G \mathbb{E}$ related by

$$\nabla^\varphi_\xi = \varphi_{\mathbb{E}}^{-1} \circ \nabla_\xi \circ \varphi_{\mathbb{E}}, \tag{3.41}$$

where $\xi \in \mathfrak{X}_L(M)$ and $\varphi_{\mathbb{E}}\colon E \to E$ is a uniquely determined homomorphism of vector bundles via $q \circ (\varphi \times \mathrm{id}_{\mathbb{E}}) = \varphi_{\mathbb{E}} \circ q$.

Proof. Denote by $\Phi\colon \Gamma(M, E) \to C^\infty(M, \mathbb{E})$ a $C^\infty(M, \mathbb{R})$-module isomorphism then we get

$$\begin{aligned}\nabla^\varphi_\xi s &= \Phi^{-1}(\Phi^{TP}(\omega_{\eta^\varphi}(\xi))\Phi(s)) = \Phi^{-1}(\Phi^{TP}(\varphi_*^{-1} \circ \omega_\eta(\xi))\Phi(s))\\ &= \Phi^{-1}(\Phi^{TP}(\omega_\eta(\xi))(\Phi(s) \circ \varphi^{-1}) \circ \varphi)\\ &= \Phi^{-1}(\Phi^{TP}(\omega_\eta(\xi))\Phi(\varphi_{\mathbb{E}} \circ s) \circ \varphi)\\ &= \varphi_{\mathbb{E}}^{-1} \circ \Phi^{-1}(\Phi^{TP}(\omega_\eta(\xi))\Phi(\varphi_{\mathbb{E}} \circ s))\\ &= \varphi_{\mathbb{E}}^{-1} \circ \nabla_\xi(\varphi_{\mathbb{E}} \circ s),\end{aligned}$$

therefore we have obtained the transformation rule for the induced L-connections. ♠

3.4 Holonomy

Let (P, p, M, G) be a principal fiber bundle and let $(L \xrightarrow{\pi} M, [\cdot,\cdot], a)$ be a Lie algebroid. Consider a principal Lie algebroid connection $\eta\colon p^*L \to TP$ with the connection form $\omega_\eta \in \Omega^1_L(M, \mathcal{A}(P))$.

If $\alpha\colon [0,1] \to L$ is an L-path with the base path $\gamma\colon [0,1] \to M$ then for any $u_0 \in P_{\gamma(0)}$ there exists a unique horizontal lift $\tilde{\gamma}\colon [0,1] \to P$ of α satisfying the system

$$\dot{\tilde{\gamma}}(t) = \eta(\tilde{\gamma}(t), \alpha(t)), \tag{3.42}$$

$$\tilde{\gamma}(0) = u_0. \tag{3.43}$$

For the proof see [10]. It is easy to see that $\gamma = p \circ \tilde{\gamma}$, i.e., $\tilde{\gamma}$ is a lift of γ to P. Therefore we can define a mapping $P_\alpha\colon P_{\gamma(0)} \to P_{\gamma(1)}$, called the *parallel transport along* α with respect to the connection η, as follows. If $u_0 \in P_{\gamma(0)}$ then we define

$$P_\alpha(u_0) = \tilde{\gamma}(1), \tag{3.44}$$

where $\tilde{\gamma}(t)$ is the unique horizontal lift of $\alpha(t)$ with $\tilde{\gamma}(0) = u_0$.

Let $\tilde{\gamma}\colon [0,1] \to P$ be a horizontal lift of α then $\tilde{\gamma}^g = r^g \circ \tilde{\gamma}\colon [0,1] \to P$ is also a horizontal lift of α, because

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\gamma}^g(t) = Tr^g.\dot{\tilde{\gamma}}(t) = Tr^g.\eta(\tilde{\gamma}(t), \alpha(t)) = \eta(\tilde{\gamma}(t).g, \alpha(t)) = \eta(\tilde{\gamma}^g(t), \alpha(t)),$$

where we used the fact that η is G-equivariant, i.e., $Tr^g \circ \eta = \eta \circ \hat{r}^g$. Now assume that $\tilde{\gamma}(0) = u_0$ then $\tilde{\gamma}^g(0) = u_0.g$ and we get $P_\alpha(u_0.g) = \tilde{\gamma}^g(1) = r^g(\tilde{\gamma}(1)) = P_\alpha(u_0).g$. Thus

$$P_\alpha \circ r^g = r^g \circ P_\alpha, \tag{3.45}$$

i.e., $P_\alpha\colon P_{\gamma(0)} \to P_{\gamma(1)}$ is a G-equivariant mapping and therefore a diffeomorphism.

