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Doctoral Thesis

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Chiral perturbation theory and the low energy phenomenology of pseudoscalar mesons

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Contents

Introduction

How could be a theory taken seriously if none of its basic building blocks has ever been directly observed? Science has made stunning progress during the 20th century. Today, one needs to put much more effort in gaining theoretical insight and technical background to be able to at least understand the forefront of scientific exploration in any of its fields. Yet, science is still faithful to its founding principles and it's as deeply rooted in empirical observation of Nature as ever. And yes, growing amount of experimental data and theoretical apprehension during the past 30 years has made such a theory a fundamental part of our knowledge.

Quantum chromodynamics is a beautiful example of a theory based on just a few simple principles leading to nontrivial results. The degrees of freedom used to introduce the model, the quark and gluon fields, do not seem to be the right ones to describe our ordinary world, despite the apparent simplicity. They are appropriate in the high energy domain, but in the low energy limit the usual perturbation expansion fails and no predictions can be made directly. However, that does not automatically disprove the theory. We are only trying to use an inconvenient parametrization in an area it's not very useful and more sophisticated methods have to be developed to study the nonperturbative aspects of the theory.

The effective theory framework is one of the approaches used to explore the low energy behavior of a more fundamental model. The basic idea is used throughout physics, or even all natural sciences in broader sense - many details of the underlining theory often do not explicitly manifest themselves in specific domains. Thus an effective theory can be introduced by reformulating the model in terms of only the relevant degrees of freedom, while the rest will get expressed merely in the background structure of the theory, i.e. some constants. Such an approach will be valid only in the area where the assumption about the relevancy of the degrees of freedom is correct, but inside these boundaries the predictions should be equivalent, while given more easily. The understanding of the behavior is usually more clear and intuitive.

Such an approach can be introduced systematically in quantum field theory. The chosen high energy degrees of freedom are integrated out in the Feynman's path integral and remain part of the theory as coupling constants tied to effective vertices. The effective theory is then organized as a perturbative expansion in momenta and therefore it contains infinite number of terms, each with a coupling constant. This guarantees that in the low energy limit the lower order terms should be numerically more important than the higher order ones.

Chiral perturbation theory is the result of this program applied to Quantum chromodynamics in the lowest energy domain. It intends to describe the interactions of the lightest hadrons - the pseudoscalar mesons. However, because the explicit out-integration of the heavy degrees of freedom was not successfully done yet, the coupling constants are unknown and can be derived only from experiments. This leads to the fact that not only the relative importance of the coupling constants is not well known but also the complete character of the theory is debated, allowing for several scenarios how to organize the perturbative expansion. The goal of this work is to theoretically investigate these approaches, to look into the possibly important subtle details and to compare predictions in several specific cases.

The thesis is divided into two parts. The first is a summary of the well known theoretical foundations, starting from Quantum chromodynamics and proceeding to the introduction of Chiral perturbation theory as an effective theory of QCD and its several versions. Background in several other related topics is discussed next, the phase structure of QCD with varying numbers of light quark flavors, dispersion representation of the scattering amplitude and Resonance chiral theory. This part intends to be a simple introduction of the theory to a reader unfamiliar with the topic, a short overview focusing on what will be needed throughout the rest of the work including a broader list of references for more in-depth study.

The second part contains our own results discussing three concrete cases on which the topic is demonstrated. The core of each one is an article reprinted very close to its original form. The articles are completed with a commentary which provides further information about the background of the research.

Part I

Theoretical background

Chapter 1

Basics of Chiral perturbation theory

1.1 QCD and its symmetries

Let us do a very short review of the basics of the theory of strong interactions. We won't go into any details, the aim is only to reflect the basic ideas. More detailed treatment can be found e.g. in textbooks [1, 2].

Quantum chromodynamics is constructed as the fundamental theory of strong interactions. It is based on the principle of *local gauge invariance*, which connects all modern theories of fundamental interactions. In the case of QCD, invariance under local transformations of the *color group* $SU(3)$ is assumed.

The basic building blocks of the theory are spin 1/2 fermion fields, the *quarks*. It is assumed that they come in six *flavors*, which differ, from the point of view of the strong interaction, only in their masses. Each of the quarks can have one of three *colors*, which play a role similar to charge in electromagnetism. In a noninteracting theory, the colors are indistinguishable and the Lagrangian is globally symmetric under $SU(3)$ transformations.

The principle of local gauge invariance assumes that the interaction is locally equivalent to a space-time dependent, i.e. *local*, transformation of the fermion *matter fields*. The form of the interaction terms in the Lagrangian have to be such that they can be transformed away in the surrounding of any arbitrarily chosen space-time point. This implies that the interaction has to be mediated by a set of vector fields, the *gauge fields*, which are then defined up to such a *gauge* transformation. The resulting Lagrangian will be invariant under any local transformations of the associated symmetry group.

When this program is worked out for the strong interaction, where quarks are the matter fields, color is the local symmetry group associated with the interaction and *gluons* arise as the gauge fields, the complete QCD Lagrangian is obtained

$$\mathcal{L}_{QCD} \;=\; -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} \;+\; \bar{q}(i\gamma_\mu D^\mu - \mathcal{M})q\,. \tag{1.1}$$

$G^{\mu\nu}$ is the antisymmetric gluon field tensor

$$G^{\mu\nu} \;=\; \partial^\mu G^\nu - \partial^\nu G^\mu + ig\,[\,G^\mu, G^\nu\,], \quad G_\mu = \frac{1}{2}\lambda^a G^a_\mu\,, \tag{1.2}$$

G_μ^a are the gluon fields, a the gluon index and λ^a the Gell-Mann matrices. The symbol q denotes the quark field flavor vector[1]

$$q = \begin{pmatrix} u \\ d \\ s \\ c \\ b \\ t \end{pmatrix}, \tag{1.3}$$

$\mathcal{M}$ the quark mass matrix and D^μ is the *covariant* derivative

$$D^\mu = \partial^\mu + igG^\mu, \tag{1.4}$$

where g is the strong interaction coupling constant.

Let's take a look at the basic predictions following from this theory. From the third term in the definition of the gluon field tensor $G^{\mu\nu}$ in (1.2) it is clear, that the non–Abelian character of the theory gives rise to self-interactions of the massless gluons. This leads, due to the renormalization procedure, to a unique behavior of the coupling constant g: it decreases with the scale. This peculiarity, *asymptotic freedom*, has fatal implications for the applicability of the perturbation theory - in the low energy domain the coupling constant grows and the expansion fails. On the other hand, in the high energy region the interaction weakens and the quarks should become essentially free.

The value of any theory is determined by its ability to describe reality, to predict and agree with the results of experiments. In the real world, rather hadrons are observed, not quarks and gluons. However, in the high energy domain the data can be interpreted as if there were free *partons* inside the hadrons. Perturbative QCD works very well here and its predictions agree, e.g., with the results of deep inelastic scattering experiments.

Hadronization and *quark confinement* occur in the region, where nonperturbative methods have to be applied. This is a highly nontrivial task and various methods are being worked out. We will focus on the effective theory approach in this work.

Symmetries of the QCD Lagrangian play a decisive role in exploring the low energy behavior of the theory. Apparently, the Lagrangian (1.1) has only one additional symmetry besides the $SU(3)$ color group - it is invariant under the phase changes of each of the quark fields. This $U(1)$ symmetry is tied to the baryon number conservation, which is very well supported by the experiments.

As it is well known, the hadron spectrum possesses approximate symmetries - isospin symmetry and eightfold way. These should follow from the QCD Lagrangian if we wanted to interpret the hadrons as its low energy bound states.

When one realizes that the role of the different quark flavors differs only in the mass term, several possible approximate symmetries can be seen. If the difference in the masses of some of the flavors was small, compared to the scale of the theory, an additional $SU(N_f)$ symmetry would arise. Let's suppose that this is the case for the three "light" quarks, u, d and s. Then we can rewrite the mass term in the QCD Lagrangian (1.1)

[1]the quark fields are also color triplets

$$\begin{aligned} m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s &= \frac{1}{3}(m_u + m_d + m_s)(\bar{u}u + \bar{d}d + \bar{s}s) + \\ &+\frac{1}{3}(m_u - m_d)(\bar{u}u - \bar{d}d) + \frac{1}{3}(m_u - m_s)(\bar{u}u - \bar{s}s) + \frac{1}{3}(m_d - m_s)(\bar{d}d - \bar{s}s), \end{aligned} \tag{1.5}$$

and divide the Lagrangian into a symmetry conserving and a symmetry breaking part

$$\mathcal{L}_{QCD} = \mathcal{L}_0^V + \mathcal{L}_{sb}^V \tag{1.6}$$

$$\mathcal{L}_0^V = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + i\bar{q}\gamma_\mu D^\mu q - \frac{1}{3}\bar{q}\mathcal{M}_{u+d+s}q + \mathcal{L}_{heavy\, quarks} \tag{1.7}$$

$$\mathcal{L}_{sb}^V = -\frac{1}{3}(m_u - m_d)(\bar{u}u - \bar{d}d) - \cdots , \tag{1.8}$$

where q now includes only the light quark flavors and $\mathcal{M}_{u+d+s}$ is a 3×3 identity matrix multiplied by the sum of the light masses $m_u + m_d + m_s$.

