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Analysis of Kinetic Equations and their Fluid Dynamic Limits

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Preface

The mechanics of fluids is a very large and diverse discipline where every problem can be investigated from many different points of view. These views vary even in how detailed description is involved, i.e., we can describe a fluid in terms of the statistical physics, the thermodynamics of continuum, the quantum mechanics, . . . , they all have their own models of fluids. Then a question arises: “Are there any relations between these models?”

The process of the derivation of one model from the other one is often called the fluid dynamic limit. Both models can be on the same level of description or both can be from different levels. This work is focused on the relationship between the kinetic models and the macroscopic models of fluids, thus connects the statistical physics and the thermodynamics of continuum.

The work is divided into the three parts. Each part has its own introduction of the issue, use a slightly different notation, has its own bibliography and all terms are explained independently of the other parts. This is mainly due to the different natures of the investigated problems.

On the other side, all problems have some common features, and the most important one is the kinetic equation. We introduce the different modifications of the Boltzmann equation where the main role plays the Boltzmann collision operator and its properties. Here we would like to refer the reader to the first two sections of Chapter 2 where the precise and detailed derivation of the Boltzmann equation from the Liouville equation is demonstrated. The Boltzmann (kinetic) equation is a significant simplification of the huge N -particle problem (N is very large, typically of the order of Avogadro’s number) because instead of solving $6N$ equations of the motion we solve the one equation for the statistical distribution of particles in a fluid.

The first part is devoted to the Prandtl equation developed in order to approximate the behavior of the fluid near the boundary. It was Claude Bardos who pointed out to me the importance of this problem because of the blowup solution of the Prandtl equation¹. The result from this chapter completes the results obtained by C. Bardos, F. Golse and collaborators. The connection of the Boltzmann equation and the Prandtl equation is examined on the appropriately rescaled Boltzmann equation.

The second part investigates the model of multipolar viscous fluids and it is discovered that the higher order approximations of the collisional transfer originate the multipolar character of fluids. A slightly modified Boltzmann equation is introduced and the equations of balance of mass, momentum and energy of multipolar viscous fluids are found. In addition, the constitutive relations for the multipolar stress tensors and the heat flux of the linear dipolar viscous fluids are obtained, as well as the sign, magnitude and dependence on the density and the temperature of the coefficients in constitutive relations. The thermodynamic theory of multipolar viscous fluids according to J. Nečas and M. Šilhavý

¹Weinan E and Bjorn Enquist, Blowup of solutions of the unsteady Prandtl’s equation, *Comm. Pure and Appl. Math.*, Volume 50, Issue 12, Pages 1287 - 1293.

is introduced on the most simple model of multipolar viscous fluids – the linear dipolar viscous fluids. The general constitutive relations implied by the Clausius–Duhem inequality and the principle of material frame-indifference are developed. Both types of the constitutive relations are compared at the end of the chapter.

The third part was inspired by Jindřich Nečas at the sixth Paseky school of Mathematical theory in fluid mechanics) who gave me the article about a generalization of the Boltzmann equation. On the advice of Claude Bardos I started to investigate the acoustic limit of the generalized Boltzmann equation and I obtained some formal and analytical results. At the same time I tried to understand the physical background of such a generalization and I realized that the problem with the opposite sign is more natural. Hence the problems with the both signs are examined.

Contents

1	Compressible Boundary Layer and the Boltzmann Equation	4
1.1	Introduction	4
1.2	Prandtl Idea	6
1.3	Rescaled Boltzmann equation	7
1.4	Conclusion	10
1.5	Acknowledgements	10
	Bibliography	11
2	Theory of Multipolar Viscous Fluids from the Kinetic Scope	12
2.1	BBGKY hierarchy	13
2.2	Kinetic equation	16
2.3	Fluid dynamic limit	22
2.4	Enskog equation	30
2.5	Thermodynamics of dipolar viscous fluids – First approach	32
2.6	Thermodynamics of dipolar viscous fluids – Second approach	38
2.7	Conclusion	41
2.8	Acknowledgements	45
	Bibliography	46
3	Acoustic Regime of the Generalized Boltzmann Equation	48
3.1	Introduction	48
3.2	Sign of the additional term	50
3.3	Formal solution and the fluid dynamic limit	51
3.4	Initial value problem	55
3.5	Boundary value problem	57
3.6	Conclusion	61
3.7	Acknowledgements	62
	Bibliography	63
A	Appendix	64
A.1	Tensors formed from a unit vector	64
A.2	Orthogonal conditions	65

Chapter 1

Compressible Boundary Layer and the Boltzmann Equation

We consider flow of ideal gas in half space described by the system of compressible Navier–Stokes equations. We apply the Prandtl scaling and we obtain the system of compressible Prandtl equations. In this article a modification of the classical Chapman–Enskog method is proposed, which allows us to derive the system of compressible Prandtl equations directly from the Boltzmann equation without the use of the Knudsen-layer correction. Different types of boundary conditions are discussed.

1.1 Introduction

To describe the flow of ideal gas in R^D , one can choose two different descriptions – macroscopic and kinetic.

On the molecular (kinetic) level, the fundamental concept is based on the density of particles $f(x, v, t)$ in a single particle phase space $R^D \times R^D$. Here x and v denote the position and velocity of the particle at time t . Assuming zero external forces, the dimensionless kinetic equation for this non-negative function $f(x, v, t)$ takes the form

$$\partial_t f + (v \cdot \nabla_x) f = \frac{1}{\text{Kn}} \mathcal{C}(f). \quad (1.1)$$

On the right hand side there is the collision operator $\mathcal{C}(f)$, which acts only in the space of velocities, i.e. particles are assumed to be mass points and collisions instantaneous. According to the particular choice of the collision operator we call (1.1) the Boltzmann equation, the BGK model, etc.

On the macroscopic level the basic principles are the laws of conservation of mass, momentum and energy, together with the second law of thermodynamics. The description is based on macroscopic (measurable) quantities — density ρ , velocity u and temperature θ . For ideal (viscous) gas we obtain the system of dimensionless compressible Navier–Stokes equations:

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot (\rho u) &= 0, \\ \rho(\partial_t + u \cdot \nabla_x) u + \nabla_x(\rho \theta) &= \text{Kn} \nabla_x \cdot [\mu \sigma(u)], \\ \frac{D}{2} \rho(\partial_t + u \cdot \nabla_x) \theta + \rho \theta (\nabla_x \cdot u) &= \text{Kn} \frac{1}{2} \sigma(u) : \sigma(u) + \text{Kn} \nabla_x \cdot [\kappa \nabla_x \theta], \end{aligned} \quad (1.2)$$

where $\sigma(u) = (\nabla_x u + \nabla_x^T u) - \frac{2}{D} (\nabla_x \cdot u) I$ and μ, κ are the coefficients of viscosity and thermal conductivity of the gas.

The dimensionless parameter Kn , the Knudsen number, plays the key role in the study of the connection of these two models. From the kinetic formulation it follows that the Knudsen number is the ratio of the mean free path λ of the particle to the characteristic macroscopic length L . On the macroscopic level the Knudsen number is the ratio of the Mach number Ma to the Reynolds number Re (von Karman's relation)

$$\text{Kn} = \frac{\lambda}{L} = \frac{\text{Ma}}{\text{Re}}. \quad (1.3)$$

For sufficiently small Knudsen numbers or sufficiently dense gases both models can be used. Every macroscopic quantity has its corresponding molecular quantity and their relation is based on taking the mean value of the molecular quantity with respect to the density f :

$$\rho = \int_{R^D} f \, dv \equiv \langle f \rangle, \quad \rho u = \langle v f \rangle, \quad \rho \left(\frac{1}{2} |u|^2 + \frac{D}{2} \theta \right) = \langle \frac{1}{2} |v|^2 f \rangle.$$

In order to not have to distinguish between particular models of collision operator, we summarize their common and essential properties:

- collision invariants: $\langle \mathcal{C}(f) \psi \rangle = 0$ for $\psi = 1, v_1, \dots, v_D, |v|^2$, which formally lead to the system of macroscopic equations:

$$\begin{aligned} \partial_t \langle f \rangle + \nabla_x \cdot \langle v f \rangle &= 0, \\ \partial_t \langle v f \rangle + \nabla_x \cdot \langle v \otimes v f \rangle &= 0, \\ \partial_t \langle \frac{1}{2} |v|^2 f \rangle + \nabla_x \cdot \langle v \frac{1}{2} |v|^2 f \rangle &= 0. \end{aligned} \quad (1.4)$$

- fundamental inequality: $\langle \mathcal{C}(f) \ln f \rangle \leq 0$, which guarantees the validity of the second law of thermodynamics and enables us to characterize the gas in equilibrium by one of the following equivalent propositions:

$$\mathcal{C}(f) = 0 \iff \langle \mathcal{C}(f) \ln f \rangle = 0 \iff f = m = \frac{\rho}{(2\pi\theta)^{D/2}} \exp\left(-\frac{|v-u|^2}{2\theta}\right). \quad (1.5)$$

The equilibrium density m is called the Maxwellian distribution.

The formal system (1.4) of the macroscopic equations is not closed and several methods of its closure were proposed. We will deal with the Chapman–Enskog method ([ChC39] and [BGL91]) based on the expansion of the state close to equilibrium with respect to a small parameter Kn . It is known that the Chapman–Enskog method if applied to (1.4) results in the system (1.2) of compressible Navier–Stokes equations.

In the presence of the boundary both formulations have to be completed by stating the boundary conditions. For simplicity we assume that the domain is the half space $x' = (x_1, \dots, x_{D-1}) \in R^{D-1}$ and $x_D > 0$ with an impenetrable wall at $x_D = 0$. Density distribution of particles leaving the wall ($v' \in R^{D-1}$ and $v_D > 0$) can be prescribed in different ways. We shall deal with the following three types

- specular reflection:

$$f(v', v_D) = f(v', -v_D); \quad (1.6)$$

- reverse reflection:

$$f(v', v_D) = f(-v' + 2u'_w, -v_D); \quad (1.7)$$

- diffusive reflection:

$$f(v', v_D) = \frac{-\int_{v_D < 0} v_D f(v', v_D) dv' dv_D}{(2\pi\theta_w)^{(D-1)/2} \theta_w} \exp\left(-\frac{|v' - u'_w|^2 + v_D^2}{2\theta_w}\right). \quad (1.8)$$

Here $(u'_w, 0)$ and θ_w are the velocity and temperature of the wall. For the system of Navier–Stokes equations we consider

$$\partial_{x_D} u' = 0, \quad u_D = 0 \quad \text{and} \quad \partial_{x_D} \theta = 0; \quad (1.9)$$

$$u' = u'_w, \quad u_D = 0 \quad \text{and} \quad \partial_{x_D} \theta = 0; \quad (1.10)$$

$$u' = u'_w, \quad u_D = 0 \quad \text{and} \quad \theta = \theta_w. \quad (1.11)$$

1.2 Prandtl Idea

In the rest of this paper we shall deal with the flow of fluids with small viscosity and we shall focus on the thin boundary layer. Let ε be the thickness of such a region. From the Prandtl theory [ChM93] we know that such a flow is characterized by a high Reynolds number $\text{Re} = \varepsilon^{-2}$. Because the gas is compressible, its Mach number is finite ($\text{Ma} = 1$) and consequently the Knudsen number $\text{Kn} = \varepsilon^2$. To derive a simpler system of macroscopic equations in the boundary layer we rescale the independent variables

$$\tilde{x}' = x', \quad \tilde{x}_D = \frac{x_D}{\varepsilon} \quad \text{and} \quad \tilde{t} = t.$$

From the continuity equation we see that the normal component u_D of the velocity is also very small, therefore we set

$$\tilde{\rho}(\tilde{x}, \tilde{t}) = \rho(x, t), \quad \tilde{u}'(\tilde{x}, \tilde{t}) = u'(x, t), \quad \tilde{u}_D(\tilde{x}, \tilde{t}) = \frac{u_D(x, t)}{\varepsilon} \quad \text{and} \quad \tilde{\theta}(\tilde{x}, \tilde{t}) = \theta(x, t).$$

For these new quantities we obtain the system of compressible Prandtl equations provided higher order terms in the system of compressible Navier–Stokes equations can be neglected. Thus we derive the system (written without tilde):

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot (\rho u) &= 0, \\ \rho(\partial_t + u \cdot \nabla_x) u' + \nabla_{x'} \cdot (\rho \theta) &= \partial_{x_D} (\mu \partial_{x_D} u'), \\ \partial_{x_D} (\rho \theta) &= 0, \\ \frac{D}{2} \rho (\partial_t + u \cdot \nabla_x) \theta + \rho \theta (\nabla_x \cdot u) &= \mu |\partial_{x_D} u'|^2 + \partial_{x_D} (\kappa \partial_{x_D} \theta), \end{aligned} \quad (1.12)$$

together with the set of boundary conditions considered above. From the third equation it follows that the pressure depends only on the variables x' and t , so it can be considered to be a given function. Because also values of the velocity u' and the temperature θ have to be prescribed as $x_D \rightarrow \infty$, we suppose

$$\begin{aligned} \rho \theta(x', t) &= \rho_E \theta_E(x', x_D = 0, t), \\ u'(x_D = \infty) &= u'_E(x_D = 0), \\ \theta(x_D = \infty) &= \theta_E(x_D = 0), \end{aligned} \quad (1.13)$$

where ρ_E , u_E and θ_E is the solution of the compressible Euler equations on the halfspace with the slip boundary conditions.

1.3 Rescaled Boltzmann equation

In this section we establish the connection between the appropriately rescaled Boltzmann equation and the system of compressible Prandtl equations. For this purpose we (slightly) modify the standard Chapman–Enskog method ([ChC39] and [BGL91]). Before doing so we introduce convenient notation.

In order to find a solution not far from equilibrium it is advantageous to linearize the collision operator by Taylor’s formula

$$\begin{aligned} \frac{2}{m} \mathcal{C}(m(1 + \varepsilon g)) &= \underbrace{\frac{2}{m} \mathcal{C}(m)}_{= 0, \text{ see (1.5)}} + \varepsilon \underbrace{\frac{2}{m} D\mathcal{C}(m)(mg)}_{\equiv \varepsilon \mathcal{L}(g)} + \varepsilon^2 \underbrace{\frac{1}{m} D^2\mathcal{C}(m)(mg)^2}_{\equiv \varepsilon^2 \mathcal{Q}(g, g)} + O(\varepsilon^3). \end{aligned}$$

The linear operator \mathcal{L} is supposed to be non-negative, self-adjoint and to satisfy the Fredholm alternative with $(D+2)$ -dimensional kernel spanned by the collision invariants $\{1, v_1, \dots, v_D, |v|^2\}$ on the Hilbert space

$$L_m^2(R^D) = \{g(v) : R^D \rightarrow R \mid g m^{1/2} \in L^2(R^D)\}$$

with the scalar product

$$(g, h)_{L_m^2} = \int_{R^D} gh m dv.$$

Further we use vector (tensor) notation:

$$\mathcal{V} = \left(\frac{v' - u'}{\sqrt{\theta}}, \frac{v_D}{\sqrt{\theta}} \right), \quad \mathcal{A}(\mathcal{V}) = \mathcal{V} \otimes \mathcal{V} - \frac{1}{D} |\mathcal{V}|^2 I, \quad \mathcal{B}(\mathcal{V}) = \left(\frac{1}{2} |\mathcal{V}|^2 - \frac{D+2}{2} \right) \mathcal{V},$$

where \otimes is the tensor product. We assume that the solution of

$$\mathcal{L}(\mathcal{A}') = \mathcal{A} \quad \text{and} \quad \mathcal{L}(\mathcal{B}') = \mathcal{B}$$

can be written (we know that \mathcal{A}, \mathcal{B} satisfy Fredholm alternative) in the form

$$\mathcal{A}'(\mathcal{V}) = -\alpha(\rho, \theta, |\mathcal{V}|) \mathcal{A}(\mathcal{V}) \quad \text{and} \quad \mathcal{B}'(\mathcal{V}) = -\beta(\rho, \theta, |\mathcal{V}|) \mathcal{B}(\mathcal{V}),$$

where α, β are positive scalar functions. We denote by m_0 a particular type of Maxwellian distribution with zero mean velocity in x_D direction, i.e.

$$m_0 = \frac{\rho}{(2\pi\theta)^{D/2}} \exp\left(-\frac{|v' - u'|^2 + v_D^2}{2\theta}\right).$$

Following the Prandtl idea of scaling we introduce the rescaled Boltzmann equation (tilde omitted)

$$\partial_t f + (v' \cdot \nabla_{x'}) f + \frac{1}{\varepsilon} v_D \partial_{x_D} f = \frac{1}{\varepsilon^2} \mathcal{C}(f). \quad (1.14)$$

Now we can state our main result.

Theorem 1.1 (Formal). *Assume that ρ, u, θ solve the system of compressible Prandtl equations (1.12) with the coefficients of viscosity and thermal conductivity given by¹*

$$\mu(\rho, \theta) = \theta \langle \alpha(\rho, \theta, |\mathcal{V}|) \mathcal{A}_{1D}^2(\mathcal{V}) m_0 \rangle, \quad (1.15)$$

$$\kappa(\rho, \theta) = \theta \langle \beta(\rho, \theta, |\mathcal{V}|) \mathcal{B}_D^2(\mathcal{V}) m_0 \rangle, \quad (1.16)$$

¹we can observe the absence of the parameter ε in front of these coefficients in the system (1.12). This differs from the result of the classical Chapman–Enskog expansion.

with one set of the boundary conditions (1.9)–(1.11) and with additional conditions (1.13). Then there exist functions g and w such that

$$h^\varepsilon = m_0 (1 + \varepsilon g + \varepsilon^2 w) \quad (1.17)$$

is an approximative solution of order 1 to the rescaled Boltzmann equation (1.14), i.e.

$$\partial_t h^\varepsilon + (v' \cdot \nabla_{x'}) h^\varepsilon + \frac{1}{\varepsilon} v_D \partial_{x_D} h^\varepsilon = \frac{1}{\varepsilon^2} \mathcal{L}(h^\varepsilon) + O(\varepsilon) \quad (1.18)$$

and specular reflection boundary condition corresponding to macroscopic conditions (1.9) is approximated up to the order 2, while reverse reflection resp. diffusive reflection corresponding to macroscopic conditions (1.10) resp. (1.11) are satisfied approximately to the order 1.

Moreover, w lies in the orthogonal complement to the kernel of linear operator \mathcal{L} and g is given by the formula

$$g = -\alpha(\rho, \theta, |\mathcal{V}|) \sum_{j=1}^{D-1} \mathcal{A}_{jD}(\mathcal{V}) \partial_{x_D} u_j - \beta(\rho, \theta, |\mathcal{V}|) \mathcal{B}_D(\mathcal{V}) \frac{\partial_{x_D} \theta}{\sqrt{\theta}} + v_D \frac{u_D}{\theta}. \quad (1.19)$$

Proof. The above mentioned modification of the standard Chapman–Enskog method is based on the relations of orthogonality for g . This function does not lie in the orthogonal space to the kernel of operator \mathcal{L} but for $\psi = v_D$ this condition is:

$$\langle g v_D m_0 \rangle = \rho u_D. \quad (1.20)$$

There is no dispute due to particular choice of Maxwellian m_0 .

We substitute approximative solution h^ε into the rescaled Boltzmann equation (1.14). Comparing the terms of the same order and omitting higher order terms we obtain

$$\mathcal{L}(g) = v_D \frac{\partial_{x_D} m_0}{m_0}, \quad (1.21)$$

$$\mathcal{L}(w) = \frac{(\partial_t + v' \cdot \nabla_{x'}) m_0}{m_0} + \frac{v_D \partial_{x_D} (m_0 g)}{m_0} - \mathcal{Q}(g, g). \quad (1.22)$$

The right hand sides of both the equations have to satisfy conditions of Fredholm alternative with collision invariants $\{1, v' - u', v_D, \frac{1}{2}(|v' - u'|^2 + v_D^2)\}$. For partial derivative ∂ with respect to t or x_j we have the formula

$$\frac{\partial m_0}{m_0} = \frac{\partial(\rho\theta)}{\rho\theta} + \left(\frac{|v' - u'|^2 + v_D^2}{2\theta} - \frac{D+2}{2} \right) \frac{\partial\theta}{\theta} + \frac{(v' - u') \cdot \partial u'}{\theta}.$$

From the equation (1.21) for the collision invariant $\psi = v_D$ we see that

$$\partial_{x_D}(\rho\theta) = 0 \quad (1.23)$$

and consequently this equation can be rewritten as

$$\mathcal{L}(g) = \sum_{j=1}^{D-1} \mathcal{A}_{jD}(\mathcal{V}) \partial_{x_D} u_j + \mathcal{B}_D(\mathcal{V}) \frac{\partial_{x_D} \theta}{\sqrt{\theta}}.$$

Solution g is exactly (1.19) if we suppose additional condition (1.20). Solvability conditions for (1.22) reduce to the remaining Prandtl equations. The term $\mathcal{Q}(g, g)$ has a property of collision invariants. The second term gives the relations:

$$\begin{aligned} \langle v_D \partial_{x_D}(m_0 g) \rangle &= \partial_{x_D}(\rho u_D), \\ \langle (v_j - u_j) v_D \partial_{x_D}(m_0 g) \rangle &= \partial_{x_D}(\theta \langle \mathcal{A}_{jD} m_0 g \rangle) + \rho u_D \partial_{x_D} u_j, \quad j = 1, \dots, D-1, \\ \langle v_D^2 \partial_{x_D}(m_0 g) \rangle &= 0, \\ \left\langle \frac{|v' - u'|^2 + v_D^2}{2} v_D \partial_{x_D}(m_0 g) \right\rangle &= \partial_{x_D}(\theta^{3/2} \langle \mathcal{B}_D m_0 g \rangle) + \frac{D+2}{2} \rho \theta \partial_{x_D} u_D \\ &\quad + \sum_{j=1}^{D-1} (\partial_{x_D} u_j) \theta \langle \mathcal{A}_{jD} m_0 g \rangle. \end{aligned}$$

From the first term of (1.22) we deduce

$$\begin{aligned} \langle (\partial_t + v' \cdot \nabla_{x'}) m_0 \rangle &= \partial_t \rho + \nabla_{x'} \cdot (\rho u'), \\ \langle (v_j - u_j) (\partial_t + v' \cdot \nabla_{x'}) m_0 \rangle &= \rho (\partial_t + u' \cdot \nabla_{x'}) u_j + \partial_{x_j}(\rho \theta), \quad j = 1, \dots, D-1, \\ \langle v_D (\partial_t + v' \cdot \nabla_{x'}) m_0 \rangle &= 0, \\ \left\langle \frac{|v' - u'|^2 + v_D^2}{2} (\partial_t + v' \cdot \nabla_{x'}) m_0 \right\rangle &= \frac{D}{2} \rho (\partial_t + u' \cdot \nabla_{x'}) \theta + \frac{D}{2} \theta (\partial_t \rho + \nabla_{x'} \cdot (\rho u')) \\ &\quad + \rho \theta (\nabla_{x'} \cdot u'). \end{aligned}$$

The expressions containing \mathcal{A}_{jD} and \mathcal{B}_D can still be simplified using definitions (1.15) and (1.16) of the coefficients μ and κ , due to the symmetry

$$\theta \langle \mathcal{A}_{jD}(\mathcal{V}) m_0 g \rangle = -\mu \partial_{x_D} u_j \quad \text{and} \quad \theta^{3/2} \langle \mathcal{B}_D(\mathcal{V}) m_0 g \rangle = -\kappa \partial_{x_D} \theta.$$

Let us notice that from (1.23) we have $\frac{D}{2} \theta u_D \partial_{x_D} \rho + \frac{D}{2} \rho u_D \partial_{x_D} \theta = 0$ and the Prandtl equations follow.

To prove the statement about boundary conditions we have to show that relations (1.6)–(1.8) hold approximately with h^ε instead of f up to the order mentioned above. In all these cases the wall is impenetrable, so

$$g(x_D = 0) = -\alpha \sum_{j=1}^{D-1} \mathcal{A}_{jD}(\mathcal{V}) \partial_{x_D} u_j - \beta \mathcal{B}_D(\mathcal{V}) \frac{\partial_{x_D} \theta}{\sqrt{\theta}}.$$

For specular reflection Maxwellians on both sides of (1.6) are equal and $g(x_D = 0) = 0$. In the case of diffusive reflection we can evaluate

$$-\int_{v_D < 0} v_D h^\varepsilon(x_D = 0) dv = \frac{\rho \theta}{(2\pi\theta)^{1/2}} + O(\varepsilon^2),$$

so for $\theta = \theta_w$ and $u' = u'_w$ we obtain equal Maxwellians too. Reverse reflection has order of approximation one, because we have

$$g(x_D = 0) = -\alpha \sum_{j=1}^{D-1} \mathcal{A}_{jD}(\mathcal{V}) \partial_{x_D} u_j.$$

□

1.4 Conclusion

Our result completes a list of several recent studies. In the article [BAGS99] the authors introduced scaling, which gives the incompressible Prandtl equations from the Boltzmann equation by the moments method. In the article [SBGS00] the stationary compressible Prandtl equations are derived by a modified Hilbert method. In this paper, we propose a modification of the Chapman–Enskog method that enables us to perform the limit leading to the evolutionary compressible Prandtl equations without the use of the Knudsen-layer correction.

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Chapter 2

Theory of Multipolar Viscous Fluids from the Kinetic Scope

The theory of multipolar fluids, originally introduced in the papers by Green and Rivlin (see [GR64a] and [GR64b]) and later on extended by Nečas and Šilhavý ([NS91]), results to the system of equations where certain constitutive quantities, in particular the Cauchy stress, the heat flux and the higher order stress-like tensors depend on the higher order spatial derivatives of the velocity and the temperature.

This approach leads to the system of partial differential equations generalizing the classical Navier–Stokes and Navier–Stokes–Fourier equations for fluids, and also the classical, yet not at all understood, equations of finite (visco)-elasticity. The advantage of such systems lies in their accessibility by current tools of mathematical analysis. Thus, the articles by Nečas, Novotný and Šilhavý ([NNS89] and [NNS92]) were the first ones where the long-time and large-data well-posedness (existence, uniqueness and continuous dependence on data) was established in the case of (dipolar respectively tripolar) compressible heat-conducting fluids. Similarly, Nečas and Růžička ([NR92]) developed a consistent mathematical theory for dynamical problems of finite elasticity (finite visco-elasticity). While for classical compressible fluids (“classical” means without a need to incorporate the dependence of the constitutive relations on the higher spatial derivatives), there are nowadays alternative and deeper mathematical studies that concern long-time and large-data existence theory (see the monographies by P. L. Lions [Li98], E. Feireisl [Fe03] or a very recent paper by E. Feireisl [Fe07]), in the case of elastic materials subjected to large time-dependent deformations the long-time and large-data theory is not available.

Although the accessibility of the models to current methods of mathematical analysis, the well-posedness of some models as well as consistency of the models with the classical principles of continuum thermodynamics are the positive features of multipolar fluid-like models, these models are not popular and this is due to several reasons.

First of all, there are no experimental data that would confirm the higher gradient theory and give insight what kind of boundary conditions should be taken into account for higher derivatives.

Another reason is the lack of the details of physical micro-processes which in fluids are passing, and which are originating macroscopic phenomena expressed and described by the theory of multipolar fluids.

Also, even for linear constitutive equations, there are too many material coefficients that it is difficult to identify and we know almost nothing neither about their dependence

on density and temperature nor about their sign.

The aim of this chapter is to provide somehow different insight on the multipolar fluid models. We use the approach of the fluid dynamic limit from the kinetic level to the macroscopic one and we take the advantage of this technique to derive the sign, and the density and temperature dependence of coefficients in constitutive relations. We also slightly reveal the secret of micro-processes in the fluid and we show that it is the higher order approximation of the collisional transfer which is originating their multipolar character. The key role in this study plays the binary collision – a non-local and non-instant process in a fluid.

The scheme of this chapter is the following. In the first section we recall the BBGKY hierarchy, a usual step in the derivation of kinetic equations. In the next section, after making several simplifications, we conclude with the kinetic equation which incorporates the higher order approximation of the collisional transfer.

Developing such a kinetic equation we then take in Section 2.3 its fluid dynamic limit and we obtain the equations of balance of mass, momentum and energy for dipolar viscous fluids. In addition we get the constitutive relations for such a fluid.

In the next section 2.4 we recall the Enskog kinetic equation and show some similar features of his kinetic equation and our one.

In section 2.5, we recall the foundations of dipolar fluids within the framework of continuum thermodynamics how it has been done at the early nineties. In section 2.6 we provide foundations of dipolar fluid model using completely different approach developed by K. R. Rajagopal and his coworkers.

In the concluding section, we compare the results of all these approaches.

2.1 BBGKY hierarchy

Let us start with the closed system of N identical particles: every particle obeys the Newtonian mechanics and $\mathbf{x}_i = \mathbf{x}_i(t)$ and $\mathbf{v}_i = \mathbf{v}_i(t)$ denote the position and the velocity of the particle i at time t . Such a system is described by a N -particle distribution function $f^N = f^N(\mathbf{x}_1, \mathbf{v}_1, \dots, \mathbf{x}_N, \mathbf{v}_N, t)$ fulfilling

$$\int f^N(\mathbf{x}_1, \mathbf{v}_1, \dots, \mathbf{x}_N, \mathbf{v}_N, t) \prod_{i=1}^N d\mathbf{x}_i d\mathbf{v}_i = N!. \quad (2.1)$$

The norm $N!$ exactly corresponds to the fact that the particles are indistinguishable, so exactly the same system can be realized in $N!$ ways. The Hamiltonian of such a system consists of the sum of the kinetic energy of all particles and the potential energy of their mutual interactions (we assume that the inter-particle potential φ_{ij} acting between particles i and j is spherically symmetric and depends only on the distance $|\mathbf{x}_j - \mathbf{x}_i|$ of both particles); thus

$$H^N = \sum_{i=1}^N \frac{1}{2} m |\mathbf{v}_i|^2 + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \varphi_{ij}, \quad (2.2)$$

where m is the mass of one particle. The time evolution of such a closed system is driven by the Liouville equation which represents the conservation of the probability density in

a phase space

$$\partial_t f^N + \sum_{i=1}^N \mathbf{v}_i \cdot \nabla_{\mathbf{x}_i} f^N - \frac{1}{m} \sum_{\substack{i,j=1 \\ i \neq j}}^N \nabla_{\mathbf{x}_i} \varphi_{ij} \cdot \nabla_{\mathbf{v}_i} f^N = 0. \quad (2.3)$$

Here $\nabla_{\mathbf{x}}$ denotes a gradient with respect to three coordinates of the vector \mathbf{x} , and $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^3 x_i y_i$ is a scalar product of two vectors.

