Faculty of Mathematics and Physics
Charles University, Prague

Department of Mathematical Analysis

Abstract of the Doctoral Thesis

Fine properties of Sobolev embeddings

Jan Vybíral

Thesis supervisor: Doc. RNDr. Luboš Pick, DSc.

Prague, 2008

Matematicko-fyzikální fakulta
Univerzita Karlova, Praha

Katedra matematické analýzy

Autoreferát doktorské disertační práce

Jemné vlastnosti Sobolevova vnoření

Jan Vybíral

Školitel: Doc. RNDr. Luboš Pick, DSc.

Praha, 2008

Disertační práce byla vypracována v rámci doktorandského studia, které uchazeč absolvoval na Katedře matematické analýzy MFF UK v letech 2002-2008.

Doktorand: Jan Vybíral

Školitel: Doc. RNDr. Luboš Pick, DSc.
Katedra matematické analýzy
Matematicko-fyzikální fatkulta UK
Sokolovská 83, 186 75, Praha 8

Školící pracoviště: Katedra matematické analýzy
Matematicko-fyzikální fatkulta UK
Sokolovská 83, 186 75, Praha 8

Oponenti: Doc. RNDr. Petr Gurka, CSc.
Katedra matematiky
Technická fakulta ČZU
Kamýcká 129, 165 21 Praha 6 - Suchdol

Doc. RNDr. Aleš Nekvinda, CSc.
Katedra matematiky
Stavební fakulta ČVUT
Thákurova 7, 166 29 Praha 6

Autoreferát byl rozeslán dne ....

Contents

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $1 \le p \le \infty$ and let k be a natural number. We denote by $W_p^k(\Omega)$ the Sobolev spaces of functions from $L_p(\Omega)$ with all distributive derivatives of order smaller or equal to k in $L_p(\Omega)$. If

$$1 \le p_1, p_2 < \infty, \qquad k_1 - k_2 \ge n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ \tag{1.1}$$

and the boundary of Ω is Lipschitz then $W_{p_1}^{k_1}(\Omega)$ is continuously embedded into $W_{p_2}^{k_2}(\Omega)$, i.e.

$$W_{p_1}^{k_1}(\Omega) \hookrightarrow W_{p_2}^{k_2}(\Omega). \tag{1.2}$$

This theorem goes back to Sobolev [22].

If the inequality in (1.1) is strict, the embedding is even compact, cf. [20] and [15]. During the second half of the last century, this fact (and its numerous generalisations) found its applications in many areas of modern analysis, especially in connection with partial differential (and pseudo-differential) equations. For this reason, the study of spaces of smooth functions became an important part of functional analysis with (1.2) playing a central role. There is a vast literature on function spaces of Sobolev type and all of them deal also with many variants of the Sobolev embedding. We refer at least to [1], [19], [16], [23], [17] and [10].

The thesis is composed of 5 papers [27]–[31]. In these papers we studied several aspects of the Sobolev embedding (and some of its generalisations) and presented some new results.

In the following sections, we describe our achievements.

2 *Optimal Sobolev embeddings on* $\mathbb{R}^n$ Publ. Mat. 51 (2007), 17-44.

Let us first recall the concept of the non-increasing rearrangement.

We denote by $\mathfrak{M}(\mathbb{R}^n)$ the set of real-valued Lebesgue-measurable functions on $\mathbb{R}^n$ finite almost everywhere and by $\mathfrak{M}_+(\mathbb{R}^n)$ the class of non-negative functions in $\mathfrak{M}(\mathbb{R}^n)$. Finally, $\mathfrak{M}_+(0,\infty,\downarrow)$ denotes the set of all non-increasing functions from $\mathfrak{M}_+(0,\infty)$. Given $f \in \mathfrak{M}(\mathbb{R}^n)$ we define its non-increasing rearrangement by

$$f^*(t) = \inf\{\lambda > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| \le t\}, \quad 0 < t < \infty. \tag{2.1}$$

For a set $A \subset \mathbb{R}^n$ we denote by $|A|$ its Lebesgue measure. A detailed treatment of rearrangements may be found in [3].

We also recall some basic aspects of the theory of Banach function norms. For details, see again [3].

Definition 2.1. A functional $\varrho : \mathfrak{M}_+(0,\infty) \to [0,\infty]$ is called a *Banach function norm* on $(0,\infty)$ if, for all $f, g, f_n, (n = 1, 2, \dots)$, in $\mathfrak{M}_+(0,\infty)$, for all constants $a \ge 0$ and for all measurable subsets E of $(0,\infty)$, it satisfies the following axioms

$$\begin{aligned}
&(A_1) \quad \varrho(f)=0 \quad \text{if and only if} \quad f=0 \text{ a.e.};\\
&\qquad\;\; \varrho(af)=a\varrho(f);\\
&\qquad\;\; \varrho(f+g)\le\varrho(f)+\varrho(g);\\
&(A_2) \quad \text{if} \quad 0\le g\le f \text{ a.e. then} \quad \varrho(g)\le\varrho(f);\\
&(A_3) \quad \text{if} \quad 0\le f_n\uparrow f \text{ a.e. then} \quad \varrho(f_n)\uparrow\varrho(f);\\
&(A_4) \quad \text{if} \quad |E|<\infty \quad \text{then} \quad \varrho(\chi_E)<\infty;\\
&(A_5) \quad \text{if} \quad |E|<\infty \quad \text{then} \quad \int_E f\le C_E\varrho(f)
\end{aligned}$$

with some constant $0<C_E<\infty$, depending on ϱ and E but independent of f.

If, in addition, $\varrho(f)=\varrho(f^*)$, we say that ϱ is a *rearrangement-invariant (r.i.) Banach function norm.* We often use the notions *norm* and *r.i. norm* to shorten the notation.

Definition 2.2. Let ϱ_R and ϱ_D be two r. i. norms. We set

$$L^{\varrho_R}(\mathbb{R}^n)=\left\{u\in L^1_{\text{loc}}(\mathbb{R}^n): ||u|L^{\varrho_R}(\mathbb{R}^n)||=\varrho_R(u^*)<\infty\right\} \tag{2.2}$$

and

$$W^1_{\varrho_D}(\mathbb{R}^n)=\left\{u\in L^1_{\text{loc}}(\mathbb{R}^n): ||u|W^1_{\varrho_D}(\mathbb{R}^n)||=\varrho_D(u^*)+\varrho_D(|\nabla u|^*)<\infty\right\}. \tag{2.3}$$

The space L^{ϱ_R} is called a *rearrangement-invariant Banach function space.* It follows directly from its definition that if $u^*=v^*$ for two measurable functions u and v, then $||u|L^{\varrho_R}(\mathbb{R}^n)||=||v|L^{\varrho_R}(\mathbb{R}^n)||$. Hence, the norm depends only on the size of the function values, not on a specific distribution of these values. The space $W^1_{\varrho_D}(\mathbb{R}^n)$ is called the *Sobolev space associated to* L^{ϱ_D}. Here, ∇u denotes the gradient of a function u.

Our aim is to study the embedding

$$W^1_{\varrho_D}(\mathbb{R}^n)\hookrightarrow L^{\varrho_R}(\mathbb{R}^n). \tag{2.4}$$

The embedding (2.4) is equivalent to

$$\varrho_R(u^*)\le c[\varrho_D(u^*)+\varrho_D(|\nabla u|^*)], \quad u\in W^1_{\varrho_D}(\mathbb{R}^n). \tag{2.5}$$

The inequality (2.5) is the main subject of our study.

We are interested in two main questions:

1. Suppose that the 'range' norm ϱ_R is given. We want to find the optimal (that is, essentially smallest) norm ϱ_D for which (2.5) holds. The optimality means that if (2.5) holds with ϱ_D replaced by some other rearrangement-invariant norm σ, then there exists a constant $C>0$ such that $\varrho_D(u^*)\le C\sigma(u^*)$ for all functions $u\in L^1_{\text{loc}}(\mathbb{R}^n)$.

2. Suppose that the 'domain' norm ϱ_D is given. We would like to construct the corresponding optimal 'range' norm ϱ_R. This means that the ϱ_R will be the essentially largest rearrangement-invariant norm for which (2.5) holds.

The first step in the study of (2.5) is a reduction of (2.5) to the boundedness of certain Hardy operators.

Theorem 2.3. *Let ϱ_D, ϱ_R be two r.i. Banach function norms on $(0,\infty)$. Then the inequality*

$$\varrho_R(u^*) \le c[\varrho_D(u^*) + \varrho_D(|\nabla u|^*)], \quad u \in W^1_{\varrho_D}(\mathbb{R}^n), \tag{2.6}$$

holds if and only if there is a constant $K > 0$ such that

$$\varrho_R\left(\int_t^\infty f(s)s^{1/n-1}\mathrm{d}s\right) \le K\varrho_D\left(f(t) + \int_t^\infty f(s)s^{1/n-1}\mathrm{d}s\right) \tag{2.7}$$

for all $f \in \mathfrak{M}_+(0,\infty)$.

The main tool in the proof is the following generalisation of the Pólya—Szegö principle from [7, (4.3)]:

$$\int_0^t \left[-s^{1-1/n}\frac{\mathrm{d}u^*}{\mathrm{d}s}\right]^*(s)\mathrm{d}s \le c\int_0^t |\nabla u|^*(s)\mathrm{d}s, \tag{2.8}$$

which holds for every $t > 0$ and every weakly differentiable function u such that $(\nabla u) \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ and

$$|\{x \in \mathbb{R}^n : |u(x)| > s\}| < \infty \quad \text{for all} \quad s > 0.$$

Up to this place, our approach follows [10]. But unlike there, (2.7) involves two different integral operators and therefore it is still not suitable for further investigation. Therefore we will derive another equivalent version of (2.6). In (2.7) we substitute

$$g(t) = f(t) + \int_t^\infty f(s)s^{1/n-1}\mathrm{d}s, \quad f \in \mathfrak{M}_+(0,\infty), \quad t > 0. \tag{2.9}$$

We shall need also the inverse substitution. Namely, if g is defined by (2.9), then

$$f(t) = g(t) - e^{nt^{1/n}}\int_t^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s. \tag{2.10}$$

Finally, we sum up (2.9) and (2.10) and obtain

$$\int_t^\infty f(s)s^{1/n-1}\mathrm{d}s = e^{nt^{1/n}}\int_t^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u \text{ for a.e. } t > 0. \tag{2.11}$$

This substitution can now be used to reformulate (2.6).

Theorem 2.4. *Let ϱ_D, ϱ_R be two r.i. Banach function norms on $(0,\infty)$. Then, (2.6) is equivalent to*

$$\varrho_R\left(e^{nt^{1/n}}\int_t^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\right) \le c\varrho_D(g) \text{ for all } g \in \mathbf{G}, \tag{2.12}$$

where $\mathbf{G}$ is a new class of functions, defined by

$$\begin{aligned}\mathbf{G} = \Big\{g \in \mathfrak{M}_+(0,\infty) : \ &\text{there is a function } f \in \mathfrak{M}_+(0,\infty) \text{ such that} \\ &g(t) = f(t) + \int_t^\infty f(s)s^{1/n-1}\mathrm{d}s \text{ for all } t > 0\Big\} \\ = \Big\{g \in \mathfrak{M}_+(0,\infty) : g(t) - e^{nt^{1/n}}&\int_t^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s \ge 0 \text{ for all } t > 0\Big\}.\end{aligned} \tag{2.13}$$

Hence the inequality (2.6) is equivalent to the boundedness of the Hardy-type operator

$$(Gg)(u) = e^{nu^{1/n}} \int_u^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s, \quad u > 0, \tag{2.14}$$

on the set $\mathbf{G}$, the image of the positive cone $\mathfrak{M}_+(0,\infty)$ under the operator

$$f \to f(t) + \int_t^\infty f(s)s^{1/n-1}\mathrm{d}s.$$

Before we proceed any further we shall state some basic properties of the class $\mathbf{G}$.

Remark 2.5. (i) $\mathbf{G}$ contains all non-negative non-increasing functions.

(ii) For every g from $\mathbf{G}$, Gg is non-increasing.

(iii) The set $\mathbf{G}$ is a *convex cone*, that is, for every $\alpha, \beta > 0$ and $g_1, g_2 \in \mathbf{G}$, we have $\alpha g_1 + \beta\alpha_2 \in \mathbf{G}$.

Remark 2.6. (i) To show some applications we prove that $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{\frac{np}{n-p},p}(\mathbb{R}^n)$ for $1 \le p < n$. In this case, we have $\varrho_R(f) = ||f^*(t)t^{-1/n}||_p$ and $\varrho_D(f) = ||f||_p$. Using Remark 2.5 (ii) and the boundedness of classical Hardy operators on L^p we get for every function $g \in \mathbf{G}$ that

$$\begin{aligned}\varrho_R(Gg) = ||t^{-1/n}(Gg)^*(t)||_p &= \left|\left|t^{-1/n}e^{nt^{1/n}}\int_t^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\right|\right|_p \\ &\le \left|\left|t^{-1/n}\int_t^\infty g(u)u^{1/n-1}\mathrm{d}u\right|\right|_p \le c||t^{-1/n}g(t)t^{1/n}||_p = c||g||_p = c\varrho_D(g).\end{aligned}$$

(ii) Another application of the obtained results is the embedding $W^1(L^{n,1})(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$. In this case

$$\begin{aligned}\varrho_R(Gg) = \sup_{t>0}(Gg)(t) = (Gg)(0) &= \int_0^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u \\ &\le \int_0^\infty g(u)u^{1/n-1}\mathrm{d}u \le \int_0^\infty g^*(u)u^{1/n-1}\mathrm{d}u = \varrho_D(g)\end{aligned}$$

for every function $g \in \mathbf{G}$. Now we used Remark 2.5 (ii).

(iii) Both these applications recover well-known results. They demonstrate some important aspects of this method. First, the second basic property of the class $\mathbf{G}$ (c.f. Remark 2.5, (ii)) lies in the roots of every Sobolev embedding. Second, the boundedness of Hardy operators plays a crucial role in this theory.

Now we can describe the solution of one of the main problems stated before. We shall construct the optimal domain norm ϱ_D to a given range norm ϱ_R.

Theorem 2.7. *Let the norm ϱ_R satisfy*

$$\varrho_R\left(G(g^{**})\right) \le c\varrho_R\left(G(g^*)\right), \quad g \in \mathfrak{M}_+(0,\infty). \tag{2.15}$$

Then the optimal domain norm ϱ_D corresponding to ϱ_R is defined by

$$\varrho_D(g) := \varrho_R\left(G(g^{**})\right), \quad g \in \mathfrak{M}_+(0,\infty). \tag{2.16}$$

Next, we solve the converse problem. Namely, the norm ϱ_D is now considered to be fixed and we are searching for the optimal ϱ_R. First of all we shall introduce some notation. We recall (2.14) and define

$$(Gg)(t) = e^{nt^{1/n}} \int_t^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s, \quad g \in \mathfrak{M}_+(0,\infty), \quad t > 0, \tag{2.17}$$

$$(Hh)(t) = t^{1/n-1}e^{-nt^{1/n}} \int_0^t h(s)e^{ns^{1/n}}\mathrm{d}s, \quad h \in \mathfrak{M}_+(0,\infty), \quad t > 0, \tag{2.18}$$

$$E(s) = e^{-ns^{1/n}} \int_0^s e^{nu^{1/n}}\mathrm{d}u, \quad s > 0. \tag{2.19}$$

The operators G and H are mutually dual in the following sense

$$\int_0^\infty h(t)Gg(t)\mathrm{d}t = \int_0^\infty g(u)Hh(u)\mathrm{d}u \quad \text{for all} \quad g, h \in \mathfrak{M}_+(0,\infty). \tag{2.20}$$

Theorem 2.8. *Assume that the r.i. norm ϱ_D satisfies*

$$\varrho_D\left(\int_s^\infty f(u)\frac{E(u)}{u}u^{1/n-1}\mathrm{d}u\right) \le c\varrho_D(f), \quad f \in \mathfrak{M}_+(0,\infty). \tag{2.21}$$

and that its dual norm ϱ'_D satisfies

$$\varrho'_D(H(h^{**})) \le c\varrho'_D(H(h^*)), \quad h \in \mathfrak{M}_+(0,\infty). \tag{2.22}$$

Then the optimal range norm in (2.12) *associated to ϱ_D is given as the dual norm to $\varrho'_D(H(f^{**}))$. Or, equivalently, the dual of the optimal range norm can be described by $\varrho'_R(f) := \varrho'_D(H(f^{**}))$.*

We also derive sufficient conditions for (2.21) and (2.22). In general, we follow the idea of [10, Theorem 4.4]. First of all, for every function $f \in \mathfrak{M}_+(0,\infty)$, we define the dilation operator E by

$$(E_s f)(t) = f(st), \quad t > 0, \quad s > 0.$$

It is well known, [3, Chapter 3, Prop. 5.11], that for every r.i. norm ϱ on $\mathfrak{M}_+(0,\infty)$ and every $s > 0$ the operator E_s satisfies

$$\varrho(E_s f) \le c\varrho(f), \quad f \in \mathfrak{M}_+(0,\infty).$$

The smallest possible constant c in this inequality (which depends of course on s) is denoted by $h_\varrho(s)$. Hence

$$h_\varrho(s) = \sup_{f \not\equiv 0} \frac{\varrho(E_s f)}{\varrho(f)}.$$

Using this notation, we may give a characterisation of (2.15) and (2.22).

Theorem 2.9. *If a rearrangement–invariant norm ϱ_R satisfies $\int_0^1 s^{-1/n}h_{\varrho_R}(s)\mathrm{d}s < \infty$, then it also satisfies* (2.15).

Theorem 2.10. *If an r.i. norm σ satisfies $\int_0^1 s^{-1/n} h_\sigma(s)\mathrm{d}s < \infty$ then it satisfies also (2.22) with ϱ_D' replaced by σ.*

We will now present some applications of our results.

Example 2.11. Let

$$\varrho_R(f) = \varrho_\infty(f) = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x)|.$$

Then $h_{\varrho_R}(s) = 1$ and, according to Theorem 2.9, (2.15) is satisfied and the optimal domain norm is given by

$$\varrho_D(f) \approx \sup_{t>0} (Gf^*)(t) = \int_0^\infty f^*(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s, \quad f \in \mathfrak{M}(\mathbb{R}^n).$$

This norm is essentially smaller than $\varrho_{n,1}(f) = \int_0^\infty t^{1/n-1}f^*(t)\mathrm{d}t$, hence this result improves the second example from Remark 2.6. Now, an easy calculation shows that

$$\varrho_D(f) \approx f^*(1) + \int_0^1 f^*(t)t^{1/n-1}\mathrm{d}t \approx \varrho_\infty(f^*\chi_{(1,\infty)}) + \varrho_{n,1}(f^*\chi_{(0,1)}), \quad f \in \mathfrak{M}(\mathbb{R}^n).$$

Example 2.12. Let

$$\varrho_D(f) = \varrho_1(f) = \int_{\mathbb{R}^n} |f(x)|\mathrm{d}x.$$

In that case, $\varrho_D' = \varrho_\infty$, whence $h_{\varrho_D'}(s) = 1$. So, by Theorem 2.10, (2.22) is satisfied. It is a simple exercise to verify (2.21). Using Theorem 2.8, the optimal range norm can be described as the dual norm to

$$\sigma(f) = \varrho_\infty(Hf^*) = \varrho_\infty\Big(t^{1/n-1}e^{-nt^{1/n}}\int_0^t f^*(s)e^{ns^{1/n}}\mathrm{d}s\Big).$$

The optimal range norm

$$\varrho_R(g) = \sigma'(g) = \sup_{f:\varrho_\infty(Hf^*)\le 1} \int_0^\infty f^*(t)g^*(t)\mathrm{d}t,$$

is equivalent to

$$\varrho_R(g) = \sup_{f:\varrho_\infty(Hf^*)\le 1} \int_0^\infty f^*(t)g^*(t)\mathrm{d}t \approx \int_0^1 g^*(t)t^{-1/n}\mathrm{d}t + \int_1^\infty g^*(t)\mathrm{d}t.$$

Finally, we consider the case of limiting Sobolev embedding, where ϱ_D is set to be $\varrho_D(f) = \varrho_n(f) = \Big(\int_{\mathbb{R}^n} |f(x)|^n\mathrm{d}x\Big)^{1/n}$. In that case, $\varrho_D'(f) = \varrho_{n'}(f)$, where n' is the conjugated exponent to n, namely $\frac{1}{n} + \frac{1}{n'} = 1$. Direct calculation shows that $h_{\varrho_D'}(s) = s^{-1/n'}$ and $\int_0^1 s^{-1/n}h_{\varrho_D'}(s)\mathrm{d}s = \infty$. Moreover, standard examples ($h(s) = \frac{1}{s|\log s|^2}\chi_{(0,1/2)}(s)$) show that (2.22) is not satisfied.

To include this important case into the frame of our work, we develop a finer theory of an optimal range space. This is described in the following assertion.

Theorem 2.13. *Let ϱ_D be a given r.i. norm such that* (2.21) *holds and*

$$\varrho_D'(H\chi_{(0,1)}) < \infty. \tag{2.23}$$

Set

$$\sigma(h) = \varrho_D'(Hh^*), \quad h \in \mathfrak{M}_+(0,\infty).$$

Then,

$$\varrho_R := \sigma' \tag{2.24}$$

is an r.i. norm which satisfies (2.12) *and which is optimal for* (2.12).

Let us apply Theorem 2.13 to the limiting Sobolev embeddings with

$$\varrho_D(f) = \varrho_n(f) = \Big(\int_0^\infty |f^*(t)|^n \mathrm{d}t\Big)^{1/n}.$$

It may be shown, that (2.23) and (2.21) are satisfied in this case. So, Theorem 2.13 is applicable and gives the optimal range norm. The result is presented in the next Theorem.

Theorem 2.14. *Let $\varrho_D = \varrho_n$. Then, the optimal range norm, ϱ_R, satisfies*

$$\varrho_R(f) \approx \varrho_n(f) + \lambda(f^*\chi_{(0,1)}), \tag{2.25}$$

where

$$\lambda(g) := \left(\int_0^1 \left(\frac{g^*(t)}{\log(\frac{e}{t})}\right)^n \frac{\mathrm{d}t}{t}\right)^{\frac{1}{n}}, \qquad g \in \mathfrak{M}(0,1).$$

Remark 2.15. We note that λ from Theorem 2.14 is the well-known norm discovered in various contexts independently by Maz'ya [17], Hanson [13] and Brézis–Wainger [5].

3 *A remark on better-lambda inequality* Math. Ineq. Appl. 10 (2007), 335-341.

The classical Riesz potentials are defined for every real number $0 < \gamma < n$ as a convolution operators $(I_\gamma f)(x) = (\tilde{I}_\gamma * f)(x)$, where $x \in \mathbb{R}^n$ and $\tilde{I}_\gamma(x) = |x|^{\gamma-n}$. This definition coincides with the usual one up to some multiplicative constant c_γ which is not interesting for our purpose. Burkholder and Gundy invented in [6] the technique involving distribution function later known as *good λ-inequality*. This inequality dealt with level sets of singular integral operators and of maximal operator. Later, Bagby and Kurtz discovered in [2] that the reformulation of good λ-inequality in terms of non-increasing rearrangement contains more information.

We generalise their approach in the following way. For every Young's function Φ satisfying the Δ_2-condition we define the Riesz potential

$$(I_\Phi f)(x) = \int_{\mathbb{R}^n} \tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right) f(y)\mathrm{d}y,$$

where $\tilde{\Phi}$ is the Young's function conjugated to Φ and $\tilde{\Phi}^{-1}$ is its inverse. Instead of the classical Hardy-Littlewood maximal operator we work with a generalised maximal operator

$$(M_\varphi f)(x) = \sup_{Q \ni x} \frac{1}{\varphi(|Q|)} \int_Q |f(y)| \mathrm{d}y,$$

where φ is a given nonnegative function on $(0, \infty)$ and the supremum is taken over all cubes Q containing x with sides parallel to the coordinate axes such that $\varphi(|Q|) > 0$. For every measurable set $\Omega \subset \mathbb{R}^n$ we denote by $|\Omega|$ its Lebesgue measure.

We prove that under some restrictive conditions on function Φ one can obtain an inequality combining the nonincreasing rearrangement of $I_\Phi f$ and $M_{\tilde{\Phi}^{-1}} f$. We also show that this restrictive condition cannot be left out.

Definition 3.1. 1. Let $\phi : [0, \infty) \to [0, \infty)$ be a non-decreasing and right-continuous function with $\phi(0) = 0$ and $\lim_{t \to \infty} \phi(t) = \infty$. Then the function Φ defined by

$$\Phi(t) = \int_0^t \phi(s) \mathrm{d}s, \quad t \geq 0$$

is said to be a *Young's function*.

2. A Young's function is said to satisfy the Δ_2—*condition* if there is $c > 0$ such that

$$\Phi(2t) \leq c\, \Phi(t), \quad t \geq 0.$$

3. A Young's function is said to satisfy the ∇_2—*condition* if there is $l > 1$ such that

$$\Phi(t) \leq \frac{1}{2l} \Phi(lt), \quad t \geq 0.$$

4. Let Φ be a Young's function, represented as the indefinite integral of ϕ. Let

$$\psi(s) = \sup\{u : \phi(u) \leq s\}, \quad s \geq 0.$$

Then the function

$$\tilde{\Phi}(t) = \int_0^t \psi(s) \mathrm{d}s, \quad t \geq 0,$$

is called the *complementary Young's function* of Φ.

Assume now that a Young's function Φ satisfies the Δ_2—condition. Using the classical O'Neil inequality (see [18]) we obtain

$$(I_\Phi f)^*(t) \leq c \left\{ \tilde{\Phi}^{-1}\left(\frac{1}{t}\right) \int_0^t f^*(u) \mathrm{d}u + \int_t^\infty f^*(u) \tilde{\Phi}^{-1}\left(\frac{1}{u}\right) \mathrm{d}u \right\}, \tag{3.1}$$

We shall derive a better λ-inequality connecting the operators I_Φ and $M_{\tilde{\Phi}^{-1}}$.

Theorem 3.2. *Let us suppose that a Young's function* Φ *satisfies the* Δ_2*—condition. Let us further suppose that there is a constant* $c_1 > 0$ *such that*

$$\tilde{\Phi}^{-1}(s)\tilde{\Phi}^{-1}(1/s) < c_1, \quad s > 0. \tag{3.2}$$

Then there is a constant $c_2 > 0$*, such that for every function* f *and every positive number* t

$$(I_\Phi f)^*(t) \leq (I_\Phi|f|)^*(t) \leq c_2\,(M_{\tilde{\Phi}^{-1}} f)^*(t/2) + (I_\Phi|f|)^*(2t) \tag{3.3}$$

In the following example we will show that the assumption (3.2) cannot be omitted.

Theorem 3.3. *There is a Young's function* Φ *satisfying the* Δ_2*—condition for which*

$$\sup_{f,t>0} \frac{(I_\Phi f)^*(t) - (I_\Phi f)^*(2t)}{(M_{\tilde{\Phi}^{-1}} f)^*(t/2)} = \infty.$$

4 *A new proof of Jawerth-Franke embedding* to appear in Rev. Mat. Complut.

In this paper, we considered an analogue of a Sobolev embedding generalised to Besov and Triebel-Lizorkin spaces. Let us first give their definition.

Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$ and let $S'(\mathbb{R}^n)$ be its dual - the space of all tempered distributions. endowed with the norm For $\psi \in S(\mathbb{R}^n)$ we denote by

$$\widehat{\psi}(\xi) = (F\psi)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^n} e^{-i<x,\xi>}\psi(x)\mathrm{d}x, \quad x \in \mathbb{R}^n,$$

its Fourier transform and by $\psi^\vee$ or $F^{-1}\psi$ its inverse Fourier transform.

We give a Fourier-analytic definition of Besov and Triebel-Lizorkin spaces, which relies on the so-called *dyadic resolution of unity*. Let $\varphi \in S(\mathbb{R}^n)$ with

$$\varphi(x) = 1 \quad \text{if} \quad |x| \leq 1 \quad \text{and} \quad \varphi(x) = 0 \quad \text{if} \quad |x| \geq \frac{3}{2}. \tag{4.1}$$

We put $\varphi_0 = \varphi$ and $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$ for $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$. This leads to the identity

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1, \qquad x \in \mathbb{R}^n.$$

Definition 4.1. (i) Let $s \in \mathbb{R}, 1 \leq p, q \leq \infty$. Then $B^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$||f|B^s_{pq}(\mathbb{R}^n)|| = \left(\sum_{j=0}^{\infty} 2^{jsq}||(\varphi_j\widehat{f})^\vee|L_p(\mathbb{R}^n)||^q\right)^{1/q} < \infty \tag{4.2}$$

(with the usual modification for $q = \infty$).

(ii) Let $s \in \mathbb{R}, 1 \le p < \infty, 1 \le q \le \infty$. Then $F^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$||f|F^s_{pq}(\mathbb{R}^n)|| = \left|\left|\left(\sum_{j=0}^{\infty} 2^{jsq}|(\varphi_j \widehat{f})^\vee(\cdot)|^q\right)^{1/q} |L_p(\mathbb{R}^n)\right|\right| < \infty \tag{4.3}$$

(with the usual modification for $q = \infty$).

Remark 4.2. These spaces have a long history. In this context we recommend [19], [24], [25] and [26] as standard references. We point out that the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ are independent of the choice of ψ in the sense of equivalent norms. Special cases of these two scales include Lebesgue spaces, Sobolev spaces, Hölder-Zygmund spaces and many other important function spaces. We omit any detailed discussion.

The classical Sobolev embedding theorem can be extended to these two scales.

Theorem 4.3. *Let* $-\infty < s_1 < s_0 < \infty$ *and* $0 < p_0 < p_1 \le \infty$ *with*

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}. \tag{4.4}$$

(i) If $0 < q_0 \le q_1 \le \infty$*, then*

$$B^{s_0}_{p_0q_0}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1q_1}(\mathbb{R}^n). \tag{4.5}$$

(ii) If $0 < q_0, q_1 \le \infty$ *and* $p_1 < \infty$*, then*

$$F^{s_0}_{p_0q_0}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1q_1}(\mathbb{R}^n). \tag{4.6}$$

We observe that there is no condition on the fine parameters q_0, q_1 in (4.6). This surprising effect was first observed in full generality by Jawerth, [14]. Using (4.6), we may prove

$$F^{s_0}_{p_0q}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1p_1}(\mathbb{R}^n) = B^{s_1}_{p_1p_1}(\mathbb{R}^n) \quad \text{and} \quad B^{s_0}_{p_0p_0}(\mathbb{R}^n) = F^{s_0}_{p_0p_0}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1q}(\mathbb{R}^n)$$

for every $0 < q \le \infty$. But Jawerth ([14]) and Franke ([12]) showed that these embeddings are not optimal and may be improved.

Theorem 4.4. *Let* $-\infty < s_1 < s_0 < \infty$*,* $0 < p_0 < p_1 \le \infty$ *and* $0 < q \le \infty$ *with* (4.4).

(i) Then

$$F^{s_0}_{p_0q}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1p_0}(\mathbb{R}^n). \tag{4.7}$$

(ii) If $p_1 < \infty$*, then*

$$B^{s_0}_{p_0p_1}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1q}(\mathbb{R}^n). \tag{4.8}$$

The original proofs (see [14] and [12]) use interpolation techniques. We rely on a different method. First, we observe that using (for example) the wavelet decomposition method, (4.7) and (4.8) is equivalent to

$$f^{s_0}_{p_0q} \hookrightarrow b^{s_1}_{p_1p_0} \quad \text{and} \quad b^{s_0}_{p_0p_1} \hookrightarrow f^{s_1}_{p_1q} \tag{4.9}$$

under the same restrictions on parameters s_0, s_1, p_0, p_1, q as in Theorem 4.4. Here, b^s_{pq} and f^s_{pq} stands for the sequence spaces of Besov and Triebel-Lizorkin type. We prove (4.9) directly using the technique of the non-increasing rearrangement on a rather elementary level.

We introduce the sequence spaces associated with the Besov and Triebel-Lizorkin spaces. Let $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$. Then $Q_{\nu m}$ denotes the closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, centred at $2^{-\nu}m$, and with side length $2^{-\nu}$. By $\chi_{\nu m} = \chi_{Q_{\nu m}}$ we denote the characteristic function of $Q_{\nu m}$. If

$$\lambda = \{\lambda_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\},$$

$-\infty < s < \infty$ and $0 < p, q \leq \infty$, we set

$$||\lambda|b^s_{pq}|| = \Big(\sum_{\nu=0}^{\infty} 2^{\nu(s-\frac{n}{p})q} \Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} \tag{4.10}$$

appropriately modified if $p = \infty$ and/or $q = \infty$. If $p < \infty$, we define also

$$||\lambda|f^s_{pq}|| = \Big|\Big|\Big(\sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} |2^{\nu s}\lambda_{\nu m}\chi_{\nu m}(\cdot)|^q\Big)^{1/q} |L_p(\mathbb{R}^n)\Big|\Big|. \tag{4.11}$$

The connection between the function spaces $B^s_{pq}(\mathbb{R}^n)$, $F^s_{pq}(\mathbb{R}^n)$ and the sequence spaces b^s_{pq}, f^s_{pq} may be given by various decomposition techniques, we refer to [26, Chapters 2 and 3] for details and further references.

As a result of these characterisations, (4.7) and (4.8) are equivalent to (4.9).

We gave a new proof of Theorem 4.4. Instead of interpolation, we used the technique of the non-increasing rearrangement on a rather elementary level. It means, we gave the direct proof of the following embedding theorems for sequence spaces of Besov and Triebel-Lizorkin type.

Theorem 4.5. *Let $-\infty < s_1 < s_0 < \infty$, $0 < p_0 < p_1 \leq \infty$ and $0 < q \leq \infty$. Then*

$$f^{s_0}_{p_0 q} \hookrightarrow b^{s_1}_{p_1 p_0} \quad \text{if} \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}. \tag{4.12}$$

Theorem 4.6. *Let $-\infty < s_1 < s_0 < \infty, 0 < p_0 < p_1 < \infty$ and $0 < q \leq \infty$. Then*

$$b^{s_0}_{p_0 p_1} \hookrightarrow f^{s_1}_{p_1 q} \quad \text{if} \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}. \tag{4.13}$$

Theorems 4.5 and 4.6 are sharp in the following sense.

Theorem 4.7. *Let $-\infty < s_1 < s_0 < \infty$, $0 < p_0 < p_1 \leq \infty$ and $0 < q_0, q_1 \leq \infty$ with*

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

(i) If

$$f^{s_0}_{p_0 q_0} \hookrightarrow b^{s_1}_{p_1 q_1}, \tag{4.14}$$

then $q_1 \geq p_0$.

(ii) If $p_1 < \infty$ and

$$b^{s_0}_{p_0 q_0} \hookrightarrow f^{s_1}_{p_1 q_1}, \tag{4.15}$$

then $q_0 \leq p_1$.

Remark 4.8. Using (any of) the usual decomposition techniques, the same statements hold true also for the function spaces. These results were first proved in [21].

5 *Sampling numbers and function spaces* J. Compl. 23 (2007), 773-792.

If the inequality in (1.1) is strict, then the embedding (1.2) is compact. The quality of this compactness may be in some sense described by many techniques. We mention at least the approximation numbers, Gelfand numbers or entropy numbers. We shall concentrate on other approximation quantities, namely the so-called *sampling numbers*.

First, we give the definition of Besov and Triebel-Lizorkin spaces on domains.

Let Ω be a bounded domain. Let $D(\Omega) = C_0^\infty(\Omega)$ be the collection of all complex-valued infinitely-differentiable functions with compact support in Ω and let $D'(\Omega)$ be its dual - the space of all complex-valued distributions on Ω.

Let $g \in S'(\mathbb{R}^n)$. Then we denote by $g|\Omega$ its restriction to Ω:

$$(g|\Omega) \in D'(\Omega), \qquad (g|\Omega)(\psi) = g(\psi) \quad \text{for} \quad \psi \in D(\Omega).$$

Definition 5.1. Let Ω be a bounded domain in $\mathbb{R}^n$. Let $s \in \mathbb{R}$, $1 \le p, q \le \infty$ with $p < \infty$ in the F-case. Let A^s_{pq} stand either for B^s_{pq} or F^s_{pq}. Then

$$A^s_{pq}(\Omega) = \{f \in D'(\Omega) : \exists g \in A^s_{pq}(\mathbb{R}^n) : g|\Omega = f\}$$

and

$$||f|A^s_{pq}(\Omega)|| = \inf ||g|A^s_{pq}(\mathbb{R}^n)||,$$

where the infimum is taken over all $g \in A^s_{pq}(\mathbb{R}^n)$ such that $g|\Omega = f$.

We now introduce the concept of sampling numbers.

Definition 5.2. Let Ω be a bounded Lipschitz domain. Let $G_1(\Omega)$ be a space of continuous functions on Ω and $G_2(\Omega) \subset D'(\Omega)$ be a space of distributions on Ω. Suppose that the embedding

$$id : G_1(\Omega) \hookrightarrow G_2(\Omega)$$

is compact.

For $\{x_j\}_{j=1}^k \subset \Omega$ we define the *information map*

$$N_k : G_1(\Omega) \to \mathbb{C}^n, \qquad N_k f = (f(x_1), \dots, f(x_k)), \quad f \in G_1(\Omega).$$

For any (linear or nonlinear) mapping $\varphi_n : \mathbb{C}^k \to G_2(\Omega)$ we consider

$$S_k : G_1(\Omega) \to G_2(\Omega), \qquad S_k = \varphi_k \circ N_k.$$

(i) Then, for all $k \in \mathbb{N}$, the $k-$th *sampling number* $g_k(id)$ is defined by

$$g_k(id) = \inf_{S_k} \sup\{||f - S_k f|G_2(\Omega)|| : ||f|G_1(\Omega)|| \le 1\}, \tag{5.1}$$

where the infimum is taken over all k-tuples $\{x_j\}_{j=1}^k \subset \Omega$ and all (linear or nonlinear) φ_k.
(ii) For all $k \in \mathbb{N}$ the $k-$th *linear sampling number* $g_k^{\text{lin}}(id)$ is defined by (5.1), where now only linear mappings φ_k are admitted.

The study of sampling numbers of the Sobolev embeddings of spaces of Besov and Triebel-Lizorkin type is dived into three steps.

Step 1: **The case** $s_2 > 0$

In this subsection, we discuss the case where $\Omega = I^n = (0,1)^n$ is the unit cube, $G_1(\Omega) = A^{s_1}_{p_1q_1}(\Omega)$ and $G_2(\Omega) = A^{s_2}_{p_2q_2}(\Omega)$ with $s_1 > \frac{n}{p_1}$ and $s_1 - n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+ > s_2 > 0$. Here, $A^s_{pq}(\Omega)$ stands either for a Besov space $B^s_{pq}(\Omega)$ or a Triebel-Lizorkin space $F^s_{pq}(\Omega)$, see Definition 5.1 for details. We start with the most simple and most important case, namely when $p_1 = p_2 = q_1 = q_2$.