Now it's transparent that $\mathcal{L}_0^V$ is invariant under a $q \to U^V q$ transformation of the light quark fields, where U^V is a 3×3 unitary matrix. We can exclude the baryon number conservation symmetry discussed above with the condition $\det U^V = 1$ and hence isolate an approximate $SU(3)$ symmetry. The symmetry breaking part $\mathcal{L}_{sb}^V$ would vanish if the light quark masses were identical. In that case the symmetry would be exact.

It is fairly straightforward to explain the isospin symmetry as being due to a small difference between the masses of the two lightest quarks, u and d, while the eightfold way resulting from the s quark's mass being fairly close too. QCD predicts nearly degenerate states forming $SU(2)$ and $SU(3)$ multiplets and this is exactly the case when one looks at the hadron spectrum.

If not only the differences but also the masses themselves were small compared to the scale of the theory, left and right components of the quark fields could transform independently while preserving the symmetry. The symmetry breaking part then includes the complete mass term of the involved quarks. To see this, let's rewrite the Lagrangian (1.6) to the form

$$\mathcal{L}_{QCD} = \mathcal{L}_0^A + \mathcal{L}_{sb}^A \tag{1.9}$$

$$\mathcal{L}_0^A = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + i\bar{q}_L\gamma_\mu D^\mu q_L + i\bar{q}_R\gamma_\mu D^\mu q_R + \mathcal{L}_{heavy\, quarks} \tag{1.10}$$

$$\mathcal{L}_{sb}^A = -\bar{q}\mathcal{M}q = -\bar{q}_R\mathcal{M}q_L + \text{h.c.} , \tag{1.11}$$

where

$$q_{L,R} = \frac{1}{2}(1 \mp \gamma_5)\, q, \quad q = q_L + q_R = \gamma_5(q_R - q_L) \tag{1.12}$$

and $\mathcal{M}$ is the light quark mass matrix

$$\mathcal{M} \;=\; \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \,. \tag{1.13}$$

The left- and the right-handed components now transform under the $SU(N_f)$ rotations separately

$$q_L \;\rightarrow\; U_L\, q_L, \qquad q_R \;\rightarrow\; U_R\, q_R, \tag{1.14}$$

so the symmetry group of $\mathcal{L}_0^A$ is

$$G \;=\; SU(N_f)_L \times SU(N_f)_R \times U(1)_L \times U(1)_R \;, \tag{1.15}$$

where N_f is the number of the quark flavors included (2 or 3).

There is no left-right symmetry observed in the hadron spectrum. On the other hand, there are several indications that the approximate symmetry tied to the smallness of the quark masses, *chiral symmetry*, is indeed present. The most important one is that the mass of the lowest bound states, the pion isospin triplet, is very small compared to the scale of the theory, while the rest of the pseudoscalar octet masses are fairly small too. This observed pattern can be explained if it is assumed that the mass of u, d and s quarks is not only similar, but very small too and moreover, the symmetry is spontaneously broken. The absence of the explicit left-right doublets is then expected and the pseudoscalar mesons are interpreted as *Goldstone bosons*.

1.2 Spontaneous symmetry breaking

Because $SU(2)$ and $SU(3)$ are Lie groups[2], they can be parametrized as

$$U^V \;=\; e^{-iv^a Q^{aV}}, \quad U_L \;=\; e^{-iv_L^a Q^{aV}}, \quad U_R \;=\; e^{-iv_R^a Q^{aV}}, \tag{1.16}$$

where v^a, v_L^a and v_R^a are real parameters. The hermitian matrices Q^{aV} are the *generators* of the group obeying Lie algebra

$$[\, Q^{aV}, Q^{bV}\,] \;=\; if^{abc}\, Q^{cV} \;. \tag{1.17}$$

f^{abc} are called *structure constants*. The $SU(2)$ has three independent generators, therefore it is a three parametric group. Analogously, the $SU(3)$ has eight generators.

In the fundamental matrix representation, the generators can be conventionally chosen

$$Q^{aV} \;=\; \frac{1}{2}\,\lambda^a \;, \tag{1.18}$$

the λ^a are the Pauli or the Gell-Mann matrices.

[2]for a short review of the theory of Lie groups and general references see [1]

The $SU(N_f)_L \times SU(N_f)_R$ group elements can be displayed in a doublet notation

$$U_{L\times R} \;=\; (\, U_L, U_R\,)\;, \tag{1.19}$$

$$U_{L\times R} \;=\; e^{-iv_L^a L^a - iv_R^a R^a}, \tag{1.20}$$

$$L^a \;=\; (\, Q^{aV}, 0\,), \quad R^a \;=\; (\, 0\,, Q^{aV}\,), \tag{1.21}$$

where the left component acts on q_L, while the right on q_R. L^a and R^a are the generators of $SU(N_f)_L$ and $SU(N_f)_R$ in this notation. The Lie algebra is then of the form

$$[\, L^a, L^b\,] \;=\; if^{abc}\, L^c, \tag{1.22}$$

$$[\, R^a, R^b\,] \;=\; if^{abc}\, R^c, \tag{1.23}$$

$$[\, L^a, R^b\,] \;=\; 0\;. \tag{1.24}$$

The isospin/eightfold way and the chiral symmetry can be found by a reparametrization

$$V^a \;=\; R^a + L^a \;=\; (\, Q^{aV}, Q^{aV}\,), \tag{1.25}$$

$$A^a \;=\; R^a - L^a \;=\; (\, -Q^{aV}, Q^{aV}\,)\;. \tag{1.26}$$

The general group element takes the form

$$U_{L\times R} \;=\; e^{-iv^a V^a - i\xi^a A^a} \tag{1.27}$$

and the commutators of the Lie algebra can be expressed as

$$[\, V^a, V^b\,] \;=\; if^{abc}\, V^c, \tag{1.28}$$

$$[\, A^a, A^b\,] \;=\; if^{abc}\, V^c, \tag{1.29}$$

$$[\, V^a, A^b\,] \;=\; if^{abc}\, A^c. \tag{1.30}$$

We can see that the generators V^a form a subalgebra, hence they generate a subgroup

$$U^V \;=\; e^{-iv^a V^a} \;=\; (\, e^{\,-iv^a Q^{aV}}, e^{-iv^a Q^{aV}}\,) \tag{1.31}$$

of the whole symmetry group of the theory. We find

$$U^V q \;=\; e^{-iv^a Q^{aV}} q_L + e^{-iv^a Q^{aV}} q_R \;=\; e^{-iv^a Q^{aV}} q. \tag{1.32}$$

So indeed, compared to (1.16), this is the isospin/eightfold way symmetry.

On the other hand, the generators A^a does not form a subalgebra and so the chiral symmetry, with elements expressed as

$$U^A \;=\; e^{-i\xi^a A^a} \;=\; (\, e^{\,i\xi^a Q^{aV}}, e^{-i\xi^a Q^{aV}} \,), \tag{1.33}$$

is not a subgroup.

Hence we see that the symmetry group of the theory can be rewritten

$$G \;=\; SU(N_f)_V \times SU(N_f)_A \times U(1)_V \times U(1)_A \;. \tag{1.34}$$

The $SU(N_f)_V$ and $U(1)_V$ are the symmetries discussed in the previous section, the axial $U(1)_A$ fails to be a symmetry on quantum level due to the Abelian anomaly.

The Noether theorem states that if a Lagrangian is invariant under the transformations of an n–parametric group, then n conserved vector currents exist. In our cases the vector currents are

$$J^a_{\mu L,R} \;=\; \frac{1}{2}\bar{q}_{L,R}\gamma_\mu\lambda^a q_{L,R} \;=\; \frac{1}{2}\,(\,\frac{1}{2}\bar{q}\gamma_\mu\lambda^a q \;\mp\; \frac{1}{2}\bar{q}\gamma_\mu\gamma_5\lambda^a q\,), \tag{1.35}$$

$$J^{aV}_\mu \;=\; J^a_{\mu R} \;+\; J^a_{\mu L} \;=\; \frac{1}{2}\bar{q}\gamma_\mu\lambda^a q\,, \tag{1.36}$$

$$J^{aA}_\mu \;=\; J^a_{\mu R} \;-\; J^a_{\mu L} \;=\; \frac{1}{2}\bar{q}\gamma_\mu\gamma_5\lambda^a q\,. \tag{1.37}$$

Then the generators in the operator representation acting on the vectors from the Hilbert space of states can be written

$$\mathcal{U}^V \;=\; e^{-iv^a\mathcal{V}^a}, \qquad \mathcal{V}^a \;=\; \int \mathrm{d}^3x\; J^{aV}_0(x), \tag{1.38}$$

$$\mathcal{U}^A \;=\; e^{-i\xi^a\mathcal{A}^a}, \qquad \mathcal{A}^a \;=\; \int \mathrm{d}^3x\; J^{aA}_0(x)\;. \tag{1.39}$$

