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DIPLOMOVÁ PRÁCE

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Fyzikální modelování dynamiky peněz ve směnné ekonomice

Ústav teoretické fyziky

Vedoucí diplomové práce: RNDr. František Slanina, CSc.
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Na tomto místě bych rád poděkoval svému vedoucímu práce RNDr. Františku Slaninovi, CSc., který mi byl po celé dva roky průvodcem ve světě ekonofyziky a rádcem při přípravě tohoto textu.

Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

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Abstrakt: Předkládaná diplomová práce patří do interdisciplinárního oboru zvaného ekonofyzika. V této práci se snažíme o modelování přechodu od prosté vzájemné výměny zboží ke komoditním penězům, které byly předchůdci dnešních bankovek. K popisu takovéhoto přechodu používáme mikroskopickou strukturu interagujících agentů, z jejichž chování povstávají měřitelné makroekonomické veličiny. Ukazujeme, že vznik komoditních peněz je možný jako důsledek kooperativního chování jednotlivých účastníků trhu. Sledujeme různé statistické charakteristiky procesu vzniku a trvání komoditních peněz a model dále vhodně modifikujeme tak, aby byl co nejvíce obecný a umožňoval nám na něm měřit další pozorovatelné veličiny. Počítačové simulace doplňujeme analytickým výpočtem, při kterém využíváme teorie středního pole a reprezentativního agenta. S pomocí metod statistické fyziky vypočteme stacionární bod pro případ homogenního rozložení produktů v portfoliu reprezentativního agenta.
Klíčová slova: ekonofyzika, vznik peněz, agentní model

Title: Physical modelling of the dynamics of money in exchange economy
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Abstract: The presented diploma thesis belongs to the interdisciplinary area called econophysics. In this work we try to model the transition from simple barter trade to commodity money which was the forerunner of today's bank-notes. For the description of such a transition we use microscopic structure of interacting agents which can give rise to measurable macroeconomic quantities. We show that the emergence of commodity money is possible as a result of the cooperative behaviour of individual traders. We observe several statistical features of the process of money emergence and its duration and we modify the model in such a way that it would be as general as possible and allow us to measure other observable quantities. We supplement the computer simulations with analytical approach in which we use the mean field theory and the theory of a representative agent. Using the methods of statistical physics we calculate equilibrium for the case of the homegeneous distribution of commodities in representative agent's portfolio.
Keywords: econophysics, money emergence, agent-based model

Chapter 1

Introduction

In our present work we study the dynamics of money in exchange economy. This topic belongs to a quite new interdisciplinary field called econophysics. For those who are unfamiliar with the wide variety of problems econophysicists deal with we will summarize here the most important problems and their possible solutions.

Physics is a discipline with long history and highly sophisticated mathematical methods which were often used by physicists earlier than by mathematicians themselves. The methods developed primarily for the description of physical systems could be used in many other fields of research. Especially the approaches of statistical physics find many applications in areas like sociology, computer science, finance and economics. Physical applications in the two latter became so vast that it resulted in establishment of the new specialization - econophysics.

A great deal of econophysics (but not all as we will see later) is based on the simulations of economy and financial markets as a system of interacting agents. These agent-based models stem from an idea that every market, however complicated, is composed of elementary components (usually the traders) whose interaction results in the dynamics of the financial market. The background of this approach is quite obvious because real-life markets' evolution is driven by the investors who buy and sell financial instruments in order to achieve their goals. When the investors collectively buy then the price increases and vice versa when the investors sell their shares the price of the corresponding asset diminishes.

There are several general characteristics of a financial market which every model should take into account. These characteristics are common for all markets with no regard to type or location of the particular market. We can say that they play the role of a certain arbiter for the proposed models. For example it can be empirically observed that the distributions of returns (changes of the prices) exhibit power-law tails or that the volatility (deviations of the prices) is strongly long-time autocorrelated. This phenomenon is called volatility clustering. Another typical feature of the market is the regular emergence of bubbles and their subsequent crashes. By chance we live nowadays (2009) in one of these recurrent critical periods. In the following text we present the most influential models in econophysics and compare their features with the above mentioned stylized facts. For more information about models of financial markets see [1].

The first numerical experiment dealing with the stock exchange market was G. J. Stigler's model [2], 1964, with agents who randomly place orders to sell or to buy. The price of the stock is then calculated from the performed trades. Even though the agents don't follow any strategy, the stock price fluctuates as in the real market and we can see the volatility clustering. The distributions of the returns do not exhibit fat tails so we cannot take the model too seriously but Stigler's work was really pioneering and another more successful simulations followed.

The first model dealing with crashes was proposed by Kim and Markowitz [3]. In their paper the authors tried to explore the stock market crash in 1987 when a sudden decrease of U.S. stock without any serious reason in the form of a new information influencing the market occured. Most economists blamed hedgers and portfolio insurers because it was thought that they contributed to the crash by increasing volatility. For this reason Kim and Markowitz attempted to investigate the relationship between hedging strategy and volatility with a simulated market containing two types of investors, so called rebalancers and portfolio insurers. The rebalancers' strategy is to keep one half of their wealth in stock and the other half in cash. This strategy stabilizes the market because when the price is increasing the traders sell stock to keep their wealth half to half and vice versa when the price is decreasing. The aim of portfolio insurers is to preserve the wealth at least on some minimal level (called floor). When the price is decreasing the portfolio insurer sells his stock in order to keep more of his fortune in riskless cash. This behaviour has evidently destabilizing effect. Although the model doesn't cover other important features of the crash and possesses a few imperfections, it shows that portfolio insurance strategies have destabilizing potential and could cause higher deviations of prices.

Kim and Markowitz used for the description of the behaviour of their agents different complicated strategies commonly applied in real financial markets. More recent models use much simpler rules of agents' behaviour. For example the approach proposed by M. Levy, H. Levy and Solomon [4] derives the behaviour of the agents from the traditional economic theory of utility which all participants of the market should maximize. Each agent dynamically allocates his whole wealth between two possible financial assets - risky shares or riskless bonds - according to the recent changes of prices. A random variable is added to the utility function in order to prevent the system to get stuck which also corresponds with individual preferences of each agent.

M. Levy, H. Levy and Solomon primarily attempted to answer questions about the nature and mechanisms of bubbles and crashes (and also were quite successful in explaining the crashes as a result of synchronous response of the investors to the past evolution of the market and information they were given) but the main problem of their model is that it lacks scaling laws (power-law tails). This inconvenience was partly solved in extensions of the model in [5] and [6].

Another interesting model which includes the emergence of scaling laws was devised by Lux and Marchesi [7]. These authors divide the agents randomly into 3 groups, the first group countains fundamentalists who regard the price fluctuations as deviations from its intrinsic value, the second and third group are chartists who believe they can guess the future prices from the past price series and either buy or sell according to their "knowledge". The substantial ingredient of the model is that actions of the agents affect the price changes which subsequently influence the ratio of agents in the groups. The result is that the model is able to permanently generate bubbles, caused by the overvaluation of assets, and market crashes.

The approach when agents act in bunches is surely a way to construct a system with properties of the power-law tail. First references about this phenomenom could be found for example in the sandpile model [8] which structually resembles avalanches on a pile of sand when we add grains on the top. The alternative to the sandpile model is the Cont-Bouchaud model [9] closely related to the effects of bond percolation.

Not all econophysical articles are interested only in the financial markets. For example a remarkable strategy describing the behaviour of ants was introduced by Kirman [10]. His stochastic model of binary choice simulates how the ants act to maintain continual supply of food. This model served Kirman as a paradigm for describing the behaviour of human agents imitating each other (so-called zero inteligence model).

It is interesting to note that some researchers performed experiments on humans. For example Jasmina Arifovic [11] simulated the market with the help of a group of students and some basic market rules.

All the models described above deal with situations from relatively recent history. On the contrary Sneppen and Donangelo introduced a model simulating economy at its birth [12] and in our thesis we would like to reassume their work, investigate it in depth and make some interesting extensions.

Chapter 2

Model of money emergence

Before money was established the usual mean of trade was barter, i.e. people exchanged goods in their property for another goods they needed. This way of trade is very ineffective for many reasons, especially because it makes storing wealth very difficult. Consequently the ancient civilizations usually used a very specific commodity as a mean of trade. This commodity was for example gold, silver, diamont, different fruits, tulips and so on. Even until the 1970's the money in the USA were partially based on the gold standard and up to this day the return to the gold standard is supported by many economists, especially those following the Austrian school of economics. All of these means of trade have common features like that they are very hard to find or grow and that they are very durable. We can pose a question how it is possible that so many people agreed on one preferred commodity and what is the nature of the emergence of such a pattern in trading.

The literature concerning money emergence is quite voluminous. In this work we will follow papers by Donangelo and Sneppen who draw their inspirations from Yasutomi [13] and Kiyotaki and Wright [14]. Another contribution to this theme was made for example by Jones [15]. Except the work [12] we are going to investigate in more detail Donangelo and Sneppen also wrote many other articles dealing with money emergence - we should at least mention [16] and [17].

In the paper [12] called Self-organization of value and demand Donangelo and Snepen simulate the early economy as a system of interacting agents who desire to complete their portfolios with products they lack by bartering with other traders. Beside this strategy they keep records about products which were asked from them. In case pure barter trade is not possible, the agents can buy products they already possess in their portfolios if the products are significantly present in the list of previous demand. The product with the greatest demand then plays the role of money, i.e. using money we are able to buy anything we want to because every seller accepts money. It was found out that the emergence of money is induced by the trading activity itself without any forces outside the system or any special properties imposed on the products. We will follow this direction and construct our model of economy with similar conditions and patterns of behaviour which we will later modify for better understanding of the nature of the emergence process.

2.1 Model description

We assume an economic system which will simulate an ancient civilization. This system will be isolated without any wealth coming in or out and with constant number of people (agents) who interact among themselves according to given rules. We model the behaviour of the agents as very selfish because they don't cooperate among themselves and pursue their own goals. There will be two kinds of a goal. First, each agent is envious and therefore if he finds something the other agent has and he doesn't, he attempts to make a deal to obtain this desired commodity. The second reason for trade will be if the agent doesn't find anything interesting but on the other hand he has an opportunity to buy something valuable and have this commodity prepared for a future exchange of the first type. The second way of trade can be interpreted as savings or delayed consumption.

The parameters of our simulated economy are the number of agents N_{ag}, the number of all different products in the system N_{pr} which can be traded with, the number of products each agent is given at the beginning N_{un} and the length of memory N_{mem} which limits the agent's local memory of trade opportunities. It is important to stress that the agents remember every attempt to trade and even unperformed trades are remembered. This feature describes common behaviour of market participants and without it the model loses its main properties.

The evolution of the system is fully governed by two matrices Mem and Por, both of them depending on discrete time $t \in \mathbf{N}$. Por is a portfolio matrix with dimensions $N_{ag} \times N_{pr}$ and it stores the information about what each agent owns. Each row is therefore composed of frequencies of commodities some agent possesses, Por_{ij} being the count of the j-th commodity for the i-th agent. When two agents agree on a trade, the matrix Por changes in two rows, in each row one element is increased by 1 unit and another one by 1 unit descreased. Mem is a matrix $N_{ag} \times N_{mem}$ which stores the information about prior encounters and in the i-th row articles attempted to trade with the i-th agent are saved.

The dynamics of the system is following. At first we fill the matrices Mem

and Por with suitable random data. The matrix Por is filled with nonnegative whole numbers which must fulfil the condition $\sum_{j=1}^{N_{pr}} Por_{ij} = N_{un}$ for all i if we want the wealth to be fairly distributed. The matrix Mem is filled with numbers $0, 1, 2, ..., (N_{pr} - 1)$, each number represents one product. After this initiation we take two randomly chosen agents and let them interact. The trade is performed if both of them agree on the exchange. After the trade they both erase the last item in their memory, remember the latest demand and change their portfolios according to what each one bought and sold. In this setting we enable the exchange of one commodity for another only which is different from the original paper. If the agents don't come to an agreement, they only put the demand into their memory and nothing else happens. This scheme we repeat and at each time step there is one attempt to trade until we stop the run.

2.2 Simulation

The first thing we would like to know about the evolution of the system is whether it reaches some kind of a stationary state. The initial conditions are random because we don't have any reason or an a priori pattern for doing it otherwise. The system should reach some stable state independently on the realization of the random varible building the initial distribution of commodities in matrices otherwise the investigation of our problem wouldn't make any sense.

We can for example study whether the system has a steady quotient between the first and the second way of trade after some time from the initiation as in [12]. The result can be seen in Figure (2.1). It is evident that after some time the quotient between the first and second kind of trade becomes constant. This means that there is a constant probability that when two agents meet, the trade will be of the second type for example.

The system evolves and each agent has some commodities loaded in his memory. How could be recognized the most favourite commodity and how should we measure such a thing? There could be many methods but the following one seems to be the most appropriate. We find for each agent an item which is most present in his memory and then select the item with the "money property" which most agents prefer. The number of agents who prefer this money commodity we denote as Q, Q/N_{ag} is apparently percentage evaluation of the populality of the money commodity among the agents.

More precisely, let us define function of two variables $g(i, k)$,

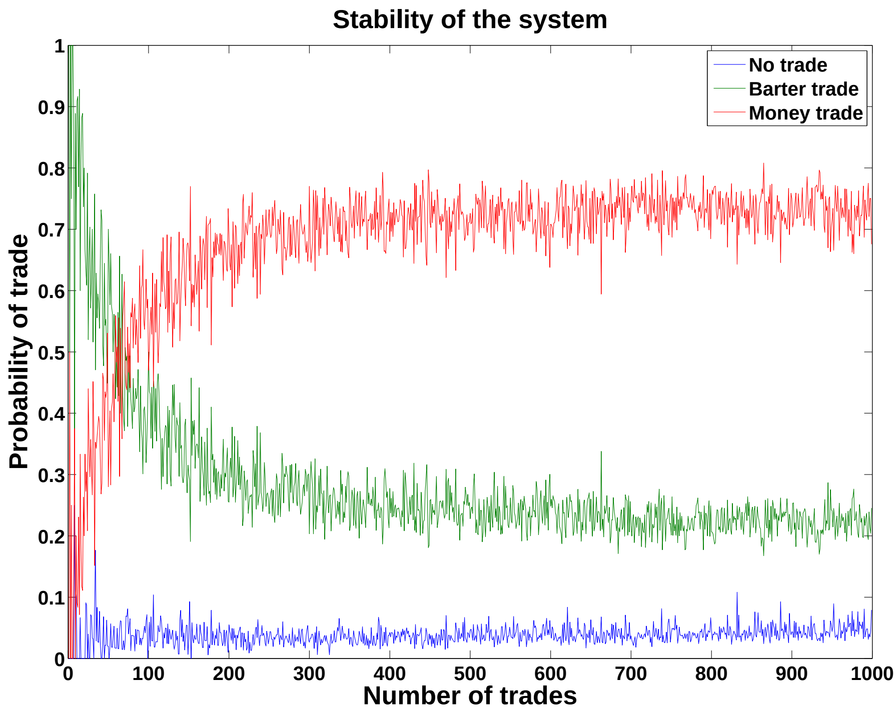

Figure 2.1: The system tends to the stationary state with constant ratio between money and barter trade. We set the parameters $N_{ag} = 20$, $N_{pr} = 20$, $N_{mem} = 40$ and $N_{un} = 40$. As can be seen the system stabilizes very quickly and the quotient between the money and barter exchange is approximately 0.75:0.25. For our setting of parameters it takes about 1000 iterations before the system reaches its stationary state, for bigger number of agents and products this unstable period would be of course much longer.