Proposition 5.3. *Let* $\Omega = I^n = (0,1)^n$. *Let* $G_1(\Omega) = B^{s_1}_{pp}(\Omega)$ *and* $G_2(\Omega) = B^{s_2}_{pp}(\Omega)$ *with* $1 \le p \le \infty$,

$$s_1 > \frac{n}{p}, \quad \textit{and} \quad s_1 > s_2 > 0.$$

Then

$$g_k^{\text{lin}}(id) \lesssim k^{-\frac{s_1 - s_2}{n}}.$$

The proof of this statement requires unfortunately several techniques from the theory of function spaces like characterisation by differences, local polynomial approximation and multiplier assertions. See [29] for details.

Using the real interpolation method, the results could be easily extended.

Proposition 5.4. *Let* $\Omega = I^n = (0,1)^n$. *Let* $G_1(\Omega) = A^{s_1}_{p_1q_1}(\Omega)$ *and* $G_2(\Omega) = A^{s_2}_{p_2q_2}(\Omega)$ *with* $1 \le p_1, p_2, q_1, q_2 \le \infty$ *(*$p_1, p_2 < \infty$ *in the* F*-case),*

$$s_1 > \frac{n}{p_1}, \quad \textit{and} \quad s_1 - n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+ > s_2 > 0. \tag{5.2}$$

Then

$$g_k^{\text{lin}}(id) \lesssim k^{-\frac{s_1 - s_2}{n} + \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+}. \tag{5.3}$$

It turns out, that these estimates are sharp. Namely, we have

Theorem 5.5. *Let* $\Omega = I^n = (0,1)^n$. *Let* $G_1(\Omega) = A^{s_1}_{p_1q_1}(\Omega)$ *and* $G_2(\Omega) = A^{s_2}_{p_2q_2}(\Omega)$ *with* $1 \le p_1, p_2, q_1, q_2 \le \infty$ *(*$p_1, p_2 < \infty$ *in the* F*-case) and* (5.2) *Then*

$$g_k(id) \approx g_k^{\text{lin}}(id) \approx k^{-\frac{s_1 - s_2}{n} + \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+}. \tag{5.4}$$

Step 2: **The case** $s_2 = 0$

In the case $s_2 = 0$, new phenomena come into play. The same method can be applied also in this case. Unfortunately, there appears a gap between the estimates from below and from above. The exact formulation is as follows.

Theorem 5.6. *Let $\Omega = I^n = (0,1)^n$. Let*

$$id : G_1(\Omega) \hookrightarrow G_2(\Omega)$$

with

$$G_1(\Omega) = B^s_{p_1q_1}, \quad G_2(\Omega) = B^0_{p_2q_2}$$

and

$$1 \le p_1, q_1, p_2, q_2 \le \infty, \quad s > \frac{n}{p_1}.$$

Then

$$k^{-\frac{s}{n}+(\frac{1}{p_1}-\frac{1}{p_2})_+} \lesssim g_k(id) \lesssim g_k^{\mathrm{lin}}(id) \lesssim k^{-\frac{s}{n}+(\frac{1}{p_1}-\frac{1}{p_2})_+}(1+\log k)^{1/q_2}, \qquad k \in \mathbb{N}. \tag{5.5}$$

This effect was studied in detail in [30], see below.

Step 3: **The case $s_2 < 0$**

As in the last case, we consider the situation when $s_2 < 0$.

Theorem 5.7. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let*

$$id : G_1(\Omega) = A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow G_2(\Omega) = A^{s_2}_{p_2q_2}(\Omega)$$

with $1 \le p_1, p_2, q_1, q_2 \le \infty$ (with $p_1, p_2 < \infty$ in the F-case) and

$$s_1 > \frac{n}{p_1}, \qquad s_2 < 0.$$

If $p_1 \ge p_2$, then

$$g_k(id) \approx g_k^{\mathrm{lin}}(id) \approx k^{-\frac{s_1}{n}}. \tag{5.6}$$

If $p_1 < p_2$ and $s_2 > \frac{n}{p_2} - \frac{n}{p_1}$, then

$$g_k(id) \approx g_k^{\mathrm{lin}}(id) \approx k^{-\frac{s_1}{n}+\frac{s_2}{n}+\frac{1}{p_1}-\frac{1}{p_2}}. \tag{5.7}$$

If $p_1 < p_2$ and $\frac{n}{p_2} - \frac{n}{p_1} > s_2$, then

$$g_k(id) \approx g_k^{\mathrm{lin}}(id) \approx k^{-\frac{s_1}{n}}. \tag{5.8}$$

These estimates can be applied in connection with elliptic differential operators, which was the actual motivation for this research, c.f. [8] and [9]. Let us briefly introduce this setting. Let

$$\mathcal{A} : H \to G$$

be a bounded linear operator from a Hilbert space H to another Hilbert space G. We assume that $\mathcal{A}$ is boundedly invertible, hence

$$\mathcal{A}(u) = f$$

has a unique solution for every $f \in G$. A typical application is an operator equation, where $\mathcal{A}$ is an elliptic differential operator, and we assume that

$$\mathcal{A} : H_0^s(\Omega) \to H^{-s}(\Omega),$$

where Ω is a bounded Lipschitz domain, $H_0^s(\Omega)$ is a function space of Sobolev type with fractional order of smoothness $s > 0$ of functions vanishing on the boundary and H^{-s} is a function space of Sobolev type with negative smoothness $-s < 0$. The classical example is the Poisson equation

$$-\Delta u = f \quad \text{in} \quad \Omega \qquad \text{and} \qquad u = 0 \quad \text{on} \quad \partial\Omega.$$

Here, $s = 1$ and

$$\mathcal{A} = -\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$$

is bounded and boundedly invertible. We want to approximate the *solution operator* $u = S(f)$ using only function values of f.

We define the k-th linear sampling number of the identity $id : H^{-1+t}(\Omega) \to H^{-1}(\Omega)$ by

$$g_k^{\text{lin}}(id : H^{-1+t}(\Omega) \to H^{-1}(\Omega)) = \inf_{S_k} ||id - S_k|\mathcal{L}(H^{-1+t}(\Omega), H^{-1}(\Omega))||, \qquad (5.9)$$

where t is a positive real number with $-1 + t > \frac{n}{2}$, and the k-th linear sampling number of $S : H^{-1+t}(\Omega) \to H^1(\Omega)$ by

$$g_k^{\text{lin}}(S : H^{-1+t}(\Omega) \to H^1(\Omega)) = \inf_{S_k} ||S - S_k|\mathcal{L}(H^{-1+t}(\Omega), H^1(\Omega))||. \qquad (5.10)$$

The infimum in (5.9) and (5.10) runs over all linear operators S_k of the form (1.1) and $\mathcal{L}(X, Y)$ stands for the space of bounded linear operators between two Banach spaces X and Y, equipped with the classical operator norm.

It turns out that these quantities are equivalent (up to multiplicative constants which do not depend neither on f nor on k) and are of the asymptotic order

$$g_k^{\text{lin}}(S : H^{-1+t}(\Omega) \to H^1(\Omega)) \approx g_k^{\text{lin}}(id : H^{-1+t}(\Omega) \to H^{-1}(\Omega)) \approx k^{-\frac{-1+t}{n}}.$$

We refer to [8] and [9] for a detailed discussion of this approach. The estimates of sampling numbers of an embedding between two function spaces translates therefor into estimates of sampling numbers of the solution operator S. We observe that the more regular f, the faster is the decay of the linear sampling numbers of the solution operator S. Let us also point out that optimal linear methods (not restricted to use only the function values of f) achieve asymptotically a better rate of convergence, namely $k^{-\frac{t}{n}}$. Hence, the limitation to the sampling operators results in a serious restriction. One has to pay at least $k^{1/n}$ in comparison with optimal linear methods.

Using our estimates of sampling numbers of identities between Besov and Triebel-Lizorkin spaces, this result may be generalised as follows.[1] If $p \geq 2$, $1 \leq q \leq \infty$ and $-1 + t > \frac{d}{p}$ then

$$g_k^{\text{lin}}(S : B_{pq}^{-1+t}(\Omega) \to H^1(\Omega)) \approx g_k^{\text{lin}}(id : B_{pq}^{-1+t}(\Omega) \to H^{-1}(\Omega)) \approx k^{-\frac{-1+t}{n}}.$$

[1]Although the results are stated only for Besov spaces, they are proved also for Triebel-Lizorkin spaces, which include also fractional Sobolev spaces as a special case.

If $p < 2$ with $\dfrac{1}{p} > \dfrac{1}{n} + \dfrac{1}{2}$, $1 \le q \le \infty$ and $-1+t > \frac{n}{p}$ then

$$g_k^{\text{lin}}(S : B_{pq}^{-1+t}(\Omega) \to H^1(\Omega)) \approx g_k^{\text{lin}}(id : B_{pq}^{-1+t}(\Omega) \to H^{-1}(\Omega)) \approx k^{-\frac{t}{n}+\frac{1}{p}-\frac{1}{2}}.$$

Finally, if $p < 2$ with $\dfrac{1}{p} < \dfrac{1}{n} + \dfrac{1}{2}$, $1 \le q \le \infty$ and $-1+t > \frac{n}{p}$ then

$$g_k^{\text{lin}}(S : B_{pq}^{-1+t}(\Omega) \to H^1(\Omega)) \approx g_k^{\text{lin}}(id : B_{pq}^{-1+t}(\Omega) \to H^{-1}(\Omega)) \approx k^{-\frac{-1+t}{n}}.$$

We prove the same results also for the nonlinear sampling numbers $g_k(S)$. Altogether, the regularity information of f may now be described by an essentially broader scale of function spaces.

6 *Dilation operators and sampling numbers* to appear in J. of Function Spaces and Appl.

This paper is divided into two parts. In the first part, we consider the dilation operators

$$T_k : f \to f(2^k \cdot), \quad k \in \mathbb{N},$$

in the framework of Besov spaces $B_{pq}^s(\mathbb{R}^n)$. Their behaviour is well known if $1 \le p, q \le \infty$ and $s > 0$, cf. [11, 2.3.1]. As mentioned there, the case $s = 0$ remained open. Some partial results can be found in [4]. For $1 \le p, q \le \infty$ we supply the final answer to this problem showing that

$$||T_k|\mathcal{L}(B_{pq}^0(\mathbb{R}^n))|| \approx 2^{-k\frac{d}{p}} \cdot \begin{cases} k^{\frac{1}{q}-\frac{1}{p}}, & \text{if } 1 < p < \infty \text{ and } p \ge \max(q,2), \\ k^{\frac{1}{q}-\frac{1}{2}}, & \text{if } 1 < p < \infty \text{ and } 2 \ge \max(p,q), \\ 1, & \text{if } 1 < p < \infty \text{ and } q \ge \max(p,2), \\ k^{\frac{1}{q}}, & \text{if } p = 1 \text{ or } p = \infty, \end{cases} \tag{6.1}$$

where $||T_k|\mathcal{L}(B_{pq}^0(\mathbb{R}^n))||$ denotes the norm of the operator T_k from $B_{pq}^0(\mathbb{R}^n)$ into itself. One observes that for $1 < p < \infty$ the number 2 plays an exceptional role. This effect has its origin in the Littlewood-Paley decomposition theorem.

The second part of the paper deals with applications to estimates of sampling numbers. Let us briefly sketch this approach.

Let $\Omega = (0,1)^n$ and let $B_{pq}^s(\Omega)$ denote the Besov spaces on Ω, see Definition 5.1 for details. We try to approximate $f \in B_{p_1q_1}^{s_1}(\Omega)$ in the norm of another Besov space, say $B_{p_2q_2}^{s_2}(\Omega)$, by a linear sampling method

$$S_k f = \sum_{j=1}^{n} f(x_j) h_j, \tag{6.2}$$

where $h_j \in B_{p_2q_2}^{s_2}(\Omega)$ and $x_j \in \Omega$. To give a meaning to the pointwise evaluation in (6.2), we suppose that

$$s_1 > \frac{n}{p_1}.$$

Then the embedding $B^{s_1}_{p_1q_1}(\Omega) \hookrightarrow C(\bar{\Omega})$ holds true and the pointwise evaluation represents a bounded operator. Second, we always assume that the embedding $B^{s_1}_{p_1q_1}(\Omega) \hookrightarrow B^{s_2}_{p_2q_2}(\Omega)$ is compact. This is true if, and only if,

$$s_1 - s_2 > n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ .$$

Concerning the parameters p_1, p_2, q_1, q_2 we always assume that they belong to $[1, \infty]$.

We measure the worst case error of $S_k f$ on the unit ball of $B^{s_1}_{p_1q_1}(\Omega)$, given by

$$\sup\{||f - S_k f|B^{s_2}_{p_2q_2}(\Omega)|| : ||f|B^{s_1}_{p_1q_1}(\Omega)|| \leq 1\}. \tag{6.3}$$

The same worst case error may be considered also for nonlinear sampling methods

$$S_k f = \varphi(f(x_1), \ldots, f(x_k)), \tag{6.4}$$

where $\varphi : \mathbb{C}^k \to B^{s_2}_{p_2q_2}(\Omega)$ is an arbitrary mapping. We shall discuss the decay of (6.3) for linear (6.2) and nonlinear (6.4) sampling methods.

The case $s_2 \neq 0$ was considered in [29], but the interesting limiting case $s_2 = 0$ was left open so far. It is the aim of this paper to close this gap. It was already pointed out in [29], see especially (2.6) in [29] for details, that the estimates from above for the dilation operators T_k on the target space $B^{s_2}_{p_2q_2}(\mathbb{R}^n)$ have their direct counterparts in estimates from above for the decay of sampling numbers. Using this method, which will not be repeated here, a direct application of (6.1) supplies the estimates

$$g^{\mathrm{lin}}_k(id) \lesssim k^{-\frac{s}{d}} \cdot \begin{cases} (\log k)^{\frac{1}{q_2} - \frac{1}{p}}, & \text{if } 1 < p < \infty \text{ and } p \geq \max(q_2, 2), \\ (\log k)^{\frac{1}{q_2} - \frac{1}{2}}, & \text{if } 1 < p < \infty \text{ and } 2 \geq \max(p, q_2), \\ 1, & \text{if } 1 < p < \infty \text{ and } q_2 \geq \max(p, 2), \\ (\log k)^{\frac{1}{q_2}}, & \text{if } p = 1 \text{ or } p = \infty, \end{cases} \tag{6.5}$$

where $g^{\mathrm{lin}}_k(id)$ with $2 \leq k \in \mathbb{N}$ are the linear sampling numbers of the embedding

$$id : B^s_{pq_1}(\Omega) \to B^0_{pq_2}(\Omega), \qquad s > \frac{n}{p}.$$

Surprisingly, all estimates in (6.5) are sharp.

References

[1] R. A. Adams, *Sobolev spaces*, Pure and Applied Mathematics, Vol. 65. Academic Press, New York-London, 1975.

[2] R. J. Bagby and D. S. Kurtz, *A Rearranged Good λ-Inequality*, Trans. Amer. Math. Soc. 293 (1986), 71-81.

[3] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, San Diego, 1988.

[4] G. Bourdaud, Sur les opérateurs pseudo-différentiels à coefficinets peu réguliers, Habilitation thesis, Université de Paris-Sud, Paris, 1983.

[5] H. Brézis and S. Wainger, *A note on limiting cases of Sobolev embeddings and convolution inequalities*, Comm. Partial Diff. Eq. 5 (1980), 773-789.

[6] D. L. Burkholder and R. F. Gundy, *Extrapolation and interpolation of quasilinear operators on martingales*, Acta Math.,124(1970), 249-304.

[7] A. Cianchi and L. Pick, *Sobolev embeddings into BMO, VMO, and L_∞*, Ark. Mat. 36 (1998), 317-340.

[8] S. Dahlke, E. Novak, W. Sickel, Optimal approximation of elliptic problems by linear and nonlinear mappings I, J. Complexity 22 (2006), 29-49.

[9] S. Dahlke, E. Novak, W. Sickel, Optimal approximation of elliptic problems by linear and nonlinear mappings II, J. Complexity 22 (2006), 549-603.

[10] D. E. Edmunds, R. Kerman, L. Pick, *Optimal Sobolev Imbeddings Involving Rearrangement-Invariant Quasinorms*, J. Funct. Anal. 170 (2000), 307-355.

[11] D. E. Edmunds, H. Triebel, Function spaces, entropy numbers, differential operators, Cambridge Univ. Press, Cambridge, 1996.

[12] J. Franke, On the spaces F^s_{pq} of Triebel-Lizorkin type: pointwise multipliers and spaces on domains, Math. Nachr. 125 (1986), 29-68.

[13] K. Hansson, *Imbedding theorems of Sobolev type in potential theory*, Math. Scand. 45 (1979), 77-102.

[14] B. Jawerth, Some observations on Besov and Lizorkin-Triebel spaces, Math. Scand. 40 (1977), 94-104.

[15] V. I. Kondrashov, *Sur certaines propriétés des fonctions dans l'espace L^ν_p*, Doklady Akad. Nauk SSSR 48 (1945), 535-538.

[16] A. Kufner, O. John and S. Fučík, *Function spaces*, Academia, Prague, 1977

[17] V. Maz'ya, *Sobolev spaces*, Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985.

[18] R. O'Neil, *Convolution Operators and L(p,q) spaces*, Duke Math. J. 30(1963), 129-142.

[19] J. Peetre, *New thoughts on Besov spaces*, Duke Univ. Math. Series, Durham, Univ., 1976.

[20] F. Rellich, *Ein Satz über mittlere Konvergenz*, Nach. Wiss. Gesell. Göttingen, Math.-Phys. Kl. (1930) 30-35.

[21] W. Sickel and H. Triebel, Hölder inequalities and sharp embeddings in function spaces of B^s_{pq} and F^s_{pq} type, Z. Anal. Anwendungen, 14 (1995), 105-140.

[22] S. L. Sobolev, *Об одной теореме функционального анализа*, Mat. Sbornik 4 (1938), 471-497;
Engl. transl.:On a theorem of functional analysis, Amer. Math. Soc. Transl. (2), 34 (1963), 39-68.

[23] H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland, Amsterdam, 1978.

[24] H. Triebel, Theory of function spaces, Birkhäuser, Basel, 1983.

[25] H. Triebel, Theory of function spaces II, Birkhäuser, Basel, 1992.

[26] H. Triebel, Theory of function spaces III, Birkhäuser, Basel, 2006.

[27] J. Vybíral, *A remark on better-lambda inequality*, Math. Ineq. Appl. 10 (2007), 335-341.

[28] J. Vybíral, *Optimal Sobolev embeddings on* $\mathbb{R}^n$, Publ. Mat. 51 (2007), 17-44.

[29] J. Vybíral, *Sampling numbers and function spaces*, J. Compl. 23 (2007), 773-792.

[30] J. Vybíral, *Dilation operators and sampling numbers*, to appear in J. of Function Spaces and Appl.

[31] J. Vybíral, *A new proof of Jawerth-Franke embedding*, to appear in Rev. Mat. Complut.

7 Publications relevant to the thesis

- *Optimal Sobolev embeddings on* $\mathbb{R}^n$
  Publ. Mat. 51 (2007), 17-44.
- *A remark on better-lambda inequality*
  Math. Ineq. Appl. 10 (2007), 335-341.
- *A new proof of Jawerth-Franke embedding*
  to appear in Rev. Mat. Complut.
- *Sampling numbers and function spaces*
  J. Compl. 23 (2007), 773-792.
- *Dilation operators and sampling numbers*
  to appear in J. of Function Spaces and Appl.

OPTIMAL SOBOLEV EMBEDDINGS ON $\mathbb{R}^n$

JAN VYBÍRAL

Abstract. We study Sobolev-type embeddings involving rearrangement-invariant norms. In particular, we focus on the question when such embeddings are optimal. We concentrate on the case when the functions involved are defined on $\mathbb{R}^n$. This subject has been studied before, but only on bounded domains. We first establish the equivalence of the Sobolev embedding to a new type of inequality involving two integral operators. Next, we show this inequality to be equivalent to the boundedness of a certain Hardy operator on a specific new type of cone of positive functions. This Hardy operator is then used to provide optimal domain and range rearrangement-invariant norm in the embedding inequality. Finally, the limiting case of the Sobolev embedding on $\mathbb{R}^n$ is studied in detail.

2000 Mathematics Subject Classification: 46E35, 46E30.
Key words: Sobolev embeddings, rearrangement-invariant norms, Hardy operators, cones of positive functions.

1. Introduction

Embeddings of spaces of smooth functions into other spaces of integrable functions form an important field of study in the theory of function spaces. Consider, for example, the classical Sobolev inequality [13] on bounded domains Ω in $\mathbb{R}^n$, $n \geq 2$. This states that, given $1 < p < n$ and setting $q = np/(n-p)$,

$$\mathring{W}^1_p(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for} \quad 1 < p < n. \tag{1.1}$$

(Here $L^q(\Omega)$ is the classical Lebesgue space, $W^1_p(\Omega)$ denotes the usual Sobolev space, $\mathring{W}^1_p(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^1_p(\Omega)$ and $\hookrightarrow$ denotes a continuous embedding.)

Now, (1.1) is the so-called sublimiting case of the Sobolev embedding (since p is strictly less than the dimension of Ω). The limiting case $p = n$ is of crucial importance and great interest. Standard examples show that although $np/(n-p)$ tends to infinity as p approaches n from the left, we may not replace the L^q-norm on the right side of (1.1) by the L^∞-norm.

It has been proved in many situations that the scale of Lebesgue spaces, although of primary interest, is not rich enough to describe all the important situations. Especially in limiting situations things can be very delicate and we have to consider finer scales of function spaces. It turns out to be very rewarding to study Sobolev-type embeddings in a broader context of general rearrangement-invariant spaces. These involve Lebesgue spaces, but also Lorentz and Orlicz spaces together with their numerous mutations, and more.

On bounded domains, a comprehensive study of Sobolev-type inequalities involving rearrangement-invariant function spaces has been carried out in [4].

In this paper, we study (1.1) with Ω replaced by the entire $\mathbb{R}^n$. In such situation, the techniques which have been successfully used for bounded domains do not work. We develop a new method suitable to deal with such problems.

Let us now briefly outline our approach. Let ϱ_R and ϱ_D be rearrangement-invariant Banach function norms on $(0, \infty)$ (precise definitions will be given in Section 2).

Our aim is to study the embedding

$$W^1_{\varrho_D}(\mathbb{R}^n) \hookrightarrow L^{\varrho_R}(\mathbb{R}^n), \tag{1.2}$$

with

$$L^{\varrho_R}(\mathbb{R}^n) = \left\{u \in L^1_{\text{loc}}(\mathbb{R}^n) : ||u|L^{\varrho_R}(\mathbb{R}^n)|| = \varrho_R(u^*) < \infty\right\} \tag{1.3}$$

and

$$W^1_{\varrho_D}(\mathbb{R}^n) = \left\{u \in L^1_{\text{loc}}(\mathbb{R}^n) : ||u|W^1_{\varrho_D}(\mathbb{R}^n)|| = \varrho_D(u^*) + \varrho_D(|\nabla u|^*) < \infty\right\}, \tag{1.4}$$

where u^* is the non-increasing rearrangement of u.

The embedding (1.2) is then equivalent to

$$\varrho_R(u^*) \le c[\varrho_D(u^*) + \varrho_D(|\nabla u|^*)], \quad u \in W^1_{\varrho_D}(\mathbb{R}^n). \tag{1.5}$$

The inequality (1.5) is the main subject of our study. Let us mention that a similar question in the frame of Bessel potential spaces was studied recently in [9].

We are interested in two main questions:

1. Suppose that the 'range' norm ϱ_R is given. We want to find the optimal (that is, essentially smallest) norm ϱ_D for which (1.5) holds. The optimality means that if (1.5) holds with ϱ_D replaced by some other rearrangement-invariant norm σ, then there exists a constant $C > 0$ such that $\varrho_D(u^*) \le C\sigma(u^*)$ for all functions $u \in L^1_{\text{loc}}(\mathbb{R}^n)$.

2. Suppose that the 'domain' norm ϱ_D is given. We would like to construct the corresponding optimal 'range' norm ϱ_R. This means that the ϱ_R will be the essentially largest rearrangement-invariant norm for which (1.5) holds.

In Section 3, we reduce (1.5) to a certain new type of inequality involving two different Hardy-type operators. Similar inequalities appeared recently in [5], but in a completely different context. In Section 4 we prove another equivalent version of (1.5), namely inequality (4.4), which connects certain specific Hardy operator with an interesting cone of positive functions. The delicate interplay between this operator on the one side and the cone on the other side plays a crucial role in the subsequent sections, and is of independent interest. Especially, we emphasise that knowledge of both of these notions is indispensable in most of the results yet to come. We refer to Lemma 5.1 and Lemma 6.1 for details. The action of Hardy operators on cones of positive functions was very recently studied in [11] and [12] in a different context. It seems to be a very promising subject of study which opens interesting new directions of research and which might provide new ways how to approach to various difficult problems.

In Section 5 and 6 we find optimal domain and optimal range spaces for (1.2) under two rather restrictive conditions (5.2) and (6.10). In Section 7 we show that these conditions are satisfied in sub-limiting cases and give a complete answer in these situations.

In order to be able to give definitive answer in the limiting case as well, we have to develop a yet finer method. This is done in Section 8, where the limiting case is investigated in detail.

A crucial step is provided by Lemmas 5.1 and 6.1. The rather technical proofs of these results are given in the Appendix. These lemmas play a substantial role in our approach as they describe the wonderful interplay between the Hardy operator (4.6) and the convex cone (4.5).

In [14] we studied the inequality

$$\varrho_R(u^*) \le c\varrho_D(|\nabla u|^*), \quad u \in W^1_{\varrho_D}(\mathbb{R}^n),$$

which corresponds to one part of (1.5). As we shall see, the study of (1.5) requires several new techniques to be developed.

Throughout the paper, c stands for a positive constant, not necessarily the same at each occurrence. Sometimes we abbreviate the inequality $A \le c\,B$ to $A \lesssim B$. The same applies to symbols "$\gtrsim$" and "$\approx$".

2. Rearrangement-invariant norms

We denote by $\mathfrak{M}(\mathbb{R}^n)$ the set of real-valued Lebesgue-measurable functions on $\mathbb{R}^n$ finite almost everywhere and by $\mathfrak{M}_+(\mathbb{R}^n)$ the class of non-negative functions in $\mathfrak{M}(\mathbb{R}^n)$. Finally, $\mathfrak{M}_+(0,\infty,\downarrow)$ denotes the set of all non-increasing functions from $\mathfrak{M}_+(0,\infty)$. Given $f \in \mathfrak{M}(\mathbb{R}^n)$ we define its non-increasing rearrangement by

$$f^*(t) = \inf\{\lambda > 0 : |\{|f(x)| > \lambda\}| \le t\}, \quad 0 < t < \infty. \tag{2.1}$$

For a set $A \subset \mathbb{R}^n$ we denote by $|A|$ its Lebesgue measure. A detailed treatment of rearrangements may be found in [1]. Furthermore, we set

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\mathrm{d}s, \quad 0 < t < \infty. \tag{2.2}$$

We point out two important properties, namely

$$(f+g)^*(t) \le f^*\left(\frac{t}{2}\right) + g^*\left(\frac{t}{2}\right), \quad 0 < t < \infty, \tag{2.3}$$

and

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t), \quad 0 < t < \infty, \quad f, g \in \mathfrak{M}(\mathbb{R}^n). \tag{2.4}$$

We briefly recall some basic aspects of the theory of Banach function norms. For details, see [1].

Definition 2.1. A functional $\varrho : \mathfrak{M}_+(0,\infty) \to [0,\infty]$ is called a *Banach function norm* on $(0,\infty)$ if, for all $f, g, f_n, (n = 1, 2, \dots)$, in $\mathfrak{M}_+(0,\infty)$, for all constants $a \ge 0$ and for all measurable subsets E of $(0,\infty)$, it satisfies the following axioms

$$\begin{aligned}
&(A_1) \quad \varrho(f) = 0 \quad \text{if and only if} \quad f = 0 \text{ a.e.};\\
& \quad \varrho(af) = a\varrho(f);\\
& \quad \varrho(f+g) \le \varrho(f) + \varrho(g);\\
&(A_2) \quad \text{if} \quad 0 \le g \le f \text{ a.e. then} \quad \varrho(g) \le \varrho(f);\\
&(A_3) \quad \text{if} \quad 0 \le f_n \uparrow f \text{ a.e. then} \quad \varrho(f_n) \uparrow \varrho(f);\\
&(A_4) \quad \text{if} \quad |E| < \infty \quad \text{then} \quad \varrho(\chi_E) < \infty;\\
&(A_5) \quad \text{if} \quad |E| < \infty \quad \text{then} \quad \int_E f \le C_E\,\varrho(f)
\end{aligned}$$

with some constant $0 < C_E < \infty$, depending on ϱ and E but independent of f.

If, in addition, $\varrho(f) = \varrho(f^*)$, we say that ϱ is *rearrangement-invariant* (*r.i.*) *Banach function norm.* We often use the notions *norm* and *r.i. norm* to shorten the notation.

Definition 2.2. The *dilation operator* E_s, $0 < s < \infty$, is defined by

$$(E_s f)(t) = f(st), \quad 0 < t < \infty, \quad f \in \mathfrak{M}(0,\infty). \tag{2.5}$$

The *dual* of a norm ϱ is the functional

$$\varrho'(g) = \sup_{h:\varrho(h)=1} \int_0^\infty g(t)h(t)\mathrm{d}t, \quad g, h \in \mathfrak{M}_+(0,\infty). \tag{2.6}$$

Theorem 2.3. (G. H. Hardy, J. E. Littlewood). *If $f, g \in \mathfrak{M}(\mathbb{R}^n)$ then*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\mathrm{d}x \le \int_0^\infty f^*(s)g^*(s)\mathrm{d}s. \tag{2.7}$$

Theorem 2.4. (G. G. Lorentz, W. A. J. Luxemburg). *Let ϱ be a Banach function norm. Then*

$$\varrho'' = \varrho. \tag{2.8}$$

Theorem 2.5. (G. H. Hardy-J. E. Littlewood-G. Pólya). *Let ϱ be an r.i. norm on $(0,\infty)$ and let $f_1, f_2 \in \mathfrak{M}(\mathbb{R}^n)$ satisfy*

$$\int_0^t f_1^*(s)\mathrm{d}s \leq \int_0^t f_2^*(s)\mathrm{d}s, \quad s > 0.$$

Then

$$\varrho(f_1^*) \leq \varrho(f_2^*).$$

Lemma 2.6. (Hardy's Lemma). *Let f_1 and f_2 be non-negative measurable functions on $(0,\infty)$ and suppose*

$$\int_0^t f_1(s)\mathrm{d}s \leq \int_0^t f_2(s)\mathrm{d}s$$

for all $t > 0$. Let $h \in \mathfrak{M}_+(0,\infty,\downarrow)$. Then

$$\int_0^\infty f_1(s)h(s)\mathrm{d}s \leq \int_0^\infty f_2(s)h(s)\mathrm{d}s.$$

If $1 \leq p \leq \infty$, we define

$$\varrho_p(g) = ||g||_p := \begin{cases} \left(\int_{\mathbb{R}^n} |g(x)|^p \mathrm{d}x\right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{x\in\mathbb{R}^n} |g(x)| & \text{if } p = \infty. \end{cases}$$

3. Reduction to Hardy Operators

In this section we present the first step in the study of (1.5), namely a reduction of (1.5) to the boundedness of certain Hardy operators.

Theorem 3.1. *Let ϱ_D, ϱ_R be two r.i. Banach function norms on $(0,\infty)$. Then the inequality*

$$\varrho_R(u^*) \leq c[\varrho_D(u^*) + \varrho_D(|\nabla u|^*)], \quad u \in W^1_{\varrho_D}(\mathbb{R}^n), \tag{3.1}$$

holds if and only if there is a constant $K > 0$ such that

$$\varrho_R\left(\int_t^\infty f(s)s^{1/n-1}\mathrm{d}s\right) \leq K\varrho_D\left(f(t) + \int_t^\infty f(s)s^{1/n-1}\mathrm{d}s\right) \tag{3.2}$$

for all $f \in \mathfrak{M}_+(0,\infty)$.

Proof. Step 1.

Let us suppose that (3.1) holds and that a function $f \in \mathfrak{M}_+(0,\infty)$ is given. We define a new function u by

$$u(x) = \int_{\omega_n|x|^n}^\infty f(t)t^{1/n-1}\mathrm{d}t, \quad x \in \mathbb{R}^n,$$

where ω_n is the volume of unit ball in $\mathbb{R}^n$. We may assume, that $u(x)$ is finite a.e. (otherwise both sides of (3.2) are identically infinite and there is nothing to prove). Considering level sets of u we obtain

$$u^*(t) = \int_t^\infty f(s)s^{1/n-1}\mathrm{d}s, \ |(\nabla u)(x)| = n\omega_n^{1/n} f(\omega_n|x|^n), \ |(\nabla u)|^*(t) = n\omega_n^{1/n} f^*(t).$$

We point out, that if $u \notin W^1_{\varrho_D}(\mathbb{R}^n)$, then (3.1) holds trivially. Therefore we may apply (3.1) and obtain

$$\varrho_R\left(\int_t^\infty f(s)s^{1/n-1}\mathrm{d}s\right) = \varrho_R(u^*(t)) \leq c\left[\varrho_D(f) + \varrho_D\left(\int_t^\infty f(s)s^{1/n-1}\mathrm{d}s\right)\right],$$

which is equivalent to (3.2).

Step 2.

Let us now assume that (3.2) is true and $u \in W^1_{\varrho_D}(\mathbb{R}^n)$ with compact support is given. First note that

$$u^*(t) = -\int_t^\infty \frac{\mathrm{d}u^*(s)}{\mathrm{d}s}\mathrm{d}s. \tag{3.3}$$

Next, we recall the following generalization of the Pólya—Szegö principle from [3, (4.3)]:

$$\int_0^t \left[-s^{1-1/n}\frac{\mathrm{d}u^*}{\mathrm{d}s}\right]^*(s)\mathrm{d}s \le c\int_0^t |\nabla u|^*(s)\mathrm{d}s, \tag{3.4}$$

which holds for every $t > 0$ and every weakly differentiable function u such that $(\nabla u) \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ and

$$|\{x \in \mathbb{R}^n : |u(x)| > s\}| < \infty \quad \text{for all} \quad s > 0.$$

As $\nabla u \in L_{\varrho_D}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ and u has compact support, these assumptions are satisfied and (3.4) applies to u.

Using Theorem of Hardy, Littlewood and Pólya (Theorem 2.5) on (3.4) we obtain

$$\varrho_D\left(-s^{1-1/n}\frac{\mathrm{d}u^*(s)}{\mathrm{d}s}\right) \le \varrho_D(|(\nabla u)|^*(t)). \tag{3.5}$$

We combine our assumption with these observations and use (3.3), (3.2) with $f = s^{1-1/n}\frac{\mathrm{d}u^*(s)}{\mathrm{d}s}$ and (3.5) to obtain

$$\begin{aligned}\varrho_R(u^*(t)) &= \varrho_R\left(-\int_t^\infty \frac{\mathrm{d}u^*(s)}{\mathrm{d}s}\mathrm{d}s\right)\\ &\le c\left[\varrho_D\left(-\int_t^\infty \frac{\mathrm{d}u^*(s)}{\mathrm{d}s}\mathrm{d}s\right) + \varrho_D\left(-s^{1-1/n}\frac{\mathrm{d}u^*(s)}{\mathrm{d}s}\right)\right]\\ &\le c\left[\varrho_D(u^*(t)) + \varrho_D(|\nabla u|^*(t))\right].\end{aligned}$$

Hence, (3.1) holds for every $u \in W^1_{\varrho_D}(\mathbb{R}^n)$ with compact support. For a general $u \in W^1_{\varrho_D}(\mathbb{R}^n)$ we define

$$u_n = u\varphi_n, \qquad \varphi_n(x) = \begin{cases} 1 & \text{if } |x| < n,\\ n+1-|x| & \text{if } n \le |x| \le n+1,\\ 0 & \text{if } |x| > n+1.\end{cases}$$

We apply (3.1) to u_n and use

$$|u_n(x)| \le |u(x)|, \qquad |(\nabla u_n)(x)| \le c[|(\nabla u)(x)| + |u(x)|], \qquad x \in \mathbb{R}^n, \quad n \in \mathbb{N}.$$

This leads to

$$\varrho_R(u_n^*) \le c\left[\varrho_D(u_n^*) + \varrho_D(|\nabla u_n|^*)\right] \le c\left[\varrho_D(u^*) + \varrho_D(|\nabla u|^*)\right]. \tag{3.6}$$

The monotone convergence of $|u_n|$ to $|u|$ and axiom (A_3) show that the left side of (3.6) tends to $\varrho_R(u)$ as n tends to infinity. □

4. Another equivalent version of (1.5)

The inequality (3.2) obtained in Theorem 3.1 is still not suitable for further investigation. Therefore we will derive another equivalent version of (3.1). In (3.2) we substitute

$$g(t) = f(t) + \int_t^\infty f(s)s^{1/n-1}\mathrm{d}s, \quad f \in \mathfrak{M}_+(0,\infty), \quad t > 0. \tag{4.1}$$

We shall need also the inverse substitution. Namely, if g is defined by (4.1), then

$$f(t) = g(t) - e^{nt^{1/n}} \int_t^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s. \tag{4.2}$$

If f is differentiable, then it may be proven by differentiation of (4.1). For a general f we observe, that the equation (4.1) has only one solution f for a fixed $g \in \mathfrak{M}_+(0,\infty)$. And a direct computation shows that it is given by (4.2).

Finally, we sum up (4.1) and (4.2) and obtain

$$\int_t^\infty f(s)s^{1/n-1}\mathrm{d}s = e^{nt^{1/n}} \int_t^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u \text{ for a.e. } t > 0. \tag{4.3}$$

This substitution can now be used to reformulate (3.1).

Theorem 4.1. *Let ϱ_D, ϱ_R be two r.i. Banach function norms on $(0,\infty)$. Then, (3.1) is equivalent to*

$$\varrho_R\left(e^{nt^{1/n}} \int_t^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\right) \le c\varrho_D(g) \text{ for all } g \in \mathbf{G}, \tag{4.4}$$

where $\mathbf{G}$ is the new class of functions, defined by

$$\begin{aligned}\mathbf{G} = \Big\{g \in \mathfrak{M}_+(0,\infty) : \quad &\text{there is a function } f \in \mathfrak{M}_+(0,\infty) \text{ such that} \\ &g(t) = f(t) + \int_t^\infty f(s)s^{1/n-1}\mathrm{d}s \text{ for all } t > 0\Big\} \\ = \left\{g \in \mathfrak{M}_+(0,\infty) : g(t) - e^{nt^{1/n}} \int_t^\infty \right. & \left. g(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s \ge 0 \text{ for all } t > 0\right\}.\end{aligned} \tag{4.5}$$

Proof. The assertion follows immediately from Theorem 3.1, (4.2) and (4.3). □

Hence the inequality (3.1) is equivalent to the boundedness of the Hardy-type operator

$$(Gg)(u) = e^{nu^{1/n}} \int_u^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s, \quad u > 0 \tag{4.6}$$

on the set $\mathbf{G}$. Using this notation, we may rewrite (4.3). If g is defined by (4.1), we have $Gg(t) = \int_t^\infty f(s)s^{1/n-1}\mathrm{d}s$. Furthermore, the set $\mathbf{G}$ is the image of the positive cone $\mathfrak{M}_+(0,\infty)$ under the operator

$$f \to f(t) + \int_t^\infty f(s)s^{1/n-1}\mathrm{d}s.$$

Before we proceed any, further we shall derive some basic properties of the class $\mathbf{G}$.