Consider a diffeomorphism $\tau\colon [0,1] \to [0,1]$ and an L-path $\alpha\colon [0,1] \to L$ with its reparametrization given by $\alpha^\tau(t) = \tau'(t)\alpha(\tau(t))$. Further if $\widetilde{\gamma}\colon [0,1] \to P$ is a horizontal lift of α then $\widetilde{\gamma}^\tau = \widetilde{\gamma} \circ \tau$ is a horizontal lift of α^τ, because

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\gamma}^\tau(t) = \tau'(t)\dot{\widetilde{\gamma}}(\tau(t)) = \tau'(t)\eta(\widetilde{\gamma}(\tau(t)), \alpha(\tau(t))) = \eta(\widetilde{\gamma}^\tau(t), \alpha^\tau(t)).$$

Now suppose that $\tau(0) = 0$ and $\tau(1) = 1$, i.e., τ is an orientation preserving diffeomorphism. In case $\widetilde{\gamma}(0) = u_0$ then $\widetilde{\gamma}^\tau(0) = u_0$ and we have

$$P_{\alpha^\tau}(u_0) = \widetilde{\gamma}^\tau(1) = \widetilde{\gamma}(\tau(1)) = \widetilde{\gamma}(1) = P_\alpha(u_0).$$

On the other hand if $\tau(0) = 1$ and $\tau(1) = 0$, i.e., τ is an orientation non-preserving diffeomorphism then provided that $\widetilde{\gamma}(0) = u_0$ and $\widetilde{\gamma}(1) = u_1$ we get

$$P_{\alpha^\tau}(u_1) = \widetilde{\gamma}^\tau(1) = \widetilde{\gamma}(\tau(1)) = \widetilde{\gamma}(0) = u_0 = P_\alpha^{-1}(u_1).$$

Because P_α is a bijection, so $P_{\alpha^\tau} = P_\alpha^{-1}$. For any L-path α we will denote by $\bar{\alpha}$ an L-path defined by $\bar{\alpha}(t) = -\alpha(1-t)$. From the previous we have $P_{\bar{\alpha}} = (P_\alpha)^{-1}$.

Further if α_0 and α_1 are composable L-paths, i.e., $\pi(\alpha_0(1)) = \pi(\alpha_1(0))$, and $\alpha = \alpha_1 \cdot \alpha_0$, then $P_\alpha = P_{\alpha_1} \circ P_{\alpha_0}$. Let $\widetilde{\gamma}_0$ be a horizontal lift of α_0 with $\widetilde{\gamma}_0(0) = u_0$ and $\widetilde{\gamma}_1$ be a horizontal lift of α_1 with $\widetilde{\gamma}_1(0) = \widetilde{\gamma}_0(1)$. Then $\widetilde{\gamma}\colon [0,1] \to P$ defined by

$$\widetilde{\gamma}(t) = \begin{cases} \widetilde{\gamma}_0^\tau(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ \widetilde{\gamma}_1^\tau(2t-1) & \text{for } \frac{1}{2} \le t \le 1, \end{cases}$$

is a horizontal lift of $\alpha = \alpha_1 \cdot \alpha_0 = \alpha_1^\tau \odot \alpha_0^\tau$, where τ is any cutoff function. Because $\widetilde{\gamma}(0) = \widetilde{\gamma}_0^\tau(0) = u_0$, so

$$P_\alpha(u_0) = \widetilde{\gamma}(1) = \widetilde{\gamma}_1(1) = P_{\alpha_1}(\widetilde{\gamma}_1(0)) = P_{\alpha_1}(\widetilde{\gamma}_0(1)) = P_{\alpha_1}(P_{\alpha_0}(u_0)).$$

Moreover we see that P_α does not depend on a cutoff function τ.

An L-path α for which the base path γ is a loop based at x, i.e, $x = \gamma(0) = \gamma(1)$, will be called an L-loop based at x. For any L-loop α based at x we have a G-equivariant diffeomorphism $P_\alpha\colon P_x \to P_x$.

For fixed $x_0 \in M$ we define the *holonomy group* $\mathrm{Hol}(\eta, x_0) \subset \mathrm{Diff}(P_{x_0})$ as the group of all $P_\alpha\colon P_{x_0} \to P_{x_0}$ for α any L-loop based at x_0. If we consider only those L-loops which are L-homotopic to the constant trivial L-loop 0_{x_0} based at x_0 then we obtain the *restricted holonomy group* $\mathrm{Hol}_0(\eta, x_0)$.

Let us fix $u_0 \in P_{x_0}$ then the elements $\tau^G(u_0, P_\alpha(u_0)) \in G$ for all L-loops based at x_0 form a subgroup of the structure group G. We will denote it by $\mathrm{Hol}(\omega_\eta, u_0)$ and call it the *holonomy group*. Restricting only to the L-loops which are L-homotopic to the constant trivial L-loop 0_{x_0} we get the *restricted holonomy group* $\mathrm{Hol}_0(\omega_\eta, u_0)$.