$i \in \{1, 2, .., N_{ag}\}$, $k \in \{0, 1, .., N_{pr} - 1\}$,

$$g(i, k) \equiv \sum_j \delta(Mem_{ij} - k) \tag{2.1}$$

where δ denotes the Kronecker symbol. Next we define function $G(i)$

$$G(i) = l \qquad if \qquad g(i, l) \geq g(i, x) \quad \forall x \neq l. \tag{2.2}$$

The definition of $G(i)$ is ambiguous and so if there are several variables which fulfil the inequality we take the smallest one. Commodity l^* which occurs most frequently as the value of function G we call the commodity with money property and number of agents who regard l^* most valuable we denote as popularity index Q. Again, if there are several possibilities we take as money the smallest l^*.

How the evolution of index Q looks like and how the money property passes from one item to another one is shown in Figures (2.2) and (2.3). We can notice the standard behaviour of the system. The initial state is random and therefore the index Q (green line) is very low at the beginning. After some time index Q increases due to trade activity of the agents who begin to regard one item very valuable, the quotient Q/N_{ag} can even reach one and all agents can come to a full agreement about the money property. Later index Q decreases due to the random turbulence caused again by the traders. When it reaches some boundary, it enables another product to become more valuable, to be the main mean of trade and therefore to gain money property. Another rise of index Q follows and the whole course begins to repeat. It is very interesting to notice that the change of the holder of money property is accompanied by an abrupt decrease of popularity index Q. It will be shown later that we are able to find out probability that after such a steep decrease there will be a transition of money property.

2.3 Commodities heterogeneity

So far we have composed a system of agents with such a prescribed pattern of behaviour that it matches up with our notion of a situation in real ancient society. However, it is a little strange that the commodities in the system are totally equal and each commodity can become valuable. Goods in the real world differ in many aspects, for example the quantity of the commodity available on the market is very important and thus we should investigate our system supplied with such a feature. To see if it makes sense to add such a new attribute, let one commodity (the 10-th) be rare, say half distributed than other items, and see what effect it has. Two specimen of the evolution with this new condition are presented in Figures (2.4) and (2.5). It is clear that the rare commodity prevails over all other commodities and index Q reaches its maximum and remains there. This means that the only way to perform money trade is with goods #10. Figure (2.5) displays the same sample evolution as in Figure (2.4), just supplied with a longer time period, and confirms our suspicion that the deviation of Q is very low and never leads to the change of money property.

Rare products therefore seem to be more valuable than products plentifully represented in agents' portfolios. To illustrate this proposition we will define density function $D(x)$ which will determine the probability of occurence of commodities in agents' portfolios and which enables us to model

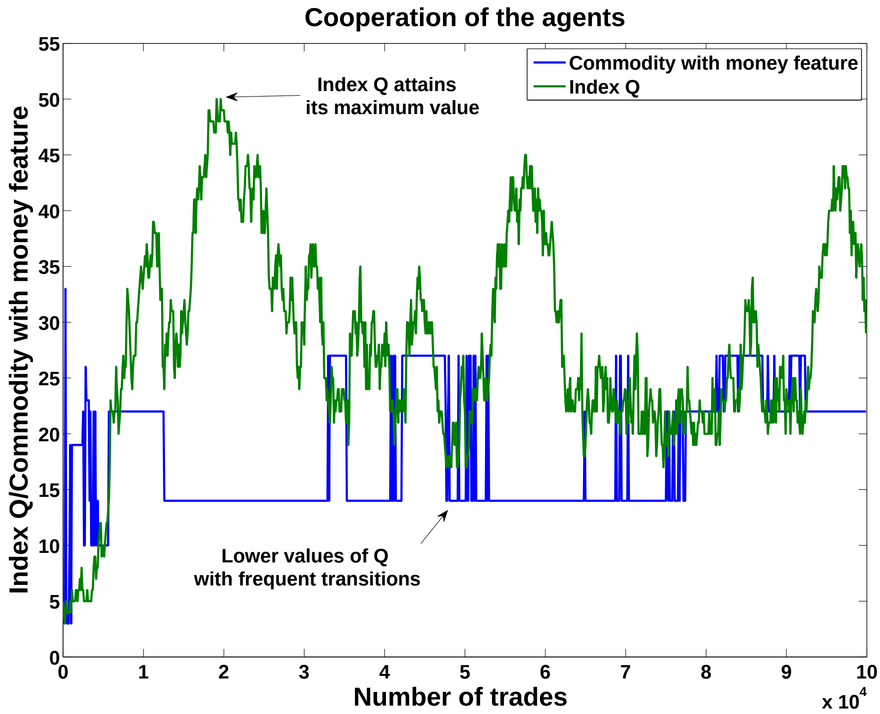

Figure 2.2: Establishment of money property and its evolution. The parameters we set $N_{ag} = 50$, $N_{pr} = 50$, $N_{mem} = 100$, $N_{un} = 100$. The role of money plays at first commodity number 22, after that commodity number 14 whose dominance commodity number 27 disrupts with its two lower peaks. At the end commodity number 22 is again most preferred. At one moment the system is fully ordered with index Q equal to 50.

the uneven presence of commodities in the system. We will run our program many times to obtain distribution of money property for all goods. Our primary aim is to describe the dependence of the frequency plot on density $D(x)$.

Let us begin with the simpliest form of $D(x)$ - linear function. We model the rareness that we take commodity after commodity and let it be in the system with probability $D(x)$ which is dependent on the commodity number x ranging from 0 to 49. We erase one unit of the chosen commodity with supplementary probability $1 - D(x)$. The formula of $D(x)$ could be for example

$$D_1(x) = 0.7 + \frac{0.3x}{N_{pr} - 1} \tag{2.3}$$

if we want to erase the first goods with number 0 with probability 0.3 and

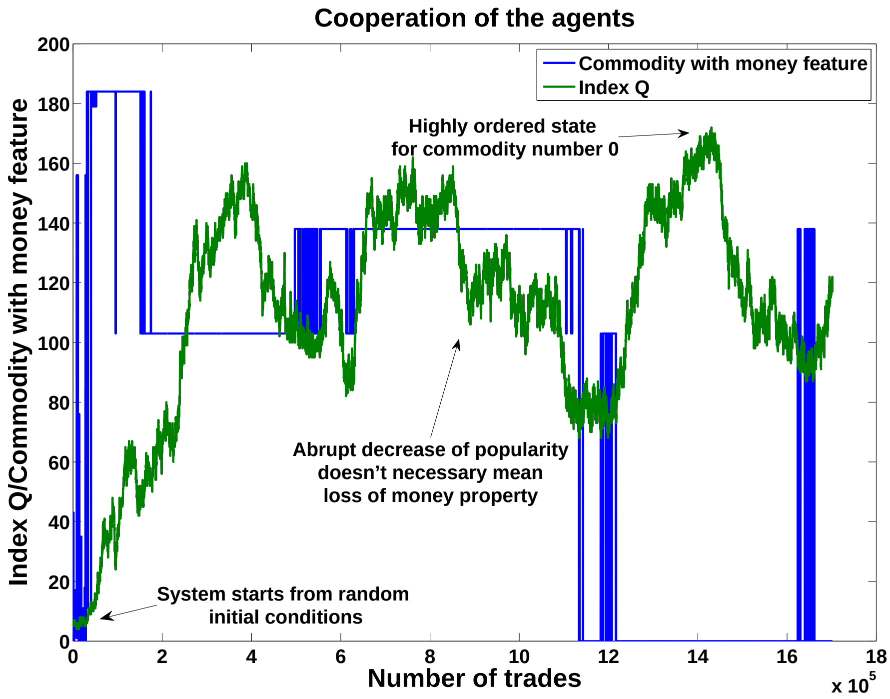

Figure 2.3: Establishment of money property and its evolution. The parameters we set $N_{ag} = 200$, $N_{pr} = 200$, $N_{mem} = 400$, $N_{un} = 400$. Money property is possessed by commodities number 104, 140 and 0, successively.

don't want to erase the last item 49 at all. A slighter modification could be

$$D_2(x) = 0.9 + \frac{0.1x}{N_{pr} - 1} \tag{2.4}$$

with evident interpretation of the line of smaller slope which doesn't erase the last item. Figure (2.6) shows the frequency plots of money property for these two densities. In both cases we see that the more rare the commodity is the more frequent it becomes money. In the first case the frequency plot is fully ruled by commodities with smaller numbers and commodities with higher numbers don't occure at all. In the second case the frequency plot is also dominated by commodities with smaller numbers but here also other commodities are given a chance to become money. This is because the function $D_2(x)$ is not so steep. This indicates that the rate between the values of $D(x)$ is proportional to the occurence in the frequency plot.

Let us try another four functions $D_3(x)$, $D_4(x)$, $D_5(x)$ and $D_6(x)$

$$D_3(x) = 0.7 + \frac{0.3e^x}{e^{N_{pr}-1}} \tag{2.5}$$

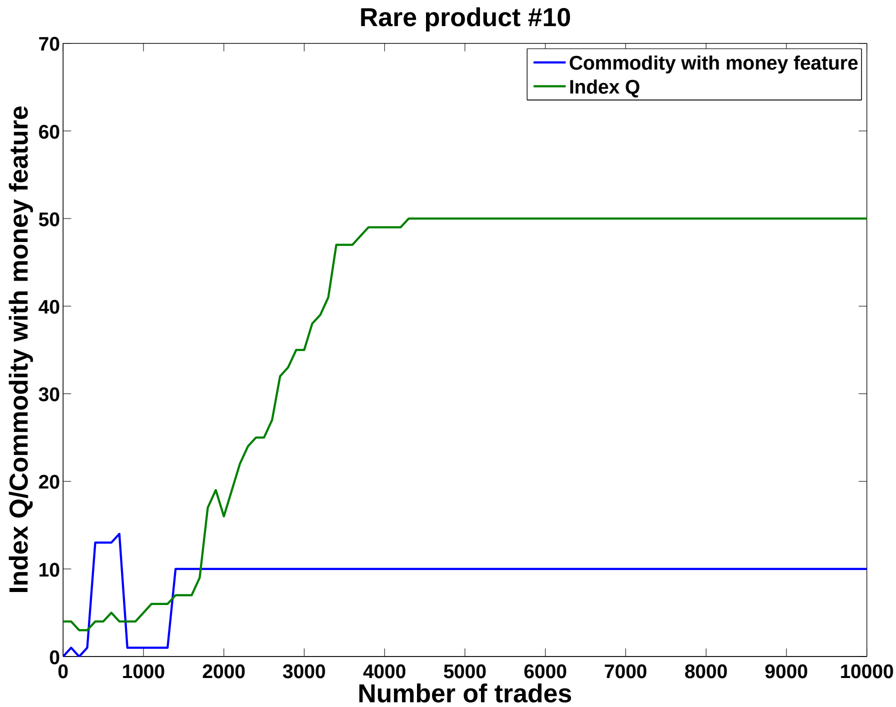

Figure 2.4: Agents prefer rare commodity #10. The parameters we set $N_{ag} = 50$, $N_{pr} = 50$, $N_{mem} = 100$, $N_{un} = 100$. We can notice that after leaving the initial random state there are several commodities with money property but when index Q is increasing, the commodity number #10 dominates. Index Q reaches its maximum after approximately 4500 trades.

$$D_4(x) = 0.9 + \frac{0.1e^x}{e^{N_{pr}-1}} \tag{2.6}$$

$$D_5(x) = 0.7 + \frac{0.3\sqrt{x}}{\sqrt{N_{pr}-1}} \tag{2.7}$$

$$D_6(x) = 0.9 + \frac{0.1\sqrt{x}}{\sqrt{N_{pr}-1}}. \tag{2.8}$$

Frequency plots for density functions $D_3(x)$, $D_4(x)$ are shown in Figure (2.7). These two density functions are exponentials and from their shapes we can estimate that almost all commodities in the system are more or less equal except a few latest ones which has proportionally much higher occurence in agents' portfolios. Therefore we can see such an abrupt change in the behaviour of the curves near points 45.

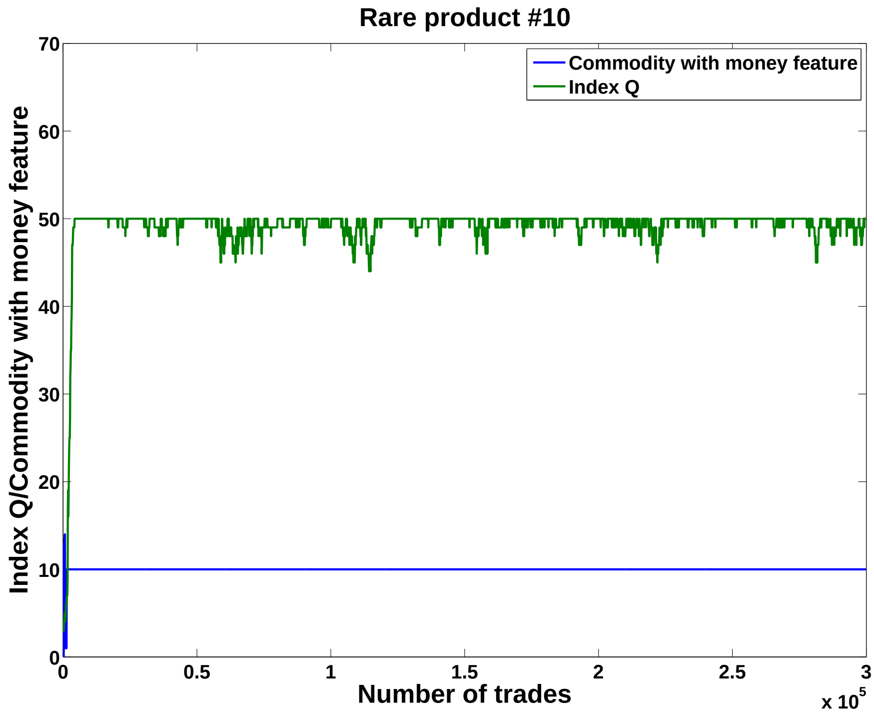

Figure 2.5: Small deviations of index Q in the case of rare commodity #10. The parameters we set $N_{ag} = 50$, $N_{pr} = 50$, $N_{mem} = 100$, $N_{un} = 100$.

Functions $D_5(x)$ and $D_6(x)$ consist of square root and therefore contain a few very rare elements at the beginning of the set of commodities. Because of this property the frequency plots are much steeper from the beginning than those for linear density functions as we can see in Figure (2.8).