The Liouville equation is often expressed in terms of the Poisson brackets

$$\partial_t f^N + [f^N, H^N]^N = 0, \quad (2.4)$$

where

$$[\psi, \varphi]^N = \sum_{i=1}^N \{ \nabla_{\mathbf{x}_i} \psi \cdot \nabla_{m\mathbf{v}_i} \varphi - \nabla_{m\mathbf{v}_i} \psi \cdot \nabla_{\mathbf{x}_i} \varphi \}. \quad (2.5)$$

We assume that the initial conditions are symmetric due to the interchange of particles. Because the Hamiltonian is also symmetric, we have the symmetry of the density f^N all the time. It is then reasonable to define the h -particle distribution function f^h through the relation

$$f^h = \frac{1}{(N-h)!} \int f^N(\mathbf{x}_1, \mathbf{v}_1, \dots, \mathbf{x}_N, \mathbf{v}_N, t) \prod_{i=h+1}^N d\mathbf{x}_i d\mathbf{v}_i, \quad h = 1, 2, \dots, N-1. \quad (2.6)$$

The norm of the density f^h has a similar interpretation as the norm of the density f^N

$$\int f^h(\mathbf{x}_1, \mathbf{v}_1, \dots, \mathbf{x}_h, \mathbf{v}_h, t) \prod_{i=1}^h d\mathbf{x}_i d\mathbf{v}_i = \frac{N!}{(N-h)!}. \quad (2.7)$$

Upon integration of the Liouville equation over the positions and the velocities of $N-h$ particles where $h = 1, \dots, N-1$ we obtain the chain of equations – the so-called BBGKY hierarchy which is named by the authors Bogoliubov [Bo46], Born and Green [BG49], Kirkwood [Ki46] and Yvon [Yv35]:

$$\partial_t f^h + [f^h, H^h]^h = \frac{1}{m} \sum_{i=1}^h \int \nabla_{\mathbf{v}_i} f^{h+1} \cdot \nabla_{\mathbf{x}_i} \varphi_{i,h+1} d\mathbf{x}_{h+1} d\mathbf{v}_{h+1}, \quad h = 1, \dots, N-1. \quad (2.8)$$

The BBGKY hierarchy is not a closed system. Thus, in order to solve the equation for f^h we have to know the solution f^{h+1} . Fortunately, we can adopt and combine additional physical assumptions/restrictions which help us to evaluate the right hand side of the equation for f^h in terms of f^1, f^2, \dots, f^h , and we get the closed chain of the equations for f^1, f^2, \dots, f^h .

Let us mention some of these assumptions/restrictions:

binary collisions — in a rarefied gas the probability of the collisions of 3 or more particles is negligible. The number n of particles per unit volume and the particle diameter σ have to satisfy

$$n\sigma^3 \ll 1. \quad (2.9)$$

repulsive potential — the particles can form more complex objects like dimers, ... in the case of the potential with an attractive part (e.g., the Lennard–Jones potential). Then the closure must include both the probability density of one particle f^1 and the probability density of dimers f^2, \dots (See [RHB82] and [Ra84].)

instantaneous collisions — the phase space of two interacting particles can be divided into the pre-collision, post-collision and interactive part. If the potential of interaction is strictly repulsive and has a short range – the concept of *cutoff potential* – we can neglect the interactive part of the phase space and evaluate only the pre-collision and post-collision parts.

point particles — the internal structure of particles is omitted. Very simple model of particles with internal (quantum) states is the mixture where each internal state of particle generates a new specie of the mixture.

neutral particles — the charged particles require a specific approach. It is mainly due to the softness of the Coulomb’s forces (they decrease with the square of the distance), hence each particle interacts with each other and the concept of cutoff potential is useless. In addition, the particles generate the electromagnetic field and from the macroscopic point of view the system have to obey the Maxwell’s laws. A gas of charged particles described by the Landau equation is the basic model of plasma.

molecular chaos — it is assumed that the particles entering a collision are in the independent states. It means that the material has no memory and the probability density in the pre-collision part of the phase space can be factorized, i.e., $f^2 = f^1 \cdot f^1$. We can imagine that the particle hits many of other particles between two subsequent collisions with the same particle. We notice that the principle of factorization is directly contained in the Liouville equation. Let us assume that the s -particle system can be divided into two independent groups with q and $s - q$ particles. The Hamiltonian of such system is $H^s = H^q + H^{s-q}$ and for the Poisson brackets we have

$$[\cdot, H^s]^s = [\cdot, H^q]^q + [\cdot, H^{s-q}]^{s-q}. \quad (2.10)$$

As the groups are independent, the probability density $f^s = f^q \cdot f^{s-q}$ and the Liouville equation splits into two parts

$$f^q \{ \partial_t f^{s-q} + [f^{s-q}, H^{s-q}]^{s-q} \} + f^{s-q} \{ \partial_t f^q + [f^q, H^q]^q \} = 0. \quad (2.11)$$

hard spheres — the model of many “billiard balls” where the collisions of particles (balls) are instantaneous and elastic.

collisional transfer — the transfer of mass, momentum and energy between two regions, i.e., across an imaginary surface, has different origins. At first, there is the free motion of particles between collisions, and the particles carry the mass, the momentum and the energy themselves. At second, there are the binary collisions of point particles on the imaginary surface. The total amount of mass, momentum and energy of both particles is divided due to the details of the collision. At third, these quantities are transferred over the distance separating the two colliding particles. The transfer of momentum and energy (not of mass) originates in the shell around the imaginary surface of the thickness comparable with the interaction range. This mechanism is called the *collisional transfer*.

initial conditions — initial conditions must also obey the chosen physical assumptions/restrictions, for example, the molecular chaos must be present in the initial conditions.

no mixture, no external forces, no chemical reactions — the assumptions which reduce the complexity of the problem significantly.

2.2 Kinetic equation

In this section we deal with the closure of the equation for the 1-particle distribution function

$$\partial_t f^1 + \mathbf{v}_1 \cdot \nabla_{\mathbf{x}_1} f^1 = \frac{1}{m} \iint \nabla_{\mathbf{v}_1} f^2 \cdot \nabla_{\mathbf{x}_1} \varphi_{12} d\mathbf{x}_2 d\mathbf{v}_2. \quad (2.12)$$

We assume the potential $\varphi_{12} = \varphi(|\mathbf{x}_1 - \mathbf{x}_2|)$ is a strictly repulsive cutoff potential which is negligible on distances larger than σ – the range of inter-particle forces. We also postulate the assumption of binary collisions – “rarefied gas”. Furthermore, we assume that no external forces act on the system of particles and no chemical reactions take place inside the system.

We would like to emphasize that we do not impose any additional assumptions on the binary collisions (e.g., instantaneous collisions, point particles, hard spheres, ...) and we investigate the binary collisions as a non-local and non-instant processes, i.e., collisions have a duration time and particles interact over a separating distance.

We follow the considerations of H. S. Green [Gr52]. As we assume only the binary collisions, the Liouville equation holds for the 2-particle distribution function f^2 during the collision. It implies that f^2 stays constant along the trajectory of colliding particles in a phase space. Especially, it is equal to the value at time t_0 when particles are entering the collision (the distance $|\mathbf{x}_1(t_0) - \mathbf{x}_2(t_0)|$ between two particles at this time is exactly σ). In a rarefied gas we can assume that both particles have not any interactions in the past and therefore their velocities are completely independent of each other. This crucial step – the principle of molecular chaos – changes the reversible process into the irreversible one. (We have chosen the past and the future.) Mathematically, it means that

$$f^2(\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{v}_1(t), \mathbf{v}_2(t), t) = f^1(\mathbf{x}_1(t_0), \mathbf{v}_1(t_0), t_0) f^1(\mathbf{x}_2(t_0), \mathbf{v}_2(t_0), t_0).$$

The formula above evaluates the 2-particle distribution function in the terms of the 1-particle distribution function, and using the kinetic equation (2.12), we have closed the kinetic equation for f^1 :

$$\partial_t f^1 + \mathbf{v}_1 \cdot \nabla_{\mathbf{x}_1} f^1 = \frac{1}{m} \iint \nabla_{\mathbf{v}_1} [f^1(\mathbf{x}_1(t_0), \mathbf{v}_1(t_0), t_0) f^1(\mathbf{x}_2(t_0), \mathbf{v}_2(t_0), t_0)] \cdot \nabla_{\mathbf{x}_1} \varphi_{12} d\mathbf{x}_2 d\mathbf{v}_2. \quad (2.13)$$

This equations differs from the Boltzmann equation because of the evolution of space and time variables. But the concept of instantaneous point collisions is so useful, and we have also good knowledge of the classical Boltzmann equation, that we reformulate the above formula in such a way

$$\begin{aligned} & f^1(\mathbf{x}_1(t_0), \mathbf{v}_1(t_0), t_0) f^1(\mathbf{x}_2(t_0), \mathbf{v}_2(t_0), t_0) \\ &= f^1(\mathbf{x}_1(t), \mathbf{v}_1(t_0), t) f^1(\mathbf{x}_1(t), \mathbf{v}_2(t_0), t) + E_1 + E_2, \end{aligned} \quad (2.14)$$

where

$$\begin{aligned} E_1 &= f^1(\mathbf{R}(t), \mathbf{v}_1(t_0), t) f^1(\mathbf{R}(t), \mathbf{v}_2(t_0), t) \\ &\quad - f^1(\mathbf{x}_1(t), \mathbf{v}_1(t_0), t) f^1(\mathbf{x}_1(t), \mathbf{v}_2(t_0), t), \end{aligned} \quad (2.15)$$

and

$$E_2 = f^1(\mathbf{x}_1(t_0), \mathbf{v}_1(t_0), t_0) f^1(\mathbf{x}_2(t_0), \mathbf{v}_2(t_0), t_0) - f^1(\mathbf{R}(t), \mathbf{v}_1(t_0), t) f^1(\mathbf{R}(t), \mathbf{v}_2(t_0), t). \quad (2.16)$$

The first term E_1 emphasizes the nature of the *collisional transfer*, and it moves the collision into its real physical center – the mass center $\mathbf{R}(t) = (\mathbf{x}_1(t) + \mathbf{x}_2(t))/2$. The second term E_2 emphasizes that a binary collision is a non-local and non-instant process.

Let us simplify the current notation $\mathbf{x}_1(t)$, $\mathbf{v}_1(t)$, $\mathbf{x}_2(t)$, $\mathbf{v}_2(t)$ will be \mathbf{x} , \mathbf{v} , \mathbf{y} , \mathbf{w} , and $\mathbf{v}_1(t_0)$, $\mathbf{v}_2(t_0)$ will be \mathbf{v}_0 , \mathbf{w}_0 , f^1 will be f , $\mathbf{r} = \mathbf{y} - \mathbf{x}$ and $\varphi_{12} = \varphi$. The equation (2.13) becomes

$$\begin{aligned} \partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f &= \frac{1}{m} \iint \nabla_{\mathbf{v}} [f(\mathbf{x}, \mathbf{v}_0, t) f(\mathbf{x}, \mathbf{w}_0, t)] \cdot \nabla_{\mathbf{x}} \varphi d\mathbf{y} d\mathbf{w} \\ &+ \frac{1}{m} \iint \nabla_{\mathbf{v}} [E_1 + E_2] \cdot \nabla_{\mathbf{x}} \varphi d\mathbf{y} d\mathbf{w}. \end{aligned} \quad (2.17)$$

The system of two interacting particles has the equations of the motion:

$$\frac{d\mathbf{r}}{dt} = \mathbf{w} - \mathbf{v}, \quad m \frac{d\mathbf{v}}{dt} = \nabla_{\mathbf{r}} \varphi, \quad m \frac{d\mathbf{w}}{dt} = -\nabla_{\mathbf{r}} \varphi, \quad (2.18)$$

$$\mathbf{r}(t_0) = \mathbf{r}_0, \quad \mathbf{v}(t_0) = \mathbf{v}_0, \quad \mathbf{w}(t_0) = \mathbf{w}_0, \quad (2.19)$$

for the nine unknown functions of time $\mathbf{r}(t)$, $\mathbf{v}(t)$ and $\mathbf{w}(t)$. This system has the eight constants of the motion $I_p = I_p(\mathbf{r}, \mathbf{v}, \mathbf{w})$, i.e., the functions of \mathbf{r} , \mathbf{v} and \mathbf{w} which stay constant during the collision of particles. The following equation holds for each I_p ;

$$\begin{aligned} \frac{dI_p}{dt} &= 0 \\ &= \frac{d\mathbf{r}}{dt} \cdot \nabla_{\mathbf{r}} I_p + \frac{d\mathbf{v}}{dt} \cdot \nabla_{\mathbf{v}} I_p + \frac{d\mathbf{w}}{dt} \cdot \nabla_{\mathbf{w}} I_p \\ &= (\mathbf{w} - \mathbf{v}) \cdot \nabla_{\mathbf{r}} I_p + \frac{1}{m} \nabla_{\mathbf{r}} \varphi \cdot (\nabla_{\mathbf{v}} I_p - \nabla_{\mathbf{w}} I_p), \quad p = 1, \dots, 8. \end{aligned} \quad (2.20)$$

Usually, the total momentum vector, total angular momentum vector and total energy are the seven constants of the motion. Any sufficiently smooth function of the constants of the motion I_p is also a constant and also satisfies the equation (2.20). (The ninth constant of the motion is the invariance of the system (2.18) against a time shifting.)

We can inverse the problem and evaluate the initial conditions \mathbf{r}_0 , \mathbf{v}_0 and \mathbf{w}_0 as functions of the constants of the motion, hence solve the nine equations

$$\begin{aligned} I_p(\mathbf{r}_0, \mathbf{v}_0, \mathbf{w}_0) &= I_p, \quad p = 1, \dots, 8, \\ |\mathbf{r}_0| &= \sigma. \end{aligned} \quad (2.21)$$

It implies that the term $f(\mathbf{x}, \mathbf{v}_0, t) f(\mathbf{x}, \mathbf{w}_0, t)$ is a function of the constants of the motion (and \mathbf{x} , t in addition), thus we can apply the formula (2.20) on this term. Therefore

$$\frac{1}{m} \nabla_{\mathbf{r}} \varphi \cdot \nabla_{\mathbf{v}} [f(\mathbf{v}_0) f(\mathbf{w}_0)] = \nabla_{\mathbf{r}} \cdot [f(\mathbf{v}_0) f(\mathbf{w}_0) (\mathbf{w} - \mathbf{v})] - \frac{1}{m} \nabla_{\mathbf{w}} \cdot [f(\mathbf{v}_0) f(\mathbf{w}_0) \nabla_{\mathbf{r}} \varphi].$$

If we substitute this relation in the kinetic equation (2.17), the second term vanish due to the Gauss theorem and the first term can be converted to a surface integral on the unit

sphere S^2 ($\boldsymbol{\eta}$ is a outer unit normal vector to S^2):

$$\begin{aligned} \frac{1}{m} \iint \nabla_{\mathbf{v}} [f(\mathbf{x}, \mathbf{v}_0, t) f(\mathbf{x}, \mathbf{w}_0, t)] \cdot \nabla_{\mathbf{x}} \varphi \, d\boldsymbol{\eta} d\mathbf{w} \\ = \sigma^2 \iint_{R^3 \times S^2} [f(\mathbf{v}_0) f(\mathbf{w}_0)] (\mathbf{w} - \mathbf{v}) \cdot \boldsymbol{\eta} \, d\boldsymbol{\eta} d\mathbf{w}. \end{aligned} \quad (2.22)$$

We split the unit sphere S^2 into the two hemispheres

$$S_+^2 = \{\boldsymbol{\eta} : (\mathbf{w} - \mathbf{v}) \cdot \boldsymbol{\eta} > 0\}, \quad \text{and} \quad S_-^2 = \{\boldsymbol{\eta} : (\mathbf{w} - \mathbf{v}) \cdot \boldsymbol{\eta} < 0\}. \quad (2.23)$$

Now, when $(\mathbf{w} - \mathbf{v}) \cdot \boldsymbol{\eta} < 0$, the particles are entering the collision, hence $\mathbf{v}_0 = \mathbf{v}$ and $\mathbf{w}_0 = \mathbf{w}$. On the other hand, when $(\mathbf{w} - \mathbf{v}) \cdot \boldsymbol{\eta} > 0$, the particles are leaving the collision and the pre-collision velocities $\mathbf{v}_0, \mathbf{w}_0$ are related to the post-collision velocities \mathbf{v}, \mathbf{w} by the relations:

$$\begin{aligned} \mathbf{v}_0 &= \mathbf{v} - [\boldsymbol{\eta} \cdot (\mathbf{v} - \mathbf{w})] \boldsymbol{\eta}, \\ \mathbf{w}_0 &= \mathbf{w} + [\boldsymbol{\eta} \cdot (\mathbf{v} - \mathbf{w})] \boldsymbol{\eta}. \end{aligned} \quad (2.24)$$

The relation (2.22) reduces further to

$$\begin{aligned} \frac{1}{m} \iint \nabla_{\mathbf{v}} [f(\mathbf{x}, \mathbf{v}_0, t) f(\mathbf{x}, \mathbf{w}_0, t)] \cdot \nabla_{\mathbf{x}} \varphi \, d\boldsymbol{\eta} d\mathbf{w} \\ = \sigma^2 \iint_{R^3 \times S_+^2} [f(\mathbf{v}_0) f(\mathbf{w}_0)] (\mathbf{w} - \mathbf{v}) \cdot \boldsymbol{\eta} \, d\boldsymbol{\eta} d\mathbf{w} \\ + \sigma^2 \iint_{R^3 \times S_-^2} [f(\mathbf{v}) f(\mathbf{w})] (\mathbf{w} - \mathbf{v}) \cdot \boldsymbol{\eta} \, d\boldsymbol{\eta} d\mathbf{w} \\ = \iint_{R^3 \times S_+^2} [f(\mathbf{v}_0) f(\mathbf{w}_0) - f(\mathbf{v}) f(\mathbf{w})] [\sigma^2 (\mathbf{w} - \mathbf{v}) \cdot \boldsymbol{\eta}] \, d\boldsymbol{\eta} d\mathbf{w}. \end{aligned}$$

The last equality relation is due to the substitution $\boldsymbol{\eta} = -\boldsymbol{\eta}'$ in the second integral.

Let us define the Boltzmann collision operator¹

$$\mathcal{Q}(f, f) = \iint_{R^3 \times S_+^2} [f(\mathbf{v}_0) f(\mathbf{w}_0) - f(\mathbf{v}) f(\mathbf{w})] b(|\mathbf{v} - \mathbf{w}|, \boldsymbol{\eta}) \, d\boldsymbol{\eta} d\mathbf{w}, \quad (2.25)$$

where $b(|\mathbf{v}_1 - \mathbf{v}_2|, \boldsymbol{\eta})$ is a measurable non-negative function called the Boltzmann kernel. Then we have the kinetic equation

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = \mathcal{Q}(f, f) - \frac{1}{m} \iint \nabla_{\mathbf{v}} [E_1 + E_2] \cdot \nabla_{\mathbf{r}} \varphi \, d\mathbf{r} d\mathbf{w}. \quad (2.26)$$

The Boltzmann collision operator has the property of collision invariants (see (2.52) below) which reflects that the mass, the momentum and the energy is conserved during the instantaneous point collisions. These quantities are also conserved in the case of non-local and non-instant collisions, but, as the quantities are transferred between two different

¹The Boltzmann collision operator is the reduced form of the symmetric bilinear quantity:

$$\mathcal{Q}(f, g) = \frac{1}{2} \iint_{R^3 \times S_+^2} [f(\mathbf{v}_0) g(\mathbf{w}_0) + g(\mathbf{v}_0) f(\mathbf{w}_0) - f(\mathbf{v}) g(\mathbf{w}) - g(\mathbf{v}) f(\mathbf{w})] b(|\mathbf{v} - \mathbf{w}|, \boldsymbol{\eta}) \, d\boldsymbol{\eta} d\mathbf{w}.$$

spatial points, the new flux sources originate. For this reason the terms E_1 and E_2 are sometimes called the source terms.

We expand the term E_1 into the Taylor series

$$E_1(\mathbf{x}, \mathbf{r}, \mathbf{v}_0, \mathbf{w}_0, t) = \sum_{n=1}^{\infty} \frac{1}{n!} \left[\otimes^n \left(\frac{1}{2} \mathbf{r} \right) \right] : \nabla_{\mathbf{x}}^n [f(\mathbf{x}, \mathbf{v}_0, t) f(\mathbf{x}, \mathbf{w}_0, t)].$$

Here $\otimes^n \mathbf{r}$ denotes a n th-order tensor which is a tensor product of the same vectors \mathbf{r} , i.e., $(\otimes^n \mathbf{r})_{i_1 \dots i_n} = r_{i_1} \dots r_{i_n}$, and $:$ denotes a scalar product of two tensors $A : B = \sum_{i_1, \dots, i_n} A_{i_1 \dots i_n} B_{i_1 \dots i_n}$.

We use this formula to rearrange the first source term

$$\iint \nabla_{\mathbf{v}} E_1 \cdot \nabla_{\mathbf{r}} \varphi \, d\mathbf{r} d\mathbf{w} = \sum_{n=1}^{\infty} \frac{1}{2^n n!} \nabla_{\mathbf{x}}^n : \left\{ \int \left[\otimes^n \mathbf{r} \right] \left[\nabla_{\mathbf{v}} \int f(\mathbf{v}_0) f(\mathbf{w}_0) d\mathbf{w} \right] \cdot \nabla_{\mathbf{r}} \varphi \, d\mathbf{r} \right\}.$$

The source term E_2 was closely studied in the papers [Gr52] and [SC58]. Here we would like to give the sketch of these results and to explain why we will omit its contribution. In the simplified notation it reads

$$E_2 = f(\mathbf{x}_0, \mathbf{v}_0, t_0) f(\mathbf{y}_0, \mathbf{w}_0, t_0) - f(\mathbf{R}, \mathbf{v}_0, t) f(\mathbf{R}, \mathbf{w}_0, t), \quad (2.27)$$

and applying the Taylor series expansions to, we obtain

$$\begin{aligned} E_2 &= (t_0 - t) [\partial_{t_0} + \mathbf{v}_0 \cdot \nabla_{\mathbf{x}_0} + \mathbf{w}_0 \cdot \nabla_{\mathbf{y}_0}] [f(\mathbf{x}_0, \mathbf{v}_0, t_0) f(\mathbf{y}_0, \mathbf{w}_0, t_0)] \\ &\quad + [(t - t_0) \mathbf{v}_0 + (\mathbf{x}_0 - \mathbf{R})] \nabla_{\mathbf{x}_0} f(\mathbf{x}_0, \mathbf{v}_0, t_0) f(\mathbf{y}_0, \mathbf{w}_0, t_0) \\ &\quad + [(t - t_0) \mathbf{w}_0 + (\mathbf{y}_0 - \mathbf{R})] f(\mathbf{x}_0, \mathbf{v}_0, t_0) \nabla_{\mathbf{y}_0} f(\mathbf{y}_0, \mathbf{w}_0, t_0). \end{aligned}$$

The first term on the right hand side is zero because it is the Liouville equation for two particles entering a collision. In such a case $\varphi(|\mathbf{y}_0 - \mathbf{x}_0|) = 0$ and the assumption of binary collisions implies that the probability density of such two particles is independent of the rest of the system during the collision.

The mass center of two colliding particles moves constantly

$$\mathbf{R}(t) = \frac{1}{2}(\mathbf{x}_0 + \mathbf{y}_0) + \frac{1}{2}(\mathbf{v}_0 + \mathbf{w}_0)(t - t_0),$$

and if we neglect any further partial derivatives with respect to time and the spatial coordinates, we obtain

$$E_2 = \frac{1}{2} \mathbf{r}' \cdot [\nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{v}_0, t) f(\mathbf{x}, \mathbf{w}_0, t) - f(\mathbf{x}, \mathbf{v}_0, t) \nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{w}_0, t)], \quad (2.28)$$

where

$$\mathbf{r}' = (\mathbf{y}_0 - \mathbf{x}_0) + (\mathbf{w}_0 - \mathbf{v}_0)(t - t_0). \quad (2.29)$$

In order to estimate the magnitude of the influence of the source terms we develop the dimensionless form of the kinetic equation (2.26). As usual, we rescale the variables with respect to their characteristic values. The spatial variable \mathbf{x} has the characteristic length Λ_* of a surrounding box, the relative position \mathbf{r} of two colliding particles has the characteristic length of the mean free path λ_* because of the range $0 < |\mathbf{r}| < \sigma$ of cutoff potential. The mass of the initial data gives the characteristic density ϱ_* and the mean square velocity of the initial data, where velocities are relative to the movement of the mass center of system, gives the characteristic temperature θ_* . The time scale

t_* is deduced from the concept of mean free path $t_* = \lambda_*/\sqrt{\theta_*}$. The *Knudsen number* $\varepsilon = \lambda_*/\Lambda_*$ is defined as the ratio between the micro-space and the macro-space.

Then the dimensionless form of the kinetic equation (2.26) reads

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = \frac{1}{\varepsilon} \mathcal{Q}(f, f) - I_{E_1}(f) - I_{E_2}(f), \quad (2.30)$$

where

$$I_{E_1}(f) = \sum_{n=1}^{\infty} \frac{\varepsilon^{n-1}}{2^n n!} \nabla_{\mathbf{x}}^n : \left\{ \int \left[\overset{n}{\otimes} \mathbf{r} \right] \left[\nabla_{\mathbf{v}} \int f(\mathbf{x}, \mathbf{v}_0, t) f(\mathbf{x}, \mathbf{w}_0, t) d\mathbf{w} \right] \cdot \nabla_{\mathbf{r}} \varphi d\mathbf{r} \right\} \quad (2.31)$$

$$I_{E_2}(f) = \iint \nabla_{\mathbf{v}} \left\{ \frac{1}{2} \mathbf{r}' \cdot \left[\nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{v}_0, t) f(\mathbf{y}, \mathbf{w}_0, t) - f(\mathbf{x}, \mathbf{v}_0, t) \nabla_{\mathbf{y}} f(\mathbf{y}, \mathbf{w}_0, t) \right] \right\} \cdot \nabla_{\mathbf{r}} \varphi d\mathbf{r} d\mathbf{w}. \quad (2.32)$$

Let us multiply the equation (2.30) by ε and take the formal limit $\varepsilon \rightarrow 0+$. We obtain the equilibrium equation

$$\mathcal{Q}(f, f) = 0. \quad (2.33)$$

It is well known, see [Ce88, Section II.7], that the solution of this integral equation is the Maxwellian

$$M(\mathbf{v}) = \frac{\varrho}{(2\pi\theta)^{3/2}} \exp\left(-\frac{|\mathbf{v}|^2}{2\theta}\right), \quad (2.34)$$

where $\varrho = \varrho(\mathbf{x}, t)$, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and $\theta = \theta(\mathbf{x}, t)$ are the density, the bulk velocity and the absolute temperature of the system. In fact there is the equivalence between the equilibrium solution and the Maxwellian

$$\mathcal{Q}(f, f) = 0 \quad \text{if and only if} \quad f = M. \quad (2.35)$$

Since the additional terms $I_{E_1}(f)$ and $I_{E_2}(f)$ have higher orders in powers of ε , we suggest the further simplification of the equation (2.30), and we evaluate both terms at the Maxwellian

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = \frac{1}{\varepsilon} \mathcal{Q}(f, f) - I_{E_1}(M) - I_{E_2}(M). \quad (2.36)$$

Such a consideration underlines the meaning of the both terms in the sense of the source terms. It also significantly simplifies the problem of the solution because the equation (2.36) is then the Boltzmann equation plus some additional (source) terms.

Since the energy is conserved during a binary collision, we have

$$|\mathbf{v}_0|^2 + |\mathbf{w}_0|^2 = |\mathbf{v}|^2 + |\mathbf{w}|^2 + 2\varphi, \quad (2.37)$$

and we apply it to the product of Maxwellians ($\mathbf{v}_0 = \mathbf{v}_0 - \mathbf{u}$, $\mathbf{w} = \mathbf{w} - \mathbf{u}$, ...)

$$M(\mathbf{v}_0)M(\mathbf{w}_0) = M(\mathbf{v})M(\mathbf{w})e^{-\varphi/\theta}. \quad (2.38)$$

The first term on the right hand side of the equation (2.36) transforms into

$$I_{E_1}(M) = - \sum_{n=1}^{\infty} \frac{\varepsilon^{n-1}}{2^n n!} \nabla_{\mathbf{x}}^n : \left\{ \frac{\varrho M(\mathbf{v})}{\theta} \left[\int \left(\overset{n}{\otimes} \mathbf{r} \otimes \nabla_{\mathbf{r}} \varphi \right) e^{-\varphi/\theta} d\mathbf{r} \right] \mathbf{v} \right\}. \quad (2.39)$$

We would like to emphasize the dependence on ε of the right hand side. We will observe its consequences later.