Theorem 17. Let (P, p, M, G) be a principal fiber bundle and $(L \xrightarrow{\pi} M, [\cdot,\cdot], a)$ be a Lie algebroid. Consider a principal Lie algebroid connection η and fix $x_0 \in M$ and $u_0 \in P_{x_0}$.

i) We have an isomorphism $\mathrm{Hol}(\omega_\eta, u_0) \xrightarrow{\sim} \mathrm{Hol}(\eta, x_0)$ given by

$$g \mapsto (u \mapsto f_g(u) = u_0.g.\tau^G(u_0, u)) \quad \text{with the inverse} \quad f \mapsto g_f = \tau^G(u_0, f(u_0)).$$

ii) We have $\mathrm{Hol}(\omega_\eta, u_0.g) = \mathrm{conj}_{g^{-1}} \mathrm{Hol}(\omega_\eta, u_0)$ and $\mathrm{Hol}_0(\omega_\eta, u_0.g) = \mathrm{conj}_{g^{-1}} \mathrm{Hol}_0(\omega_\eta, u_0)$.

iii) We have $\mathrm{Hol}(\omega_\eta, P_\alpha(u_0)) = \mathrm{Hol}(\omega_\eta, u_0)$ and $\mathrm{Hol}_0(\omega_\eta, P_\alpha(u_0)) = \mathrm{Hol}_0(\omega_\eta, u_0)$ for each L-path α with $\pi(\alpha(0)) = x_0$.

Proof. i) If $g \in \mathrm{Hol}(\omega_\eta, u_0)$ then there exists an L-loop α based at x_0 such that $\tau^G(u_0, P_\alpha(u_0)) = g$ or in other words $P_\alpha(u_0) = u_0.g$. Because P_α is G-equivariant, we get $P_\alpha(u) = P_\alpha(u_0.\tau^G(u_0,u)) = P_\alpha(u_0).\tau^G(u_0,u) = u_0.g.\tau^G(u_0,u) = f_g(u)$. Further it is easy to see that $g \mapsto f_g$ is a group homomorphism. The rest of the proof follows from the definition of $\mathrm{Hol}(\omega_\eta, u_0)$ and $\mathrm{Hol}(\eta, x_0)$.
ii) This follows from the properties of the mapping τ^G and from the G-equivariance of the parallel transport. Since we have

$$\tau^G(u_0.g, P_\alpha(u_0.g)) = \tau^G(u_0.g, P_\alpha(u_0).g) = g^{-1}.\tau^G(u_0, P_\alpha(u_0)).g.$$

iii) Denote $u_1 = P_\alpha(u_0)$, then by definition $g \in \mathrm{Hol}(\omega_\eta, u_1)$ if and only if $g = \tau^G(u_1, P_\beta(u_1))$ for some L-loop β based at $x_1 = \pi(\alpha(1)) = p(u_1)$. Moreover we have

$$P_\alpha(u_0.g) = P_\alpha(u_0).g = u_1.g = P_\beta(u_1) = P_\beta(P_\alpha(u_0)),$$
$$u_0.g = ((P_\alpha)^{-1} \circ P_\beta \circ P_\alpha)(u_0) = P_{\bar\alpha\cdot\beta\cdot\alpha}(u_0)$$

and this is equivalent to $g \in \mathrm{Hol}(\omega_\eta, u_0)$. Furthermore β is L-homotopic to the trivial constant L-loop 0_{x_1} based at x_1 if and only if $\bar\alpha \cdot \beta \cdot \alpha$ is L-homotopic to the trivial constant L-loop 0_{x_1} based at x_1, so we also have $\mathrm{Hol}_0(\omega_\eta, P_\alpha(u_0)) = \mathrm{Hol}_0(\omega_\eta, u_0)$. ♠

Lemma 30. Let $\varphi \in \mathrm{Gau}(P)$ be a gauge transformation and α be an L-path. If P^η_α and $P^{\eta^\varphi}_\alpha$ denotes the parallel transport along α with respect to the connection η and η^φ then

$$\varphi_{\gamma(1)} \circ P^{\eta^\varphi}_\alpha = P^\eta_\alpha \circ \varphi_{\gamma(0)}, \tag{3.46}$$

where γ is the base path of α.

Proof. Let $\tilde\gamma$ be a horizontal lift of α with respect to the connection η then $\tilde\gamma^\varphi = \varphi^{-1} \circ \tilde\gamma$ is a horizontal lift of α with respect to the connection η^φ, as

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde\gamma^\varphi(t) = T\varphi^{-1}.\dot{\tilde\gamma}(t) = T\varphi^{-1}.\eta(\tilde\gamma(t),\alpha(t)) = (T\varphi^{-1} \circ \eta \circ \hat\varphi)(\varphi^{-1}(\tilde\gamma(t)),\alpha(t)) = \eta^\varphi(\tilde\gamma^\varphi(t),\alpha(t)).$$

Therefore in case $\tilde\gamma(0) = u_0 \in P_{\gamma(0)}$ we have

$$(\varphi^{-1}_{\gamma(1)} \circ P^\eta_\alpha)(u_0) = \varphi^{-1}(\tilde\gamma(1)) = \tilde\gamma^\varphi(1) = P^{\eta^\varphi}_\alpha(\tilde\gamma^\varphi(0)) = P^{\eta^\varphi}_\alpha(\varphi^{-1}(u_0)) = (P^{\eta^\varphi}_\alpha \circ \varphi^{-1}_{\gamma(0)})(u_0),$$

thus we are done. ♠

From now on we will assume that $(L \xrightarrow{\pi} M, [\cdot\,,\cdot], a)$ is a transitive Lie algebroid, i.e., $a\colon L \to TM$ is surjective, and that M is a connected manifold. Then M is an orbit of L, i.e., for any two points $x, y \in M$ there exists an L-path α, with base path γ, such that $\gamma(0) = x$ and $\gamma(1) = y$.