2.4 Memory corrosion and its temperature

We have assumed so far that the memory of agents is perfect in the sense the agents remember every trade attempted to be performed in the history until the cut-off N_{mem}. One may find it strange that we have taken almost all processes as random and the memory is modelled deterministically. In some sense this arrangement resembles a thermodynamic system at zero temperature. After the agent performed a trade the memory "freezes" and stays the same until another chance for the trade comes. The alternative approach would be to let the agents forget. We can take the inspiration in the Ising model for magnets and introduce the concept of temperature into our sys-

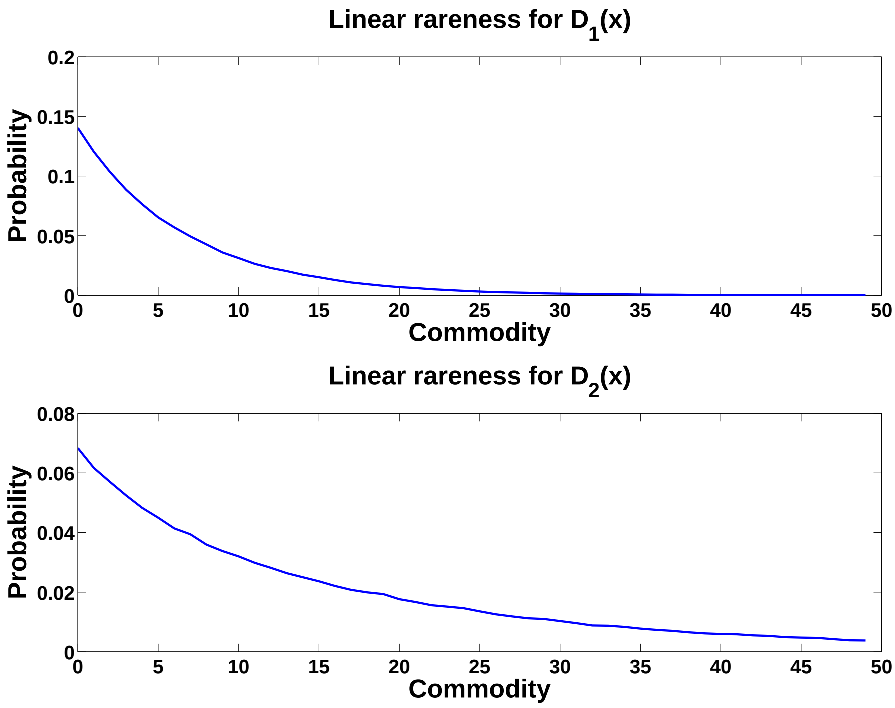

Figure 2.6: Frequency plots for densities $D_1(x)$ and $D_2(x)$. The parameters we set $N_{ag} = 50$, $N_{pr} = 50$, $N_{mem} = 100$, $N_{un} = 100$. Plots are calculated from 200 000 sample evolutions of length 50 000 trades. $D_1(x)$ is a steep linear function and thus the probability of money occurence is much higher for commodities with lower index than for the rest of the goods. $D_2(x)$ is gentler than $D_1(x)$ so its plot begins on lower value at zero and gives a chance even to the latest commodities.

tem. In statistical physics the probability of the state S with energy E at temperature T is given by the Boltzmann distribution

$$P(S) = \frac{e^{-E/kT}}{Z} \tag{2.9}$$

where Z is the partition function and k the Boltzmann constant. We have already seen that our system is usually in a state where the memory is dominated by only one commodity which is common to almost all agents. This state is highly organized and we can call this state the ground state and assign him zero energy. Let us introduce "noise" into our system and say that when we change 1 item in the memory of the agent, we have applied work to the memory and its energy increased. In view of this we can say that the system

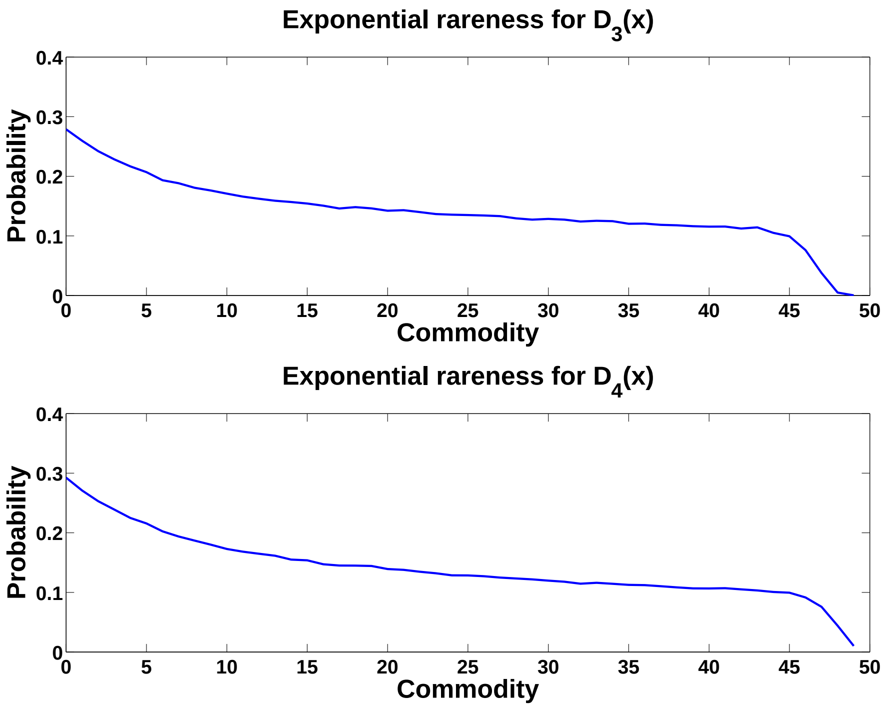

Figure 2.7: Frequency plots for densities $D_3(x)$ and $D_4(x)$. The parameters we set $N_{ag} = 50$, $N_{pr} = 50$, $N_{mem} = 100$, $N_{un} = 100$. Plots are calculated from 200 000 sample evolutions of length 50 000 trades. The abrupt change near point 45 is caused by the shape of the exponential density function which quickly rises for the latest commodities.

has temperature T if we change 1 item in agent's memory with probability $e^{-1/T}$ at each step (this formula arises from Boltzmann distribution after appropriate rescaling of constants, the units we take dimensionless). The increase of temperature therefore causes the increase of agents' lapses. In Figure (2.9) we can compare the evolution at four differrent temperatures 0, 0.145, 0.221 and 0.62, successively. This picture shows that increasing temperature disrupts the order in the system which is demonstrated by both the lower value of coefficient Q and the unstability of the commodity with money property. Just like in other physical systems the temperature here causes greater disorder.

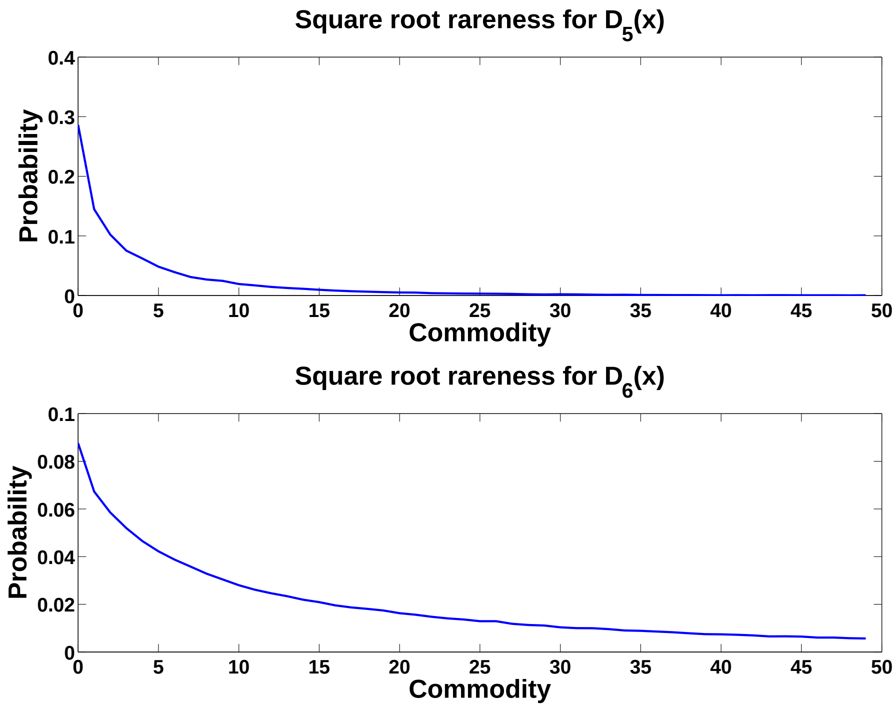

Figure 2.8: Frequency plots for densities $D_5(x)$ and $D_6(x)$. The parameters we set $N_{ag} = 50$, $N_{pr} = 50$, $N_{mem} = 100$, $N_{un} = 100$. Plots are calculated from 500 000 sample evolutions of length 50 000 trades.

2.5 Probability of money transition

We can get the impression from Figure (2.2) that the trasition of money property from one commodity to another happens at a specific value of Q. To investigate if our notion is right we run the simulation many times and record the values of index Q at which the commodity loses its money property. We should incorporate the condition that this commodity has money property long enough to follow the characteristic history in the shape of reverse "U" because we would like to avoid those unordered regions before another money establishes. The resulting frequency plot we see at the first plot of Figure (2.10) (minimal waiting time 10 000) and we can make the conclusion that the loss of money property happens in a certain interval of index Q. If the system drops for example into some state with $Q = 22$, then we have a 50 percent chance that the transition happens (supposing parameters of Figure (2.10)).

When we decrease the prescribed minimum waiting time of duration of

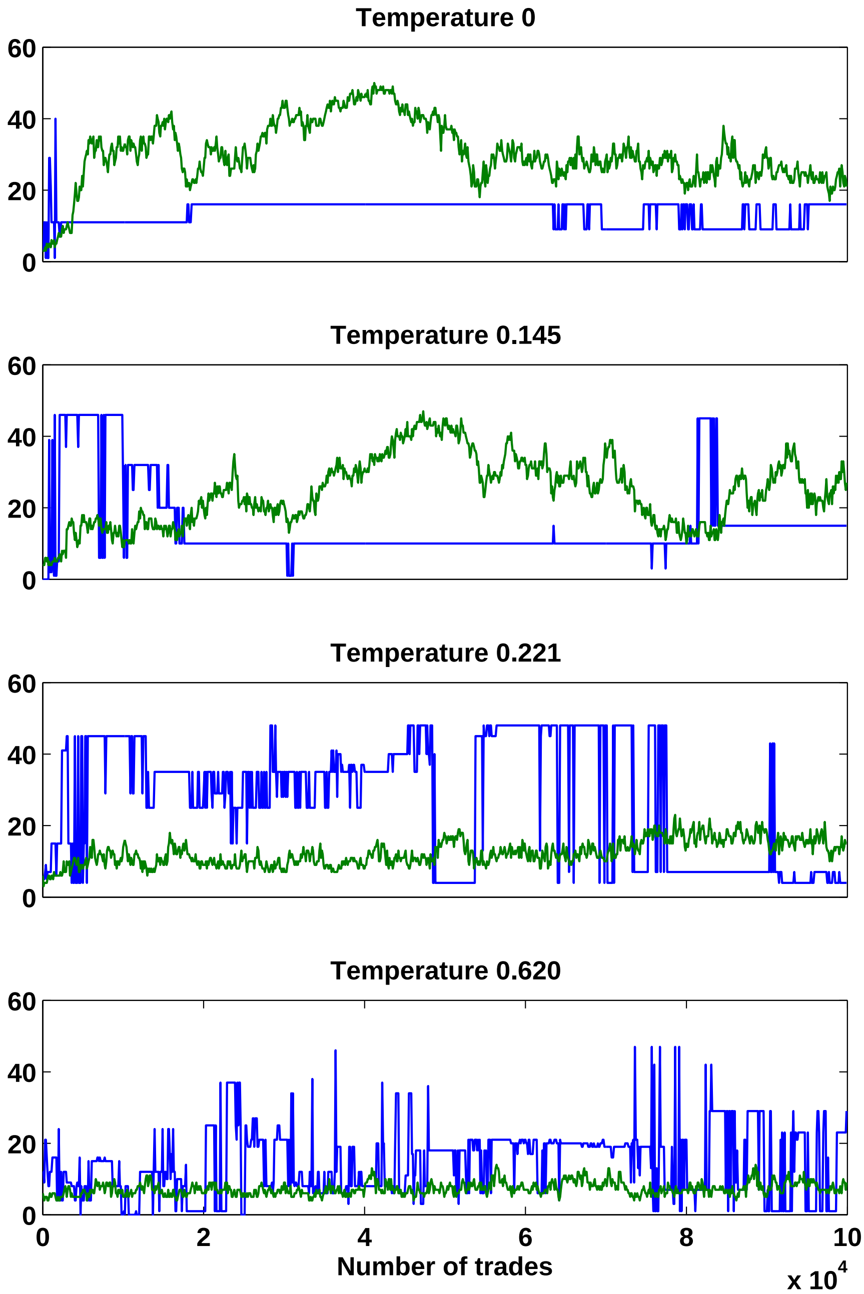

Figure 2.9: Higher temperatures disrupt the trade organization. The parameters we set $N_{ag} = 50$, $N_{pr} = 50$, $N_{mem} = 100$, $N_{un} = 100$. Notice the decrease of index Q when temperature increases.

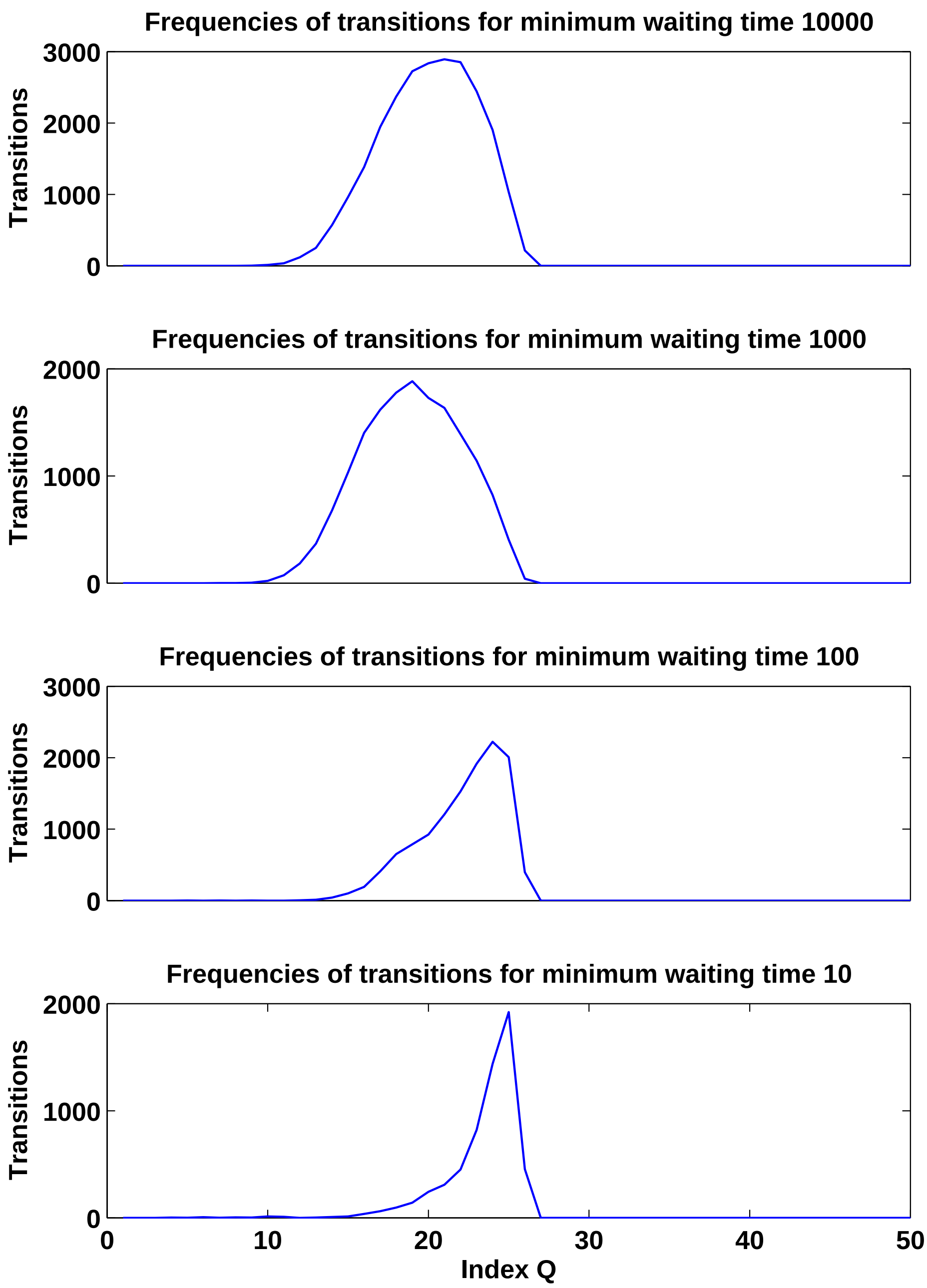

Figure 2.10: The transition of money property occurs only when the value of Q is around 22. The parameters we set $N_{ag} = 50$, $N_{pr} = 50$, $N_{mem} = 100$, $N_{un} = 100$, number of performed trades is in the range of millions.

index Q we see that the expected value of transition increases. The interpretation could be that in the case of lower minimum waiting time there might be two or more commodities strugling for the money property and the drop of Q doesn't have to be so significant in order to cause the loss of money property. The important characteristic is that the frequency plot is not substantially dependent on our choice of the minimum waiting time.