We apply the symmetry with respect to the vector \mathbf{r} . The integrand is the tensor product of the same vectors \mathbf{r} and we use the results from the Appendix A.1. All terms with n even vanish, and consequently

$$I_{E_1}(M) = - \sum_{n=1}^{n \text{ odd}} \frac{\varepsilon^{n-1}}{2^n n!} \nabla_{\mathbf{x}}^n : \left\{ \frac{\varrho M(\mathbf{v})}{\theta} \left[\int_{S^2} \otimes \boldsymbol{\eta} d\boldsymbol{\eta} \right] \left[\int_0^\sigma r^{n+2} \varphi' e^{-\varphi/\theta} dr \right] \mathbf{v} \right\}. \quad (2.40)$$

Furthermore, we truncate the sum to the first two terms corresponding to $n = 1, 3$. Such a simplification is sufficient to introduce the model of linear dipolar viscous fluids and, clearly, it reduces an algebra indispensably. Hence we have

$$I_{E_1}(M) \doteq - \nabla_{\mathbf{x}} \cdot \left\{ \varrho M(\mathbf{v}) \left[\frac{2\pi}{3\theta} \int_0^\sigma r^3 \varphi' e^{-\varphi/\theta} dr \right] \mathbf{v} \right\} \\ - \varepsilon^2 \nabla_{\mathbf{x}}^3 : \left\{ \varrho M(\mathbf{v}) \left[\frac{\pi}{180\theta} \int_0^\sigma r^5 \varphi' e^{-\varphi/\theta} dr \right] (\mathcal{I} \otimes \mathbf{v} + \llbracket \mathbf{v} \rrbracket + \mathbf{v} \otimes \mathcal{I}) \right\}.$$

We define the *positive* coefficients (φ is a strictly repulsive potential):

$$\alpha(\theta) = - \frac{2\pi}{3\theta} \int_0^\sigma r^3 \varphi' e^{-\varphi/\theta} dr, \quad (2.41)$$

and

$$\beta(\theta) = - \frac{\pi}{60\theta} \int_0^\sigma r^5 \varphi' e^{-\varphi/\theta} dr, \quad (2.42)$$

and we notice that $\nabla_{\mathbf{x}}^2 : \mathcal{I} = \Delta_{\mathbf{x}}$. The resulting expression for the first source term is in the form

$$I_{E_1}(M) \doteq \nabla_{\mathbf{x}} \cdot \{ \alpha(\theta) \varrho M(\mathbf{v}) \mathbf{v} \} + \varepsilon^2 \nabla \Delta_{\mathbf{x}} \cdot \{ \beta(\theta) \varrho M(\mathbf{v}) \mathbf{v} \}. \quad (2.43)$$

Let us start to discuss the role of the source term $I_{E_2}(M)$. We introduce the new variables

$$\mathbf{G} = \frac{1}{2}(\mathbf{v} + \mathbf{w}), \quad \mathbf{g} = \mathbf{w} - \mathbf{v},$$

and we calculate the formula for the gradient of the Maxwellian:

$$\nabla_{\mathbf{x}} M(\mathbf{v}) = M(\mathbf{v}) \left\{ \frac{\nabla_{\mathbf{x}} \varrho}{\varrho} + \frac{\mathbf{v} \cdot \nabla_{\mathbf{x}} \mathbf{u}}{\theta} + \left(\frac{1}{2} |\mathbf{v}|^2 - \frac{3}{2} \theta \right) \frac{\nabla_{\mathbf{x}} \theta}{\theta^2} \right\}. \quad (2.44)$$

On substituting this formula, and using (2.38), we simplify the expression for $E_2(M)$:

$$E_2(M) = - \frac{1}{2\theta} M(\mathbf{v}) M(\mathbf{w}) e^{-\varphi/\theta} \left[\nabla_{\mathbf{x}} \mathbf{u} + \frac{\nabla_{\mathbf{x}} \theta}{\theta} \otimes \mathbf{G} \right] : (\mathbf{r}' \otimes \mathbf{g}_0). \quad (2.45)$$

Here we used the conservation law of momentum during a binary collision, i.e., $\mathbf{G} = \mathbf{G}_0$.

The details of a binary collision can be used to explicitly evaluate $\mathbf{r}' = \mathbf{r}'(\mathbf{r}, \mathbf{g})$ and $\mathbf{g}_0 = \mathbf{g}_0(\mathbf{r}, \mathbf{g})$, and then perform the integration over the variable \mathbf{r} in $I_{E_2}(M)$. The result has the following form (see [SC58, p. 126]):

$$I_{E_2}(M) = - \int \nabla_{\mathbf{g}} \cdot \left\{ M(\mathbf{v}) M(\mathbf{w}) \left[\nabla_{\mathbf{x}} \mathbf{u} + \frac{\nabla_{\mathbf{x}} \theta}{\theta} \otimes \mathbf{G} \right] \right. \\ \left. : [F_1 \mathbf{g} \otimes \mathcal{I} + F_2 \llbracket \mathbf{g} \rrbracket + F_3 \mathcal{I} \otimes \mathbf{g} + F_4 \mathbf{g} \otimes \mathbf{g} \otimes \mathbf{g}] \right\} d\mathbf{w}, \quad (2.46)$$

where $F_i, i = 1, \dots, 4$ are the coefficients independent of the variables \mathbf{v} and \mathbf{w} .

Formula (2.46) is very useful because it enables us to verify the solvability conditions of the linearized problem (see below) and even to find out the solution in terms of the Sonine polynomials. From the quantitative point of view this source term contributes to the coefficients of viscosity and thermal conductivity in the Stokes and the Fourier law. (Such a result can be expected as the source term depends only on the first gradients of the bulk velocity \mathbf{u} and of the first gradient of temperature θ .)

Because of the quantitative meaning of the source term $I_{E_2}(M)$, we omit this term in our calculations and we conclude with the kinetic equation

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = \frac{1}{\varepsilon} \mathcal{Q}(f, f) - \nabla_{\mathbf{x}} \cdot s(M), \quad (2.47)$$

where

$$s(M) = \alpha(\theta) \varrho M(\mathbf{v}) \mathbf{v} + \varepsilon^2 \Delta_{\mathbf{x}} [\beta(\theta) \varrho M(\mathbf{v}) \mathbf{v}]. \quad (2.48)$$

2.3 Fluid dynamic limit

The probability density $f = f(\mathbf{x}, \mathbf{v}, t)$ and the macroscopic quantities such as the density $\varrho = \varrho(\mathbf{x}, t)$, the bulk velocity $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and the absolute temperature $\theta = \theta(\mathbf{x}, t)$ are tied up together. The macroscopic quantities are averaged properties of particles over all possible velocities \mathbf{v} with respect to the density f . We denote the integration of any scalar-, vector- or tensor-valued function $y(\mathbf{v})$ over all velocities \mathbf{v} by $\langle y \rangle$:

$$\langle y \rangle = \int_{R^3} y(\mathbf{v}) d\mathbf{v}. \quad (2.49)$$

The dependence of y on \mathbf{x}, t is omitted.

The properties of particles corresponding to the macroscopic density ϱ , the velocity \mathbf{u} and the energy e are the particle's mass 1, the particle's velocity \mathbf{v} and the particle's kinetic energy $\frac{1}{2}|\mathbf{v}|^2$. We have

$$\langle f \rangle = \varrho, \quad \langle \mathbf{v} f \rangle = \varrho \mathbf{u}, \quad \langle \frac{1}{2} |\mathbf{v}|^2 f \rangle = \varrho \left(\frac{1}{2} |\mathbf{u}|^2 + \frac{3}{2} \theta \right). \quad (2.50)$$

We can observe that the process of averaging has the two consequences. On one side it hides many details about particles and simplifies the problem. On the other side we have to introduce new macroscopic quantities to interpret the macroscopic model. For example, the pure kinetic energy of particles has to be interpreted as the sum of kinetic energy $\frac{1}{2} \varrho |\mathbf{u}|^2$ plus the internal energy $\frac{3}{2} \varrho \theta$ on the macroscopic level because of the relation

$$\langle \frac{1}{2} |\mathbf{v}|^2 f \rangle = \frac{1}{2} \varrho |\mathbf{u}|^2 + \langle \frac{1}{2} |\mathbf{v} - \mathbf{u}|^2 f \rangle. \quad (2.51)$$

The Boltzmann collision operator has the property of collision invariants (e.g., [Ce88, Section II.6]):

$$\langle \psi(\mathbf{v}) \mathcal{Q}(f, f) \rangle = 0 \quad \text{if and only if} \quad \psi(\mathbf{v}) \in \text{span}\{1, v_1, v_2, v_3, |\mathbf{v}|^2\} \quad (2.52)$$

which means that the invariants of binary collisions (the mass, the momentum and the energy) are also conserved in average by the Boltzmann collision operator regardless of any details of collisions.

We successively multiply the kinetic equation (2.47) by the collision invariants 1, \mathbf{v} and $\frac{1}{2}|\mathbf{v}|^2$ and integrate over all velocities \mathbf{v} , we find the local conservation laws:

$$\partial_t \langle f \rangle + \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} f \rangle + \nabla_{\mathbf{x}} \cdot \langle s(M) \rangle = 0, \quad (2.53)$$

$$\partial_t \langle \mathbf{v} f \rangle + \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \otimes \mathbf{v} f \rangle + \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} s(M) \rangle = 0, \quad (2.54)$$

$$\partial_t \langle \frac{1}{2} |\mathbf{v}|^2 f \rangle + \nabla_{\mathbf{x}} \cdot \langle \frac{1}{2} |\mathbf{v}|^2 \mathbf{v} f \rangle + \nabla_{\mathbf{x}} \cdot \langle \frac{1}{2} |\mathbf{v}|^2 s(M) \rangle = 0. \quad (2.55)$$

Such a process is called the *fluid dynamic limit* and helps us discover the hydrodynamic equations of fluids from the kinetic equation.

We expand the solution of the equation (2.47) in a series of powers of ε :

$$f = M(1 + \varepsilon g + \varepsilon^2 h + \dots), \quad (2.56)$$

where M is the Maxwellian defined in (2.34). Because of the properties of the Maxwellian

$$\langle M \rangle = \varrho, \quad \langle \mathbf{v} M \rangle = \varrho \mathbf{u}, \quad \langle \frac{1}{2} |\mathbf{v}|^2 M \rangle = \varrho (\frac{1}{2} |\mathbf{u}|^2 + \frac{3}{2} \theta), \quad (2.57)$$

and the relations (2.50), we impose the following conditions on all remaining terms in the expansion (2.56)

$$\langle \Psi M \rangle = 0, \quad \langle \mathbf{v} \Psi M \rangle = 0, \quad \langle \frac{1}{2} |\mathbf{v}|^2 \Psi M \rangle = 0, \quad \Psi = g, h, \dots \quad (2.58)$$

This expansion is called the Chapman–Enskog method, and the different orders of approximations give us the different fluid dynamic limits (see [BGL91], [ChC39], [FK72]). In the case of the Boltzmann equation the first order approximation gives us the compressible Euler equations, the second order gives us the compressible Navier–Stokes equations and even higher order approximations were studied and the systems of Burnett and Superburnett equations were developed.

The investigated kinetic equation (2.47) only differs from the Boltzmann equation by the source term $s(M)$, thus the Chapman–Enskog method is suitable even in this case. The general equations of multipolar viscous fluids (2.115)–(2.117) are close to the system of compressible Navier–Stokes equations, hence we use the second order approximation of the solution and substitute $f = M(1 + \varepsilon g + \varepsilon^2 h)$ in the kinetic equation (2.47). The result is

$$\begin{aligned} \frac{(\partial_t + \mathbf{v} \cdot \nabla_{\mathbf{x}})M}{M} + \frac{\nabla_{\mathbf{x}} \cdot s(M)}{M} + \varepsilon \frac{(\partial_t + \mathbf{v} \cdot \nabla_{\mathbf{x}})Mg}{M} \\ = \mathcal{L}(g) + \varepsilon [\mathcal{L}(h) + \Gamma(g, g)] + O(\varepsilon^2), \end{aligned} \quad (2.59)$$

where \mathcal{L} is the linear Boltzmann collision operator

$$\mathcal{L}(g) = \frac{2}{M} \mathcal{Q}(Mg, M) \quad (2.60)$$

and Γ is the bilinear operator

$$\Gamma(g, g) = \frac{1}{M} \mathcal{Q}(Mg, Mg). \quad (2.61)$$

The linear Boltzmann collision operator has the equivalent form

$$\mathcal{L}(g) = \iint_{R^3 \times S^2} [g(\mathbf{v}') + g(\mathbf{w}') - g(\mathbf{v}) - g(\mathbf{w})] M(\mathbf{w}) b(|\mathbf{v} - \mathbf{w}|, \boldsymbol{\eta}) d\boldsymbol{\eta} d\mathbf{w}, \quad (2.62)$$

and it is a non-negative, self-adjoint operator which satisfies the Fredholm alternative with 5-dimensional kernel $\text{Ker } \mathcal{L} = \text{span}\{1, v_1, v_2, v_3, |\mathbf{v}|^2\}$ on the Hilbert space $L_M^2 = \{y(\mathbf{v}) : R^3 \rightarrow R \mid y(\mathbf{v})M^{1/2} \in L^2(R^3)\}$ equipped with the scalar product

$$\langle y, z \rangle_M = \langle yz \rangle_M = \int_{R^3} yz M d\mathbf{v}. \quad (2.63)$$

It means that there exists the inverse operator $\mathcal{L}^{-1} : \text{Ker } \mathcal{L}^\perp \rightarrow \text{Ker } \mathcal{L}^\perp$ which yields the unique solution of the problem

$$\mathcal{L}y = z \quad \implies \quad \exists! y = \mathcal{L}^{-1}z \in \text{Ker } \mathcal{L}^\perp \quad \text{for all } z \in \text{Ker } \mathcal{L}^\perp. \quad (2.64)$$

The inverse operator \mathcal{L}^{-1} has the symmetry

$$\langle y, \mathcal{L}^{-1}z \rangle_M = \langle \mathcal{L}^{-1}y, z \rangle_M \quad \text{for all } y, z \in \text{Ker } \mathcal{L}^\perp. \quad (2.65)$$

We compare the terms of the same powers of ε in the equation (2.59) and we obtain the system of linear equations for the unknown functions g and h . The equation for g reads

$$\mathcal{L}(g) = \frac{(\partial_t + \mathbf{v} \cdot \nabla_{\mathbf{x}})M}{M} + \frac{\nabla_{\mathbf{x}} \cdot s(M)}{M}. \quad (2.66)$$

We should emphasize that the source term $s(M)$ given by the equation (2.48) depends internally on the Knudsen number ε and such inconsistency is the crucial step in the derivation of multipolar viscous fluids from the kinetic level. From the mathematical point of view it is the correction of the order $O(\varepsilon^2)$ and it should be negligible in the range of our approximation (2.59).

The functions g and h have to be in $\text{Ker } \mathcal{L}^\perp$ because of the relations (2.58), thus we have the unique solution of the equation (2.66) if and only if the right hand side is in $\text{Ker } \mathcal{L}^\perp$, too. We adjust each term of the right hand side separately but the solvability conditions are verified altogether; see (2.82).

Using the formula (2.34), we directly obtain that

$$\begin{aligned} \frac{(\partial_t + \mathbf{v} \cdot \nabla_{\mathbf{x}})M}{M} &= \frac{1}{\varrho} \{ \partial_t \varrho + \nabla_{\mathbf{x}} \cdot (\varrho \mathbf{u}) \} \\ &+ \frac{\mathbf{v}}{\varrho \theta} \cdot \{ \varrho (\partial_t + \mathbf{u} \cdot \nabla_{\mathbf{x}}) \mathbf{u} + \nabla_{\mathbf{x}} (\varrho \theta) \} \\ &+ \frac{\frac{1}{2} |\mathbf{v}|^2 - \frac{3}{2} \theta}{\varrho \theta^2} \{ \varrho (\partial_t + \mathbf{u} \cdot \nabla_{\mathbf{x}}) \theta + \frac{2}{3} \varrho \theta \nabla_{\mathbf{x}} \cdot \mathbf{u} \} \\ &+ \frac{\mathbf{v} \otimes \mathbf{v} - \frac{1}{3} |\mathbf{v}|^2 \mathcal{I}}{2 \varrho \theta^2} : \{ \varrho \theta \sigma(\mathbf{u}) \} \\ &+ \frac{(\frac{1}{2} |\mathbf{v}|^2 - \frac{5}{2} \theta) \mathbf{v}}{\varrho \theta^3} \cdot \{ \varrho \theta \nabla_{\mathbf{x}} \theta \}, \end{aligned} \quad (2.67)$$

where the symbol $\sigma(\mathbf{u})$ denotes the deviatoric part of the symmetric velocity gradient tensor

$$\sigma(\mathbf{u}) = \nabla_{\mathbf{x}} \mathbf{u} + \nabla_{\mathbf{x}}^T \mathbf{u} - \frac{2}{3} (\nabla_{\mathbf{x}} \cdot \mathbf{u}) \mathcal{I}. \quad (2.68)$$

All terms on the right hand side of (2.67) are mutually orthogonal with respect to the scalar product on the Hilbert space L_M^2 . We would like to point at the first three curly braces on the right hand side of (2.67). These terms are the left hand sides of the compressible Euler equations of an ideal gas, i.e., $p = \varrho \theta$.

We also calculate directly the first part of the source term $s(M)$

$$\begin{aligned}
\frac{\nabla_{\mathbf{x}} \cdot \{\alpha(\theta) \varrho M(\mathbf{V}) \mathbf{V}\}}{M} &= \frac{\mathbf{V}}{\varrho\theta} \cdot \nabla_{\mathbf{x}} (\alpha(\theta) \varrho^2\theta) \\
&+ \frac{\frac{1}{2}|\mathbf{V}|^2 - \frac{3}{2}\theta}{\varrho\theta^2} \left\{ \frac{2}{3}\alpha(\theta) \varrho^2\theta \nabla_{\mathbf{x}} \cdot \mathbf{u} \right\} \\
&+ \frac{\mathbf{V} \otimes \mathbf{V} - \frac{1}{3}|\mathbf{V}|^2 \mathcal{I}}{2\varrho\theta^2} : \{ \alpha(\theta) \varrho^2\theta \sigma(\mathbf{u}) \} \\
&+ \frac{(\frac{1}{2}|\mathbf{V}|^2 - \frac{5}{2}\theta)\mathbf{V}}{\varrho\theta^3} \cdot \{ \alpha(\theta) \varrho^2\theta \nabla_{\mathbf{x}}\theta \}.
\end{aligned} \tag{2.69}$$

The second part of the source term $s(M)$ requires much effort as the calculations are algebraically complicated and they result in the tents terms. Fortunately, we need only know the part belonging to the $\text{Ker } \mathcal{L}$ and we can calculate the projection on this subspace. If we assume that all partial derivatives of ψ with respect to the spatial variable \mathbf{x} are continuous, we deduce the subsequent formula

$$\begin{aligned}
\int \psi \nabla \Delta_{\mathbf{x}} \cdot \vartheta d\mathbf{V} &= \nabla \Delta_{\mathbf{x}} \cdot \left(\int \psi \vartheta d\mathbf{V} \right) \\
&- \nabla_{\mathbf{x}}^2 : \left(\int \nabla_{\mathbf{x}} \psi \otimes \vartheta d\mathbf{V} + \int \vartheta \otimes \nabla_{\mathbf{x}} \psi d\mathbf{V} + \int \nabla_{\mathbf{x}} \psi \cdot \vartheta d\mathbf{V} \mathcal{I} \right) \\
&+ \nabla_{\mathbf{x}} \cdot \left(\int 2[\nabla_{\mathbf{x}}^2 \psi] \vartheta d\mathbf{V} + \int \Delta_{\mathbf{x}} \psi \vartheta d\mathbf{V} \right) \\
&- \int \nabla \Delta_{\mathbf{x}} \psi \cdot \vartheta d\mathbf{V}.
\end{aligned} \tag{2.70}$$

Let us calculate the projection of the term $\nabla \Delta_{\mathbf{x}} \cdot [\beta(\theta) \varrho M(\mathbf{V}) \mathbf{V}]$ with the help of the above formula.

At first, we use the constant $\psi = 1$. Because each component of $\beta(\theta) \varrho M(\mathbf{V}) \mathbf{V}$ is odd in some component \mathcal{V}_i , $i = 1, 2, 3$, we obtain

$$\begin{aligned}
\int \nabla \Delta_{\mathbf{x}} \cdot [\beta(\theta) \varrho M(\mathbf{V}) \mathbf{V}] d\mathbf{V} &= \nabla \Delta_{\mathbf{x}} \cdot \int \beta(\theta) \varrho M(\mathbf{V}) \mathbf{V} d\mathbf{V} \\
&= 0.
\end{aligned} \tag{2.71}$$

At second, we use the functions $\psi = \mathcal{V}_i$ for $i = 1, 2, 3$. As all spatial derivatives of \mathcal{V}_i are independent of any component of the vector \mathbf{V} , we have

$$\begin{aligned}
\int \mathbf{V} \nabla \Delta_{\mathbf{x}} \cdot [\beta(\theta) \varrho M(\mathbf{V}) \mathbf{V}] d\mathbf{V} &= \nabla \Delta_{\mathbf{x}} \cdot \left(\int \mathbf{V} \otimes \mathbf{V} \beta(\theta) \varrho M(\mathbf{V}) d\mathbf{V} \right) \\
&= \nabla \Delta_{\mathbf{x}} K(\varrho, \theta),
\end{aligned} \tag{2.72}$$

where

$$K(\varrho, \theta) = \beta(\theta) \varrho^2 \theta. \tag{2.73}$$

Let us look at the magnitude of the quantity $K(\varrho, \theta)$ in the case of the hard sphere potential. We use the approximation of the hard sphere potential as proposed in paper [SC58]:

$$\varphi_n(r) = \kappa(\sigma/r)^n. \tag{2.74}$$

The following limit, in the sense of distributions, can be proved

$$\frac{1}{\theta} \varphi_n' e^{-\varphi_n/\theta} \rightarrow -\delta(r - \sigma) \quad \text{as} \quad n \rightarrow \infty, \tag{2.75}$$

where δ is the Dirac function.

We take the relations (2.41) and (2.42) for the coefficients $\alpha(\theta)$ and $\beta(\theta)$, using the above formula, we get

$$\alpha = \frac{2}{3}\pi\sigma^3 \quad \text{and} \quad \beta = \frac{1}{60}\pi\sigma^5. \quad (2.76)$$

Both coefficients are independent of the absolute temperature θ and are small because of the particle's diameter σ . The quantity $K(\varrho, \theta)$ depends linearly on the temperature and on the density square in the case of the hard sphere potential.

In the rest of this section we will assume that *all spatial derivatives of the quantity $K(\varrho, \theta)$ are negligible*. We will discuss such a consideration later from the point of view of the state equation of gas and we will compare it with the assumptions made in paper [NS91], hence

$$\int \boldsymbol{\nu} \nabla \Delta_{\mathbf{x}} \cdot [\beta(\theta)\varrho M(\boldsymbol{\nu})\boldsymbol{\nu}] d\boldsymbol{\nu} \doteq \mathbf{0}. \quad (2.77)$$

Further calculations with the formula (2.70) are more technical and they are described in details in Appendix A.2. The following formulas hold

$$\int (\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \theta \mathcal{I}) \nabla \Delta_{\mathbf{x}} \cdot [\beta(\theta)\varrho M(\boldsymbol{\nu})\boldsymbol{\nu}] d\boldsymbol{\nu} = K \Delta_{\mathbf{x}} \sigma(\mathbf{u}) + \frac{2}{3} K \nabla \Delta_{\mathbf{x}} \cdot \mathbf{u} \mathcal{I}, \quad (2.78)$$

and

$$\int \left(\frac{1}{2}|\boldsymbol{\nu}|^2 - \frac{5}{2}\theta\right) \boldsymbol{\nu} \nabla \Delta_{\mathbf{x}} \cdot [\beta(\theta)\varrho M(\boldsymbol{\nu})\boldsymbol{\nu}] d\boldsymbol{\nu} = K \boldsymbol{\pi}(\mathbf{u}) + \frac{5}{2} K \nabla \Delta_{\mathbf{x}} \theta, \quad (2.79)$$

where $\boldsymbol{\pi}(\mathbf{u})$ is a vector which has the components ($k = 1, 2, 3$):

$$\begin{aligned} \pi_k(\mathbf{u}) = \sum_{i,j=1}^3 \left(\Delta_{\mathbf{x}} u_i \delta_{jk} + \Delta_{\mathbf{x}} u_j \delta_{ik} + \Delta_{\mathbf{x}} u_k \delta_{ij} \right. \\ \left. + 2 \frac{\partial^2 u_i}{\partial x_j \partial x_k} + 2 \frac{\partial^2 u_k}{\partial x_i \partial x_j} + 2 \frac{\partial^2 u_m}{\partial x_j \partial x_m} \delta_{ik} \right) \frac{\partial u_i}{\partial x_j}. \end{aligned} \quad (2.80)$$

Let us write down the above formulas in the form similar to (2.69)

$$\begin{aligned} \frac{\nabla \Delta_{\mathbf{x}} \cdot [\beta(\theta)\varrho M(\boldsymbol{\nu})\boldsymbol{\nu}]}{M} &= \frac{\frac{1}{2}|\boldsymbol{\nu}|^2 - \frac{3}{2}\theta}{\varrho\theta^2} \left\{ \frac{2}{3} K \nabla \Delta_{\mathbf{x}} \cdot \mathbf{u} \right\} \\ &+ \frac{\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \frac{1}{3}|\boldsymbol{\nu}|^2 \mathcal{I}}{2\varrho\theta^2} : \left\{ K \Delta_{\mathbf{x}} \sigma(\mathbf{u}) \right\} \\ &+ \frac{\left(\frac{1}{2}|\boldsymbol{\nu}|^2 - \frac{5}{2}\theta\right) \boldsymbol{\nu}}{\varrho\theta^3} \cdot \left\{ K \nabla \Delta_{\mathbf{x}} \theta + \frac{2}{5} K \boldsymbol{\pi}(\mathbf{u}) \right\} \\ &+ q(\varrho, \mathbf{u}, \theta, \boldsymbol{\nu}). \end{aligned} \quad (2.81)$$

Here the function $q(\varrho, \mathbf{u}, \theta, \boldsymbol{\nu})$ represents the unknown part which is orthogonal to $\text{Ker } \mathcal{L}$ and is also orthogonal to any linear combinations of the functions $\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \frac{1}{3}|\boldsymbol{\nu}|^2 \mathcal{I}$ and $\left(\frac{1}{2}|\boldsymbol{\nu}|^2 - \frac{5}{2}\theta\right) \boldsymbol{\nu}$.

Finally, we express the right hand side of the equation (2.66), using the results (2.67),

(2.69) and (2.81), it reads

$$\begin{aligned}
\mathcal{L}(g) &= \frac{1}{\varrho} \{ \partial_t \varrho + \nabla_{\mathbf{x}} \cdot (\varrho \mathbf{u}) \} \\
&+ \frac{\mathbf{V}}{\varrho \theta} \cdot \{ \varrho (\partial_t + \mathbf{u} \cdot \nabla_{\mathbf{x}}) \mathbf{u} + \nabla_{\mathbf{x}} (\varrho \theta + \alpha(\theta) \varrho^2 \theta) \} \\
&+ \frac{\frac{1}{2} |\mathbf{V}|^2 - \frac{3}{2} \theta}{\varrho \theta^2} \{ \varrho (\partial_t + \mathbf{u} \cdot \nabla_{\mathbf{x}}) \theta + \frac{2}{3} (\varrho \theta + \alpha(\theta) \varrho^2 \theta) \nabla_{\mathbf{x}} \cdot \mathbf{u} + \varepsilon^2 \frac{2}{3} K \nabla \Delta_{\mathbf{x}} \cdot \mathbf{u} \} \\
&+ \frac{\mathbf{V} \otimes \mathbf{V} - \frac{1}{3} |\mathbf{V}|^2 \mathcal{I}}{2 \varrho \theta^2} : \{ (\varrho \theta + \alpha(\theta) \varrho^2 \theta) \sigma(\mathbf{u}) + \varepsilon^2 K \Delta_{\mathbf{x}} \sigma(\mathbf{u}) \} \\
&+ \frac{(\frac{1}{2} |\mathbf{V}|^2 - \frac{5}{2} \theta) \mathbf{V}}{\varrho \theta^3} \cdot \{ (\varrho \theta + \alpha(\theta) \varrho^2 \theta) \nabla_{\mathbf{x}} \theta + \varepsilon^2 K \nabla \Delta_{\mathbf{x}} \theta + \varepsilon^2 \frac{2}{5} K \boldsymbol{\pi}(\mathbf{u}) \} \\
&+ \varepsilon^2 q(\varrho, \mathbf{u}, \theta, \mathbf{V}).
\end{aligned} \tag{2.82}$$

Let us introduce the hydrodynamic equations of linear dipolar viscous fluids (the Einstein convention of summation over the same indices is used):

$$\partial_t \varrho + \nabla_{\mathbf{x}} \cdot (\varrho \mathbf{u}) = 0, \tag{2.83}$$

$$\partial_t (\varrho \mathbf{u}) + \nabla_{\mathbf{x}} \cdot (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_{\mathbf{x}} p = \varepsilon \nabla_{\mathbf{x}} \cdot \mathcal{T}^{(0,V)}, \tag{2.84}$$

$$\partial_t (\varrho e) + \nabla_{\mathbf{x}} \cdot (\varrho e \mathbf{u} + p \mathbf{u}) = \varepsilon \frac{\partial}{\partial x_k} \left\{ \mathcal{T}_{jk}^{(0,V)} u_j + \mathcal{T}_{ijk}^{(1,V)} \frac{\partial u_i}{\partial x_j} \right\} + \varepsilon \nabla_{\mathbf{x}} \cdot \mathbf{q}, \tag{2.85}$$

where

$$p = \varrho \theta + \alpha(\theta) \varrho^2 \theta, \tag{2.86}$$

$$e = \frac{1}{2} |\mathbf{u}|^2 + \frac{3}{2} \theta, \tag{2.87}$$

$$\mathcal{T}^{(0,V)} = \mu_{\alpha} \sigma(\mathbf{u}) + \varepsilon^2 \mu_{\beta} \Delta_{\mathbf{x}} \sigma(\mathbf{u}), \tag{2.88}$$

$$\begin{aligned}
\mathcal{T}_{ijk}^{(1,V)} &= \varepsilon^2 \mu_{\gamma} \left(\Delta_{\mathbf{x}} u_i \delta_{jk} + \Delta_{\mathbf{x}} u_j \delta_{ik} + \Delta_{\mathbf{x}} u_k \delta_{ij} \right. \\
&\quad \left. + 2 \frac{\partial^2 u_i}{\partial x_j \partial x_k} + 2 \frac{\partial^2 u_k}{\partial x_i \partial x_j} + 2 \frac{\partial^2 u_m}{\partial x_j \partial x_m} \delta_{ik} \right),
\end{aligned} \tag{2.89}$$

$$\mathbf{q} = \kappa_{\alpha} \nabla_{\mathbf{x}} \theta - \varepsilon \kappa_{\gamma} \Delta_{\mathbf{x}} \mathbf{u} + \varepsilon^2 \kappa_{\beta} \nabla \Delta_{\mathbf{x}} \theta. \tag{2.90}$$

The coefficients of viscosity and thermal conductivity are functions of the density ϱ and the absolute temperature θ

$$\mu_{\alpha}(\varrho, \theta) = \mu(\theta) (1 + \alpha(\theta) \varrho), \quad \mu_{\beta}(\varrho, \theta) = \mu(\theta) \beta(\theta) \varrho, \quad \mu_{\gamma}(\varrho, \theta) = \frac{3}{2} \mu_{\beta}(\varrho, \theta), \tag{2.91}$$

$$\kappa_{\alpha}(\varrho, \theta) = \kappa(\theta) (1 + \alpha(\theta) \varrho), \quad \kappa_{\beta}(\varrho, \theta) = \kappa(\theta) \beta(\theta) \varrho, \quad \kappa_{\gamma}(\varrho, \theta) = \beta(\theta) \varrho^2 \theta. \tag{2.92}$$

Here the coefficients $\mu(\theta)$ and $\kappa(\theta)$ are the first approximations of the classical coefficients of an ideal gas obtained from the Boltzmann equation and usually expressed in terms of so-called Ω -integrals (see [ChC39] and [FK72]). Both coefficients $\mu(\theta)$, $\kappa(\theta)$ are independent of the density and they are related by the Eucken ratio ($c_v = 3/2$ is the specific heat)

$$\kappa(\theta) / \mu(\theta) = \frac{5}{2} c_v = \frac{15}{4}. \tag{2.93}$$

We can observe that the coefficients of the viscosity μ_{α} , μ_{β} , μ_{γ} and the thermal conductivity κ_{α} , κ_{β} of linear dipolar viscous fluids are linear functions of the density ϱ , and that κ_{γ} is proportional to ϱ^2 .