The Hamiltonian can be separated to the symmetry conserving and the symmetry braking part as well

$$H_{QCD} \;=\; H^A_0 \;+\; H^A_{sb} \tag{1.40}$$

$$H^A_{sb} \;=\; \int \mathrm{d}^3x\; \bar{q}\mathcal{M}q \;=\; -\int \mathrm{d}^3x\; \mathcal{L}^A_{sb}\,, \tag{1.41}$$

the generators of the group $\mathcal{V}^a$, $\mathcal{A}^a$ commute with the symmetry conserving part of the Hamiltonian

$$[\,\mathcal{V}^a, H_0^A\,] \;=\; 0, \qquad [\,\mathcal{A}^a, H_0^A\,] \;=\; 0\;. \tag{1.42}$$

There are generally two different ways by which the symmetries, as expressed in (1.42), can manifest themselves in the properties of the physical states. It is easy to see that if a generator of a symmetry $\mathcal{Q}^a$ commutes with the Hamiltonian H, one expects to see a degenerate multiplet of energy eigenstates

$$H\,|\,m\,\rangle \;=\; E_m\,|\,m\,\rangle, \quad \mathcal{Q}^a\,|\,m\,\rangle \;=\; |\,m'\,\rangle, \tag{1.43}$$

$$E_{m'}\,|\,m'\,\rangle \;=\; H\,|\,m'\,\rangle \;=\; H\,\mathcal{Q}^a\,|\,m\,\rangle = \;\mathcal{Q}^a\,H\,|\,m\,\rangle \;=\; E_m\,|\,m'\,\rangle. \tag{1.44}$$

However, this is true in quantum field theory only if the lowest bound state, the vacuum, is unique. That means that it has to be a singlet under any transformation of the symmetry group

$$\mathcal{U}\,|\,0\,\rangle \;=\; |\,0\,\rangle \tag{1.45}$$

which is equivalent to

$$\mathcal{Q}^a\,|\,0\,\rangle \;=\; 0\;, \tag{1.46}$$

for all generators $\mathcal{Q}^a$. This is the familiar Wigner-Weyl realization of the symmetry.

On the other hand, if some generators $\mathcal{Q}^a_B$ does not annihilate the vacuum, the states $\mathcal{Q}^a_B\;|\,0\,\rangle$ should have the same energy as the physical ground state. In other words, the ground state is not symmetric with respect to the complete symmetry group and the corresponding states in the excited multiplets, obtained by the perturbative expansion around the vacuum, are missing. Instead of that, as is shown by the *Goldstone theorem* [3], for every such symmetry breaking generator there is exactly one independent state of zero energy. These are observable as massless particles with spin 0, the *Goldstone bosons*. This is the Nambu-Goldstone realization of the symmetry. It is important to note that the 'well-behaved' generators which annihilate the vacuum form a subalgebra

$$\mathcal{Q}^a\,|\,0\,\rangle \;=\; 0\;,\;\; \mathcal{Q}^b\,|\,0\,\rangle \;=\; 0 \quad \Rightarrow \quad [\,\mathcal{Q}^a, \mathcal{Q}^b\,]\,|\,0\,\rangle \;=\; 0\;, \tag{1.47}$$

therefore they generate a subgroup (*little group*) of the whole symmetry group.

In the case of QCD, the spontaneous breaking of the approximate symmetry group

$$G \;=\; SU(N_f)_V \times SU(N_f)_A \times U(1)_V \tag{1.48}$$

to the isospin or eightfold way subgroup

$$H \;=\; SU(N_f)_V \times U(1)_V \tag{1.49}$$

explains the pattern of the observed hadron multiplets very well. Particularly, there should be one containing three or alternatively eight pseudoscalar Goldstone bosons with approximately zero mass, i.e. small compared to the scale of the theory, in the spectrum. This prediction of the pion triplet and the pseudoscalar octet of pions, kaons and eta still remains the only successful direct prediction of hadron bound states from QCD up to date.

According to the Goldstone theorem [3], for every generator of the symmetry group $\mathcal{Q}^a_B$, for which there is an operator $\mathcal{O}$, such that

$$\langle\, 0 \,|[\, \mathcal{Q}^a_B, \mathcal{O}\,]\,|\, 0 \,\rangle \;\neq\; 0 \tag{1.50}$$

there must be one independent massless state $|\, G.boson \,\rangle$ with

$$\langle\, 0 \,| J_0^{aB}(0) \,|\, G.boson \,\rangle\langle\, G.boson \,|\mathcal{O}\,|\, 0 \,\rangle \;\neq\; 0\ . \tag{1.51}$$

In the case of QCD, the non-vanishing of the first element in (1.51), which involves only the symmetry current and it is independent of the operator $\mathcal{O}$ considered, is both the necessary and sufficient condition for the occurrence of the spontaneous symmetry breakdown. Thus this element is the fundamental *order parameter*.

For the case of the chiral symmetry, denoting the Goldstone bosons ϕ^a, in a convenient normalization we have

$$\langle\, 0 \,| J_\mu^{aA}(x) \,|\, \phi^b(p) \,\rangle \;=\; \langle\, 0 \,| \frac{1}{2}\bar{q}\gamma_\mu\gamma_5\lambda^a q \,|\, \phi^b(p) \,\rangle = i p_\mu N_p F_\phi e^{-ipx}\delta^{ab}, \tag{1.52}$$

as a consequence of Lorentz invariance and linear realization of $SU(N_f)_V$. If we identify the Goldstone bosons with the pseudoscalar mesons, e.g. the pions, the constant F_ϕ becomes the familiar pion decay constant and because it is not equal to zero, the condition is fulfilled and the symmetry breaking really seems to occur.

1.3 Effective field theory

The effective field theory approach [4, 5] has become a very strong tool in high energy particle physics, especially when dealing with theories with nontrivial structure, such as QCD. We will demonstrate the basic ideas it's laid on.

As we have already seen, the elementary practical problem of QCD is that the degrees of freedom used to formulate the theory, the quark and gluon fields, cannot be considered as the interpolating fields for the particles in the spectrum, the hadrons. The standard perturbation expansion works only in the high energy domain. The natural desire is thus to reformulate the QCD Lagrangian and express it in terms of fields from the spectrum. However, such a direct reparametrization is highly nontrivial and has not been done yet. The construction of the effective field theory based on the known symmetry properties partly bypasses this problem.

Let's have a Lagrangian depending on fields collectively denoted as $\Phi(x)$

$$\mathcal{L} \;=\; \mathcal{L}(\Phi, \partial_\mu\Phi) \tag{1.53}$$

describing a theory with such a particle content that there is a mass gap between some light and heavy states

$$m_L < \Lambda \leq m_H \ . \tag{1.54}$$

This means that for all center of mass energies $\sqrt{s} < \Lambda$, only the light particles can appear in the initial and final states.

We'll suppose that one is interested in the interactions of the light particles only. In general, the original fields $\Phi(x)$ do not have to directly correspond to the particles in the spectrum and thus it would be convenient to re-express the Lagrangian in terms of some new fields Φ_L and Φ_H

$$\Phi \ = \ F(\Phi_L, \Phi_H) \ , \tag{1.55}$$

$$\mathcal{L}(\Phi) \ = \ \mathcal{L}(\Phi_L, \Phi_H) \ , \tag{1.56}$$

such that Φ_L should be the interpolating fields for the particles in which one is interested in. However, even without the direct knowledge of (1.55), much can be done under the assumption that the particles corresponding to Φ_L describe all the lowest energy bound states under the cutoff Λ. In this energy region, only the intermediate states may depend on Φ_H, the heavy degrees of freedom. It is possible to define *effective vertices* by summing all the contributions of the intermediate states of the heavy fields Φ_H in the elements of S-matrix. Thus the *effective Lagrangian* can be introduced

$$e^{i\int \mathrm{d}^4x \ \mathcal{L}_{eff}(\Phi_L)} \ = \ \int \mathcal{D}\Phi_H \ e^{i\int \mathrm{d}^4x \ \mathcal{L}(\Phi_L,\Phi_H)} \ . \tag{1.57}$$

The S-matrix elements calculated from this effective Lagrangian are equivalent to those from the original one for all energies smaller than Λ.

The effective vertices, as defined in (1.57), are nonlocal and thus are not polynomials in external momenta. However, the contribution of the heavy field intermediate states is suppressed for $E << \Lambda$. Therefore an expansion of the effective vertices in terms of momenta, which should converge reasonably quickly for $E << \Lambda$, can be done. Equivalently, the whole effective Lagrangian can be organized as a derivative expansion.

If any symmetry of the original theory is known, it constrains the form of the effective Lagrangian very strongly, as both the original and the effective Lagrangian have to possess the same symmetry properties. Even more generally, a phenomenological Lagrangian can be constructed purely as a most general form compatible with the presumed symmetry properties (Weinberg "theorem" [5]). In fact, all physical theories having the same symmetry and particle content have the same derivative expansion of their effective Lagrangians. They only differ in the value of *low energy coupling constants*, which appear with each of the expansion terms. It's interesting that the details of the underlying physics has virtually no influence on the low energy behavior of the system.