2.6 Correlation of waiting times

Another interesting question is whether two succesive waiting times of money property are correlated. Can we for example claim that after some money commodity lasted relatively long time, the subsequent money commodity will have higher probability to last longer than usual? In order to answer this question we record waiting times l_i of money commoditities which reaches some minimum length and then plot points (l_i, l_{i+1}). If the waiting times were correlated, we would observe the typical line intersecting the origin. When we look at Figures (2.11) and (2.12), we don't recognize such a line, the data seem to be totally uncorrelated.

2.7 Distribution of waiting times

We can also draw the frequency plot of waiting times l_i, see Figure (2.13). The data are plotted in the logaritmic scale and in this scale the frequencies decrease for longer waiting times as a line with slope -1.8. This indicates that the frequency plot of waiting times has the fat tail with coefficient 1.8, i.e. it behaves like function $1/x^{1.8}$ for waiting times long enough. This could be compared to a pure random walk whose coefficient is 1.5. We can therefore conclude that our system behaves nontrivially.

2.8 Conclusion

In order to conclude this section we repeat that we have designed a few basic rules for the behaviour of the agents and these rules we have applied to our simulation. The evolution of the system exhibits the main characteristic similar to that in [12], i.e. we can observe how the money emerges as a result of trading activities of the agents. This proves that the concept of the simulated market is robust as was already noticed by Sneppen and Donangelo themselves.

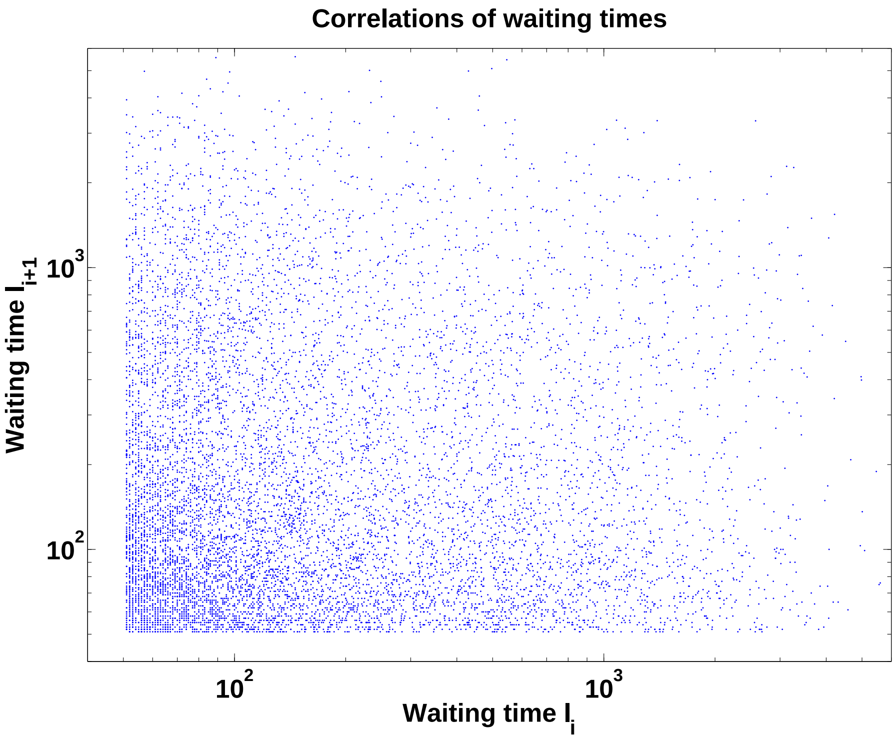

Figure 2.11: Waiting times of money property seem to be uncorrelated. The parameters we set $N_{ag} = 20$, $N_{pr} = 20$, $N_{mem} = 40$, $N_{un} = 40$. For the statistics it was performed 5 mil. trades and the data are plotted in the logaritmic scale in order to recognize any possible pattern of dependance. We ask for the waiting times to be at least 50 trades long in order that it could be taken into account.

We made several significant modifications in both commodities and agents' behaviour. When we deviate from the uniform distribution of commodities in the system, then the commodities become unequal in the sense that more rare commodities have higher probability of becoming money as was shown in section (2.3). This result is not suprising because every commodity which played the role of money in real-world early economies was very hard to find and there was only small amount of it in circulation.

When we introduce temperature into the system and allow the agents to forget, then the system becomes less ordered, behaves more chaotically and at high temperatures the phenomenon of money emergence can totally disappear. The indicator of this confusion is the low value of index Q. From the economist's point of view the temperature captures the individual deviation from the rational decision-making based on the experience of each agent. We

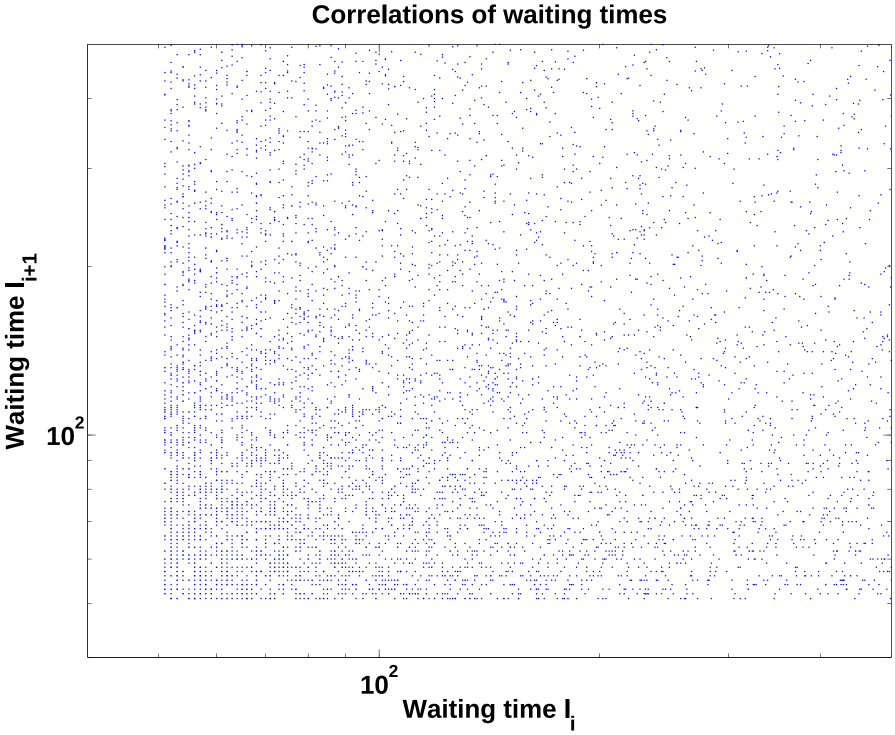

Figure 2.12: Waiting times of money property seem to be uncorrelated. Enlargement of Figure (2.11).

can say that if every agent deviated too much from the best trading strategy, the money would never emerge.

In section (2.5) we discovered that as soon as some commodity attains money property it loses its money position around certain value of index Q whose distribution is shown in Figure (2.10). This transition is usually preceded by an abrupt decrease of the popularity index Q which may, when reduced to some value from the critical interval, lead to the end of one money era. It would be worth the effort to collect data to which we could link our model because in that case we would for example have the tool for finding possible weak moments of the world's leading currency (which is at present the US dolar). We hope that this survey will be undergone sometimes in the future.

The latest simulation analysis deals with the study of waiting times of money property. We discovered that the waiting times of money duration are uncorrelated and the frequencies of waiting times exhibit a tail with the exponent 1.8 which indicates nontrivial behaviour.

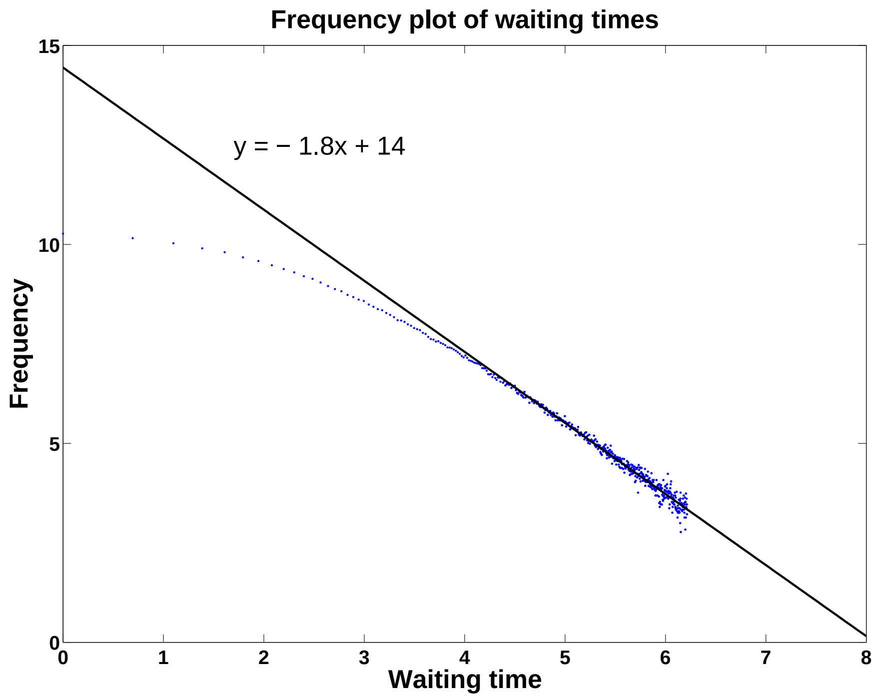

Figure 2.13: The frequency plot for waiting times l_i. The paremeters we set $N_{ag} = 50$, $N_{pr} = 50$, $N_{mem} = 100$, $N_{un} = 100$. Both axes are plotted in the logaritmic scale, in this scale the frequency of waiting times decrease as $1/x^{1.8}$.

Chapter 3

Analytical solution

In the simulations we had N_{ag} agents who interacted between themselves with the strategy we imposed on them. Now we would like to investigate the system analytically. Our strategy will be to transcribe the if-then conditions into mathematical equations and try to solve them. The main idea is to observe the portfolio of just one agent called the representative agent and the influence of other agents on the representative agent approximate by an averaged "mean field". With this approximation we can assume that the evolution of our system is simililar to the evolution of the system with only two agents - the representative agent and a rival who substitutes for the rest of the original system.

3.1 Markov approximation

Denote by $\vec{R}$ one possible configuration of the portfolio of the representative agent; that is when for example there are only two products and the agent owns k-times the first product and n-times the second product, $\vec{R}$ is a two dimensional vector (k, n). The coordinates of vector $\vec{R}$ will be denoted by $\{r_i\}_{i=1}^{N_{pr}}$ so if for example r_i equals 4, then the representative agent owns 4 items of the i-th product. We see immediatelly that every vector $\vec{R}$ is a subject to the constraint $\sum_{j=1}^{Npr} r_j = N_{un}$ (N_{un} is the wealth of each agent) so its l_1 norm is always constant. It could be shown that using some basic combinatorial operations the number of all possible states $card\{\vec{R}\}$ is $\binom{N_{un}+N_{pr}-1}{N_{pr}-1}$ (trick with balls and dividers, see book [18], section 3.4). The composition of agent's portfolio changes in time and we can imagine this as a stochastic process $\{X_t\}$, $t \in \mathbf{N_0}$, taking values in the set $\{\vec{R}\}$ (hence $\{\vec{R}\}$ is the state space of $\{X_t\}$). In the simulations the value of X_i for some i depends strongly on the history of the process but in our approximation with only two agents and no recorded history X_i depends only on the last realized value X_{i-1}. Thus the problem possesses the Markov property and we can think of it as a Markov chain with discrete time and the finite set of states. Moreover, the behaviour of the agents is unvarying and so the chain is homogeneous. To complete our analysis we must add that all states are persistent non-null because the chain is irreducible (every state must be accessible from any state) and with finite number of states. For more information about Markow chains see [20], chapter 6.

3.2 Transition matrix

The homogeneous Markov chain is fully described by the transition matrix $w = card\{\vec{R}\} \times card\{\vec{R}\}$. Every trade consists of one purchase and one sale and therefore the portfolio vector $\vec{R}$ changes at 2 positions in one step - the i-th becomes bigger by 1 unit and the j-th becomes smaller by 1 unit. This interchange could be described by the operator L_{ij}:

$$L_{ij}\left(r_1, r_2, \ldots, r_i, \ldots, r_j, \ldots\right) \equiv \left(r_1, r_2, \ldots, r_i + 1, \ldots, r_j - 1, \ldots\right). \qquad (3.1)$$

Because we have constraints on the form of the vector $\vec{R}$ which can attain only nonnegative integer values and its biggest value cannot exceed N_{un}, operator L_{ij} is defined on all those $\vec{R}$'s for which the transformation $L_{ij}\vec{R}$ makes sense. We can also notice that L_{ij} is a nonlinear operator. Probability of transition from the state $\vec{R}$ to another state $L_{ij}\vec{R}$

$$w\left(\vec{R} \to L_{ij}\vec{R}\right) = P\left(X_{n+1} = L_{ij}\vec{R} | X_n = \vec{R}\right) \qquad (3.2)$$

is supposed to be a function of values r_i and r_j only. This simplifies things a lot and without this assumption the model would be practically unsolvable. In fact the transition between any two elements from the state space depends on all components of the vectors but the intuition tells us that the i-th and j-th position play the crucial role.

3.3 Different means of trade

The agents will trade either because they both lack some product the other agent owns (so called barter trade) or because the other agent has something they imagine valuable, this case we call money trade. The third option is the mixed barter-money or money-barter trade. The probabilities for each possibility will be as follows:

$$w_{BB}\left(\vec{R} \to L_{ij}\vec{R}\right) = \delta(r_i) \times (1-\delta(r_j)) \times P_0(r_j) \times (1-P_0(r_i)) \tag{3.3}$$

for barter-barter trade,

$$w_{MM}\left(\vec{R} \to L_{ij}\vec{R}\right) = (1-\delta(r_j)) \times m(r_i) \times m(r_j) \times (1-P_0(r_i)) \tag{3.4}$$

for money-money trade,

$$w_{MB}\left(\vec{R} \to L_{ij}\vec{R}\right) = (1-\delta(r_j)) \times P_0(r_j) \times (1-P_0(r_i)) \times m(r_i) \tag{3.5}$$

for money-barter trade,

$$w_{BM}\left(\vec{R} \to L_{ij}\vec{R}\right) = (1-\delta(r_j)) \times m(r_j) \times \delta(r_i) \times (1-P_0(r_i)) \tag{3.6}$$

for barter-money trade.