The convective form of the above equations is

$$(\partial_t + \mathbf{u} \cdot \nabla_{\mathbf{x}})\varrho + \varrho \nabla_{\mathbf{x}} \cdot \mathbf{u} = 0, \quad (2.94)$$

$$\varrho(\partial_t + \mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} + \nabla_{\mathbf{x}} p = \varepsilon \nabla_{\mathbf{x}} \cdot \mathcal{T}^{(0,V)}, \quad (2.95)$$

$$\begin{aligned} \frac{3}{2}\varrho(\partial_t + \mathbf{u} \cdot \nabla_{\mathbf{x}})\theta + p \nabla_{\mathbf{x}} \cdot \mathbf{u} = \\ \varepsilon \mathcal{T}_{jk}^{(0,V)} \frac{\partial u_j}{\partial x_k} + \varepsilon \frac{\partial}{\partial x_k} \left[\mathcal{T}_{ijk}^{(1,V)} \frac{\partial u_i}{\partial x_j} \right] + \varepsilon \nabla_{\mathbf{x}} \cdot \mathbf{q}. \end{aligned} \quad (2.96)$$

Comparing the equations (2.94)–(2.96) with the right hand side of the equation (2.82), we get that the solvability conditions are satisfied if we eliminate the time derivatives of the functions ϱ , \mathbf{u} and θ with the help of the hydrodynamic equations as the right hand sides of the equations (2.95) and (2.96) are at least of order one with respect to ε .

Therefore, we have the reduced form of the equation for g :

$$\begin{aligned} \mathcal{L}(g) = & + \frac{\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \frac{1}{3}|\boldsymbol{\nu}|^2 \mathcal{I}}{2\varrho\theta^2} : \{(\varrho\theta + \alpha(\theta)\varrho^2\theta)\sigma(\mathbf{u}) + \varepsilon^2 K \Delta_{\mathbf{x}}\sigma(\mathbf{u})\} \\ & + \frac{(\frac{1}{2}|\boldsymbol{\nu}|^2 - \frac{5}{2}\theta)\boldsymbol{\nu}}{\varrho\theta^3} \cdot \{(\varrho\theta + \alpha(\theta)\varrho^2\theta)\nabla_{\mathbf{x}}\theta + \varepsilon^2 K \nabla_{\mathbf{x}}\Delta_{\mathbf{x}}\theta + \varepsilon^2 \frac{2}{5}K \boldsymbol{\pi}(\mathbf{u})\} \\ & + \varepsilon^2 q(\varrho, \mathbf{u}, \theta, \boldsymbol{\nu}), \end{aligned} \quad (2.97)$$

while the remaining terms from the hydrodynamic equations were moved into the equation for h :

$$\mathcal{L}(h) = \frac{(\partial_t + \mathbf{v} \cdot \nabla_{\mathbf{x}})(Mg)}{M} - \Gamma(g, g) + R. \quad (2.98)$$

The remainder R is

$$\begin{aligned} R = \frac{\boldsymbol{\nu}}{\varrho\theta} \cdot (\nabla_{\mathbf{x}} \cdot \mathcal{T}^{(0,V)}) + \frac{\frac{1}{2}|\boldsymbol{\nu}|^2 - \frac{3}{2}\theta}{\varrho\theta^2} \cdot \left\{ \mathcal{T}_{jk}^{(0,V)} \frac{\partial u_j}{\partial x_k} \right. \\ \left. + \frac{\partial}{\partial x_k} \left[\mathcal{T}_{ijk}^{(1,V)} \frac{\partial u_i}{\partial x_j} \right] + \nabla_{\mathbf{x}} \cdot (\mathbf{q} + \varepsilon K \Delta_{\mathbf{x}}\mathbf{u}) \right\}. \end{aligned} \quad (2.99)$$

To verify the solvability conditions of the equation (2.98) we successively multiply this equation by $\psi = 1, v_1, v_2, v_3$ and $\frac{1}{2}|\mathbf{v}|^2$. We use the property $g \in \text{Ker } \mathcal{L}^\perp$ and we recall that the operator $\Gamma(g, g) = \frac{1}{M}\mathcal{Q}(Mg, Mg)$ has the property of collision invariants. It implies that the solvability conditions are equivalent to

$$0 = \nabla_{\mathbf{x}} \cdot \langle \psi \mathbf{v} g \rangle_M + \langle \psi R \rangle_M \quad \text{for each} \quad \psi = 1, v_1, v_2, v_3, \frac{1}{2}|\mathbf{v}|^2. \quad (2.100)$$

It holds obviously for $\psi = 1$.

The terms with the remainder R may be evaluated directly

$$\langle \mathbf{v} R \rangle_M = \langle \boldsymbol{\nu} R \rangle_M = \nabla_{\mathbf{x}} \cdot \mathcal{T}^{(0,V)}, \quad (2.101)$$

$$\begin{aligned} \langle \frac{1}{2}|\mathbf{v}|^2 R \rangle_M &= \langle (\frac{1}{2}|\boldsymbol{\nu}|^2 - \frac{3}{2}\theta) R \rangle_M + \langle \boldsymbol{\nu} R \rangle_M \cdot \mathbf{u} \\ &= \frac{\partial}{\partial x_k} \left\{ \mathcal{T}_{jk}^{(0,V)} u_j + \mathcal{T}_{ijk}^{(1,V)} \frac{\partial u_i}{\partial x_j} \right\} + \nabla_{\mathbf{x}} \cdot (\mathbf{q} + \varepsilon K \Delta_{\mathbf{x}}\mathbf{u}). \end{aligned} \quad (2.102)$$

To calculate the terms with g we use the following algebraic relations

$$\begin{aligned} \mathbf{v} \otimes \mathbf{v} &= \frac{\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \frac{1}{3}|\boldsymbol{\nu}|^2 \mathcal{I}}{3} + \frac{1}{3}|\boldsymbol{\nu}|^2 \mathcal{I} + \mathbf{u} \otimes \boldsymbol{\nu} + \boldsymbol{\nu} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{u}, \\ \frac{1}{2}|\mathbf{v}|^2 \mathbf{v} &= \frac{(\frac{1}{2}|\boldsymbol{\nu}|^2 - \frac{5}{2}\theta)\boldsymbol{\nu}}{2} + \frac{(\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \frac{1}{3}|\boldsymbol{\nu}|^2 \mathcal{I})\mathbf{u}}{3} + \frac{5}{2}\theta\boldsymbol{\nu} + \frac{1}{3}|\boldsymbol{\nu}|^2 \mathbf{u} + \frac{1}{2}|\mathbf{v}|^2 \mathbf{u} - \frac{1}{2}|\mathbf{u}|^2 \boldsymbol{\nu}, \end{aligned}$$

where the underlined parts are in $\text{Ker } \mathcal{L}^\perp$. We obtain

$$\nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \otimes \mathbf{v} g \rangle_M = \nabla_{\mathbf{x}} \cdot \langle (\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \frac{1}{3} |\boldsymbol{\nu}|^2 \mathcal{I}) g \rangle_M, \quad (2.103)$$

$$\begin{aligned} \nabla_{\mathbf{x}} \cdot \langle \frac{1}{2} |\mathbf{v}|^2 \mathbf{v} g \rangle_M &= \nabla_{\mathbf{x}} \cdot \langle (\frac{1}{2} |\boldsymbol{\nu}|^2 - \frac{5}{2} \theta) \boldsymbol{\nu} g \rangle_M \\ &+ \nabla_{\mathbf{x}} \cdot [\langle (\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \frac{1}{3} |\boldsymbol{\nu}|^2 \mathcal{I}) g \rangle_M \mathbf{u}]. \end{aligned} \quad (2.104)$$

Due to the symmetry of the operator \mathcal{L}^{-1} we can avoid to calculate the solution g and transform the task into

$$\langle (\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \frac{1}{3} |\boldsymbol{\nu}|^2 \mathcal{I}) g \rangle_M = \langle \mathcal{L}^{-1} (\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \frac{1}{3} |\boldsymbol{\nu}|^2 \mathcal{I}) \mathcal{L}(g) \rangle_M, \quad (2.105)$$

$$\langle (\frac{1}{2} |\boldsymbol{\nu}|^2 - \frac{5}{2} \theta) \boldsymbol{\nu} g \rangle_M = \langle \mathcal{L}^{-1} ((\frac{1}{2} |\boldsymbol{\nu}|^2 - \frac{5}{2} \theta) \boldsymbol{\nu}) \mathcal{L}(g) \rangle_M. \quad (2.106)$$

The special technique of the Sonine polynomials and the so-called bracket integrals was developed (see [FK72],[ChC39]) in order to solve the inverses $\mathcal{L}^{-1} (\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \frac{1}{3} |\boldsymbol{\nu}|^2 \mathcal{I})$ and $\mathcal{L}^{-1} ((\frac{1}{2} |\boldsymbol{\nu}|^2 - \frac{5}{2} \theta) \boldsymbol{\nu})$. Here, we follow the proposition of Burnett and we suppose that the first order approximations of these solutions should be sufficient. In fact, we are interested in the qualitative aspects of these terms and the error of such approximations is only a few percent. The expressions for the solutions becomes

$$\mathcal{L}^{-1} (\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \frac{1}{3} |\boldsymbol{\nu}|^2 \mathcal{I}) \doteq -\frac{\mu(\theta)}{\rho\theta} (\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \frac{1}{3} |\boldsymbol{\nu}|^2 \mathcal{I}), \quad (2.107)$$

$$\mathcal{L}^{-1} ((\frac{1}{2} |\boldsymbol{\nu}|^2 - \frac{5}{2} \theta) \boldsymbol{\nu}) \doteq -\frac{2\kappa(\theta)}{5\rho\theta} (\frac{1}{2} |\boldsymbol{\nu}|^2 - \frac{5}{2} \theta) \boldsymbol{\nu}, \quad (2.108)$$

and using these results in combination with (2.97), equations (2.105) and (2.106) reduce to the form

$$\begin{aligned} &\langle (\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \frac{1}{3} |\boldsymbol{\nu}|^2 \mathcal{I}) g \rangle_M \\ &= -\frac{1}{\rho\theta^2} \langle (\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \frac{1}{3} |\boldsymbol{\nu}|^2 \mathcal{I}) \otimes (\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \frac{1}{3} |\boldsymbol{\nu}|^2 \mathcal{I}) \rangle_M : \frac{1}{2} [\mu_\alpha \sigma(\mathbf{u}) + \varepsilon^2 \mu_\beta \Delta_{\mathbf{x}} \sigma(\mathbf{u})] \\ &= -\frac{1}{10\rho\theta^2} \langle (\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \frac{1}{3} |\boldsymbol{\nu}|^2 \mathcal{I}) : (\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \frac{1}{3} |\boldsymbol{\nu}|^2 \mathcal{I}) \rangle_M [\mu_\alpha \sigma(\mathbf{u}) + \varepsilon^2 \mu_\beta \Delta_{\mathbf{x}} \sigma(\mathbf{u})] \\ &= -[\mu_\alpha \sigma(\mathbf{u}) + \varepsilon^2 \mu_\beta \Delta_{\mathbf{x}} \sigma(\mathbf{u})], \end{aligned} \quad (2.109)$$

and similarly (the Eucken ratio gives us $\frac{2}{5} \kappa_\beta = \frac{3}{2} \mu_\beta$ in the last equation)

$$\begin{aligned} &\langle (\frac{1}{2} |\boldsymbol{\nu}|^2 - \frac{5}{2} \theta) \boldsymbol{\nu} g \rangle_M \\ &= -\frac{2}{5\rho\theta^3} \langle (\frac{1}{2} |\boldsymbol{\nu}|^2 - \frac{5}{2} \theta) \boldsymbol{\nu} \otimes (\frac{1}{2} |\boldsymbol{\nu}|^2 - \frac{5}{2} \theta) \boldsymbol{\nu} \rangle_M [\kappa_\alpha \nabla_{\mathbf{x}} \theta + \varepsilon^2 \kappa_\beta \nabla \Delta_{\mathbf{x}} \theta + \varepsilon^2 \frac{2}{5} \kappa_\beta \boldsymbol{\pi}(\mathbf{u})] \\ &= -\frac{2}{15\rho\theta^3} \langle (\frac{1}{2} |\boldsymbol{\nu}|^2 - \frac{5}{2} \theta) \boldsymbol{\nu} \cdot (\frac{1}{2} |\boldsymbol{\nu}|^2 - \frac{5}{2} \theta) \boldsymbol{\nu} \rangle_M [\kappa_\alpha \nabla_{\mathbf{x}} \theta + \varepsilon^2 \kappa_\beta \nabla \Delta_{\mathbf{x}} \theta + \varepsilon^2 \frac{2}{5} \kappa_\beta \boldsymbol{\pi}(\mathbf{u})] \\ &= -[\kappa_\alpha \nabla_{\mathbf{x}} \theta + \varepsilon^2 \kappa_\beta \nabla \Delta_{\mathbf{x}} \theta + \varepsilon^2 \frac{3}{2} \mu_\beta \boldsymbol{\pi}(\mathbf{u})]. \end{aligned} \quad (2.110)$$

The constitutive relations for $\mathcal{T}^{(0,V)}$, $\mathcal{T}^{(1,V)}$ and \mathbf{q} result now from the solvability

conditions for h , i.e., using (2.101), (2.103), (2.109) and (2.102), (2.104), (2.110), we get

$$\begin{aligned} & \nabla_{\mathbf{x}} \cdot \{ \mathcal{T}^{(0,V)} - [\mu_{\alpha} \sigma(\mathbf{u}) + \varepsilon^2 \mu_{\beta} \Delta_{\mathbf{x}} \sigma(\mathbf{u})] \} = 0, \\ & \nabla_{\mathbf{x}} \cdot \left\{ \mathbf{q} + \varepsilon K \Delta_{\mathbf{x}} \mathbf{u} - [\kappa_{\alpha} \nabla_{\mathbf{x}} \theta + \varepsilon^2 \kappa_{\beta} \nabla \Delta_{\mathbf{x}} \theta] \right. \\ & \quad + \frac{\partial}{\partial x_k} \left[\mathcal{T}_{ijk}^{(1,V)} \frac{\partial u_i}{\partial x_j} - \varepsilon^2 \frac{3}{2} \mu_{\beta} \boldsymbol{\pi}(\mathbf{u}) \right] \\ & \quad \left. + \frac{\partial}{\partial x_k} \left[\mathcal{T}_{jk}^{(0,V)} u_j - [\mu_{\alpha} \sigma_{jk}(\mathbf{u}) + \varepsilon^2 \mu_{\beta} \Delta_{\mathbf{x}} \sigma_{jk}(\mathbf{u})] u_j \right] \right\} = 0. \end{aligned}$$

Remark. The system of hydrodynamic equations can also be obtained by averaging the kinetic equation (2.47) if we substitute f by the approximation solution $f = M(1 + \varepsilon g)$

$$\partial_t \langle \psi(1 + \varepsilon g) \rangle_M + \nabla_{\mathbf{x}} \cdot \langle \psi \mathbf{v}(1 + \varepsilon g) \rangle_M + \nabla_{\mathbf{x}} \cdot \langle \psi s(M) \rangle_M = 0.$$

We successively apply $\psi = 1, v_1, v_2, v_3, \frac{1}{2} |\mathbf{v}|^2$ and we obtain the system

$$\begin{aligned} & \partial_t \varrho + \nabla_{\mathbf{x}} \cdot (\varrho \mathbf{u}) = 0, \\ & \partial_t (\varrho \mathbf{u}) + \nabla_{\mathbf{x}} \cdot (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_{\mathbf{x}} p = -\varepsilon \nabla_{\mathbf{x}} \cdot \langle (\boldsymbol{\mathcal{V}} \otimes \boldsymbol{\mathcal{V}} - \frac{1}{3} |\boldsymbol{\mathcal{V}}|^2 \mathcal{I}) g \rangle, \\ & \partial_t (\varrho e) + \nabla_{\mathbf{x}} \cdot (\varrho e \mathbf{u} + p \mathbf{u}) + \varepsilon^2 K \nabla \Delta_{\mathbf{x}} \cdot \mathbf{u} = \\ & \quad -\varepsilon \nabla_{\mathbf{x}} \cdot [\langle (\boldsymbol{\mathcal{V}} \otimes \boldsymbol{\mathcal{V}} - \frac{1}{3} |\boldsymbol{\mathcal{V}}|^2 \mathcal{I}) g \rangle \mathbf{u}] - \varepsilon \nabla_{\mathbf{x}} \cdot \langle (\frac{1}{2} |\boldsymbol{\mathcal{V}}|^2 - \frac{5}{2} \theta) \boldsymbol{\mathcal{V}} g \rangle. \end{aligned}$$

Finally, we use the above formulas (2.109) and (2.110) again.

2.4 Enskog equation

Enskog was the first who studied the collisional transfer of momentum and energy on the model of hard spheres. In this section we would like to point out the similarity between his theory and the source term $s(M)$; see (2.48).

The Enskog equation for the model of hard spheres reads (see, e.g., [ChC39, Chapter 16], [BLTP91])

$$(\partial_t + \mathbf{v} \cdot \nabla_{\mathbf{x}}) f = \mathbf{E}(f; f, f), \quad (2.111)$$

where

$$\begin{aligned} \mathbf{E}(f; f, f)(\mathbf{x}, \mathbf{v}, t) &= \sigma^2 \iint_{(\mathbf{v}-\mathbf{w}) \cdot \boldsymbol{\eta} > 0} [\chi(\mathbf{x} - \frac{1}{2} \sigma \boldsymbol{\eta}) f(\mathbf{x}, \mathbf{v}_0, t) f(\mathbf{x} - \sigma \boldsymbol{\eta}, \mathbf{w}_0, t) \\ & \quad - \chi(\mathbf{x} + \frac{1}{2} \sigma \boldsymbol{\eta}) f(\mathbf{x}, \mathbf{v}, t) f(\mathbf{x} + \sigma \boldsymbol{\eta}, \mathbf{w}, t)] (\mathbf{v} - \mathbf{w}) \cdot \boldsymbol{\eta} d\boldsymbol{\eta} d\mathbf{w}. \end{aligned} \quad (2.112)$$

Here χ is the so-called *pair correlation function*.

The Enskog equation is a modification of the Boltzmann equation for the model of hard spheres, i.e., $b(|\mathbf{v} - \mathbf{w}|, \boldsymbol{\eta}) = \sigma^2 (\mathbf{v} - \mathbf{w}) \cdot \boldsymbol{\eta}$, where the additional *ad hoc* assumptions are taken into account:

- 1) the centers of two colliding particles are at the distance σ which is equal to the diameter of hard spheres. (The different signs are implied by the relation $(\mathbf{v} - \mathbf{w}) \cdot \boldsymbol{\eta} = -(\mathbf{v}_0 - \mathbf{w}_0) \cdot \boldsymbol{\eta}$.)

2) the probability of the collision of two particles is increased by the factor χ . This factor is evaluated at the contact point of two spheres and it represents the following considerations of Enskog. Let n be the number of particles per unit volume. The probability of the collision of two particles should be increased due to the fact that the particles occupy a volume themselves, i.e., the unit volume in which the center of particle can lie is reduced to $1 - \frac{4}{3}\pi\sigma^3 n$. Furthermore, if the centers of two particles are at the distance larger than σ , but less than 2σ , then the particles are shielding each other, and it reduces the probability of collisions. Enskog included these considerations into the factor χ :

$$\chi(\mathbf{x}) = \frac{1 - \frac{11}{8} n(\mathbf{x})\tilde{\alpha}}{1 - 2n(\mathbf{x})\tilde{\alpha}}, \quad \text{where} \quad \tilde{\alpha} = \frac{2}{3}\pi\sigma^3. \quad (2.113)$$

The factor χ is equal to unity for a rarefied gas and increases with increasing density of the gas. There exists the maximum density n_c where χ tends to infinity (hard spheres are packed so closely that the motion is impossible).

For brevity, we assume a rarefied gas, i.e., $\chi = 1$, and we expand $f(\mathbf{x} - \sigma\boldsymbol{\eta}, \mathbf{w}_0, t)$ and $f(\mathbf{x} + \sigma\boldsymbol{\eta}, \mathbf{w}, t)$ in the relation (2.112) by the Taylor theorem. Then we replace all the gradients of f by the gradients of M , and we get

$$\begin{aligned} \mathbf{E}(f; f, f) &\doteq \mathcal{Q}(f, f) + \sum_{k=1} \frac{\sigma^{k+2}}{k!} \iint_{(\mathbf{v}-\mathbf{w}) \cdot \boldsymbol{\eta} > 0} \otimes^{k+1} \boldsymbol{\eta} : \\ &\{ [(-1)^k M(\mathbf{v}_0) \nabla_{\mathbf{x}}^k M(\mathbf{w}_0) - M(\mathbf{v}) \nabla_{\mathbf{x}}^k M(\mathbf{w})] \otimes (\mathbf{v} - \mathbf{w}) \} d\boldsymbol{\eta} d\mathbf{w}. \end{aligned} \quad (2.114)$$

The second term on the right hand side corresponds ‘‘in some ways’’ to the source term $I_{E_1}(M)$ in the case of the model of hard spheres. There are the same powers of σ , the same powers and symmetry of tensors $\otimes^{k+1} \boldsymbol{\eta}$. But the Maxwellians also depends on the vector $\boldsymbol{\eta}$ through the relations (2.24), hence the even terms do not vanish. Any further calculations are very technical.

We conclude this section with the result obtained by Enskog [ChC39, Eq. 16.32,3] for $k = 1$ which reads

$$\begin{aligned} \mathbf{E}(f; f, f) &= \mathcal{Q}(f, f) - \sigma^3 \iint_{(\mathbf{v}-\mathbf{w}) \cdot \boldsymbol{\eta} > 0} \boldsymbol{\eta} \otimes \boldsymbol{\eta} : \\ &\{ M(\mathbf{v}) M(\mathbf{w}) [\nabla_{\mathbf{x}} \ln M(\mathbf{w}_0) + \nabla_{\mathbf{x}} \ln M(\mathbf{w})] \otimes (\mathbf{v} - \mathbf{w}) \} d\boldsymbol{\eta} d\mathbf{w} \\ &= \mathcal{Q}(f, f) \\ &\quad - \frac{\mathbf{v}}{\rho\theta} \cdot \nabla_{\mathbf{x}} (\tilde{\alpha} \rho^2 \theta) \\ &\quad - \frac{\frac{1}{2} |\mathbf{v}|^2 - \frac{3}{2} \theta}{\rho\theta^2} \left\{ \frac{2}{3} \tilde{\alpha} \rho^2 \theta \nabla_{\mathbf{x}} \cdot \mathbf{u} \right\} \\ &\quad - \frac{\mathbf{v} \otimes \mathbf{v} - \frac{1}{3} |\mathbf{v}|^2 \mathcal{I}}{2\rho\theta^2} : \left\{ \frac{2}{5} \tilde{\alpha} \rho^2 \theta \sigma(\mathbf{u}) \right\} \\ &\quad - \frac{(\frac{1}{2} |\mathbf{v}|^2 - \frac{5}{2} \theta) \mathbf{v}}{\rho\theta^3} \cdot \left\{ \frac{3}{5} \tilde{\alpha} \rho^2 \theta \nabla_{\mathbf{x}} \theta \right\}. \end{aligned}$$

The right hand side is qualitatively comparable to the relations of orthogonality (2.69) as the coefficient $\alpha(\theta)$ is equal to the coefficient $\tilde{\alpha}$ in the case of hard spheres; see (2.76).

2.5 Thermodynamics of dipolar viscous fluids – First approach

The general theory of multipolar materials was introduced by Green and Rivlin ([GR64a] and [GR64b]) and further extended and generalized by Nečas and Šilhavý ([NS91]). Here we explain their concept and we introduce the most simple example of such fluids – the linear dipolar viscous fluid.

The multipolar viscous fluids are characterized as materials for which the thermodynamic processes satisfy the local forms of balance equations (see below), the constitutive quantities are functions of higher order spatial derivatives of the density, the velocity and the temperature², and for which the Clausius–Duhem inequality and the principle of material frame-indifference holds. The multipolar viscous fluids of grade N are materials where the constitutive quantities depend on the velocity gradients up to order N , i.e., the case $N = 1$ are the monopolar viscous fluids where the constitutive quantities depend only on the first gradient of velocity (e.g., the Stokes law for the stress tensor), and the case $N = 2$ are the dipolar viscous fluids where the constitutive quantities depend on the first and also the second gradient of velocity. The local balance equations of mass, momentum, energy and angular momentum, in the case of the absence of external momentum and heat sources, have the following form (the Einstein convention of summation over the same indices is used and the dot over the terms denotes the material time derivative $\partial_t + \mathbf{u} \cdot \nabla_{\mathbf{x}}$):

$$\dot{\rho} + \rho \nabla_{\mathbf{x}} \cdot \mathbf{u} = 0, \quad (2.115)$$

$$\rho \dot{\mathbf{u}} = \nabla_{\mathbf{x}} \cdot \mathcal{T}^{(0)}, \quad (2.116)$$

$$\rho(\nu + \frac{1}{2}|\mathbf{u}|^2) \dot{} = \sum_{n=0}^{N-1} \frac{\partial}{\partial x_k} \left[\mathcal{T}_{ij_1 \dots j_n k}^{(n)} \frac{\partial^n u_i}{\partial x_{j_1} \dots \partial x_{j_n}} \right] + \nabla_{\mathbf{x}} \cdot \mathbf{q}, \quad (2.117)$$

$$\rho(\mathcal{E}_{ijk} x_j v_k) \dot{} = \frac{\partial}{\partial x_p} (\mathcal{E}_{ijk} x_j \mathcal{T}_{kp}^{(0)} + \mathcal{E}_{ijk} \mathcal{T}_{kjp}^{(1)}), \quad i = 1, 2, 3. \quad (2.118)$$

Here ρ is the density, \mathbf{u} is the velocity, ν is the internal energy, θ is the absolute temperature, η is the entropy, \mathbf{q} is the heat flux and \mathcal{E} is the Levi–Civita tensor

$$\mathcal{E}_{ijk} = \begin{cases} +1 & (i, j, k) \text{ is an even permutation of } (1, 2, 3), \\ -1 & (i, j, k) \text{ is an odd permutation of } (1, 2, 3), \\ 0 & \text{otherwise: } i = j \text{ or } j = k \text{ or } i = k. \end{cases}$$

The tensors $\mathcal{T}^{(n)}$ are the multipolar stress tensors of order $n + 2$ which are symmetric in the indices j_1, \dots, j_n . The convention $\mathcal{T}^{(n)} = 0$ for $n \geq N$ is used.

The Clausius–Duhem inequality usually reads as

$$\rho \dot{\eta} \geq \nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{q}}{\theta} \right). \quad (2.119)$$

The principle of material frame-indifference postulates the independence of the physical quantities on the observers. It postulates that if the coordinates are transformed by any Galilean transformation

$$\bar{x}_p = \mathcal{O}_{pi}(t)x_i + c_p(t), \quad (2.120)$$

²Nečas and Šilhavý proposed only the first gradient of the temperature but we have the dependence of the heat flux on the third gradient of the temperature; see (2.90)

where $\mathcal{O}(t)$ is a time-dependent orthogonal matrix and $\mathbf{c}(t)$ is a time-dependent vector, then the physical quantities ϱ , θ , η , e , \mathbf{q} and $\mathcal{T}^{(n)}$ are transformed in the following way

$$\bar{\varrho} = \varrho, \quad \bar{\theta} = \theta, \quad \bar{\eta} = \eta, \quad \bar{e} = e, \quad (2.121)$$

$$\bar{q}_p = \mathcal{O}_{pi} q_i, \quad (2.122)$$

$$\bar{\mathcal{T}}_{p_1 \dots p_{n+2}}^{(n)} = \mathcal{O}_{p_1 i_1} \dots \mathcal{O}_{p_{n+2} i_{n+2}} \mathcal{T}_{i_1 \dots i_{n+2}}^{(n)}. \quad (2.123)$$

The arguments on the left hand side are $(\bar{\mathbf{x}}, t)$ and on the right hand side are (\mathbf{x}, t) .

The multipolar viscous fluids of type (N, M, K) are the materials where the constitutive equations for e , η , \mathbf{q} and $\mathcal{T}^{(n)}$, $n = 0, \dots, N-1$ are functions of ϱ , \mathbf{u} and θ and their gradients

$$f = f(\varrho, \nabla_{\mathbf{x}} \varrho, \dots, \nabla_{\mathbf{x}}^{M-1} \varrho, \nabla_{\mathbf{x}} \mathbf{u}, \dots, \nabla_{\mathbf{x}}^K \mathbf{u}, \theta, \nabla_{\mathbf{x}} \theta). \quad (2.124)$$

Let us state, without the proof, the consequences of the principle of material frame-indifference as stated in paper [NS91].

Proposition 2.1. A multipolar viscous fluid of type (N, M, K) satisfies the principle of material frame-indifference if and only if the following two conditions are satisfied:

- (1) The functions e , η , \mathbf{q} and $\mathcal{T}^{(n)}$ depend on the first spatial gradient of velocity only through its symmetric part $\mathcal{D} = \frac{1}{2}(\nabla_{\mathbf{x}} \mathbf{u} + \nabla_{\mathbf{x}}^T \mathbf{u})$, i.e.,

$$\begin{aligned} f(\varrho, \nabla_{\mathbf{x}} \varrho, \dots, \nabla_{\mathbf{x}}^{M-1} \varrho, \nabla_{\mathbf{x}} \mathbf{u}, \dots, \nabla_{\mathbf{x}}^K \mathbf{u}, \theta, \nabla_{\mathbf{x}} \theta) \\ = f(\varrho, \nabla_{\mathbf{x}} \varrho, \dots, \nabla_{\mathbf{x}}^{M-1} \varrho, \mathcal{D}, \nabla_{\mathbf{x}}^2 \mathbf{u}, \dots, \nabla_{\mathbf{x}}^K \mathbf{u}, \theta, \nabla_{\mathbf{x}} \theta), \end{aligned} \quad (2.125)$$

where f stands for any of the functions e , η , \mathbf{q} and $\mathcal{T}^{(n)}$.