Let (P, p, M, G) be a principal fiber bundle and $x_0 \in M$. Then we consider the group $\mathrm{Gau}_{x_0}(P)$, called the *restricted group of gauge transformations*, of those gauge transformations which are the identity on P_{x_0}. It is easy to see that this group is a normal subgroup of the group of gauge transformations $\mathrm{Gau}(P)$. Further for any $u_0 \in P_{x_0}$ we define a group homomorphism $\lambda_{u_0}\colon \mathrm{Gau}(P) \to G$ by

$$\lambda_{u_0}(\varphi) = \tau^G(u_0, \varphi(u_0)) = g_\varphi(u_0). \tag{3.47}$$

Because λ_{u_0} is surjective, we get an exact sequence

$$\{e\} \longrightarrow \mathrm{Gau}_{x_0}(P) \longrightarrow \mathrm{Gau}(P) \xrightarrow{\lambda_{u_0}} G \longrightarrow \{e\} \tag{3.48}$$

of groups. Hence we have an isomorphism $\mathrm{Gau}(P)/\,\mathrm{Gau}_{x_0}(P) \simeq G$ of groups.

Now we take up the question of reducible connections. Given a principal Lie algebroid connection η then the *stabilizer* or the *isotropy subgroup* of η is the subgroup $\mathrm{Gau}(P)_\eta$ of $\mathrm{Gau}(P)$ that leaves η fixed, i.e.,

$$\mathrm{Gau}(P)_\eta = \{\varphi \in \mathrm{Gau}(P);\ \eta\cdot\varphi = \eta\}. \tag{3.49}$$

Denote by $Z(G)$ the center of the structure group G. Then for any $h \in Z(G)$ we have $r^h \in \mathrm{Gau}(P)$ and because η is G-equivariant we obtain $r^h \in \mathrm{Gau}(P)_\eta$. Therefore any isotropy subgroup $\mathrm{Gau}(P)_\eta$ contains the subgroup isomorphic to $Z(G)$.

Definition 20. A principal Lie algebroid connection η with the connection form ω_η is called *irreducible*, if $\mathrm{Gau}(P)_\eta = \{r^h;\, h \in Z(G)\} \simeq Z(G)$, otherwise η is called *reducible.*[1] Further we will denote the set of all irreducible connection forms by $\mathcal{A}^*(P, L)$ and the set of all irreducible flat connection forms by $\mathcal{H}^*(P, L)$.

Lemma 31. The restricted group of gauge transformations $\mathrm{Gau}_{x_0}(P)$ acts freely on the space of connection forms $\mathcal{A}(P, L)$.

Proof. Let η be a principal Lie algebroid connection and consider $\varphi \in \mathrm{Gau}_{x_0}(P)$ satisfying $\eta^\varphi = \eta$. Because M is an orbit of L, for any $x \in M$ there exists an L-path α such that $\pi(\alpha(0)) = x_0$ and $\pi(\alpha(1)) = x$. Using Lemma 30 we obtain

$$\varphi_x \circ P^\eta_\alpha = \varphi_x \circ P^{\eta^\varphi}_\alpha = P^\eta_\alpha \circ \varphi_{x_0}.$$

Therefore we have $\varphi_x = P^\eta_\alpha \circ \varphi_{x_0} \circ (P^\eta_\alpha)^{-1} = P^\eta_\alpha \circ (P^\eta_\alpha)^{-1} = \mathrm{id}_{P_x}$, because $\varphi_{x_0} = \mathrm{id}_{P_{x_0}}$. Thus we have proved that $\varphi = \mathrm{id}_P$, hence $\mathrm{Gau}_{x_0}(P)$ acts on $\mathcal{A}(P, L)$ freely. ♠

Theorem 18. Let (P, p, M, G) be a principal fiber bundle and $(L \xrightarrow{\pi} M, [\cdot\,,\cdot], a)$ be a Lie algebroid. Consider a principal Lie algebroid connection η and fix $u_0 \in P_{x_0}$. Then $\lambda_{u_0} \colon \mathrm{Gau}(P)_\eta \xrightarrow{\sim} Z_G(\mathrm{Hol}(\omega_\eta, u_0))$ is a group isomorphism.