In these equations we have used $\delta(x)$ for the Kronecker delta

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{else,} \end{cases}$$

with $P_0(r_i)$ we estimate probability that the representative agent with r_i units of some commodity will encounter an agent who doesn't possess this commodity. In order to calculate the value of $P_0(r_i)$ we assume a big system in which the total count of some product is approximately $\gamma \equiv \frac{N_{un}N_{ag}}{N_{pr}}$ and the distribution of this one product among the agents is absolutely random. The search for $P_0(r_i)$ we can therefore link to the Maxwell-Boltzmann model (Maxwell-Boltzmann model could be found for example in [18], section 3.3) where distinguishable particles are deposited into boxes. In the limit when the number of particles and boxes grow at the same rate we can analytically express the probability that there will be k particles in some box as $p_k = \frac{\lambda^k}{k!}e^{-\lambda}$ where λ is the ratio between number of particles and boxes. In this formula we recognize the well-known Poisson distribution. When the representative agent owns r_i units of some product, there are only $\gamma - r_i$ units of the same product in the rest of the system and therefore in the application for our model $\lambda = \frac{\gamma - r_i}{N_{ag}-1}$ and $P_0(r_i) = p_0 = e^{-\lambda}$.

The function $m(r_j)$ stands for the memory and its value tells us with what probability the j-th commodity will be chosen as the article of trade during money exchange. Memory in the simulations depends on N_{mem} previous encounters but we have chosen Markow chain approach and thus we need an approximate formula which describes the memory on the basis of the actual state. From the simulations we know that most of the time the system is ordered and most agents' memories are similar in the sense of index Q so it is reasonable not to distinguish representative agent's memory from the memory of his rival. Next, the rival agent substitutes for all agents except one and when this rival agent doesn't have some commodity, from the condition of wealth preservation it must be concentrated in the representative agent's portfolio. Concentration of one commodity implies higher demand for this commodity on the basis of barter exchange and therefore we set $m(r_i) \equiv P_0(r_i)$. This assumption is very convenient and makes further calculations much easier as will be seen later.

The origin of equations (3.3)-(3.6) is following. First take the probability w_{BB}. Because the trade has barter-barter property, we need to ensure that the agent possesses the commodity he is supposed to sell and on the other hand that he doesn't possess the commodity he buys. For this reason we used two Kronecker deltas. The term $P_0(r_j) \times (1 - P_0(r_i))$ tells us with what probability the rival agent doesn't have the j-th product so the representative agent can sell and at the same time that the rival agent has the i-th product so the representative agent can buy. The two probabilities $P_0(r_i)$ and $P_0(r_j)$ are not independet because the agents possess constant quantity of products and hence the number of one product influences the number of other product. Then again, when there are a lot of products and agents, the influence is not so strong and we can take the two random variables as independent and assume that the probability density factorises. It is also important to note that the sum of all probabilities of transitions beginning in one fixed state $\vec{R}$ is not equal to one. That is because in our approximation we take the particular transition for granted and don't count in other possibilities the barter trade could also be possible. But we remind that the agents meet randomly according to the uniform distribution and therefore the probability of one state $\vec{R}$ turning into another state $\vec{\underline{\mathrm{R}}}$ is in the right ratio with the probability of different state $\vec{T}$ turning into another different state $\vec{\underline{\mathrm{T}}}$. At the end of the calculation we will normalize all probabilities. Other types of trade would be done in the same way with the use of memory function $m(r_i)$ which models the memory of the agent from the current state of the portfolio.

So now we are able to characterize the process as a Markov chain with some initial distribution and the matrix of transition $w = a_{BB}w_{BB} + a_{BM}w_{BM} + a_{MB}w_{MB} + a_{MM}w_{MM}$ where the coefficients before w's denote the probabilities of occurence of the very type of trade. We can estimate these coefficients from the simulation.

3.4 Master equation

Vectors $\vec{R}$ are very instructive tool how to describe portfolios of the agents but it is quite difficult to manipulate them. We will now introduce another representation of the portfolios which simplifies further algebraic manipulations.

Every $\vec{R}$ can be uniquely transformed into a matrix S which is a matrix $(N_{un}+1)\times N_{pr}$, every element in S is zero except N_{pr} positions $S\binom{r_i}{i}$ with the value of one. The symbol $S\binom{r_i}{i}$ denotes the element in the r_i-th row and i-th column, this notation will be useful in the following sections. Let $p_t(S)$ denote the probability of state S at time t, then the matrix Π_t defined as the averaged occupancy of agent's portfolio is

$$\Pi_t = \sum_{\{S\}} p_t(S)S. \tag{3.7}$$

This will be the basic quantity whose evolution will be investigated analytically. The elements of this matrix tell us probability that the representative agent owns given number of some product at time t.

Now we would like to compute the derivative of Π_t at some element $\binom{x}{y}$ so we could be able to observe the dynamics of the system. When we look at the equation above, we can differentiate the sum term by term and therefore we need to know the time derivative of $p_t(S)$. This can be done easily with equations (3.3)-(3.6). The probability increases when one state S' very near S changes to S and on the contrary the probability decreases when S changes to something else. In view of this we can write

$$\frac{dp_t(S)}{dt} = \sum_{i\neq j} w(L_{ij}\vec{R}\to\vec{R})p_t(L_{ij}\vec{R}) - w(\vec{R}\to L_{ij}\vec{R})p_t(\vec{R}), \tag{3.8}$$

in the equation we sum over all indices which make sense for given $\vec{R}$. Thanks to the unique equivalence between the representations $\vec{R}$ and S we can write $p_t(\vec{R})$ as well as $p_t(S)$ because we consider the same state, just in different representations. By taking the derivative of Π_t term by term and substituting for $\dot{p}_t(\vec{R})$ we get

$$\frac{d\Pi}{dt}\binom{x}{y} = \sum_S \sum_{i\neq j}\left[w(L_{ij}\vec{R}\to\vec{R})p_t(L_{ij}\vec{R}) - w(\vec{R}\to L_{ij}\vec{R})p_t(\vec{R})\right] S\binom{x}{y}. \tag{3.9}$$

We can notice that the composite operator $L_{ij}L_{ji}$ is the identity on the set of all possible states and because we sum over all states and all possible transitions we are able to rewrite the equation as

$$\frac{d\Pi}{dt}\binom{x}{y} = \sum_S \sum_{i\neq j} w(\vec{R} \to L_{ji}\vec{R})p_t(\vec{R})(\widehat{L_{ji}\vec{R}})\binom{x}{y} - \sum_S \sum_{i\neq j} w(\vec{R} \to L_{ij}\vec{R})p_t(\vec{R})S\binom{x}{y} \tag{3.10}$$

Here the symbol $\widehat{L_{ji}\vec{R}}$ represents the matrix form of $L_{ji}\vec{R}$ and we again sum over all possible states and indices. By changing the indices and factorizing we get

$$\frac{d\Pi_t}{dt}\binom{x}{y} = \sum_S \sum_{i\neq j} w(\vec{R} \to L_{ij}\vec{R})p_t(\vec{R})\left(\widehat{L_{ij}\vec{R}}\binom{x}{y} - S\binom{x}{y}\right). \tag{3.11}$$

Now let's have a look what w means - it consists of four terms defined above. Let us take the first term for the barter-barter trade. It tells us that the i-th coordinate of $\vec{R}$ should be zero and the j-th coordinate non-zero. From what was already said the parenthesis $\left(\widehat{L_{ij}R}\binom{x}{y} - S\binom{x}{y}\right)$ is always zero except these four cases

1. if $S\binom{x}{y} = 1$, $x \neq 0$, $y = j$ and $S\binom{0}{i} = 1$, i.e. the representative agent sells commodity y, we call this the donor case
2. if $S\binom{x}{y} = 1$, $x = 0$, $y = i$ and $S\binom{0}{j} = 0$, i.e. the representative agent buys commodity y, we call this the receiver case
3. if $S\binom{x}{y} = 0$, $S\binom{x+1}{y} = 1$, $y = j$ and $S\binom{0}{i} = 1$, i.e. the donor case
4. if $S\binom{x}{y} = S\binom{1}{y} = 0$, $S\binom{0}{y} = 1$, $y = i$ and $S\binom{0}{j} = 0$, i.e. the receiver case

This conditions have very clear interpretation. In the first case the element of S with coordinates (x, y) is one, i.e. the representative agent owns x units of product y and wants to sell it (thus $x \neq 0$, $y = j$), reminding we operate with barter-barter trade the agent must not have the second product meant for the purchase and thus $S\binom{0}{i} = 1$. In similar fashion we proceed in the second case when the agent owns zero units of commodity y and wants to buy it (position $x = 0$, $y = i$), subsequently he has to sell the commodity number j and therefore he has to own it (condition $S\binom{0}{j} = 0$). The third case shows the situation when the representative agent owns $(x+1)$ units of product y and sells. That results in inserting one into position (x, y) where was originally zero before the application of operator L_{ij}. This operation is again accompanied with the purchase of commodity i and thus $S\binom{0}{i} = 1$. In the fourth case we receive commodity y and therefore we cannot own it. Similar reasoning as before follows.

The first two events gives -1 in the parenthesis and the last two events gives 1 in the parenthesis. We will now try to compose the equations for derivative of Π_t at the first row of this matrix. At the first row $x = 0$ and therefore we must use our prepared cases number 2 and 3. When we combine them with the expression (3.3), we get

$$\begin{aligned}\dot{\Pi}\binom{0}{y} =& \sum_S p_t(S) \sum_{i,j\neq y} \left(-2S\binom{0}{y}+1\right) S\binom{0}{y} P_0(i)\,(1-P_0(0)) \left(1-S\binom{0}{j}\right) S\binom{i}{j} + \\ &+ \sum_S p_t(S) \sum_{i\neq y} \left(-2S\binom{0}{y}+1\right)\left(1-S\binom{0}{y}\right) S\binom{1}{y} P_0(1)\,(1-P_0(0))\, S\binom{0}{i}\end{aligned} \tag{3.12}$$

In the first term (second case) the expression $P_0(i)\,(1-P_0(0))$ is taken from (3.3), we have the receiver case with $x = 0$ and therefore $r_i = 0$, the value of r_j we denoted by index i. Formula $\left(-2S\binom{0}{y}+1\right) S\binom{0}{y}$ ensures condition on position $x = 0$ and the sign -1, formula $\left(1-S\binom{0}{j}\right) S\binom{i}{j}$ stands for the condition $S\binom{0}{j} = 0$ and it enables to interact with other commodities with one at the position $\binom{i}{j}$. The δ's from equation (3.3) are already incorporated in all cases. We sum over all row indices i and over all column indices j except y because we forbid to trade the commodity for itself. The second term was made similarly and according to the third condition but for reader's convenience we will explain its structure. The term $\left(-2S\binom{0}{y}+1\right)\left(1-S\binom{0}{y}\right)$ ensures $S\binom{x}{y} = 0$ and may take the only nonzero value $+1$ which is the value of the parenthesis $\left(\widehat{L_{ij}R}\binom{x}{y} - S\binom{x}{y}\right)$. The term $S\binom{1}{y}$ ensures $S\binom{x+1}{y} = 1$, $P_0(1)\,(1-P_0(0))$ again from equation (3.3), $S\binom{0}{i}$ is the interaction with other commodity during the trade. We again cannot trade the commodity for itself.

We see that $S\binom{x}{y}S\binom{x}{y} = S\binom{x}{y}$, the sums can be interchanged (finite number of states and indices), some of the sums vanish (for example $S(\binom{o}{y})S(\binom{1}{y}) = 0$ always) and when we introduce the notation for the correlation function $R\left(\binom{x}{y},\binom{w}{z}\right)$

$$R\left(\binom{x}{y},\binom{w}{z}\right) = \sum_S S\binom{x}{y} S\binom{w}{z} p_t(S) \tag{3.13}$$

we can write

$$\begin{aligned}\dot{\Pi}\binom{0}{y} =& \sum_{i\neq 0, j\neq y} -R\left(\binom{0}{y},\binom{i}{j}\right) P_0(i)(1-P_0(0)) + \\ &+ \sum_{i\neq y} R\left(\binom{1}{y},\binom{0}{i}\right) P_0(1)(1-P_0(0)).\end{aligned} \tag{3.14}$$

Now we have simplified the time derivative of matrix Π at the first row. We must still divide following calculation into two branches: the second row and the rest of the matrix. The conditions on the shape of matrix S expressed in former four cases force us to do it so.

For the second row we need to use the first, third and fourth condition. In the same manner as before we get

$$\begin{aligned}\dot{\Pi}\binom{1}{y} =& \sum_S p_t(S) \sum_{i \neq y} \left(-2S\binom{1}{y} + 1\right) S\binom{1}{y} P_0(1)\,(1 - P_0(0))\, S\binom{0}{i} + \\ &+ \sum_S p_t(S) \sum_{i \neq y} \left(-2S\binom{1}{y} + 1\right) \left(1 - S\binom{1}{y}\right) S\binom{2}{y} S\binom{0}{i} (1 - P_0(0))\, P_0(2) + \\ &+ \sum_S p_t(S) \sum_{i \neq 0, j \neq y} \left(-2S\binom{1}{y} + 1\right) \left(1 - S\binom{1}{y}\right) S\binom{0}{y} P_0(i)\,(1 - P_0(0))\, S\binom{i}{j}\end{aligned} \tag{3.15}$$

and in the notation of correlation functions

$$\begin{aligned}\dot{\Pi}\binom{1}{y} =& \sum_{i \neq y} -R\left(\binom{1}{y}, \binom{0}{i}\right) P_0(1)(1 - P_0(0)) + \\ &+ \sum_{i \neq y} R\left(\binom{2}{y}, \binom{0}{i}\right) P_0(2)(1 - P_0(0)) + \\ &+ \sum_{i \neq 0, j \neq y} R\left(\binom{0}{y}, \binom{i}{j}\right) P_0(i)(1 - P_0(0))\end{aligned} \tag{3.16}$$

From the first and third condition we get for $x \geq 2$

$$\begin{aligned}\dot{\Pi}\binom{x}{y} =& \sum_{i \neq y} -R\left(\binom{x}{y}, \binom{0}{i}\right) P_0(x)(1 - P_0(0)) + \\ &+ \sum_{i \neq y} R\left(\binom{x+1}{y}, \binom{0}{i}\right) P_0(x+1)(1 - P_0(0)).\end{aligned} \tag{3.17}$$

3.5 Master equation for money-money trade

We have just finished the analysis of the barter-barter trade in the previous section. Let us have a look at the money-money trade. Before we begin to compose the equations we will first simplify the transition probability $w_{MM}(.)$ by substituting for memory $m(.)$ which we have set equal to $P_0(.)$

$$w_{MM}(\vec{R} \to L_{ij}\vec{R}) = (1 - \delta(r_j)) \times P_0(r_i) \times (1 - P_0(r_i)) \times P_0(r_j). \tag{3.18}$$

We can perform the same operations as we already did for the barter-barter trade and again discuss the value of $\left(\widehat{L_{ij}\vec{R}}\binom{x}{y} - S\binom{x}{y}\right)$. The list of all possible nonzero values of the parenthesis is

1. $S\binom{x}{y} = 1$, $x \neq 0$, $y = j$, the donor case
2. $S\binom{x}{y} = 1$, $y = i$, the receiver case
3. $S\binom{x}{y} = 0$, $S\binom{x+1}{y} = 1$, $y = j$, the donor case
4. $S\binom{x}{y} = 0$, $S\binom{x-1}{y} = 1$, $y = i$, the receiver case.