- (2) The constitutive functions e , η , \mathbf{q} and $\mathcal{T}^{(n)}$ are isotropic scalar-, vector-, or tensor-valued functions of the scalar, vector, tensor arguments ϱ , $\nabla_{\mathbf{x}} \varrho$, \dots , $\nabla_{\mathbf{x}}^{M-1} \varrho$, \mathcal{D} , $\nabla_{\mathbf{x}}^2 \mathbf{u}$, \dots , $\nabla_{\mathbf{x}}^K \mathbf{u}$, θ , $\nabla_{\mathbf{x}} \theta$.

The isotropic function is a function which satisfies the postulates of the principle of material frame-indifference through its arguments, i.e.,

$$\begin{aligned} \mathcal{T}_{p_1 \dots p_{n+2}}^{(n)}(\bar{\varrho}, \nabla_{\bar{\mathbf{x}}} \bar{\varrho}, \dots, \nabla_{\bar{\mathbf{x}}}^{M-1} \bar{\varrho}, \bar{\mathcal{D}}, \nabla_{\bar{\mathbf{x}}}^2 \bar{\mathbf{u}}, \dots, \nabla_{\bar{\mathbf{x}}}^K \bar{\mathbf{u}}, \bar{\theta}, \nabla_{\bar{\mathbf{x}}} \bar{\theta}) \\ = \mathcal{O}_{p_1 i_1} \dots \mathcal{O}_{p_{n+2} i_{n+2}} \mathcal{T}_{i_1 \dots i_{n+2}}^{(n)}(\varrho, \nabla_{\mathbf{x}} \varrho, \dots, \nabla_{\mathbf{x}}^{M-1} \varrho, \mathcal{D}, \nabla_{\mathbf{x}}^2 \mathbf{u}, \dots, \nabla_{\mathbf{x}}^K \mathbf{u}, \theta, \nabla_{\mathbf{x}} \theta). \end{aligned} \quad (2.126)$$

In the rest of this section we focus on the most simple multipolar fluid – the *linear dipolar viscous fluid*. The constitutive equations of such a fluid are independent of the gradients of density ($M = 1$) and the multipolar stress tensors and the heat flux depend linearly on the gradients of velocity and on the gradient of temperature. The word dipolar means there are only two stress tensors ($N = 2$), so we interested in the multipolar viscous fluids of type $(2, 1, K)$, where

$$f = f(\varrho, \nabla_{\mathbf{x}} \mathbf{u}, \dots, \nabla_{\mathbf{x}}^K \mathbf{u}, \theta, \nabla_{\mathbf{x}} \theta). \quad (2.127)$$

We introduce the division of the multipolar stress tensor $\mathcal{T}^{(n)}$ into the equilibrium part $\mathcal{T}^{(n,E)}$ corresponding to the zero values of the gradients of velocity and the zero value of the gradient of temperature

$$\mathcal{T}^{(n,E)}(\varrho, \theta) = \mathcal{T}^{(n)}(\varrho, 0, \dots, 0, \theta, 0) \quad (2.128)$$

and the viscous part $\mathcal{T}^{(n,V)}$ of the multipolar stress tensor

$$\mathcal{T}^{(n,V)} = \mathcal{T}^{(n)} - \mathcal{T}^{(n,E)}. \quad (2.129)$$

The equilibrium part and the viscous part of the multipolar stress tensor have to obey the principle of material frame-indifference independently, thus they must be isotropic functions and the relation

$$\mathcal{T}_{prs}^{(1,E)}(\varrho, \theta) = \mathcal{O}_{pi}\mathcal{O}_{rj}\mathcal{O}_{sk}\mathcal{T}_{ijk}^{(1,E)}(\varrho, \theta) \quad (2.130)$$

have to hold even for $\mathcal{O} = -\mathcal{I}$. It implies

$$\mathcal{T}^{(1,E)} = 0. \quad (2.131)$$

From the equation (2.120) we get the transformation law

$$\frac{\partial \bar{x}_p}{\partial x_i} = \mathcal{O}_{pi}, \quad \text{or equivalently} \quad \frac{\partial x_i}{\partial \bar{x}_p} = \mathcal{O}_{pi}, \quad (2.132)$$

and this gives us the transformation law for the gradient of the temperature θ :

$$\frac{\partial \bar{\theta}}{\partial \bar{x}_p} = \frac{\partial \theta}{\partial x_i} \frac{\partial x_i}{\partial \bar{x}_p} = \mathcal{O}_{pi} \frac{\partial \theta}{\partial x_i}. \quad (2.133)$$

It is a little bit complicated with the transformation law of the gradients of velocity because

$$\bar{u}_p = \dot{\bar{x}}_p = \dot{\mathcal{O}}_{pi}x_i + \mathcal{O}_{pi}u_i + \dot{c}_p, \quad (2.134)$$

thus

$$\frac{\partial \bar{u}_p}{\partial \bar{x}_r} = \frac{\partial}{\partial x_j} (\dot{\mathcal{O}}_{pi}x_i + \mathcal{O}_{pi}u_i + \dot{c}_p) \frac{\partial x_j}{\partial \bar{x}_r} = \dot{\mathcal{O}}_{pj}\mathcal{O}_{rj} + \mathcal{O}_{pi}\mathcal{O}_{rj} \frac{\partial u_i}{\partial x_j}, \quad (2.135)$$

where the matrix $\dot{\mathcal{O}}_{pj}\mathcal{O}_{rj}$ is antisymmetric. (By the way, it implies the dependence of the constitutive equations through the symmetric part of the gradient of velocity \mathcal{D} because $\bar{\mathcal{D}}_{pr} = \mathcal{O}_{pi}\mathcal{O}_{rj}\mathcal{D}_{ij}$.) All higher gradients of velocity are transformed as usual

$$\frac{\partial^n \bar{u}_p}{\partial \bar{x}_{r_1} \cdots \partial \bar{x}_{r_n}} = \mathcal{O}_{pi}\mathcal{O}_{r_1j_1} \cdots \mathcal{O}_{r_nj_n} \frac{\partial^n u_i}{\partial x_{j_1} \cdots \partial x_{j_n}} \quad n \geq 2. \quad (2.136)$$

As the dependence on the gradients of velocity and the gradient of temperature is supposed to be linear, the transformation laws have to hold for each linear term separately. For example, suppose that the heat flux \mathbf{q} depends linearly on \mathcal{D} , then this part transforms as \mathcal{O}^2 because of $\text{sym}G$, but transforms as \mathcal{O} because \mathbf{q} in an isotropic vector-valued function. Hence the trick $\mathcal{O} = -\mathcal{I}$ shows that the heat flux must be independent of \mathcal{D} . More generally:

$$\mathcal{T}^{(0,V)} = \mathcal{T}^{(0,V)}(\varrho, \theta, \nabla_{\mathbf{x}}\mathbf{u}, \nabla_{\mathbf{x}}^3\mathbf{u}, \nabla_{\mathbf{x}}^5, \dots), \quad (2.137)$$

$$\mathcal{T}^{(1,V)} = \mathcal{T}^{(1,V)}(\varrho, \theta, \nabla_{\mathbf{x}}^2\mathbf{u}, \nabla_{\mathbf{x}}^4\mathbf{u}, \dots, \nabla_{\mathbf{x}}\theta), \quad (2.138)$$

$$\mathbf{q} = \mathbf{q}(\varrho, \theta, \nabla_{\mathbf{x}}^2\mathbf{u}, \nabla_{\mathbf{x}}^4\mathbf{u}, \dots, \nabla_{\mathbf{x}}\theta). \quad (2.139)$$

The linear dipolar viscous fluid has the following reduced equation of balance of energy (the equation is reduced by the equation of balance of momentum)

$$\varrho \dot{\nu} = \mathcal{T}_{ij}^{(0)} \frac{\partial u_i}{\partial x_j} + \frac{\partial \mathcal{T}_{ijk}^{(1)}}{\partial x_k} \frac{\partial u_i}{\partial x_j} + \mathcal{T}_{ijk}^{(1)} \frac{\partial^2 u_i}{\partial x_j \partial x_k} + \frac{\partial q_i}{\partial x_i}. \quad (2.140)$$

We introduce the Helmholtz free energy

$$\xi = \nu - \theta\eta, \quad (2.141)$$

using the above equations (2.140), (2.141), we obtain the equivalent form of the Clausius–Duhem inequality — the dissipation inequality which reads

$$\varrho \dot{\xi} \leq -\varrho\eta\dot{\theta} + \mathcal{T}_{ij}^{(0)} \frac{\partial u_i}{\partial x_j} + \frac{\partial \mathcal{T}_{ijk}^{(1)}}{\partial x_k} \frac{\partial u_i}{\partial x_j} + \mathcal{T}_{ijk}^{(1)} \frac{\partial^2 u_i}{\partial x_j \partial x_k} + \frac{q_i}{\theta} \frac{\partial \theta}{\partial x_i}. \quad (2.142)$$

We use the constitutive relations (2.127) to express the material time derivative of the Helmholtz free energy and the gradient $\partial \mathcal{T}^{(1,V)} / \partial x_k$ in more detailed form

$$\begin{aligned} & \varrho \left(-\varrho \frac{\partial \xi}{\partial \varrho} \frac{\partial u_i}{\partial x_i} + \sum_{n=1}^K \frac{\partial \xi}{\partial \nabla_{\mathbf{x}}^n u_i} \nabla_{\mathbf{x}}^n u_i + \frac{\partial \xi}{\partial \theta} \dot{\theta} + \frac{\partial \xi}{\partial \nabla_{\mathbf{x}} \theta} \nabla_{\mathbf{x}} \theta \right) \\ & \leq -\varrho\eta\dot{\theta} + \mathcal{T}_{ij}^{(0,E)} \frac{\partial u_i}{\partial x_j} + \mathcal{T}_{ij}^{(0,V)} \frac{\partial u_i}{\partial x_j} \\ & \quad + \left(\frac{\partial \mathcal{T}_{ijk}^{(1,V)}}{\partial \varrho} \frac{\partial \varrho}{\partial x_k} + \sum_{n=1}^K \frac{\partial \mathcal{T}_{ijk}^{(1,V)}}{\partial \nabla_{\mathbf{x}}^n u_p} \frac{\partial \nabla_{\mathbf{x}}^n u_p}{\partial x_k} + \frac{\partial \mathcal{T}_{ijk}^{(1,V)}}{\partial \theta} \frac{\partial \theta}{\partial x_k} + \frac{\partial \mathcal{T}_{ijk}^{(1,V)}}{\partial \nabla_{\mathbf{x}} \theta} \frac{\partial \nabla_{\mathbf{x}} \theta}{\partial x_k} \right) \frac{\partial u_i}{\partial x_j} \\ & \quad + \mathcal{T}_{ijk}^{(1,V)} \frac{\partial^2 u_i}{\partial x_j \partial x_k} + \frac{q_i}{\theta} \frac{\partial \theta}{\partial x_i}. \end{aligned} \quad (2.143)$$

The first term on the left hand side was adjusted with the help of the continuity equation $\dot{\varrho} = -\varrho \nabla_{\mathbf{x}} \cdot \mathbf{u}$ and we applied the identity $\mathcal{T}^{(1,E)} = 0$; see (2.131).

This inequality is required to hold for every process in a fluid, hence all the derivatives of velocity and temperature and also their material time derivatives are completely arbitrary. We compare the terms with the material time derivatives and we see immediately

$$\frac{\partial \xi}{\partial \nabla_{\mathbf{x}}^n u_i} = 0, \quad \frac{\partial \xi}{\partial \nabla_{\mathbf{x}} \theta} = 0, \quad \frac{\partial \xi}{\partial \theta} = -\eta. \quad (2.144)$$

In other words, the Helmholtz free energy ξ , from the definition (2.141) also the internal energy ν and the entropy η , depends only on the density ϱ and the temperature θ :

$$\xi = \xi(\varrho, \theta). \quad (2.145)$$

For the same reason, i.e., $\partial \nabla_{\mathbf{x}} \theta / \partial x_k$ has an arbitrary value, we have $\mathcal{T}^{(1,V)}$ independent of the gradient of temperature

$$\frac{\partial \mathcal{T}_{ijk}^{(1,V)}}{\partial \nabla_{\mathbf{x}} \theta} = 0. \quad (2.146)$$

Furthermore, we can systematically replace all the gradients of velocity and temperature by their $\alpha > 0$ multiples and we obtain the inequality

$$\begin{aligned} -\varrho^2 \frac{\partial \xi}{\partial \varrho} \alpha \frac{\partial u_i}{\partial x_i} & \leq \mathcal{T}_{ij}^{(0,E)} \alpha \frac{\partial u_i}{\partial x_j} + \mathcal{T}_{ij}^{(0,V)} \alpha \frac{\partial u_i}{\partial x_j} \\ & \quad + \left(\frac{\partial \mathcal{T}_{ijk}^{(1,V)}}{\partial \varrho} \frac{\partial \varrho}{\partial x_k} + \sum_{n=1}^K \frac{\partial \mathcal{T}_{ijk}^{(1,V)}}{\partial \nabla_{\mathbf{x}}^n u_p} \alpha \frac{\partial \nabla_{\mathbf{x}}^n u_p}{\partial x_k} + \frac{\partial \mathcal{T}_{ijk}^{(1,V)}}{\partial \theta} \alpha \frac{\partial \theta}{\partial x_k} \right) \alpha \frac{\partial u_i}{\partial x_j} \\ & \quad + \mathcal{T}_{ijk}^{(1,V)} \alpha \frac{\partial^2 u_i}{\partial x_j \partial x_k} + \frac{q_i}{\theta} \alpha \frac{\partial \theta}{\partial x_i}. \end{aligned}$$

Some terms are linear and some terms are quadratic in α , so we divide the equation by α and take the limit $\alpha \rightarrow 0+$. We notice that the point at which the stress tensors and the heat flux are evaluated $(\varrho, \alpha \nabla_{\mathbf{x}} \mathbf{u}, \dots, \alpha \nabla_{\mathbf{x}}^K \mathbf{u}, \theta, \nabla_{\mathbf{x}} \theta)$ tends to the equilibrium point $(\varrho, 0, \dots, 0, \theta, 0)$, thus the tensors $\mathcal{T}^{(0,V)}$, $\mathcal{T}^{(1,V)}$ and also the derivative $\partial \mathcal{T}^{(1,V)} / \partial \varrho$ vanish. The limit inequality reads

$$-\varrho^2 \frac{\partial \xi}{\partial \varrho} \frac{\partial u_i}{\partial x_j} \delta_{ij} \leq \mathcal{T}_{ij}^{(0,E)} \frac{\partial u_i}{\partial x_j} + \frac{q_i}{\theta} \frac{\partial \theta}{\partial x_i}.$$

The gradient of temperature can vary independently, it implies that

$$\mathbf{q}(\varrho, 0, \dots, 0, \theta, 0) = 0. \quad (2.147)$$

The remaining two terms depend linearly on the gradient of velocity, hence the equality have to hold and we get

$$\mathcal{T}^{(0,E)} = -p(\varrho, \theta) \mathcal{I}, \quad (2.148)$$

where p is the pressure

$$p(\varrho, \theta) = \varrho^2 \frac{\partial \xi}{\partial \varrho}. \quad (2.149)$$

We know the interpretation of both partial derivatives of the Helmholtz free energy, the equations (2.149) and (2.144), therefore we can derive the Gibbs equation

$$\begin{aligned} \varrho \dot{\xi} &= \varrho \frac{\partial \xi}{\partial \varrho} \dot{\varrho} + \varrho \frac{\partial \xi}{\partial \theta} \dot{\theta} \\ &= \mathcal{T}_{ij}^{(0,E)} \frac{\partial u_i}{\partial x_j} - \varrho \eta \dot{\theta}. \end{aligned} \quad (2.150)$$

Finally, we subtract the Gibbs equation from the dissipation inequality (2.142) and we obtain the residual dissipation inequality

$$\mathcal{T}_{ij}^{(0,V)} \frac{\partial u_i}{\partial x_j} + \frac{\partial \mathcal{T}_{ijk}^{(1,V)}}{\partial x_k} \frac{\partial u_i}{\partial x_j} + \mathcal{T}_{ijk}^{(1,V)} \frac{\partial^2 u_i}{\partial x_j \partial x_k} + \frac{q_i}{\theta} \frac{\partial \theta}{\partial x_i} \geq 0. \quad (2.151)$$

According to paper [NS91, Theorem 5.1], the viscous parts $\mathcal{T}^{(0,V)}$ and $\mathcal{T}^{(1,V)}$ of the linear multipolar stress tensors and the heat flux \mathbf{q} can be explicitly stated. Here we restrict their general forms to the case $K = 3$:

$$\begin{aligned} \mathcal{T}_{ij}^{(0,V)} &= \bar{\lambda} \frac{\partial u_k}{\partial x_k} \delta_{ij} + \bar{\mu} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \\ &\quad + \bar{\alpha} \Delta_{\mathbf{x}} \frac{\partial u_k}{\partial x_k} \delta_{ij} + \bar{\beta}_1 \Delta_{\mathbf{x}} \frac{\partial u_i}{\partial x_j} + \bar{\beta}_2 \Delta_{\mathbf{x}} \frac{\partial u_j}{\partial x_i} + \bar{\gamma} \frac{\partial^3 u_k}{\partial x_i \partial x_j \partial x_k}, \end{aligned} \quad (2.152)$$

$$\begin{aligned} \mathcal{T}_{ijk}^{(1,V)} &= \bar{c}_1 \Delta_{\mathbf{x}} u_k \delta_{ij} + \bar{c}_2 \frac{\partial^2 u_p}{\partial x_k \partial x_p} \delta_{ij} \\ &\quad + \bar{c}_3 \Delta_{\mathbf{x}} u_j \delta_{ik} + \bar{c}_4 \frac{\partial^2 u_p}{\partial x_j \partial x_p} \delta_{ik} \\ &\quad + \bar{c}_5 \Delta_{\mathbf{x}} u_i \delta_{jk} + \bar{c}_6 \frac{\partial^2 u_p}{\partial x_i \partial x_p} \delta_{jk} \end{aligned} \quad (2.153)$$

$$\begin{aligned} &\quad + \bar{c}_7 \frac{\partial^2 u_i}{\partial x_j \partial x_k} + \bar{c}_8 \frac{\partial^2 u_k}{\partial x_i \partial x_j} + \bar{c}_9 \frac{\partial^2 u_j}{\partial x_i \partial x_k}, \\ q_i &= \bar{k} \frac{\partial \theta}{\partial x_i} + \bar{d}_1 \frac{\partial^2 u_p}{\partial x_i \partial x_p} + \bar{d}_2 \Delta_{\mathbf{x}} u_i, \end{aligned} \quad (2.154)$$

where all coefficients $\bar{\lambda}$, $\bar{\mu}$, $\bar{\alpha}$, $\bar{\beta}_1$, $\bar{\beta}_2$, $\bar{\gamma}$, $\bar{c}_1, \dots, \bar{c}_9$, \bar{k} , \bar{b}_1 and \bar{b}_2 are scalar functions of ϱ and θ .

We would like to notice that there should be three additional terms in formula (2.153) according to paper [NS91, Equation 6.12]

$$\bar{c}_{10} \frac{\partial \theta}{\partial x_k} \delta_{ij} + \bar{c}_{12} \frac{\partial \theta}{\partial x_j} \delta_{ik} + \bar{c}_{13} \frac{\partial \theta}{\partial x_i} \delta_{jk},$$

but we have restriction (2.146), so these terms are zero.

The equation of balance of angular momentum (2.118) has the reduced form (applying the equation of balance of momentum and energy)

$$\mathcal{E}_{ijk} \left(\mathcal{T}_{kj}^{(0)} + \frac{\partial \mathcal{T}_{kjp}^{(1)}}{\partial x_p} \right) = 0, \quad i = 1, 2, 3. \quad (2.155)$$

We split the multipolar stress tensors into the equilibrium and viscous parts and we use the explicit forms of $\mathcal{T}^{(0,E)} = -p\mathcal{I}$ and $\mathcal{T}^{(1,E)} = 0$, then

$$\mathcal{E}_{ijk} \left(\mathcal{T}_{kj}^{(0,V)} + \frac{\partial \mathcal{T}_{kjp}^{(1,V)}}{\partial x_p} \right) = 0 \quad (2.156)$$

because $\mathcal{E}_{ijk} \delta_{jk} = 0$.

The Levi–Civita tensor has some interesting properties. For each tensor A_{ij} it holds $\mathcal{E}_{ijk} A_{jk} = -\mathcal{E}_{ijk} A_{kj}$. Especially, if A is a symmetric tensor (e.g., δ_{ij} or the second partial derivatives $\partial^2/\partial x_i \partial x_j$ of a sufficiently regular function) then $\mathcal{E}_{ijk} A_{jk} = 0$.

We substitute the above formulas (2.152), (2.153) for the linear multipolar stress tensors $\mathcal{T}^{(0,V)}$, $\mathcal{T}^{(1,V)}$ into the equation (2.155) and we apply the properties of Levi–Civita tensor. Then

$$\begin{aligned} \mathcal{E}_{ijk} \mathcal{T}_{kj}^{(0,V)} &= \mathcal{E}_{ijk} (\bar{\beta}_1 - \bar{\beta}_2) \Delta_{\mathbf{x}} \frac{\partial u_k}{\partial x_j}, \\ \mathcal{E}_{ijk} \frac{\partial \mathcal{T}_{kjp}^{(1,V)}}{\partial x_p} &= \mathcal{E}_{ijk} (-\bar{c}_3 + \bar{c}_5 - \bar{c}_7 + \bar{c}_9) \Delta_{\mathbf{x}} \frac{\partial u_k}{\partial x_j} + \bar{r}_i (\nabla_{\mathbf{x}}^2 \mathbf{u}), \end{aligned}$$

where the vector \bar{r}_i depends linearly on the second partial derivatives of velocity

$$\begin{aligned} \bar{r}_i &= \mathcal{E}_{ijk} \left(\frac{\partial \bar{c}_3}{\partial x_k} \Delta_{\mathbf{x}} u_j + \frac{\partial \bar{c}_4}{\partial x_k} \frac{\partial^2 u_p}{\partial x_j \partial x_p} + \frac{\partial \bar{c}_5}{\partial x_j} \Delta_{\mathbf{x}} u_k \right. \\ &\quad \left. + \frac{\partial \bar{c}_6}{\partial x_j} \frac{\partial^2 u_p}{\partial x_k \partial x_p} + \frac{\partial \bar{c}_7}{\partial x_p} \frac{\partial^2 u_k}{\partial x_j \partial x_p} + \frac{\partial \bar{c}_8}{\partial x_p} \frac{\partial^2 u_p}{\partial x_j \partial x_k} + \frac{\partial \bar{c}_9}{\partial x_p} \frac{\partial^2 u_j}{\partial x_k \partial x_p} \right). \end{aligned}$$

The velocity gradients of all orders may be chosen in a completely arbitrary way, hence the coefficients of $\mathcal{T}^{(0,V)}$, $\mathcal{T}^{(1,V)}$ are in the relation:

$$\bar{\beta}_1 + \bar{c}_5 + \bar{c}_7 = \bar{\beta}_2 + \bar{c}_3 + \bar{c}_9, \quad (2.157)$$

and this relation holds for any linear dipolar viscous fluid of type (2, 1, 3) which satisfies the local law of balance of angular momentum.

2.6 Thermodynamics of dipolar viscous fluids – Second approach

In this section, we provide an alternative approach developed by K. R. Rajagopal and his coworkers [RS04] for deriving the constitutive relations of dipolar viscous fluids. Instead of assuming a priori the structure for the stresses $\mathcal{T}^{(0)}$, $\mathcal{T}^{(1)}$, and the heat flux \mathbf{q} , we will start assuming forms of two scalars: the Helmholtz free energy ξ and the rate of entropy production h , and we determine the forms for $\mathcal{T}^{(0)}$, $\mathcal{T}^{(1)}$ and \mathbf{q} by maximizing h (defined in (2.163) and (2.165) below) considered as a non-negative function of gradients of the state variables keeping track of all relevant constraints. We will first illustrate the efficiency of the whole procedure on the classical compressible (Navier–Stokes–Fourier) fluid, and then we apply this methodology to dipolar viscous fluid.

We start, analogously as in Section 2.5, with the system of balance equations (2.115)–(2.118) or rather with their “equivalent” forms (2.115), (2.116), (2.140) and (2.155), and we introduce the rate of entropy production ζ through

$$\rho\zeta = \rho\dot{\eta} - \nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{q}}{\theta} \right). \quad (2.158)$$

Then by (2.119), the second law of thermodynamics is automatically met if

$$\rho\zeta \geq 0. \quad (2.159)$$

Using (2.158) and recalling the definition of the Helmholtz free energy ξ in (2.141) we obtain the following system of governing equations

$$\dot{\rho} + \rho \nabla_{\mathbf{x}} \cdot \mathbf{u} = 0, \quad (2.160)$$

$$\rho \dot{\mathbf{u}} = \nabla_{\mathbf{x}} \cdot \mathcal{T}^{(0)}, \quad (2.161)$$

$$\rho \dot{\nu} = \mathcal{T}_{ij}^{(0)} \frac{\partial u_i}{\partial x_j} + \frac{\partial \mathcal{T}_{ijk}^{(1)}}{\partial x_k} \frac{\partial u_i}{\partial x_j} + \mathcal{T}_{ijk}^{(1)} \frac{\partial^2 u_i}{\partial x_j \partial x_k} + \frac{\partial q_i}{\partial x_i}, \quad (2.162)$$

$$\rho \theta \zeta = \mathcal{T}_{ij}^{(0)} \frac{\partial u_i}{\partial x_j} + \frac{\partial \mathcal{T}_{ijk}^{(1)}}{\partial x_k} \frac{\partial u_i}{\partial x_j} + \mathcal{T}_{ijk}^{(1)} \frac{\partial^2 u_i}{\partial x_j \partial x_k} + \frac{q_i}{\theta} \frac{\partial \theta}{\partial x_i} - \rho \dot{\xi} - \rho \eta \dot{\theta}. \quad (2.163)$$

Note that the equation (2.163) together with the inequality (2.159) forms a stronger variant of the Clausius–Duhem inequality, as stated in (2.119).

We assume that the second order tensor

$$\mathcal{T}^{(0)} + \nabla_{\mathbf{x}} \cdot \mathcal{T}^{(1)} = \left[\mathcal{T}_{ij}^{(0)} + \frac{\partial \mathcal{T}_{ijk}^{(1)}}{\partial x_k} \right]_{i,j=1}^3 \quad (2.164)$$

is symmetric, hence the balance equation of angular momentum (2.155) holds.

The thermodynamic framework developed by K. R. Rajagopal and his coworkers (see a nice survey article [RS04] where the references to earlier works and to articles in which the framework is applied to modelling responses pertinent to visco-elasticity, solid to solid transformation, twinning crystallization in polymers, single crystal super alloys etc., can be found) starts with the constitutive assumptions for the rate of entropy production

$$h = \rho \theta \zeta, \quad (2.165)$$

and the Helmholtz free energy ξ and requires that (2.163) (or its consequences (2.171) resp. (2.180) specified bellow) holds in all admissible processes. The forms for $\mathcal{T}^{(0)}$, $\mathcal{T}^{(1)}$

and \mathbf{q} are then determined by requiring that the rate of entropy production is maximal for the selected forms. We might refer to Málek, Rajagopal [MR05, p. 390–391] or [RS04] for a discussion concerning this assumption.

In what follows, we firstly apply this approach to classical (monopolar) Navier–Stokes–Fourier fluids, and then we use it to develop the constitutive relations for dipolar viscous fluids.

Case I: Navier–Stokes–Fourier (monopolar) fluid

In order to explain the main ideas of the framework as simple as possible we first restrict ourselves to monopolar fluid, i.e.,

$$\mathcal{T}^{(1)} \equiv 0.$$

Next, we assume that

$$\xi = \tilde{\xi}(\varrho, \theta) \tag{2.166}$$

and

$$\begin{aligned} h &= \tilde{h}(\mathcal{D}^\delta(\mathbf{u}), \nabla_{\mathbf{x}} \cdot \mathbf{u}, \nabla_{\mathbf{x}} \theta) = \tilde{\mu} |\mathcal{D}^\delta(\mathbf{u})|^2 + \tilde{\lambda} |\nabla_{\mathbf{x}} \cdot \mathbf{u}|^2 + \tilde{\kappa} |\nabla_{\mathbf{x}} \theta|^2, \\ \tilde{\mu} = \tilde{\mu}(\varrho, \theta) &\geq 0, \quad \tilde{\lambda} = \tilde{\lambda}(\varrho, \theta) \geq 0, \quad \tilde{\kappa} = \tilde{\kappa}(\varrho, \theta) \geq 0. \end{aligned} \tag{2.167}$$

Here we use the index δ to denote the deviatoric part of (3x3) tensors $A^\delta := A - \frac{1}{3}(\text{Tr } A)\mathcal{I}$, and we recall that \mathcal{D} is the symmetric velocity gradient $\mathcal{D}(\mathbf{u}) = \frac{1}{2}(\nabla_{\mathbf{x}} \mathbf{u} + \nabla_{\mathbf{x}}^T \mathbf{u})$.

With such a choice of h the second law of thermodynamics (in the form (2.159)) is automatically fulfilled.

Inserting (2.166) into (2.163), we conclude with help of (2.160) and (2.164) that

$$h = \mathcal{T}^{(0)} : \mathcal{D}(\mathbf{u}) + \mathbf{q} \cdot \frac{\nabla_{\mathbf{x}} \theta}{\theta} + \varrho^2 \frac{\partial \tilde{\xi}(\varrho, \theta)}{\partial \varrho} \nabla_{\mathbf{x}} \cdot \mathbf{u} - \varrho \left(\eta + \frac{\partial \tilde{\xi}(\varrho, \theta)}{\partial \theta} \right) \dot{\theta}. \tag{2.168}$$

Setting

$$\eta = -\frac{\partial \tilde{\xi}(\varrho, \theta)}{\partial \theta}, \tag{2.169}$$

and denoting

$$p = p(\varrho, \theta) := \varrho^2 \frac{\partial \tilde{\xi}(\varrho, \theta)}{\partial \varrho}, \tag{2.170}$$

the identity (2.168) simplifies to

$$h = [\mathcal{T}^{(0)}]^\delta : \mathcal{D}^\delta(\mathbf{u}) + \mathbf{q} \cdot \frac{\nabla_{\mathbf{x}} \theta}{\theta} + \left(\frac{1}{3} \text{Tr } \mathcal{T}^{(0)} + p \right) \nabla_{\mathbf{x}} \cdot \mathbf{u}. \tag{2.171}$$

Next, the constitutive equations for $\mathcal{T}^{(0)}$ and \mathbf{q} are determined by maximizing \tilde{h} with respect to $\mathcal{D}^\delta(\mathbf{u})$, $\nabla_{\mathbf{x}} \cdot \mathbf{u}$ and $\nabla_{\mathbf{x}} \theta$ provided that (2.171) holds as the constraint.