Proof. First we prove that $\lambda_{u_0} \colon \mathrm{Gau}(P)_\eta \to G$ is injective. Consider $\varphi_1, \varphi_2 \in \mathrm{Gau}(P)_\eta$ such that $\lambda_{u_0}(\varphi_1) = \lambda_{u_0}(\varphi_2)$. Because $\lambda_{u_0}(\varphi_1^{-1} \circ \varphi_2) = e$, using exactness of the sequence (3.48), we get $\varphi_1^{-1} \circ \varphi_2 \in \mathrm{Gau}_{x_0}(P)$. Furthermore we have $\eta \cdot (\varphi_1^{-1} \circ \varphi_2) = \eta$, but from Lemma 31 we know that $\mathrm{Gau}_{x_0}(P)$ acts freely on $\mathcal{A}(P, L)$ hence $\varphi_1^{-1} \circ \varphi_2 = \mathrm{id}_P$. Thus λ_{u_0} restricted on $\mathrm{Gau}(P)_\eta$ is injective.

Now for any $g \in \mathrm{Hol}(\omega_\eta, u_0)$ there exists an L-loop α based at x_0 satisfying $P^\eta_\alpha(u_0) = u_0.g$. In case $\varphi \in \mathrm{Gau}(P)_\eta$ then from Lemma 30 we obtain $\varphi_{x_0} \circ P^\eta_\alpha = P^\eta_\alpha \circ \varphi_{x_0}$. Therefore we have $(\varphi_{x_0} \circ P^\eta_\alpha)(u_0) = \varphi(u_0.g) = \varphi(u_0).g = u_0.\lambda_{u_0}(\varphi).g$ and $(P^\eta_\alpha \circ \varphi_{x_0})(u_0) = P^\eta_\alpha(u_0.\lambda_{u_0}(\varphi)) = u_0.g.\lambda_{u_0}(\varphi)$. Because the principal right action G on P is free, from $u_0.\lambda_{u_0}(\varphi).g = u_0.g.\lambda_{u_0}(\varphi)$ we obtain $\lambda_{u_0}(\varphi).g = g.\lambda_{u_0}(\varphi)$, i.e., $\lambda_{u_0}(\varphi) \in Z_G(\mathrm{Hol}(\omega_\eta, u_0))$.

To prove the whole statement we need to verify that for any $g \in Z_G(\mathrm{Hol}(\omega_\eta, u_0))$ there exists $\varphi \in \mathrm{Gau}(P)_\eta$ satisfying $\lambda_{u_0}(\varphi) = g$. First we define $\varphi_{x_0} \colon P_{x_0} \to P_{x_0}$ by $\varphi_{x_0}(u) = u_0.g.\tau^G(u_0, u)$ for any $u \in P_{x_0}$. Because φ_{x_0} is G-equivariant, we have $\varphi_{x_0} \in \mathrm{Diff}(P_{x_0})$. Further for any $x \in M$ there exists an L-path α such that $\pi(\alpha(0)) = x_0$ and $\pi(\alpha(1)) = x$. Hence we define $\varphi_x \colon P_x \to P_x$ by $\varphi_x = P^\eta_\alpha \circ \varphi_{x_0} \circ (P^\eta_\alpha)^{-1}$. It is easy to see that φ_x is G-equivariant and thus $\varphi_x \in \mathrm{Diff}(P_x)$. But we need to verify that this definition of φ_x does not depend on the choice of an L-path form x_0 to x. Thus let β be another L-path satisfying $\pi(\beta(0)) = x_0$ and $\pi(\beta(1)) = x$. Then $P^\eta_\alpha \circ \varphi_{x_0} \circ (P^\eta_\alpha)^{-1} = P^\eta_\beta \circ \varphi_{x_0} \circ (P^\eta_\beta)^{-1}$ if and only if $\varphi_{x_0} \circ P^\eta_{\bar\alpha\cdot\beta} = P^\eta_{\bar\alpha\cdot\beta} \circ \varphi_{x_0}$. Because $P^\eta_{\bar\alpha\cdot\beta} \in \mathrm{Hol}(\eta, x_0)$, we have $P^\eta_{\bar\alpha\cdot\beta}(u_0) = u_0.h$, where $h \in \mathrm{Hol}(\omega_\eta, u_0)$. Further for any $u \in P_{x_0}$ we may write

$$\begin{aligned}
(\varphi_{x_0} \circ P^\eta_{\bar\alpha\cdot\beta})(u) &= (\varphi_{x_0} \circ P^\eta_{\bar\alpha\cdot\beta})(u_0.\tau^G(u_0, u)) = (\varphi_{x_0}(P^\eta_{\bar\alpha\cdot\beta}(u_0))).\tau^G(u_0, u)\\
&= (\varphi_{x_0}(u_0.h)).\tau^G(u_0, u) = u_0.g.h.\tau^G(u_0, u)\\
&= u_0.h.g.\tau^G(u_0, u) = (P^\eta_{\bar\alpha\cdot\beta}(u_0.g)).\tau^G(u_0, u)\\
&= (P^\eta_{\bar\alpha\cdot\beta}(\varphi_{x_0}(u_0))).\tau^G(u_0, u) = (P^\eta_{\bar\alpha\cdot\beta} \circ \varphi_{x_0})(u_0.\tau^G(u_0, u))\\
&= (P^\eta_{\bar\alpha\cdot\beta} \circ \varphi_{x_0})(u),
\end{aligned}$$

therefore $\varphi_x \colon P_x \to P_x$ is well defined. Thence we have constructed a G-equivariant mapping $\varphi \colon P \to P$ such that $p \circ \varphi = p$. We have to verify that φ is a diffeomorphism.