Such a program can be realized in the case of very low energy QCD relatively straightforwardly. We know that the few lightest bound states are the Goldstone bosons of the spontaneously broken approximate symmetry $SU(N_f)_L \times SU(N_f)_R$. Therefore it is enough to find the most general form of such a Lagrangian, that the degrees of freedom are the pseudoscalar mesons and the symmetry properties are identical to those of the original theory.

1.4 Construction of the effective Lagrangian

In this section we will proceed to recapitulate the main points of the effective field theory procedure application on QCD, leading to *Chiral perturbation theory (χPT)*. We will focus on the $SU(3)_L \times SU(3)_R$ case, which will be our main interest later.

The standard start, as used in the classical papers [6, 7], is to write down a generating functional of vector currents, axial vector currents, scalar and pseudoscalar densities in the form

$$e^{iZ[v,a,s,p,\theta]} = \langle 0_{out} \,|\, 0_{in} \rangle_{v,a,s,p} = \int \mathcal{D}\Phi \; e^{i \int \mathrm{d}^4x \; \mathcal{L}[v,a,s,p,\theta]} \tag{1.58}$$

$$\mathcal{L}[v,a,s,p,\theta] = \mathcal{L}_0^A + i\bar{q}\gamma^\mu[v_\mu(x) + \gamma_5 a_\mu(x)]q - \bar{q}[s(x) - i\gamma_5 p(x)]q - \frac{1}{16\pi^2}\theta(x) G^a_{\mu\nu} G^{a\mu\nu}. \tag{1.59}$$

The already introduced $\mathcal{L}_0^A$ is the QCD Lagrangian in *chiral limit*, i.e. the symmetry conserving part, where the light quark masses are set to zero. The physical value of external fields v, a, s, p, θ, which are 3×3 hermitian matrices, therefore is

$$v_\mu = a_\mu = p = 0, \quad s(x) = \mathcal{M}, \quad \theta(x) = \theta_0 \,. \tag{1.60}$$

An advantage of introducing the external sources is also the easy incorporation of electroweak interactions of the pseudoscalar mesons. In particular, the electromagnetic interaction can be included with the choice

$$v^\mu = -eA^\mu Q = -eA^\mu(\frac{1}{2}\lambda^3 + \frac{1}{2\sqrt{3}}\lambda^8), \quad a^\mu = 0 \,, \tag{1.61}$$

where Q is the quark charge matrix

$$Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}. \tag{1.62}$$

The Lagrangian (1.59) is invariant under local $U(3)_L \times U(3)_R$ transformations, apart from the axial anomaly, if the external fields transform in the following way

$$v'_\mu = \frac{1}{2}[\, U_R v_\mu U_R^+ + U_L v_\mu U_L^+ + i\, U_R \partial_\mu U_R^+ + i\, U_L \partial_\mu U_L^+ \,] \tag{1.63}$$

$$a'_\mu \;=\; \frac{1}{2}[\,U_R v_\mu U_R^+ \,+\, U_L v_\mu U_L^+ \,+\, i\,U_R \partial_\mu U_R^+ \,-\, i\,U_L \partial_\mu U_L^+\,] \tag{1.64}$$

$$s' + ip' \;=\; U_R(s+ip)U_L^+ \;. \tag{1.65}$$

The form for θ can be found in [7]. This directly implies for the light quark mass matrix

$$\mathcal{M}' \;=\; U_R \mathcal{M} U_L^+ \,. \tag{1.66}$$

To be able to reconstruct the effective version of the generating functional (1.58), one has to know the transformation properties of the Goldstone bosons, which will replace the quarks and gluons as the low energy degrees of freedom. Conventionally, these are collected in a 3×3 unitary matrix

$$U(x)^+ U(x) \;=\; 1 \tag{1.67}$$

for which

$$U'(x) \;=\; U_R U(x) U_L^+ \,. \tag{1.68}$$

This form of the transformation follows from the classical work [4].

As shown in [7], one can get set the physical value of the vacuum angle $\theta_0 = 0$ if parity conservation of the strong interaction is required. As we won't be interested in the Green's functions coupled to $\theta(x)$, we will generally put the external field $\theta(x)$ equal to zero

$$\theta(x) \;=\; 0\,, \quad \det U(x) \;=\; 1\,. \tag{1.69}$$

The matrix field $U(x)$ can then be parametrized

$$U(x) \;=\; e^{\frac{i}{F_0}\phi^a(x)\lambda^a}. \tag{1.70}$$

To allow the proper normalization of the Goldstone boson fields, F_0 is a parameter left to be fixed later. (1.69) permits to make v_μ and a_μ traceless and introduce the covariant derivative as

$$D_\mu U \;=\; \partial_\mu U - i[\,v_\mu, U] - i\{\,a_\mu, U\} \tag{1.71}$$

$$v_\mu \;=\; \frac{1}{2}v_\mu^a \lambda^a, \quad a_\mu \;=\; \frac{1}{2}a_\mu^a \lambda^a\,. \tag{1.72}$$

Using the covariant derivative is a convenient way to guarantee the correct transformational properties of the effective Lagrangian under the local symmetry group.

As an alternative possibility, which we will not use, one can retain the ninth degree of freedom associated with the vacuum angle and the $U(1)_A$ anomaly and incorporate it in the effective theory

$$U(x) \;=\; e^{\frac{1}{3}i\phi_0(x)}e^{\frac{i}{F_0}\phi^a(x)\lambda^a}. \tag{1.73}$$

This gives rise to a ninth pseudoscalar meson which can be identified with the η'.

The next step is to find the most general form of the effective Lagrangian with the same transformation properties as the original generating functional (1.58). Its building blocks will be the matrix field $U(x)$ and as an effective theory, it has to be constructed as an expansion in covariant derivatives of the fields. Lorentz invariance permits terms with only even number of derivatives

$$\mathcal{L}_{eff} \;=\; \mathcal{L}^{(0)} \;+\; \mathcal{L}^{(2)} \;+\; \mathcal{L}^{(4)} \;+\; \ldots\,. \tag{1.74}$$

These lead to terms of order $O(p^n)$ in the S-matrix elements. Chiral invariance also limits the occurrence of vector and axial vector sources to the covariant derivatives and field strength tensors. However, the densities, and the quark mass matrix $\mathcal{M}$ in particular, can appear due to (1.66) in various combinations with the meson matrix field $U(x)$. This is expected - the symmetry is only an approximate one and the terms depending on the quark masses correspond to the symmetry breaking part of the QCD Lagrangian. The resulting effective Lagrangian is then a double expansion, in derivatives and quark masses. We will have a closer look on how to incorporate the quark masses into the power counting in the next chapter. As we will see, in the Generalized scheme they break the 'even only terms' character of the expansion.

From the resulting generating functional in the effective theory framework

$$e^{iZ_{eff}[\phi,v,a,s,p]} \;=\; \int \mathcal{D}\phi \; e^{i\int \mathrm{d}^4x \;\mathcal{L}_{eff}[\phi,v,a,s,p]} \tag{1.75}$$

it's generally possible to obtain contributions to the Green functions and S-matrix elements with arbitrary number of loops. The traditionally used loop expansion [7] rests on the reparametrization

$$U(\Phi) \;=\; u(\Phi)e^{i\xi}u(\Phi)\,, \tag{1.76}$$

where Φ in $U = u^2$ are the classical fields as solutions of the leading order equations of motion, while ξ are the quantum perturbations.

The generating functional (1.75) as a whole is non-renormalizable, as is immediately clear from the fact that it contains infinite number of terms. However, it's easily renormalizable up to any given finite order in momenta. The order of a connected graph, chiral dimension, was derived in [5]

$$D \;=\; 2 \;+\; 2L \;+\; \sum_n (n-2)N_n\,. \tag{1.77}$$

L is the number of loops and N_n the number of vertices of order $O(p^n)$ contained in the graph. This formula gives a guide how to count different graphs and which of them contribute to the required order. From (1.77) follows, that to the divergences of every loop graph, the relevant counter terms can always be found in contributions from some higher order terms in the chiral expansion of the effective Lagrangian, which have the same chiral dimension. Order by order, the divergences are then renormalized by the effective low energy coupling constants.

Chapter 2

χPT power counting schemes

This chapter proceeds to the next step in building the effective theory framework - the construction of the concrete form of the effective Lagrangian and its subsequent practical handling. As indicated in the previous section, there is no ambiguity in the derivative part of the expansion governed by Lorentz invariance. On the other hand, the symmetry breaking part proportional to powers of quark masses is a different matter entirely and a power counting scheme has to be introduced to sew the two expansions together.

We will discuss the substance from a wider point to view, with the focus being to offer some understanding about the motivations of introducing the competing approaches to χPT, maybe with a bit of historical perspective as well. The technical details will be largely postponed to the second part of the work, where the differences will be demonstrated together with our own calculations and results.