Setting

$$\begin{aligned} \Phi(\mathcal{D}^\delta(\mathbf{u}), \nabla_{\mathbf{x}} \cdot \mathbf{u}, \nabla_{\mathbf{x}} \theta) &= \tilde{h}(\mathcal{D}^\delta(\mathbf{u}), \nabla_{\mathbf{x}} \cdot \mathbf{u}, \nabla_{\mathbf{x}} \theta) \\ &+ \ell \left[\tilde{h}(\mathcal{D}^\delta(\mathbf{u}), \nabla_{\mathbf{x}} \cdot \mathbf{u}, \nabla_{\mathbf{x}} \theta) - [\mathcal{T}^{(0)}]^\delta : \mathcal{D}^\delta(\mathbf{u}) - \mathbf{q} \cdot \frac{\nabla_{\mathbf{x}} \theta}{\theta} - \left(\frac{1}{3} \text{Tr } \mathcal{T}^{(0)} + p \right) \nabla_{\mathbf{x}} \cdot \mathbf{u} \right], \end{aligned}$$

this constrained maximization leads to the following set of necessary (and due to a convex character of \tilde{h} also sufficient) conditions for extrema:

$$\frac{1 + \ell}{\ell} \frac{\partial \tilde{h}}{\partial \mathcal{D}_{ij}^\delta(\mathbf{u})} = [\mathcal{T}^{(0)}]_{ij}^\delta, \quad i, j = 1, 2, 3, \quad (2.172)$$

$$\frac{1 + \ell}{\ell} \frac{\partial \tilde{h}}{\partial \nabla_{\mathbf{x}} \cdot \mathbf{u}} = \frac{1}{3} \text{Tr} \mathcal{T}^{(0)} + p, \quad (2.173)$$

$$\frac{1 + \ell}{\ell} \frac{\partial \tilde{h}}{\partial \frac{\partial \theta}{\partial x_i}} = \frac{q_i}{\theta}, \quad i = 1, 2, 3. \quad (2.174)$$

In order to evaluate the Lagrange multiplier ℓ , we multiply each equation in (2.172) by $\mathcal{D}_{ij}^\delta(\mathbf{u})$, in (2.174) by $\frac{\partial \theta}{\partial x_i}$, and (2.173) by $\nabla_{\mathbf{x}} \cdot \mathbf{u}$, take a sum of them and compare the result with (2.171). We conclude that

$$\frac{1 + \ell}{\ell} = \frac{\tilde{h}}{\frac{\partial \tilde{h}}{\partial \mathcal{D}_{ij}^\delta(\mathbf{u})} \mathcal{D}_{ij}^\delta(\mathbf{u}) + \frac{\partial \tilde{h}}{\partial \nabla_{\mathbf{x}} \cdot \mathbf{u}} \nabla_{\mathbf{x}} \cdot \mathbf{u} + \frac{\partial \tilde{h}}{\partial \frac{\partial \theta}{\partial x_i}} \frac{\partial \theta}{\partial x_i}} = \frac{1}{2}, \quad (2.175)$$

where the last equality is a consequence of directly calculated derivatives of \tilde{h} of the form (2.167).

Using these direct calculations together with (2.175), we obtain that

$$\mathcal{T}^{(0)} = [\mathcal{T}^{(0)}]^\delta + \frac{1}{3} \text{Tr} \mathcal{T}^{(0)} \mathcal{I} = -p(\varrho, \theta) \mathcal{I} + \tilde{\lambda}(\varrho, \theta) \nabla_{\mathbf{x}} \cdot \mathbf{u} \mathcal{I} + \tilde{\mu}(\varrho, \theta) \mathcal{D}^\delta(\mathbf{u}), \quad (2.176)$$

and

$$\mathbf{q} = \theta \tilde{\kappa}(\varrho, \theta) \nabla_{\mathbf{x}} \theta, \quad (2.177)$$

which are the constitutive equations for Newtonian compressible fluids, where the heat flux is given by the Fourier law.

Case II: Dipolar viscous fluid

From the point of view discussed in Case I, it seems natural to call a fluid *dipolar viscous fluid* if the Helmholtz free energy (taking into account non-dissipative (elastic) processes that occur inside the fluid) remains as in (2.166), and the rate of entropy production depends both on the quantities stated in (2.167) and their first derivatives. Because of the identity

$$\frac{\partial^2 u_i}{\partial x_j \partial x_k} = \frac{\partial \mathcal{D}_{ij}(\mathbf{u})}{\partial x_k} + \frac{\partial \mathcal{D}_{ik}(\mathbf{u})}{\partial x_j} + \frac{\partial \mathcal{D}_{jk}(\mathbf{u})}{\partial x_i},$$

we for simplicity consider h in the form

$$\begin{aligned} h &= \tilde{h}(\mathcal{D}^\delta(\mathbf{u}), \nabla_{\mathbf{x}} \cdot \mathbf{u}, \nabla_{\mathbf{x}} \theta, \nabla_{\mathbf{x}}^2 \mathbf{u}, \nabla_{\mathbf{x}}^2 \theta) \\ &= \tilde{\mu} |\mathcal{D}^\delta(\mathbf{u})|^2 + \tilde{\lambda} |\nabla_{\mathbf{x}} \cdot \mathbf{u}|^2 + \tilde{\kappa} |\nabla_{\mathbf{x}} \theta|^2 + \tilde{\mu}_1 |\nabla_{\mathbf{x}}^2 \mathbf{u}|^2 + \tilde{\kappa}_1 |\nabla_{\mathbf{x}}^2 \theta|^2. \end{aligned} \quad (2.178)$$

Inserting ξ of the form (2.166) into (2.163), using again (2.160), (2.169), (2.170), and also (2.164), we observe that (2.166) simplifies to

$$h = \left(\mathcal{T}_{ij}^{(0)} + \frac{\partial \mathcal{T}_{ijk}^{(1)}}{\partial x_k} \right) \mathcal{D}_{ij}(\mathbf{u}) + \mathcal{T}_{ijk}^{(1)} \frac{\partial^2 u_i}{\partial x_j \partial x_k} + \frac{q_i}{\theta} \frac{\partial \theta}{\partial x_i} + p \nabla_{\mathbf{x}} \cdot \mathbf{u}. \quad (2.179)$$

Denoting just for simplicity, $\mathcal{S}_{ij} := \mathcal{T}_{ij}^{(0)} + \frac{\partial \mathcal{T}_{ijk}^{(1)}}{\partial x_k}$, and decomposing \mathcal{S} and $\mathcal{D}(\mathbf{u})$ into their deviatoric parts and the rest, we can rewrite (2.179) as

$$h = \mathcal{S}_{ij}^\delta \mathcal{D}_{ij}^\delta + \mathcal{T}_{ijk}^{(1)} \frac{\partial^2 u_i}{\partial x_j \partial x_k} + \frac{q_i}{\theta} \frac{\partial \theta}{\partial x_i} + \left(\frac{1}{3} \text{Tr } \mathcal{S} + p\right) \nabla_{\mathbf{x}} \cdot \mathbf{u}. \quad (2.180)$$

Proceeding as in Case I, i.e., maximizing \tilde{h} of the form (2.178) with respect to $\mathcal{D}^\delta(\mathbf{u})$, $\nabla_{\mathbf{x}} \cdot \mathbf{u}$, $\nabla_{\mathbf{x}} \theta$, $\nabla_{\mathbf{x}}^2 \mathbf{u}$, and $\nabla_{\mathbf{x}}^2 \theta$ over all possible values fulfilling (2.180) considered as the constraint, we arrive at the following set of conditions

$$\frac{1+\ell}{\ell} \frac{\partial \tilde{h}}{\partial \mathcal{D}_{ij}^\delta(\mathbf{u})} = \mathcal{S}_{ij}^\delta = [\mathcal{T}^{(0)}]_{ij}^\delta + \left[\frac{\partial \mathcal{T}_{ijk}^{(1)}}{\partial x_k} \right]^\delta, \quad i, j = 1, 2, 3, \quad (2.181)$$

$$\frac{1+\ell}{\ell} \frac{\partial \tilde{h}}{\partial \nabla_{\mathbf{x}} \cdot \mathbf{u}} = \frac{1}{3} \text{Tr } \mathcal{S} + p = \frac{1}{3} [\text{Tr } \mathcal{T}^{(0)} + \text{Tr}(\nabla_{\mathbf{x}} \cdot \mathcal{T}^{(1)})] + p, \quad (2.182)$$

$$\frac{1+\ell}{\ell} \frac{\partial \tilde{h}}{\partial \frac{\partial \theta}{\partial x_i}} = \frac{q_i}{\theta}, \quad i = 1, 2, 3, \quad (2.183)$$

$$\frac{1+\ell}{\ell} \frac{\partial \tilde{h}}{\partial \frac{\partial^2 u_i}{\partial x_j \partial x_k}} = \mathcal{T}_{ijk}^{(1)}, \quad i, j, k = 1, 2, 3, \quad (2.184)$$

$$\frac{1+\ell}{\ell} \frac{\partial \tilde{h}}{\partial \frac{\partial^2 \theta}{\partial x_i \partial x_j}} = 0, \quad i, j = 1, 2, 3, \quad (2.185)$$

It clearly implies that \tilde{h} should be independent of $\nabla_{\mathbf{x}}^2 \theta$, i.e., $\tilde{\kappa}_1 = 0$, and again $\frac{1+\ell}{\ell}$ given by the equation similar to (2.175) satisfies

$$\frac{1+\ell}{\ell} = \frac{1}{2}. \quad (2.186)$$

Using this and computing explicitly the derivatives of \tilde{h} we come to the following forms of the constitutive equations:

$$\mathcal{T}_{ijk}^{(1)} = \tilde{\mu}_1(\varrho, \theta) \frac{\partial^2 u_i}{\partial x_j \partial x_k}, \quad i, j, k = 1, 2, 3, \quad (2.187)$$

$$\frac{\partial \mathcal{T}_{ijk}^{(1)}}{\partial x_k} = \frac{\partial}{\partial x_k} \left(\tilde{\mu}_1(\varrho, \theta) \frac{\partial^2 u_i}{\partial x_j \partial x_k} \right), \quad i, j = 1, 2, 3, \quad (2.188)$$

$$q_i = \theta \kappa(\varrho, \theta) \frac{\partial \theta}{\partial x_i}, \quad i = 1, 2, 3, \quad (2.189)$$

$$\begin{aligned} \mathcal{T}_{ij}^{(0)} &= -p(\varrho, \theta) \delta_{ij} + \tilde{\lambda}(\varrho, \theta) \nabla_{\mathbf{x}} \cdot \mathbf{u} + \tilde{\mu}(\varrho, \theta) \mathcal{D}_{ij}^\delta(\mathbf{u}) \\ &\quad - \frac{\partial}{\partial x_k} \left(\tilde{\mu}_1(\varrho, \theta) \frac{\partial^2 u_i}{\partial x_j \partial x_k} \right), \quad i = 1, 2, 3. \end{aligned} \quad (2.190)$$

The set of equations (2.187), (2.189) and (2.190) forms the set of constitutive relations we were looking for.

2.7 Conclusion

Comparison of results

In this chapter we have demonstrated the three different approaches to the theory of multipolar viscous fluids, i.e., fluids where the higher order gradients of the density, the

velocity and the temperature are involved in the constitutive quantities.

The first, kinetic approach includes some higher order details of binary collisions in the kinetic equation (2.47) and the fluid dynamic limit of this equation leads to the balance equations of mass (2.83), momentum (2.84) and energy (2.85) for bipolar viscous fluids. Also the constitutive relations (2.88)–(2.90) for linear bipolar viscous fluids are obtained.

The second, classical approach of the continuum thermodynamics assumes very general constitutive relations (2.127) and their further simplifications are based on the second law of thermodynamics (2.143) in combination with the principle of material frame indifference and the balance equations. Finally, we obtain the general constitutive relations for linear bipolar viscous fluids (2.152)–(2.154).

The third one, takes into account the higher order gradients of the velocity and the temperature as the new origins of entropy production (2.178). The constitutive relations (2.187), (2.189) and (2.190) for bipolar viscous fluids are obtained by the maximization of the rate of entropy production (2.178) influenced by the constraint (2.180).

The main difference between the second and third approach is in the complexity of obtained constitutive relations. The second one gives us very general relations where a lot of coefficients plays the role and it is difficult to identify the influence of each coefficient individually. The third one introduces only the necessary set of additional coefficients in the constitutive relations, hence their influences and meanings are easier to understand.

Let us compare the constitutive relations (2.88)–(2.90) obtained from the kinetic model with the general constitutive relations (2.152)–(2.154) of the linear dipolar viscous fluid of type (2, 1, 3). We find out that

$$\begin{aligned}
\bar{\lambda} &= -\frac{2}{3}\varepsilon\mu_\alpha, \\
\bar{\mu} &= \varepsilon\mu_\alpha, \\
\bar{\alpha} &= -\frac{2}{3}\varepsilon^3\mu_\beta, \\
\bar{\beta}_1 &= \bar{\beta}_2 = \varepsilon^3\mu_\beta, \\
\bar{\gamma} &= 0, \\
\bar{c}_1 &= \bar{c}_3 = \bar{c}_5 = \varepsilon^3\mu_\gamma, \\
\bar{c}_4 &= \bar{c}_7 = \bar{c}_8 = 2\varepsilon^3\mu_\gamma, \\
\bar{c}_2 &= \bar{c}_6 = \bar{c}_9 = 0, \\
\bar{k} &= \varepsilon\kappa_\alpha, \\
\bar{d}_1 &= 0, \\
\bar{d}_2 &= -\varepsilon^2\kappa_\gamma,
\end{aligned}$$

where μ_α , μ_β , μ_γ , κ_α , κ_β and κ_γ are positive coefficients given by relations (2.91)–(2.92).

The additional term $\varepsilon^2\kappa_\beta\nabla\Delta_{\mathbf{x}}\theta$ in the constitutive relation (2.90) of the heat flux \mathbf{q} depends linearly on the third gradient of temperature. It is in accordance with the principle of material frame-indifference (see the paragraph above the formula (2.139)), but there is not any corresponding term in the theory of J. Nečas and M. Šilhavý [NS91] as they considered only the first gradient of temperature.

The term $-\varepsilon\kappa_\gamma\Delta_{\mathbf{x}}\mathbf{u}$ in the constitutive relation for the heat flux \mathbf{q} has an exclusive character because it is the only term of the order ε in all relations, the coefficient of which depends on the density square and it is directly implied by the solvability conditions for g .

We recall that in the scope of this work there are the three fundamental ways of the transport of mass, momentum and energy between two regions, i.e., across an imaginary

boundary. The first way is the free motion of particles, and the particles carry the mass, the momentum and the energy itself. This process plays the most important role in the equilibrium, and leads to the system of compressible Euler equations. The second way are the collisions of point particles occurring on the imaginary surface. This process is also presented in the equilibrium, but the rate of change of mass, momentum and energy between two regions is non-zero in the non-equilibrium state. From the macroscopic point of view it originates the stress tensor and the heat flux, and it leads to the Navier–Stokes model of fluids. The third way is the collisional transfer of momentum and energy across the distance between two interacting particles. The first approximation of such a phenomena was studied by Enskog who showed that it quantitatively specifies the constitutive relations of the pressure, the stress tensor and the heat flux.

In this paper we showed that the higher order approximations of the collisional transfer call for introducing of the multipolar stress tensors and this leads to the model of the multipolar fluids. This can be immediately observed from the source term (2.48) originating at the term E_1 ; see (2.16). In fact, we considered only the first two non-zero terms in the Taylor series (2.40), as the structure of that expansion and Appendix A.1 predict that all remaining terms are of the form

$$\varepsilon^{2r} \nabla_{\mathbf{x}}^r \cdot \{ \gamma^{(r)}(\theta) \varrho M(\mathbf{V}) \mathbf{V} \}, \quad r = 2, 3, \dots$$

Such a form is very close to the Cauchy stress tensor for the linear multipolar fluid tensor $\mathcal{T}^{(0,V)}$ (see [NS91, Equation 5.15]):

$$\begin{aligned} \mathcal{T}_{ij}^{(0,V)} = & \bar{\lambda} \frac{\partial u_k}{\partial x_k} \delta_{ij} + \bar{\mu} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \\ & + \sum_{r=0} \left(\bar{\alpha}^{(r)} \Delta_{\mathbf{x}}^{r+1} \frac{\partial u_k}{\partial x_k} \delta_{ij} + \bar{\beta}_1^{(r)} \Delta_{\mathbf{x}}^{r+1} \frac{\partial u_i}{\partial x_j} + \bar{\beta}_2^{(r)} \Delta_{\mathbf{x}}^{r+1} \frac{\partial u_j}{\partial x_i} + \bar{\gamma}^{(r)} \Delta_{\mathbf{x}}^r \frac{\partial^3 u_k}{\partial x_i \partial x_j \partial x_k} \right). \end{aligned}$$

Notices on the fluid dynamic limit

The important step in the derivation of the model of dipolar viscous fluids from the kinetic equation (2.47) is the modified Chapman–Enskog method which overlooks the dependence of the source term $s(M)$ on the Knudsen number ε and keeps the higher order terms in the equation (2.66) for g – the first order approximate solution. We do not have very strong support for that, but we can make the observation to motivate it. The compressible Navier–Stokes equations

$$\begin{aligned} \partial_t \varrho + \nabla_{\mathbf{x}} \cdot (\varrho \mathbf{u}) &= 0, \\ \varrho (\partial_t + \mathbf{u} \cdot \nabla_{\mathbf{x}}) \mathbf{u} + \nabla_{\mathbf{x}} (\varrho \theta) &= \varepsilon \nabla_{\mathbf{x}} \cdot [\mu \sigma(\mathbf{u})], \\ \frac{3}{2} \varrho (\partial_t + \mathbf{u} \cdot \nabla_{\mathbf{x}}) \theta + \varrho \theta \nabla_{\mathbf{x}} \cdot \mathbf{u} &= \varepsilon \frac{1}{2} \mu \sigma(\mathbf{u}) : \sigma(\mathbf{u}) + \varepsilon \nabla_{\mathbf{x}} \cdot [\kappa \nabla_{\mathbf{x}} \theta], \end{aligned}$$

are invariant due to the transformation of the variables t and \mathbf{x} given by

$$t' = \varepsilon t, \quad \text{and} \quad \mathbf{x}' = \varepsilon \mathbf{x}.$$

The equations of the multipolar fluids (2.83)–(2.85) are a generalization of the compressible Navier–Stokes equations and they are invariant due to the same transformation. As the equations of the multipolar fluids are more close to the compressible Navier–Stokes equations than the Burnett or Superburnett equations we have not any other choice and we have to keep the whole source term $s(M)$ in the equation for g .

The moot point of the derivation is the quantity $K = \beta(\theta)\varrho^2\theta$; see (2.73). Because of the solvability condition (2.72), this quantity should contribute to the pressure \widehat{p} , i.e., the state equation of a gas should be

$$\widehat{p} = \varrho\theta(1 + \alpha(\theta)\varrho) + \varepsilon^2\Delta_{\mathbf{x}}[\beta(\theta)\varrho^2\theta].$$

But such state equation is not suitable for our purposes as it is out of the common frame of the thermodynamics (see below).

The fundamental element of the thermodynamics of fluids (fluids without chemical reactions) is the Helmholtz free energy ξ given by relation (2.141).

From the point of view of the statistical physics (see [Ba91], [Kv98]), the fluid is a canonical ensemble, and the absolute temperature θ and the density ϱ are its natural parameterization. The Helmholtz free energy $\xi = \xi(\varrho, \theta)$ is a thermodynamic potential, tightly connected with the canonical ensemble, which has a minimum at an equilibrium and whose partial derivatives with respect to the parameters ϱ, θ have simple interpretations. The partial derivative with respect to the density gives us the pressure (2.149) and the partial derivative with respect to the temperature gives us the entropy (2.144).

In the paper [NS91, Theorem 4.2] the authors J. Nečas and M. Šilhavý proved that the most general constitutive relation for the Helmholtz free energy of multipolar viscous fluids depends only on the density, the gradients of density and the temperature, and is independent of the gradients of velocity and of the gradient of temperature. (We have demonstrated their result by relations (2.144) in the case of linear multipolar viscous fluids and these relations determine $\xi = \xi(\varrho, \theta)$ again; see (2.145).)

The pressure \widehat{p} has to be independent of the gradients of temperature and gradients of density because of relation (2.149), and this hints us to make the assumption about the spatial derivatives of K . The direct benefit of this assumption is the linearization of the multipolar stress tensors and of the heat flux.

We would like to notice that the equation (2.86) for pressure p is in the form of the virial series [Ba91, Part 9.2.4] because of the term with the square of the density. It demonstrates the importance of the collisional transfer again, and it is in the accordance with the assumption of binary collisions because of the correspondence of powers of the density and the number of particles which are involved in collisions.

Drawbacks

Recently, there is a drawback in the kinetic approach because it leads to the constitutive relations where some coefficients have bad signs. It is mainly the coefficient μ_β which sign is problematic from the point of view of the second law of thermodynamics. This can be observed comparing the coefficient μ_β with the coefficient $\tilde{\mu}_1$ from the third approach, see relations (2.88) and (2.190). Also the mathematical theory of multipolar viscous fluids is based on the opposite sign of μ_β .

The approximations (2.107) and (2.108) in the derivation should be discussed, too. We believe that this approximation is the reason why the coefficients of the linear multipolar stress tensors do not satisfy relation (2.157) implied by the law of balance of angular momentum. In our approach, the terms \bar{c}_2, \bar{c}_6 and \bar{c}_9 of the constitutive relation (2.153) of $\mathcal{T}^{(1, \mathbf{V})}$ are missing. We can observe that these terms have such a combinations of indices which cannot be obtained by partial derivation of the term $|\mathbf{V}|^2\mathbf{V}$, but the higher order moments of velocity \mathbf{V} , for example $\mathbf{V} \otimes \mathbf{V} \otimes \mathbf{V}$, have to be included. In addition, we have

no idea how to obtain the local law of balance of angular momentum in the form (2.118) where both multipolar stress tensors $\mathcal{T}^{(0)}$ and $\mathcal{T}^{(1)}$ are presented.

The evaluation of the coefficient $\beta(\theta) = \frac{1}{60}\pi\sigma^5$ for the hard sphere potential implies that the coefficients in the constitutive relations of linear dipolar viscous fluids are really small as they are proportional to $\varepsilon^3\sigma^5$ where $\varepsilon = Ma/Re$ equals to the ratio of the Mach number Ma and the Reynolds number Re (von Karman's relation) and σ is the diameter of particles.

We believe that this is a serious reason why the influence of the multipolar terms is so hard to observe and measure.

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Chapter 3

Acoustic Regime of the Generalized Boltzmann Equation

The generalized Boltzmann equation takes into account the changes of the probability density on the scale of the collision time order. We study the fluctuations of such system near the spatially homogeneous state and we establish the fluid dynamic limit of these fluctuations to the system of the generalized acoustic equations. Also some analytical aspects of the linearized problem are presented.

3.1 Introduction

Let us introduce a generalization of the Boltzmann equation proposed by B. V. Alexeev [Al94]. His idea can be briefly described from the following heuristic point of view.

Let $f(\mathbf{x}, \mathbf{v}, t)$ be the probability density of one particle in the phase space – the position \mathbf{x} , the velocity \mathbf{v} compose its coordinates. We assume that there are not external forces, so the element of the phase space $d\mathbf{x}d\mathbf{v}$ stays constant. The changes of the probability density f along a streamline are caused by the collisions of particles, thus the evolution of f is driven by the equation

$$f(\mathbf{x} + \mathbf{v} dt, \mathbf{v}, t + dt) - f(\mathbf{x}, \mathbf{v}, t) = \mathcal{Q}(f, f)(\mathbf{x}, \mathbf{v}, t) dt \quad (3.1)$$

where \mathcal{Q} is the *Boltzmann collision operator* (e.g., [Ce88, Chapter II])

$$\mathcal{Q}(f, f) = \iint_{R^3 \times S_+^2} [f(\mathbf{v}')f(\mathbf{v}'_*) - f(\mathbf{v})f(\mathbf{v}_*)] b(|\mathbf{v} - \mathbf{v}_*|, \boldsymbol{\eta}) d\boldsymbol{\eta} d\mathbf{v}_*. \quad (3.2)$$

The vector $\boldsymbol{\eta}$ is a outer unit normal vector to the half sphere $S_+^2 = \{\boldsymbol{\eta} \in R^3 : |\boldsymbol{\eta}| = 1 \text{ and } (\mathbf{v} - \mathbf{v}_*) \cdot \boldsymbol{\eta} > 0\}$. The vectors \mathbf{v}' and \mathbf{v}'_* are the pre-collision velocities related to the post-collision ones through the relations

$$\mathbf{v}' = \mathbf{v} - \boldsymbol{\eta}[(\mathbf{v} - \mathbf{v}_*) \cdot \boldsymbol{\eta}], \quad (3.3)$$

$$\mathbf{v}'_* = \mathbf{v}_* + \boldsymbol{\eta}[(\mathbf{v} - \mathbf{v}_*) \cdot \boldsymbol{\eta}]. \quad (3.4)$$

We will even use the same notation for functions $f_* = f(\mathbf{x}, \mathbf{v}_*, t)$, $f'_* = f(\mathbf{x}, \mathbf{v}'_*, t)$, ...

The term $b(|\mathbf{v} - \mathbf{v}_*|, \boldsymbol{\eta})$ is the Boltzmann kernel, it is a non-negative measurable function, and it hides the details of particles interactions. Here we assume that the particles interact with a spherically symmetric potential, and the interactions have a short range (cutoff potential). The additional assumptions on b will be imposed later.

The Boltzmann collision operator only has regard to the binary collisions. Two particles entering collision must be in the spatial volume $b(|\mathbf{v} - \mathbf{v}_*|, \boldsymbol{\eta})d\boldsymbol{\eta}dt$ in order to hit each other during the time interval $(t, t + dt)$, and only the collisions where one of the pre-collision or post-collision velocities is \mathbf{v} are taken into account. The binary collisions $(\mathbf{v}', \mathbf{v}'_*) \rightarrow (\mathbf{v}, \mathbf{v}_*)$ increase f , and the binary collisions $(\mathbf{v}, \mathbf{v}_*) \rightarrow (\mathbf{v}', \mathbf{v}'_*)$ decrease it. We integrate the probability of such collisions over all possible values of the velocity \mathbf{v}_* and the unit vector $\boldsymbol{\eta}$, i.e., $(\mathbf{v} - \mathbf{v}_*) \cdot \boldsymbol{\eta} > 0$ holds, and we obtain the right hand side of equation (3.1).

We expand the left hand side of (3.1) into the Taylor series. The terms up to the first order give us the Boltzmann equation

$$\partial_t f + (\mathbf{v} \cdot \nabla_{\mathbf{x}})f = \mathcal{Q}(f, f), \quad (3.5)$$

and the terms up to the second order give us the *generalized Boltzmann equation*

$$\partial_t f + (\mathbf{v} \cdot \nabla_{\mathbf{x}})f \pm \tau \{ \partial_t^2 f + 2 \partial_t (\mathbf{v} \cdot \nabla_{\mathbf{x}})f + (\mathbf{v} \otimes \mathbf{v}) : \nabla_{\mathbf{x}}^2 f \} = \mathcal{Q}(f, f). \quad (3.6)$$

Here $\tau = dt/2 > 0$ and the operator \otimes denotes a tensor product.

The heuristic arguments mentioned above give us the positive sign in front of the additional terms in the generalized Boltzmann equation, but according to Alexeev there should be the negative one. We discuss this question in more details, in the next section, and we believe that the right sign is the plus. In any case, the generalized Boltzmann equation is a perturbation of the classical Boltzmann equation, and the different signs give us the different characters of these perturbations. The plus sign is a hyperbolic perturbation which keeps the evolutionary character of our problem. The minus sign is an elliptic perturbation which changes the evolutionary character to a boundary value problem.

Alexeev described more precise derivation of the generalized Boltzmann equation in his paper [Al94]. He had started from the BBGKY hierarchy and he had obtained the same result by the appropriate scaling of that hierarchy on the level of the collision time order. For further details, we refer the reader to his paper.

Let us list some important items of the theory of the Boltzmann equation (see [Ce88], [CIP94]).

We define the *transport operator* Λ

$$\Lambda \equiv \partial_t + \mathbf{v} \cdot \nabla_{\mathbf{x}}, \quad (3.7)$$

then the generalized Boltzmann equation (3.6) has the compact form

$$\Lambda (f \pm \tau \Lambda f) = \mathcal{Q}(f, f). \quad (3.8)$$

The equilibrium distribution is the *Maxwellian*

$$M = \frac{\varrho}{(2\pi\theta)^{3/2}} \exp\left(-\frac{|\mathbf{v} - \mathbf{u}|^2}{2\theta}\right), \quad (3.9)$$

where ϱ , \mathbf{u} and θ are the density, the bulk velocity and the temperature of the system.

The Boltzmann collision operator has the *equilibrium* property

$$\mathcal{Q}(f, f) = 0 \quad \text{if and only if} \quad f = M \quad (3.10)$$

and the property of the *collision invariants*

$$\int_{R^3} \psi(\mathbf{v}) \mathcal{Q}(f, f) d\mathbf{v} = 0 \quad \text{if and only if} \quad \psi(\mathbf{v}) \in \text{span}(1, v_1, v_2, v_3, |\mathbf{v}|^2). \quad (3.11)$$

In the next section, the right sign of the additional terms in the generalized Boltzmann equation is discussed, and it is followed by the section where the formal fluid dynamic limit from the generalized Boltzmann equation to the system of the generalized acoustic equations is described. Furthermore, the linearized problem is studied, and the formal results are specified in the next two sections where both signs are treated separately. The paper is concluded with the comparison of our results with the results obtained for the Boltzmann equation.

3.2 Sign of the additional term

According to Alexeev [Al94] there can be a minus sign in front of the additional terms because of the possibility to take the backward difference instead of the forward one in equation (3.1)

$$f(\mathbf{x}, \mathbf{v}, t) - f(\mathbf{x} - \mathbf{v} dt, \mathbf{v}, t - dt) = \mathcal{Q}(f, f)(\mathbf{x}, \mathbf{v}, t) dt. \quad (3.12)$$

His further arguments are based on the properties of the entropy given below.