Remark 4.2. (i) $\mathbf{G}$ contains all non-negative non-increasing functions. To see this, note that for all $g \in \mathfrak{M}_+(0,\infty,\downarrow)$

$$\begin{aligned} g(t) - e^{nt^{1/n}} \int_t^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s \ge \\ \ge g(t)\left\{1 - e^{nt^{1/n}} \int_t^\infty s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s\right\} = 0.\end{aligned} \tag{4.7}$$

(ii) For every g from $\mathbf{G}$, Gg is non-increasing. Indeed, let $g \in \mathbf{G}$ and let f be defined by (4.2), then

$$(Gg)'(t) = \left[e^{nt^{1/n}} \int_t^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\right]' = -t^{1/n-1}f(t) \le 0. \tag{4.8}$$

(iii) The set $\mathbf{G}$ is a *convex cone*, that is, for every $\alpha, \beta > 0$ and $g_1, g_2 \in \mathbf{G}$, we have $\alpha g_1 + \beta\alpha_2 \in \mathbf{G}$. The proof of this statement is trivial.

Remark 4.3. (i) To show some applications we prove that $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{\frac{np}{n-p},p}(\mathbb{R}^n)$ for $1 \leq p < n$. In this case, we have $\varrho_R(f) = ||f^*(t)t^{-1/n}||_p$ and $\varrho_D(f) = ||f||_p$. Using Remark 4.2 (ii) and the boundedness of classical Hardy operators on L^p we get for every function $g \in \mathbf{G}$ that

$$\begin{aligned}\varrho_R(Gg) &= ||t^{-1/n}(Gg)^*(t)||_p \\ &= \left|\left| t^{-1/n} e^{nt^{1/n}} \int_t^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u \right|\right|_p \\ &\leq \left|\left| t^{-1/n} \int_t^\infty g(u)u^{1/n-1}\mathrm{d}u \right|\right|_p \\ &\leq c||t^{-1/n}g(t)t^{1/n}||_p = c||g||_p = c\varrho_D(g).\end{aligned}$$

(ii) Another application of the obtained results is the embedding $W^1(L^{n,1})(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$. In this case

$$\begin{aligned}\varrho_R(Gg) = \sup_{t>0}(Gg)(t) = (Gg)(0) &= \int_0^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u \\ \leq \int_0^\infty g(u)u^{1/n-1}\mathrm{d}u &\leq \int_0^\infty g^*(u)u^{1/n-1}\mathrm{d}u = \varrho_D(g)\end{aligned}$$

for every function $g \in \mathbf{G}$. Now we used Remark 4.2 (ii) and Theorem 2.3.

(iii) Both these applications recover well-known results. They demonstrate some important aspects of this method. First, the second basic property of the class $\mathbf{G}$ (c.f. Remark 4.2, (ii)) lies in the roots of every Sobolev embedding. Second, the boundedness of Hardy operators plays a crucial role in this theory.

(iv) We haven't used the property (4.7) yet. It will play a crucial role in the study of optimality of obtained results.

5. Optimal domain space

In this section we are going to solve one of the main problems stated in the Introduction. We shall construct the optimal domain norm ϱ_D to a given range norm ϱ_R.

We start with a crucial lemma describing one important property of the class $\mathbf{G}$ which shall be useful later on. We postpone its proof to Appendix.

Lemma 5.1. *The inequality*

$$\int_t^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u \leq c\int_t^\infty g^{**}(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u, \quad t \geq 0. \tag{5.1}$$

holds for every $g \in \mathbf{G}$ with c independent of g.

Now we may solve the problem of the optimal domain space.

Theorem 5.2. *Let the norm ϱ_R satisfy*

$$\varrho_R\left(G(g^{**})\right) \leq c\varrho_R\left(G(g^*)\right), \quad g \in \mathfrak{M}_+(0,\infty). \tag{5.2}$$

Then the optimal domain norm ϱ_D corresponding to ϱ_R in the sense described in the Introduction is defined by

$$\varrho_D(g) := \varrho_R\left(G(g^{**})\right), \quad g \in \mathfrak{M}_+(0,\infty). \tag{5.3}$$

Proof. First, we point out that the functional ϱ_D defined by (5.3) is a norm. The axioms $(A_1)-(A_3)$ are trivially satisfied. To prove (A_4) for ϱ_D we fix a set $E \subset (0,\infty)$ with $|E| < \infty$. Then we get $G\chi_E^*(t) \leq \chi_{(0,|E|)}(t)$ for every $t > 0$, and using (5.2) and (A_4) for ϱ_R, we get

$$\varrho_D(\chi_E) = \varrho_R(G\chi_E^{**}) \leq c\varrho_R(G\chi_E^*) \leq c\varrho_R(\chi_{(0,|E|)}) < \infty.$$

To verify (A_5) for ϱ_D we fix also a set $E \subset (0,\infty)$ with $|E| = a < \infty$ and use (A_5) for ϱ_R. Consequently,

$$\begin{aligned}
\varrho_D(g) = \varrho_R(Gg^{**}) &\geq c \int_0^{a/2} (Gg^{**})(t)\mathrm{d}t \\
&\geq c \int_0^{a/2} e^{nt^{1/n}} \int_{a/2}^{a} g^{**}(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s\mathrm{d}t \\
&\geq c\, g^{**}(a) \int_0^{a/2} e^{nt^{1/n}}\mathrm{d}t \int_{a/2}^{a} s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s \\
&\geq c_E \int_0^a g^*(s)\mathrm{d}s \geq c_E \int_E g.
\end{aligned}$$

Now we have to verify that (4.4) really holds. Let us fix a $g \in \mathbf{G}$. Then, by (5.1) and (5.3)

$$\begin{aligned}
\varrho_R\left(e^{nt^{1/n}} \int_t^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\right) &\leq c\varrho_R\left(e^{nt^{1/n}} \int_t^\infty g^{**}(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\right) \\
&= c\varrho_D(g).
\end{aligned}$$

Finally, we have to show that ϱ_D is optimal. Let us suppose that (4.4) holds with some other r.i. norm σ instead of ϱ_D. We want to show that $\varrho_D(g) \leq c\sigma(g)$ for every function $g \in \mathfrak{M}_+(0,\infty)$. Using (5.2) and the first property of the class $\mathbf{G}$ from Remark 4.2, namely that $g^* \in \mathbf{G}$ for every function $g \geq 0$, we get

$$\begin{aligned}
\varrho_D(g) &= \varrho_R\left(e^{nt^{1/n}} \int_t^\infty g^{**}(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\right) \\
&\leq c\varrho_R\left(e^{nt^{1/n}} \int_t^\infty g^{*}(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\right) \\
&\leq c\sigma(g^*) = c\sigma(g).
\end{aligned}$$

□

6. Optimal Range Space

In this section we solve the converse problem. Namely, the norm ϱ_D is now considered to be fixed and we are searching for the optimal ϱ_R. First of all we shall introduce some notation.

We recall (4.6) and define

$$(6.1)\qquad (Gg)(t) = e^{nt^{1/n}} \int_t^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s, \quad g \in \mathfrak{M}_+(0,\infty), \quad t > 0,$$

$$(6.2)\qquad (Hh)(t) = t^{1/n-1}e^{-nt^{1/n}} \int_0^t h(s)e^{ns^{1/n}}\mathrm{d}s, \quad h \in \mathfrak{M}_+(0,\infty), \quad t > 0,$$

$$(6.3)\qquad E(s) = e^{-ns^{1/n}} \int_0^s e^{nu^{1/n}}\mathrm{d}u, \quad s > 0.$$

The operators G and H are dual in the following sense

$$(6.4)\qquad \int_0^\infty h(t)Gg(t)\mathrm{d}t = \int_0^\infty g(u)Hh(u)\mathrm{d}u \quad \text{for all} \quad g,h \in \mathfrak{M}_+(0,\infty).$$

As in [4], we would like to use duality to define ϱ_R. Using the notation introduced above, we can rewrite (4.4) as

$$(6.5)\qquad \sup_{g\in\mathbf{G}} \frac{\varrho_R(Gg)}{\varrho_D(g)} < \infty.$$

We may employ the duality in the following way:

$$\sup_{g\in\mathbf{G}}\frac{\varrho_R(Gg)}{\varrho_D(g)}=\sup_{g\in\mathbf{G},h\in\mathfrak{M}_+(0,\infty,\downarrow)}\frac{\int_0^\infty(Gg)(t)h(t)\mathrm{d}t}{\varrho_D(g)\varrho_R'(h)}$$
$$=\sup_{g\in\mathbf{G},h\in\mathfrak{M}_+(0,\infty,\downarrow)}\frac{\int_0^\infty(Hh)(t)g(t)\mathrm{d}t}{\varrho_D(g)\varrho_R'(h)}.$$

We have used Remark 4.2 (ii), (6.4) and the so-called *resonance* of the measure space $((0,\infty),\mathrm{d}x)$. We refer to [1, Chapter 2, Def. 2.3. and Chapter 2, Theorem 2.7.] for details.

Let us now suppose for a moment that extending the supremum over all $g\in\mathfrak{M}_+(0,\infty)$ gives an equivalent quantity. Then we could continue the calculation

(6.6)
$$\sup_{g\in\mathbf{G}}\frac{\varrho_R(Gg)}{\varrho_D(g)}\approx\sup_{g\in\mathfrak{M}_+(0,\infty),h\in\mathfrak{M}_+(0,\infty,\downarrow)}\frac{\int_0^\infty(Hh)(t)g(t)\mathrm{d}t}{\varrho_D(g)\varrho_R'(h)}=\sup_{h\in\mathfrak{M}_+(0,\infty,\downarrow)}\frac{\varrho_D'(Hh)}{\varrho_R'(h)},$$

and the inequality (4.4) would be equivalent to

$$\varrho_D'(Hh)\le c\varrho_R'(h),\qquad h\in\mathfrak{M}_+(0,\infty,\downarrow) \tag{6.7}$$

A sufficient condition that would enable us to extend the supremum is given in the following lemma. We postpone its proof to Appendix.

Lemma 6.1. *Assume that the r.i. norm ϱ_D satisfies*

$$\varrho_D\left(\int_s^\infty f(u)\frac{E(u)}{u}u^{1/n-1}\mathrm{d}u\right)\le c\varrho_D(f),\quad f\in\mathfrak{M}_+(0,\infty). \tag{6.8}$$

Then

(6.9)
$$\sup_{g\in\mathbf{G}}\frac{\int_0^\infty(Hh)(t)g(t)\mathrm{d}t}{\varrho_D(g)}\approx\sup_{g\in\mathfrak{M}_+(0,\infty)}\frac{\int_0^\infty(Hh)(t)g(t)\mathrm{d}t}{\varrho_D(g)},\ \textit{for all } h\in\mathfrak{M}_+(0,\infty,\downarrow)$$

The constants of equivalence do not depend on the choice of $h\in\mathfrak{M}_+(0,\infty,\downarrow)$.

As we shall see, the condition (6.8) is satisfied in all important examples, including the limiting Sobolev embedding. Equipped with this tool, we can now easily solve our problem.

Theorem 6.2. *Assume that the r.i. norm ϱ_D satisfies* (6.8) *and that its dual norm ϱ_D' satisfies*

$$\varrho_D'(H(h^{**}))\le c\varrho_D'(H(h^*)),\quad h\in\mathfrak{M}_+(0,\infty). \tag{6.10}$$

Then the optimal range norm in (4.4) *associated to ϱ_D is given as a dual norm to $\varrho_D'(H(f^{**}))$. Or, equivalently, the dual of the optimal range norm can be described by $\varrho_R'(f):=\varrho_D'(H(f^{**}))$.*

Proof. According to Lemma 6.1 and the calculation above, (4.4) is equivalent to (6.7) But for our choice of ϱ_R' this inequality is trivially true.

To prove the optimality, suppose, again, that there is another r.i. norm σ, such that (6.7) is true when we substitute its dual norm σ' in place of ϱ_R'. Then,

$$\sigma'(f)=\sigma'(f^*)\ge c\varrho_D'(H(f^*))\ge c\varrho_D'(H(f^{**}))=c\varrho_R'(f),\ \text{for all } f\in\mathfrak{M}_+(0,\infty),$$

proving the optimality of ϱ_R.

Finally, we have to prove that the functional $\varrho(f)=\varrho_D'(H(f^{**}))$ is a norm. Again, the axioms $(A_1)-(A_3)$ are trivially satisfied. Using (6.10), Hardy's Lemma 2.6 and axiom (A_4) for ϱ_D' we get also (A_4) for ϱ. (A_5) follows from the same axiom for ϱ_D'. □

7. The study of (5.2) and (6.10)

In this section we derive sufficient conditions for (6.8) and (6.10). In general, we follow the idea of [4, Theorem 4.4]. First of all, for every function $f \in \mathfrak{M}_+(0,\infty)$, we define the dilation operator E by

$$(E_s f)(t) = f(st), \quad t > 0, \quad s > 0.$$

It is well known, [1, Chapter 3, Prop. 5.11], that for every r.i. norm ϱ on $\mathfrak{M}_+(0,\infty)$ and every $s > 0$ the operator E_s satisfies

$$\varrho(E_s f) \le c\varrho(f), \quad f \in \mathfrak{M}_+(0,\infty).$$

The smallest possible constant c in this inequality (which depends of course on s) is denoted by $h_\varrho(s)$. Hence

$$h_\varrho(s) = \sup_{f \not\equiv 0} \frac{\varrho(E_s f)}{\varrho(f)}.$$

Now we are ready to prove the following result.

Theorem 7.1. *If a rearrangement–invariant norm ϱ_R satisfies $\int_0^1 s^{-1/n} h_{\varrho_R}(s)\mathrm{d}s < \infty$, then it also satisfies* (5.2).

Proof. Step 1.
Let us suppose that the positive real numbers s, t, y satisfy $st < y$ and $0 < s < 1$. Then $t^{1/n} < (y/s)^{1/n}$ and, consequently,

$$e^{nt^{1/n} - n(y/s)^{1/n}} \le \left[e^{nt^{1/n} - n(y/s)^{1/n}}\right]^{s^{1/n}} = e^{n(st)^{1/n} - ny^{1/n}}.$$

So, for every function $f \in \mathfrak{M}_+(0,\infty)$, we obtain

$$e^{nt^{1/n}} \int_{st}^\infty f^*(y) y^{1/n-1} e^{-n(y/s)^{1/n}} \mathrm{d}y \le e^{n(st)^{1/n}} \int_{st}^\infty f^*(y) y^{1/n-1} e^{-ny^{1/n}} \mathrm{d}y.$$

Step 2.
We may now come to the proof of the Theorem. Fix a function $g \in \mathfrak{M}_+(0,\infty)$, with $\varrho_R'(g) = 1$. Then we use several times Fubini's Theorem, the change of variables, and inequality from Step 1 and obtain

$$\begin{aligned}
\int_0^\infty g^*(t) G f^{**}(t)\mathrm{d}t &= \int_0^\infty g^*(t) e^{nt^{1/n}} \int_t^\infty f^{**}(s) s^{1/n-1} e^{-ns^{1/n}} \mathrm{d}s\mathrm{d}t \\
&= \int_0^\infty s^{1/n-1} e^{-ns^{1/n}} \int_0^s g^*(u) e^{nu^{1/n}} \mathrm{d}u \int_0^1 f^*(st)\mathrm{d}t\mathrm{d}s \\
&= \int_0^1 \int_0^\infty f^*(st) s^{1/n-1} e^{-ns^{1/n}} \int_0^s g^*(u) e^{nu^{1/n}} \mathrm{d}u\mathrm{d}s\mathrm{d}t \\
&= \int_0^1 \int_0^\infty g^*(u) e^{nu^{1/n}} \int_u^\infty f^*(st) s^{1/n-1} e^{-ns^{1/n}} \mathrm{d}s\mathrm{d}u\mathrm{d}t \\
&= \int_0^1 t^{-1/n} \int_0^\infty g^*(u) e^{nu^{1/n}} \int_{tu}^\infty f^*(y) y^{1/n-1} e^{-n(y/t)^{1/n}} \mathrm{d}y\mathrm{d}u\mathrm{d}t \\
&= \int_0^1 s^{-1/n} \int_0^\infty g^*(t) e^{nt^{1/n}} \int_{st}^\infty f^*(y) y^{1/n-1} e^{-n(y/s)^{1/n}} \mathrm{d}y\mathrm{d}t\mathrm{d}s \\
&\le \int_0^1 s^{-1/n} \int_0^\infty g^*(t) e^{n(st)^{1/n}} \int_{st}^\infty f^*(y) y^{1/n-1} e^{-ny^{1/n}} \mathrm{d}y\mathrm{d}t\mathrm{d}s \\
&= \int_0^1 s^{-1/n} \int_0^\infty g^*(t) (Gf^*)(st)\mathrm{d}t\mathrm{d}s.
\end{aligned}$$

Taking a supremum over g, we obtain that the left-hand side of (5.2) can be estimated from above by

$$\begin{aligned}\sup_{g\geq 0:\varrho_R'(g)=1}\int_0^1 s^{-1/n}\int_0^\infty g^*(t)(Gf^*)(st)\mathrm{d}t\mathrm{d}s \\ &= \int_0^1 s^{-1/n}\varrho_R((Gf^*)(s\cdot))\mathrm{d}s \\ &\leq \int_0^1 s^{-1/n}h_{\varrho_R}(s)\varrho_R(Gf^*)\mathrm{d}s \\ &= \left(\int_0^1 s^{-1/n}h_{\varrho_R}(s)\mathrm{d}s\right)\varrho_R(Gf^*).\end{aligned}$$

□

An analogous result can be obtained also for (6.10). The proof is omitted as it uses the same ideas as the preceding one.

Theorem 7.2. *If an r.i. norm σ satisfies $\int_0^1 s^{-1/n}h_\sigma(s)\mathrm{d}s<\infty$ then it satisfies also* (6.10) *with ϱ_D' replaced by σ.*

We will now present some applications of our results.

Example 7.3. Let

$$\varrho_R(f)=\varrho_\infty(f)=\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}|f(x)|.$$

Then $h_{\varrho_R}(s)=1$ and, according to Theorem 7.1, (5.2) is satisfied and the optimal domain norm is given by

$$\varrho_D(f)\approx\sup_{t>0}(Gf^*)(t)=\int_0^\infty f^*(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s,\quad f\in\mathfrak{M}(\mathbb{R}^n).$$

This norm is essentially smaller than $\varrho_{n,1}(f)=\int_0^\infty t^{1/n-1}f^*(t)\mathrm{d}t$, hence this result improves the second example from Remark 4.3. Now, an easy calculation shows that

$$\varrho_D(f)\approx f^*(1)+\int_0^1 f^*(t)t^{1/n-1}\mathrm{d}t\approx\varrho_\infty(f^*\chi_{(1,\infty)})+\varrho_{n,1}(f^*\chi_{(0,1)}),\quad f\in\mathfrak{M}(\mathbb{R}^n).$$

Example 7.4. Let

$$\varrho_D(f)=\varrho_1(f)=\int_{\mathbb{R}^n}|f(x)|\mathrm{d}x.$$

In that case, $\varrho_D'=\varrho_\infty$, whence $h_{\varrho_D'}(s)=1$. So, by Theorem 7.2, (6.10) is satisfied. It is a simple exercise to verify (6.8). Using Theorem 6.2, the optimal range norm can be described as the dual norm to

$$\sigma(f)=\varrho_\infty(Hf^*)=\varrho_\infty\left(t^{1/n-1}e^{-nt^{1/n}}\int_0^t f^*(s)e^{ns^{1/n}}\mathrm{d}s\right).$$

To simplify

$$\varrho_R(g)=\sigma'(g)=\sup_{f:\varrho_\infty(Hf^*)\leq 1}\int_0^\infty f^*(t)g^*(t)\mathrm{d}t,$$

we take $f(t)=t^{-1/n}\chi_{(0,1)}(t)+\chi_{(1,\infty)}(t)$. Calculation shows that then Hf^* is bounded on $(0,\infty)$. This choice leads to

$$\varrho_R(g)\gtrsim\int_0^1 g^*(t)t^{-1/n}\mathrm{d}t+\int_1^\infty g^*(t)\mathrm{d}t.$$

To prove the converse estimate, take $f \in \mathfrak{M}_+(0,\infty,\downarrow)$ bounded and $g \in \mathfrak{M}_+(0,\infty,\downarrow)$ bounded, with bounded support and differentiable. Then a direct calculation using only integration by parts and Fubini's Theorem shows that

$$\begin{aligned}\int_0^\infty f^*(t)g^*(t)\mathrm{d}t &= \int_0^\infty t^{1/n-1}e^{-nt^{1/n}}\int_0^t f^*(s)e^{ns^{1/n}}\mathrm{d}s\cdot\left[g^*(t)-t^{1-1/n}\frac{\mathrm{d}g^*}{\mathrm{d}t}(t)\right]\mathrm{d}t\\ &\le \varrho_\infty(Hf^*)\int_0^\infty\left[g^*(t)-t^{1-1/n}\frac{\mathrm{d}g^*}{\mathrm{d}t}(t)\right]\mathrm{d}t \qquad (7.1)\\ &\lesssim \varrho_\infty(Hf^*)\left[\int_0^1 g^*(t)t^{-1/n}\mathrm{d}t+\int_1^\infty g^*(t)\mathrm{d}t\right]\end{aligned}$$

If $f \in \mathfrak{M}_+(0,\infty,\downarrow)$ is not bounded, it may be approximated by a monotone sequence $f_n \nearrow f$, $f_n \in \mathfrak{M}_+(0,\infty,\downarrow)$. This procedure shows that (7.1) holds for every $f \in \mathfrak{M}_+(0,\infty,\downarrow)$ and g as above. Finally, every $g \in \mathfrak{M}_+(0,\infty,\downarrow)$ may also be approximated by differentiable functions $g_n \nearrow g$, $g_n \in \mathfrak{M}_+(0,\infty,\downarrow)$ with bounded supports. This provides (7.1) for all $f, g \in \mathfrak{M}_+(0,\infty,\downarrow)$.

Hence,

$$\varrho_R(g) = \sup_{f:\varrho_\infty(Hf^*)\le 1}\int_0^\infty f^*(t)g^*(t)\mathrm{d}t \approx \int_0^1 g^*(t)t^{-1/n}\mathrm{d}t+\int_1^\infty g^*(t)\mathrm{d}t.$$

8. The limiting embedding

In this section we consider the case of limiting Sobolev embedding, where ϱ_D is set to be $\varrho_D(f)=\varrho_n(f)=\left(\int_{\mathbb{R}^n}|f(x)|^n\mathrm{d}x\right)^{1/n}$. In that case, $\varrho_D'(f)=\varrho_{n'}(f)$, where n' is the conjugated exponent to n, namely $\frac{1}{n}+\frac{1}{n'}=1$. Direct calculation shows that $h_{\varrho_D'}(s)=s^{-1/n'}$ and $\int_0^1 s^{-1/n}h_{\varrho_D'}(s)\mathrm{d}s=\infty$. Moreover, standard examples ($h(s)=\frac{1}{s|\log s|^2}\chi_{(0,1/2)}(s)$) show that (6.10) is not satisfied.

To include this important case into the frame of our work, we will develop a finer theory of optimal range space. This is described in the following assertion.

Theorem 8.1. *Let ϱ_D be a given r.i. norm such that* (6.8) *holds and*

$$\varrho_D'(H\chi_{(0,1)})<\infty. \qquad (8.1)$$

Set

$$\sigma(h)=\varrho_D'(Hh^*),\quad h\in\mathfrak{M}_+(0,\infty).$$

Then,

$$\varrho_R := \sigma' \qquad (8.2)$$

is an r.i. norm which satisfies (4.4) *and which is optimal for* (4.4).

Proof. Step 1.

We will prove that ϱ_R is an r.i. norm. The axioms (A_2) and (A_3) are easy to verify. Let us assume that $\varrho_R(f)=0$ for some $f\in\mathfrak{M}_+(0,\infty)$. Then

$$0=\varrho_R(f)=\sup_{\sigma(g)=1}\int_0^\infty f(t)g(t)\mathrm{d}t. \qquad (8.3)$$

According to (8.1), $\sigma(\chi_E)$ is finite for every measurable set $E\subset(0,\infty)$ with $|E|<\infty$. Together with (8.3) this implies that $\int_E f=0$ for every such set E and, consequently, $f=0$ almost everywhere, which proves (A_1).

To verify (A_5), take a set $E\subset(0,\infty)$ with $|E|<\infty$. Then, for every $f\in\mathfrak{M}_+(0,\infty)$,

$$\varrho_R(f)=\sup_{\sigma(h)\ne 0}\frac{\int fh}{\sigma(h)}\ge\frac{\int f\chi_E}{\sigma(\chi_E)}=c_E\int_E f.$$

The axiom (A_4) is an easy consequence of (8.2) and the estimate

$$\sigma(g) \geq c_E \int_0^{|E|} g^*(u)\mathrm{d}u, \quad g \in \mathfrak{M}_+(0,\infty). \tag{8.4}$$

To prove (8.4), we use Fubini's Theorem

$$\begin{aligned}
\sigma(g) = \varrho_D'(Hg^*) &= \varrho_D'\left(t^{1/n-1}e^{-nt^{1/n}} \int_0^t g^*(u)e^{nu^{1/n}}\mathrm{d}u\right) \\
&\geq \frac{\int_0^{2|E|} t^{1/n-1}e^{-nt^{1/n}} \int_0^t g^*(u)e^{nu^{1/n}}\mathrm{d}u\mathrm{d}t}{\varrho_D(\chi_{(0,2|E|)})} \\
&= c\int_0^{2|E|} g^*(u)e^{nu^{1/n}} \int_u^{2|E|} t^{1/n-1}e^{-nt^{1/n}}\mathrm{d}t\mathrm{d}u \\
&\geq c_E \int_0^{|E|} g^*(u)\mathrm{d}u.
\end{aligned}$$

Step 2.
We show that ϱ_R and ϱ_D satisfy (4.4). As in Section 6, we obtain

$$\begin{aligned}
\sup_{g\in\mathbf{G}} \frac{\varrho_R(Gg)}{\varrho_D(g)} = \sup_{g\in\mathbf{G}} \frac{\sigma'(Gg)}{\varrho_D(g)} &= \sup_{g\in\mathbf{G},h\in\mathfrak{M}_+(0,\infty,\downarrow)} \frac{\int_0^\infty (Gg)(t)h(t)\mathrm{d}t}{\varrho_D(g)\sigma(h)} \\
&= \sup_{g\in\mathbf{G},h\in\mathfrak{M}_+(0,\infty,\downarrow)} \frac{\int_0^\infty (Hh)(t)g(t)\mathrm{d}t}{\varrho_D(g)\sigma(h)}.
\end{aligned} \tag{8.5}$$

Together with Lemma 6.1, this yields

$$\begin{aligned}
\sup_{g\in\mathbf{G}} \frac{\varrho_R(Gg)}{\varrho_D(g)} &= \sup_{h\in\mathfrak{M}_+(0,\infty,\downarrow)} \frac{1}{\sigma(h)} \sup_{g\in\mathbf{G}} \frac{\int_0^\infty (Hh)(t)g(t)\mathrm{d}t}{\varrho_D(g)} \\
&\approx \sup_{h\in\mathfrak{M}_+(0,\infty,\downarrow)} \frac{1}{\sigma(h)} \sup_{g\in\mathfrak{M}_+(0,\infty)} \frac{\int_0^\infty (Hh)(t)g(t)\mathrm{d}t}{\varrho_D(g)} \\
&= \sup_{h\in\mathfrak{M}_+(0,\infty,\downarrow)} \frac{\varrho_D'(Hh^*)}{\sigma(h)} = 1.
\end{aligned}$$

Step 3.
Finally, we prove the optimality of ϱ_R. Let the r.i. norms ν and ϱ_D satisfy (4.4) with ν instead of ϱ_R, that is

$$\sup_{g\in\mathbf{G}} \frac{\nu(Gg)}{\varrho_D(g)} < \infty.$$

Then, proceeding as above,

$$\infty > \sup_{g\in\mathbf{G}} \frac{\nu(Gg)}{\varrho_D(g)} = \sup_{g\in\mathbf{G},h\in\mathfrak{M}_+(0,\infty,\downarrow)} \frac{\int_0^\infty (Gg)(t)h(t)\mathrm{d}t}{\varrho_D(g)\nu'(h)} \approx \sup_{h\in\mathfrak{M}_+(0,\infty,\downarrow)} \frac{\varrho_D'(Hh^*)}{\nu'(h)}.$$

Hence, for every $h \in \mathfrak{M}_+(0,\infty)$,

$$\sigma(h) = \varrho_D'(Hh^*) \leq c\,\nu'(h).$$

Consequently,

$$\nu(f) = \nu''(f) \leq c\,\sigma'(f) = c\,\varrho_R(f), \quad \text{for all} \quad f \in \mathfrak{M}_+(0,\infty).$$

□

Let us apply Theorem 8.1 to the limiting Sobolev embeddings with

$$\varrho_D(f) = \varrho_n(f) = \left(\int_0^\infty |f^*(t)|^n\mathrm{d}t\right)^{1/n}$$

or

$$\varrho_D(f) = \varrho_{n,1}(f) = \int_0^\infty t^{1/n-1} f^*(t)\mathrm{d}t,$$

respectively. Direct calculation shows that (8.1) is satisfied in both these cases.
To verify (6.8), we point out that

$$E(s) \approx \begin{cases} s, & \text{for} \quad s \in (0,1], \\ s^{1-1/n}, & \text{for} \quad s \in (1,\infty). \end{cases} \tag{8.6}$$

Hence, Fubini's Theorem, (8.6) and Lemma 2.6 imply that

$$\begin{aligned} \varrho_{n,1}\left(\int_t^\infty f(u)\frac{E(u)}{u}u^{1/n-1}\mathrm{d}u\right) &= n\int_0^\infty f(u)\frac{E(u)}{u}u^{1/n-1}u^{1/n}\mathrm{d}u \\ &\le c\int_0^\infty t^{1/n-1} f(t)\mathrm{d}t \le c\int_0^\infty t^{1/n-1} f^*(t)\mathrm{d}t = c\varrho_{n,1}(f). \end{aligned}$$

When $\varrho_D = \varrho_n$, (6.8) is a consequence of Hardy's inequality. We refer to [8] for details. So, in both the cases, Theorem 8.1 is applicable and gives the optimal range norm. The result is presented in the next Theorem.

Theorem 8.2. *Let $\varrho_D = \varrho_n$. Then, the optimal range norm, ϱ_R, satisfies*

$$\varrho_R(f) \approx \varrho_n(f) + \lambda(f^*\chi_{(0,1)}), \tag{8.7}$$

where

$$\lambda(g) := \left(\int_0^1 \left(\frac{g^*(t)}{\log(\frac{e}{t})}\right)^n \frac{\mathrm{d}t}{t}\right)^{\frac{1}{n}}, \qquad g \in \mathfrak{M}(0,1).$$

Proof. We first recall that for $\varrho_D = \varrho_n$, both (6.8) and (8.1) are satisfied. Thus, by Theorem 8.1,

$$\begin{aligned} \varrho_R'(h) \approx \varrho_{n'}(Hh^*) &= \varrho_{n'}\left(t^{1/n-1}e^{-nt^{1/n}}\int_0^t h^*(s)e^{ns^{1/n}}\,\mathrm{d}s\right) \\ &\approx \varrho_{n'}\left(\chi_{(0,1)}(t)t^{1/n-1}e^{-nt^{1/n}}\int_0^t h^*(s)e^{ns^{1/n}}\,\mathrm{d}s\right) \\ &\quad + \varrho_{n'}\left(\chi_{(1,\infty)}(t)t^{1/n-1}e^{-nt^{1/n}}\int_0^t h^*(s)e^{ns^{1/n}}\,\mathrm{d}s\right) \\ &=: I + II. \end{aligned}$$

Since

$$e^{-n} \le e^{n(s^{1/n}-t^{1/n})} \le 1 \quad \text{for} \quad 0 \le s \le t \le 1$$

we obtain

$$I \approx \varrho_{n'}\left(\chi_{(0,1)}(t)t^{1/n-1}\int_0^t h^*(s)\,\mathrm{d}s\right) = \left(\int_0^1 \left(\int_0^t h^*(s)\,\mathrm{d}s\right)^{n'} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}}.$$

As for II, we use monotonicity of h^*, (6.3) and (8.6)

$$\begin{aligned}
II &= \left(\int_1^\infty \left(\int_0^t h^*(s) e^{ns^{1/n}}\,\mathrm{d}s\right)^{n'} e^{-nn't^{1/n}} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}}\\
&\geq \left(\int_1^\infty h^*(t)^{n'} \left(e^{-nt^{1/n}} \int_0^t e^{ns^{1/n}}\,\mathrm{d}s\right)^{n'} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}}\\
&\approx \left(\int_1^\infty h^*(t)^{n'} \left(t^{1-1/n}\right)^{n'} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}}\\
&= \left(\int_1^\infty h^*(t)^{n'}\,\mathrm{d}t\right)^{\frac{1}{n'}}.
\end{aligned}$$

Conversely, by the weighted Hardy inequality (cf. [8]),

$$\begin{aligned}
II &\approx \left(\int_1^\infty \left(\int_0^1 h^*(s) e^{ns^{1/n}}\mathrm{d}s\right)^{n'} e^{-nn't^{1/n}} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}}\\
&\quad + \left(\int_1^\infty \left(\int_1^t h^*(s) e^{ns^{1/n}}\,\mathrm{d}s\right)^{n'} e^{-nn't^{1/n}} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}}\\
&\leq c\left[\int_0^1 h^*(s)\mathrm{d}s + \left(\int_1^\infty h^*(t)^{n'}\,\mathrm{d}t\right)^{\frac{1}{n'}}\right]\\
&\leq c\left[\left(\int_0^1 \left(\int_0^t h^*(s)\mathrm{d}s\right)^{n'} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}} + \left(\int_1^\infty h^*(t)^{n'}\mathrm{d}t\right)^{\frac{1}{n'}}\right]
\end{aligned}$$

Altogether,

$$\varrho_R'(g) \approx \left(\int_0^1 \left(\int_0^t h^*(s)\,\mathrm{d}s\right)^{n'} \frac{\mathrm{d}t}{t} + \int_1^\infty h^*(t)^{n'}\,\mathrm{d}t\right)^{\frac{1}{n'}}.$$

Now, set

$$\nu(g) := \left(\int_0^\infty g^*(t)^n v(t)\,\mathrm{d}t\right)^{\frac{1}{n}},$$

where

$$v(t) = \begin{cases} t^{-1}\left(\log \frac{e}{t}\right)^{-n}, & t \in (0,1),\\ 1, & t \in (1,\infty). \end{cases}$$

Then, by [10, Theorem 4], ν is an r.i. norm. More precisely, it is a special case of a classical Lorentz norm whose Köthe dual has been characterised in [10, Theorem 1]. Thus,

$$\nu'(f) \approx \left(\int_0^\infty \left(\int_0^t f^*(s)\,\mathrm{d}s\right)^{n'} \frac{v(t)}{\left(\int_0^t v(s)\,\mathrm{d}s\right)^{n'}}\,\mathrm{d}t\right)^{\frac{1}{n'}} \approx \varrho_R'(f),$$

as an easy calculation shows.

Finally, since both ν and ϱ_R are r.i. norms, it follows from the Principle of Duality (2.8) that

$$\varrho_R \approx \nu,$$

as desired. □

Remark 8.3. We note that λ from Theorem 8.2 is the well-known norm discovered in various contexts independently by Maz'ya [7], Hanson [6] and Brézis–Wainger [2].

Appendix A. proofs of lemmas

As we have promised, we deliver here the proofs of Lemma 5.1 and Lemma 6.1.
Proof of Lemma 5.1
We fix $g \in \mathbf{G}$ and $t \geq 0$. Then, according to (4.5), there is a function $f \geq 0$ such that (4.1) holds. Thus the left-hand side of (5.1) can be rewritten as

$$\begin{aligned}
\text{(A.1)} \qquad & \int_t^\infty \left(f(u) + \int_u^\infty f(s)s^{1/n-1}\mathrm{d}s \right) u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u \\
&= \int_t^\infty f(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u + \int_t^\infty f(s)s^{1/n-1}\int_t^s u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\mathrm{d}s \\
&= e^{-nt^{1/n}}\int_t^\infty f(s)s^{1/n-1}\mathrm{d}s.
\end{aligned}$$

The right-hand side of (5.1) is more complicated. Using (2.4), (4.1) and Fubini's Theorem we get

$$\begin{aligned}
\text{(A.2)} \qquad g^{**}(u) &\approx f^{**}(u) + \left(\int_t^\infty f(s)s^{1/n-1}\mathrm{d}s \right)^{**}(u) \\
&= f^{**}(u) + \int_u^\infty f(s)s^{1/n-1}\mathrm{d}s + \frac{1}{u}\int_0^u f(s)s^{1/n}\mathrm{d}s.
\end{aligned}$$

We insert the formula (A.2) in (5.1) and use Fubini's Theorem to arrive at

$$\begin{aligned}
\int_t^\infty g^{**}(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u \approx & \underbrace{\int_t^\infty f^{**}(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u}_{\text{I}} \\
& + \underbrace{\int_t^\infty \left(\int_u^\infty f(s)s^{1/n-1}\mathrm{d}s \right) u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u}_{\text{II}} \\
& + \underbrace{\int_t^\infty \left(\int_0^u f(s)s^{1/n-1}\mathrm{d}s \right) u^{1/n-2}e^{-nu^{1/n}}\mathrm{d}u}_{\text{III}}.
\end{aligned}$$

Each of these three integrals can be further estimated. We start with the second one:

$$\text{II} = e^{-nt^{1/n}}\int_t^\infty f(s)s^{1/n-1}\mathrm{d}s - \int_t^\infty f(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s.$$

To deal with integrals I and III, we use the notation $h(s) := \int_s^\infty u^{1/n-2}e^{-nu^{1/n}}\mathrm{d}u$. Then, by Fubini's Theorem,

$$\text{I} \geq \int_t^\infty \frac{1}{u}\left(\int_t^u f(s)\mathrm{d}s \right) u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u = \int_t^\infty f(s)h(s)\mathrm{d}s$$

and

$$\text{III} \geq \int_t^\infty \int_t^u f(s)s^{1/n}\mathrm{d}s u^{1/n-2}e^{-nu^{1/n}}\mathrm{d}u = \int_t^\infty f(s)s^{1/n}h(s)\mathrm{d}s.$$

The last three estimates give us

$$\begin{aligned}
\text{I} + \text{II} + \text{III} \geq & \int_t^\infty f(s)h(s)(s^{1/n}+1)\mathrm{d}s + e^{-nt^{1/n}}\int_t^\infty f(s)s^{1/n-1}\mathrm{d}s \\
& - \int_t^\infty f(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s.
\end{aligned}$$

This estimate and (A.1) imply that it is enough to prove that

$$\int_t^\infty f(s)h(s)(s^{1/n}+1)\mathrm{d}s \geq \int_t^\infty f(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s.$$

But the last inequality is a trivial consequence of the pointwise estimate

$$h(s)(s^{1/n}+1) \geq s^{1/n-1}e^{-ns^{1/n}}, \quad s>0,$$

which may be proved by direct calculation. □

Proof of Lemma 6.1

As $\mathbf{G} \subset \mathfrak{M}_+(0,\infty)$, the estimate "$\lesssim$" in (6.9) follows immediately. To prove the reverse one, take a $h \in \mathfrak{M}_+(0,\infty,\downarrow)$. Moreover, if $f \in \mathfrak{M}_+(0,\infty)$, we put $\tilde{f}(s) = f(s)\frac{E(s)}{s}$ for all $s>0$, where E is defined by (6.3), and $g(t) = \tilde{f}(t) + \int_t^\infty \tilde{f}(s)s^{1/n-1}\mathrm{d}s$, $t>0$. We claim, that the following two conditions are satisfied:

I. $\varrho_D(g) \leq c\varrho_D(f)$,

II. $\int_0^\infty (Hh)(t)g(t)\mathrm{d}t \geq c\int_0^\infty (Hh)(t)f(t)\mathrm{d}t$.

Indeed, to prove I, we use the fact that $s^{-1}E(s) \leq 1$ for all $s>0$. We get (c.f. (8.6) and (6.8))

$$\begin{aligned}\varrho_D(g) &= \varrho_D\left(f(s)\frac{E(s)}{s} + \int_s^\infty f(u)\frac{E(u)}{u}u^{1/n-1}\mathrm{d}u\right)\\ &\leq \varrho_D\left(f(s)\frac{E(s)}{s}\right) + \varrho_D\left(\int_s^\infty f(u)\frac{E(u)}{u}u^{1/n-1}\mathrm{d}u\right)\\ &\leq \varrho_D(f) + c\varrho_D(f) = c\varrho_D(f),\end{aligned}$$

where we used (6.8).