2.1 Standard χPT

First it should be noted that there are several distinct χPT versions depending on the symmetry group considered. Two flavor χPT [6] is based on the $SU(2)_L \times SU(2)_R$ approximate symmetry and describes the interactions of pions solely. It is valid only up to the kaon production threshold, but on the other hand the expansion in quark masses is expected to converge well as only the two lightest quarks are taken into account.

If not stated otherwise, we will concentrate on the $SU(3)_L \times SU(3)_R$ framework [7], which involves the s-quark too. The theory also includes kaon and eta states and the range of validity is pushed up to the lightest resonances, unless they are included explicitly (see chapter 5). The effective Lagrangian contains more terms at each order due to the higher complexity of the symmetry group and the much heavier s-quark mass might cause the symmetry breaking part to converge more slowly.

The $U(3)_L \times U(3)_R$ approach [7, 8] also includes η' as a degree of freedom and is based on a simultaneous chiral and large N_c expansions. Though we will touch it in chapter 7, it is largely out of the scope of this work.

Focusing on the $SU(3)$ three flavor case [7], the form of the Lagrangian is obtained by finding all possible terms compatible with constraints dictated by the symmetry properties. As follows from the previous chapter, when using the building blocks

$$U(x) \;=\; \exp\frac{i}{F_0}\,\phi^a(x)\lambda^a, \;\; \mathcal{M} \;=\; \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}, \tag{2.1}$$

the Lagrangian has to be invariant under the transformation (1.68,1.66)

$$U'(x) \;=\; U_R U(x) U_L^+ \tag{2.2}$$

$$\mathcal{M}' \;=\; U_R \mathcal{M} U_L^+ . \tag{2.3}$$

This leads to the following leading order contributions in both expansions

$$\mathcal{L}_{der}^{lead} \;=\; \frac{F_0^2}{4}\mathrm{Tr}[\partial_\mu U \partial^\mu U^+] \tag{2.4}$$

$$\mathcal{L}_{sb}^{lead} \;=\; \frac{F_0^2}{4}\,\mathrm{Tr}[2B_0(U^+\mathcal{M} + \mathcal{M}^+ U)]\,. \tag{2.5}$$

It is straightforward to calculate

$$M_\pi^2 \;=\; B_0(m_u + m_d) + O(m_q^2). \tag{2.6}$$

As we can see, because $M_\phi^2 \sim B_0 m_q$, $B_0\mathcal{M}$ can be used as a an expansion parameter of order $O(p^2)$ rather than $\mathcal{M}$ itself, which is much harder to grasp at. This is the basic motivation behind the *Standard power counting scheme (Standard χPT, SχPT)*. Including now also the external sources introduced in the previous section, it can be written down as

$$\mathcal{L}^{(2(k+l))} \;\sim p^{2k}\chi^l, \quad \chi = 2B_0(s + ip). \tag{2.7}$$

The covariant derivative is also of order $O(p)$ in momenta, thus the vector and axial vector sources are counted as $O(p)$ as well.

'Odd order' terms are thus not allowed

$$\mathcal{L}_{eff} \;=\; \mathcal{L}^{(2)} \,+\, \mathcal{L}^{(4)} \,+\, \mathcal{L}^{(6)} \,+\, \ldots \tag{2.8}$$

$$\mathcal{L}^{(2)} \;=\; \frac{F_0^2}{4}\mathrm{Tr}[D_\mu U D^\mu U^+ \,+\, (U^+\chi + \chi^+ U)] \tag{2.9}$$

$$\mathcal{L}^{(4)} \;=\; \mathcal{L}^{(4)}(L_1 \ldots L_{10}) \;+\; \mathcal{L}_{WZ}^{(4)} \tag{2.10}$$

$$\mathcal{L}^{(6)} \;=\; \mathcal{L}^{(6)}(C_1 \ldots C_{90}) \;+\; \mathcal{L}_{WZ}^{(6)}(C_1^W \ldots C_{23}^W)\,. \tag{2.11}$$

The explicit form of $\mathcal{L}^{(4)}$ can be found in [7], the non-anomalous part reads

$$\begin{aligned}
\mathcal{L}^{(4)}(L_1 \dots L_{10}) \;=\;& L_1 \,\mathrm{Tr}[D_\mu U^+ D^\mu U]^2 \,+\, L_2 \,\mathrm{Tr}[D_\mu U^+ D_\nu U]\,\mathrm{Tr}[D^\mu U^+ D^\nu U] \,+ \\
&+ L_3 \,\mathrm{Tr}[D_\mu U^+ D^\mu U D_\nu U^+ D^\nu U] \,+\, L_4 \,\mathrm{Tr}[D_\mu U^+ D^\mu U]\,\mathrm{Tr}[\chi^+ U + \chi\, U^+] \\
&+ L_5 \,\mathrm{Tr}[D_\mu U^+ D^\mu U(\chi^+ U + U^+ \chi)] \,+\, L_6 \,\mathrm{Tr}[\chi^+ U + \chi\, U^+]^2 \,+\, L_7 \,\mathrm{Tr}[\chi^+ U - \chi\, U^+]^2 \\
&+ L_8 \,\mathrm{Tr}[\chi^+ U \chi^+ U + \chi\, U^+ \chi\, U^+] \,-\, iL_9 \,\mathrm{Tr}[F_R^{\mu\nu} D_\mu U D_\nu U^+ + F_L^{\mu\nu} D_\mu U^+ D_\nu U] \\
&+ L_{10} \,\mathrm{Tr}[U^+ F_R^{\mu\nu} U F^L_{\mu\nu}], \qquad (2.12)
\end{aligned}$$

where

$$F_{R,L}^{\mu\nu} \;=\; \partial^\mu (v^\nu \pm a^\nu) - \partial^\nu (v^\mu \pm a^\mu) - i[v^\mu \pm a^\mu, v^\nu \pm a^\nu]. \qquad (2.13)$$

The contributions to $\mathcal{L}^{(6)}$ were published in [9] and [10].

Using the formula (1.77), the renormalized $O(p^4)$ couplings can be introduced [7]

$$L_i \;=\; L_i^r(\mu) + \frac{\Gamma_i}{4\pi^2}\,\mu^{d-4}\left[\frac{1}{d-4} - \frac{1}{2}(\ln 4\pi + \Gamma'(1) + 1)\right] \qquad (2.14)$$

and similarly for the rest of the *low energy coupling constants (LEC's)* defined in this and the next section. The numerical factors Γ_i can be found in the cited references.

Predictions for desired observables can be obtained from the Lagrangian in the usual way. They will be expressed in the form of a *chiral expansion* in terms of the low energy coupling constants parameterizing the Lagrangian. However, because the values of LEC's are not known from the first principles, the theory would have no predictive power without an additional input. Therefore some experimentally well known quantities, such as pseudoscalar meson masses and decay constants, are used to fix the LEC's. This is most often done in the following way - the chiral expansions of the known observables are inverted to express the LEC's as functions of the observables.

In calculations done to the $O(p^4)$ order, usually all $O(p^2)$ LEC's are fitted in this way. More specifically, in the isospin limit there are three free parameters at the leading order: F_0, $2B_0\hat{m} = B_0(m_u + m_d)$ and $r = \hat{m}/m_s$. These can be expressed in terms of F_π, M_π and M_K, for example. Similar fits for most of the $O(p^4)$ LEC's $L_1 \dots L_{10}$ exist [7, 11, 12]. At the $O(p^6)$ order the things are different, the number of low energy constants is too large to be manageable in such a way. These must be estimated using other methods, for example resonance saturation [12].

2.2 Generalized counting

In the first half of the nineties, roughly a decade after the foundations of χPT were laid down [5, 6, 7], a new way of power counting was proposed in a series of papers [13, 14, 15, 16]. It was noticed that taking $\chi = 2B_0\mathcal{M}$ as an expansion parameter included an underlying assumption that could not be fully justified. Clearly, the Standard counting works if $B_0 \sim O(1)$ and $\mathcal{M} \sim O(p^2)$. However, if B_0, related to the chiral condensate as

$$B_0(N_f)F_0^2(N_f) \;=\; \Sigma(N_f) \;=\; -\,\langle\, 0\,|\bar{q}q\,|\,0\,\rangle_{m_q\to 0}, \quad q = u, d(, s), \tag{2.15}$$

was sufficiently small itself, terms quadratic in the quark masses could be comparable to those proportional to $B_0 m_q$ in the chiral expansion. For example, $2B_0\hat{m}$ would no longer dominate the pion mass expansion (2.6). It was argued that the smallness of the condensate does not contradict any known physical principle as the spontaneous symmetry breaking is guaranteed by the nonzero value of the pseudoscalar decay constant in the chiral limit F_0 (1.52). The behavior of the theory with a large value of B_0 with its linear response to nonzero quark masses was compared to a ferromagnet, while the nonlinear case with a small quark condensate reminded the antiferromagnet.