As usual in the kinetic theory of gases, the *entropy* of a spatially homogeneous gas that occupies a bounded container Ω is given by the relation

$$\mathcal{H} = \iint_{\Omega \times R^3} f \ln f d\mathbf{v} d\mathbf{x}. \quad (3.13)$$

Recall that the multiplication of the classical Boltzmann equation

$$\Lambda f = \mathcal{Q}(f, f) \quad (3.14)$$

by the term $1 + \ln f$, followed by the integration over the whole phase space, leads to the *\mathcal{H} -theorem*: the entropy \mathcal{H} of a system never increases with time and it is steady if and only if the distribution function f is the Maxwellian

$$\frac{d\mathcal{H}}{dt} = \iint_{\Omega \times R^3} \ln f \mathcal{Q}(f, f) d\mathbf{v} d\mathbf{x} \leq 0. \quad (3.15)$$

The \mathcal{H} -theorem implies that the entropy should be a convex function with the global minimum at the equilibrium, therefore we have

$$\frac{d^2\mathcal{H}}{dt^2} \geq 0. \quad (3.16)$$

This inequality is the *Prigogine principle* of the minimum entropy production.

We can use the same procedure and multiply the generalized Boltzmann equation by the term $1 + \ln f$. Upon integration over the whole phase space, we obtain the following relation

$$\frac{d\mathcal{H}}{dt} \pm \tau \left(\frac{d^2\mathcal{H}}{dt^2} - \iint_{\Omega \times R^3} \frac{(\partial_t f)^2}{f} d\mathbf{v} d\mathbf{x} \right) = \iint_{\Omega \times R^3} \ln f \mathcal{Q}(f, f) d\mathbf{v} d\mathbf{x}. \quad (3.17)$$

Alexeev pointed out that if we choose the minus sign in this relation, we obtain the inequality $0 \leq \tau d^2\mathcal{H}/dt^2 - d\mathcal{H}/dt$ which combines the \mathcal{H} -theorem with the Prigogine principle.

Let us look at equation (3.12) again. There is $\mathcal{Q}(f, f)(\mathbf{x}, \mathbf{v}, t) dt$, not $\mathcal{Q}(f, f)(\mathbf{x} - \mathbf{v}t, \mathbf{v}, t - dt) dt$, on the right hand side. It means that the collision operator is expressed in terms of the post-collision velocities instead of the pre-collision ones. Such approach is formally possible, but it is in the contradiction with the derivation of the Boltzmann equation where the principle of the molecular chaos is assumed.

From the mathematical point of view, the collision operator expressed in terms of the post-collision velocities is not the Boltzmann collision operator, but the minus Boltzmann collision operator. Hence both inequalities (3.15), (3.16) are reversed, and the argument of Alexeev does not hold.

The other point of view can be the formal limit $\tau \rightarrow 0+$ from the generalized Boltzmann equation to the Boltzmann equation. Let us look at the problem

$$\begin{aligned} (f_\tau + \tau \Lambda f_\tau)(\mathbf{x}, t) &= h(\mathbf{x}, t), \\ f_\tau(\mathbf{x}, t)|_{t=0} &= h_0(\mathbf{x}). \end{aligned}$$

We can write the formal solution of this problem

$$\begin{aligned} f_\tau(\mathbf{x}, t) &= h_0(\mathbf{x} - \mathbf{v}t) e^{-t/\tau} + \frac{1}{\tau} \int_0^t e^{-(t-s)/\tau} h(\mathbf{x} - \mathbf{v}(t-s), s) ds \\ &= h_0(\mathbf{x} - \mathbf{v}t) e^{-t/\tau} + h(\mathbf{x}, t) - h_0(\mathbf{x} - \mathbf{v}t) e^{-t/\tau} \\ &\quad - \int_0^t e^{-(t-s)/\tau} \frac{d}{ds} h(\mathbf{x} - \mathbf{v}(t-s), s) ds. \end{aligned}$$

Such family of solutions $\{f_\tau\}$ converges to h for any $t > 0$ if $\tau \rightarrow 0+$, but it is not true for the opposite sign.

The arguments mentioned above express our opinion that the right sign in the generalized Boltzmann equation (3.8) is the plus as this approach keeps the evolutionary character of the problem, and it is in the formal accordance with the heuristic derivation of the Boltzmann equation.

The opposite, i.e., minus sign is also possible, and it poses a well-formed problem (see Section 3.5). But we would rather interpret it as a possible perturbation than the right one.

3.3 Formal solution and the fluid dynamic limit

The dimensionless form of the generalized Boltzmann equation reads

$$\Lambda(f_\varepsilon \pm \tau \Lambda f_\varepsilon) = \frac{1}{\varepsilon} \mathcal{Q}(f_\varepsilon, f_\varepsilon), \quad (3.18)$$

where $\varepsilon > 0$ is a small parameter called the *Knudsen number*, and it is the ratio of the mean free path to the macroscopic length scale of the system. The dependency of the solution f_ε on the parameter τ will not be explicitly stated as the parameter $\tau > 0$ is fixed.

Several methods based on the expansion of the solution in a series of powers of ε were developed. The most famous methods are the Hilbert method and the Chapman–Enskog

method (see, e.g., [Ce88, Chapter IV]). The main difference between both methods is that the Hilbert method also expands the corresponding macroscopic quantities, like the density ϱ , the bulk velocity \mathbf{u} and the absolute temperature θ , into the series of powers of ε , while the Chapman–Enskog method does not.

The formal solution and the fluid dynamic limit of the generalized Boltzmann equation was investigated by Alexeev. He used the Chapman–Enskog method and developed the system of the generalized compressible Euler equations and also the system of the generalized compressible Navier–Stokes equations.

Here we study the acoustic limit for the generalized Boltzmann equation and the results described here are similar to ones obtained by C. Bardos, F. Golse and C. D. Levermore [BGL00] for the Boltzmann equation.

The acoustic limit investigates the fluctuations about a spatially homogeneous fluid state – the state near the *absolute Maxwellian*

$$m(\mathbf{v}) = \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{1}{2}|\mathbf{v}|^2\right). \quad (3.19)$$

If we express the macroscopic quantities, the density ϱ , the bulk velocity \mathbf{u} and the absolute temperature θ , in the form of fluctuations (the homogeneous parts of these quantities are the same as in the absolute Maxwellian)

$$\varrho = 1 + \varepsilon^s \varrho', \quad \mathbf{u} = \varepsilon^s \mathbf{u}', \quad \theta = 1 + \varepsilon^s \theta', \quad s > 0, \quad (3.20)$$

and we substitute these quantities into the compressible Euler equations, we derive the system of acoustic equations for the fluctuations ϱ' , \mathbf{u}' and θ' :

$$\begin{aligned} \partial_t \varrho' + \nabla_{\mathbf{x}} \cdot \mathbf{u}' &= 0, \\ \partial_t \mathbf{u}' + \nabla_{\mathbf{x}}(\varrho' + \theta') &= 0, \\ \frac{3}{2} \partial_t \theta' + \nabla_{\mathbf{x}} \cdot \mathbf{u}' &= 0. \end{aligned} \quad (3.21)$$

This system of equations has an adjective “acoustic” because from the first and the third equation we obtain the equation for the pressure $p' = \varrho' + \theta'$ which together with the second equation forms the system

$$\begin{aligned} \partial_t p' + \frac{5}{3} \nabla_{\mathbf{x}} \cdot \mathbf{u}' &= 0, \\ \partial_t \mathbf{u}' + \nabla_{\mathbf{x}} p' &= 0. \end{aligned}$$

Then we have the wave equation for the pressure p' (or similarly for the velocity \mathbf{u}')

$$\partial_t^2 p' - \frac{5}{3} \Delta_{\mathbf{x}} p' = 0, \quad (3.22)$$

where the speed of waves (the speed of sound) is $c = \sqrt{5/3}$.

We assume that there exists a family $\{f_\varepsilon\}$ of the non-negative solutions of the generalized Boltzmann equation (3.18) in the form

$$f_\varepsilon = m(1 + \varepsilon^s g_\varepsilon)$$

which implies that g_ε is the formal solution of the problem

$$\Lambda(g_\varepsilon \pm \tau \Lambda g_\varepsilon) = \frac{1}{\varepsilon} \mathcal{L} g_\varepsilon + \varepsilon^{s-1} \Gamma(g_\varepsilon, g_\varepsilon). \quad (3.23)$$

The nonlinear operator Γ is

$$\Gamma(g, h) = \int_{R^3 \times S_+^2} (g' h'_* + g'_* h' - g h_* - g_* h) b(|\mathbf{v} - \mathbf{v}_*|, \boldsymbol{\eta}) d\boldsymbol{\eta} m(\mathbf{v}_*) d\mathbf{v}_*,$$

and the linear operator \mathcal{L} is

$$\mathcal{L}g = 2 \int_{R^3 \times S_+^2} (g' + g'_* - g - g_*) b(|\mathbf{v} - \mathbf{v}_*|, \boldsymbol{\eta}) d\boldsymbol{\eta} m(\mathbf{v}_*) d\mathbf{v}_*.$$

We denote by $L_{\mathbf{v}}^2$ the Hilbert space of all square integrable complex-valued functions with respect to the Borel measure $m(\mathbf{v})d\mathbf{v}$ on R^3 and for the inner product we use the notation

$$\langle g, h \rangle = \int_{R^3} g(\mathbf{v}) \overline{h(\mathbf{v})} m(\mathbf{v}) d\mathbf{v}. \quad (3.24)$$

H. Grad proved [Gr63] that the linear operator \mathcal{L} is a closed self-adjoint and non-positive operator on the space $L_{\mathbf{v}}^2$ with the domain of definition

$$D(\mathcal{L}) = \{ g \in L_{\mathbf{v}}^2 \mid \nu g \in L_{\mathbf{v}}^2 \}$$

and that the operator can be decomposed into two parts

$$\mathcal{L} = \mathcal{K} - \nu I,$$

where ν is a multiplication operator called the *collision frequency*

$$\nu(\mathbf{v}) = \int_{R^3 \times S_+^2} b(|\mathbf{v} - \mathbf{v}_*|, \boldsymbol{\eta}) d\boldsymbol{\eta} m(\mathbf{v}_*) d\mathbf{v}_*,$$

and \mathcal{K} is a compact self-adjoint operator on $L_{\mathbf{v}}^2$. Such decomposition is valid if we impose the additional requirements on the Boltzmann kernel b (so-called the Grad's cutoff). Particularly, we assume the collision frequency $\nu(\mathbf{v})$ is a monotone function in $|\mathbf{v}|$ and

$$0 < \nu_0 \leq \nu(\mathbf{v}) \leq \nu_1(1 + |\mathbf{v}|^\alpha) \quad \text{where} \quad 0 \leq \alpha \leq 1. \quad (3.25)$$

(Case $\alpha = 0$ corresponds to the Maxwellian potential and case $\alpha = 1$ to the hard spheres; see [Ce88, Chapter IV].)

The linear operator \mathcal{L} has the kernel $\text{Ker}(\mathcal{L}) = \text{span}(1, v_1, v_2, v_3, |\mathbf{v}|^2)$, and the following formulas hold for each $g \in D(\mathcal{L})$:

$$\begin{aligned} \langle \mathcal{L}g, g \rangle &\leq 0 \quad \text{with equality iff} \quad g \in \text{Ker}(\mathcal{L}), \\ \langle \mathcal{L}g, g \rangle &\leq -\mu \|g\|_{L_{\mathbf{v}}^2}^2 \quad \text{whenever} \quad g \in \text{Ker}(\mathcal{L})^\perp \quad (\mu > 0). \end{aligned}$$

The Hilbert space L_t^2 and $L_{\mathbf{x}}^2$ is the set of all square integrable functions with respect to the Lebesgue measure dt on $(0, T)$ and $d\mathbf{x}$ on R^3 . The indices $t, \mathbf{x}, \mathbf{v}$ will be combined in the straightforward way.

We start with a formal theorem about the fluid dynamic limit of our system (3.18) in the acoustic regime.

Theorem 3.1. (Formal) Let $s > 0$ and $T > 0$. Let f_ε be a family of weak solutions to the generalized Boltzmann equation (3.18), whose fluctuations g_ε converge, as $\varepsilon \rightarrow 0$, in the sense of distributions to a function $g \in L^\infty(0, T; L^2_{\mathbf{x}\mathbf{v}})$. Furthermore, assume that the moments

$$\langle g_\varepsilon \rangle, \quad \langle \mathbf{v} g_\varepsilon \rangle, \quad \langle \mathbf{v} \otimes \mathbf{v} g_\varepsilon \rangle, \quad \langle \mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v} g_\varepsilon \rangle, \quad \langle |\mathbf{v}|^2 \mathbf{v} \otimes \mathbf{v} g_\varepsilon \rangle$$

satisfy the local conservations laws and converge in the sense of distributions as $\varepsilon \rightarrow 0$ to the corresponding moments

$$\langle g \rangle, \quad \langle \mathbf{v} g \rangle, \quad \langle \mathbf{v} \otimes \mathbf{v} g \rangle, \quad \langle \mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v} g \rangle, \quad \langle |\mathbf{v}|^2 \mathbf{v} \otimes \mathbf{v} g \rangle$$

and assume that

$$\mathcal{L}g_\varepsilon \rightarrow \mathcal{L}g = 0, \quad \varepsilon^s \Gamma(g_\varepsilon, g_\varepsilon) \rightarrow 0$$

in the sense of distributions. Then g has the form of the infinitesimal Maxwellian, i.e.,

$$g = \varrho + \mathbf{u} \cdot \mathbf{v} + \left(\frac{1}{2} |\mathbf{v}|^2 - \frac{3}{2} \right) \theta, \quad (3.26)$$

where $(\varrho, \mathbf{u}, \theta)$ solve the generalized acoustic equations

$$\begin{aligned} \partial_t \varrho + \nabla_{\mathbf{x}} \cdot \mathbf{u} \pm \tau \left\{ \partial_t^2 \varrho + 2 \partial_t \nabla_{\mathbf{x}} \cdot \mathbf{u} + \Delta_{\mathbf{x}} (\varrho + \theta) \right\} &= 0, \\ \partial_t \mathbf{u} + \nabla_{\mathbf{x}} (\varrho + \theta) \pm \tau \left\{ \partial_t^2 \mathbf{u} + 2 \partial_t \nabla_{\mathbf{x}} (\varrho + \theta) + \Delta_{\mathbf{x}} \mathbf{u} + 2 \nabla_{\mathbf{x}} (\nabla_{\mathbf{x}} \cdot \mathbf{u}) \right\} &= 0, \\ \frac{3}{2} \partial_t \theta + \nabla_{\mathbf{x}} \cdot \mathbf{u} \pm \tau \left\{ \frac{3}{2} \partial_t^2 \theta + 2 \partial_t \nabla_{\mathbf{x}} \cdot \mathbf{u} + \Delta_{\mathbf{x}} (\varrho + \frac{7}{2} \theta) \right\} &= 0, \end{aligned} \quad (3.27)$$

and satisfy the initial conditions (the case with positive sign)

$$\begin{aligned} \varrho(0, \mathbf{x}) &= \varrho_0(\mathbf{x}), & \partial_t \varrho(0, \mathbf{x}) &= \varrho_1(\mathbf{x}), \\ \mathbf{u}(0, \mathbf{x}) &= \mathbf{u}_0(\mathbf{x}), & \partial_t \mathbf{u}(0, \mathbf{x}) &= \mathbf{u}_1(\mathbf{x}), \\ \theta(0, \mathbf{x}) &= \theta_0(\mathbf{x}), & \partial_t \theta(0, \mathbf{x}) &= \theta_1(\mathbf{x}), \end{aligned}$$

or boundary conditions (the case with negative sign)

$$\begin{aligned} \varrho(0, \mathbf{x}) &= \varrho^{in}(\mathbf{x}), & \varrho(T, \mathbf{x}) &= \varrho^{out}(\mathbf{x}), \\ \mathbf{u}(0, \mathbf{x}) &= \mathbf{u}^{in}(\mathbf{x}), & \mathbf{u}(T, \mathbf{x}) &= \mathbf{u}^{out}(\mathbf{x}), \\ \theta(0, \mathbf{x}) &= \theta^{in}(\mathbf{x}), & \theta(T, \mathbf{x}) &= \theta^{out}(\mathbf{x}). \end{aligned}$$

Proof. We multiply the equation (3.23) by ε and we take the limit of $\varepsilon \rightarrow 0$. From the assumptions of the theorem we obtain $\mathcal{L}g = 0$. Since we assume $g \in L^2_{\mathbf{v}}$ we have $g \in \text{Ker}(\mathcal{L})$, so g must have the form of the infinitesimal Maxwellian. Due to the property of collision invariants the fluctuations g_ε satisfy

$$\begin{aligned} \partial_t \langle g_\varepsilon \rangle + \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} g_\varepsilon \rangle \pm \tau \left\{ \partial_t^2 \langle g_\varepsilon \rangle + 2 \partial_t \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} g_\varepsilon \rangle + \nabla_{\mathbf{x}}^2 : \langle \mathbf{v} \otimes \mathbf{v} g_\varepsilon \rangle \right\} &= 0, \\ \partial_t \langle \mathbf{v} g_\varepsilon \rangle + \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \otimes \mathbf{v} g_\varepsilon \rangle \\ \pm \tau \left\{ \partial_t^2 \langle \mathbf{v} g_\varepsilon \rangle + 2 \partial_t \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \otimes \mathbf{v} g_\varepsilon \rangle + \nabla_{\mathbf{x}}^2 : \langle \mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v} g_\varepsilon \rangle \right\} &= 0, \\ \partial_t \langle \frac{1}{2} |\mathbf{v}|^2 g_\varepsilon \rangle + \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \frac{1}{2} |\mathbf{v}|^2 g_\varepsilon \rangle \\ \pm \tau \left\{ \partial_t^2 \langle \frac{1}{2} |\mathbf{v}|^2 g_\varepsilon \rangle + 2 \partial_t \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \frac{1}{2} |\mathbf{v}|^2 g_\varepsilon \rangle + \nabla_{\mathbf{x}}^2 : \langle \mathbf{v} \otimes \mathbf{v} \frac{1}{2} |\mathbf{v}|^2 g_\varepsilon \rangle \right\} &= 0. \end{aligned}$$

The theorem now follows by letting $\varepsilon \rightarrow 0$ in these equations, using the convergences assumed above and finally using the limiting form of g given by (3.26). \square

Remark. The system of the generalized acoustic equations (3.27) can be obtained by the linearization of the generalized compressible Euler equations, introduced in [Al94, eq.(3.52)–(3.54)]

$$\begin{aligned}
& \partial_t \varrho + \nabla_{\mathbf{x}} \cdot (\varrho \mathbf{u}) \pm \tau \{ \partial_t^2 \varrho + 2 \partial_t \nabla_{\mathbf{x}} \cdot (\varrho \mathbf{u}) + \nabla_{\mathbf{x}}^2 : (\varrho \mathbf{u} \otimes \mathbf{u}) + \Delta_{\mathbf{x}} p \} = 0, \\
& \partial_t (\varrho \mathbf{u}) + \nabla_{\mathbf{x}} \cdot (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_{\mathbf{x}} p \pm \tau \{ \partial_t^2 (\varrho \mathbf{u}) + 2 \partial_t \nabla_{\mathbf{x}} \cdot (\varrho \mathbf{u} \otimes \mathbf{u}) \\
& \quad + 2 \partial_t \nabla_{\mathbf{x}} p + \Delta_{\mathbf{x}} (p \mathbf{u}) + 2 \nabla_{\mathbf{x}} \nabla_{\mathbf{x}} \cdot (p \mathbf{u}) + \nabla_{\mathbf{x}}^2 : (\varrho \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u}) \} = 0, \\
& \partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{3}{2} p \right) + \nabla_{\mathbf{x}} \cdot \left(u \frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{5}{2} p u \right) \\
& \quad \pm \tau \{ \partial_t^2 \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{3}{2} p \right) + 2 \partial_t \nabla_{\mathbf{x}} \cdot \left(u \frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{5}{2} p u \right) \\
& \quad + \Delta_{\mathbf{x}} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{5}{2} p^2 / \varrho \right) + \nabla_{\mathbf{x}}^2 : \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{7}{2} p \right) \mathbf{u} \otimes \mathbf{u} \right] \} = 0,
\end{aligned}$$

around the equilibrium state with the zero bulk velocity and the density and the temperature equal to 1, see (3.19):

$$\varrho = 1 + \varepsilon^s \varrho', \quad \mathbf{u} = \varepsilon^s \mathbf{u}' \quad \text{and} \quad \theta = 1 + \varepsilon^s \theta'.$$

The pressure p is governed by the state equation of an ideal gas $p = \varrho \theta$.

In the next two sections we present the analytical results for both signs in the generalized Boltzmann equation. Because of the different fundamentals of such problems we present both problems separately.

3.4 Initial value problem

The initial value problem of the Boltzmann equation was extensively studied in a context of the theory of semigroups by authors [Gr63], [Uk74], [EP75], [Uk76] and [NI76]. Here we would like to prove the existence of the Boltzmann semigroup of our problem.

We omit the nonlinear operator Γ in the equation (3.23) and define the initial value problem (the parameters $\tau > 0$ and $\varepsilon > 0$ are fixed)

$$\begin{aligned}
& \tau \Lambda^2 g_{\varepsilon} + \Lambda g_{\varepsilon} - \frac{1}{\varepsilon} \mathcal{L} g_{\varepsilon} = 0, \quad t \geq 0 \text{ and } (\mathbf{x}, \mathbf{v}) \in R^3 \times R^3, \\
& g_{\varepsilon}(0) = g_0, \quad (\mathbf{x}, \mathbf{v}) \in R^3 \times R^3, \\
& \partial_t g_{\varepsilon}(0) = g_1, \quad (\mathbf{x}, \mathbf{v}) \in R^3 \times R^3.
\end{aligned} \tag{3.28}$$

We assume the initial conditions to be independent of the parameter ε .

We denote

$$h_{\varepsilon} = (\varepsilon / \tau \nu)^{1/2} (g_{\varepsilon} + \tau \Lambda g_{\varepsilon}) \tag{3.29}$$

and we transform the equation (3.28) into the linear system

$$\partial_t \begin{pmatrix} g_{\varepsilon} \\ h_{\varepsilon} \end{pmatrix} = \begin{pmatrix} -\mathbf{v} \cdot \nabla_{\mathbf{x}} - \tau^{-1} I & (\nu / \tau \varepsilon)^{1/2} I \\ -(\nu / \tau \varepsilon)^{1/2} I + (\nu \tau \varepsilon)^{-1/2} \mathcal{K} & -\mathbf{v} \cdot \nabla_{\mathbf{x}} \end{pmatrix} \begin{pmatrix} g_{\varepsilon} \\ h_{\varepsilon} \end{pmatrix}. \tag{3.30}$$

The Fourier transform of a complex-valued Lebesgue integrable function

$$\widehat{f}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int_{R^3} f(\mathbf{x}) e^{-i \mathbf{k} \cdot \mathbf{x}} d\mathbf{x} \quad \text{for } \mathbf{k} \in R^3$$

gives us

$$\partial_t \begin{pmatrix} \widehat{g_{\varepsilon}} \\ \widehat{h_{\varepsilon}} \end{pmatrix} = (\mathcal{E}_{\mathbf{k}} + \mathcal{F}) \begin{pmatrix} \widehat{g_{\varepsilon}} \\ \widehat{h_{\varepsilon}} \end{pmatrix} = \mathcal{G}_{\mathbf{k}} \begin{pmatrix} \widehat{g_{\varepsilon}} \\ \widehat{h_{\varepsilon}} \end{pmatrix},$$

where \mathcal{E}_k is the multiplication operator

$$\mathcal{E}_k = \begin{pmatrix} -i \mathbf{v} \cdot \mathbf{k} & (\nu/\tau\varepsilon)^{1/2} \\ -(\nu/\tau\varepsilon)^{1/2} & -i \mathbf{v} \cdot \mathbf{k} \end{pmatrix}$$

and \mathcal{F} is a bounded perturbation (independent of \mathbf{k}) of the operator \mathcal{E}_k

$$\mathcal{F} = \begin{pmatrix} -\tau^{-1} I & 0 \\ (\nu\tau\varepsilon)^{-1/2} \mathcal{K} & 0 \end{pmatrix}.$$

We consider the Hilbert space

$$\widehat{M}_0 = L^2_{\mathbf{v}} \times L^2_{\mathbf{v}}$$

of complex valued functions with the scalar product

$$\left\langle \begin{pmatrix} g_1 \\ h_1 \end{pmatrix} \middle| \begin{pmatrix} g_2 \\ h_2 \end{pmatrix} \right\rangle = \int_{R^3 \times R^3} g_1 \bar{g}_2 m(\mathbf{v}) d\mathbf{v} + \int_{R^3 \times R^3} h_1 \bar{h}_2 m(\mathbf{v}) d\mathbf{v},$$

and we define

$$\widehat{M}_q = \left\{ \begin{pmatrix} g \\ h \end{pmatrix} \in \widehat{M}_0 \middle| |\mathbf{v}|^q \begin{pmatrix} g \\ h \end{pmatrix} \in \widehat{M}_0 \right\}$$

The adjoint operator \mathcal{E}_k^* to the multiplication operator \mathcal{E}_k is

$$\mathcal{E}_k^* = \overline{\mathcal{E}_k^T} = \begin{pmatrix} i \mathbf{v} \cdot \mathbf{k} & -(\nu/\tau\varepsilon)^{1/2} \\ (\nu/\tau\varepsilon)^{1/2} & i \mathbf{v} \cdot \mathbf{k} \end{pmatrix} = -\mathcal{E}_k \quad (3.31)$$

Definition 3.2 ([Pazy83], definition 4.1). Let X be a Banach space and let X^* be its dual. For every $x \in X$ we define $x^* \in X^*$ such that $\langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2$. A linear operator A in X is *dissipative* if for every $x \in D(A)$ there is a x^* such that $\operatorname{Re} \langle Ax, x^* \rangle \leq 0$.

Proposition 3.3 ([Pazy83], corollary 4.4). Let A be a densely defined closed linear operator. If both A and its adjoint operator A^* are dissipative, then A is the infinitesimal generator of the strongly continuous \mathcal{C}_0 semigroup of contractions on X .

Lemma 3.4. Let $\mathbf{k} \in R^3$ be a fixed parameter. The operator \mathcal{E}_k is a closed linear operator with the domain densely defined in the space \widehat{M}_0

$$D(\mathcal{E}_k) = \left\{ \begin{pmatrix} \widehat{g} \\ \widehat{h} \end{pmatrix} \in \widehat{M}_0 \middle| (\nu^{1/2} + i \mathbf{v} \cdot \mathbf{k}) \begin{pmatrix} \widehat{g} \\ \widehat{h} \end{pmatrix} \in \widehat{M}_0 \right\} \supseteq \widehat{M}_1.$$

This operator is an infinitesimal generator of \mathcal{C}_0 semigroup of contractions on the space \widehat{M}_0 .

Proof. By Proposition 3.3, it suffices to prove that both operators \mathcal{E}_k and \mathcal{E}_k^* are dissipative. It however follows from

$$\begin{aligned} \left\langle \mathcal{E}_k \begin{pmatrix} \widehat{g} \\ \widehat{h} \end{pmatrix} \middle| \begin{pmatrix} \widehat{g} \\ \widehat{h} \end{pmatrix} \right\rangle &= -i \int_{R^3 \times R^3} (\mathbf{k} \cdot \mathbf{v}) (|\widehat{g}|^2 + |\widehat{h}|^2) m(\mathbf{v}) d\mathbf{v} \\ &\quad - 2i \int_{R^3 \times R^3} (\nu/\tau\varepsilon)^{1/2} \operatorname{Im}(\widehat{g} \overline{\widehat{h}}) m(\mathbf{v}) d\mathbf{v} \\ &= - \left\langle \mathcal{E}_k^* \begin{pmatrix} \widehat{g} \\ \widehat{h} \end{pmatrix} \middle| \begin{pmatrix} \widehat{g} \\ \widehat{h} \end{pmatrix} \right\rangle. \end{aligned}$$

□

Since the operator \mathcal{F} is a bounded perturbation of an infinitesimal generator of \mathcal{C}_0 semigroup ([Pazy83]), we can state the following lemma about the operator \mathcal{G}_k

Lemma 3.5. Operator \mathcal{G}_k is an infinitesimal generator of \mathcal{C}_0 semigroup $\widehat{T}_k(t)$ with the estimate

$$\|\widehat{T}_k(t)\| \leq e^{C_B t}.$$

The constant C_B is independent of the parameter $\mathbf{k} \in R^3$.

Proof. For an arbitrary $\widehat{f} \in D(\mathcal{G}_k) \equiv D(\mathcal{E}_k)$ we have

$$\begin{aligned} \frac{d}{dt} \|\widehat{T}_k(t)\widehat{f}\|^2 &= \langle \mathcal{G}_k \widehat{T}_k(t)\widehat{f} \mid \widehat{T}_k(t)\widehat{f} \rangle + \langle \widehat{T}_k(t)\widehat{f} \mid \mathcal{G}_k \widehat{T}_k(t)\widehat{f} \rangle = \langle (\mathcal{G}_k + \mathcal{G}_k^*) \widehat{T}_k(t)\widehat{f} \mid \widehat{T}_k(t)\widehat{f} \rangle \\ &\leq \|\mathcal{G}_k + \mathcal{G}_k^*\| \cdot \|\widehat{T}_k(t)\widehat{f}\|^2. \end{aligned}$$

As the operator $\mathcal{G}_k + \mathcal{G}_k^*$ is independent of \mathbf{k}

$$\mathcal{G}_k + \mathcal{G}_k^* = \begin{pmatrix} -2/\tau I & (\tau\nu\varepsilon)^{-1/2}\mathcal{K} \\ (\tau\nu\varepsilon)^{-1/2}\mathcal{K} & 0 \end{pmatrix},$$

the estimate now follows for $C_B = \|\mathcal{G}_k + \mathcal{G}_k^*\|/2 \leq \|\mathcal{F}\|$. □

Let us define

$$M_{p,q} = \left\{ \begin{pmatrix} g \\ h \end{pmatrix} = \frac{1}{(2\pi)^{3/2}} \int_{R^3} \begin{pmatrix} \widehat{g} \\ \widehat{h} \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{v}} d\mathbf{k} \mid |\mathbf{k}|^p |\mathbf{v}|^q \begin{pmatrix} \widehat{g} \\ \widehat{h} \end{pmatrix} \in L_{\mathbf{k}\mathbf{v}}^2 \times L_{\mathbf{k}\mathbf{v}}^2 \right\}.$$

For $f \in M_{0,0}$ we can define the Boltzmann semigroup

$$T(t)f = \frac{1}{(2\pi)^{3/2}} \int_{R^3} \widehat{T}_k(t)\widehat{f}(\mathbf{k}, \mathbf{v}) e^{i\mathbf{k}\cdot\mathbf{v}} d\mathbf{k} \quad (3.32)$$

and we have obtained

Theorem 3.6. *The operator $T(t)$ is a \mathcal{C}_0 semigroup on $M_{0,0}$ with the norm*

$$\|T(t)\| \leq e^{C_B t}$$

and it gives the unique solution of the system (3.30) with the initial conditions $(g_0, h_0) \in M_{1,1}$.