[1] In the literature the them 'irreducible' is sometimes used only for connections with maximal holonomy; such connections have in particular a trivial stabilizer.

Let $(U_\alpha, \varphi_\alpha)$ be a principal bundle atlas for P with transitions functions $\varphi_{\alpha\beta}\colon U_{\alpha\beta} \to G$ and assume, by shrinking U_α if necessary, that U_α are contractible. Consider the local sections $s_\alpha \in \Gamma(U_\alpha, P)$ given by $\varphi_\alpha(s_\alpha(x)) = (x, e)$. Because for $x \in U_{\alpha\beta}$ we have

$$s_\alpha(x).\varphi_{\alpha\beta}(x) = \varphi_\alpha^{-1}(x,e).\varphi_{\alpha\beta}(x) = \varphi_\alpha^{-1}(x, e.\varphi_{\alpha\beta}(x)) = \varphi_\beta^{-1}(x,e) = s_\beta(x),$$

thus $s_\alpha.\varphi_{\alpha\beta} = s_\beta$. Further for $\varphi_\alpha \circ \varphi \circ \varphi_\beta^{-1}\colon U_{\alpha\beta} \times G \to U_{\alpha\beta} \times G$ we can write

$$(\varphi_\alpha \circ \varphi \circ \varphi_\beta^{-1})(x, g) = (\varphi_\alpha \circ \varphi)(\varphi_\beta^{-1}(x,e).g) = (\varphi_\alpha \circ \varphi)(s_\beta(x).g) = \varphi_\alpha(\varphi(s_\beta(x)).g).$$

Therefore to prove the smoothness of φ it is enough to show that $\varphi \circ s_\alpha$ is a smooth local section. Now fix $x \in U_\alpha$, because U_α is contractible, thus there exists a smooth homotopy $\gamma\colon [0,1] \times U_\alpha \to U_\alpha$ such that $\gamma(0, y) = y$ and $\gamma(1, y) = x$ for all $y \in U_\alpha$. Since $a\colon L \to TM$ is surjective, there exists a smooth mapping $\alpha\colon [0,1] \times U_\alpha \to L$ satisfying

$$a(\alpha(t,y)) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t,y),$$

i.e., $\alpha(\cdot, y)\colon [0,1] \to L$ is an L-path with the base path $\gamma(\cdot, y)\colon [0,1] \to M$ such that $\pi(\alpha(0,y)) = y$ and $\pi(\alpha(1,y)) = x$. Further there exists a unique horizontal lift $\widetilde{\gamma}\colon [0,1] \times U_\alpha \to P$ satisfying

$$\begin{aligned}\dot{\widetilde{\gamma}}(t,y) &= \eta(\widetilde{\gamma}(t,y), \alpha(t,y))\\ \widetilde{\gamma}(0,y) &= s_\alpha(y).\end{aligned}$$

Now let β be an L-path from x_0 to x. Thus for any $y \in U_\alpha$ we can write

$$\varphi_y = P^\eta_{\bar{\alpha}^y} \circ P^\eta_\beta \circ \varphi_{x_0} \circ P^\eta_{\bar{\beta}} \circ P^\eta_{\alpha^y} = P^\eta_{\bar{\alpha}^y} \circ \varphi_x \circ P^\eta_{\alpha^y}$$

and we obtain

$$\begin{aligned}(\varphi \circ s_\alpha)(y) &= \varphi_y(s_\alpha(y)) = (P^\eta_{\bar{\alpha}^y} \circ \varphi_x \circ P^\eta_{\alpha^y})(s_\alpha(y)) = (P^\eta_{\bar{\alpha}^y} \circ \varphi_x)(\widetilde{\gamma}(1,y))\\ &= P^\eta_{\bar{\alpha}^y}(\widetilde{\gamma}(1,y)).\tau^G(\widetilde{\gamma}(1,y), \varphi_x(\widetilde{\gamma}(1,y))\\ &= \widetilde{\gamma}(0,y).\tau^G(\widetilde{\gamma}(1,y), \varphi_x(\widetilde{\gamma}(1,y))\\ &= s_\alpha(y).\tau^G(\widetilde{\gamma}(1,y), \varphi_x(\widetilde{\gamma}(1,y)).\end{aligned}$$

Because $\varphi_x\colon P_x \to P_x$ is a smooth mapping, so $\varphi \circ s_\alpha$ is also smooth. Therefore we have prove the smoothness of φ. As $\varphi_y^{-1} = P^\eta_{\bar{\alpha}^y} \circ \varphi_x^{-1} \circ P^\eta_{\alpha^y}$, by the same argument we obtain that $\varphi^{-1} \circ s_\alpha$ is smooth since φ_x^{-1} is a smooth mapping. Therefore we have $\varphi \in \mathrm{Gau}(P)$.