The proof of II is more complicated. The left-hand side of the condition II can be simplified by

$$\int_0^\infty (Hh)(t)g(t)\mathrm{d}t = \int_0^\infty (Gg)(t)h(t)\mathrm{d}t = \int_0^\infty h(t)\left(\int_t^\infty \tilde{f}(s)s^{1/n-1}\mathrm{d}s\right)\mathrm{d}t$$

and the right-hand side by

$$\begin{aligned}\int_0^\infty (Hh)(t)f(t) &= \int_0^\infty f(u)u^{1/n-1}e^{-nu^{1/n}}\left(\int_0^u h(t)e^{nt^{1/n}}\mathrm{d}t\right)\mathrm{d}u\\ &= \int_0^\infty h(t)\left(e^{nt^{1/n}}\int_t^\infty f(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\right)\mathrm{d}t.\end{aligned}$$

By Hardy's Lemma 2.6, the result will follow if we show that, for all $\xi>0$ and for all $f \in \mathfrak{M}_+(0,\infty)$,

$$\int_0^\xi\int_t^\infty \tilde{f}(s)s^{1/n-1}\mathrm{d}s\mathrm{d}t \geq \int_0^\xi e^{nt^{1/n}}\int_t^\infty f(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\mathrm{d}t. \tag{A.3}$$

Using Fubini's Theorem we can rewrite the right-hand side of (A.3) as

(A.4)

$$\int_0^\xi f(s)s^{1/n-1}e^{-ns^{1/n}}\left(\int_0^s e^{nt^{1/n}}\mathrm{d}t\right)\mathrm{d}s + \int_\xi^\infty f(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s\int_0^\xi e^{nt^{1/n}}\mathrm{d}t,$$

and the left-hand side of (A.3) as

$$\begin{aligned}&\int_0^\xi \tilde{f}(s)s^{1/n}\mathrm{d}s + \xi\int_\xi^\infty \tilde{f}(s)s^{1/n-1}\mathrm{d}s\\ &\qquad = \int_0^\xi f(s)s^{1/n-1}E(s)\mathrm{d}s + \xi\int_\xi^\infty f(s)s^{1/n-2}E(s)\mathrm{d}s.\end{aligned} \tag{A.5}$$

The first integral in the last sum in (A.5) is equal to the first integral in (A.4). So, we shall deal with the second integrals. We shall use the following observation

$$\frac{1}{s}\int_0^s e^{nu^{1/n}}\mathrm{d}u \geq \frac{1}{\xi}\int_0^\xi e^{nu^{1/n}}\mathrm{d}u, \quad s>\xi,$$

and finish the proof by

$$\begin{aligned}\xi\int_\xi^\infty f(s)s^{1/n-2}E(s)\mathrm{d}s &= \xi\int_\xi^\infty f(s)s^{1/n-2}e^{-ns^{1/n}}\int_0^s e^{nu^{1/n}}\mathrm{d}u\mathrm{d}s\\ &\geq \xi\int_\xi^\infty f(s)s^{1/n-1}e^{-ns^{1/n}}\frac{1}{\xi}\int_0^\xi e^{nu^{1/n}}\mathrm{d}u\mathrm{d}s\\ &= \int_\xi^\infty f(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s\int_0^\xi e^{nu^{1/n}}\mathrm{d}u.\end{aligned}$$

□

Acknowledgement. I would like to express my deep gratitude to Luboš Pick for his interest and valuable comments and to anonymous referee for many remarks, which helped to improve this paper.

References

[1] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, San Diego, 1988.

[2] H. Brézis and S. Wainger, *A note on limiting cases of Sobolev embeddings and convolution inequalities*, Comm. Partial Diff. Eq. 5 (1980), 773-789.

[3] A. Cianchi and L. Pick, *Sobolev embeddings into BMO, VMO, and L_∞*, Ark. Mat. 36 (1998), 317-340.

[4] D. E. Edmunds, R. Kerman, L. Pick, *Optimal Sobolev Imbeddings Involving Rearrangement-Invariant Quasinorms*, J. Funct. Anal. 170 (2000), 307-355.

[5] M. Carro, A. Gogatishvili, J. Martín, L. Pick, *Weighted inequalities involving two Hardy operators with applications to embeddings of function spaces*, to appear in J. Operator Theory.

[6] K. Hansson, *Imbedding theorems of Sobolev type in potential theory*, Math. Scand. 45 (1979), 77-102.

[7] V. G. Maz'ya, *Sobolev Spaces*, Springer, Berlin, 1985.

[8] B. Muckenhoupt, *Hardy's inequality with weights*, Studia Math. 44 (1972), 31-38.

[9] B. Opic and W. Trebels, *Sharp embeddings of Bessel potential spaces with logarithmic smoothness*, Math. Proc. Camb. Phil. Soc. (2003), 347-384.

[10] E. Sawyer, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math. 96 (1990), 145-158.

[11] G. Sinnamon, *Embeddings of concave functions and duals of Lorentz spaces*, Publ. Math. 46 (2002), 489-515.

[12] G. Sinnamon, *Transferring monotonicity in weighted norm inequalities*, Collect. Math. 54 (2003), 181-216.

[13] S. L. Sobolev, *Applications of Functional Analysis in Mathematical Physics*, Transl. of Mathem. monographs, Amer. Math. Soc. Vol. 7, Providence, 1963.

[14] J. Vybíral, *Optimality of Function Spaces for Boundedness of Integral Operators and Sobolev Embeddings*, Diploma Thesis, MFF UK, Prague, 2002.

A REMARK ON BETTER λ-INEQUALITY

JAN VYBÍRAL

Abstract

We generalize the inequality of R. J. Bagby and D. S. Kurtz [BK] to a wider class of potentials defined in terms of Young's functions. We make use of a certain submultiplicativity condition. We show that this condition cannot be omited.

Key words: Riesz potentials, Better λ-inequality, Nonincreasing rearrangement, Young's functions

2000 Mathematics Subject Classification: 31C15, 42B20

1. Introduction

The classical Riesz potentials are defined for every real number $0 < \gamma < n$ as a convolution operators $(I_\gamma f)(x) = (\tilde{I}_\gamma * f)(x)$, where $\tilde{I}_\gamma(x) = |x|^{\gamma - n}$. This definition coincides with the usual one up to some multiplicative constant c_γ which is not interesting for our purpose. Burkholder and Gundy invented in [BG] the technique involving distribution function later known as *good λ-inequality.* This inequality dealt with level sets of singular integral operators and of maximal operator. Later, Bagby and Kurtz discovered in [BK] that the reformulation of good λ-inequality in terms of non-increasing rearrangement contains more information.

We generalize their approach in the following way. For every Young's function Φ satisfying the Δ_2-condition we define the Riesz potential

$$(I_\Phi f)(x) = \int_{\mathbf{R}^n} \tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right) f(y)\mathrm{d}y,$$

where $\tilde{\Phi}$ is Young's function conjugated to Φ and $\tilde{\Phi}^{-1}$ is its inverse. Instead of the classical Hardy-Littlewood maximal operator we work with a generalized maximal operator

$$(M_\varphi f)(x) = \sup_{Q \ni x} \frac{1}{\varphi(|Q|)} \int_Q |f(y)|\mathrm{d}y,$$

where φ is a given nonnegative function on $(0, \infty)$ and the supremum is taken over all cubes Q containing x with sides parallel to the coordinate axes such that $\varphi(|Q|) > 0$. For every measurable set $\Omega \subset \mathbf{R}^n$ we denote by $|\Omega|$ its Lebesgue measure.

We prove that under some restrictive condition on function Φ one can obtain an inequality combining the nonincreasing rearrangement of $I_\Phi f$ and $M_{\tilde{\Phi}^{-1}} f$. We also show that this restrictive condition cannot be left out.

2. Better λ—inequality

Before we state our main result, we give some definitions and recall some very well known results about Young's functions and non-increasing rearrangements.

Lebesgue measure will be denoted by μ or simply be an absolute value. Let Ω be a subset of $\mathbf{R}^n$, $n \geq 1$. We denote by $\mathfrak{M}$ the collection of all extended scalar-valued Lebesgue measurable functions on Ω and by $\mathfrak{M}_0$ the class of functions in $\mathfrak{M}$ that are finite μ-a.e. Further let $\mathfrak{M}^+$ be the cone of nonnegative functions from $\mathfrak{M}$ and $\mathfrak{M}_0^+$ the class of nonnegative functions from $\mathfrak{M}_0$. We shall also write $\mathfrak{M}(\Omega), \mathfrak{M}^+(\Omega)$ and so on when we want to emphasize the underlying space Ω.

The letter c denotes a general constant which doesn't depend on the parameters involved. It may change from one occurrence to another.

Definition 2.1. 1. Let $\phi : [0,\infty) \to [0,\infty)$ be a non-decreasing and right-continuous function with $\phi(0) = 0$ and $\phi(\infty) = \lim_{t\to\infty} \phi(t) = \infty$. Then the function Φ defined by

$$\Phi(t) = \int_0^t \phi(s)\mathrm{d}s, \quad t \geq 0$$

is said to be a *Young's function*.

2. A Young's function is said to satisfy the Δ_2—*condition* if there is $c > 0$ such that

$$\Phi(2t) \leq c\,\Phi(t), \quad t \geq 0.$$

3. A Young's function is said to satisfy the ∇_2—*condition* if there is $l > 1$ such that

$$\Phi(t) \leq \frac{1}{2l}\Phi(lt), \quad t \geq 0.$$

4. Let Φ be a Young's function, represented as the indefinite integral of ϕ. Let

$$\psi(s) = \sup\{u : \phi(u) \leq s\}, \quad s \geq 0.$$

Then the function

$$\tilde{\Phi}(t) = \int_0^t \psi(s)\mathrm{d}s, \quad t \geq 0,$$

is called the *complementary Young's function* of Φ.

The following theorem puts these three notions together. For the proof see [KR].

Theorem 2.2. *Let Φ be a Young's function and $\tilde{\Phi}$ be its complementary Young's function. Then Φ satisfies the Δ_2—condition if and only if $\tilde{\Phi}$ satisfies the ∇_2—condition.*

We shall need following lemma.

Lemma 2.3. *Let Φ be a Young's function satisfying the Δ_2—condition. Then there is a constant $c > 0$ such that*

$$\int_0^t \tilde{\Phi}^{-1}\left(\frac{1}{u}\right)\mathrm{d}u \leq c\,t\tilde{\Phi}^{-1}\left(\frac{1}{t}\right), \quad 0 < t < \infty$$

Proof. If Φ satisfies the Δ_2—condition, then $\tilde{\Phi}$ satisfies the ∇_2—condition. It means that there is a real number $k > 1$ such that $\tilde{\Phi}(t) \leq \frac{1}{2k}\tilde{\Phi}(kt)$ for every $t > 0$. When we pass to inverses we get $\tilde{\Phi}^{-1}\left(\frac{1}{u}\right) \leq \frac{l}{2}\tilde{\Phi}^{-1}\left(\frac{1}{lu}\right)$, where $l = 2k > 2$ and $u > 0$. Now setting $h(s) = \tilde{\Phi}^{-1}\left(\frac{1}{s}\right)$ and $H(u) = \int_0^u h(s)\mathrm{d}s$ we get $2h(s) \leq lh(ls)$ and integrating this inequality from 0 to t we obtain $2H(t) \leq H(lt)$. To show that $H(t)$ is finite for all $t > 0$, write

$$H(t) = \int_0^t h(s)\mathrm{d}s = \sum_{k=0}^{\infty} \int_{t/l^{k+1}}^{t/l^k} h(s)\mathrm{d}s \leq \sum_{k=0}^{\infty} \int_{t/l^{k+1}}^{t/l^k} \frac{l^k}{2^k} h(l^k s)\mathrm{d}s =$$

$$= \sum_{k=0}^{\infty} \frac{1}{2^k} \int_{t/l}^{t} h(u)\mathrm{d}u < \infty.$$

Because h is a decreasing function, we can calculate

$$lth(t) \geq \int_t^{lt} h(s)\mathrm{d}s = H(lt) - H(t) \geq 2H(t) - H(t) = H(t),$$

which can be rewritten as

$$lt\tilde{\Phi}^{-1}\left(\frac{1}{t}\right) \geq \int_0^t \tilde{\Phi}^{-1}\left(\frac{1}{u}\right) \mathrm{d}u.$$

□

Definition 2.4. The *distribution function* μ_f of a function f in $\mathfrak{M}_0(\Omega)$ is given by

$$\mu_f(\lambda) = \mu(\{x \in \Omega : |f(x)| > \lambda\}), \quad \lambda \geq 0.$$

For every $f \in \mathfrak{M}_0(\Omega)$ we define its *nonincreasing rearrangement* f^* by

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leq t\}, \quad 0 \leq t < \infty$$

and its *maximal function* f^{**} by

$$f^{**}(t) = t^{-1} \int_0^t f^*(u)\mathrm{d}u, \quad 0 < t < \infty.$$

Assume now that Young's function Φ satisfies the Δ_2—condition. Using the classical O'Neil inequality (see [O]) and lemma 2.3 we obtain

$$(1) \qquad (I_\Phi f)^*(t) \leq c \left\{ \tilde{\Phi}^{-1}\left(\frac{1}{t}\right) \int_0^t f^*(u)\mathrm{d}u + \int_t^\infty f^*(u)\tilde{\Phi}^{-1}\left(\frac{1}{u}\right) \mathrm{d}u \right\},$$

We shall derive a better λ-inequality connecting the operators I_Φ and $M_{\tilde{\Phi}^{-1}}$.

Theorem 2.5. *Let us suppose that a Young's function Φ satisfies the Δ_2—condition. Let us further suppose that there is a constant $c_1 > 0$ such that*

$$(2) \qquad \tilde{\Phi}^{-1}(s)\tilde{\Phi}^{-1}(1/s) < c_1, \quad s > 0.$$

Then there is a constant $c_2 > 0$, such that for every function f and every positive number t

$$(3) \qquad (I_\Phi f)^*(t) \leq (I_\Phi|f|)^*(t) \leq c_2 \left(M_{\tilde{\Phi}^{-1}} f\right)^*(t/2) + (I_\Phi|f|)^*(2t)$$

Proof. We may assume that given function f is nonnegative.

First we shall estimate the size of the level set $G = \{x \in \mathbf{R}^n : (I_\Phi g)(x) > \lambda\}$ for function $g \in L^1(\mathbf{R}^n)$. According to (1), $|G| < \infty$. Hence we can find a real number $R \geq 0$ such that $|G| = |B(0, R)|$. We can write

$$\lambda|G| = \int_G \lambda \leq \int_G (I_\Phi g)(x)\mathrm{d}x = \int_G \int_{\mathbf{R}^n} g(y)\tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right) \mathrm{d}y\mathrm{d}x =$$
$$= \int_{\mathbf{R}^n} \int_G \tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right) \mathrm{d}x g(y)\mathrm{d}y \leq$$
$$||g||_1 \int_{B(0,R)} \tilde{\Phi}^{-1}\left(\frac{1}{|x|^n}\right) \mathrm{d}x = ||g||_1 \alpha_n \int_0^{|G|/\alpha_n} \tilde{\Phi}^{-1}(1/s)\mathrm{d}s.$$

Dividing this inequality by $|G|$ and using the lemma 2.3 we obtain

$$\lambda \leq ||g||_1 \frac{\alpha_n}{|G|} \int_0^{|G|/\alpha_n} \tilde{\Phi}^{-1}(1/s)\mathrm{d}s \leq \tilde{c}\, ||g||_1 \tilde{\Phi}^{-1}\left(\frac{1}{|G|}\right).$$

This can be rewritten as

$$(4) \qquad |G| \leq \frac{1}{\tilde{\Phi}\left(\frac{\lambda}{\tilde{c}||g||_1}\right)},$$

where $\tilde{c}$ is independent of g and λ.

We can now pass to the proof of our theorem which is mainly based on [BK]. For a given function $f \geq 0$ and a real number $t > 0$ we shall denote by E the set $\{x \in \mathbf{R}^n : (I_\Phi f)(x) > (I_\Phi f)^*(2t)\}$. Then $|E| \leq 2t$ and we can find an open set Ω,

$|\Omega| < 3t, E \subset \Omega$. Now using Whitney covering theorem (see [S]) we can find cubes Q_k with disjoint interiors, such that $\Omega = \cup_{k=1}^{\infty} Q_k$ and $\operatorname{diam} Q_k \leq \operatorname{dist}(Q_k, \mathbf{R}^n \setminus \Omega) \leq 4 \operatorname{diam} Q_k$.

We want to show that there is a constant $C > 0$ such that for every f, t and for every corresponding cube Q_k

$$|\{x \in Q_k : I_\Phi f(x) > C(M_{\tilde{\Phi}^{-1}} f)(x) + (I_\Phi f)^*(2t)\}| \leq \frac{1}{6}|Q_k|. \tag{5}$$

Then we would have $|\{x \in \mathbf{R}^n : I_\Phi f(x) > C(M_{\tilde{\Phi}^{-1}} f)(x) + (I_\Phi f)^*(2t)\}| \leq 1/6 \sum |Q_k| \leq t/2$ and thus

$$\begin{aligned} &|\{x \in \mathbf{R}^n : I_\Phi f(x) > C(M_{\tilde{\Phi}^{-1}} f)^*(t/2) + (I_\Phi f)^*(2t)\}| \leq \\ &\quad \leq |\{x \in \mathbf{R}^n : I_\Phi f(x) > C(M_{\tilde{\Phi}^{-1}} f)(x) + (I_\Phi f)^*(2t)\}| + \\ &\quad + |\{x \in \mathbf{R}^n : (M_{\tilde{\Phi}^{-1}} f)(x) > (M_{\tilde{\Phi}^{-1}} f)^*(t/2)\}| \leq t/2 + t/2 = t, \end{aligned}$$

which finishes the proof.

To prove (5) fix k and choose $x_k \in (\mathbf{R}^n \setminus \Omega)$ so that $\operatorname{dist}(x_k, Q_k) \leq 4 \operatorname{diam}(Q_k)$. Let Q be a cube with center at x_k having diameter $20 \operatorname{diam}(Q_k)$. Split $f = g + h = f\chi_Q + f\chi_{\mathbf{R}^n \setminus Q}$. We may assume that $g \in L^1(\mathbf{R}^n)$, otherwise the right-hand side of (3) would be infinite.

We shall prove that for C_1 and C_2 large enough

$$|\{x \in Q_k : (I_\Phi g)(x) > C_1(M_{\tilde{\Phi}^{-1}} f)(x)\}| \leq 1/6|Q_k|, \tag{6}$$

and, for every $x \in Q_k$,

$$I_\Phi h(x) \leq C_2(M_{\tilde{\Phi}^{-1}} f)(x) + I_\Phi f(x_k) \leq C_2(M_{\tilde{\Phi}^{-1}} f)(x) + (I_\Phi f)^*(2t), \tag{7}$$

which together gives (5).

For the first inequality, notice that for $x \in Q_k$

$$(M_{\tilde{\Phi}^{-1}} f)(x) \geq \frac{1}{\tilde{\Phi}^{-1}(|Q|)} \int_Q g = \frac{||g||_1}{\tilde{\Phi}^{-1}(|Q|)}.$$

Using (4) now gives

$$\begin{aligned} &|\{x \in Q_k : (I_\Phi g)(x) > C_1(M_{\tilde{\Phi}^{-1}} f)(x)\}| \leq \\ &\qquad \left| \left\{ x \in Q_k : (I_\Phi g)(x) > \frac{C_1 ||g||_1}{\tilde{\Phi}^{-1}(|Q|)} \right\} \right| \leq \frac{1}{\tilde{\Phi}\left(\frac{C_1}{\tilde{c}\tilde{\Phi}^{-1}(|Q|)}\right)}, \end{aligned}$$

where $\tilde{c}$ is the constant from (4). The last expression is less then $|Q_k|/6$ for C_1 big enough (here we use (2) again).

In the proof of the second inequality we shall use two observations. The first is that

$$\left| \tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right) - \tilde{\Phi}^{-1}\left(\frac{1}{|x_k-y|^n}\right) \right| \leq c \frac{|x_k - x|}{|x-y|} \tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right) \tag{8}$$

with c independent of k, $y \in (\mathbf{R}^n \setminus Q)$ and $x \in Q_k$.

The second is that for any $\delta > 0$ and any $x \in \mathbf{R}^n$

$$\int_{y:|x-y|>\delta} \delta \frac{f(y)}{|x-y|} \tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right) \mathrm{d}y \leq c\, M_{\tilde{\Phi}^{-1}} f(x). \tag{9}$$

The proof of (7) now follows easily. For every $x \in Q_k$ we get

$$\begin{aligned} &I_\Phi h(x) - I_\Phi f(x_k) \leq I_\Phi h(x) - I_\Phi h(x_k) \leq \\ &\int_{\mathbf{R}^n \setminus Q} \left| \tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right) - \tilde{\Phi}^{-1}\left(\frac{1}{|x_k-y|^n}\right) \right| f(y)\mathrm{d}y \leq \\ &c|x_k - x| \int_{\mathbf{R}^n \setminus Q} \frac{1}{|x-y|} \tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right) f(y)\mathrm{d}y \leq \\ &cM_{\tilde{\Phi}^{-1}} f(x). \end{aligned}$$

It remains to prove (8) and (9). Proof of (9) is a combination of definition of $M_{\tilde{\Phi}^{-1}}$ and (2).

To prove (8) let us write $\tilde{\Phi}(t) = \int_0^t \tilde{\varphi}(u)\mathrm{d}u$ and $A(t) = \tilde{\Phi}^{-1}(t^{-n})$ for $t > 0$. Then

$$\frac{1}{s}\int_0^s \tilde{\varphi}(u)\mathrm{d}u \leq \tilde{\varphi}(s), \quad s > 0$$

or, equivalently, $\tilde{\Phi}(s) \leq s\tilde{\Phi}'(s)$ for $s > 0$. Now we set $s = A(t)$ and obtain

$$-tA'(t) = \frac{nt^{-n}}{\tilde{\Phi}'(A(t))} \leq cA(t).$$

Finally the left hand side of (8) can be estimated by

$$|A(|x-y|) - A(|x_k - y|)| \leq c\left|\int_{|x-y|}^{|x_k-y|} \frac{A(t)}{t}\mathrm{d}t\right| \leq c\frac{|x_k - x|}{|x-y|}A(|x-y|).$$

□

In the following example we will show that the assumption (2) cannot be omitted.

Theorem 2.6. *There is a Young's function Φ satisfying the Δ_2—condition for which*

$$\sup_{f,t>0} \frac{(I_\Phi f)^*(t) - (I_\Phi f)^*(2t)}{(M_{\tilde{\Phi}^{-1}} f)^*(t/2)} = \infty$$

Proof. Set

$$\tilde{\Phi}(u) = \begin{cases} u^3 & \text{if } 0 < u < 1 \\ \frac{3}{2}u^2 - \frac{1}{2} & \text{if } 1 < u < \infty \end{cases}, \qquad \tilde{\varphi}(u) = \begin{cases} 3u^2 & \text{if } 0 < u < 1 \\ 3u & \text{if } 1 < u < \infty \end{cases}.$$

Then

$$\Phi(u) = \begin{cases} \frac{2}{3\sqrt{3}}u^{3/2} & \text{if } 0 < u < 3 \\ \frac{u^2}{6} + \frac{1}{2} & \text{if } 3 < u < \infty \end{cases}, \qquad \varphi(u) = \begin{cases} \sqrt{\frac{u}{3}} & \text{if } 0 < u < 3 \\ \frac{u}{3} & \text{if } 3 < u < \infty \end{cases}.$$

Finally $\tilde{\Phi}^{-1}(u) = \sqrt[3]{u}$ for $0 < u < 1$ and $\tilde{\Phi}^{-1}(u) = \sqrt{2/3(u+1/2)}$ for $u > 1$.

Let $n = 1$. For any integer $m > 0$ set $t_m = 1/m$, $f_m(x) = \chi_{(0,t_m)}(x)$. Then

$$(M_{\tilde{\Phi}^{-1}} f_m)^*(t_m/2) = (M_{\tilde{\Phi}^{-1}} f_m)(0) = \sup_{0<s<1/m} \frac{1}{\tilde{\Phi}^{-1}(s)}\int_0^s 1 = m^{-2/3},$$

$$(I_\Phi f_m)^*(t_m) = (I_\Phi f_m)(0) = \int_0^{1/m} \tilde{\Phi}^{-1}(1/s)\mathrm{d}s = \sqrt{\frac{2}{3}}\int_0^{1/m}\sqrt{\frac{1}{u} + \frac{1}{2}}\mathrm{d}u,$$

$$(I_\Phi f_m)^*(2t_m) = (I_\Phi f_m)\left(\frac{3}{2}t_m\right) = \int_{1/(2m)}^{3/(2m)} \tilde{\Phi}^{-1}(1/s)\mathrm{d}s = \sqrt{\frac{2}{3}}\int_{1/(2m)}^{3/(2m)}\sqrt{\frac{1}{u} + \frac{1}{2}}\mathrm{d}u.$$

We can now estimate

$$\frac{(I_\Phi f_m)^*(t_m) - (I_\Phi f_m)^*(2t_m)}{(M_{\tilde{\Phi}^{-1}} f_m)^*(t_m/2)} \geq$$

$$\sqrt{\frac{2}{3}} m^{2/3} \left\{ \int_0^{1/(2m)} \sqrt{\frac{1}{u}} \mathrm{d}u - \int_{1/m}^{3/(2m)} \sqrt{m + \frac{1}{2}} \mathrm{d}u \right\} =$$

$$\sqrt{\frac{2}{3}} m^{2/3} \left\{ \frac{\sqrt{2}}{\sqrt{m}} - \frac{\sqrt{m+\frac{1}{2}}}{2m} \right\} = \sqrt{\frac{2}{3}} m^{1/6} \left\{ \sqrt{2} - \frac{1}{2}\sqrt{1 + \frac{1}{2m}} \right\}.$$

The last expression tends to infinity as m tends to infinity. □

References

[BK] R. J. Bagby and D. S. Kurtz, *A Rearranged Good λ-Inequality*, Trans. Amer. Math. Soc.,293 (1986),71-81.

[BG] D. L. Burkholder and R. F. Gundy, *Extrapolation and interpolation of quasilinear operators on martingales*, Acta Math.,124(1970), 249-304.

[KR] M. A. Krasnosel'skii and Ya.B. Rutickii, *Convex functions and Orlicz spaces*, GITTL, Moscow, 1958; English transl., Noordhoff, Groningen, 1961.

[O] R. O'Neil, *Convolution Operators and L(p,q) spaces*, Duke Math. J. 30(1963), 129-142.

[S] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N. J., 1970.

A new proof of the Jawerth-Franke embedding

Jan Vybíral

Abstract

We present an alternative proof of the Jawerth embedding

$$F^{s_0}_{p_0q}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1p_0}(\mathbb{R}^n),$$

where

$$-\infty < s_1 < s_0 < \infty, \quad 0 < p_0 < p_1 \le \infty, \quad 0 < q \le \infty$$

and

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

The original proof given in [3] uses interpolation theory. Our proof relies on wavelet decompositions and transfers the problem from function spaces to sequence spaces. Using similar techniques, we also recover the embedding of Franke, [2].

AMS Classification: 46E35

Keywords and phrases: Besov spaces, Triebel-Lizorkin spaces, Sobolev embedding, Jawerth-Franke embedding

1 Introduction

Let $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ denote the Besov and Triebel-Lizorkin function spaces, respectively. The classical Sobolev embedding theorem can be extended to these two scales.

Theorem 1.1. *Let* $-\infty < s_1 < s_0 < \infty$ *and* $0 < p_0 < p_1 \le \infty$ *with*

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}. \tag{1.1}$$

(i) If $0 < q_0 \le q_1 \le \infty$*, then*

$$B^{s_0}_{p_0q_0}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1q_1}(\mathbb{R}^n). \tag{1.2}$$

(ii) If $0 < q_0, q_1 \le \infty$ *and* $p_1 < \infty$*, then*

$$F^{s_0}_{p_0q_0}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1q_1}(\mathbb{R}^n). \tag{1.3}$$

We observe, that there is no condition on the fine paramters q_0, q_1 in (1.3). This surprising effect was first observed in full generality by Jawerth, [3]. Using (1.3), we may prove

$$F^{s_0}_{p_0q}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1p_1}(\mathbb{R}^n) = B^{s_1}_{p_1p_1}(\mathbb{R}^n) \quad \text{and} \quad B^{s_0}_{p_0p_0}(\mathbb{R}^n) = F^{s_0}_{p_0p_0}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1q}(\mathbb{R}^n)$$

for every $0 < q \le \infty$. But Jawerth ([3]) and Franke ([2]) showed, that these embeddings are not optimal and may be improved.

Theorem 1.2. *Let* $-\infty < s_1 < s_0 < \infty$, $0 < p_0 < p_1 \le \infty$ *and* $0 < q \le \infty$ *with* (1.1).
(i) Then

$$F^{s_0}_{p_0q}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1p_0}(\mathbb{R}^n). \tag{1.4}$$

(ii) If $p_1 < \infty$*, then*

$$B^{s_0}_{p_0p_1}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{p_1q}(\mathbb{R}^n). \tag{1.5}$$

The original proofs (see [3] and [2]) use interpolation techniques. We rely on a different method. First, we observe that using (for example) the wavelet decomposition method, (1.4) and (1.5) is equivalent to

$$f^{s_0}_{p_0q} \hookrightarrow b^{s_1}_{p_1p_0} \quad \text{and} \quad b^{s_0}_{p_0p_1} \hookrightarrow f^{s_1}_{p_1q} \tag{1.6}$$

under the same restrisctions on parameters s_0, s_1, p_0, p_1, q as in Theorem 1.2. Here, b^s_{pq} and f^s_{pq} stands for the sequence spaces of Besov and Triebel-Lizorkin type. We prove (1.6) directly using the technique of non-increasing rearrangement on a rather elementary level.

All the unimportant constants are denoted by the letter c, whose meaning may differ from one occurrence to another. If $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are two sequences of positive real numbers, we write $a_n \lesssim b_n$ if, and only if, there is a positive real number $c > 0$ such that $a_n \le c\, b_n, n \in \mathbb{N}$. Furthermore, $a_n \approx b_n$ means that $a_n \lesssim b_n$ and simultaneously $b_n \lesssim a_n$.

2 Notation and definitions

We introduce the sequence spaces associated with the Besov and Triebel-Lizrokin spaces. Let $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$. Then $Q_{\nu m}$ denotes the closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, centred at $2^{-\nu}m$, and with side length $2^{-\nu}$. By $\chi_{\nu m} = \chi_{Q_{\nu m}}$ we denote the characteristic function of $Q_{\nu m}$. If

$$\lambda = \{\lambda_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\},$$

$-\infty < s < \infty$ and $0 < p, q \le \infty$, we set

$$||\lambda|b^s_{pq}|| = \Big(\sum_{\nu=0}^{\infty} 2^{\nu(s-\frac{n}{p})q}\Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu\, m}|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} \tag{2.1}$$

appropriately modified if $p = \infty$ and/or $q = \infty$. If $p < \infty$, we define also

$$||\lambda|f^s_{pq}|| = \Big|\Big|\Big(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} |2^{\nu s}\lambda_{\nu\, m}\chi_{\nu\, m}(\cdot)|^q\Big)^{1/q}|L_p(\mathbb{R}^n)\Big|\Big|. \tag{2.2}$$

The connection between the function spaces $B^s_{pq}(\mathbb{R}^n)$, $F^s_{pq}(\mathbb{R}^n)$ and the sequence spaces b^s_{pq}, f^s_{pq} may be given by various decomposition techniques, we refer to [7, Chapters 2 and 3] for details and further references.

As a result of these characterisations, (1.4) is equivalent to (1.6).

We use the technique of non-increasing rearrangement. We refer to [1, Chapter 2] for details.

Definition 2.1. Let μ be the Lebesgue measure in $\mathbb{R}^n$. If h is a measurable function on $\mathbb{R}^n$, we define the non-increasing rearrangement of h through

$$h^*(t) = \sup\{\lambda > 0 : \mu\{x \in \mathbb{R}^n : |h(x)| > \lambda\} > t\}, \qquad t \in (0,\infty). \tag{2.3}$$

We denote its averages by

$$h^{**}(t) = \frac{1}{t}\int_0^t h^*(s)ds, \quad t > 0.$$

We shall use the following properties. The first two are very well known and their proofs may be found in [1], Proposition 1.8 in Chapter 2, and Theorem 3.10 in Chapter 3.

Lemma 2.2. *If $0 < p \le \infty$, then*

$$||h|L_p(\mathbb{R}^n)|| = ||h^*|L_p(0,\infty)||$$

for every measurable function h.

Lemma 2.3. *If $1 < p \le \infty$, then there is a constant c_p such that*

$$||h^{**}|L_p(0,\infty)|| \le c_p||h^*|L_p(0,\infty)||$$

for every measurable function h.

Lemma 2.4. *Let h_1 and h_2 be two non-negative measurable functions on $\mathbb{R}^n$. If $1 \le p \le \infty$, then*

$$||h_1 + h_2|L_p(\mathbb{R}^n)|| \le ||h_1^* + h_2^*|L_p(0,\infty)||.$$

Proof. The proof follows from Theorems 3.4 and 4.6 in [1, Chapter 2] □

3 Main results

In this part, we present a direct proof of the discrete versions of Jawerth and Franke embedding. We start with the Jawerth embedding.