3.5 Boundary value problem

We omit the nonlinear operator Γ in the equation (3.23) and we define the boundary value problem (the parameters $T > 0$, $\tau > 0$ and $\varepsilon > 0$ are fixed)

$$\begin{aligned} -\tau\Lambda^2 g_\varepsilon + \Lambda g_\varepsilon - \frac{1}{\varepsilon} \mathcal{L}g_\varepsilon &= f & \text{in } U = (0, T) \times R^3 \times R^3, \\ g_\varepsilon(0) = g_\varepsilon^{in} \quad \text{and} \quad g_\varepsilon(T) &= g_\varepsilon^{out} & (\mathbf{x}, \mathbf{v}) \in R^3 \times R^3, \end{aligned} \quad (3.33)$$

We introduce the following notation and recall some well-known facts:

- The Hilbert space

$$W^1 = \{ g \in L^2_{t\mathbf{x}\mathbf{v}} \mid \Lambda g \in L^2_{t\mathbf{x}\mathbf{v}} \}$$

which is endowed with the inner product and the norm

$$((g, h)) = (\Lambda g, \Lambda h) + (g, h) \quad \text{and} \quad |||g||| = ((g, g))^{1/2}.$$

Proposition 3.7 ([CIP94], p. 246). For piecewise \mathcal{C}^1 boundary, there exists trace operator from W^1 to $L^2_{\mathbf{x}\mathbf{v}}$.

In our case, there exists $C > 0$ such that

$$\|g(0)\|_{L^2_{\mathbf{x}\mathbf{v}}} + \|g(T)\|_{L^2_{\mathbf{x}\mathbf{v}}} \leq C |||g|||.$$

The space W^1 is continuously embedded into the space $L^\infty(0, T; L^2_{\mathbf{x}\mathbf{v}})$, from the identity

$$\frac{1}{2}\|g(t)\|_{L^2_{\mathbf{x}\mathbf{v}}}^2 - \frac{1}{2}\|g(0)\|_{L^2_{\mathbf{x}\mathbf{v}}}^2 = \int_0^t \int_{R^3} \int_{R^3} g \Lambda g m(\mathbf{v}) d\mathbf{v} d\mathbf{x} dt \quad \text{for all } 0 \leq t \leq T,$$

we see immediately

$$\|g\|_{L^\infty(0, T; L^2_{\mathbf{x}\mathbf{v}})} \leq C \left(\|g(0)\|_{L^2_{\mathbf{x}\mathbf{v}}} + \|\Lambda g\|_{L^2_{t\mathbf{x}\mathbf{v}}} \right) \leq C |||g|||. \quad (3.34)$$

- The Hilbert space W_0^1 is the closure of $\mathcal{D}(U)$ with respect to the norm $|||\cdot|||$. There $\mathcal{D}(U)$ is the set of all infinitely differentiable functions with the compact support. The space W_0^1 is identical with the set of all functions from W^1 with the zero trace. Due to the previous inequality we have the equivalence of the norms

$$\frac{1}{C} \|\Lambda g\|_{L^2_{t\mathbf{x}\mathbf{v}}} \leq |||g||| \leq C \|\Lambda g\|_{L^2_{t\mathbf{x}\mathbf{v}}} \quad \text{for all } g \in W_0^1.$$

- The bilinear form $\mathcal{B} : (W^1 \cap D(\mathcal{L})) \times W_0^1 \rightarrow R$ is defined through

$$\mathcal{B}[g, \varphi] := \tau(\Lambda g, \Lambda \varphi) + (\Lambda g, \varphi) - \frac{1}{\varepsilon}(\mathcal{L}g, \varphi).$$

To maximally simplify our treatment we suppose the boundary conditions to be ε independent.

Assumption. The boundary conditions (g^{in}, g^{out}) are in the form

$$\begin{aligned} g^{in} &= \varrho^{in} + v \cdot \mathbf{u}^{in} + \left(\frac{1}{2}|v|^2 - \frac{3}{2}\right)\theta^{in}, \\ g^{out} &= \varrho^{out} + v \cdot \mathbf{u}^{out} + \left(\frac{1}{2}|v|^2 - \frac{3}{2}\right)\theta^{out}, \end{aligned} \quad (3.35)$$

where $(\varrho^{in}, \varrho^{out})$, $(\mathbf{u}^{in}, \mathbf{u}^{out})$ and $(\theta^{in}, \theta^{out})$ are the traces of some functions ϱ^\bullet , \mathbf{u}^\bullet and θ^\bullet from the Sobolev space $H^1((0, T) \times R^3)$.

Then we can choose the non-homogeneous part of the solution g_ε to be

$$g^\bullet = \varrho^\bullet + v \cdot \mathbf{u}^\bullet + \left(\frac{1}{2}|v|^2 - \frac{3}{2}\right)\theta^\bullet$$

and we see $g^\bullet \in W^1 \cap D(\mathcal{L})$ with $\mathcal{L}g^\bullet = 0$ independently of ε .

Definition 3.8. We say that $g_\varepsilon \in W^1 \cap D(\mathcal{L})$ is the weak solution of (3.33) iff:

- (i) $\mathcal{B}[g_\varepsilon, \varphi] = (f, \varphi) \quad \text{for all } \varphi \in W_0^1$
- (ii) $g_\varepsilon(0) = g_\varepsilon^{in} \quad \text{a.e.} \quad \text{and} \quad g_\varepsilon(T) = g_\varepsilon^{out} \quad \text{a.e.}$

Theorem 3.9. Let $f \in L^2_{txv}$. Problem (3.33) has the unique weak solution $g \in L^\infty(0, T; L^2_{xv})$.

Proof. (based on the Lax–Milgram theorem)

1. We split $g_\varepsilon = g^\bullet + g_\varepsilon^\circ$, where $g_\varepsilon^\circ \in W_0^1 \cap D(\mathcal{L})$ than our equation reads

$$\mathcal{B}[g_\varepsilon^\circ, \varphi] = \langle\langle f^*, \varphi \rangle\rangle \quad \text{for all } \varphi \in W_0^1.$$

$\langle\langle f^*, \cdot \rangle\rangle$ is the continuous functional on W_0^1 given by the formula

$$\langle\langle f^*, \varphi \rangle\rangle = (f, \varphi) - \tau(\Lambda g^\bullet, \Lambda \varphi) - (\Lambda g^\bullet, \varphi).$$

2. For fixed $g_\varepsilon^\circ \in W_0^1 \cap D(\mathcal{L})$ we see $\mathcal{B}[g_\varepsilon^\circ, \cdot]$ is the continuous functional on W_0^1 and by the Riesz Representation theorem, we can introduce the operator $\mathcal{A} : D_{\mathcal{A}} \rightarrow W_0^1$

$$\mathcal{B}[g_\varepsilon^\circ, \varphi] = ((\mathcal{A}g_\varepsilon^\circ, \varphi)) \quad \text{for all } \varphi \in W_0^1. \quad (3.36)$$

The domain of definition $D_{\mathcal{A}} = W_0^1 \cap D(\mathcal{L})$ is dense in W_0^1 and the operator \mathcal{A} is closed. Indeed, assume $h_k \rightarrow h \in W_0^1$, $\{h_k\} \subset D_{\mathcal{A}}$ and $\mathcal{A}h_k \rightarrow w \in W_0^1$. From the definition of the operator \mathcal{A} for fixed φ

$$|(\mathcal{L}h_k - \mathcal{L}h_\ell, \varphi)| \leq C (\|h_k - h_\ell\| + \|\mathcal{A}h_k - \mathcal{A}h_\ell\|) \|\varphi\|,$$

so $(\mathcal{L}h_k, \varphi)$ is the Cauchy sequence. Operator $(\mathcal{L}\cdot, \varphi) : D_{\mathcal{A}} \rightarrow R$ is closed (composed map of a bounded and a closed operator), which gives

$$h \in D_{\mathcal{A}} \quad \text{and} \quad \lim_{k \rightarrow \infty} (\mathcal{L}h_k, \varphi) = (\mathcal{L}h, \varphi).$$

Now we can pass to the limit $\varepsilon \rightarrow 0$ in (3.36)

$$\mathcal{B}[h, \varphi] = \lim_{k \rightarrow \infty} \mathcal{B}[h_k, \varphi] = \lim_{k \rightarrow \infty} ((\mathcal{A}h_k, \varphi)) = ((w, \varphi))$$

and we have get $\mathcal{A}h = w$, too.

3. The operator \mathcal{A} is 1-1 with $\mathcal{R}(\mathcal{A}) = W_0^1$. As the operator \mathcal{L} is non-positive and as we have the equivalence of the norms, there exists $\beta > 0$, such that

$$\mathcal{B}[h, h] \geq \tau \|\Lambda h\|^2 \geq \beta \|h\|^2 \quad \text{for all } h \in D_{\mathcal{A}}. \quad (3.37)$$

It follows that the operator \mathcal{A} and also the adjoint operator \mathcal{A}^* are 1-1, $\overline{\mathcal{R}(\mathcal{A})} = \text{Ker}(\mathcal{A}^*)^\perp = W_0^1$. The positive constant β is the minimum modulus of the closed operator \mathcal{A} and it assures that range is closed, $\mathcal{R}(\mathcal{A}) = W_0^1$. The inverse \mathcal{A}^{-1} is the bounded operator by the theorem of closed graphs.

4. By the Riesz Representation theorem each element f^* of the dual space $(W_0^1)^*$ can be uniquely associated with $w_{f^*} \in W_0^1$ and we found the solution of our problem $g_\varepsilon^\circ = \mathcal{A}^{-1}w_{f^*}$. Inequality (3.37) gives even the uniqueness of the weak solution.

□

Next step in the rigorous proof of our formal Theorem 3.1 is the existence and the uniqueness of the weak solution of acoustic system (3.27).

Theorem 3.10. *Let $(\varrho^{in}, \varrho^{out})$, $(\mathbf{u}^{in}, \mathbf{u}^{out})$ and $(\theta^{in}, \theta^{out})$ be the traces of the functions from the Sobolev space $H((0, T) \times \mathbb{R}^3)$. Then there exists the unique weak solution of the generalized acoustic equations (3.27).*

Proof. We can split density, velocity and temperature into the homogeneous^o and the non-homogeneous[•] part:

$$\varrho = \varrho^\bullet + \varrho^\circ, \quad \mathbf{u} = \mathbf{u}^\bullet + \mathbf{u}^\circ \quad \text{and} \quad \theta = \theta^\bullet + \theta^\circ.$$

We test the first equation by ϱ° , the second by \mathbf{u}° and the third by θ° and sum together. It gives the identity $\|\cdot\| = \|\cdot\|_{L^2_{tx}}$:

$$\begin{aligned} & \frac{1}{2} \|\varrho^\circ(t)\|_{L^2_x}^2 + \frac{1}{2} \|\mathbf{u}^\circ(t)\|_{L^2_x}^2 + \frac{3}{4} \|\theta^\circ(t)\|_{L^2_x}^2 + \tau \left\{ \|\partial_t \varrho^\circ\|^2 + \|\partial_t \mathbf{u}^\circ\|^2 + \frac{3}{2} \|\partial_t \theta^\circ\|^2 \right. \\ & \left. + \|\nabla_x \varrho^\circ\|^2 + \|\nabla_x \mathbf{u}^\circ\|^2 + 2\|\nabla_x \cdot \mathbf{u}^\circ\|^2 + \frac{7}{2} \|\nabla_x \theta^\circ\|^2 + 2(\nabla_x \varrho^\circ, \nabla_x \theta^\circ) + 4(\nabla_x \cdot \mathbf{u}^\circ, \partial_t \varrho^\circ + \partial_t \theta^\circ) \right\} = I_0 \end{aligned}$$

where I_0 is the sum of the inner products of the non-homogeneous part and the homogeneous part and can be proceeded by the Young and Gronwall inequality

$$\begin{aligned} I_0 = & -(\partial_t \varrho^\bullet + \nabla_x \cdot \mathbf{u}^\bullet, \varrho^\circ) - (\partial_t \mathbf{u}^\bullet + \nabla_x(\varrho^\bullet + \theta^\bullet), \mathbf{u}^\circ) - \left(\frac{3}{2} \partial_t \theta^\bullet + \nabla_x \cdot \mathbf{u}^\bullet, \theta^\circ\right) \\ & - \tau \left\{ (\partial_t \varrho^\bullet + 2\nabla_x \cdot \mathbf{u}^\bullet, \partial_t \varrho^\circ) + (\partial_t \mathbf{u}^\bullet + 2\nabla_x(\varrho^\bullet + \theta^\bullet), \partial_t \mathbf{u}^\circ) + \left(\frac{3}{2} \partial_t \theta^\bullet + 2\nabla_x \cdot \mathbf{u}^\bullet, \partial_t \theta^\circ\right) \right. \\ & \left. + (\nabla_x(\varrho^\bullet + \theta^\bullet), \nabla_x \varrho^\circ) + (\nabla_x \mathbf{u}^\bullet, \nabla_x \mathbf{u}^\circ) + 2(\nabla_x \cdot \mathbf{u}^\bullet, \nabla_x \cdot \mathbf{u}^\circ) + (\nabla_x(\varrho^\bullet + \frac{7}{2} \theta^\bullet), \nabla_x \theta^\circ) \right\}. \end{aligned}$$

The main difficulty here is caused by the last two terms on the left hand side. As the coefficients in front of the L^2 -norms of the derivatives of the density and of the temperature are non-symmetric, we introduce the parameter $\epsilon \in (0, 1)$ and split the last two terms as

$$2(\nabla_x \varrho^\circ, \nabla_x \theta^\circ) + 4(\nabla_x \cdot \mathbf{u}^\circ, \partial_t \varrho^\circ + \partial_t \theta^\circ) = I_1 + I_2,$$

where

$$\begin{aligned} I_1 = & 2(\epsilon \nabla_x \varrho^\circ, \epsilon^{-1} \nabla_x \theta^\circ) + 2\epsilon(\partial_t \mathbf{u}^\circ, \epsilon \nabla_x \varrho^\circ + \epsilon^{-1} \nabla_x \theta^\circ) \\ = & \epsilon \|\partial_t \mathbf{u}^\circ + \epsilon \nabla_x \varrho^\circ + \epsilon^{-1} \nabla_x \theta^\circ\|^2 + (1 - \epsilon) \|\epsilon \nabla_x \varrho^\circ + \epsilon^{-1} \nabla_x \theta^\circ\|^2 \\ & - \epsilon \|\partial_t \mathbf{u}^\circ\|^2 - \epsilon^2 \|\nabla_x \varrho^\circ\|^2 - \epsilon^{-2} \|\nabla_x \theta^\circ\|^2 \end{aligned}$$

and

$$I_2 = (4 - 2\epsilon^2)(\nabla_x \mathbf{u}^\circ, \partial_t \varrho^\circ) + 2(\nabla_x \cdot \mathbf{u}^\circ, \partial_t \theta^\circ).$$

Now we have to assure the term I_2 can be proceeded with the Young inequality on the right hand side. Let $\gamma, \delta \in (0, 1)$ then

$$|I_2| \leq (4 - 2\epsilon^2)\gamma \|\partial_t \varrho^\circ\|^2 + 2\delta \|\partial_t \theta^\circ\|^2 + \left(\frac{2 - \epsilon^2}{2\gamma} + \frac{1}{2\delta}\right) \|\nabla_x \cdot \mathbf{u}^\circ\|^2.$$

We compare the coefficients and we obtain

$$\delta < \frac{3}{4} \text{ and } \gamma < \frac{1}{4 - 2\epsilon^2} \implies 2 - \epsilon^2 < \frac{8\gamma}{3} < \frac{4}{3(2 - \epsilon^2)} \implies \epsilon > \sqrt{2(1 - 1/\sqrt{3})} \doteq 0.92.$$

We see that $\varrho^\circ, \mathbf{u}^\circ$ and θ° are in the space $H_0^1((0, T) \times \mathbb{R}^3) \cap L^\infty(0, T; L^2_x)$. For the difference of two distinct solutions we have the same estimate with $I_0 = 0$, so the solution has to be unique. □

Let g_ε be the weak solution of (3.33) with $f = 0$, so

$$\mathcal{B}[g_\varepsilon, \varphi] = 0 \quad \text{for all } \varphi \in W_0^1.$$

We have the estimates:

$$\begin{aligned} \frac{\beta}{2} \|\|g_\varepsilon^\circ\|\|^2 + \frac{\mu}{\varepsilon} \|g_\varepsilon^{\circ\perp}\|^2 &\leq C \|\|g_\varepsilon^\bullet\|\|^2, \\ |(\mathcal{L}g_\varepsilon, \varphi)| &\leq \varepsilon \|\|g_\varepsilon\|\| \|\|\varphi\|\| \quad \text{for all } \varphi \in W_0^1, \end{aligned}$$

with the constant $C > 0$ independent of ε . The function $g_\varepsilon^{\circ\perp}$ is the part of g_ε° from the orthogonal complement to the $\text{Ker}(\mathcal{L})$. We see immediately, that there is weak convergence $g_\varepsilon^\circ \rightharpoonup g^\circ$ in W_0^1 with $g^\circ \in W_0^1 \cap \text{Ker}(\mathcal{L})$ and part $g_\varepsilon^{\circ\perp}$ converges even strongly to zero in L_{txv}^2 . The choice of the test functions

$$\varphi(t, x, v) = \psi(t, x)\omega(v),$$

where $\psi \in H_0^1((0, T) \times R^3)$ and $\omega(v) \in \{1, v, |v|^2\}$, shows the validity of the acoustic system.

Theorem 3.11. *With the additional assumptions (3.35) on the initial conditions we proved that*

$$\begin{aligned} g_\varepsilon \rightharpoonup g = \varrho + \mathbf{v} \cdot \mathbf{u} + \left(\frac{1}{2}|\mathbf{v}|^2 - \frac{3}{2}\right)\theta & \quad \text{in } W^1, \\ \langle g_\varepsilon \omega \rangle \rightharpoonup \langle g \omega \rangle & \quad \text{in } H^1((0, T) \times R^3), \\ g_\varepsilon^\perp \rightarrow 0 & \quad \text{in } L_{txv}^2, \\ \mathcal{L}g_\varepsilon \rightarrow 0 & \quad \text{in } \mathcal{D}'(U), \end{aligned}$$

where $\varrho, \mathbf{u}, \theta$ is the weak solution of the acoustic system (3.27).

3.6 Conclusion

The generalized Boltzmann equation (3.6) is the perturbation of the Boltzmann equation (3.5) where the changes on the small time scale $\tau > 0$ (τ has order of the mean free time) are counted in.

The character of this perturbed equation depends on the sign of the additional terms. The positive sign keeps the evolutionary character, the negative sign switch to the boundary problem. The arguments for and against of both signs were discussed in Section 3.2, but we believe that the right sign is the positive one.

The analogy (3.17) of the \mathcal{H} -theorem (3.15) can be derived, but the interpretation is not so clear as in the case of the Boltzmann equation. There arise new sources of entropy which are compensated by the smallness of the perturbation parameter $\tau > 0$ and by the strict convexity of the entropy.

Theorem 3.1 about the acoustic limit for the generalized Boltzmann equation is given and the final system of the generalized acoustic equations (3.27) is, as can be expected, the perturbation of the acoustic equations (3.21). The additional terms in the generalized Boltzmann equation requires the knowledge of the higher velocity moments of solution g_ε and this breaks the symmetry between the fluctuations of the density ϱ and the temperature θ in the generalized acoustic equations. Therefore the analogy of the wave equation for the pressure (3.22) is difficult to obtain.

The formal results are specified in the case of the linearized problem. In the case of the positive sign the existence of the Boltzmann semigroup (3.32) is proved which corresponds to the results obtained for the linearized Boltzmann equation. In the case of the negative sign the existence and uniqueness of the weak solution is proved, the proof is based on the Lax–Milgram theorem, and some convergence assumptions of the formal theorem are proved.

3.7 Acknowledgements

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Appendix A

Appendix

A.1 Tensors formed from a unit vector

In this appendix we refer the results¹ about the integration of n th-order tensor formed from a unit vector over all orientations of such a unit vector, i.e., integration over the unit sphere $S^2 = \{\boldsymbol{\eta} \in R^3 : |\boldsymbol{\eta}| = 1\}$:

$$\int_{S^2} \overset{n}{\otimes} \boldsymbol{\eta} \, d\boldsymbol{\eta} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \frac{4\pi (n/2)! 2^{n/2}}{(n+1)!} \cdot \left[\begin{array}{c} \text{all } n\text{th-order tensor} \\ \text{combinations of unit tensors} \end{array} \right] & \text{if } n \text{ is even.} \end{cases} \quad (\text{A.1})$$

Particularly useful integrals are

$$\int_{S^2} d\boldsymbol{\eta} = 4\pi, \quad (\text{A.2})$$

$$\int_{S^2} \boldsymbol{\eta} \otimes \boldsymbol{\eta} \, d\boldsymbol{\eta} = \frac{4\pi}{3} \mathcal{I}, \quad (\text{A.3})$$

$$\int_{S^2} \boldsymbol{\eta} \otimes \boldsymbol{\eta} \otimes \boldsymbol{\eta} \otimes \boldsymbol{\eta} \, d\boldsymbol{\eta} = \frac{4\pi}{15} (\mathcal{I} \otimes \mathcal{I} + \llbracket \mathcal{I} \rrbracket), \quad (\text{A.4})$$

where the last relation in the components reads:

$$\int_{S^2} \eta_i \eta_j \eta_k \eta_l \, d\boldsymbol{\eta} = \frac{4\pi}{15} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad i, j, k, l = 1, 2, 3.$$

¹Snider, R. F. and Curtiss, C. F.: *Kinetic Theory of Moderately Dense Gases*. The Physics of Fluids, Vol. 1, No. 2, 122–138 (1958); Appendix A

A.2 Ortogonal conditions

In this appendix we calculate the equations (2.78) and (2.79). We use the Einstein convention of summation over the same indices. The upper index denotes a component of vector or tensor and the lower index denotes the partial derivative with respect to the space coordinate corresponding to the index. In this notation the equation (2.70) becomes

$$\begin{aligned} \int \psi \vartheta^i{}_{ijj} d\mathbf{V} &= \left(\int \psi \vartheta^i d\mathbf{V} \right)_{ijj} - \left(\int \psi_i \vartheta^j d\mathbf{V} + \int \psi_j \vartheta^i d\mathbf{V} + \int \psi_m \vartheta^m \delta^{ij} d\mathbf{V} \right)_{ij} \\ &\quad + \left(\int \psi_{ij} \vartheta^j d\mathbf{V} + \int \psi_{ji} \vartheta^j d\mathbf{V} + \int \psi_{mm} \vartheta^j \delta^{ij} d\mathbf{V} \right)_i - \left(\int \psi_{ijj} \vartheta^i d\mathbf{V} \right), \end{aligned}$$

where $\vartheta^i = \beta(\theta)\varrho M(\mathbf{V})\mathcal{V}^i$.

At first, we consider $\boldsymbol{\psi} = \mathbf{V} \otimes \mathbf{V} - \theta \mathcal{I}$ and we calculate its spatial derivatives. We denote c a generic constant with respect to the variable \mathbf{V} and we remember the assumption that all derivatives of the quantity $K = \beta(\theta)\varrho^2\theta$ are negligible

$$\begin{aligned} \psi^{kl} &= \mathcal{V}^k \mathcal{V}^l + c \\ \psi^{kl}{}_i &= -u^k{}_i \mathcal{V}^l - \mathcal{V}^k u^l{}_i + c \\ \psi^{kl}{}_{ij} &= -u^k{}_{ij} \mathcal{V}^l - \mathcal{V}^k u^l{}_{ij} + c \\ \psi^{kl}{}_{ijj} &= -u^k{}_{ijj} \mathcal{V}^l - \mathcal{V}^k u^l{}_{ijj} + c \end{aligned}$$

it follows

$$\begin{aligned} \int \psi^{kl} \vartheta^i{}_{ijj} d\mathbf{V} &\doteq + K(u^k{}_i \delta^{lj} + u^l{}_i \delta^{kj} + u^k{}_j \delta^{li} + u^l{}_j \delta^{ki} + u^k{}_m \delta^{lm} \delta^{ij} + u^l{}_m \delta^{km} \delta^{ij})_{ij} \\ &\quad - K(2u^k{}_{ij} \delta^{lj} + 2u^l{}_{ij} \delta^{kj} + u^k{}_{mm} \delta^{lj} \delta^{ij} + u^l{}_{mm} \delta^{kj} \delta^{ij})_i \\ &\quad + K(u^k{}_{ijj} \delta^{li} + u^l{}_{ijj} \delta^{ki}) \\ &= K \Delta_{\mathbf{x}}(u^k{}_l + u^l{}_k). \end{aligned}$$

This result can be rewritten in more comprehensive form. Let \mathcal{P}_ψ be the projection on the subspace generated by ψ^{kl} , $1 \leq k \leq l \leq 3$, then

$$\mathcal{P}_\psi \boldsymbol{\vartheta} = \sum_{1 \leq k \leq l \leq 3} \frac{\psi^{kl} \tau^{kl}}{(1 + \delta^{kl})\varrho\theta^2} = \sum_{k,l=1}^3 \frac{\psi^{kl} \tau^{kl}}{2\varrho\theta^2}$$

where

$$\tau^{kl} = K \Delta_{\mathbf{x}}(u^k{}_l + u^l{}_k).$$

Tensor $\boldsymbol{\tau}$ has the following trace and traceless part

$$\begin{aligned} \text{Tr } \boldsymbol{\tau} &= 2K \nabla_{\mathbf{x}} \cdot \mathbf{u}, \\ \boldsymbol{\tau} - \frac{1}{3} \text{Tr } \boldsymbol{\tau} \mathcal{I} &= K \Delta_{\mathbf{x}} \boldsymbol{\sigma}(\mathbf{u}). \end{aligned}$$

Applying the symmetry of tensors $\boldsymbol{\tau}$ and $\boldsymbol{\psi}$, we develop the formula

$$\mathcal{P}_\psi \boldsymbol{\vartheta} = \frac{\mathbf{V} \otimes \mathbf{V} - \frac{1}{3} |\mathbf{V}|^2 \mathcal{I}}{2\varrho\theta^2} : \left(\boldsymbol{\tau} - \frac{1}{3} \text{Tr } \boldsymbol{\tau} \mathcal{I} \right) + \frac{\frac{1}{2} |\mathbf{V}|^2 - \frac{3}{2} \theta}{2\varrho\theta^2} \cdot \frac{2}{3} \text{Tr } \boldsymbol{\tau}$$

or equally with respect to the above formulas

$$\mathcal{P}_\psi \boldsymbol{\vartheta} = \frac{\mathbf{V} \otimes \mathbf{V} - \frac{1}{3} |\mathbf{V}|^2 \mathcal{I}}{2\varrho\theta^2} : K \Delta_{\mathbf{x}} \boldsymbol{\sigma}(\mathbf{u}) + \frac{\frac{1}{2} |\mathbf{V}|^2 - \frac{3}{2} \theta}{\varrho\theta^2} \cdot \frac{2}{3} K \nabla_{\mathbf{x}} \cdot \mathbf{u}$$

Next, we consider $\boldsymbol{\psi} = (\frac{1}{2}|\boldsymbol{\mathcal{V}}|^2 - \frac{5}{2}\theta)\boldsymbol{\mathcal{V}}$ and we calculate its spatial derivatives

$$\begin{aligned}
\psi^k &= (\frac{1}{2}|\boldsymbol{\mathcal{V}}|^2 - \frac{5}{2}\theta)\mathcal{V}^k \\
\psi^k_{,i} &= -u^l_i \mathcal{V}^l \mathcal{V}^k - \frac{5}{2}\theta_i \mathcal{V}^k - \frac{1}{2}|\boldsymbol{\mathcal{V}}|^2 u^k_i + c \\
\psi^k_{,ij} &= -u^l_{ij} \mathcal{V}^l \mathcal{V}^k + u^l_i u^k_j \mathcal{V}^l + u^l_i u^l_j \mathcal{V}^k - \frac{5}{2}\theta_{ij} \mathcal{V}^k + u^l_j u^k_i \mathcal{V}^l - \frac{1}{2}|\boldsymbol{\mathcal{V}}|^2 u^k_{ij} + c \\
\psi^k_{,ijj} &= -u^l_{ijj} \mathcal{V}^l \mathcal{V}^k + 2u^l_{ij} u^l_j \mathcal{V}^k + 2u^l_{ij} u^k_j \mathcal{V}^l + u^l_i u^l_{jj} \mathcal{V}^k \\
&\quad + u^l_i u^k_{jj} \mathcal{V}^l - \frac{5}{2}\theta_{ijj} \mathcal{V}^k + u^l_{jj} u^k_i \mathcal{V}^l + 2u^l_j u^k_{ij} \mathcal{V}^l - \frac{1}{2}|\boldsymbol{\mathcal{V}}|^2 u^k_{ijj} + c
\end{aligned}$$

it follows

$$\begin{aligned}
&\int \psi^k \vartheta^i_{ijj} d\boldsymbol{\mathcal{V}} \doteq \\
&\quad - K(\frac{5}{2}\theta_i \delta^{kj} + \frac{5}{2}\theta_j \delta^{ki} + \frac{5}{2}\theta_m \delta^{km} \delta^{ij})_{ij} \\
&\quad + K(2u^l_i u^l_j \delta^{kj} + 2u^l_i u^k_j \delta^{lj} + 2u^l_j u^k_i \delta^{lj} - \frac{5}{2}\theta_{ij} \delta^{kj} + u^l_j u^l_j \delta^{ki} + 2u^l_j u^k_j \delta^{li} - \frac{5}{2}\theta_{jj} \delta^{ki})_i \\
&\quad - K(2u^l_{ij} u^l_j \delta^{ki} + 2u^l_{ij} u^k_j \delta^{li} + 2u^k_{ij} u^l_j \delta^{li} + u^l_{jj} u^l_i \delta^{ki} + u^k_{jj} u^l_i \delta^{li} + u^l_{jj} u^k_i \delta^{li} - \frac{5}{2}\theta_{ijj} \delta^{ki}) \\
&\quad = \frac{5}{2}K \nabla \Delta_{\boldsymbol{x}} \theta_k + K(\Delta_{\boldsymbol{x}} u^i \delta^{jk} + \Delta_{\boldsymbol{x}} u^j \delta^{ik} + \Delta_{\boldsymbol{x}} u^k \delta^{ij}) u^i_j + 2K(u^i_{jk} + u^k_{ij} + u^m_{mj} \delta^{ik}) u^i_j.
\end{aligned}$$