These cases are easier than for the barter-barter trade and we need to distiguish only the first row and the rest of matrix S. The simplification is that we don't need to bother with commodity possession during the buying mode. For the first row we must use the second and the third case so

$$\begin{aligned}\dot{\Pi}\begin{pmatrix}0\\y\end{pmatrix} =& \sum_S p_t(S) \sum_{j\neq 0, i\neq y} S\begin{pmatrix}0\\y\end{pmatrix}\left(-2 + S\begin{pmatrix}0\\y\end{pmatrix}\right) P_0(0)(1-P_0(0))P_0(j)S\begin{pmatrix}j\\i\end{pmatrix} + \\ &+ \sum_S p_t(S) \sum_{i;j\neq y} S\begin{pmatrix}1\\y\end{pmatrix} P_0(i)(1-P_0(i))P_0(1)S\begin{pmatrix}i\\j\end{pmatrix}\end{aligned} \tag{3.19}$$

and in the language of correlation functions

$$\begin{aligned}\dot{\Pi}\begin{pmatrix}0\\y\end{pmatrix} =& \sum_{j\neq 0, i\neq y} -R\left(\begin{pmatrix}0\\y\end{pmatrix}, \begin{pmatrix}j\\i\end{pmatrix}\right) P_0(0)(1-P_0(0))P_0(j) + \\ &+ \sum_{i;j\neq y} R\left(\begin{pmatrix}1\\y\end{pmatrix}, \begin{pmatrix}i\\j\end{pmatrix}\right) P_0(i)(1-P_0(i))P_0(1).\end{aligned} \tag{3.20}$$

For other rows ($x \geq 1$) we must use all scenarios

$$\begin{aligned}\dot{\Pi}\begin{pmatrix}x\\y\end{pmatrix} =& \sum_S p_t(S) \sum_{i;j\neq y} S\begin{pmatrix}x\\y\end{pmatrix}\left(-2 + S\begin{pmatrix}x\\y\end{pmatrix}\right) P_0(i)(1-P_0(i))P_0(x)S\begin{pmatrix}i\\j\end{pmatrix} + \\ &+ \sum_S p_t(S) \sum_{i\neq y, j\neq 0} S\begin{pmatrix}x\\y\end{pmatrix}\left(-2 + S\begin{pmatrix}x\\y\end{pmatrix}\right) P_0(x)(1-P_0(x))P_0(j)S\begin{pmatrix}j\\i\end{pmatrix} + \\ &+ \sum_S p_t(S) \sum_{i;j\neq y} S\begin{pmatrix}x+1\\y\end{pmatrix} P_0(i)(1-P_0(i))P_0(x+1)S\begin{pmatrix}i\\j\end{pmatrix} + \\ &+ \sum_S p_t(S) \sum_{j\neq 0, i\neq y} S\begin{pmatrix}x-1\\y\end{pmatrix} P_0(x-1)(1-P_0(x-1))P_0(j)S\begin{pmatrix}j\\i\end{pmatrix}\end{aligned} \tag{3.21}$$

and again with autocorrelation functions

$$\begin{aligned}\dot{\Pi}\binom{x}{y} = &\sum_{i,j\neq y} -R\left(\binom{x}{y},\binom{i}{j}\right) P_0(i)(1-P_0(i))P_0(x)+\\&+\sum_{i\neq y, j\neq 0} -R\left(\binom{x}{y},\binom{j}{i}\right) P_0(x)(1-P_0(x))P_0(j)+\\&+\sum_{i,j\neq y} R\left(\binom{x+1}{y},\binom{i}{j}\right) P_0(i)(1-P_0(i))P_0(x+1)+\\&+\sum_{j\neq 0, i\neq y} R\left(\binom{x-1}{y},\binom{j}{i}\right) P_0(x-1)(1-P_0(x-1))P_0(j).\end{aligned} \tag{3.22}$$

When we look at the transition probabilities (3.5) and (3.6) for transactions barter-money and money-barter more carefully, we notice that we can save ourselves a lot of work. Because we set $m(.) = P_0(.)$, it holds $w_{BB}(.) = w_{BM}(.)$ and $w_{MM}(.) = w_{MB}(.)$. Thus it is fully sufficient to calculate with w_{BB} and w_{MM} only! Thanks to these very helpful identities we can therefore immediately proceed to the next step.

3.6 Model simplification

Looking back at the equations we have expressed the time derivate of the averaged occupancy Π_t as a function of correlation functions of the second order $R\left(\binom{x}{y},\binom{i}{j}\right)$. This set of equations isn't closed. We could have tried to take the time derivative of correlation functions but we would only obtain correlation functions of higher order (specifically up to the fourth order, for more detail see Appendix A). With this approach we could go on and on but the system would never close. The only way how to move forward is to apply the approximation. In the following we use the Kirkwood approximation (which could be found in [19]) and we approximate each correlation function $R(\binom{x}{y},\binom{w}{z})$ by $\Pi\binom{x}{y}\Pi\binom{w}{z}$. This would be true if (for example) the elements (x, y) and (w, z) were independent. This scenario, evidently, is not fulfilled because the elements influence each other but in the case of a big system this dependence is not too strong and we can therefore hope that the error induced by this approximatin is small enough.

To make the calculation easier we will try to find such a solution for which all products are equally distributed first, i.e. the homogeneous solution for which holds $\Pi\binom{i}{x} = \Pi\binom{i}{y}$ for all x, y. To stress the independence from the commodity number we use notation $\Pi\binom{i}{.}$. The differential equation for the barter-barter trade after all these simplifications looks

$$\begin{aligned}\dot{\Pi}\binom{0}{\cdot} =&(N_{pr}-1)\sum_{i=1}^{N_{un}} -\Pi\binom{0}{\cdot}\Pi\binom{i}{\cdot}P_0(i)(1-P_0(0))+\\&+(N_{pr}-1)\Pi\binom{1}{\cdot}\Pi\binom{0}{\cdot}P_0(1)(1-P_0(0))\end{aligned}\tag{3.23}$$

$$\begin{aligned}\dot{\Pi}\binom{1}{\cdot} =&-(N_{pr}-1)\Pi\binom{1}{\cdot}\Pi\binom{0}{\cdot}P_0(1)(1-P_0(0))+\\&+(N_{pr}-1)\Pi\binom{2}{\cdot}\Pi\binom{0}{\cdot}P_0(2)(1-P_0(0))+\\&+(N_{pr}-1)\sum_{i=1}^{N_{un}}\Pi\binom{0}{\cdot}\Pi\binom{i}{\cdot}P_0(i)(1-P_0(0))\end{aligned}\tag{3.24}$$

$$\begin{aligned}\dot{\Pi}\binom{x}{\cdot} =&-(N_{pr}-1)\Pi\binom{x}{\cdot}\Pi\binom{0}{\cdot}P_0(x)(1-P_0(0))+\\&+(N_{pr}-1)\Pi\binom{x+1}{\cdot}\Pi\binom{0}{\cdot}P_0(x+1)(1-P_0(0))\end{aligned}\tag{3.25}$$

where $x \geq 2$. For the money-money trade similarly

$$\begin{aligned}\dot{\Pi}\binom{0}{\cdot} =&(N_{pr}-1)\sum_{j=1}^{N_{un}} -\Pi\binom{0}{\cdot}\Pi\binom{j}{\cdot}P_0(0)(1-P_0(0))P_0(j)+\\&+(N_{pr}-1)\sum_{i=0}^{N_{un}}\Pi\binom{1}{\cdot}\Pi\binom{i}{\cdot}P_0(i)(1-P_0(i))P_0(1)\end{aligned}\tag{3.26}$$

$$\begin{aligned}\dot{\Pi}\binom{x}{\cdot} =&(N_{pr}-1)\sum_{i=0}^{N_{un}} -\Pi\binom{x}{\cdot}\Pi\binom{i}{\cdot}P_0(i)(1-P_0(i))P_0(x)+\\&+(N_{pr}-1)\sum_{j=1}^{N_{un}} -\Pi\binom{x}{\cdot}\Pi\binom{j}{\cdot}P_0(x)(1-P_0(x))P_0(j)+\\&+(N_{pr}-1)\sum_{i=0}^{N_{un}}\Pi\binom{x+1}{\cdot}\Pi\binom{i}{\cdot}P_0(i)(1-P_0(i))P_0(x+1)+\\&+(N_{pr}-1)\sum_{j=1}^{N_{un}}\Pi\binom{x-1}{\cdot}\Pi\binom{j}{\cdot}P_0(x-1)(1-P_0(x-1))P_0(j)\end{aligned}\tag{3.27}$$

where $x \geq 1$. The derivative of complete averaged occupancy will be some linear combination of the derivatives for barter-barter and money-money trade, the coefficient belonging to barter-barter trade we denote as α, the coefficient belonging to money-money trade we denote as β.

Let us now investigate the equilibrium point, i.e. the solution which does not change in time. We lay the derivative of the averaged occupancy equal to zero and get a system of nonlinear difference equations with variables $\Pi\binom{i}{.}$, $i = 0, 1, .., N_{un}$. Our goal in the next chapter will be of course the exact solution to this system of equations.

3.7 Equilibrium

In the equations derived above sums and products of overaged occupancies occure. We can notice that many of the sums are similar. In fact there are only two types of sums which we denote by A and B

$$A \equiv (N_{pr} - 1) \sum_{i=0}^{N_{un}} \Pi \binom{i}{.} P_0(i)(1 - P_0(i)) \tag{3.28}$$

$$B \equiv (N_{pr} - 1) \sum_{j=1}^{N_{un}} \Pi \binom{j}{.} P_0(j). \tag{3.29}$$

Next, because we deal with probabilities, the normalization condition must be met. Therefore we can choose $\Pi\binom{0}{.} = 1$ and all the probabilities normalize at the end of the calculation. With this choice we are able to eliminate the product between the averaged occupancies. To make the equations more simple let us introduce these three new symbols

$$\begin{aligned} \theta(x) &\equiv \Pi \binom{x}{.} P_0(x) \\ f(x) &\equiv 1 - P_0(x) \\ \sigma &\equiv (N_{pr} - 1)(1 - P_0(0))\alpha. \end{aligned}$$

After all these transformations the final equations look

$$0 = \dot{\Pi}\binom{0}{.} = -\alpha B f(0) + \sigma\theta(1) - \beta B P_0(0) f(0) + \beta A \theta(1) \tag{3.30}$$

$$\begin{aligned} 0 = \dot{\Pi}\binom{1}{.} &= -\sigma\theta(1) + \sigma\theta(2) + \alpha B f(0) - \beta A \sigma(1) - \\ &- \beta B \theta(1) f(1) + \beta A \theta(2) + \beta B \theta(0) f(0) \end{aligned} \tag{3.31}$$

$$\begin{aligned} 0 = \dot{\Pi}\binom{x}{.} &= -\sigma\theta(x) + \sigma\theta(x+1) - \beta A \sigma(x) - \beta B \theta(x) f(x) + \\ &+ \beta A \theta(x+1) + \beta B \theta(x-1) f(x-1) \end{aligned} \tag{3.32}$$

where again $x \geq 2$. In equation (3.32) we obtain the relation between three successive values $\theta(x-1)$, $\theta(x)$ and $\theta(x+1)$, this difference equation of second order is equipped with initial conditions (3.30) and (3.31). Equation (3.32) can be rewritten into a more suitable form

$$\theta(x+1)\left[\sigma + A\beta\right] = \theta(x)\left[\sigma + A\beta + f(x)B\beta\right] - \theta(x-1)\left[f(x-1)B\beta\right]. \tag{3.33}$$

When we apply this relation at the term $\theta(x+k)$ for some natural k, we get

$$\begin{aligned}\theta(x+k) =& \sum_{i=0}^{k}\left(\frac{\beta B}{\sigma+\beta A}\right)^{k-i}\frac{f(x+k-1)!}{f(x+i-1)!}\theta(x)- \\ &-\sum_{i=0}^{k-1}\left(\frac{\beta B}{\sigma+\beta A}\right)^{k-i}\frac{f(x+k-1)!}{f(x+i)!}f(x-1)\theta(x-1)\end{aligned} \tag{3.34}$$

where we have used this notation similar to the definition of factorials

$$\frac{f(x)!}{f(y)!} = f(x).f(x-1)...f(y+1). \tag{3.35}$$

The value of $x \geq 2$ is arbitrary, we can set $x = 2$ a because $k = 1, 2, ...$, then all $\theta(x)$ for $x \geq 3$ are expressed as a function of $\theta(1)$ and $\theta(2)$. Equation (3.30) gives us the value of $\theta(1)$

$$\theta(1) = \frac{f(0)\alpha B + \beta B f(0) P_0(0)}{\sigma + \beta A}. \tag{3.36}$$

Substituting from (3.36) into (3.31) we obtain the value of $\theta(2)$

$$\begin{aligned}\theta(2) =& \frac{f(0)\alpha B + \beta B f(0) P_0(0)}{(\sigma+\beta A)^2}\left[\sigma + \beta(A + f(1)B)\right] - \\ &- \frac{P_0(0) f(0)\beta B + f(0) B\alpha}{(\sigma+\beta A)}.\end{aligned} \tag{3.37}$$

Substituting (3.36) and (3.37) into (3.34) with $x = 2$ gives us the final set of equations

$$\begin{aligned}\theta(2+k) =& -\frac{\beta B f(0) P_0(0) + B f(0)\alpha}{\sigma+\beta A}\sum_{i=0}^{k}\left(\frac{\beta B}{\sigma+\beta A}\right)^{k-i}\frac{f(2+k-1)!}{f(2+i-1)!}+ \\ &+ \frac{f(0)\alpha B + \beta B f_0(0) P_0(0)}{\sigma+\beta A}\left[\frac{\sigma+\beta A+\beta f(1)B}{\sigma+\beta A}\sum_{i=0}^{k}\left(\frac{\beta B}{\sigma+\beta A}\right)^{k-i}\frac{f(2+k-1)!}{f(2+i-1)!} - \right. \\ &\left. - \sum_{i=0}^{k-1}\left(\frac{\beta B}{\sigma+\beta A}\right)^{k-i}\frac{f(2+k-1)!}{f(2+i)!}f(1)\right].\end{aligned} \tag{3.38}$$

Every transformed averaged occupancy $\theta(x) = \Pi\binom{x}{\cdot}P_0(x)$ can be expressed in terms of A and B. When we introduce another coefficient Z

$$Z \equiv \frac{B}{\sigma + \beta A}, \tag{3.39}$$

all θ's can be expressed as functions of this coefficient

$$\theta(0) = P_0(0) \tag{3.40}$$

$$\theta(1) = Z(f(0)\alpha + \beta f(0)P_0(0)) \tag{3.41}$$

$$\theta(2) = Z^2\left(f(0)\alpha + \beta f(0)P_0(0)\right)\beta f(1) \tag{3.42}$$

$$\begin{aligned}\theta(2+k) = &-Z\left(\beta f(0)P_0(0) + f(0)\alpha\right)\sum_{i=0}^{k}(\beta Z)^{k-i}\frac{f(2+k-1)!}{f(2+i-1)!}+\\ &+Z\left(f(0)\alpha + \beta f(0)P_0(0)\right)\left[(1+Z\beta f(1))\sum_{i=0}^{k}(\beta Z)^{k-i}\frac{f(2+k-1)!}{f(2+i-1)!}-\right.\\ &\left.-\sum_{i=0}^{k-1}(\beta Z)^{k-i}\frac{f(2+k-1)!}{f(2+i)!}f(1)\right].\end{aligned} \tag{3.43}$$

Coefficient Z is the function of transformed averaged occupancies $\theta(x)$ and these occupancies are on the other hand functions of input parameters and Z. We have therefore come to a single equation in Z, its solution can be substituted into (3.40)-(3.43) and after the change to original overaged occupancies $\Pi\binom{x}{\cdot}$ and the normalization we obtain our desired solution.