Theorem 3.1. *Let $-\infty < s_1 < s_0 < \infty$, $0 < p_0 < p_1 \le \infty$ and $0 < q \le \infty$. Then*

$$f^{s_0}_{p_0q} \hookrightarrow b^{s_1}_{p_1p_0} \quad \text{if} \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}. \tag{3.1}$$

Proof. Using the elementary embedding

$$f^s_{pq_0} \hookrightarrow f^s_{pq_1} \quad \text{if} \quad 0 < q_0 \le q_1 \le \infty \tag{3.2}$$

and the lifting property of Besov and Triebel-Lizorkin spaces (which is even simpler in the language of sequence spaces), we may restrict ourselves to the proof of

$$f^s_{p_0\infty} \hookrightarrow b^0_{p_1p_0}, \quad \text{where} \quad s = n\Big(\frac{1}{p_0} - \frac{1}{p_1}\Big). \tag{3.3}$$

Let $\lambda \in f^s_{p_0\infty}$ and set

$$h(x) = \sup_{\nu\in\mathbb{N}_0} 2^{\nu s} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|\chi_{\nu m}(x).$$

Hence

$$|\lambda_{\nu m}| \le 2^{-\nu s} \inf_{x\in Q_{\nu m}} h(x), \quad \nu \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n.$$

Using this notation,

$$||\lambda|f^s_{p_0\infty}|| = ||h|L_{p_0}(\mathbb{R}^n)||$$

and

$$||\lambda|b^0_{p_1p_0}||^{p_0} \le \sum_{\nu=0}^{\infty} 2^{-\nu n}\Big(\sum_{m\in\mathbb{Z}^n} \inf_{x\in Q_{\nu m}} h(x)^{p_1}\Big)^{p_0/p_1} \le \sum_{\nu=0}^{\infty} 2^{-\nu n}\Big(\sum_{k=1}^{\infty} h^*(2^{-\nu n}k)^{p_1}\Big)^{p_0/p_1}.$$

Using the monotonicity of h^* and $p_0 < p_1$ we get

$$\begin{aligned} ||\lambda|b^0_{p_1p_0}||^{p_0} &\lesssim \sum_{\nu=0}^{\infty} 2^{-\nu n}\Big(\sum_{l=0}^{\infty} 2^{nl}\cdot(2^n-1)\cdot h^*(2^{-\nu n}2^{nl})^{p_1}\Big)^{p_0/p_1} \\ &\lesssim \sum_{\nu=0}^{\infty} 2^{-\nu n}\sum_{l=0}^{\infty} 2^{nl\frac{p_0}{p_1}} h^*(2^{-\nu n}2^{nl})^{p_0}. \end{aligned}$$

We substitute $j = l - \nu$ and obtain

$$\begin{aligned} ||\lambda|b^0_{p_1p_0}||^{p_0} &\lesssim \sum_{j=-\infty}^{\infty}\sum_{\nu=-j}^{\infty} 2^{-\nu n}2^{n(\nu+j)\frac{p_0}{p_1}} h^*(2^{jn})^{p_0} \\ &= \sum_{j=-\infty}^{\infty} 2^{nj\frac{p_0}{p_1}} h^*(2^{jn})^{p_0} \sum_{\nu=-j}^{\infty} 2^{n\nu\left(\frac{p_0}{p_1}-1\right)} \\ &\approx \sum_{j=-\infty}^{\infty} 2^{nj} h^*(2^{nj})^{p_0} \approx ||h^*|L_{p_0}(0,\infty)||^{p_0} = ||h|L_{p_0}(\mathbb{R}^n)||^{p_0}. \end{aligned}$$

If $p_1 = \infty$, only notational changes are necessary. □

Theorem 3.2. *Let* $-\infty < s_1 < s_0 < \infty, 0 < p_0 < p_1 < \infty$ *and* $0 < q \le \infty$. *Then*

$$b^{s_0}_{p_0p_1} \hookrightarrow f^{s_1}_{p_1q} \quad \text{if} \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}. \tag{3.4}$$

Proof. Using the lifting property and (3.2), we may suppose that $s_1 = 0$ and $0 < q < p_0$.
By Lemma 2.4, we observe that

$$||\lambda|f^0_{p_1q}|| = \left|\left|\left(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^q\chi_{\nu m}(x)\right)^{1/q}|L_{p_1}(\mathbb{R}^n)\right|\right|$$

may be estimated from above by

$$\left|\left|\sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty}\tilde{\lambda}^q_{\nu m}\tilde{\chi}_{\nu m}(\cdot)|L_{\frac{p_1}{q}}(0,\infty)\right|\right|^{1/q}, \tag{3.5}$$

where $\tilde{\lambda}_\nu = \{\tilde{\lambda}_{\nu m}\}_{m=0}^{\infty}$ is a non-increasing rearrangement of $\lambda_\nu = \{\lambda_{\nu m}\}_{m\in\mathbb{Z}^n}$ and $\tilde{\chi}_{\nu m}$ is a characteristic function of the interval $(2^{-\nu n}m, 2^{-\nu n}(m+1))$.
Using duality, (3.5) may be rewritten as

$$\sup_g\left(\int_0^\infty g(x)\left(\sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty}\tilde{\lambda}^q_{\nu m}\tilde{\chi}_{\nu m}(x)\right)dx\right)^{1/q} = \sup_g\left(\sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty}2^{-\nu n}\tilde{\lambda}^q_{\nu m}g_{\nu m}\right)^{1/q}, \tag{3.6}$$

where the supremum is taken over all non-increasing non-negative measurable functions g with $||g|L_\beta(0,\infty)|| \le 1$ and $g_{\nu m} = 2^{\nu n}\int g(x)\tilde{\chi}_{\nu m}(x)dx$. Here, β is the conjugated index to $\frac{p_1}{q}$. Similarly, α stands for the conjugated index to $\frac{p_0}{q}$.
We use twice Hölder's inequality and estimate (3.6) from above by

$$\left(\sum_{\nu=0}^{\infty}2^{-\nu n}\left(\sum_{m=0}^{\infty}\tilde{\lambda}^{p_0}_{\nu m}\right)^{\frac{p_1}{p_0}}\right)^{1/p_1}\cdot\sup_g\left(\sum_{\nu=0}^{\infty}2^{-\nu n}\left(\sum_{m=0}^{\infty}g^\alpha_{\nu m}\right)^{\frac{\beta}{\alpha}}\right)^{\frac{1}{\beta q}} \tag{3.7}$$

Since $s_0 = n\Big(\dfrac{1}{p_0} - \dfrac{1}{p_1}\Big)$ and $p_1\Big(s_0 - \dfrac{n}{p_0}\Big) = -n$, the first factor in (3.7) is equal to $||\lambda|b^{s_0}_{p_0p_1}||$. To finish the proof, we have to show that there is a number $c > 0$ such that

$$\left(\sum_{\nu=0}^{\infty}2^{-\nu n}\left(\sum_{m=0}^{\infty}g^\alpha_{\nu m}\right)^{\frac{\beta}{\alpha}}\right)^{\frac{1}{\beta q}} \le c \tag{3.8}$$

holds for every non-increasing non-negative measurable functions g with $||g|L_\beta(0,\infty)|| \le 1$. We fix such a function g. Using the monotonicity of g, we get

$$\begin{aligned}\sum_{m=0}^{\infty}g^\alpha_{\nu m} &= \sum_{l=0}^{\infty}\sum_{m=2^{ln}-1}^{2^{(l+1)n}}\left(2^{\nu n}\int\limits_{2^{-\nu n}m}^{2^{-\nu n}(m+1)}g(x)dx\right)^\alpha\\ &\lesssim \sum_{l=0}^{\infty}2^{ln}\left(2^{\nu n}\int\limits_{2^{-\nu n}(2^{ln}-1)}^{2^{-\nu n}2^{ln}}g(x)dx\right)^\alpha \le \sum_{l=0}^{\infty}2^{ln}(g^{**})^\alpha(2^{(l-\nu)n}).\end{aligned}$$

We use $1 < \beta < \alpha$, Lemma 2.3 and obtain

$$\begin{aligned}
\left(\sum_{\nu=0}^{\infty} 2^{-\nu n}\left(\sum_{m=0}^{\infty} g_{\nu m}^{\alpha}\right)^{\frac{\beta}{\alpha}}\right)^{1/\beta} &\leq \left(\sum_{\nu=0}^{\infty} 2^{-\nu n}\left(\sum_{l=0}^{\infty} 2^{ln}(g^{**})^{\alpha}(2^{(l-\nu)n})\right)^{\frac{\beta}{\alpha}}\right)^{1/\beta}\\
&\leq \left(\sum_{\nu=0}^{\infty} 2^{-\nu n}\sum_{l=0}^{\infty} 2^{ln\frac{\beta}{\alpha}}(g^{**})^{\beta}(2^{(l-\nu)n})\right)^{1/\beta}\\
&\leq \left(\sum_{k=-\infty}^{\infty} 2^{kn\frac{\beta}{\alpha}}\sum_{\nu=-k}^{\infty} 2^{\nu n(\frac{\beta}{\alpha}-1)}(g^{**})^{\beta}(2^{kn})\right)^{1/\beta}\\
&\lesssim \left(\sum_{k=-\infty}^{\infty} 2^{kn}(g^{**})^{\beta}(2^{kn})\right)^{1/\beta}\\
&\lesssim ||g^{**}|L_{\beta}(0,\infty)|| \leq c\,||g|L_{\beta}(0,\infty)|| \leq c\,.
\end{aligned}$$

Taking the $\frac{1}{q}$−power of this estimate, we finish the proof of (3.8). □

The Theorems 3.1 and 3.2 are sharp in the following sense.

Theorem 3.3. *Let* $-\infty < s_1 < s_0 < \infty$*,* $0 < p_0 < p_1 \leq \infty$ *and* $0 < q_0, q_1 \leq \infty$ *with*

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

(i) If

$$f^{s_0}_{p_0q_0} \hookrightarrow b^{s_1}_{p_1q_1}, \tag{3.9}$$

then $q_1 \geq p_0$*.*
(ii) If $p_1 < \infty$ *and*

$$b^{s_0}_{p_0q_0} \hookrightarrow f^{s_1}_{p_1q_1}, \tag{3.10}$$

then $q_0 \leq p_1$*.*

Remark 3.4. Using (any of) the usual decomposition techniques, the same statements hold true also for the function spaces. These results were first proved in [4].

Proof. (i) Suppose that $0 < q_1 < p_0 < \infty$ and set

$$\lambda_{\nu m} = \begin{cases} \nu^{-\frac{1}{q_1}} 2^{\nu(\frac{n}{p_1}-s_1)} & \text{if} \quad \nu \in \mathbb{N}_0 \quad \text{and} \quad m = 0,\\ 0, & \text{otherwise.}\end{cases}$$

A simple calculation shows, that $||\lambda|f^{s_0}_{p_0q_0}|| < \infty$ and $||\lambda|b^{s_1}_{p_1q_1}|| = \infty$. Hence, (3.9) does not hold.
(ii) Suppose that $0 < p_1 < q_0 \leq \infty$ and set

$$\lambda_{\nu m} = \begin{cases} \nu^{-\frac{1}{p_1}} 2^{\nu(\frac{n}{p_1}-s_1)} & \text{if} \quad \nu \in \mathbb{N}_0 \quad \text{and} \quad m = 0,\\ 0, & \text{otherwise.}\end{cases}$$

Again, it is a matter of simple calculation to show, that $||\lambda|b^{s_0}_{p_0q_0}|| < \infty$ and $||\lambda|f^{s_1}_{p_1q_1}|| = \infty$. Hence, (3.10) is not true. □

References

[1] C. Bennett and R. Sharpley, Interpolation of Operators, Academic Press, San Diego, 1988.

[2] J. Franke, On the spaces F^s_{pq} of Triebel-Lizorkin type: pointwise multipliers and spaces on domains, Math. Nachr. 125 (1986), 29-68.

[3] B. Jawerth, Some observations on Besov and Lizorkin-Triebel spaces, Math. Scand. 40 (1977), 94-104.

[4] W. Sickel and H. Triebel, Hölder inequalities and sharp embeddings in function spaces of B^s_{pq} and F^s_{pq} type, Z. Anal. Anwendungen, 14 (1995), 105-140.

[5] H. Triebel, Theory of function spaces, Birkhäuser, Basel, 1983.

[6] H. Triebel, Theory of function spaces II, Birkhäuser, Basel, 1992.

[7] H. Triebel, Theory of function spaces III, Birkhäuser, Basel, 2006.

Sampling numbers and function spaces

Jan Vybíral

Abstract

We want to recover a continuous function $f : (0,1)^d \to \mathbb{C}$ using only its function values. Let us assume, that f is from the unit ball of some function space (for example a fractional Sobolev space or a Besov space) and the precision of the reconstruction is measured in the norm of another function space of this type. We describe the rate of convergence of the optimal sampling method (linear as well as nonlinear) in this setting.

AMS Classification: 41A25, 41A46, 46E35

Keywords and phrases: Linear and nonlinear approximation methods; Besov and Triebel-Lizorkin spaces; Sampling operators

1 Introduction

We study the following question. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $B^s_{pq}(\Omega)$ denote the scale of Besov spaces on Ω, see Definition A.1 and Definition A.3 for details. We try to approximate $f \in B^{s_1}_{p_1q_1}(\Omega)$ in the norm of another Besov space, say $B^{s_2}_{p_2q_2}(\Omega)$, by a linear sampling method

$$S_n f = \sum_{j=1}^{n} f(x_j) h_j, \tag{1.1}$$

where $h_j \in B^{s_2}_{p_2q_2}(\Omega)$ and $x_j \in \Omega$. First of all, we have to give a meaning to the pointwise evaluation in (1.1). For this reason, we shall restrict ourselves to the case

$$s_1 > \frac{d}{p_1},$$

which guarantees the continuous embedding $B^{s_1}_{p_1q_1}(\Omega) \hookrightarrow C(\bar{\Omega})$. Second, we always assume that the embedding $B^{s_1}_{p_1q_1}(\Omega) \hookrightarrow B^{s_2}_{p_2q_2}(\Omega)$ is compact, which holds if and only if

$$s_1 - s_2 > d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ .$$

We measure the worst case error of $S_n f$ by

$$\sup\{||f - S_n f|B^{s_2}_{p_2q_2}(\Omega)|| : ||f|B^{s_1}_{p_1q_1}(\Omega)|| \leq 1\}. \tag{1.2}$$

The same worst case error may also be considered for nonlinear sampling methods

$$S_n f = \varphi(f(x_1), \dots, f(x_n)), \tag{1.3}$$

where $\varphi : \mathbb{C}^n \to B^{s_2}_{p_2q_2}(\Omega)$ is an arbitrary mapping. In this paper, we discuss the decay of (1.2) for linear (1.1) and nonlinear (1.3) sampling methods.

In some cases we restrict ourselves to the case $\Omega = I^d = (0,1)^d$. This allows to describe the optimal sampling operator more explicitly. However, we conjecture, that many of these results can be generalised to general bounded Lipschitz domains.

Let $L_p(\Omega)$ stand for the usual Lebesgue space and $W^k_p(\Omega), k \in \mathbb{N}$, denotes the classical Sobolev space over Ω. Then it is well known that

$$\inf_{S_n} \sup\{||f - S_n f|L_{p_2}(\Omega)|| : ||f|W^k_{p_1}(\Omega)|| \leq 1\} \approx n^{-\frac{k}{d} + (\frac{1}{p_1} - \frac{1}{p_2})_+}, \tag{1.4}$$

where the infimum in (1.4) runs over all linear sampling operators S_n, see (1.1) (cf. [5] or [10]). The result remains true if we switch to the general situation where nonlinear methods S_n are allowed. In [12], this statement has been proved for arbitrary bounded Lipschitz domain, but with the Sobolev spaces replaced by the more general scales of Besov and Triebel-Lizorkin spaces. The target space was always given by $L_{p_2}(\Omega)$. The proof given there uses the simple structure of the Lebesgue space. It is the main aim of this paper to generalise (1.4) and to investigate also other “target” spaces.

Let us present our main results. If $s_2 > 0$, then the quantity

$$\inf_{S_n} \sup\{||f - S_n f|B^{s_2}_{p_2q_2}(\Omega)|| : ||f|B^{s_1}_{p_1q_1}(\Omega)|| \leq 1\} \tag{1.5}$$

behaves like

$$n^{-\frac{s_1-s_2}{d}+(\frac{1}{p_1}-\frac{1}{p_2})_+}$$

in both, the linear as well as the nonlinear setting. We prove this result only for the special case of $\Omega = (0,1)^d$. However in this situation we are able to give an explicit description of in order optimal operator which we are going to introduce now. Namely, if $n \approx 2^{kd}$, where $k \in \mathbb{N}$ is fixed, we use a smooth decomposition of unity $\{\psi_{k,\nu}\}$ such that $\sum_\nu \psi_{k,\nu}(x) = 1$ for $x \in (0,1)^d$ where the support of $\psi_{k,\nu}$ is concentrated around $2^{-k}\nu$. Then we approximate f locally on $\operatorname{supp} \psi_{k,\nu}$ by a polynomial $g_{k,\nu}$ and define

$$S_n f = \sum_\nu g_{k,\nu} \psi_{k,\nu}.$$

To calculate each of the $2^{(k+2)d}$ functions $g_{k,\nu}$ we need to combine $\binom{M+d-1}{d}$ function values of f in a linear way. Altogether, we need $2^{(k+2)d}\binom{M+d-1}{d} \approx 2^{kd} \approx n$ function values of f to obtain $S_n f$. Here, $M > s_1$ is a fixed natural number. The generalisation of this construction to bounded Lipschitz domains remains a subject of further study.

If $s_2 < 0$, we give the following characterisation of (1.5). If $p_1 \geq p_2$ or $p_1 < p_2$ and $\dfrac{d}{p_2} - \dfrac{d}{p_1} > s_2$, then (1.5) decays like

$$n^{-\frac{s_1}{d}}$$

and if $p_1 < p_2$ and $0 > s_2 > \dfrac{d}{p_2} - \dfrac{d}{p_1}$, then (1.5) behaves like

$$n^{-\frac{s_1}{d}+\frac{s_2}{d}+\frac{1}{p_1}-\frac{1}{p_2}}.$$

All these results hold for linear as well as nonlinear methods S_n.

These estimates can be applied in connection with elliptic differential operators, which was the actual motivation for this research, c.f. [6] and [7]. Let us briefly introduce this setting. Let

$$\mathcal{A} : H \to G$$

be a bounded linear operator from a Hilbert space H to another Hilbert space G. We assume that $\mathcal{A}$ is boundedly invertible, hence

$$\mathcal{A}(u) = f$$

has a unique solution for every $f \in G$. A typical application is an operator equation, where $\mathcal{A}$ is an elliptic differential operator, and we assume that

$$\mathcal{A} : H_0^s(\Omega) \to H^{-s}(\Omega),$$

where Ω is a bounded Lipschitz domain, $H_0^s(\Omega)$ is a function space of Sobolev type with fractional order of smoothness $s > 0$ of functions vanishing on the boundary and H^{-s} is a function space of Sobolev type with negative smoothness $-s < 0$. The classical example is the Poisson equation

$$-\Delta u = f \quad \text{in} \quad \Omega \qquad \text{and} \qquad u = 0 \quad \text{on} \quad \partial\Omega.$$

Here, $s = 1$ and

$$\mathcal{A} = -\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$$

is bounded and boundedly invertible. We want to approximate the *solution operator* $u = S(f)$ using only function values of f.
We define the n-th linear sampling number of the identity $id : H^{-1+t}(\Omega) \to H^{-1}(\Omega)$ by

$$g_n^{\text{lin}}(id : H^{-1+t}(\Omega) \to H^{-1}(\Omega)) = \inf_{S_n} ||id - S_n|\mathcal{L}(H^{-1+t}(\Omega), H^{-1}(\Omega))||, \qquad (1.6)$$

where t is a positive real number with $-1 + t > \frac{d}{2}$, and the n-th linear sampling number of $S : H^{-1+t}(\Omega) \to H^1(\Omega)$ by

$$g_n^{\text{lin}}(S : H^{-1+t}(\Omega) \to H^1(\Omega)) = \inf_{S_n} ||S - S_n|\mathcal{L}(H^{-1+t}(\Omega), H^1(\Omega))||. \qquad (1.7)$$

The infimum in (1.6) and (1.7) runs over all linear operators S_n of the form (1.1) and $\mathcal{L}(X, Y)$ stands for the space of bounded linear operators between two Banach spaces X and Y, equipped with the classical operator norm.
It turns out that these quantities are equivalent (up to multiplicative constants which do not depend neither on f nor on n) and are of the asymptotic order

$$g_n^{\text{lin}}(S : H^{-1+t}(\Omega) \to H^1(\Omega)) \approx g_n^{\text{lin}}(id : H^{-1+t}(\Omega) \to H^{-1}(\Omega)) \approx n^{-\frac{-1+t}{d}}.$$

We refer to [6] and [7] for a detailed discussion of this approach. The estimates of sampling numbers of embedding between two function spaces translates therefor into estimates of sampling numbers of the solution operator S. We observe that the more regular f, the faster is the decay of the linear sampling numbers of the solution operator S. Let us also point out that optimal linear methods (not restricted to use only the function values of f) achieve asymptotically a better rate of convergence, namely $n^{-\frac{t}{d}}$. Hence, the limitation to the sampling operators results in a serious restriction. One has to pay at least $n^{1/d}$ in comparison with optimal linear methods.
Using our estimates of sampling numbers of identities between Besov and Triebel-Lizorkin spaces, this result may be generalised as follows.[1] If $p \geq 2$, $1 \leq q \leq \infty$ and $-1 + t > \frac{d}{p}$ then

$$g_n^{\text{lin}}(S : B^{-1+t}_{pq}(\Omega) \to H^1(\Omega)) \approx g_n^{\text{lin}}(id : B^{-1+t}_{pq}(\Omega) \to H^{-1}(\Omega)) \approx n^{-\frac{-1+t}{d}}.$$

If $p < 2$ with $\dfrac{1}{p} > \dfrac{1}{d} + \dfrac{1}{2}$, $1 \leq q \leq \infty$ and $-1 + t > \frac{d}{p}$ then

$$g_n^{\text{lin}}(S : B^{-1+t}_{pq}(\Omega) \to H^1(\Omega)) \approx g_n^{\text{lin}}(id : B^{-1+t}_{pq}(\Omega) \to H^{-1}(\Omega)) \approx n^{-\frac{t}{d}+\frac{1}{p}-\frac{1}{2}}.$$

Finally, if $p < 2$ with $\dfrac{1}{p} < \dfrac{1}{d} + \dfrac{1}{2}$, $1 \leq q \leq \infty$ and $-1 + t > \frac{d}{p}$ then

$$g_n^{\text{lin}}(S : B^{-1+t}_{pq}(\Omega) \to H^1(\Omega)) \approx g_n^{\text{lin}}(id : B^{-1+t}_{pq}(\Omega) \to H^{-1}(\Omega)) \approx n^{-\frac{-1+t}{d}}.$$

We prove the same results also for the nonlinear sampling numbers $g_n(S)$. Altogether, the regularity information of f may now be described by an essentially broader scale of function spaces.

[1]Although the results are stated only for Besov spaces, they are proved also for Triebel-Lizorkin spaces, which include also fractional Sobolev spaces as a special case.

All the unimportant constants are denoted by the letter c, whose meaning may differ from one occurrence to another. If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive real numbers, we write $a_n \lesssim b_n$ if, and only if, there is a positive real number $c > 0$ such that $a_n \le c\, b_n, n \in \mathbb{N}$. Furthermore, $a_n \approx b_n$ means that $a_n \lesssim b_n$ and simultaneously $b_n \lesssim a_n$.

I would like to thank to Erich Novak, Winfried Sickel, Hans Triebel and to the anonymous referee for many valuable discussions and comments on the topic.

2 Sampling numbers

The notation and basic facts about function spaces, which we shall need later on, are included in the Appendix.

We now introduce the concept of sampling numbers.

Definition 2.1. Let Ω be a bounded Lipschitz domain. Let $G_1(\Omega)$ be a space of continuous functions on Ω and $G_2(\Omega) \subset D'(\Omega)$ be a space of distributions on Ω. Suppose, that the embedding

$$id : G_1(\Omega) \hookrightarrow G_2(\Omega)$$

is compact.

For $\{x_j\}_{j=1}^n \subset \Omega$ we define the *information map*

$$N_n : G_1(\Omega) \to \mathbb{C}^n, \qquad N_n f = (f(x_1), \dots, f(x_n)), \quad f \in G_1(\Omega).$$

For any (linear or nonlinear) mapping $\varphi_n : \mathbb{C}^n \to G_2(\Omega)$ we consider

$$S_n : G_1(\Omega) \to G_2(\Omega), \qquad S_n = \varphi_n \circ N_n.$$

(i) Then, for all $n \in \mathbb{N}$, the $n-$th *sampling number* $g_n(id)$ is defined by

$$g_n(id) = \inf_{S_n} \sup\{||f - S_n f|G_2(\Omega)|| : ||f|G_1(\Omega)|| \le 1\}, \tag{2.1}$$

where the infimum is taken over all n-tuples $\{x_j\}_{j=1}^n \subset \Omega$ and all (linear or nonlinear) φ_n.

(ii) For all $n \in \mathbb{N}$ the $n-$th *linear sampling number* $g_n^{\text{lin}}(id)$ is defined by (2.1), where now only linear mappings φ_n are admitted.

2.1 The case $s_2 > 0$

In this subsection, we discuss the case where $\Omega = I^d = (0,1)^d$ is the unit cube, $G_1(\Omega) = A^{s_1}_{p_1 q_1}(\Omega)$ and $G_2(\Omega) = A^{s_2}_{p_2 q_2}(\Omega)$ with $s_1 > \dfrac{d}{p_1}$ and $s_1 - d\Big(\dfrac{1}{p_1} - \dfrac{1}{p_2}\Big)_+ > s_2 > 0$. Here, $A^s_{pq}(\Omega)$ stands either for a Besov space $B^s_{pq}(\Omega)$ or a Triebel-Lizorkin space $F^s_{pq}(\Omega)$, see Definition A.3 for details. We start with the most simple and most important case, namely when $p_1 = p_2 = q_1 = q_2$.

Proposition 2.2. *Let $\Omega = I^d = (0,1)^d$. Let $G_1(\Omega) = B^{s_1}_{pp}(\Omega)$ and $G_2(\Omega) = B^{s_2}_{pp}(\Omega)$ with $1 \le p \le \infty$,*

$$s_1 > \frac{d}{p}, \quad \textit{and} \quad s_1 > s_2 > 0.$$

Then

$$g_n^{\text{lin}}(id) \lesssim n^{-\frac{s_1 - s_2}{d}}.$$

Proof. First, we introduce necessary notation. Let $a > 0$, $z \in \mathbb{R}^d$ and $U \subset \mathbb{R}^d$. Then

$$aU = \{ax : x \in U\} \text{ and } z + aU = \{z + ax : x \in U\}. \tag{2.2}$$

Furthermore, if $k \in \mathbb{N}_0$ and $\nu \in \mathbb{Z}^d$, we set

$$Q_{k,\nu} = \{x \in \mathbb{R}^d : 2^{-k}\nu_i < x_i < 2^{-k}(\nu_i + 1)\},$$
$$Q^{k,\nu} = \{x \in I^d : 2^{-k}\Big(\nu_i - \frac{1}{2}\Big) < x_i < 2^{-k}\Big(\nu_i + \frac{3}{2}\Big)\}.$$

We point out, that (up to a set of measure zero)

$$I^d = \bigcup \{Q_{k,\nu} : 0 \le \nu_i \le 2^k - 1, i = 1, 2, \dots, d\}.$$

Next, we introduce smooth decomposition of unity, first on $\mathbb{R}^d$ and then its restriction to I^d. Let $\tilde{\psi} \in S(\mathbb{R}^d)$ with

$$\operatorname{supp} \tilde{\psi} \subset \Big(-\frac{1}{2}, \frac{3}{2}\Big)^d \quad \text{and} \quad \sum_{\nu \in \mathbb{Z}^d} \tilde{\psi}(x - \nu) = 1, \quad x \in \mathbb{R}^d.$$

Then we define

$$\psi_{k,\nu}(x) = \begin{cases} \tilde{\psi}(2^k x - \nu), & \text{if } x \in I^d, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Let us denote $A_k = \{-1, 0, \dots, 2^k\}^d$. By (2.3), the following identities are true for every $k \in \mathbb{N}$:

$$\sum_{\nu \in \mathbb{Z}^d} \psi_{k,\nu}(x) = \sum_{\nu \in A_k} \psi_{k,\nu}(x) = \chi_{I_d}(x) = \begin{cases} 1, & \text{if } x \in I^d, \\ 0 & \text{otherwise,} \end{cases}$$
$$\operatorname{supp} \psi_{k,\nu} \subset Q^{k,\nu}, \qquad \nu \in A_k.$$

Now we define linear approximation operators $\tilde{S}_k$. Take $f \in G_1(I^d)$ and consider the decomposition

$$f = \sum_{\nu \in A_k} f \psi_{k,\nu}.$$

To each $Q_{k,\nu}$ we associate $g_{k,\nu} \in \mathcal{P}^M(Q^{k,\nu})$ such that $g_{k,\nu}(2^{-k}\cdot)$ approximates $f(2^{-k}\cdot)$ on $2^k Q^{k,\nu}$ according to Corollary A.6, see the Appendix,

$$||(f - g_{k,\nu})(2^{-k}\cdot)|B^{s_1}_{pp}(2^k Q^{k,\nu})|| \lesssim \left(\int_0^1 t^{-s_1 p} ||d_t^{M,2^k Q^{k,\nu}}(f(2^{-k}\cdot))(x)|L_p(2^k Q^{k,\nu})||^p \frac{\mathrm{d}t}{t}\right)^{1/p}. \tag{2.4}$$

The operators $\tilde{S}_k : G_1(I^d) \to G_2(I^d)$ are defined by

$$\tilde{S}_k f = \sum_{\nu \in A_k} g_{k,\nu} \psi_{k,\nu}, \qquad k \in \mathbb{N}. \tag{2.5}$$

Trivially, the right-hand side of (2.5) belongs to $G_1(I^d)$ and hence also to $G_2(I^d)$. The operators $\tilde{S}_k$ use $\binom{M+d-1}{d} \cdot (2^k + 2)^d \approx 2^{kd}$ points. So, it is enough to prove the estimate

$$||\sum_{\nu \in A_k} (f - g_{k,\nu}) \psi_{k,\nu} | B^{s_2}_{pp}(I^d)|| \lesssim 2^{-k(s_1 - s_2)} ||f|B^{s_1}_{pp}(I^d)||.$$

We use the dilation property (cf. [9, Prop. 2.2.1]) as well as the embedding $B^{s_1}_{pp}(\mathbb{R}^d) \hookrightarrow B^{s_2}_{pp}(\mathbb{R}^d)$ and obtain

$$
\begin{aligned}
&\Big|\Big|\sum_{\nu\in A_k}(f-g_{k,\nu})\psi_{k,\nu}|B^{s_2}_{pp}(I^d)\Big|\Big| \\
&\qquad \lesssim 2^{k\left(s_2-\frac{d}{p}\right)}\Big|\Big|\sum_{\nu\in A_k}(f-g_{k,\nu})(2^{-k}\cdot)\psi_{k,\nu}(2^{-k}\cdot)|B^{s_2}_{pp}(2^kI^d)\Big|\Big| \qquad (2.6)\\
&\qquad \lesssim 2^{k\left(s_2-\frac{d}{p}\right)}\Big|\Big|\sum_{\nu\in A_k}(f-g_{k,\nu})(2^{-k}\cdot)\psi_{k,\nu}(2^{-k}\cdot)|B^{s_1}_{pp}(2^kI^d)\Big|\Big|.
\end{aligned}
$$

We claim that

$$
\Big|\Big|\sum_{\nu\in A_k}(f-g_{k,\nu})(2^{-k}\cdot)\psi_{k,\nu}(2^{-k}\cdot)|B^{s_1}_{pp}(2^kI^d)\Big|\Big| \lesssim \left(\sum_{\nu\in A_k}\big|\big|(f-g_{k,\nu})(2^{-k}\cdot)|B^{s_1}_{pp}(2^kQ^{k,\nu})\big|\big|^p\right)^{1/p}. \qquad (2.7)
$$

To prove (2.7), we first decompose $\sum_{\nu\in A_k}$ into $\sum_{\alpha=1}^{K}\sum_{\nu\in A_k^\alpha}$ with the number $K\in\mathbb{N}$ (independent of $k\in\mathbb{N}$) so that

$$
\operatorname{dist}(\operatorname{supp}\psi_{k,\nu_1}(2^{-k}\cdot),\operatorname{supp}\psi_{k,\nu_2}(2^{-k}\cdot)) > 1 \qquad (2.8)
$$

for every $\nu_1,\nu_2\in A_k^\alpha$ and every $\alpha=1,\dots,K$.
To every $\nu\in A_k^\alpha$ we associate $\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}\cdot))$ defined on $\mathbb{R}^d$ such that

$$
\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}x)) = (f-g_{k,\nu})(2^{-k}x), \qquad x\in 2^kQ^{k,\nu}, \qquad (2.9)
$$

$$
\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}x)) = 0 \quad \text{if} \quad x\in\operatorname{supp}\psi_{k,\nu'}(2^{-k}\cdot) \qquad (2.10)
$$

if $\nu'\in A_k^\alpha, \nu'\neq\nu$ and

$$
||\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}x))|B^{s_1}_{pp}(\mathbb{R}^d)|| \le c\,||(f-g_{k,\nu})(2^{-k}x)|B^{s_1}_{pp}(2^kQ^{k,\nu})||. \qquad (2.11)
$$

The existence of $\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}\cdot))$ satisfying (2.9)-(2.11) follows directly from the Definition A.3, possibly combined with some smooth cut-off function and the pointwise multiplier assertion, cf. [15, Theorem 2.8.2].
Denoting

$$
\tilde\psi_{k,\nu}(x) = \tilde\psi(2^kx-\nu), \qquad x\in\mathbb{R}^d, \quad k\in\mathbb{N}, \quad \nu\in\mathbb{Z}^d, \qquad (2.12)
$$

we get

$$
\begin{aligned}
&\Big|\Big|\sum_{\nu\in A_k}(f-g_{k,\nu})(2^{-k}\cdot)\psi_{k,\nu}(2^{-k}\cdot)|B^{s_1}_{pp}(2^kI^d)\Big|\Big| \\
&\qquad \lesssim \sum_{\alpha=1}^{K}\Big|\Big|\sum_{\nu\in A_k^\alpha}(f-g_{k,\nu})(2^{-k}\cdot)\psi_{k,\nu}(2^{-k}\cdot)|B^{s_1}_{pp}(2^kI^d)\Big|\Big| \\
&\qquad \lesssim \sum_{\alpha=1}^{K}\Big|\Big|\sum_{\nu\in A_k^\alpha}\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}\cdot))\psi_{k,\nu}(2^{-k}\cdot)|B^{s_1}_{pp}(\mathbb{R}^d)\Big|\Big|.
\end{aligned}
$$

By (2.8) and the so called *localisation property*, c.f. [16, Chapter 2.4.7], we may estimate the last expression from above by

$$\sum_{\alpha=1}^{K}\Big(\sum_{\nu\in A_k^\alpha}\Big|\Big|\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}\cdot))\psi_{k,\nu}(2^{-k}\cdot)|B^{s_1}_{pp}(\mathbb{R}^d)\Big|\Big|^p\Big)^{1/p}$$
$$\lesssim\Big(\sum_{\alpha=1}^{K}\sum_{\nu\in A_k^\alpha}\Big|\Big|\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}\cdot))\psi_{k,\nu}(2^{-k}\cdot)|B^{s_1}_{pp}(\mathbb{R}^d)\Big|\Big|^p\Big)^{1/p}$$
$$=\Big(\sum_{\nu\in A_k}\Big|\Big|\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}\cdot))\psi_{k,\nu}(2^{-k}\cdot)|B^{s_1}_{pp}(\mathbb{R}^d)\Big|\Big|^p\Big)^{1/p}.$$

Together with Lemma A.7 and (2.11) this finally leads to

$$\Big|\Big|\sum_{\nu\in A_k}(f-g_{k,\nu})(2^{-k}\cdot)\psi_{k,\nu}(2^{-k}\cdot)|B^{s_1}_{pp}(2^kI^d)\Big|\Big|$$
$$\lesssim\Big(\sum_{\nu\in A_k}\Big|\Big|\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}\cdot))|B^{s_1}_{pp}(\mathbb{R}^d)\Big|\Big|^p\cdot\Big|\Big|\psi_{k,\nu}(2^{-k}\cdot)|B^{s_1}_{pp}(\mathbb{R}^d)\Big|\Big|^p\Big)^{1/p}$$
$$\lesssim\Big(\sum_{\nu\in A_k}\Big|\Big|\mathcal{E}_\nu((f-g_{k,\nu})(2^{-k}\cdot))|B^{s_1}_{pp}(\mathbb{R}^d)\Big|\Big|^p\Big)^{1/p}$$
$$\lesssim\Big(\sum_{\nu\in A_k}\Big|\Big|(f-g_{k,\nu})(2^{-k}\cdot)|B^{s_1}_{pp}(2^kQ^{k,\nu})\Big|\Big|^p\Big)^{1/p},$$

which finishes (2.7).

We insert (2.7) into (2.6) and use (2.4) together with (A.4)

$$\Big|\Big|\sum_{\nu\in A_k}(f-g_{k,\nu})\psi_{k,\nu}|B^{s_2}_{pp}(I^d)\Big|\Big|$$
$$\lesssim 2^{k\left(s_2-\frac{d}{p}\right)}\Big(\sum_{\nu\in A_k}\int_0^1 t^{-s_1p}\big|\big|(d_t^{M,2^kQ^{k,\nu}}f(2^{-k}\cdot))(x)|L_p(2^kQ^{k,\nu})\big|\big|^p\frac{\mathrm{d}t}{t}\Big)^{1/p}$$
$$\lesssim 2^{k\left(s_2-\frac{d}{p}\right)}\Big(\sum_{\nu\in A_k}\int_0^1 t^{-s_1p}\big|\big|(d_{2^{-k}t}^{M,Q^{k,\nu}}f)(2^{-k}x)|L_p(2^kQ^{k,\nu})\big|\big|^p\frac{\mathrm{d}t}{t}\Big)^{1/p}.$$

The rest is done by direct substitutions and Theorem A.4

$$\Big|\Big|\sum_{\nu\in A_k}(f-g_{k,\nu})\psi_{k,\nu}|B^{s_2}_{pp}(I^d)\Big|\Big|$$
$$\lesssim 2^{k\left(s_2-s_1-\frac{d}{p}\right)}\Big(\sum_{\nu\in A_k}\int_0^{2^{-k}}\xi^{-s_1p}\big|\big|(d_\xi^{M,Q^{k,\nu}}f)(2^{-k}x)|L_p(2^kQ^{k,\nu})\big|\big|^p\frac{\mathrm{d}\xi}{\xi}\Big)^{1/p}$$
$$\lesssim 2^{k(s_2-s_1)}\Big(\sum_{\nu\in A_k}\int_0^{2^{-k}}\xi^{-s_1p}\big|\big|(d_\xi^{M,Q^{k,\nu}}f)(x)|L_p(Q^{k,\nu})\big|\big|^p\frac{\mathrm{d}\xi}{\xi}\Big)^{1/p}$$
$$\lesssim 2^{-k(s_1-s_2)}\Big(\int_0^{2^{-k}}\xi^{-s_1p}\big|\big|(d_\xi^{M,I^d}f)(x)|L_p(I^d)\big|\big|^p\frac{\mathrm{d}\xi}{\xi}\Big)^{1/p}$$
$$\lesssim 2^{-k(s_1-s_2)}||f|B^{s_1}_{pp}(I^d)||.$$

□

Next we consider the case of general integrability and summability parameters.