In other words, in this scenario both $\mathcal{M}$ and B_0 should be counted as $O(p)$, which is the basic starting point of the *Generalized power counting (Generalized χPT, GχPT)*

$$\bar{\mathcal{L}}^{(2k+l+n)} \;\sim p^{2k}\mathcal{M}^l B_0^n. \tag{2.16}$$

The expansion now allows also odd terms in the counting

$$\mathcal{L}_{eff} \;=\; \bar{\mathcal{L}}^{(2)} \,+\, \bar{\mathcal{L}}^{(3)} \,+ \bar{\mathcal{L}}^{(4)} \;+\;\ldots \tag{2.17}$$

and the leading order includes contributions quadratic in the quark mass expansion

$$\begin{aligned}\bar{\mathcal{L}}^{(2)} \;&=\; \frac{F_0^2}{4}[\mathrm{Tr}[D_\mu U D^\mu U^+] \,+\, 2B_0\,\mathrm{Tr}[\mathcal{M}(U^+ + U)] \,+\, A_0\,\mathrm{Tr}[(U^+\mathcal{M})^2 + (\mathcal{M}^+U)^2] \\ &\qquad + Z_0^S\,\mathrm{Tr}[U^+\mathcal{M} + \mathcal{M}^+U]^2 \,+\, Z_0^P\,\mathrm{Tr}[U^+\mathcal{M} - \mathcal{M}^+U]^2]\end{aligned} \tag{2.18}$$

$$\bar{\mathcal{L}}^{(3)} \;=\; \bar{\mathcal{L}}^{(3)}(\xi, \bar{\xi}, \varrho_1 \ldots \varrho_7) \tag{2.19}$$

$$\bar{\mathcal{L}}^{(4)} \;=\; \bar{\mathcal{L}}^{(4)}(L_1, L_2, L_3, A_i \ldots F_i) \;+\; \mathcal{L}_{WZ}^{(4)}. \tag{2.20}$$

The complete form of the Generalized Lagrangian to $O(p^4)$ was never published as an intact piece, thus we include it in its full length in section 8.13.

As can be seen, the alternative counting merely reorganizes the chiral expansion, both theories are equivalent to all orders. The Standard expansion is a subset of the Generalized one, as that contains all the Standard terms at each order plus some contribution originally of higher orders in the Standard counting.

The practical handling of the expansion is analogous to the Standard case as described in the previous section with the exception that as more parameters are available at the leading order, one or two of them are left free to parametrize the difference between the Standard and the Generalized case. One possibility is to leave r free and consider the Zweig rule violating parameter $\zeta = Z_0^S/A_0$ small, as was done originally [14, 15]. To account for the possibility of large N_c limit violation, instead of using ζ, $Y = 2B_0\hat{m}/M_\pi^2$ can be left free as well. We used that approach in the second part of the work.

$\pi\pi$ scattering was proposed as a crucial experimental test of the character of the spontaneous chiral symmetry breaking [14, 15]. At the time, there was a perceived one standard deviation discrepancy between the Standard χPT prediction for the s-wave scattering length $a_0 = 0.20 \pm 0.01$ [17, 18] and the then available data $a_0 = 0.26 \pm 0.05$ [19]. The Generalized approach could account for that if the chiral condensate was smaller than expected.

However, new precision data from Brookhaven [20] gave $a_0 = 0.228 \pm 0.012 \pm 0.003$, which was in good agreement with the more precise two loop Standard calculations $a_0 = 0.219 \pm 0.005$ [12]. This was interpreted as a proof that the Standard counting is sufficient and Generalized calculations with the large number of unknown LEC's seemed redundant.

2.3 Resummed approach

The implications of the K_{e4} experimental results [20] mentioned in the previous section could be more subtle than it might seem at first sight. An important point is that they are directly related to predictions of two flavor χPT only. So the main outcome is that there is no experimental support for a small condensate scenario in this case and the two flavor Standard chiral series appear to converge in a satisfying fashion. However, whether this consequence can be stretched to the three flavor case is a more elaborate question.

Significantly, the complete Standard χPT procedure assumes more than just the large magnitude of the quark condensate. This is not only a question of the power counting but rather the subsequent treatment of the chiral expansion which in the stage of LEC reparametrization involves the inversion of the series and truncation of the higher orders. This effectively means a reordering of the expansion. Such a handling is of course usual when dealing with perturbative expansions of any sort, but can break down when the series does not converge well. Specifically, leading order terms should dominate the chiral expansion and thus none of $O(p^2)$ LEC's can be small. In terms of convenient parameters relating the LEC's to physical observables

$$Z(N_f) = \frac{F_0(N_f)^2}{F_\pi^2}, \quad X(N_f) = \frac{2\hat{m}\Sigma(N_f)}{F_\pi^2 M_\pi^2}, \tag{2.21}$$

where N_f is the number of light quark flavors, this condition for SχPT to work can be expressed as

$$Z \sim 1, \quad X \sim 1, \quad Y = \frac{X}{Z} \sim 1, \quad r \sim r_2 = 2\frac{M_K^2}{M_\pi^2} - 1 = 25.9. \tag{2.22}$$

Analysis [21] of the $\pi\pi$ experimental data [20] leads to

$$X(2) = 0.81 \pm 0.07, \quad Z(2) = 0.89 \pm 0.03 \tag{2.23}$$

while the three flavor parameters are constrained much less strictly [22]

$$X(3) \sim 0 - 0.8, \quad Z(3) \sim 0.3 - 0.9, \quad r > 14, \quad Y < 1.2. \tag{2.24}$$

Recent combined $\pi\pi$ and πK data examination [23] yield similar restrictions

$$X(3) \sim 0 - 0.8, \quad Z(3) \sim 0.2 - 1, \quad r > 15, \quad Y < 1.1. \tag{2.25}$$

Thus the possibility of a different two flavor and three flavor theory behavior is open and we will look in the background of such a scenario in detail in the next chapter. We can now simply demonstrate what would be the consequences if the three flavor theory possessed such an irregularity. Let's assume for a chiral expansion of an observable

$$A \;=\; A^{(2)} + A^{(4)} + \Delta_A^{(6)} \tag{2.26}$$

that the leading and next-to-leading terms are comparable, though the series otherwise converges well and the remainder $\Delta_A^{(6)}$ is small

$$A^{(2)} \sim A^{(4)}, \quad \Delta_A^{(6)} \ll A. \tag{2.27}$$

If we construct an expansion of the quantity $1/A$

$$\frac{1}{A} \;=\; \frac{1}{A^{(2)}} - \frac{A^{(4)}}{A^{(2)2}} + \Delta_{1/A}^{(6)}, \quad \Delta_{1/A}^{(6)} \;=\; \frac{1}{A}\left[\left(\frac{A^{(4)}}{A^{(2)}}\right)^2 + \Delta_A^{(6)}\left(\frac{A^{(4)} - A^{(2)}}{A^{(2)2}}\right)\right], \tag{2.28}$$

we can see that the higher order remainder is large $\Delta_{1/A}^{(6)} \sim 1/A$ and the series does not converge well, if at all. Therefore, under the condition (2.27), both observables A and $1/A$ cannot have a chiral expansion with satisfying convergence at the same time. Quite generally, if one dealt with a situation (2.27) with any kind of perturbative expansion, choosing the right quantity to expand is crucial and operations like (2.28), which reorder the series, should be consistently avoided.

This is the basic motivation behind the introduction of *Resummed χPT (RχPT)* [22, 23], which aims to deal with a possibility of instability in the three flavor chiral expansion. Instead of changing the power counting, as in the case of GχPT, the essence of the method is a special caution when dealing with the chiral series. The procedure can be shortly summarized as:

- Standard power counting and form of the effective Lagrangian
- Assumes possible irregularities in the expansion
- Only "bare" expansions obtained directly from the generating functional trusted
- Reparametrization of the LEC's done in a non-perturbative algebraic way
- Higher order remainders are kept and estimated, treated as sources of theoretical error

The approach will be described in more technical detail in chapter 8.

Chapter 3

Character of SBχS

When the systematic expansion of the effective Lagrangian was established in the first half of the eighties [5, 6, 7], very little was know about the structure of the QCD vacuum and the character of the spontaneous breaking of chiral symmetry *(SBχS)*. The motivation to introduce a different counting as Generalized χPT roughly a decade later [13, 14, 15, 16], was the fact that there was still neither experimental nor theoretical guide to this problem. Considerable progress was achieved since then which allows us to paint a much wider and deeper picture today.