It only remains to compute the values of coefficients α and β. From the simulation we already know that the ratio between barter and money trades is constant after the system stabilizes and therefore we can borrow these coefficients obtained in the simulation and use them in our analytical calculation. For example, we obtained for the parameters $N_{ag} = 20$, $N_{pr} = 20$, $N_{un} = 40$, $N_{mem} = 40$ values $\alpha = 0.5$ and $\beta = 0.5$ (we included no trades in α).

Unfortunately, we aren't able to solve the equation in Z analytically. We could obtain some polynomial equation but we aren't able to express its coefficients in a compact form. The problem is in the sum of terms $\frac{f(2+k-1)!}{f(2+i-1)!}$ which are very difficult to manipulate. Therefore we must try to solve the equation numerically.

We tried to solve the equation with the help of the commercial program MATLAB. We transform our equation into the form $g(Z) = 0$ and then let the program to find such Z which set function g to zero. For this purpose we have used built-in function *fsolve*. The procedure seemed to have big difficulties to find the solution for our problem. When we look at Figure (3.1) we can guess why. As we can see the numerical representation of the calculatin of function g is numerically unstable and cannot give us the value of Z.

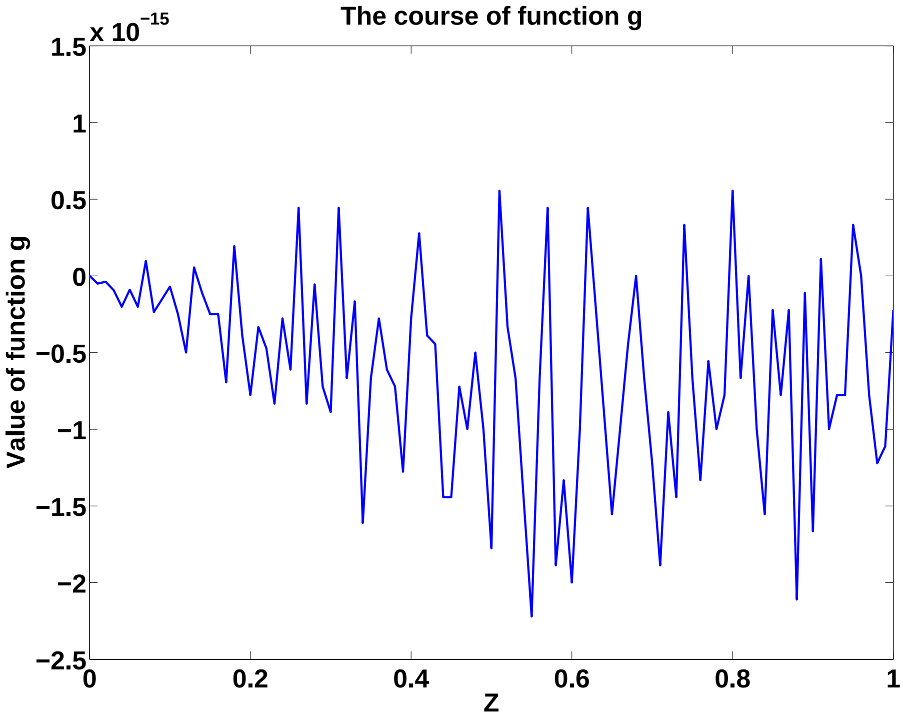

Figure 3.1: The numerical representation of function g is unstable.

We have therefore no other option then to try to find Z another way. We recall that at the beginning of this section we have assumed that the distribution of commodities in the representative agent's portfolio is the same for all commodities. This implies that the expectation is the same for all commodities. The agent is initially given N_{un} items of commodities, this number is conserved, and there are in total N_{pr} types of commodities in the system. From this we conclude that the mean value of each distribution should be N_{un}/N_{pr}.

This additional condition which is the result of our earlier restrictions can be easily incorporated into our calculations. When we investigate the dependence of the mean of the equilibrium state distribution on Z, we find that the mean is an increasing function of coefficient Z. This dependence can be seen in Figure (3.2) where we have plotted the first few averaged occupancies for different values of Z.

When we take for example the system with input parameters $N_{ag} = 20$, $N_{pr} = 20$ and $N_{un} = 40$, then the expectation should be equal to 2. This happens when we choose $Z = 0.167$. Figure (3.3) pictures the occupancies for such a choice of Z.

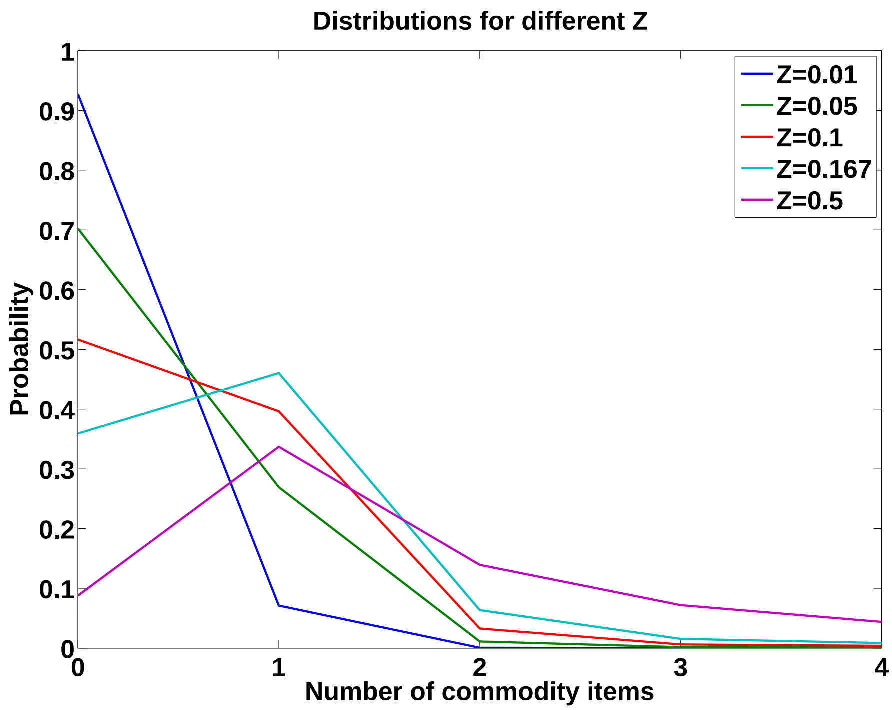

Figure 3.2: The mean of the equilibrium distribution is an increasing function of Z. Parameters were set $N_{ag} = 20$, $N_{pr} = 20$, $N_{un} = 40$, $N_{mem} = 40$. The expectations are 0.08, 0.51, 1.14, 2.01 and 5.45, respectivelly to increasing Z.

3.8 Conclusion

The analytical approach is based on the description of one representative agent and the mean field approximation for the rest of the system. The evolution of the model we describe as a stochastic process with Markow property, for quantities such as memory, which depend on the history of the process, functions of the present state were devised and we hope that these functions are good approximation of the original and more complicated history-dependent process described in chapter 2. With the help of operator L_{ij} and probabilities of transition (3.3)-(3.6) we are able to construct differential

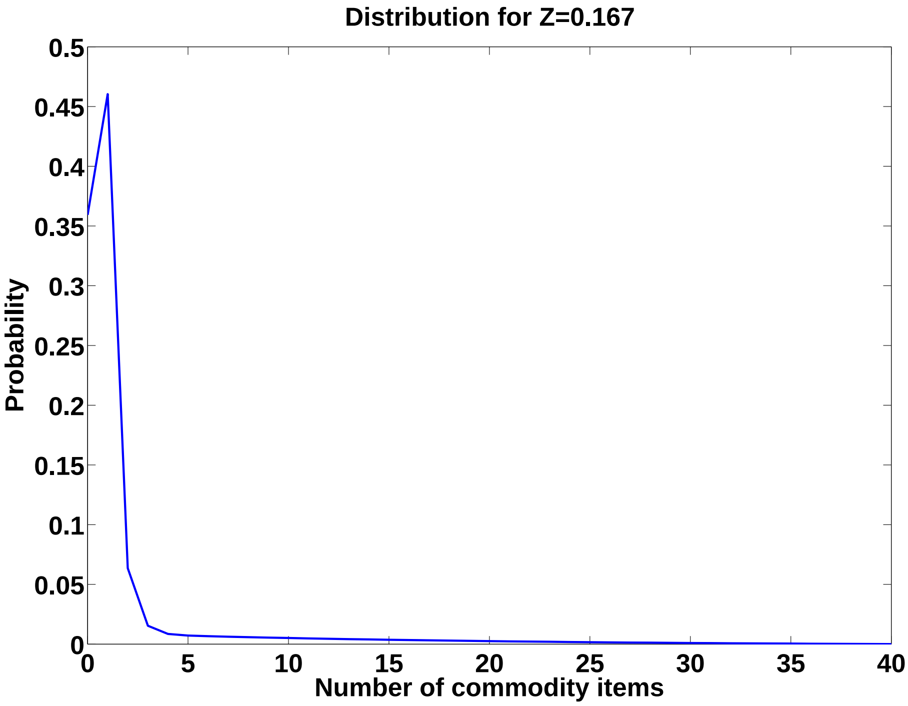

Figure 3.3: The equilibrium state distribution. Parameters were set $N_{ag} = 20$, $N_{pr} = 20$, $N_{un} = 40$ and $N_{mem} = 40$. We can notice that the probability of having more than 4 items of some commodity is highly low.

equations of the first order for the averaged occupancies $\Pi\binom{x}{y}$. During the calculation we used the Kirkwood approximation.

To make the calculations more simple and accessible we suppose that the distribution function is common for all commodities. We are interested in the equilibrium state so we lay all the derivatives equal to zero and the equations take the form of difference equations with a parameter Z. We have solved these equations and for the determination of Z we used the condition on the mean value of the distribution which can be easily calculated from the input parameters of the model. After normalization of the probabilities we obtain the final shape of the distribution function of the equilibrium state shown in Figure (3.3).

Chapter 4

Results summary

In our work we picked up threads of Donangelo and Sneppen's paper [12] which belongs to the area of econophysics. As well as the authors we have devised a model of an early economy describing the emergence of money as a result of trading activity of the agents. We have tried to investigate the features of our model both by simulations and analytically.

Let us first summarize what were the outcomes of the simulations. Besides already mentioned conclusion from [12] we have proved that for the emergence of money some level of rational decision-making of all participants is essential. Without it the system loses its interesting properties and becomes chaotic. In order to describe the level of rationality we have introduced the concept of temperature. We have clearly observed how gradual increase of temperature from the zero level induced gradual erosion of agents' memories and therefore lower level of organization.

In the original paper the commodities were equally distributed throughout the system and they had equal chance to become money. When we break this symmetry and distribute the commodities unfairly throughout the system then we can see that the rare commodity has higher potential to become money than a commodity more frequently represented in the system. This observation is in full agreement with our expectation because usually those commodities which were hard to find became money in real early economies.

We have also discovered that the loss of money property happens when the popularity index Q drops into a certain interval. This feature of our model gives us the possibility to predict the future evolution of the system and the sudden decrease of popularity might be the sign of an approaching crash.

The waiting times between the emergences and collapses of money feature seem to be uncorrelated. The frequency plot of their occurencies decay with power-law whose exponent is 1.8. Compared to the exponent of one-dimensional random walk 1.5 this indicates the nontriviality of our stochastic model.

In the analytical section we have calculated the equilibrium state for the homogeneous distribution of commodities. Our whole calculation was based on the introduction of the representative agent, who is the only agent in the system we observe, and mean field which subsitutes for the rest of the system.

We have approximated the evolution of the system with the help of Makov property. The history dependent parts of the simulation we have substituted by appropriate functions of present state. The dynamics of the system is induced by the application of operator L_{ij} on the state vector of the portfolio $\vec{R}$. Another representation for the portfolio is the matrix form $S\binom{x}{y}$ which was very useful in the following calculations.

We compute the transition matrix of the Markov chain which we further use in the derivation of our master equation. In order to close the set of equations we apply the Kirkwood approximation and replace the correlation functions by a simple product of two probabilities. In order to simplify the equations we set the first probability equal to one and obtain the master equation in the form of a system of linear differential equations of the first order. We keep in mind that we must normalize all probabilities at the end of the calculation. For simplicity we have confined ourselves to the set of solutions for which all products are equally distributed.

We are interested in the equilibrium state so we lay all derivatives in the master equation equal to zero and obtain the set of difference equations which we have subsequently solved. For the determination of parameter Z we use the condition on the mean value of the distribution function which can be easily calculated from the input parameters.

The final shape of the distribution function of the equilibrium you can see in Figure (3.3).

To sum up we have devised an economic system composed of interacting agents which simulates the emergence of money in early economies. We have explored several features of our model and all of them agree with our notion of the processes in real early economies. We have supplied the simulations with the analytical approach using the representative agent and mean field approximation.

The emergence of money was a very important step in the developement of human society and thus we find it very useful to investigate the nature of this phenomenon. Although the model simulates the situation in distant past it could reveal some economic relations applicable to present. For example, it would be very interesting to discuss the connection between our model and the regulations of the market. We have discovered that for the emergence of money no external forces are needed and it is the trade itself that induces the observed organization. This result could be very tempting for those economists who advocate liberal points of view.