From the definition of φ we get $\lambda_{u_0}(\varphi) = \tau^G(u_0, \varphi(u_0)) = \tau^G(u_0, \varphi_{x_0}(u_0)) = g$. The last step is to verify that $\eta^\varphi = \eta$. For any $x \in M$ and $\xi_x \in L_x$ there exists an L-path α such that $\pi(\alpha(0)) = x_0$, $\pi(\alpha(1)) = x$ and $\alpha(1) = \xi_x$. From Lemma 30 we obtain that $\varphi_x \circ P^{\eta^\varphi}_\alpha = P^\eta_\alpha \circ \varphi_{x_0}$ but using the definition of φ_x we have $\varphi_x \circ P^\eta_\alpha = P^\eta_\alpha \circ \varphi_{x_0}$, therefore we obtain

$$P^{\eta^\varphi}_\alpha = P^\eta_\alpha.$$

Further for any $u_x \in P_x$ there exist a unique $u_{x_0} \in P_{x_0}$ and a unique horizontal lift $\widetilde{\gamma}$, $\widetilde{\gamma}_\varphi$ of the L-path α with respect to η, η^φ respectively satisfying $\widetilde{\gamma}(0) = u_{x_0}$ and $\widetilde{\gamma}_\varphi(0) = u_{x_0}$. Let $t_0 \in (0,1]$ and define a mapping $\tau\colon [0,1] \to [0, t_0]$ by $\tau(t) = t_0 t$. Then $\alpha^\tau\colon [0,1] \to L$ given by $\alpha^\tau(t) = t_0\alpha(t_0 t)$ is an L-path. If we define $\widetilde{\gamma}^\tau = \widetilde{\gamma} \circ \tau$ and $\widetilde{\gamma}^\tau_\varphi = \widetilde{\gamma}_\varphi \circ \tau$ then $\widetilde{\gamma}^\tau$, $\widetilde{\gamma}^\tau_\varphi$ is a horizontal lift of α^τ with respect to η, η^φ respectively such that $\widetilde{\gamma}^\tau(0) = u_{x_0}$ and $\widetilde{\gamma}^\tau_\varphi(0) = u_{x_0}$. Furthermore because

$$P^{\eta^\varphi}_{\alpha^\tau} = P^\eta_{\alpha^\tau},$$

we get $\widetilde{\gamma}(t_0) = \widetilde{\gamma}^\tau(1) = \widetilde{\gamma}^\tau_\varphi(1) = \widetilde{\gamma}_\varphi(t_0)$ for any $t_0 \in (0,1]$. Moreover we have $\widetilde{\gamma}(0) = \widetilde{\gamma}_\varphi(0)$, thus we get $\widetilde{\gamma} = \widetilde{\gamma}_\varphi$ which implies

$$\dot{\widetilde{\gamma}}(t) = \dot{\widetilde{\gamma}}_\varphi(t)$$

or in other words

$$\eta(\widetilde{\gamma}(t), \alpha(t)) = \eta^\varphi(\widetilde{\gamma}_\varphi(t), \alpha(t)) = \eta^\varphi(\widetilde{\gamma}(t), \alpha(t))$$

for all $t \in [0,1]$. In case $t = 1$ we get $\eta(u_x, \xi_x) = \eta^\varphi(u_x, \xi_x)$, where $u_x = \widetilde{\gamma}(1)$ and $\xi_x = \alpha(1)$. Because u_x, ξ_x were arbitrary we have proved that $\eta^\varphi = \eta$. ♠

Let η be a principal Lie algebroid connection such that for some (equivalently any) $u_0 \in P_{x_0}$ we have $\mathrm{Hol}(\omega_\eta, u_0) = G$. From the previous theorem we obtain an isomorphism between the isotropy group $\mathrm{Gau}(P)_\eta$ and $Z_G(\mathrm{Hol}(\omega_\eta, u_0)) = Z_G(G) = Z(G)$ given by $\lambda_{u_0}(\varphi) = \tau^G(u_0, \varphi(u_0))$. Because $\{r^h;\, h \in Z(G)\} \subset \mathrm{Gau}(P)_\eta$ and $\lambda_{u_0}(\{r^h;\, h \in Z(G)\}) = Z(G)$, thus we get $\mathrm{Gau}(P)_\eta = \{r^h;\, h \in Z(G)\}$, i.e., η is irredicible.