Proposition 2.3. *Let $\Omega = I^d = (0,1)^d$. Let $G_1(\Omega) = A^{s_1}_{p_1q_1}(\Omega)$ and $G_2(\Omega) = A^{s_2}_{p_2q_2}(\Omega)$ with $1 \le p_1, p_2, q_1, q_2 \le \infty$ ($p_1, p_2 < \infty$ in the F-case),*

$$s_1 > \frac{d}{p_1}, \quad \textit{and} \quad s_1 - d\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+ > s_2 > 0. \tag{2.13}$$

Then

$$g_n^{\mathrm{lin}}(id) \lesssim n^{-\frac{s_1-s_2}{d} + \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+}. \tag{2.14}$$

Proof. First, we deal with the case $p_1 = p_2 = p$ and $p \neq q_1$ and/or $p \neq q_2$. We use the well-known real interpolation formula, c.f. [13], [1], [15] and [17]

$$B^r_{pq}(\mathbb{R}^d) = \big(B^{r_0}_{pp}(\mathbb{R}^d), B^{r_1}_{pp}(\mathbb{R}^d)\big)_{\theta,q}$$

and its counterpart

$$B^r_{pq}(I^d) = \big(B^{r_0}_{pp}(I^d), B^{r_1}_{pp}(I^d)\big)_{\theta,q}$$

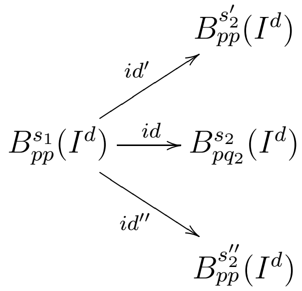
for

$$1 \le p, q \le \infty, \quad 0 < \theta < 1, \quad r_0 < r_1, \quad r = (1-\theta) r_0 + \theta r_1.$$

If, for example, $p \neq q_2$, we find two different real numbers s_2' and s_2'' such that

$$s_1 > s_2', s_2'' > 0, \qquad s_2 = (1-\theta)s_2' + \theta s_2''$$

and apply Proposition 2.2 to embeddings id' and id'' in the following diagram



Using the same approximation operator $\tilde{S}_k$, we may interpolate the estimates for $||f - \tilde{S}_k f | B^{s_2'}_{pp}(I^d)||$ and $||f - \tilde{S}_k f | B^{s_2''}_{pp}(I^d)||$ and obtain (2.14).
If also $p \neq q_1$, we proceed in the same way.
If $p_1 \le p_2$ we define s_0 by

$$s_1 > s_0 := s_2 + d\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) > s_2 > 0$$

and use the chain of embeddings

$$B^{s_1}_{p_1q_1}(I^d) \hookrightarrow B^{s_0}_{p_1q_2}(I^d) \hookrightarrow B^{s_2}_{p_2q_2}(I^d).$$

The first embedding provides the estimate

$$g_n^{\mathrm{lin}}(id) \lesssim n^{-\frac{s_1-s_0}{d}} = n^{-\frac{s_1-s_2}{d} + \frac{1}{p_1} - \frac{1}{p_2}},$$

the second one is bounded.
If $p_1 \geq p_2$, we use the embedding

$$B^{s_1}_{p_1q_1}(I^d) \hookrightarrow B^{s_2}_{p_1q_2}(I^d) \hookrightarrow B^{s_2}_{p_2q_2}(I^d).$$

The second embedding is bounded, the first one together with Proposition 2.2 gives the result.
This finishes the proof in the B-case. The F-case then follows through trivial embeddings, c.f. [15, 2.3.2]

$$F^{s_1}_{p_1q_1}(I^d) \hookrightarrow B^{s_1}_{p_1,\infty}(I^d) \hookrightarrow B^{s_2}_{p_2,1}(I^d) \hookrightarrow F^{s_2}_{p_2q_2}(I^d).$$

□

Theorem 2.4. *Let $\Omega = I^d = (0,1)^d$. Let $G_1(\Omega) = A^{s_1}_{p_1q_1}(\Omega)$ and $G_2(\Omega) = A^{s_2}_{p_2q_2}(\Omega)$ with $1 \leq p_1, p_2, q_1, q_2 \leq \infty$ ($p_1, p_2 < \infty$ in the F-case) and* (2.13) *Then*

$$g_n(id) \approx g_n^{\text{lin}}(id) \approx n^{-\frac{s_1-s_2}{d}+\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+}. \tag{2.15}$$

Proof. According to the Proposition 2.3, it is enough to prove that

$$g_n(id) \gtrsim n^{-\frac{s_1-s_2}{d}+\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+}. \tag{2.16}$$

We use the following simple observation, (c.f. [12, Proposition 20]). For $\Gamma = \{x_j\}_{j=1}^n \subset \Omega$ we denote

$$G_1^\Gamma(\Omega) = \{f \in G_1(\Omega) : f(x_j) = 0 \quad \text{for all} \quad j = 1, \ldots, n\}.$$

Then

$$g_n(id) \approx \inf_\Gamma \sup\{||f|G_2(\Omega)|| : f \in G_1^\Gamma(\Omega), ||f|G_1(\Omega)|| = 1\} \tag{2.17}$$

$$= \inf_\Gamma ||id : G_1^\Gamma(\Omega) \hookrightarrow G_2(\Omega)||, \tag{2.18}$$

where both the infima extend over all sets $\Gamma = \{x_j\}_{j=1}^n \subset \Omega$.
To prove (2.16), we construct for every $\Gamma = \{x_j\}_{j=1}^{2^{ld}}, l \in \mathbb{N}$, a function $\psi_l \in G_1^\Gamma(\Omega)$ with

$$||\psi_l|G_1(\Omega)|| \lesssim 1 \quad \text{and} \quad ||\psi_l|G_2(\Omega)|| \gtrsim 2^{l\left(s_2-s_1+d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+\right)}, \tag{2.19}$$

where the constants of equivalence do not depend on $l \in \mathbb{N}$.
We rely on the wavelet characterisation of the spaces $A^s_{pq}(\mathbb{R}^n)$, as described in [18, Section 3.1]. Let

$$\psi_F \in C^K(\mathbb{R}) \quad \text{and} \quad \psi_M \in C^K(\mathbb{R}), \quad K \in \mathbb{N},$$

be the Daubechies compactly supported K-wavelets on $\mathbb{R}$ with K large enough. Then we define

$$\Psi(x) = \prod_{i=1}^d \psi_M(x_i), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d$$

and

$$\Psi^j_m(x) = \Psi(2^j x - m), \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n.$$

Then the function

$$\psi_j(x) = \sum_m \lambda_{jm} \Psi^j_m(x), \quad j \in \mathbb{N} \tag{2.20}$$

satisfies

$$||\psi_j|A^s_{pq}(\Omega)|| \approx 2^{j(s-\frac{d}{p})} \left(\sum_m |\lambda_{jm}|^p \right)^{1/p} \tag{2.21}$$

with constants independent on $j \in \mathbb{N}$ and on the sequence $\lambda = \{\lambda_{jm}\}$. The summation in (2.20) and (2.21) runs over those $m \in \mathbb{Z}^n$ for which the support of Ψ^j_m is included in Ω. The proof of (2.21) is based on [18, Theorem 3.5]. First, this theorem tells us that the $A^s_{pq}(\Omega)$-norm of (2.20) may be estimated from above by the right-hand side of (2.21). On the other hand, considering another extension of ψ_j to $\mathbb{R}^d$ and its (unique) wavelet decomposition, we get the opposite inequality.

There is a number $k \in \mathbb{N}$ with the following property. For any $l \in \mathbb{N}$ and any $\Gamma = \{x_j\}_{j=1}^{2^{ld}}$, there are $m_j \in \mathbb{Z}^d, j = 1, \dots, 2^{ld}$ such that

$$\operatorname{supp} \Psi^{k+l}_{m_j} \subset \Omega \quad \text{and} \quad \operatorname{supp} \Psi^{k+l}_{m_j} \cap \Gamma = \emptyset, \quad \text{for} \quad j = 1, \dots, 2^{ld}.$$

Step 1: $p_1 \le p_2$. In this case, we take in (2.20) $\lambda_{k+l,m_1} = 2^{-j(s-\frac{d}{p})}$ and $\lambda_{k+l,m_n} = 0, n = 2, \dots, 2^{ld}$ and apply (2.21) twice to verify (2.19).

Step 2: $p_1 > p_2$. In this case, we take $\lambda_{k+l,m_n} = 2^{-js}$, $n = 1, \dots, 2^{ld}$ in (2.20) and apply again (2.21) twice to prove (2.19). □

2.2 The case $s_2 = 0$

In the case $s_2 = 0$, new phenomena come into play. First we point out that Lemma A.8 for $s = 0$ gives an immediate counterpart of (2.6) and this leads to the following result.

Theorem 2.5. *Let* $\Omega = I^d = (0,1)^d$. *Let*

$$id : G_1(\Omega) \hookrightarrow G_2(\Omega)$$

with

$$G_1(\Omega) = B^s_{p_1q_1}, \quad G_2(\Omega) = B^0_{p_2q_2}$$

and

$$1 \le p_1, q_1, p_2, q_2 \le \infty, \quad s > \frac{d}{p_1}.$$

Then

$$n^{-\frac{s}{d}+(\frac{1}{p_1}-\frac{1}{p_2})_+} \lesssim g_n(id) \lesssim g_n^{\mathrm{lin}}(id) \lesssim n^{-\frac{s}{d}+(\frac{1}{p_1}-\frac{1}{p_2})_+}(1+\log n)^{1/q_2}, \qquad n \in \mathbb{N}. \tag{2.22}$$

If the target space is a Lebesgue space, this can be improved, cf. [12].

Theorem 2.6. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^d$. *Let*

$$id : G_1(\Omega) = A^s_{pq}(\Omega) \hookrightarrow L_r(\Omega) = G_2(\Omega)$$

with

$$1 \le p, q \le \infty, \quad s > \frac{d}{p} \quad \text{and} \quad 1 \le r \le \infty$$

($p<\infty$ in the F-case). Then

$$g_n(id) \approx g_n^{\text{lin}}(id) \approx n^{-\frac{s}{d}+(\frac{1}{p}-\frac{1}{r})_+}, \qquad n \in \mathbb{N}.$$

Remark 2.7. We show in one example, that the logarithmic factor cannot be removed in general. Let $\Omega = I^d = (0,1)^d$ and consider the embedding

$$id : B^s_{1,1}(\Omega) \to B^0_{1,1}(\Omega).$$

Finally, take $\psi \in S(\mathbb{R}^d)$ with $\operatorname{supp}\psi \subset \Omega$ and $\widehat{\psi}(0) \neq 0$. For every $k \in \mathbb{N}$ and every $\Gamma = \{x_j\}_{j=1}^n \subset \Omega, n = 2^{kd}$, we set $f_k^\Gamma(x) = \psi(2^{k+1}(x - x^\Gamma))$, where x^Γ is chosen such that $\operatorname{supp} f_k^\Gamma \cap \Gamma = \emptyset$ and $\operatorname{supp} f_k^\Gamma \subset \Omega$. We claim that

$$||f_k^\Gamma | B^s_{1,1}(I^d)|| \le c\, 2^{k(s-d)} \tag{2.23}$$

and

$$||f_k^\Gamma | B^0_{1,1}(I^d)|| \ge c\, k\, 2^{-kd}. \tag{2.24}$$

Combining (2.23) with (2.24), it follows that

$$g_n(id) \approx g_n^{\text{lin}}(id) \approx n^{-\frac{s}{d}}(1+\log n), \quad n \in \mathbb{N}.$$

The proof of (2.23) follows directly from Lemma A.8. To prove (2.24), let $l \in \mathbb{N}$ be the smallest natural number such that

$$\widehat{\psi}(\xi) \neq 0 \quad \text{for} \quad |\xi| \le 2^{-l}$$

and write for $k \ge 2l$

$$\begin{aligned} ||f_k^\Gamma | B^0_{1,1}(I^d)|| &\ge c\, ||f_k^\Gamma | B^0_{1,1}(\mathbb{R}^d)|| = c \sum_{j=0}^{\infty} ||(\varphi_j \widehat{f_k^\Gamma})^\vee | L_1(\mathbb{R}^d)|| \\ &\ge c \sum_{j=0}^{k-l-1} ||(\varphi_1(2^{-j}\xi) 2^{(-k-1)d} \widehat{\psi}(2^{-k-1}\xi) e^{-i\xi\cdot x^\Gamma})^\vee | L_1(\mathbb{R}^d)|| \\ &= c\, 2^{(-k-1)d} \sum_{j=0}^{k-l-1} ||(\varphi_1(2^{-j}\xi)\widehat{\psi}(2^{-k-1}\xi))^\vee | L_1(\mathbb{R}^d)|| \\ &= c \sum_{j=0}^{k-l-1} ||(\varphi_1(2^{-j+k+1}\xi)\widehat{\psi}(\xi))^\vee (2^{k+1}x) | L_1(\mathbb{R}^d)|| \\ &= 2^{(-k-1)d} \sum_{j=0}^{k-l-1} ||(\varphi_1(2^{-j+k+1}\xi)\widehat{\psi}(\xi))^\vee (x) | L_1(\mathbb{R}^d)||. \end{aligned} \tag{2.25}$$

To estimate each of the summands from below, we consider the function

$$(\varphi_1(2^{-j+k+1}\cdot))^\vee = (\varphi_1(2^{-j+k+1}\cdot)\cdot \widehat{\psi} \cdot \frac{1}{\widehat{\psi}} \cdot \varphi_0(2^l\cdot))^\vee$$

and use Young's inequality to estimate its L_1-norm.

$$\begin{aligned} ||\varphi_1^\vee | L_1(\mathbb{R}^d)|| &= ||(\varphi_1(2^{-j+k+1}\cdot))^\vee | L_1(\mathbb{R}^d)|| \\ &\le ||(\varphi_1(2^{-j+k+1}\cdot)\cdot\widehat{\psi})^\vee | L_1(\mathbb{R}^d)|| \cdot ||(\frac{\varphi_0(2^l\cdot)}{\widehat{\psi}})^\vee | L_1(\mathbb{R}^d)||. \end{aligned} \tag{2.26}$$

Now, (2.24) is a combination of (2.25) and (2.26).

2.3 The case $s_2 < 0$

As the last case, we consider the situation $s_2 < 0$.

Theorem 2.8. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let*

$$id : G_1(\Omega) = A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow G_2(\Omega) = A^{s_2}_{p_2q_2}(\Omega)$$

with $1 \le p_1, p_2, q_1, q_2 \le \infty$ (with $p_1, p_2 < \infty$ in the F-case) and

$$s_1 > \frac{d}{p_1}, \qquad s_2 < 0.$$

If $p_1 \ge p_2$, then

$$g_n(id) \approx g_n^{\mathrm{lin}}(id) \approx n^{-\frac{s_1}{d}}. \tag{2.27}$$

If $p_1 < p_2$ and $s_2 > \frac{d}{p_2} - \frac{d}{p_1}$, then

$$g_n(id) \approx g_n^{\mathrm{lin}}(id) \approx n^{-\frac{s_1}{d}+\frac{s_2}{d}+\frac{1}{p_1}-\frac{1}{p_2}}. \tag{2.28}$$

If $p_1 < p_2$ and $\frac{d}{p_2} - \frac{d}{p_1} > s_2$, then

$$g_n(id) \approx g_n^{\mathrm{lin}}(id) \approx n^{-\frac{s_1}{d}}. \tag{2.29}$$

Proof. Step 1. In this step, we prove two estimates from below. First, using the method from the proof of Theorem 2.4, we obtain

$$g_n^{\mathrm{lin}}(id) \gtrsim g_n(id) \gtrsim n^{-\frac{s_1-s_2}{d}+\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}$$

exactly as in the case $s_2 > 0$. To prove the second estimate from below, namely

$$g_n^{\mathrm{lin}}(id) \gtrsim g_n(id) \gtrsim n^{-\frac{s_1}{d}}, \tag{2.30}$$

we proceed as follows. We rely on atomic decomposition of $A^{s_1}_{p_1q_1}(\mathbb{R}^d)$ spaces as described in [18, Chapter 1.5]. For every set $\Gamma \subset \Omega$ with $|\Gamma| = 2^{jd}$ we construct a function

$$\psi_j(x) = \sum_{m=1}^{M_j} \lambda_{jm} a_{jm}(x), \quad x \in \mathbb{R}^d,$$

where $M_j \approx 2^{jd}$, $\lambda_{jm} = 2^{-j\frac{d}{p_1}}$ for $m = 1, \dots, M_j$ and a_{jm} are positive atoms in the sense of [18, Definition 1.15]. As $s_1 > 0$, no moment conditions are needed. We suppose that $\operatorname{supp} a_{jm} \cap \Gamma = \emptyset$ and $\operatorname{supp} a_{jm} \subset \Omega$. Altogether, we get

$$||\psi_j|A^{s_1}_{p_1q_1}(\Omega)|| \le ||\psi_j|A^{s_1}_{p_1q_1}(\mathbb{R}^d)|| \lesssim 1$$

and

$$||\psi_j|L_1(\Omega)|| = \int_{I^d} \psi_j(x)\mathrm{d}x \approx \sum_{m=1}^{M_j} \lambda_{jm}||a_{jm}(x)|L_1(\mathbb{R}^d)|| \approx 2^{jd} \cdot 2^{-j\frac{d}{p_1}} \cdot 2^{-jd} \cdot 2^{-j(s-\frac{d}{p_1})} = 2^{-js_1}.$$

Finally, we choose a non-negative function $\varrho \in S(\mathbb{R}^d)$ such that the mapping

$$f \to \int_\Omega \varrho(x) f(x) \mathrm{d}x$$

yields a linear bounded functional on $A^{s_2}_{p_2 q_2}(\Omega)$, supp $\varrho \subset \Omega$ and $\int \varrho(x)\psi_j(x)\mathrm{d}x \gtrsim \int \psi_j(x)\mathrm{d}x$. This leads to

$$2^{-js_1} \approx ||\psi_j|L_1(\Omega)|| \lesssim \int_\Omega \varrho(x)\psi_j(x)\mathrm{d}x \lesssim ||\psi_j|A^{s_2}_{p_2q_2}(\Omega)||.$$

Hence, (2.30) is proved and it implies all estimates from below included in the theorem.
Step 2.
If $p_1 \geq p_2$ we use the following chain of embeddings

$$A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow L_{p_1}(\Omega) \hookrightarrow A^{s_2}_{p_2q_2}(\Omega) \tag{2.31}$$

and obtain

$$g^{\mathrm{lin}}_n(id) \leq g^{\mathrm{lin}}_n(id' : A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow L_{p_1}(\Omega)) \cdot ||id'' : L_{p_1}(\Omega) \hookrightarrow A^{s_2}_{p_2q_2}(\Omega)|| \lesssim n^{-\frac{s_1}{d}}. \tag{2.32}$$

If $p_1 < p_2$ and $0 > \dfrac{d}{p_2} - \dfrac{d}{p_1} > s_2$, then (2.31) holds true as well and, consequently, also (2.32) remains true.
If $p_1 < p_2$ and $0 > s_2 > \dfrac{d}{p_2} - \dfrac{d}{p_1}$, we define $r > 0$ by $\frac{1}{r} := -\frac{s_2}{d} + \frac{1}{p_2}$. It follows that $p_1 < r < p_2$. Using the embeddings

$$A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow L_r(\Omega) \hookrightarrow A^{s_2}_{p_2p_2}(\Omega) \tag{2.33}$$

we get

$$\begin{aligned} g^{\mathrm{lin}}_n(id) &\leq g^{\mathrm{lin}}_n(id' : A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow L_r(\Omega)) \cdot ||id'' : L_r(\Omega) \hookrightarrow A^{s_2}_{p_2p_2}(\Omega)|| \\ &\lesssim n^{-\frac{s_1}{d}+\frac{1}{p_1}-\frac{1}{r}} = n^{-\frac{s_1-s_2}{d}+\frac{1}{p_1}-\frac{1}{p_2}}. \end{aligned}$$

This proves the upper estimate in (2.28) if $p_2 = q_2$. The general case follows then by interpolation, similar to the proof of Proposition 2.3. □

2.4 Comparison with approximation numbers

In this closing part we wish to compare the sampling numbers of

$$id : B^{s_1}_{p_1q_1}(\Omega) \to B^{s_2}_{p_2q_2}(\Omega) \tag{2.34}$$

for $\Omega = (0, 1)^d$ with corresponding approximation numbers. Let us first recall their definition.

Definition 2.9. Let A, B be Banach spaces and let T be a compact linear operator from A to B. Then for all $n \in \mathbb{N}$ the kth approximation number $a_n(T)$ of T is defined by

$$a_n(T) = \inf\{||T - L|| : L \in L(A, B), \mathrm{rank}\ L \leq n\}, \tag{2.35}$$

where rank L is the dimension of the range of L.

Obviously, $a_n(id)$ represents the approximation of id by linear operators with the dimension of the range smaller or equal to n, in general not restricted to involve only function values. Hence

$$a_n(id) \le g_n^{\mathrm{lin}}(id), \qquad n \in \mathbb{N}.$$

We again assume that

$$s_1 > \frac{d}{p_1}, \qquad s_1 - s_2 > d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+, \tag{2.36}$$

which ensures that (2.34) is compact and its sampling numbers are well defined. The approximation numbers of (2.34) are well known, we refer to [2], [14], [4] and [18] for details. We wish to discuss, when the equivalence $a_n(id) \approx g_n^{\mathrm{lin}}(id)$ holds true. The comparison of our results with the known results for $a_n(id)$ shows, that this is the case if either

1. $s_2 > 0$ and $1 \le p_2 \le p_1 \le \infty$ or
2. $s_2 > 0$ and $1 \le p_1 \le p_2 \le 2$ or $2 \le p_1 \le p_2 \le \infty$ or
3. $0 > s_2 > d\left(\frac{1}{p_2} - \frac{1}{p_1}\right)$ and $1 \le p_1 \le p_2 \le 2$ or $2 \le p_1 \le p_2 \le \infty$.

A Function spaces on domains

A.1 Function spaces on $\mathbb{R}^d$

We use standard notation: $\mathbb{N}$ denotes the collection of all natural numbers, $\mathbb{R}^d$ is the Euclidean d-dimensional space, where $d \in \mathbb{N}$, and $\mathbb{C}$ stands for the complex plane. Let $S(\mathbb{R}^d)$ be the Schwartz space of all complex-valued rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^d$ and let $S'(\mathbb{R}^d)$ be its dual - the space of all tempered distributions.

Furthermore, $L_p(\mathbb{R}^d)$ with $1 \le p \le \infty$, are the Lebesgue spaces endowed with the norm

$$||f|L_p(\mathbb{R}^d)|| = \begin{cases} \left(\int_{\mathbb{R}^d} |f(x)|^p \mathrm{d}x\right)^{1/p}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} |f(x)|, & p = \infty. \end{cases}$$

For $\psi \in S(\mathbb{R}^d)$ we denote by

$$\widehat{\psi}(\xi) = (F\psi)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i<x,\xi>} \psi(x)\mathrm{d}x, \quad x \in \mathbb{R}^d,$$

its Fourier transform and by $\psi^\vee$ or $F^{-1}\psi$ its inverse Fourier transform.

We give a Fourier-analytic definition of Besov and Triebel-Lizorkin spaces, which relies on the so-called *dyadic resolution of unity*. Let $\varphi \in S(\mathbb{R}^d)$ with

$$\varphi(x) = 1 \quad \text{if} \quad |x| \le 1 \quad \text{and} \quad \varphi(x) = 0 \quad \text{if} \quad |x| \ge \frac{3}{2}. \tag{A.1}$$

We put $\varphi_0 = \varphi$ and $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$ for $j \in \mathbb{N}$ and $x \in \mathbb{R}^d$. This leads to identity

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1, \qquad x \in \mathbb{R}^d.$$

Definition A.1. (i) Let $s \in \mathbb{R}, 1 \le p, q \le \infty$. Then $B^s_{pq}(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|B^s_{pq}(\mathbb{R}^d)|| = \left(\sum_{j=0}^{\infty} 2^{jsq}||(\varphi_j \widehat{f})^\vee|L_p(\mathbb{R}^d)||^q\right)^{1/q} < \infty \tag{A.2}$$

(with the usual modification for $q = \infty$).
(ii) Let $s \in \mathbb{R}, 1 \le p < \infty, 1 \le q \le \infty$. Then $F^s_{pq}(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|F^s_{pq}(\mathbb{R}^d)|| = \left|\left|\left(\sum_{j=0}^{\infty} 2^{jsq}|(\varphi_j \widehat{f})^\vee(\cdot)|^q\right)^{1/q}|L_p(\mathbb{R}^d)\right|\right| < \infty \tag{A.3}$$

(with the usual modification for $q = \infty$).

Remark A.2. These spaces have a long history. In this context we recommend [13], [15], [16] and [18] as standard references. We point out that the spaces $B^s_{pq}(\mathbb{R}^d)$ and $F^s_{pq}(\mathbb{R}^d)$ are independent of the choice of ψ in the sense of equivalent norms. Special cases of these two scales include Lebesgue spaces, Sobolev spaces, Hölder-Zygmund spaces and many other important function spaces. We omit any detailed discussion.

A.2 Function spaces on domains

Let Ω be a bounded domain. Let $D(\Omega) = C_0^\infty(\Omega)$ be the collection of all complex-valued infinitely-differentiable functions with compact support in Ω and let $D'(\Omega)$ be its dual - the space of all complex-valued distributions on Ω.
Let $g \in S'(\mathbb{R}^d)$. Then we denote by $g|\Omega$ its restriction to Ω:

$$(g|\Omega) \in D'(\Omega), \qquad (g|\Omega)(\psi) = g(\psi) \quad \text{for} \quad \psi \in D(\Omega).$$

Definition A.3. Let Ω be a bounded domain in $\mathbb{R}^d$. Let $s \in \mathbb{R}$, $1 \le p, q \le \infty$ with $p < \infty$ in the F-case. Let A^s_{pq} stand either for B^s_{pq} or F^s_{pq}. Then

$$A^s_{pq}(\Omega) = \{f \in D'(\Omega) : \exists g \in A^s_{pq}(\mathbb{R}^d) : g|\Omega = f\}$$

and

$$||f|A^s_{pq}(\Omega)|| = \inf ||g|A^s_{pq}(\mathbb{R}^d)||,$$

where the infimum is taken over all $g \in A^s_{pq}(\mathbb{R}^d)$ such that $g|\Omega = f$.

We collect some important properties of spaces $A^s_{pq}(\Omega)$ which will be useful later on. For this reason, we have to restrict to bounded Lipschitz domains. We use a standard definition of the notion of Lipschitz domain, the reader may consult for example [18, Chapter 1.11.4].
Let $x \in \mathbb{R}^d, h \in \mathbb{R}^d$ and $M \in \mathbb{N}$. Then

$$(\Delta_h^{M+1} f)(x) = (\Delta_h^1 \Delta_h^M f)(x) \quad \text{with} \quad (\Delta_h^1 f)(x) = f(x+h) - f(x),$$

are the usual differences in $\mathbb{R}^d$. For $x \in \Omega$ we consider the differences with respect to Ω:

$$(\Delta^M_{h,\Omega} f)(x) = \begin{cases} (\Delta_h^M f)(x) & \text{if } x + lh \in \Omega \text{ for } l = 0, \dots, M, \\ 0 & \text{otherwise.} \end{cases}$$

We also need to adapt the classical ball means of differences to bounded domains. Let $M \in \mathbb{N}, t > 0, x \in \Omega$. Then we define

$$V^M(x,t) = \{h \in \mathbb{R}^d : |h| < t, x + \tau h \in \Omega \text{ for } 0 < \tau \le M\}$$

and

$$d_t^{M,\Omega} f(x) = t^{-d} \int\limits_{V^M(x,t)} |(\Delta_h^M f)(x)| \mathrm{d}h.$$

We shall also use the simple relation (cf. [12, (4.10)])

$$(d_t^{M,\Omega} f(\tau \cdot))(x) = (d_{\tau t}^{M,\tau\Omega} f)(\tau x), \quad x \in \Omega, \quad 0 < \tau, t < \infty. \tag{A.4}$$

The following theorem connects the classical definition of Besov and Triebel-Lizorkin spaces using differences with Definition A.3. We refer to [8] and [18, 1.11.9] for details and references to this topic.

Theorem A.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let $1 \le p, q \le \infty$ and*

$$0 < s < M \in \mathbb{N}.$$

Then $B^s_{pq}(\Omega)$ is the collection of all $f \in L_p(\Omega)$ such that

$$||f|L_p(\Omega)|| + \left(\int_0^1 t^{-sq} ||d_t^{M,\Omega} f|L_p(\Omega)||^q \frac{\mathrm{d}t}{t} \right)^{1/q} < \infty \tag{A.5}$$

in the sense of equivalent norms (usual modification if $q = \infty$).

We present a modification of the preceding theorem, which suits better for our needs.
Let $M \in \mathbb{N}$. Let $\mathcal{P}^M(\mathbb{R}^d)$ be the space of all complex-valued polynomials of degree smaller than M and let $\mathcal{P}^M(\Omega)$ be its restriction to Ω. We denote

$$D_M = \dim \mathcal{P}^M(\mathbb{R}^d) = \dim \mathcal{P}^M(\Omega) = \binom{M+d-1}{d}.$$

We say, that $\{x_j\}_{j=1}^{D_M} \subset \mathbb{R}^d$ is a $M-$*regular set* if for every $\{y_j\}_{j=1}^{D_M} \in \mathbb{R}^{D_M}$ there exists (unique) $p \in \mathcal{P}^M(\mathbb{R}^d)$ such that $p(x_j) = y_j, j = 1, \dots, D_M$. In particular, if $p(x_j) = 0$ for $p \in \mathcal{P}^M(\mathbb{R}^d)$ and all $j = 1, 2, \dots, D_M$ then $p \equiv 0$. One may observe directly (or consult [11]) that the set

$$\{m \in \mathbb{Z}^d : 0 \le m_i \le M \text{ for } i = 1, 2, \dots, d \text{ and } \sum_{i=1}^d m_i \le M\}$$

and all its translations, dilations and rotations are $M-$regular.

Theorem A.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $M \in \mathbb{N}$ and let $\{x_j\}_{j=1}^{D_M}$ be a $M-$regular set in Ω.*
Let $1 \le p, q \le \infty$ and

$$\frac{d}{p} < s < M \in \mathbb{N}. \tag{A.6}$$

Then $B^s_{pq}(\Omega)$ is the collection of all $f \in L_p(\Omega)$ such that

$$\sum_{j=1}^{D_M} |f(x_j)| + \left(\int_0^1 t^{-sq} ||d_t^{M,\Omega} f | L_p(\Omega)||^q \frac{\mathrm{d}t}{t} \right)^{1/q} < \infty \tag{A.7}$$

in the sense of equivalent norms (usual modification if $q = \infty$).

Proof. According to (A.6), the following embedding is true:

$$B^s_{pq}(\Omega) \hookrightarrow C(\bar{\Omega})$$

and for every $x \in \Omega$

$$|f(x)| \leq ||f|C(\bar{\Omega})|| \lesssim ||f|B^s_{pq}(\Omega)||.$$

This shows that the left-hand side of (A.7) is (up to some constant) smaller than the left-hand side of (A.5).

We prove the reverse inequality be contradiction. We denote the left side of (A.7) by $||f|B^s_{pq}(\Omega)||'$. We suppose, that there is no $c > 0$ such that

$$||f|L_p(\Omega)|| \leq c\, ||f|B^s_{pq}(\Omega)||' \quad \text{for all} \quad f \in B^s_{pq}(\Omega).$$

Then there is a sequence $\{f_n\}_{n=1}^\infty \subset B^s_{pq}(\Omega)$ such that

$$||f_n|L_p(\Omega)|| = 1 \quad \text{and} \quad ||f_n|B^s_{pq}(\Omega)||' < \frac{1}{n}, \quad n \in \mathbb{N}. \tag{A.8}$$

This shows, that $\{f_n\}_{n=1}^\infty$ is bounded in $B^s_{pq}(\Omega)$ and hence precompact in $C(\bar{\Omega})$. We may therefore assume that

$$f_n \to f \text{ in } C(\bar{\Omega}).$$

From (A.8) it follows that

$$\sum_{j=1}^{D_M} |f(x_j)| = 0 \quad \text{and} \quad (d_t^{M,\Omega} f)(x) = 0, \text{ for a. e. } x \in \Omega. \tag{A.9}$$

The second part of (A.9) gives that $f \in \mathcal{P}^M(\Omega)$. Furthermore, the definition of $M-$regular sets and the first part of (A.9) implies that $f = 0$. This contradicts (A.8). □

This characterisation has a direct corollary.

Corollary A.6. *Under the assumptions of Theorem A.5,*

$$\inf_{g \in \mathcal{P}^M(\Omega)} ||f - g|B^s_{pq}(\Omega)|| \approx \left(\int_0^1 t^{-sq} ||d_t^{M,\Omega} f|L_p(\Omega)||^q \frac{\mathrm{d}t}{t} \right)^{1/q}.$$

Proof. Consider some M-regular set $\{x_j\}_{j=1}^{D_M}$ and $g \in \mathcal{P}^M(\Omega)$ such that

$$g(x_j) = f(x_j), \quad j = 1, \ldots, D_M.$$

Let us mention, that the polynomial g is uniquely determined and its definition combines the function values $f(x_1), \ldots, f(x_{D_M})$ in a linear way. The rest of the proof follows directly from Theorem A.5. □

We also recall the fact that the spaces $B^s_{pq}(\mathbb{R}^d)$ are *multiplication algebras* if $s > \frac{d}{p}$, c.f. [15, 2.8.3].

Lemma A.7. *Let* $1 \le p, q \le \infty$ *and* $s > \frac{d}{p}$. *Then*

$$||h_1 \cdot h_2|B^s_{pq}(\mathbb{R}^d)|| \le c\,||h_1|B^s_{pq}(\mathbb{R}^d)|| \cdot ||h_2|B^s_{pq}(\mathbb{R}^d)||,$$

where the constant c *does not depend on* h_1 *and* h_2.

Finally, we consider the dilation operator $T_k : f \to f(2^k \cdot),\ k \in \mathbb{N}$, and its behaviour on the scale of Besov spaces. For the proof, we refer to [3, 1.7] and [9, 2.3.1].

Lemma A.8. *Let* $s \ge 0$, $1 \le p, q \le \infty$ *and* $k \in \mathbb{N}$. *Then the operator* T_k *is bounded on* $B^s_{p,q}(\mathbb{R}^d)$ *and its norm is bounded by* $c\,2^{k(s-\frac{d}{p})}$ *if* $s > 0$ *and by* $c\,2^{-k\frac{d}{p}}(1+k)^{1/q}$ *if* $s = 0$. *The constant* c *does not depend on* $k \in \mathbb{N}$.

References

[1] L. Bergh, J. Löfström, Interpolation spaces, an introduction, Berlin, Springer, 1976.

[2] M. Sh. Birman, M. Z. Solomyak, Piecewise-polynomial approximations of the classes W^α_p, Mat. Sb., Nov. Ser. 73, 331-355; English transl.: Math. USSR, Sb 2 (1967), 295-317 (1968).

[3] G. Bourdaud, Sur les opérateurs pseudo-différentiels à coefficinets peu réguliers, Habilitation thesis, Université de Paris-Sud, Paris, 1983.

[4] A. M. Caetano, About approximation numbers in function spaces, J. Approx. Theory 94 (1998), 383-395.

[5] P. G. Ciarlet, The finite element method for elliptic problems, North-Holland, Amsterdam, 1978.

[6] S. Dahlke, E. Novak, W. Sickel, Optimal approximation of elliptic problems by linear and nonlinear mappings I, J. Complexity 22 (2006), 29-49.

[7] S. Dahlke, E. Novak, W. Sickel, Optimal approximation of elliptic problems by linear and nonlinear mappings II, J. Complexity 22 (2006), 549-603.

[8] S. Dispa, Intrinsic characterisation of Besov spaces on Lipschitz domains, Math. Nachr. 260 (2003), 21-33.

[9] D. E. Edmunds, H. Triebel, Function spaces, entropy numbers, differential operators, Cambridge Univ. Press, Cambridge, 1996.

[10] S. N. Kudryavtsev, The best accuracy of reconstruction of finitely smooth functions from their values at a given number of points, Izv. Math. 62 (1) (1998), 19-53.

[11] W. Light, W. Cheney, A Course in Approximation Theory, Brooks/Cole, Pacific Grove, 1999.

[12] E. Novak, H. Triebel, Function Spaces in Lipschitz Domains and Optimal Rates of Convergence for Sampling, Constr. Approx. 23 (2006), 325-350.

[13] Peetre, J., New thoughts on Besov spaces, Duke Univ. Math. Series, Durham, Univ., 1976.

[14] V. M. Tikhomirov, Analysis II, Convex Analysis and Approximation Theory, Springer, 1990.

[15] H. Triebel, Theory of function spaces, Birkhäuser, Basel, 1983.

[16] H. Triebel, Theory of function spaces II, Birkhäuser, Basel, 1992.

[17] H. Triebel, Function spaces in Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers, Rev. Mat. Complut. 15 (2002), 475-524.

[18] H. Triebel, Theory of function spaces III, Birkhäuser, Basel, 2006.

[19] H. Triebel, Sampling numbers and embedding constants, Trudy Mat. Inst. Steklov 248 (2005), 275-284.

On dilation operators and sampling numbers

Jan Vybíral

Abstract

We consider the dilation operators $T_k : f \to f(2^k\cdot)$ in the frame of Besov spaces $B^s_{pq}(\mathbb{R}^d)$ with $1 \le p, q \le \infty$. If $s > 0$, T_k is a bounded linear operator from $B^s_{pq}(\mathbb{R}^d)$ into itself and there are optimal bounds for its norm, see [4, 2.3.1]. We study the situation in the case $s = 0$, an open problem mentioned also in [4]. It turns out, that new effects based on Littlewood-Paley theory appear.

In the second part of the paper, we apply these results to the study of the so-called *sampling numbers* of the embedding

$$id : B^{s_1}_{pq_1}(\Omega) \to B^0_{pq_2}(\Omega),$$

where $\Omega = (0,1)^d$. It was observed already in [13] that the estimates from above for the norm of the dilation operator have their immediate counterpart in the estimates from above for the sampling numbers. In this paper we show that even in the limiting case $s_2 = 0$ (left open so far), this general method supplies optimal results.

AMS Classification: 41A25, 41A46, 46E35

Keywords and phrases: Linear and nonlinear approximation methods; Besov spaces; Dilation operators; Sampling operators

1 Introduction

This paper is divided into two parts. In the first part, we consider the dilation operators

$$T_k : f \to f(2^k\cdot), \quad k \in \mathbb{N},$$

in the framework of Besov spaces $B^s_{pq}(\mathbb{R}^d)$. Their behaviour is well known if $1 \le p, q \le \infty$ and $s > 0$, cf. [4, 2.3.1]. As mentioned there, the case $s = 0$ remained open. Some partial results can be found in [1]. For $1 \le p, q \le \infty$ we supply the final answer to this problem showing that

$$||T_k|\mathcal{L}(B^0_{pq}(\mathbb{R}^d))|| \approx 2^{-k\frac{d}{p}} \cdot \begin{cases} k^{\frac{1}{q}-\frac{1}{p}}, & \text{if } 1 < p < \infty \text{ and } p \ge \max(q,2), \\ k^{\frac{1}{q}-\frac{1}{2}}, & \text{if } 1 < p < \infty \text{ and } 2 \ge \max(p,q), \\ 1, & \text{if } 1 < p < \infty \text{ and } q \ge \max(p,2), \\ k^{\frac{1}{q}}, & \text{if } p = 1 \text{ or } p = \infty, \end{cases} \tag{1.1}$$

where $||T_k|\mathcal{L}(B^0_{pq}(\mathbb{R}^d))||$ denotes the norm of the operator T_k from $B^0_{pq}(\mathbb{R}^d)$ into itself. One observes, that for $1 < p < \infty$ the number 2 plays an exceptional role. This effect has its origin in the Littlewood-Paley decomposition theorem.

The second part of the paper deals with applications to estimates of sampling numbers. Let us briefly sketch this approach.

Let $\Omega = (0,1)^d$ and let $B^s_{pq}(\Omega)$ denote the Besov spaces on Ω, see Definition 2.7 for details. We try to approximate $f \in B^{s_1}_{p_1q_1}(\Omega)$ in the norm of another Besov space, say $B^{s_2}_{p_2q_2}(\Omega)$, by a linear sampling method

$$S_n f = \sum_{j=1}^n f(x_j)h_j, \tag{1.2}$$

where $h_j \in B^{s_2}_{p_2q_2}(\Omega)$ and $x_j \in \Omega$. To give a meaning to the pointwise evaluation in (1.2), we suppose that

$$s_1 > \frac{d}{p_1}.$$

Then the embedding $B^{s_1}_{p_1q_1}(\Omega) \hookrightarrow C(\bar{\Omega})$ holds true and the pointwise evaluation represents a bounded operator. Second, we always assume that the embedding $B^{s_1}_{p_1q_1}(\Omega) \hookrightarrow B^{s_2}_{p_2q_2}(\Omega)$ is compact. This is true if, and only if,

$$s_1 - s_2 > d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+.$$

Concerning the parameters p_1, p_2, q_1, q_2 we always assume that they belong to $[1,\infty]$.