3.1 Phase structure of QCD with N_f light flavors

There has been some interest in exploring QCD and related theories for a general number of flavors of light fermions [24, 25, 26, 27, 28]. As it's known for a long time, the renormalization group equation for the QCD running coupling constant $\alpha(\mu) = g(\mu)^2/4\pi$

$$\beta(\alpha) \;=\; \mu\frac{\partial}{\partial\mu}\alpha(\mu) \;=\; -b\alpha^2(\mu) - c\alpha^3(\mu) - d\alpha^4(\mu) - \ldots \tag{3.1}$$

yields

$$b \;=\; \frac{1}{6\pi}(11N_c - 2N_f) \tag{3.2}$$

$$c \;=\; \frac{1}{24\pi^2}(34N_c^2 - 10N_cN_f - 3\frac{N_c^2-1}{N_c}N_f)\ . \tag{3.3}$$

Higher order coefficients are renormalization scheme dependent. The sign of the leading order term b determines the ultra violet behavior of the coupling, the theory is asymptotically free if $N_f < N_f^A = 11/2N_c$. There is a non-trivial infrared fixed point of the coupling for $b > 0$, $c < 0$ ($N_f \sim 8 - 16.5$ for $N_c = 3$)

$$\alpha_* \;\cong\; -\frac{b}{c}\ , \tag{3.4}$$

which can be made arbitrarily small for N_f being sufficiently close to N_f^A. This makes the perturbation theory reliable when approaching N_f^A from bellow and that means there is a conformal window in some region

$$N_f^c \; < \; N_f \; < N_f^A \; . \tag{3.5}$$

In other words, as in this case the theory is only weakly interacting in the infrared, there is asymptotic freedom, but neither confinement nor SBχS. At some N_f^c the strength of the coupling exceeds the critical value for the triggering the SBχS and a chiral phase transition occurs. This is thought to give rise to the hadronic poles in the spectrum and to the confinement of the quarks. The influence of the number of different quark flavors on the infrared character of the theory can be viewed as quark loop screening - the light quark loops screen the confining self-interactions of the gluons and as the number of flavors is rising, at first they weaken the chiral symmetry breaking, then restore it and deconfine the quarks, and finally spoil asymptotic freedom too.

Various methods has been tried out to estimate the critical number of flavors N_f^c. Analyses based on a gap equation in RG-improved ladder approximation [24, 25] give a large value $N_f^c \approx 12$ for $N_c = 3$, but they are based on an assumption that the perturbation expansion still makes sense at N_f^c. The criterion of superconvergence of the gluon propagator used by [26] gives a similarly large number $N_f^c = 9.75$, they also showed that this value is in the range where the perturbative β-function is causal. On the other hand, non-perturbative approaches provide quite different results. in [27] it was calculated the contribution of instanton–anti-instanton pairs to the vacuum energy and found that it oscillates for $N_f > 5$, which confirms earlier results that instantons alone cannot support the condensate for $N_f > 4$. Most recently, [28] found that numerical solutions of Dyson-Schwinger equations for quark, gluon and ghost propagators do not converge for $N_f \geq 5$, possibly indicating that the chiral phase transition occurs for very low $3 < N_f^c \leq 5$.

Several calculations on the lattice were done with varying number of unquenched quarks. Initial calculations of the Columbia group [29, 30] compared $N_f = 2$ and $N_f = 4$ cases and provided surprising results. Parity doublet restoration was found for both vector mesons and baryons in the chiral limit for $N_f = 4$ and the two fundamental order parameters, the Goldstone boson decay constant $F(N_f)$ and the chiral condensate $\Sigma(N_F)$ were found to be heavily suppressed - $F(4) \approx 1/2F(2)$, $\Sigma(4) \approx 1/4\Sigma(2)$. Computations were later confirmed [31]. However, latest results [32] obtained by using a larger lattice, though providing similar outcomes otherwise, showed doublet splitting of order 20% in the chiral limit. Unfortunately, the group did not publish corrected values for the pseudoscalar decay constant and the chiral condesate. Iwasaki et al. performed calculations with a largely varying number of light quarks [33, 34] and consequently tried to identify the UV and/or IR fixed point to conclude the phase structure of the theory in the continuum limit. Their conclusion is that there is a wide conformal window for $16 \geq N_f \geq 7$ and both confinement and SBχS exist only for $N_f \leq 6$.

The wider image of the light quark QCD phase structure has a close connection to the large N_c approximation too. It's straightforward to see that in this light it can be considered more as an $N_f/N_c \to 0$ limit and it should be useful in cases when the effect of the quark loops is suppressed. On the other hand, if one wanted to assume a situation when the quark loops cannot be neglected, that would consequently lead to the large N_c limit and the connected

Zweig rule to be violated. Therefore occurrence or absence of this violation can be expected as an indication whether such an assumption is based on solid ground.

3.2 Consequences for χPT

The phase structure of QCD with varying numbers of light quarks sheds a new light on the dispute about the correct χPT counting. The magnitude and character of SBχS might be different in $SU(2)$ and $SU(3)$ chiral limits. The fact that both limits are useful in practical applications is not trivial and can be tracked back to the quark mass pattern

$$m_{u,d} \ << \ m_s \sim \Lambda_{QCD} \ << \ \lambda_H \ < \ m_{c,b,t} \ . \tag{3.6}$$

The s-quark mass is small enough compared to the typical hadron scale λ_H to allow for an expansion around the $SU(3)$ chiral limit. Furthermore, it is of order of the QCD scale λ_{QCD} which means that the $\bar{s}s$ quark loops are not as suppressed as the heavy quark loops and therefore could still play a significant role in low energy hadron physics. On the other side, the mass is large enough to be considered heavy compared to the two lightest quark masses and thus an $SU(2)$ chiral limit makes good sense too. Hence the s-quark plays a quite special role in strong interaction physics.

The possible consequences of the QCD phase structure were first realized in [35], when expressing the so-called paramagnetic inequality

$$F_0(N_f+1) \ < \ F_0(N_f), \quad \Sigma(N_f+1) \ < \ \Sigma(N_f). \tag{3.7}$$

Using the spectral decomposition of the Dirac operator in a finite box it was also derived that the the chiral condensate should be more sensitive to the number of fermion flavors.

It is useful to quantify the relation (3.7) for the chiral condensate by introducing the correlation function [35]

$$\Pi(m_s) \ = \ \frac{\partial}{\partial m_2}\,\Sigma(N_f) \ = \ \lim_{m_f\to 0} \int \mathrm{d}x \langle\, 0\,|\bar{u}u(x)\bar{s}s(0)\,|\,0\,\rangle \ , \tag{3.8}$$

where s is in general the $(N_f{+}1)$-th quark. The paramagnetic inequality then takes the form

$$\Sigma(N_f) \ = \ \Sigma(N_f+1) + \int_0^{m_s} \mathrm{d}\mu\,\Pi(\mu). \tag{3.9}$$

A few comments are in order here. As it can be seen from the definition of $\Pi(m_s)$ (3.8), the difference between the $SU(N_f)$ and $SU(N_f+1)$ chiral condensates is determined by a Zweig rule violating matrix element directly tied to the scalar sector. As will be discussed in more detail in the following section, such a violation is well known to take place. The argument can also be overturned, could the Zweig rule violation observed in the decays of the scalar resonances be linked to a distinction between the character of the SBχS in the 2-flavor and 3-flavor cases? In connection, $\Pi(m_s)$ is also suppressed in the large N_c limit, which is not surprising given its relation to the N_f dependence of the chiral symmetry breaking. It does not mean it's negligible though, if its influence was found to be pronounced, it would indicate a case when the large N_c approximation fails.

The inequalities (3.7) can be directly calculated using the χPT framework

$$F_0(2)^2 = F_0(3)^2 + 16 m_s B_0 L_4^r - 2\bar{\mu}_K + O(m_s^2) \tag{3.10}$$

$$\Sigma(2) = \Sigma(3)(1 + \frac{32 m_s B_0}{F_0^2} L_6^r - 2\bar{\mu}_K - \frac{1}{3}\bar{\mu}_\eta) + O(m_s^2) \tag{3.11}$$

where L_4 and L_6 stem in the Standard $O(p^4)$ Lagrangian and can be equivalently expressed using the Generalized $\bar{\mathcal{L}}^{(2)}$ and $\bar{\mathcal{L}}^{(3)}$ constants Z_0^S and $\bar{\xi}$

$$\bar{\xi} = \frac{8B_0}{F_0^2} L_4 \tag{3.12}$$

$$Z_0^S = \frac{16B_0^2}{F_0^2} L_6 \ . \tag{3.13}$$

These LEC's were traditionally considered small and neglected as they relate to Zweig rule violation and are suppressed in the large N_c limit [7]. Their value was considered zero either at the scale of M_η [7] or M_ρ [12].

The paramagnetic inequalities (3.7) induce lower bounds for L_4^r and L_6^r, their value is $L_6^r(M_\rho) \geq -0.21.10^{-3}$ and $L_4^r(M_\rho) \geq -0.37.10^{-3}$ [36]. It was shown that the physical values have to be in a narrow band around these values in order to not to lead to a suppression of the three flavor order parameters. QCD sum rules for the scalar form factors indicate a positive value for L_6 though [36, 37].

QCD sum rules were also used to estimate the magnitude of the L_4^r and L_6^r LEC's in [38, 39]. The results were $L_4^r(M_\eta) = (0.6 \pm 0.2).10^{-3}$, $L_6^r(M_\eta) = (0.5 \pm 0.3).10^{-3}$. Moreover, utilizing a linear sigma model for scalar resonances in order to saturate the LEC's in question yielded $L_4^r = 0.55.10^{-3}$ and $L_6^r = 0.31.10^{-3}$ for the largest possible value of the κ mass $m_\kappa = 1.35$GeV allowed by the model. As the physical mass is 1430MeV and the results are quickly rising with m_κ, this can be viewed as a lower bound.