For example if $G = \mathrm{SU}(2)$ then the possibilities for the holonomy group are following. First, the holonomy group can be $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$. In both cases the centralizer is equal to the center $Z(\mathrm{SU}(2)) = \mathbb{Z}_2$. Secondly, the holonomy group may be $\mathrm{U}(1)$ and the centralizer is isomorphic to $\mathrm{U}(1)$. Finally, the holonomy group may be trivial hence the centralizer is equal to $\mathrm{SU}(2)$.

Remark. From the fact that $\mathrm{Gau}(P)_{\eta^\varphi} = \mathrm{conj}_{\varphi^{-1}} \mathrm{Gau}(P)_\eta$ for any gauge transformation φ and any principal Lie algebroid connection η, it follows that $\mathcal{A}^*(P, L)$ is invariant under the action of $\mathrm{Gau}(P)$ and the same for $\mathcal{H}^*(P, L)$. Therefore we may define, similarly like in (3.38) and (3.39), the *moduli space*

$$\mathcal{B}^*(P, L) = \mathcal{A}^*(P, L)/\mathrm{Gau}(P) \tag{3.50}$$

of gauge equivalence classes of irreducible principal Lie algebroid connections and the *moduli space*

$$\mathcal{M}^*(P, L) = \mathcal{H}^*(P, L)/\mathrm{Gau}(P) \tag{3.51}$$

of gauge equivalence classes of irreducible flat principal Lie algebroid connections.

Remark. If we define the *reduced group of gauge transformations* $\mathrm{Gau}(P)^{\mathrm{r}}$ by

$$\mathrm{Gau}(P)^{\mathrm{r}} = \mathrm{Gau}(E)/\{r^h;\, h \in Z(G)\}, \tag{3.52}$$

then the right action of $\mathrm{Gau}(P)$ on $\mathcal{A}(P, L)$ factors trough an action of the reduced group of gauge transformations $\mathrm{Gau}(P)^{\mathrm{r}}$ since $\{r^h;\, h \in Z(G)\}$ acts trivially on $\mathcal{P}(E, L)$, similarly for $\mathcal{H}(P, L)$. The set $\mathcal{A}^*(P, L)$ of all irreducible connection forms is the maximal subset of $\mathcal{A}(P, L)$ on which the reduced group of gauge transformations $\mathrm{Gau}(P)^{\mathrm{r}}$ acts freely, likewise for $\mathcal{H}^*(P, L)$.

Conclusion

It seems that Lie algebroid connections on fiber bundles, in particular on vector bundles and principal fiber bundles could have very interesting applications in mathematics and physics. Something was already outlined in the introduction. We sketch one remarkable generalization of the well-known fact for Lie algebroid connections which could be the next step in the subsequent work.

The twenty-first on the list of twenty-three problems presented by David Hilbert in 1900 was *the proof of the existence of linear differential equations having a prescribed monodromic group.* By the monodromy group of a linear differential equation we get a representation of the fundamental group of the base space. The problem asks for it converse: for any representation of the fundamental group, is there an ordinary differential equation (with regular singularities) whose monodromy representation coincides with the given one? (There exists several points of view in formulating this problem more precisely.) This problem is commonly called the *Riemann–Hilbert problem.*

A generalization of this problem to higher dimensions is called the *Riemann–Hilbert correspondence.* Let X be a connected compact manifold and let G be a Lie group. A G-local system on X is a principal fiber bundle (P, p, X, G) with a flat principal connection ω. To any flat principal connection ω on P we can assign, using the Ambrose–Singer theorem, a group homomorphism $\pi_1(X, x_0) \to G$. This is the monodromy representation given by the parallel transport. If we denote by $\mathcal{L}oc_G(X)$ the moduli space of G-local systems on X we get an isomorphism

$$\mathrm{Hom}(\pi_1(X, x_0), G)/G \simeq \mathcal{L}oc_G(X),$$

called the Riemann–Hilbert correspondence. The moduli space on the left hand side is called the *character variety.* There is now a modern (D-module and derived category) version of the Riemann–Hilbert correspondence, see [41], [42], [43], [44] and [45]. This correspondence has many applications and plays a significant role in the geometric Langlands program.

For a principal Lie algebroid connection on a principal fiber bundle we can define the parallel transport and the holomony group as we saw in Chapter 3. A natural generalization is to replace the right hand side of this correspondence by $\mathcal{L}oc_G^L(X)$ the moduli space of G-local systems on X for a fixed Lie algebroid L. A G-local system on X for the Lie algebroid L is a principal fiber bundle (P, p, X, G) with a flat principal Lie algebroid connection η. The left hand side then should be replaced by equivalence classes of homomorphism from $\mathcal{G}^L(x_0) \to G$, where $\mathcal{G}^L$ is a Lie groupoid over X associated to the Lie algebroid L, the so called *Weinstein groupoid*, and $\mathcal{G}^L(x_0)$ is a group over the corresponding point, see [46], [47].

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