We measure the worst case error of $S_n f$ on the unit ball of $B^{s_1}_{p_1q_1}(\Omega)$, given by

$$\sup\{||f - S_n f|B^{s_2}_{p_2q_2}(\Omega)|| : ||f|B^{s_1}_{p_1q_1}(\Omega)|| \le 1\}. \tag{1.3}$$

The same worst case error may be considered also for nonlinear sampling methods

$$S_n f = \varphi(f(x_1), \ldots, f(x_n)), \tag{1.4}$$

where $\varphi : \mathbb{C}^n \to B^{s_2}_{p_2q_2}(\Omega)$ is an arbitrary mapping. We shall discuss the decay of (1.3) for linear (1.2) and nonlinear (1.4) sampling methods.

The case $s_2 \ne 0$ was considered in [13], but the interesting limiting case $s_2 = 0$ was left open so far. It is the aim of this paper to close this gap. It was already pointed out in [13], see especially (2.6) in [13] for details, that the estimates from above for the dilation operators T_k on the target space $B^{s_2}_{p_2q_2}(\mathbb{R}^d)$ have their direct counterparts in estimates from above for the decay of sampling numbers. Using this method, which will not be repeated here, a direct application of (1.1) supplies the estimates

$$g_n^{\mathrm{lin}}(id) \lesssim n^{-\frac{s}{d}} \cdot \begin{cases} (\log n)^{\frac{1}{q_2}-\frac{1}{p}}, & \text{if } 1<p<\infty \text{ and } p \geq \max(q_2,2), \\ (\log n)^{\frac{1}{q_2}-\frac{1}{2}}, & \text{if } 1<p<\infty \text{ and } 2 \geq \max(p,q_2), \\ 1, & \text{if } 1<p<\infty \text{ and } q_2 \geq \max(p,2), \\ (\log n)^{\frac{1}{q_2}}, & \text{if } p=1 \text{ or } p=\infty, \end{cases} \tag{1.5}$$

where $g_n^{\mathrm{lin}}(id)$ with $2 \leq n \in \mathbb{N}$ are the linear sampling numbers of the embedding

$$id : B^s_{pq_1}(\Omega) \to B^0_{pq_2}(\Omega), \qquad s > \frac{d}{p}.$$

Surprisingly, all estimates in (1.5) are sharp.

All the unimportant constants are denoted by the letter c, whose meaning may differ from one occurrence to another. If $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are two sequences of positive real numbers, we write $a_n \lesssim b_n$ if, and only if, there is a positive real number $c > 0$ such that $a_n \leq c\, b_n, n \in \mathbb{N}$. Furthermore, $a_n \approx b_n$ means that $a_n \lesssim b_n$ and simultaneously $b_n \lesssim a_n$.

We also discuss the case when $p_1 \neq p_2$ and state some open problems connected to this question.

I would like to thank Winfried Sickel and Hans Triebel for many valuable discussions and comments on the topic.

2 Notation and definitions

2.1 Besov spaces on $\mathbb{R}^d$

We use standard notation: $\mathbb{N}$ denotes the collection of all natural numbers, $\mathbb{Z}$ is the set of all integer numbers, $\mathbb{R}^d$ is Euclidean d-dimensional space, where $d \in \mathbb{N}$, and $\mathbb{C}$ stands for the complex plane. Let $S(\mathbb{R}^d)$ be the Schwartz space of all complex-valued rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^d$ and let $S'(\mathbb{R}^d)$ be its dual - the space of all tempered distributions.

Furthermore, $L_p(\mathbb{R}^d)$ with $1 \leq p \leq \infty$, are the standard Lebesgue spaces endowed with the norm

$$||f|L_p(\mathbb{R}^d)|| = \begin{cases} \left(\int_{\mathbb{R}^d} |f(x)|^p \mathrm{d}x\right)^{1/p}, & 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} |f(x)|, & p = \infty. \end{cases}$$

For $\psi \in S(\mathbb{R}^d)$ we denote by

$$\widehat{\psi}(\xi) = (F\psi)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i<x,\xi>} \psi(x) \mathrm{d}x, \quad x \in \mathbb{R}^d, \tag{2.1}$$

its Fourier transform and by $\psi^\vee$ or $F^{-1}\psi$ its inverse Fourier transform. With the aid of duality, they are extended to $S'(\mathbb{R}^d)$.

We give a Fourier-analytic definition of the Besov spaces, which relies on the so-called *dyadic resolution of unity.* Let $\varphi \in S(\mathbb{R}^d)$ with

$$\varphi(x) = 1 \quad \text{if} \quad |x| \leq 1 \quad \text{and} \quad \varphi(x) = 0 \quad \text{if} \quad |x| \geq \frac{3}{2}. \tag{2.2}$$

We put $\varphi_0 = \varphi$ and $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$ for $j \in \mathbb{N}$ and $x \in \mathbb{R}^d$. This leads to the identity

$$\sum_{j=0}^\infty \varphi_j(x) = 1, \qquad x \in \mathbb{R}^d.$$

Definition 2.1. Let $s \in \mathbb{R}, 1 \le p, q \le \infty$. Then $B^s_{pq}(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$||f|B^s_{pq}(\mathbb{R}^d)|| = \left(\sum_{j=0}^{\infty} 2^{jsq} ||(\varphi_j \widehat{f})^\vee | L_p(\mathbb{R}^d)||^q \right)^{1/q} < \infty \tag{2.3}$$

(with the usual modification for $q = \infty$).

Remark 2.2. These spaces have a long history. In this context we recommend [7], [10], [11] and [12] as standard references. Let us mention that the spaces $B^s_{pq}(\mathbb{R}^d)$ are independent of the choice of φ in the sense of equivalent norms.

2.2 Local means and atomic decompositions

We use the characterisation of Besov spaces by *local means*. We refer to [2], [3] and [12] for further details. Let us sketch this approach.

Let $B = \{y \in \mathbb{R}^d : |y| < 1\}$ be the unit ball in $\mathbb{R}^d$ and let κ be a C^∞ function in $\mathbb{R}^d$ with $\operatorname{supp} \kappa \subset B$,

$$\kappa^\vee(\xi) \neq 0 \quad \text{if} \quad 0 < |\xi| < \epsilon \qquad \text{and} \qquad (D^\alpha \kappa^\vee)(0) = 0 \quad \text{if} \quad |\alpha| \le s.$$

for some $\epsilon > 0$. Furthermore, let κ_0 be a second C^∞ function with $\operatorname{supp} \kappa_0 \subset B$ and $\kappa_0^\vee(0) \neq 0$. Then

$$\big|\big|f|B^s_{pq}(\mathbb{R}^d)\big|\big| \approx ||\mathcal{K}_0(1,f)|L_p(\mathbb{R}^n)|| + \left(\sum_{j=1}^{\infty} 2^{jsq} \big|\big|\mathcal{K}(2^{-j}, f)|L_p(\mathbb{R}^d)\big|\big|^q \right)^{1/q}, \quad f \in S'(\mathbb{R}^d), \tag{2.4}$$

where

$$\mathcal{K}(t,f)(x) = \int_{\mathbb{R}^d} \kappa(y) f(x+ty) dy = t^{-d} \int_{\mathbb{R}^d} \kappa\left(\frac{y-x}{t}\right) f(y) dy, \quad x \in \mathbb{R}^d,$$

appropriately interpreted for $f \in S'(\mathbb{R}^d)$. The meaning of $\mathcal{K}_0(1,f)$ is defined in the same way with κ_0 instead of κ.

We shall need only one part of (2.4), namely the estimates from below of $||f|B^s_{pq}(\mathbb{R}^d)||$. In that case some of the assumptions may be omitted. The inspection of the proof of (2.4), see [8], shows that if κ is a C^∞ function in $\mathbb{R}^d$ with $\operatorname{supp} \kappa \subset B$ and $\kappa^\vee(0) = 0$, then

$$\big|\big|f|B^0_{pq}(\mathbb{R}^d)\big|\big| \gtrsim \left(\sum_{j=1}^{\infty} \big|\big|\mathcal{K}(2^{-j}, f)|L_p(\mathbb{R}^d)\big|\big|^q \right)^{1/q}, \tag{2.5}$$

Secondly we rely on *atomic decompositions.* We refer again to [12] for details.

Recall that $\mathbb{Z}^d$ stands for the lattice of all points in $\mathbb{R}^d$ with integer-valued components. Furthermore, $Q_{\nu m}$ denotes the closed cube in $\mathbb{R}^d$ with sides parallel to the axes of coordinates, centred at $2^{-\nu} m$, and with side length $2^{-\nu}$ where $m \in \mathbb{Z}^d$ and $\nu \in \mathbb{N}_0$. If Q is a cube in $\mathbb{R}^d$ and $c > 0$ then cQ is a cube in $\mathbb{R}^d$ concentric with Q and with side length c times of the side length of Q.

Definition 2.3. Let $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$, $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^d$ and $c \ge 1$. A K-times differentiable function $a(x)$ is called an (K, L) atom centred on $Q_{\nu m}$ if

$$\operatorname{supp} a \subset c Q_{\nu m}, \tag{2.6}$$

$$|D^\alpha a(x)| \le 2^{|\alpha|\nu}, \quad \text{for} \quad |\alpha| \le K \tag{2.7}$$

and

$$\int_{\mathbb{R}^d} x^\beta a(x) dx = 0, \quad \text{for} \quad |\beta| < L \quad \text{and} \quad \nu \ge 1. \tag{2.8}$$

Remark 2.4. We add a few comments on Definition 2.3. The number K denotes the smoothness of the atom (see (2.7)), L gives the number of vanishing moments, see (2.8), and the pair (ν, m) denotes the location of $\operatorname{supp} a$ (see (2.6)). Let us note that if $\nu = 0$ or $L = 0$, the condition (2.8) is empty and no moment conditions are required.

Theorem 2.5. *Let $1 \le p, q \le \infty$ and $s \in \mathbb{R}$. Let $K \in \mathbb{N}_0, L \in \mathbb{N}_0$ with*

$$K > s \quad \text{and} \quad L > -s \tag{2.9}$$

be fixed. Let $a_{\nu m}$ be (K, L) atoms centred on $Q_{\nu m}$ and let

$$\lambda = \{\lambda_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^d\}$$

be a sequence of complex numbers with

$$||\lambda|b^s_{pq}|| = \left(\sum_{\nu=0}^{\infty} 2^{\nu(s-\frac{d}{p})q}\Big(\sum_{m\in\mathbb{Z}^d} |\lambda_{\nu m}|^p\Big)^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty \tag{2.10}$$

(appropriately modified if $p = \infty$ and/or $q = \infty$).
Then the series

$$\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^d} \lambda_{\nu m} a_{\nu m},$$

converges in $S'(\mathbb{R}^d)$ to a distribution $f \in B^s_{pq}(\mathbb{R}^d)$ and

$$||f|B^s_{pq}(\mathbb{R}^d)|| \lesssim ||\lambda|b^s_{pq}||. \tag{2.11}$$

Remark 2.6. We denote by $\chi_{\nu m}$ the characteristic function of $Q_{\nu m}$. Then

$$||\lambda|b^s_{pq}|| = \left(\sum_{\nu=0}^{\infty} 2^{sq}\Big|\Big|\sum_{m\in\mathbb{Z}^d} \lambda_{\nu m}\chi_{\nu m}|L_p(\mathbb{R}^d)\Big|\Big|^q\right)^{\frac{1}{q}},$$

again appropriately modified if $q = \infty$.

2.3 Besov spaces on domains

Let Ω be a bounded domain. Let $D(\Omega) = C_0^\infty(\Omega)$ be the collection of all complex-valued infinitely-differentiable functions with compact support in Ω and let $D'(\Omega)$ be its dual - the space of all complex-valued distributions on Ω.
Let $g \in S'(\mathbb{R}^d)$. Then we denote by $g|\Omega$ its restriction to Ω:

$$(g|\Omega) \in D'(\Omega), \qquad (g|\Omega)(\psi) = g(\psi) \quad \text{for} \quad \psi \in D(\Omega).$$

Definition 2.7. Let Ω be a bounded domain in $\mathbb{R}^d$. Let $s \in \mathbb{R}$, $1 \le p, q \le \infty$. Then

$$B^s_{pq}(\Omega) = \{f \in D'(\Omega) : \exists g \in B^s_{pq}(\mathbb{R}^d) : g|\Omega = f\}$$

and

$$||f|B^s_{pq}(\Omega)|| = \inf ||g|B^s_{pq}(\mathbb{R}^d)||,$$

where the infimum is taken over all $g \in B^s_{pq}(\mathbb{R}^d)$ such that $g|\Omega = f$.

3 Dilation operators

Let $s \in \mathbb{R}$, $1 \le p, q \le \infty$ and $k \in \mathbb{N}$. Then the dyadic dilation operator

$$(T_k f)(x) = f(2^k x), \quad x \in \mathbb{R}^d, \tag{3.1}$$

is a bounded operator from $B^s_{p,q}(\mathbb{R}^d)$ into itself. Let us mention, that (3.1) has to be understood in the distributional sense. In this section we study the dependence of the norm of T_k on k.

First, we recall known results.

Lemma 3.1. *Let $s \ge 0$, $1 \le p, q \le \infty$ and $k \in \mathbb{N}$. Then the operator T_k is bounded on $B^s_{p,q}(\mathbb{R}^d)$ and its norm is bounded by $c\,2^{k(s-\frac{d}{p})}$ if $s > 0$ and by $c\,2^{-k\frac{d}{p}} k^{1/q}$ if $s = 0$. The constant c does not depend on $k \in \mathbb{N}$.*

For the proof, we refer to [1, 1.7] and [4, 2.3.1]. If $s > 0$, the estimate given by Lemma 3.1 is sharp (cf. [4]). But if $s = 0$, the result can be improved.

Proposition 3.2. *Let $1 < p < \infty$, $1 \le q \le \infty$, $k \in \mathbb{N}$ and let T_k be defined by (3.1). Then*

$$||T_k|\mathcal{L}(B^0_{pq}(\mathbb{R}^d))|| \le c\,2^{-k\frac{d}{p}} \cdot \begin{cases} k^{\frac{1}{q}-\frac{1}{p}}, & \text{if } p \ge \max(q,2), \\ k^{\frac{1}{q}-\frac{1}{2}}, & \text{if } 2 \ge \max(p,q), \\ 1, & \text{if } q \ge \max(p,2), \end{cases} \tag{3.2}$$

for some c which is independent of k.

Remark 3.3. The estimates covered by (3.2) may be summarised to

$$||T_k|\mathcal{L}(B^0_{pq}(\mathbb{R}^d))|| \le c\,2^{-k\frac{d}{p}} \cdot k^{\frac{1}{q}-\frac{1}{\max(p,q,2)}}.$$

Proof. Elementary calculation involving only (2.1) shows that

$$(\varphi_j(\xi) f(2^k\cdot)\widehat{\ }(\xi))^\vee(x) = 2^{-kd}(\varphi_j(\xi)\widehat{f}(2^{-k}\xi))^\vee(x) = (\varphi_j(2^k\xi)\widehat{f}(\xi))^\vee(2^k x). \tag{3.3}$$

From (2.3) with $f(2^k x)$ in place of $f(x)$ we obtain

$$\begin{aligned} ||f(2^k\cdot)|B^0_{p,q}(\mathbb{R}^d)|| &= \left(\sum_{j=0}^\infty ||(\varphi_j(2^k\cdot)\widehat{f})^\vee(2^k x)|L_p(\mathbb{R}^d)||^q\right)^{1/q} \\ &= 2^{-k\frac{d}{p}}\left(\sum_{j=0}^\infty ||(\varphi_j(2^k\cdot)\widehat{f})^\vee|L_p(\mathbb{R}^d)||^q\right)^{1/q}. \end{aligned} \tag{3.4}$$

If $j \ge k+1$, then $\varphi_j(2^k x) = \varphi_{j-k}(x)$. This gives

$$\begin{aligned} 2^{-k\frac{d}{p}}\left(\sum_{j=k+1}^\infty ||(\varphi_j(2^k\cdot)\widehat{f})^\vee|L_p(\mathbb{R}^d)||^q\right)^{1/q} &= 2^{-k\frac{d}{p}}\left(\sum_{j=1}^\infty ||(\varphi_j\widehat{f})^\vee|L_p(\mathbb{R}^d)||^q\right)^{1/q} \\ &\le 2^{-k\frac{d}{p}}||f|B^0_{p,q}(\mathbb{R}^d)||. \end{aligned} \tag{3.5}$$

If $j = 0$, we use (2.2) and Hausdorff-Young inequality

$$\begin{aligned} ||(\varphi_0(2^k\cdot)\widehat{f})^\vee|L_p(\mathbb{R}^d)|| &= ||(\varphi_0(2^k\cdot)\varphi_0\widehat{f})^\vee|L_p(\mathbb{R}^d)|| \\ &\approx ||\varphi_0(2^k\cdot)^\vee * (\varphi_0\widehat{f})^\vee|L_p(\mathbb{R}^d)|| \\ &\le ||\varphi_0(2^k\cdot)^\vee|L_1(\mathbb{R}^d)|| \cdot ||(\varphi_0\widehat{f})^\vee|L_p(\mathbb{R}^d)|| \\ &\le c\,||f|B^0_{p,q}(\mathbb{R}^d)||. \end{aligned} \tag{3.6}$$

In view of (3.4), (3.5) and (3.6), we have to prove that

$$\left(\sum_{j=1}^{k} ||(\varphi_j(2^k\cdot)\widehat{f})^\vee|L_p(\mathbb{R}^d)||^q\right)^{1/q} \leq c\,k^{\frac{1}{q}-\frac{1}{\max(p,q,2)}}||f|B^0_{p,q}(\mathbb{R}^d)|| \tag{3.7}$$

with the constant c independent of k and f.
To prove (3.7), denote $\alpha = \max(p,q,2)$. Using the Minkowski inequality and the Littlewood-Paley theorem one gets

$$\begin{aligned}\left(\sum_{j=1}^{k} ||(\varphi_j(2^k\cdot)\widehat{f})^\vee|L_p(\mathbb{R}^d)||^q\right)^{1/q} &\leq k^{\frac{1}{q}-\frac{1}{\alpha}}\left(\sum_{j=1}^{k} ||(\varphi_j(2^k\cdot)\widehat{f})^\vee|L_p(\mathbb{R}^d)||^\alpha\right)^{1/\alpha}\\ &\leq k^{\frac{1}{q}-\frac{1}{\alpha}}\left(\int_{\mathbb{R}^d}\left(\sum_{j=1}^{k}|(\varphi_j(2^k\cdot)\widehat{f})^\vee(\xi)|^\alpha\right)^{p/\alpha}d\xi\right)^{1/p}\\ &\leq k^{\frac{1}{q}-\frac{1}{\alpha}}\left(\int_{\mathbb{R}^d}\left(\sum_{j=1}^{k}|(\varphi_j(2^k\cdot)\widehat{f})^\vee(\xi)|^2\right)^{p/2}d\xi\right)^{1/p}\\ &\leq c\,k^{\frac{1}{q}-\frac{1}{\alpha}}||(\varphi_0\widehat{f})^\vee|L_p(\mathbb{R}^d)|| \leq c\,k^{\frac{1}{q}-\frac{1}{\alpha}}||f|B^0_{p,q}(\mathbb{R}^d)||.\end{aligned}$$

□

Next, we prove that the estimates are sharp.

Theorem 3.4. *Let $1 \leq p,q \leq \infty$, $k \in \mathbb{N}$ and let T_k be defined by* (3.1). *Then*

$$||T_k|\mathcal{L}(B^0_{pq}(\mathbb{R}^d))|| \approx 2^{-k\frac{d}{p}}\cdot\begin{cases}k^{\frac{1}{q}-\frac{1}{p}}, & \text{if } 1<p<\infty \text{ and } p\geq\max(q,2),\\ k^{\frac{1}{q}-\frac{1}{2}}, & \text{if } 1<p<\infty \text{ and } 2\geq\max(p,q),\\ 1, & \text{if } 1<p<\infty \text{ and } q\geq\max(p,2),\\ k^{\frac{1}{q}}, & \text{if } p=1 \text{ or } p=\infty,\end{cases} \tag{3.8}$$

where the constants of equivalence do not depend on k.

Remark 3.5. Let us mention, that at $p=1$, there is a jump in the exponent of k caused by the absence of the Littlewood-Paley assertion for $p=1$. At $p=\infty$, no such a jump appears.

Proof. In view of Lemma 3.1 and Proposition 3.2, we have to prove the estimates from below.
Step 1: $p = 1$.
Let $\psi \in S(\mathbb{R}^d)$ be a non-negative function with support in $\{x\in\mathbb{R}^d : |x| \leq 1/8\}$ and $\int_{\mathbb{R}^d}\psi(x)dx = 1$.
We show, that

$$||\psi(2^k\cdot)|B^0_{1,q}(\mathbb{R}^d)|| \gtrsim 2^{-kd}\cdot k^{\frac{1}{q}}, \quad k\in\mathbb{N}, \tag{3.9}$$

for $1 \leq q \leq \infty$.
We take a function $\kappa \in C^\infty(\mathbb{R})$ with

$$\operatorname{supp}\kappa \subset B = \{y\in\mathbb{R}^d : |y| < 1\}, \qquad \kappa^\vee(0) = 0,$$

$$\kappa(x) = 1 \text{ if } x \in M = \{z\in\mathbb{R}^d :\ |z-(1/2,0,\dots,0)| < 1/4\}$$

and

$$\kappa(x) \geq 0 \quad \text{if} \quad x_1 \geq 0.$$

Simple calculation shows that if $j = 1, 2, \ldots, k$ and $|x - (\frac{1}{2} \cdot \frac{1}{2^j}, 0, \ldots, 0)| < \frac{1}{2^j} \cdot \frac{1}{8}$ then

$$\operatorname{supp}_y \psi(2^k x + 2^{k-j} y) \subset M.$$

For these x we get

$$\mathcal{K}(2^{-j}, \psi(2^k \cdot))(x) = \int_{\mathbb{R}^d} \kappa(y)\psi(2^k x + 2^{k-j} y) dy = \int_{\mathbb{R}^d} \psi(2^k x + 2^{k-j} y) dy = 2^{(j-k)d}.$$

Hence,

$$||\mathcal{K}(2^{-j}, \psi(2^k \cdot))|L_1(\mathbb{R}^d)|| \gtrsim 2^{-jd} \cdot 2^{(j-k)d} = 2^{-kd}. \tag{3.10}$$

We insert (3.10) for $j = 1, 2, \ldots, k$ into (2.5). This completes the proof of (3.9).

Step 2: $p = \infty$.

We consider again a non-negative function $\psi \in S(\mathbb{R}^d)$ with $\operatorname{supp} \psi \subset \{x \in \mathbb{R}^d : |x| \leq 1/8\}$ and $\int_{\mathbb{R}^d} \psi(x) dx = 1$. Let

$$\psi_j(x) = \sum_{\substack{0 \leq l_i \leq 2^{j-\gamma} \\ i=1,2,\ldots,d}} \psi(x - (2^{2j} + l_1, l_2, \ldots, l_d)), \quad j \geq \gamma \tag{3.11}$$

and

$$f(x) = \sum_{j=\gamma}^{\infty} \psi_j(x), \qquad x \in \mathbb{R}^d, \tag{3.12}$$

where the constant $\gamma \in \mathbb{N}$ will be chosen later on depending only on d.

We observe, that (3.11) inserted into (3.12) represents an atomic decomposition of f (see Theorem 2.5 for details) and, consequently, f belongs to every space $B^0_{\infty,q}(\mathbb{R}^d), 1 \leq q \leq \infty$. We use again the local means to show that

$$||f(2^k \cdot)|B^0_{\infty,q}(\mathbb{R}^d)|| \geq c\, k^{\frac{1}{q}}, \tag{3.13}$$

with the constant c independent of k.

Namely, we choose κ as in Step 1, points

$$x_j = (2^{k-2j} - 2^{-j-1}, 0, \ldots, 0), \quad j = \gamma, \ldots, k - \gamma,$$

and show, that

$$\mathcal{K}(2^{-j}, \psi_{k-j}(2^k \cdot))(x_j) \geq 2^{-\gamma d}, \quad j = \gamma, \ldots, k - \gamma, \tag{3.14}$$

as well as

$$\mathcal{K}(2^{-j}, \psi_m(2^k \cdot))(x_j) = 0, \quad m \neq k - j. \tag{3.15}$$

From (3.14) and (3.15) it follows, that $||\mathcal{K}(2^{-j}, f(2^k \cdot))|L_\infty(\mathbb{R}^d)|| \geq 2^{-\gamma d}$, for all $j = \gamma, \ldots, k - \gamma$. Taking $q-$th power and summing up, we prove (3.13).

Let us first comment on (3.14).

$$\mathcal{K}(2^{-j}, \psi_{k-j}(2^k \cdot))(x_j) = \sum_{\substack{0 \leq l_i \leq 2^{k-j-\gamma} \\ i=1,2,\ldots,d}} \int_{\mathbb{R}^d} \kappa(y)\psi(2^k x_j + 2^{k-j} y - (2^{2(k-j)} + l_1, l_2, \ldots, l_d)) dy. \tag{3.16}$$

It is a matter of simple calculation and triangle inequality that, if $2^{-\gamma} d^{1/2} \leq \frac{1}{8}$, the following statement holds true: If the argument of ψ in (3.16) lies in the support of ψ, then $\kappa(y) = 1$.

Hence (3.16) is equal to

$$2^{(k-j-\gamma)d}\int_{\mathbb{R}^d}\psi(2^{k-j}y)dy = 2^{(k-j-\gamma)d}\cdot 2^{(j-k)d} = 2^{-\gamma d}.$$

To prove (3.15) we use an analog of (3.16)

$$\mathcal{K}(2^{-j},\psi_m(2^k\cdot))(x_j) = \sum_{\substack{0\le l_i\le 2^{m-\gamma}\\ i=1,2,\dots,d}}\int_{\mathbb{R}^d}\kappa(y)\psi(2^k x_j + 2^{k-j}y - (2^{2m}+l_1,l_2,\dots,l_d))dy.$$

It turns out that, for $m \neq k-j$ and any admissible l, there is no $y\in\mathbb{R}^d$ such that

$$|2^k x_{j,1} + 2^{k-j}y_1 - 2^{2m} + l_1| \le \frac{1}{8} \quad \text{and} \quad |y_1|\le 1.$$

Step 3. In this step, we shall prove the estimate

$$||T_k|\mathcal{L}(B^0_{pq}(\mathbb{R}^d))|| \gtrsim 2^{-\frac{kd}{p}}, \quad k\in\mathbb{N}, \tag{3.17}$$

for $1<p<\infty$ and $1\le q\le\infty$. Take any $f\in B^0_{pq}(\mathbb{R}^d)$ such that

$$\alpha := \left(\sum_{j=1}^{\infty}\big|\big|\mathcal{K}(2^{-j},f)|L_p(\mathbb{R}^d)\big|\big|^q\right)^{1/q} > 0.$$

We use (2.5) and the simple formula

$$\mathcal{K}(2^{-j},f(2^k\cdot))(x) = \mathcal{K}(2^{k-j},f)(2^k x), \quad x\in\mathbb{R}^d, \quad j\ge k+1,$$

and obtain

$$\begin{aligned}\big|\big|f(2^k\cdot)|B^0_{pq}(\mathbb{R}^d)\big|\big| &\gtrsim \left(\sum_{j=k+1}^{\infty}\big|\big|\mathcal{K}(2^{-j},f(2^k\cdot))|L_p(\mathbb{R}^d)\big|\big|^q\right)^{1/q}\\ &= \left(\sum_{j=k+1}^{\infty}\big|\big|\mathcal{K}(2^{k-j},f)(2^k\cdot)|L_p(\mathbb{R}^d)\big|\big|^q\right)^{1/q}\\ &= 2^{-\frac{kd}{p}}\alpha,\end{aligned}$$

which concludes the proof of (3.17).

Step 4.

Now we prove

$$||T_k|\mathcal{L}(B^0_{pq}(\mathbb{R}^d))|| \gtrsim k^{\frac{1}{q}-\frac{1}{2}}2^{-\frac{kd}{p}}, \quad k\in\mathbb{N}, \tag{3.18}$$

again for all $1<p<\infty$ and $1\le q\le\infty$.

First, we take a special decomposition of unity, see Definition 2.1. Namely, we suppose, that the function φ satisfies

$$\varphi(x) = 1 \quad \text{if} \quad |x|\le\frac{5}{4} \quad \text{and} \quad \varphi(x)=0 \quad \text{if} \quad |x|\ge\frac{3}{2}. \tag{3.19}$$

It is easy to see, that

$$\varphi_j(x) = \varphi(2^{-j}x)-\varphi(2^{-j+1}x) = 1, \quad \text{if} \quad \frac{3}{4}\cdot 2^j \le |x| \le \frac{5}{4}\cdot 2^j, \quad j\in\mathbb{N}.$$

Finally, we again take $\psi \in S(\mathbb{R}^d)$ with $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^d : |x| \le 1/8\}$. We define the functions f_k through their Fourier transforms:

$$\widehat{f_k}(\xi) = \sum_{j=1}^{k} \psi(2^k(\xi - \xi_j)), \quad \xi \in \mathbb{R}^d, \quad k \in \mathbb{N}, \tag{3.20}$$

where $\xi_j = (2^{-j}, 0, \ldots, 0)$. We shall show that

$$||f_k|B^0_{pq}(\mathbb{R}^d)|| \lesssim k^{\frac{1}{2}} 2^{kd(\frac{1}{p}-1)}, \quad k \in \mathbb{N} \tag{3.21}$$

and

$$||f_k(2^k\cdot)|B^0_{pq}(\mathbb{R}^d)|| \gtrsim k^{\frac{1}{q}} 2^{-kd}, \quad k \in \mathbb{N}. \tag{3.22}$$

First, we deal with (3.21). As the support of $\widehat{f_k}$ lies in the unit ball of $\mathbb{R}^d$, we may omit the terms with $j \ge 1$ in (2.3). Furthermore, since $1 < p < \infty$ we may use the Littlewood-Paley decomposition theorem to estimate

$$\begin{aligned}
||f_k|B^0_{pq}(\mathbb{R}^d)|| = ||(\varphi_0 \widehat{f_k})^\vee|L_p(\mathbb{R}^d)|| &\approx \left\| \left(\sum_{j=1}^{\infty} |(\varphi_1(2^j\cdot)\varphi_0\widehat{f_k})^\vee(x)|^2 \right)^{1/2} |L_p(\mathbb{R}^d) \right\| \\
&= \left\| \left(\sum_{j=1}^{k} |\psi(2^k(\xi - \xi_j))^\vee(x)|^2 \right)^{1/2} |L_p(\mathbb{R}^d) \right\| \\
&= \left\| \left(\sum_{j=1}^{k} |2^{-kd}\psi^\vee(2^{-k}x)e^{ix\cdot\xi_j}|^2 \right)^{1/2} |L_p(\mathbb{R}^d) \right\| \\
&= k^{\frac{1}{2}} 2^{-kd} ||\psi^\vee(2^{-k}x)|L_p(\mathbb{R}^d)|| = k^{\frac{1}{2}} 2^{kd(\frac{1}{p}-1)} ||\psi^\vee|L_p(\mathbb{R}^d)||.
\end{aligned}$$

To prove (3.22), observe that

$$f_k(2^k\cdot)\widehat{\ }(\xi) = 2^{-kd} \sum_{j=1}^{k} \psi(\xi - 2^k\xi_j), \quad \xi \in \mathbb{R}^d, \quad k \in \mathbb{N}.$$

Using again the support properties of ψ and φ_j, we arrive at

$$||f_k(2^k\cdot)|B^0_{pq}(\mathbb{R}^d)|| \approx 2^{-kd} \left(\sum_{j=1}^{k} ||\psi(\cdot - 2^k\xi_j)^\vee|L_p(\mathbb{R}^d)||^q \right)^{1/q} = k^{\frac{1}{q}} 2^{-kd} ||\psi^\vee|L_p(\mathbb{R}^d)||.$$

Step 5.

In this last step we prove the estimate

$$||T_k|\mathcal{L}(B^0_{pq}(\mathbb{R}^d))|| \gtrsim k^{\frac{1}{q}-\frac{1}{p}} 2^{-\frac{kd}{p}}, \quad k \in \mathbb{N}, \tag{3.23}$$

again for all $1 < p < \infty$ and $1 \le q \le \infty$.

Let $\psi \in S(\mathbb{R}^d)$ be a non-negative bump function with

$$\operatorname{supp}\psi \subset [0,1]^d \quad \text{and} \quad \int_{\mathbb{R}^d} \psi(x)dx = 1. \tag{3.24}$$

For a fixed $k \in \mathbb{N}$ we set

$$\psi_j(x) = \sum_{l \in N^k_j} \psi(x - l), \quad x \in \mathbb{R}^d, \quad j = 1, 2, \ldots, k-1,$$

where

$$N_j^k = \{l \in \mathbb{N}_0^d : 2^{j-1} \le l_1 - 2^j \le 2^j - 1 \quad \text{and} \quad 0 \le l_i \le 2^k - 1 \quad \text{for} \quad i = 2, \dots, d\},$$

so that the set N_j^k contains $2^{j-1+k(d-1)}$ vectors and ψ_j consists of $2^{j-1+k(d-1)}$ copies of ψ. Furthermore, we define

$$f_k(x) = \sum_{j=1}^{k-1} 2^{\frac{k-j}{p}} \psi_j(x), \quad x \in \mathbb{R}^d. \tag{3.25}$$

The proof of (3.23) is finished as soon as we prove that

$$||f_k|B^0_{pq}(\mathbb{R}^d)|| \lesssim k^{\frac{1}{p}} 2^{\frac{kd}{p}}, \quad k \in \mathbb{N}, \tag{3.26}$$

as well as

$$||f_k(2^k\cdot)|B^0_{pq}(\mathbb{R}^d)|| \gtrsim k^{\frac{1}{q}}, \quad k \in \mathbb{N}. \tag{3.27}$$

The proof of (3.26) is a rather straightforward application of Theorem 2.5. We observe, that (3.25) represents an atomic decomposition of f. This gives

$$||f_k|B^0_{pq}(\mathbb{R}^d)|| \lesssim \left(\sum_{j=1}^{k-1} 2^{j-1+k(d-1)} 2^{\frac{k-j}{p}\cdot p} \right)^{1/p} \approx k^{\frac{1}{p}} 2^{\frac{kd}{p}}.$$

In the proof of (3.27), we use again the characterisation by local means.
We choose a special kernel $\kappa \in C^\infty(\mathbb{R}^d)$ with

$$\operatorname{supp} \kappa \subset [-1,1] \times [-3,3]^{d-1}$$

and

$$\kappa(x) \ge 0 \text{ if } x_1 \le 0 \quad \text{and} \quad \kappa(x) = 1 \text{ if } x \in [-\frac{3}{4}, 0] \times [-2,2]^{d-1}.$$

We show, that for every $j = 1, \dots, k-1$ and every $x \in \mathbb{R}^d$ with

$$2^{-j+1} \le x_1 \le 2^{-j+1} + \frac{2^{-j}}{4}, \quad 0 \le x_i \le 1, i = 2, \dots, d, \tag{3.28}$$

it holds

$$\mathcal{K}(2^{-j}, f(2^k\cdot))(x) \ge c\, 2^{\frac{j}{p}}. \tag{3.29}$$

Let us point out, that this estimate is already sufficient for (3.27) since

$$\begin{aligned} ||f_k(2^k\cdot)|B^0_{pq}(\mathbb{R}^d)|| &\gtrsim \left(\sum_{j=1}^{k-1} \left|\left|\mathcal{K}(2^{-j}, f(2^k\cdot))|L_p(\mathbb{R}^d)\right|\right|^q \right)^{1/q} \\ &\gtrsim c \left(\sum_{j=1}^{k-1} \left(2^{-j} 2^{\frac{j}{p}\cdot p}\right)^{p/q} \right)^{1/q} \approx k^{\frac{1}{q}}. \end{aligned}$$

We therefore concentrate on (3.29) under the condition (3.28).
The support properties of ψ_j and κ ensure, that

$$\mathcal{K}(2^{-j}, f(2^k\cdot))(x) = 2^{\frac{j}{p}} \mathcal{K}(2^{-j}, \psi_{k-j}(2^k\cdot))(x) = 2^{\frac{j}{p}} \sum_{l \in N^k_{k-j}} \int_{\mathbb{R}^d} \kappa(y) \psi(2^k x + 2^{k-j} y - l) dy$$

for every x with (3.28). It is not difficult to verify, that (for every x) there are always at least $2^{(k-j)d}$ vectors $l \in N^k_{k-j}$ such that $\kappa(y) = 1$ on the support of $\psi(2^k x + 2^{k-j} y - l)$. Hence the last expression may be estimated from below by

$$2^{\frac{j}{p}} \cdot 2^{(k-j)d} \int_{\mathbb{R}^d} \psi(2^{k-j} y) dy = 2^{\frac{j}{p}}.$$

□

Remark 3.6. Let us observe, that Theorem 3.4 may be easily extended to $0 < q < 1$:

$$||T_k|\mathcal{L}(B^0_{pq}(\mathbb{R}^d))|| \approx 2^{-k\frac{d}{p}} \cdot \begin{cases} k^{\frac{1}{q}-\frac{1}{p}}, & \text{if } 1 < p < \infty \text{ and } p \geq \max(q, 2), \\ k^{\frac{1}{q}-\frac{1}{2}}, & \text{if } 1 < p < \infty \text{ and } 2 \geq \max(p, q), \\ k^{\frac{1}{q}}, & \text{if } p = 1 \text{ or } p = \infty, \end{cases} \tag{3.30}$$

The proof of the estimates from above may be done exactly as in the proof of Theorem 3.2. We use the Gagliardo-Nirenberg inequality, cf. [7, Chapter 5],

$$||f|B^0_{p,1}(\mathbb{R}^d)|| \leq ||f|B^0_{pq}(\mathbb{R}^d)||^{1-\theta} \cdot ||f|B^0_{p,\max(p,2)}(\mathbb{R}^d)||^{\theta} \tag{3.31}$$

with

$$1 = \frac{1-\theta}{q} + \frac{\theta}{\max(p, 2)},$$

and the construction from the proof of Theorem 3.4 to prove the estimates from below.

Remark 3.7. Theorem 3.4 may also be used to give a following comment on the atomic decomposition Theorem 2.5. If $s = 0$, we required in Theorem 2.5 that the atoms $a_{\nu m}$ satisfy the moment condition (2.8) at least for $\beta = 0$ and $\nu > 0$.

It seems to be an open question, if this restriction is really necessary. In other words, if Theorem 2.5 holds, if $s = 0$ as well as $L = -s = 0$. We show, that this is never true and that the moment conditions are indispensable.