Appendix A

Correlation functions of higher order

We saw in section (3.4) that we can express the derivative of averaged occupancy as a sum of two site correlation functions. If we don't want to apply immediately the Kirkwood approximation and would like to obtain a more accurate solution, we need to investigate the derivate of these correlation functions. When we look how the correlation function is defined, we realize that a lot of work was already done. Following the procedure from equation (3.7) to equation (3.11), i.e. differentiating term by term, inserting the derivative of $p_t(S)$, substituting and reordering the terms give the derivative of R at some points (x, y) and (a, b)

$$\frac{dR}{dt}\left(\binom{x}{y}, \binom{a}{b}\right) = \sum_S \sum_{i \neq j} w(S \to L_{ij}S) p_t(S) \left[\widehat{L_{ij}\vec{R}}\binom{x}{y} \widehat{L_{ij}\vec{R}}\binom{a}{b} - S\binom{x}{y} S\binom{a}{b} \right]. \tag{A.1}$$

Again, there are only four cases when the square bracket is nonzero but before we enlist these cases we should mention one very useful property of the correlation function $R\left(\binom{x}{y}, \binom{a}{b}\right)$. Without loss of generality we can assume that $x = 0$ because one of the positions in $R\left(\binom{x}{y}, \binom{a}{b}\right)$ has always its first coordinate zero as can be easily verified from all equations in section (3.4). The same observation tells us that $y \neq b$ and $a \neq 0$. The four cases we are interested in are:

- $S\binom{x}{y} = S\binom{a}{b} = 1$, in this scenario either commodity y is the receiver (the representative agent buys 1 item of y) or commodity b the donor (the representative agent sells b)

- $S\binom{x}{y} = 1$, $S\binom{a}{b} = 0$, here $S\binom{x}{y} S\binom{a}{b}$ equals zero and in order to make the rest of the bracket nonzero $S\binom{a}{b}$ must become one by means of operator L_{ij}. Therefore in case $a \neq 1$ commodity b must be the donor, in case $a = 1$ it can be either the donor or the receiver. Commodity b cannot be interchanged with commodity y for apparent reason that then $S\binom{x}{y}$ would be zero and the construction would fail

- $S\binom{x}{y} = 0$, $S\binom{a}{b} = 1$, here commodity y must be the donor, $S\binom{1}{y}$ has to be one and we should avoid the interchange between y and b
- $S\binom{x}{y} = S\binom{a}{b} = 0$, here we must change both positions to one and this is possible in the only way: $a = 1$, commodity b is the receiver and y the donor (we remind that the trade is barter-barter only)

We will now write the equation for the derivative of $R\left(\binom{x}{y}, \binom{a}{b}\right)$ and the explanation how this was obtained from the four cases will be given afterwards.

$$
\begin{aligned}
&\dot{R}\left(\binom{x}{y}, \binom{a}{b}\right) = \\
&\sum_S p_t(S) \sum_{i\neq 0, j\neq y} -S\binom{x}{y} S\binom{a}{b}(1-P_0(0))P_0(i)S\binom{i}{j}+ \\
&+\sum_S p_t(S) \sum_{i\neq b} -S\binom{x}{y} S\binom{a}{b} P_0(a)(1-P_0(0))S\binom{0}{i}+ \\
&+\sum_S p_t(S) S\binom{x}{y} S\binom{a}{b} P_0(a)(1-P_0(0))+ \\
&+\sum_S p_t(S) \sum_{i\neq y, i\neq b} S\binom{x}{y}\left(1-S\binom{a}{b}\right) S\binom{a+1}{b} S\binom{0}{i}(1-P_0(0))P_0(a+1)+ \\
&+\sum_S p_t(S) \sum_{i\neq 0; j\neq b,y} S\binom{x}{y}\left(1-S\binom{a}{b}\right) S\binom{0}{b} S\binom{i}{j} P_0(i)(1-P_0(0))+ \\
&+\sum_S p_t(S) \sum_{i\neq y, i\neq b} \left(1-S\binom{x}{y}\right) S\binom{a}{b} S\binom{1}{y} S\binom{0}{i}(1-P_0(0))P_0(1)+ \\
&+\sum_S p_t(S) \left(1-S\binom{x}{y}\right) S\binom{0}{y} S\binom{1}{y}(1-P_0(0))P_0(1)
\end{aligned}
\tag{A.2}
$$

The first three rows are derived from the first condition, in the first term commodity y is bought, in the second term commodity b is sold and the third term is because we included one possibility twice.

The next two rows are for the second case. In the first term commodity b is sold, in the second term b is bought (the second term is just for $a = 1$).

Another row is the third case and the last row is the last case (again, the last term is valid just for $a = 1$).

It is clear how the equation would look like when transfered into the language of correlation functions of the second, third and fourth order. More important is that the set of equations does not close. The derivative of Π is a function of correlation functions of second order and the derivative of the correlation functions of the second order depends on the correlation functions up to the fourth order. It is obvious that we could go on and we would get correlations of higher and higher order indefinitely. Again, in order to avoid this we would approximate the correlation function of higher order as a product of correlations of lower order. By doing this we would close the system and we would obtain a more accurate solution than in the main text.

After the application of the Kirkwood approximation (the underlined terms are for $a = 1$ only) we get

$$\begin{aligned}
&\dot{R}\left(\binom{x}{y},\binom{a}{b}\right) = \\
&(1-P_0(0))\left[R\left(\binom{x}{y},\binom{a}{b}\right)\left\{\sum_{i\neq 0, j\neq y} -P_0(i)\Pi\binom{i}{j} + \sum_{i\neq b} -P_0(a)\Pi\binom{0}{i} + P_0(a)\right\} + \right.\\
&+ R\left(\binom{x}{y},\binom{a+1}{b}\right)\sum_{i\neq y,b} P_0(a+1)\Pi\binom{0}{i} + \underline{R\left(\binom{x}{y},\binom{0}{b}\right)\sum_{i\neq 0; j\neq b,y} P_0(i)\Pi\binom{i}{j} +}\\
&\left. + R\left(\binom{a}{b},\binom{1}{y}\right)\sum_{i\neq y,b} P_0(1)\Pi\binom{0}{i} + \underline{R\left(\binom{0}{b},\binom{1}{y}\right)P_0(1)}\right].
\end{aligned} \tag{A.3}$$

For the money-money trade we would proceed similarly.

Appendix B

Linear stability analysis

In the chapter Analytical solution we have obtained the equilibrium for the case of the homogeneous distribution of products. Here we would like to outline the approach which could be used for the investigation of the stability of this equilibrium.

Our master equation can be written in the compact form without references to concrete elements as

$$\dot{\Pi} = F(\Pi) \tag{B.1}$$

where F is a function operating on the elements of matrix Π. This matrix equation can be expressed as a vector equation when we introduce new notation $\vec{\Pi}$ which denotes the vector composed of colums of Π

$$\vec{\Pi} \equiv \left(\Pi\binom{0}{1}, \Pi\binom{1}{1}, ..., \Pi\binom{N_{un}}{1}, \Pi\binom{0}{2}, ..., \Pi\binom{N_{un}}{N_{pr}}\right)^T. \tag{B.2}$$

We are interested in the stability of the equilibrium and therefore we need to calculate the eigenvalues of the Jacobian matrix of F

$$DF(\vec{\Pi}) \equiv \frac{\partial F(\vec{\Pi})}{\partial \vec{\Pi}}. \tag{B.3}$$

We have calculated the equilibrium point for the homogeneous distribution of products and because of this equivalence between products $DF(\vec{\Pi})$ will be a block matrix with only two kinds of blocks with dimensions $(N_{un} + 1) \times (N_{un} + 1)$. In the blocks on the diagonal we differentiate the occupancy of some product with respect to the occupancy of the same product and this first block we denote as D (diagonal). Outside the diagonal we differentiate some product's occupancy with respect to the accupancy of another product. This block we denote as N (nondiagonal).

If the blocks were single entries, our task would be to find eigenvalues of matrix M

$$M \equiv \begin{pmatrix} D & N & \cdots & N \\ N & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & N \\ N & \cdots & N & D \end{pmatrix}.$$

M is a symmetric matrix and therefore from the spectral theorem we know that it can be diagonalized by an orthogonal matrix O whose columns are perpendicular eigenvectors. From the shape of M it is clear that one eigenvector could be vector $\vec{E}_1$ which consists of one's.

$$\vec{E}_1 = (1, 1, ..., 1)^T$$

The eigenvalue corresponding to this eigenvector is the sum of elements in the row of M. Let us try vector $\vec{E}_2$ which is the simpliest vector perpendicular to $\vec{E}_1$ we can think of

$$\vec{E}_2 = (+1, -1, 0, ..., 0)^T.$$

We can check that it holds $M\vec{E}_1 = (D - N).\vec{E}_1$ so $\vec{E}_2$ is also an eigenvector with the eigenvalue $(D - N)$. Now let us try to find another eigenvector perpendicular to $\vec{E}_1$ and $\vec{E}_2$. Every vector with two first components equal to one is perpendicular to $\vec{E}_2$. We calculate its third component in order to be perpendicular to $\vec{E}_1$ and so obtained vector $\vec{E}_3$ is again an eigenvector with the same eigenvalue as $\vec{E}_2$ has.

$$\vec{E}_3 = (1, 1, -2, 0, ..., 0)^T$$

This procedure could be generalized and we get the rule for the construction of all other orthogonal eigenvectors $\vec{E}_4$, $\vec{E}_5$, $\vec{E}_6$, ..., eigenvalues coresponding to these eigenvectors have the same value $(D - N)$.

$$\begin{aligned} \vec{E}_4 &= (1, 1, 1, -3, 0, ..., 0,)^T \\ \vec{E}_5 &= (1, 1, 1, 1, -4, 0, ..., 0,)^T \\ \vec{E}_6 &= (1, 1, 1, 1, 1, -5, 0, ..., 0,)^T \\ \cdots &= \cdots \end{aligned}$$

We have discovered that matrix M can be diagonalized by orthogonal matrix O (whose columns are vectors $\vec{E}_1$, $\vec{E}_2$, $\vec{E}_3$,...) and after the transformation the first entry on the diagonal is the sum of elements in the row of M, the rest of elements on the diagonal have the same value $D - N$. Similar approach we can apply to block matrix $DF(\vec{\Pi})$ and the problem of finding the eigenvalues of $DF(\vec{\Pi})$ reduces to the problem of finding eigenvalues of much smaller matrices $D-N$ and $D+(N_{pr}-1)N$.

To keep the notation transparent we will use symbol $\frac{\partial F_i}{\partial x_j}$ for the derivative of the i-th occupancy with respect to the j-th occupancy. Whether we take the derivate with respect to the same commodity or to other commodity, we find out from the context (diagonal vs. nondiagonal block).

When we look at the equations in section 3.4 and imagine that the Kirkwood approximation is already performed, we can straightforwardly differentiate. Thus the elements in D for the barter trade will be

$$
\begin{aligned}
\frac{\partial F_0}{\partial x_0} &= \sum_{i\neq 0, j\neq y} -\Pi\binom{i}{j} P_0(i)(1-P_0(0)) = -f(0)U_2 \\
\frac{\partial F_0}{\partial x_1} &= \sum_{i\neq y} \Pi\binom{0}{i} P_0(1)(1-P_0(0)) = f(0)P_0(1)U_1 \\
\frac{\partial F_1}{\partial x_0} &= \sum_{i\neq 0, j\neq y} \Pi\binom{i}{j} P_0(i)(1-P_0(0)) = f(0)U_2 \\
\frac{\partial F_1}{\partial x_1} &= \sum_{i\neq y} -\Pi\binom{0}{i} P_0(1)(1-P_0(0)) = -f(0)P_0(1)U_1 \\
\frac{\partial F_1}{\partial x_2} &= \sum_{i\neq y} \Pi\binom{0}{i} P_0(2)(1-P_0(0)) = f(0)P_0(2)U_1 \\
\frac{\partial F_k}{\partial x_k} &= \sum_{i\neq y} -\Pi\binom{0}{i} P_0(k)(1-P_0(0)) = -f(0)P_0(k)U_1 \\
\frac{\partial F_k}{\partial x_{k+1}} &= \sum_{i\neq y} -\Pi\binom{0}{i} P_0(k+1)(1-P_0(0)) = -f(0)P_0(k+1)U_1
\end{aligned}
$$

where $k \geq 2$. In the equations above we use symbols U_1 and U_2

$$
U_1 \equiv (N_{pr}-1)\Pi\binom{0}{\cdot} \tag{B.4}
$$

$$
U_2 \equiv (N_{pr}-1)\sum_{i\neq 0}\Pi\binom{i}{\cdot} P_0(i) \tag{B.5}
$$

as we again take the advantage of the homegeneous distribution of products. All other elements of the matrix except those listed above equal zero. For the nondiagonal block N of the barter trade we obtain

$$\begin{aligned}
\frac{\partial F_0}{\partial x_0} &= \Pi\binom{1}{.}P_0(1)f(0) \\
\frac{\partial F_0}{\partial x_i} &= -\Pi\binom{0}{.}P_0(i)f(0) \\
\frac{\partial F_1}{\partial x_0} &= -\Pi\binom{1}{.}P_0(1)f(0) + \Pi\binom{2}{.}P_0(2)f(0) \\
\frac{\partial F_1}{\partial x_i} &= \Pi\binom{0}{.}P_0(i)f(0) \\
\frac{\partial F_j}{\partial x_0} &= -\Pi\binom{j}{.}P_0(j)f(0) + \Pi\binom{j+1}{.}P_0(j+1)f(0)
\end{aligned}$$

where $i \geq 1$ and $j \geq 2$. All other elements equal zero again.

When we now differentiate the master equation for the money trade from section (3.5), we have for the elements of the diagonal block D

$$\begin{aligned}
\frac{\partial F_0}{\partial x_0} &= \sum_{j\neq 0, i\neq y} -\Pi\binom{j}{i}P_0(0)(1-P_0(0))P_0(j) = -f(0)P_0(0)U_2 \\
\frac{\partial F_0}{\partial x_1} &= \sum_{i;j\neq y} \Pi\binom{i}{j}P_0(i)(1-P_0(i))P_0(1) = P_0(1)U_3 \\
\frac{\partial F_k}{\partial x_{k-1}} &= \sum_{j\neq 0, i\neq y} \Pi\binom{j}{i}P_0(k-1)(1-P_0(k-1))P_0(j) = P_0(k-1)f(k-1)U_2 \\
\frac{\partial F_k}{\partial x_k} &= \sum_{i;j\neq y} -\Pi\binom{i}{j}P_0(i)(1-P_0(i))P_0(k) + \\
&\quad + \sum_{i\neq y, j\neq 0} -\Pi\binom{j}{i}P_0(k)(1-P_0(k))P_0(j) \\
&= -P_0(k)U_3 - f(k)P_0(k)U_2 \\
\frac{\partial F_k}{\partial x_{k+1}} &= \sum_{i;j\neq y} \Pi\binom{i}{j}P_0(i)(1-P_0(i))P_0(k+1) = P_0(k+1)U_3
\end{aligned}$$

where $k \geq 1$. Here we have introduced an additional symbol U_3

$$U_3 \equiv (N_{pr}-1)\sum_i \Pi\binom{i}{.}P_0(i)(1-P_0(i)). \tag{B.6}$$

It only remains to calculate the nondiagonal block N for the money trade

$$
\begin{aligned}
\frac{\partial F_0}{\partial x_0} &= \Pi\binom{1}{.}P_0(0)f(0)P_0(1)\\
\frac{\partial F_0}{\partial x_i} &= -\Pi\binom{0}{.}P_0(0)f(0)P_0(i)+\Pi\binom{1}{.}P_0(i)f(i)P_0(1)\\
\frac{\partial F_j}{\partial x_0} &= -\Pi\binom{j}{.}P_0(0)f(0)P_0(j)+\Pi\binom{j+1}{.}P_0(0)f(0)P_0(j+1)\\
\frac{\partial F_j}{\partial x_i} &= -\Pi\binom{j}{.}P_0(i)f(i)P_0(j)-\Pi\binom{j}{.}P_0(j)f(j)P_0(i)+\\
&\quad +\Pi\binom{j+1}{.}P_0(i)f(i)P_0(j+1)+\Pi\binom{j-1}{.}P_0(j-1)f(j-1)P_0(i)
\end{aligned}
$$

where $i \geq 1$ and $j \geq 1$. Again, all other elements are zero.

Now we have everything we need to for the construction of matrices $D-N$ and $D+(N_{pr}-1)N$. Unfortunatelly, the matrices are too complicated to solve in pencil and it is inevitable to use suitable software. We tried to solve the problem in the program MATLAB and all eigenvalues seemed to have negative real parts which implies stability of our solution but deeper investigation needs to be undertaken in the future.

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