Let $1 < q \leq \infty$ and $1 \leq p \leq \infty$ and let us suppose, that Theorem 2.5 is true with $L = 0$. Hence no moment condition on $a_{\nu m}$ are needed. Let $\psi \in S(\mathbb{R}^d)$ be a non-negative function with

$$\operatorname{supp} \psi \subset \{x \in \mathbb{R}^d : |x_i| \leq 1, i = 1, \dots, d\}, \qquad \int_{\mathbb{R}^d} \psi(x) dx = 1, \tag{3.32}$$

and

$$\sum_{m \in \mathbb{Z}^d} \psi(x - m) = 1, \quad x \in \mathbb{R}^d. \tag{3.33}$$

We put

$$f_J(x) = \sum_{\nu=0}^{J} \sum_{\substack{m \in \mathbb{Z}^d \\ |m_i| \leq 2^\nu, i=1,\dots,d}} \psi(2^\nu x - m), \quad x \in \mathbb{R}^d, \quad J \in \mathbb{N}. \tag{3.34}$$

It follows by (3.32) that

$$||f_J|B^0_{pq}(\mathbb{R}^d)|| \gtrsim J, \quad J \in \mathbb{N}.$$

But if Theorem 2.5 would be true for $s = 0$ and $L = 0$, (3.34) would represent an atomic decomposition of f_J and therefore

$$||f_J|B^0_{pq}(\mathbb{R}^d)|| \lesssim \left(\sum_{\nu=0}^{J} 2^{-\nu\frac{d}{p}q} \cdot (2^{\nu+1} + 1)^{\frac{d}{p}q} \right)^{1/q} \lesssim J^{\frac{1}{q}}$$

would hold for every $J \in \mathbb{N}$. This leads to contradiction.
Let $0 < q \le 1$ and $1 \le p \le \infty$. Then every $f \in B^0_{pq}(\mathbb{R}^d)$ may be rewritten into the optimal atomic decomposition

$$f(x) = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^d} \lambda_{\nu m} a_{\nu m}(x), \quad x \in \mathbb{R}^d,$$

with

$$||\lambda|b^0_{pq}|| \lesssim ||f|B^0_{pq}(\mathbb{R}^d)||, \quad f \in B^0_{pq}(\mathbb{R}^d),$$

see [12, Chapter 1.5] for details. If Theorem 2.5 would be true for $s = 0$ and $L = 0$,

$$f(2^k x) = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^d} \lambda_{\nu m} a_{\nu m}(2^k x), \quad x \in \mathbb{R}^d,$$

would represent an atomic decomposition of $f(2^k x)$ and therefore

$$||f(2^k \cdot)|B^0_{pq}(\mathbb{R}^d)|| \lesssim 2^{k\frac{d}{p}} ||\lambda|b^0_{pq}|| \lesssim 2^{k\frac{d}{p}} ||f|B^0_{pq}(\mathbb{R}^d)||.$$

But we know by Theorem 3.4 and Remark 3.6 that this is *not* true.

For the sake of completeness, we consider also the dilation operator

$$(\tilde{T}_k f)(x) = f(2^{-k} x), \quad k \in \mathbb{N}, \quad x \in \mathbb{R}^d. \tag{3.35}$$

Its behaviour is well known if $s < 0$, see [4, p. 34] for further details:

Lemma 3.8. *Let $s < 0$, $1 \le p, q \le \infty$ and $k \in \mathbb{N}$. Then the operator $\tilde{T}_k$ is bounded on $B^s_{p,q}(\mathbb{R}^d)$ and its norm is bounded by $c\, 2^{-k(s - \frac{d}{p})}$.*

If $s = 0$, we can also characterise the norm $\tilde{T}_k$.

Theorem 3.9. *Let $1 \le p, q \le \infty$, $k \in \mathbb{N}$ and let $\tilde{T}_k$ be defined by* (3.35)*. Then*

$$||\tilde{T}_k|\mathcal{L}(B^0_{pq}(\mathbb{R}^d))|| \approx 2^{k\frac{d}{p}} \cdot \begin{cases} k^{\frac{1}{p}-\frac{1}{q}}, & \text{if } 1 < p < \infty \text{ and } p \le \min(q,2), \\ k^{\frac{1}{2}-\frac{1}{q}}, & \text{if } 1 < p < \infty \text{ and } 2 \le \min(p,q), \\ 1, & \text{if } 1 < p < \infty \text{ and } q \le \min(p,2), \\ k^{1-\frac{1}{q}}, & \text{if } p = 1 \text{ or } p = \infty, \end{cases} \tag{3.36}$$

where the constants of equivalence do not depend on k.

Remark 3.10. If $1 < p < \infty$, the estimates in (3.36) may be abbreviated to

$$||\tilde{T}_k|\mathcal{L}(B^0_{pq}(\mathbb{R}^d))|| \approx 2^{k\frac{d}{p}} \cdot k^{\frac{1}{\min(p,q,2)} - \frac{1}{q}}.$$

In this case, the jump in the exponent of k occurs by $p = \infty$.

Proof. Let $\mathring{B}^0_{pq}(\mathbb{R}^d)$ with $1 \le p, q \le \infty$ be the completion of $S(\mathbb{R}^d)$ in $B^0_{pq}(\mathbb{R}^d)$. It follows immediately from Theorem 3.4 that

$$||T_k|\mathcal{L}(B^0_{pq}(\mathbb{R}^d))|| = ||T_k|\mathcal{L}(\mathring{B}^0_{pq}(\mathbb{R}^d))||. \tag{3.37}$$

One has by [10, p. 180, (12)]

$$\mathring{B}^0_{pq}(\mathbb{R}^d)' = B^0_{p'q'}(\mathbb{R}^d), \qquad 1 \le p, q \le \infty \quad \text{and} \quad \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1. \tag{3.38}$$

Furthermore

$$||T_k|\mathcal{L}(\mathring{B}^0_{pq}(\mathbb{R}^d))|| = ||T'_k|\mathcal{L}(B^0_{p'q'}(\mathbb{R}^d))||,$$

where

$$T'_k = 2^{-kd}\tilde{T}_k$$

is the dual operator to T_k. Hence

$$||\tilde{T}_k|\mathcal{L}(B^0_{pq}(\mathbb{R}^d))|| = 2^{kd}||T_k|\mathcal{L}(\mathring{B}^0_{p'q'}(\mathbb{R}^d))||. \tag{3.39}$$

Now the proof follows by (3.37) and Theorem 3.4.

□

It is not difficult to extend Theorems 3.4 and 3.9 also to the operator

$$(T_\lambda f)(x) = f(\lambda x), \quad \lambda > 0, \quad x \in \mathbb{R}^d. \tag{3.40}$$

Theorem 3.11. *Let $1 \le p, q \le \infty$.*
(i) Then

$$||T_\lambda|\mathcal{L}(B^0_{pq}(\mathbb{R}^d))|| \approx \lambda^{-\frac{d}{p}} \cdot \begin{cases} (1+\log\lambda)^{\frac{1}{q}-\frac{1}{p}}, & \text{if } 1<p<\infty \text{ and } p \ge \max(q,2),\\ (1+\log\lambda)^{\frac{1}{q}-\frac{1}{2}}, & \text{if } 1<p<\infty \text{ and } 2 \ge \max(p,q),\\ 1, & \text{if } 1<p<\infty \text{ and } q \ge \max(p,2),\\ (1+\log\lambda)^{\frac{1}{q}}, & \text{if } p=1 \text{ or } p=\infty, \end{cases} \tag{3.41}$$

holds for every $\lambda > 1$.
(ii) Then

$$||T_\lambda|\mathcal{L}(B^0_{pq}(\mathbb{R}^d))|| \approx \lambda^{-\frac{d}{p}} \cdot \begin{cases} (1+|\log\lambda|)^{\frac{1}{p}-\frac{1}{q}}, & \text{if } 1<p<\infty \text{ and } p \le \min(q,2),\\ (1+|\log\lambda|)^{\frac{1}{2}-\frac{1}{q}}, & \text{if } 1<p<\infty \text{ and } 2 \le \min(p,q),\\ 1, & \text{if } 1<p<\infty \text{ and } q \le \min(p,2),\\ (1+|\log\lambda|)^{1-\frac{1}{q}}, & \text{if } p=1 \text{ or } p=\infty, \end{cases} \tag{3.42}$$

holds for every $0 < \lambda < 1$.

Proof. The result follows directly from the Theorems 3.4 and 3.9 and the well-known assertion

$$\sup_{\frac{1}{2}<\lambda<2} ||f(\lambda\cdot)|B^0_{pq}(\mathbb{R}^d)|| \approx ||f|B^0_{pq}(\mathbb{R}^d)||.$$

□

4 Sampling numbers

In this section we apply the estimates of the norm of the dilation operator to derive optimal estimates for the decay of sampling numbers of the identity operator between two Besov spaces. Let us first present the basic definitions and notation.

Definition 4.1. Let Ω be the unit cube $(0,1)^d$. Let $G_1(\Omega)$ be a space of continuous functions on Ω and $G_2(\Omega) \subset D'(\Omega)$ be a space of distributions on Ω. Suppose, that the embedding

$$id : G_1(\Omega) \hookrightarrow G_2(\Omega)$$

is compact.
For $\{x^j\}_{j=1}^n \subset \Omega$ we define the *information map*

$$N_n : G_1(\Omega) \to \mathbb{C}^n, \qquad N_n f = (f(x^1), \dots, f(x^n)), \quad f \in G_1(\Omega).$$

For any (linear or nonlinear) mapping $\varphi_n : \mathbb{C}^n \to G_2(\Omega)$ we consider

$$S_n : G_1(\Omega) \to G_2(\Omega), \qquad S_n = \varphi_n \circ N_n.$$

(i) Then for all $n \in \mathbb{N}$, the $n-$th *sampling number* $g_n(id)$ is defined by

$$g_n(id) = \inf_{S_n} \sup\{||f - S_n f|G_2(\Omega)|| : ||f|G_1(\Omega)|| \leq 1\}, \tag{4.1}$$

where the infimum is taken over all n-tuples $\{x^j\}_{j=1}^n \subset \Omega$ and all (linear or nonlinear) φ_n.
(ii) For all $n \in \mathbb{N}$ the $n-$th *linear sampling number* $g_n^{\text{lin}}(id)$ is defined by (4.1), where now only linear mappings φ_n are admitted.

In the following, we restrict ourselves to the scale of Besov spaces - hence $G_1(\Omega) = B^{s_1}_{p_1q_1}(\Omega)$ with

$$s_1 > \frac{d}{p_1}.$$

Then the space $B^s_{p_1q_1}(\Omega)$ is continuously embedded into the space of functions continuous on $\bar{\Omega}$ and the information map N_n is well defined. Second, we suppose that $G_2 = B^0_{p_2q_2}(\Omega)$.
The case $p_1 < p_2$ was already fully discussed in [13]. It was shown there, that both, the linear and nonlinear sampling numbers, decay asymptotically like $n^{-\frac{s}{d}+\frac{1}{p_1}-\frac{1}{p_2}}$.
We concentrate on the case $p_1 = p_2$ and give a full characterisation of the decay of g_n as well as of g_n^{lin}. This result closes some of the gaps left open in [13], which were the actual motivation for this paper. In the very end, we discuss the remaining case $p_1 > p_2$ and state several open problems connected to this question.

Theorem 4.2. *Let $\Omega = (0,1)^d$. Let $G_1(\Omega) = B^s_{pq_1}(\Omega)$ and $G_2(\Omega) = B^0_{pq_2}(\Omega)$ with $1 \leq p, q_1, q_2 \leq \infty$ and $s > \frac{d}{p}$. Then for $2 \leq n \in \mathbb{N}$*

$$g_n(id) \approx g_n^{\text{lin}}(id) \approx n^{-\frac{s}{d}} \cdot \begin{cases} (\log n)^{\frac{1}{q_2}-\frac{1}{p}}, & \textit{if } 1 < p < \infty \textit{ and } p \geq \max(q_2, 2), \\ (\log n)^{\frac{1}{q_2}-\frac{1}{2}}, & \textit{if } 1 < p < \infty \textit{ and } 2 \geq \max(p, q_2), \\ 1, & \textit{if } 1 < p < \infty \textit{ and } q_2 \geq \max(p, 2), \\ (\log n)^{\frac{1}{q_2}}, & \textit{if } p = 1 \textit{ or } p = \infty, \end{cases} \tag{4.2}$$

Proof. Step 1: Estimates from above
It follows directly from Definition 4.1 that $g_n(id) \leq g_n^{\text{lin}}(id)$. The estimates from above for g_n^{lin} are a consequence of the estimates from above obtained in Lemma 3.1 and Proposition 3.2 and summarised in Theorem 3.4 and the method presented in [13]. By this we mean especially the inequality (2.6) in [13] where now the estimate of the norm of the dilation operator has to be applied with $s_2 = 0$.
Hence, it is enough to prove the estimates from below for $g_n(id)$.
Step 2. - Estimates from below
We use the following simple observation, (c.f. [6, Proposition 20]). For $\Gamma = \{x^j\}_{j=1}^n \subset \Omega$ we denote

$$G_1^\Gamma(\Omega) = \{f \in G_1(\Omega) : f(x^j) = 0 \quad \text{for all} \quad j = 1, \dots, n\}.$$

Then

$$g_n(id) \approx \inf_\Gamma \sup\{||f|G_2(\Omega)|| : f \in G_1^\Gamma(\Omega), ||f|G_1(\Omega)|| = 1\}$$
$$= \inf_\Gamma ||id : G_1^\Gamma(\Omega) \hookrightarrow G_2(\Omega)||,$$

where both the infima are taken over all sets $\Gamma = \{x^j\}_{j=1}^n \subset \Omega$.

So, to prove the estimates from below included in (4.2), we construct for every set $\Gamma = \{x^j\}_{j=1}^{2^{kd}} \subset \Omega, k \in \mathbb{N}$ a function $f_k \in G_1^\Gamma$ such that

$$\frac{||f_k|G_2(\Omega)||}{||f_k|G_1(\Omega)||} \gtrsim 2^{-ks}k^\alpha, \tag{4.3}$$

where the power α represents the power of the logarithmic factor in each of the four cases contained in (4.2).

1. case: $g_n(id) \gtrsim n^{-\frac{s}{d}}$.

In this (most simple) case, we rely on the wavelet characterisation of the spaces $B^s_{pq}(\mathbb{R}^d)$, as described in [12, Section 3.1]. Let

$$\psi_F \in C^K(\mathbb{R}) \quad \text{and} \quad \psi_M \in C^K(\mathbb{R}), \quad K \in \mathbb{N},$$

be the Daubechies compactly supported K-wavelets on $\mathbb{R}$ with K large enough. Then we define

$$\Psi(x) = \prod_{i=1}^d \psi_M(x_i), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d$$

and

$$\Psi^j_m(x) = \Psi(2^j x - m), \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n.$$

The functions

$$\psi_j(x) = \sum_m \lambda_{jm} \Psi^j_m(x), \quad j \in \mathbb{N}, \quad x \in \mathbb{R}^d \tag{4.4}$$

satisfy

$$||\psi_j|B^s_{pq}(\mathbb{R}^d)|| \approx 2^{j(s-\frac{d}{p})} \left(\sum_m |\lambda_{jm}|^p\right)^{1/p} \tag{4.5}$$

with constants independent on $j \in \mathbb{N}$ and on the sequence $\lambda = \{\lambda_{jm}\}$. The summation in (4.4) and (4.5) runs over those $m \in \mathbb{Z}^d$ for which the support of Ψ^j_m is included in Ω. Let us comment briefly on the relationship between $||\psi_j|B^s_{pq}(\Omega)||$ and $||\psi_j|B^s_{pq}(\mathbb{R}^d)||$. Clearly, ψ_j as a function on $\mathbb{R}^d$ is an extension of $\psi_j|_\Omega$, the inequality

$$||\psi_j|B^s_{pq}(\Omega)|| \le ||\psi_j|B^s_{pq}(\mathbb{R}^d)||$$

follows trivially from Definition 2.7. On the other hand, any other extension of $\psi_j|_\Omega$ to $\mathbb{R}^d$ possesses an *unique* wavelet decomposition. The uniqueness shows, that this decomposition contains (4.4) as a proper part and has therefore a larger norm. Hence, the relation (4.5) holds also for $||\psi_j|B^s_{pq}(\Omega)||$.

There is a number $l \in \mathbb{N}_0$ such that for any $k \in \mathbb{N}$ and any $\Gamma = \{x^j\}_{j=1}^{2^{kd}}$ there is an element $m \in \mathbb{Z}^d$ such that

$$\operatorname{supp} \Psi^{k+l}_m \subset \Omega \quad \text{and} \quad \operatorname{supp} \Psi^{k+l}_m \cap \Gamma = \emptyset.$$

Taking $f_k = \Psi^{k+l}_m$ we obtain the estimate $g_n(id) \gtrsim n^{-\frac{s}{d}}$ for every $1 \le p, q_1, q_2 \le \infty$.

2. case: $g_n(id) \gtrsim n^{-\frac{s}{d}}(\log n)^{\frac{1}{q_2}}$ for $p = 1$.

We consider the function $\psi_k(x) = \psi(2^k x - m)$, where $m \in \mathbb{Z}^d$ and ψ was defined and discussed in the Step 1. of the proof of Theorem 3.4. It is possible to choose $m \in \mathbb{Z}^d$ such that

$$\operatorname{supp} \psi_k \subset \Big(\frac{1}{4}, \frac{3}{4}\Big)^d \quad \text{and} \quad \operatorname{supp} \psi_k \cap \Gamma = \emptyset.$$

To show, that this function satisfies (4.3), we argue as follows. First, we use Theorem 2.5 to get

$$||\psi_k|B^s_{1q_1}(\Omega)|| \le ||\psi_k|B^s_{1q_1}(\mathbb{R}^d)|| \lesssim 2^{k(s-d)}.$$

On the other hand, if $\tilde{\psi}_k$ is any extension of ψ_k and $\omega \in S(\mathbb{R}^d)$ satisfies

$$\operatorname{supp} \omega \subset (0,1)^d \quad \text{and} \quad \omega(x) = 1 \quad \text{for} \quad x \in \Big(\frac{1}{4}, \frac{3}{4}\Big)^d,$$

we arrive at

$$k^{\frac{1}{q_2}} 2^{-kd} \lesssim ||\psi_k|B^0_{1q_2}(\mathbb{R}^d)|| = ||\omega\tilde{\psi}_k|B^0_{1q_2}(\mathbb{R}^d)|| \lesssim ||\tilde{\psi}_k|B^0_{1q_2}(\mathbb{R}^d)||,$$

hence

$$k^{\frac{1}{q_2}} 2^{-kd} \lesssim ||\psi_k|B^0_{1q_2}(\Omega)||$$

and (4.3) follows.

3. case: $g_n(id) \gtrsim n^{-\frac{s}{d}} (\log n)^{\frac{1}{q_2} - \frac{1}{p}}$.

Let $1 \le p \le \infty$. In Step 5. of the proof of Theorem 3.4 we constructed a function $f_k(2^k \cdot)$ (see (3.25) for details). Let us point out, that this function has its support in $(0,1)^d$ and avoids the set Γ if the sampling points are uniformly distributed, hence $\Gamma = \{0, \frac{1}{2^k}, \dots, \frac{2^k-1}{2^k}, 1\}^d$. Using (3.26) and (3.27), we obtain

$$||f(2^k \cdot)|B^s_{pq_1}(\mathbb{R}^d)|| \lesssim 2^{k(s-\frac{d}{p})} ||f|B^s_{pq_1}(\mathbb{R}^d)|| \lesssim 2^{k(s-\frac{d}{p})} 2^{\frac{kd}{p}} k^{\frac{1}{p}} = k^{\frac{1}{p}} 2^{ks}$$

and

$$||f(2^k \cdot)|B^0_{pq_2}(\mathbb{R}^d)|| \gtrsim k^{\frac{1}{q_2}}.$$

Using again the cut-off function ω, we get similar estimates also for the norms on Ω. In view of (4.3), this finishes the proof for this specially chosen set Γ.

If Γ is taken arbitrary, $|\Gamma| = 2^{kd}$, we modify f_k using the Dirichlet principle. Let us sketch this modification.

First, we construct a sequence of disjoint cubes

$$\{\Omega_{j,l}\}, \quad j = 1, \dots, k, \quad l = 1, \dots, 2^{(d-1)(j-1)},$$

where each $\Omega_{j,l}$ is a cube with side length $1/2^{j+1}$ and contains in its interior at most $2^{(k-j)d}$ points from Γ.

We proceed by induction. Let $j = 1$. We divide $\Omega = (0,1)^d$ into 4^d cubes with side length $1/4$ and disjoint interiors. According to the Dirichlet principle, one of this cubes has in its interior at most $\dfrac{2^{kd}}{4^d} = 2^{(k-2)d} \le 2^{(k-1)d}$ points from Γ. We denote this cube $\Omega_{1,1}$.

Let $j = 2$. We divide each of the remaining $4^d - 1$ cubes (it means the set $\Omega \setminus \Omega_{1,1}$) into 2^d cubes with side length $1/8$ and disjoint interiors. We choose from these $2^{3d} - 2^d$ cubes 2^{d-1} cubes with the smallest number of points of Γ. The Dirichlet principle gives the estimate from above for this number by $\frac{2^{kd}}{2^{3d}-2^d-2^{d-1}+1} \le 2^{(k-2)d}$.

In next steps we always divide all remaining cubes into 2^d cubes with disjoint interiors and half the side length and choose those $2^{(j-1)(d-1)}$ of them which contain the smallest number of points of Γ. The Dirichlet principle then provides the estimate for this number.

Next, we divide each of the cubes $\Omega_{j,l}$ into 3^d cubes with disjoint interior and denote 'the middle cube' of this decomposition by $\tilde{\Omega}_{j,l}$.

As each of the cubes $\tilde{\Omega}_{j,l}$ contains at most $2^{(k-j)d}$, there is a number $m > 0$ such that we may place into each $\Omega_{j,l}$ 2^{jd} copies (i.e. dilations) of $\psi(2^{m+k}\cdot)$ with disjoint supports. We denote their sum as $\psi_{j,l}$. The number m may be chosen independent of k and Γ.

Finally, we introduce

$$g_k(x) = \sum_{j=1}^{k} \sum_{l=1}^{2^{(d-1)(j-1)}} 2^{\frac{j}{p}} \psi_{j\,l}(x). \tag{4.6}$$

The functions g_k play the role of a substitute of $f_k(2^k\cdot)$ adapted to the general sampling sets Γ.

To finish the proof, we have to show that

$$||g_k|B^0_{pq_2}(\mathbb{R}^d)|| \gtrsim k^{\frac{1}{q_2}} \tag{4.7}$$

and

$$||g_k|B^s_{pq_1}(\mathbb{R}^d)|| \lesssim k^{\frac{1}{p}} 2^{ks}. \tag{4.8}$$

The proof of (4.7) is similar to Step 5. of Theorem 3.4 and uses the characterisation by local mean. The proof of (4.8) is based on the atomic decomposition of the spaces $B^s_{pq_1}(\mathbb{R}^d)$. Let us mention, that $s > 0$ and hence no moment conditions are needed in (2.8).

4. case $g_n(id) \gtrsim n^{-\frac{s}{d}} (\log n)^{\frac{1}{q_2} - \frac{1}{2}}$.

We first present a construction which proves the result for $d = 1$, $\Omega = (-2, 2)$ and the uniform distribution of sampling points, i. e. $\Gamma = \{\frac{n}{2^k}, n = -2^{k+1} + 1, \dots 2^{k+1} - 1\}$.

We proceed as follows. First, we define a sequence of sets. Let (see Figure 1)

$$\begin{aligned}
I_1 &= \Big(-\frac{1}{2}, \frac{1}{2}\Big),\\
I_2 &= \Big(-\frac{5}{4}, -\frac{3}{4}\Big) \cup \Big(-\frac{1}{4}, \frac{1}{4}\Big) \cup \Big(\frac{3}{4}, \frac{5}{4}\Big),\\
I_3 &= \Big(-\frac{13}{8}, -\frac{11}{8}\Big) \cup \Big(-\frac{9}{8}, -\frac{7}{8}\Big) \cup \Big(-\frac{5}{8}, -\frac{3}{8}\Big) \cup \Big(-\frac{1}{8}, \frac{1}{8}\Big) \cup\\
&\qquad \cup \Big(\frac{3}{8}, \frac{5}{8}\Big) \cup \Big(\frac{7}{8}, \frac{9}{8}\Big) \cup \Big(+\frac{11}{8}, +\frac{13}{8}\Big),\\
&\vdots\\
I_n &= \bigcup \Big\{ \Big(\frac{4k-1}{2^n}, \frac{4k+1}{2^n}\Big); |k| < 2^n \Big\},\\
&\vdots
\end{aligned}$$

and

$$I_n^c = \Big(-2 + \frac{3}{2^n}, 2 - \frac{3}{2^n}\Big) \setminus I_n.$$

Let

$$\eta_i = \chi_{I_i} - \chi_{I_i^c}.$$

Observe that

$$< \eta_i; \eta_j >= \begin{cases} 0, & i \neq j,\\ 2 - \frac{1}{2^{i-1}}, & i = j. \end{cases}$$

The functions η_i are modified Rademacher functions. Slight modification of Theorem 2.b.3 in Volume I of [5] shows that Khintchin inequalities apply to these functions. Especially, for every $p < \infty$ there is a constant B_p such that

$$\Big|\Big|\sum_{i=1}^{k} \eta_i | L_p(\mathbb{R})\Big|\Big| \leq B_p k^{\frac{1}{2}} \tag{4.9}$$

for every $k \in \mathbb{N}$.

Now, take a non-negative non-trivial function $\kappa \in S(\mathbb{R})$ with $\operatorname{supp} \kappa \subset (0,1)$. As I_i contains $2^i - 1$ intervals of the length $\frac{2}{2^i}$, we may define the functions $g_{k,i}, i = 1, \dots, k$, as the sum of $2^k(2-2^{-(i-1)})$ copies of the function $\kappa(2^k\cdot)$ with disjoint supports all contained in I_i. Similarly, $g^c_{k,i}, i = 1, \dots, k$, is the sum of $2^k(2 - 2^{-(i-2)})$ copies of $\kappa(2^k\cdot)$ with disjoint supports all contained in I_i^c. We define

$$g_k = \sum_{i=1}^{k} (g_{k,i} - g^c_{k,i}).$$

The atomic decomposition theorem (cf. Theorem 2.5) together with (4.9) yields

$$||g_k | B^s_{p,q_1}(\mathbb{R})|| \lesssim ||2^{ks} \sum_{i=1}^{k} \eta_i | L_p(\mathbb{R})|| \lesssim k^{\frac{1}{2}} 2^{ks}, \quad k \in \mathbb{N}.$$

To estimate the norm of g_k in $B^0_{p,q_2}(\mathbb{R})$ from below, we use duality.
Set

$$\tilde{\kappa}_i(x) = \kappa(2^i x) - \kappa(2^i x - 1), \qquad x \in \mathbb{R}, \quad i \in \mathbb{N}. \tag{4.10}$$

We define the functions $\tilde{g}_i$ as the sum of $2^i - 2$ copies of $\tilde{\kappa}_i$ with disjoint supports all contained in $I_i \cup I_i^c$, non-negative on I_i, non-positive on I_i^c. Finally, we write

$$\tilde{g}^k = \sum_{i=2}^{k} \tilde{g}_i, \quad k \geq 2.$$

An application of the atomic decomposition theorem 2.5 leads to

$$||\tilde{g}^k | B^0_{p',q_2'}(\mathbb{R})|| \lesssim k^{\frac{1}{q_2'}} = k^{1-\frac{1}{q_2}}.$$

Let us mention, that the first moment condition $\int_{\mathbb{R}} \kappa(x)dx = 0$ is satisfied trivially by (4.10). Now we apply the functional represented by g_k to $\tilde{g}^k$. Then

$$k \approx \int_{-2}^{2} g_k(t)\tilde{g}^k(t)dt = g_k(\tilde{g}^k) \lesssim ||g_k | B^0_{p,q_2}(\mathbb{R})|| \cdot ||\tilde{g}^k | B^0_{p',q_2'}(\mathbb{R})|| \lesssim k^{1-\frac{1}{q_2}} ||g_k | B^0_{p,q_2}(\mathbb{R})||, \tag{4.11}$$

which implies

$$k^{\frac{1}{q_2}} \lesssim ||g_k | B^0_{p,q_2}(\mathbb{R})||, \qquad k \in \mathbb{N}.$$

Let us point out, that the function g_k vanishes on Γ. In view of (4.3), this finishes the proof for $d = 1$ and uniform distribution of the sampling points.

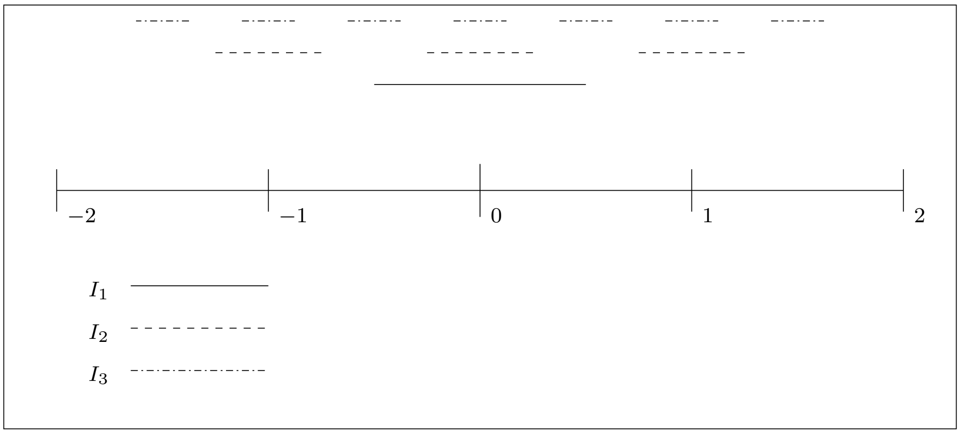


Figure 1

If the sampling points are not uniformly distributed, the construction has to be slightly modified. Let $\Omega = (0,1), k \in \mathbb{N}$ and let $\Gamma \subset (0,1)$ be an arbitrary set with $\#\Gamma \leq 2^k$. We denote by I_j^k the dyadic decomposition of $(0,1)$ into 2^k disjoint intervals of length 2^{-k}, hence

$$I_j^k = \left(\frac{j}{2^k}, \frac{j+1}{2^k}\right), \quad j = 0, \dots, 2^k - 1.$$

Furthermore, $\tilde{\Gamma}_k$ stands for the union of intervals I_j^k, which intersect Γ

$$\tilde{\Gamma}_k = \Big\{ \bigcup_j I_j^k : I_j^k \cap \Gamma \neq \emptyset \Big\}.$$

Let $r_j, j = 1, 2, \dots$ be the usual Rademacher functions

$$r_1(t) = \begin{cases} 1, & \text{if} \quad 0 < t < \frac{1}{2}, \\ -1, & \text{if} \quad \frac{1}{2} < t < 1, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad r_{j+1}(t) = r_j(2t) + r_j(2t-1), \quad j = 1, 2, \dots$$

We set

$$R_k(t) = \sum_{j=1}^{k} r_j(t), \quad k \in \mathbb{N}$$

and

$$g_{k,i}(t) = r_i(t) \cdot \sum_{j=0}^{2^k-1} \kappa(2^k t - j), \quad i = 1, 2, \dots, k,$$

where $\kappa \in C^\infty(\mathbb{R})$ is a non-trivial non-negative function with $\operatorname{supp} \kappa \subset (0,1)$. Finally, for $a \in \mathbb{N}$ we define

$$g_k^a(t) = \Big(\sum_{i=1}^{k} g_{k+a,i}(t) \Big) \cdot \big(1 - \chi_{\tilde{\Gamma}_{k+a}}(t) \big).$$

We prove that, if a is chosen sufficiently large and $1 < p \leq 2$,

$$||g_k^a | B_{p,q_1}^s(\Omega)|| \lesssim k^{\frac{1}{2}} 2^{ks}, \quad k \in \mathbb{N} \tag{4.12}$$

and

$$||g_k^a | B_{p,q_2}^0(\Omega)|| \gtrsim k^{\frac{1}{q_2}}, \quad k \in \mathbb{N}. \tag{4.13}$$

To prove (4.12), we use Theorem 2.5

$$\begin{aligned} ||g_k^a|B^s_{p,q_1}(\Omega)||^2 \le ||g_k^a|B^s_{p,q_1}(\mathbb{R})||^2 &\lesssim 2^{2(k+a)s}\Big|\Big|R_k(t)\cdot\big(1-\chi_{\tilde\Gamma_{k+a}}\big)|L_p(\mathbb{R})\Big|\Big|^2 \\ &\lesssim 2^{2ks}\Big|\Big|R_k(t)\cdot\big(1-\chi_{\tilde\Gamma_{k+a}}\big)|L_2(\mathbb{R})\Big|\Big|^2 \\ &= 2^{2ks}\sum_{i,j=1}^k (r_i,r_j) - 2^{2ks}\sum_{i,j=1}^k (r_i, r_j\chi_{\tilde\Gamma_{k+a}}). \end{aligned}$$

The first sum is obviously equal to $k\cdot 2^{2ks}$. We rewrite the second sum

$$\sum_{i,j=1}^k (r_i, r_j\chi_{\tilde\Gamma_{k+a}}) = \sum_{l:I_l^{k+a}\subset\tilde\Gamma_{k+a}} \int_{I_l^{k+a}} \sum_{i,j=1}^k r_i(t)r_j(t)dt \tag{4.14}$$

We fix an interval $I_l^{k+a}\subset\tilde\Gamma_{k+a}$ an observe that the Rademacher functions $r_i, i=1,\dots,k$, are identically $+1$ or -1 on I_l^{k+a}. We denote by β_l^+ the number of those functions, which are identically $+1$ on I_l^{k+a}, and similarly for $\beta_l^- = k-\beta_l^+$. Then

$$\sum_{i,j=1}^k r_i(t)r_j(t) = \beta_l^+\cdot\beta_l^+ + \beta_l^-\cdot\beta_l^- - 2\beta_l^+\cdot\beta_l^- = (\beta_l^+-\beta_l^-)^2 \ge 0, \qquad t\in I_l^{k+a}.$$

Hence, the last sum in (4.14) is always non-negative. This finishes the proof of (4.12).
To prove (4.13), we use duality. We prove that (for $1<p\le 2$ and $1<q_2\le 2$)

$$||R_k|B^0_{p'q_2'}(\Omega)|| \lesssim k^{\frac{1}{q_2'}}, \quad k\in\mathbb{N} \tag{4.15}$$

and

$$k \lesssim \int_0^1 g_k^a(t)R_k(t)dt, \quad k\in\mathbb{N}. \tag{4.16}$$

From (4.15) and (4.16), the result follows similarly to (4.11). For $1<p\le 2$ and $q_2=1$, we use the Gagliardo-Nirenberg inequality

$$k^{\frac{1}{\tilde q}} \lesssim ||g_k^a|B^0_{\tilde p,\tilde q}(\Omega)|| \lesssim ||g_k^a|B^0_{p,1}(\Omega)||^{1-\theta}\cdot||g_k^a|B^0_{2,2}(\Omega)||^{\theta}$$

with

$$0<\theta<1, \quad \frac{1}{\tilde p} = \frac{1-\theta}{p}+\frac{\theta}{2}, \quad \frac{1}{\tilde q} = \frac{1-\theta}{1}+\frac{\theta}{2}$$

and the estimate $||g_k^a|B^0_{2,2}(\Omega)|| \approx ||g_k^a|L_2(\Omega)|| \lesssim k^{\frac12}$.
Let us comment on (4.15) and (4.16). The proof of (4.15) may be based on local means, or the reader may consult [9]. To prove (4.16) we write

$$\begin{aligned} \int_0^1 g_k^a(t)R_k(t)dt &= \int_0^1 \Big(\sum_{i=1}^k g_{k+a,i}(t)\Big)\cdot\Big(1-\chi_{\tilde\Gamma_{k+a}}(t)\Big)\cdot\Big(\sum_{j=1}^k r_j(t)\Big)dt \\ &= \sum_{i,j=1}^k \int_0^1 g_{k+a,i}(t)r_j(t)dt - \sum_{i,j=1}^k\int_{\tilde\Gamma_{k+a}} g_{k+a,i}(t)r_j(t)dt \\ &= k||\kappa|L_1(\mathbb{R})|| - ||\kappa|L_1(\mathbb{R})||\sum_{i,j=1}^k\int_{\tilde\Gamma_{k+a}} r_i(t)r_j(t)dt. \end{aligned}$$

Using (4.14) one may show that

$$\sum_{i,j=1}^{k} \int_{\tilde{\Gamma}_{k+a}} r_i(t) r_j(t) dt \le c\, k, \quad k \in \mathbb{N}$$

with $c < 1$. This calculation gives also the only restriction on a and it turns out, that $a = 2$ will do the job. This finishes the proof in $d = 1$.

If $d > 1$, only minor modifications using tensor products are needed. We leave out the details. □

Remark 4.3. This result describes the decay of (linear and nonlinear) sampling numbers of the embedding

$$id : B^s_{p_1 q_1}(\Omega) \to B^0_{p_2 q_2}(\Omega)$$

if $p_1 = p_2$. The results for $p_1 < p_2$ may be easily derived from [13], the sampling numbers decay like $n^{-\frac{s_1}{d}+\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}$. If $p_1 > p_2$, we may use one of the embeddings

$$B^s_{p_1 q_1}(\Omega) \hookrightarrow B^0_{p_1 q_2}(\Omega) \hookrightarrow B^0_{p_2 q_2}(\Omega), \qquad B^s_{p_1 q_1}(\Omega) \hookrightarrow B^s_{p_2 q_1}(\Omega) \hookrightarrow B^0_{p_2 q_2}(\Omega)$$

and obtain (some) estimates from above. Using some of the "test functions" mentioned above, we may also provide certain estimates from below. But it should be pointed out, that in several cases, there is a logarithmic gap between the estimates from above and the estimates from below. We leave the detailed discussion opened and do not state the partial results.

References

[1] G. Bourdaud, Sur les opérateurs pseudo-différentiels à coefficinets peu réguliers, Habilitation thesis, Université de Paris-Sud, Paris, 1983.

[2] H.-Q. Bui, M. Paluszyński, M. H. Taibleson, A maximal function characterization of weighted Besov-Lipschitz and Triebel-Lizorkin spaces, Studia Math. 119 (1996), 219-246.

[3] H.-Q. Bui, M. Paluszyński, M. H. Taibleson, Characterization of weighted Besov-Lipschitz and Triebel-Lizorkin spaces. The case $q < 1$, Journ. Fourier Anal. Appl. 3, special issue, (1997), 837-846.

[4] D. E. Edmunds, H. Triebel, Function spaces, entropy numbers, differential operators, Cambridge Univ. Press, Cambridge, 1996.

[5] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces I and II, Springer, 1996.

[6] E. Novak, H. Triebel, Function Spaces in Lipschitz Domains and Optimal Rates of Convergence for Sampling, Constr. Approx. 23 (2006), 325-350.

[7] Peetre, J., New thoughts on Besov spaces, Duke Univ. Math. Series, Durham, Univ., 1976.

[8] Rychkov, V. S., On a theorem of Bui, Paluszyński, and Taibleson, Proc. Steklov Inst. Math. 227 (1999), 280-292.

[9] H. Triebel, On Haar bases in Besov spaces, Serdica 4 (1978), 330-343.

[10] H. Triebel, Theory of function spaces, Birkhäuser, Basel, 1983.

[11] H. Triebel, Theory of function spaces II, Birkhäuser, Basel, 1992.

[12] H. Triebel, Theory of function spaces III, Birkhäuser, Basel, 2006.

[13] J. Vybíral, Sampling numbers and function spaces, *to appear in J. of Complexity.*