Scalar form factors were analyzed in Standard $O(p^6)$ χPT as well [40] and a preferred region of L_4^r and L_6^r around $L_4^r(M_\rho) \sim 0.45.10^{-3}$ and $L_6^r(M_\rho) \sim 0.15.10^{-3}$ was found.

Calculations on the lattice [41, 42] found positive values for L_4 and L_6 too, latest simulations gave $L_4 = 0.4(3)(+3\text{-}1)$ and $L_6 = 0.4(2)(+2\text{-}1)$.

The parameters introduced in the previous chapter

$$Z(N_f) = \frac{F_0(N_f)^2}{F_\pi^2}, \quad X(N_f) = \frac{2m_f\Sigma(N_f)}{F_\pi^2 M_\pi^2}, \tag{3.14}$$

can conveniently measure the character of SBχS. $X(N_f)$ is a ratio by which the chiral condensate contributes to the pion mass, traditionally considered close to one (the Gell-Mann–Oakes–Renner relation). Much smaller number would be an indication for the condensate not being dominant relative to higher order corrections. Similarly, $Z(N_f)$ is the suppression ratio of the decay constant in the N_f chiral limit compared to the physical pion decay constant.

Several estimates and data analyses were made in order to constrain the value of the parameters in question. [35] showed that $X(2)$ is largely insensitive to the value of $X(3)$. It was also demonstrated [37, 36] that while both $X(2)$ and $Z(2)$ are almost independent of the value of L_4 and L_6, the modest shift of these LEC's from their lower bounds $L_4^c \geq -0.37.10^{-3}$, $L_6^c \geq -0.21.10^{-3}$, given by the paramagnetic inequalities (3.7), toward small positive values induces a profound change in the behavior of $X(3)$ and $Z(3)$, which can be as small as roughly 0.5 for $r \sim 25$. Results obtained by using the QCD sum rule method were $X(2), Z(2) \sim 0.9$ and $X(3), Z(3)$ ∼0.5-0.6 for $r \sim 25$. The two flavour parameters $X(2), Z(2)$ were found stable [43] provided that the quark mass ratio r is not too small $r < 15$.

The result of the already mentioned precise measurement of the a_0^0 π-π scattering length obtained by the E865 collaboration on the K_{e4} decay ($K^+ \to \pi^+\pi^-e^+\nu_e$) [20] $a_0^0 = 0.228 \pm 0.012 \pm 0.003$ agrees with the Standard χPT prediction very well and also confirms a dominant $SU(2)$ chiral condensate [44]. The analysis of the data [21] lead to $X(2) = 0.81 \pm 0.07$ and $Z(2) = 0.89 \pm 0.03$. A rough estimate for the $SU(2)$ effective Lagrangian LEC ℓ_3 is possible, the value $\ell_3 = -18 \pm 15$ deviates from the previously used $\ell_3 = 2.9 \pm 2.4$ by a possibly large margin. This might indicate important corrections from the L_4 and L_6 $SU(3)$ LEC's, which contribute to ℓ_3.

3.3 Scalar mesons

There is an enigmatic multiplet of particles which could be intimately tied to the character SBχS, the scalar mesons. The firmly experimentally established light scalars [45] can be organized in the following way, according to isospin

- I=1 triplets: $a_0(980)=\delta$, $a_0(1450)$
- I=1/2 doublets: $K_0^*(1430)=\kappa$
- I=0 singlets: $f_0(600)=\sigma$, $f_0(980)=S^*$, $f_0(1370)$, $f_0(1500)$, $f_0(1710)=\theta$

The physical existence of several more resonant states is hotly debated (e.g. [46]).

These resonances are special for several reasons. There is of course too much of them to fit in a single $SU(3)_V \times U(1)_V$ nonet. The underlining valence quark structure is notoriously hard to decompose, the scalars do not form an ideally mixed multiplet - the mass spectrum and the decay patterns does not allow for the usual non-relativistic constituent quark interpretation and the Zweig rule is violated. The resonances are broad, the poles in the S-matrix are far away from the real axis.

For illustrative purpose we will adopt one of the models constructed in effort to explain the intriguing character of the scalars, as introduced by Tornquist and extended by Boglionne and Pennington [47]. Several more or less similar approaches have been developed. It is assumed that there are well defined constituent quarks, as current quarks altered by the cloud of gluons, by which they gain their large constituent masses. The color singlet states, "the bare seeds", are composed from constituent quarks and have well defined propagators with poles infinitesimally close to the real axis

$$P_0 \;=\; \frac{1}{m_0^2 - s}\;. \tag{3.15}$$

The second assumption is that the influence of the sea quark loops on the bare seed states, resulting in observed hadrons, can be approximated by dressing them with hadronic interactions. The original state becomes a resonance, with the propagator shifted to

$$P(s) \;=\; \frac{1}{m^2(s) - s - im(s)\Gamma(s)} \;. \tag{3.16}$$

The physical hadron state is then a sum of the bare seed and infinite number of various interaction terms. In the case of the "well-behaved" hadrons, the sum of these terms represent only a small correction. In contrast, large contributions are supposed to dynamically generate the attributes of the scalar mesons. When the model is worked out by restricting the interactions to one-loop pseudoscalar meson ones and by fitting the bare seed mass and the coupling strength [47], it can be shown that one bare seed can give birth to more than one hadronic state and this can partly account for the excessive number of candidates for occupying the multiplet. Identifying the singlet states are still problematic though, as mixing between the bare seeds is possible and more exotic states containing valence gluons or even pure glueballs can come to play.

As a further illustration, we will consider a simplified model for the $a_0(980)$ [48]. As it's an isospin triplet, the quark composition of the bare seeds are identical to those of the pion triplet, which are then their parity partners. This means there are no s-quarks seeded, only u and d, with their various combinations denoted as $\bar{n}n$. The bare seed mass parameter is chosen as $m_0 = 1420$MeV. The result of the dressing is

$$|\, a_0(980) \,\rangle = (0.2)^{1/2} |\, \bar{n}n \,\rangle + (0.7)^{1/2} |\, \bar{K}K \,\rangle + (0.1)^{1/2} |\, \pi\eta' \,\rangle, \tag{3.17}$$

which fits the observed $a_0(980)$ well. One should not miss the strong influence of the s-quark loop hidden in the kaon contribution. It should also be noted that the dressing pushes the hadron mass substantially down.

While, to a degree, models like this can describe the observed pattern quite successfully, they rely on a fundamental phenomenological input which is left unexplained. The scalars couple to other hadrons, in particular to pseudoscalar mesons while ignoring the Zweig rule, very strongly. In other words, their content of sea quark loops seems to be high.

One should note that there doesn't seem to be anything wrong with the isospin multiplets. They are nearly degenerate as they should be. This might be an evidence that the $SU(2)$ chiral vacuum is not too much different from the quenched one with no dynamical quarks.

Chapter 4

Dispersive representation

The first two chapters concentrated on the foundations of the effective theory approach exploring the low energy properties of QCD. This one is focusing on the matter from a slightly different angle. Even if the effective theory described the interactions of the low energy bound states accurately, it might still be valuable to analyze the problem based only on the most general assumptions - the unitarity of the S-matrix, its basic analytic properties and crossing symmetry. Such a framework is being increasingly popular in recent literature dealing with low energy QCD. It was fully implemented in the case of $\pi\pi$-scattering first [14, 15], later also in the πK one [49, 50] and other cases. This chapter relies on the recent work [51, 52], where the general Reconstruction theorem valid for all two particle scattering processes was introduced.

The core of the method is the extension of the S-matrix into the complex plane in the Mandelstam variables. Along with crossing symmetry and some general assumptions about the analytic structure of the theory this allows one to write down a dispersion integral for the amplitude. Unitarity then constrains the form of the nonanalytic part of the relation up to a polynomial containing the details of the theory not accessible by this approach. Subsequently, the subtraction polynomial can be obtained by matching the dispersive and the usual χPT representations.

We applied the approach to the case of $\eta\pi$ scattering in the context of Resummed χPT (chapter 8). The goal was to fix the nonanalytic part of the amplitude which does not have the poles and cuts placed in correct physical location in its strict chiral expansion form. This chapter aims to be an introductory summary of the general procedure which will provide the necessary background to an unfamiliar reader for that part of the work.

4.1 Analyticity and crossing symmetry

Our focus will be on two particle scattering

$$\phi_1(p_1)\,\phi_2(p_2) \to \phi_3(p_3)\,\phi_4(p_4)\ , \tag{4.1}$$

with the amplitude parametrized using the standard Mandelstam variables

$$s \;=\; (p_1+p_2)^2 \;=\; (p_3+p_4)^2 \tag{4.2}$$

$$t \;=\; (p_1-p_3)^2 \;=\; (p_2-p_4)^2 \tag{4.3}$$

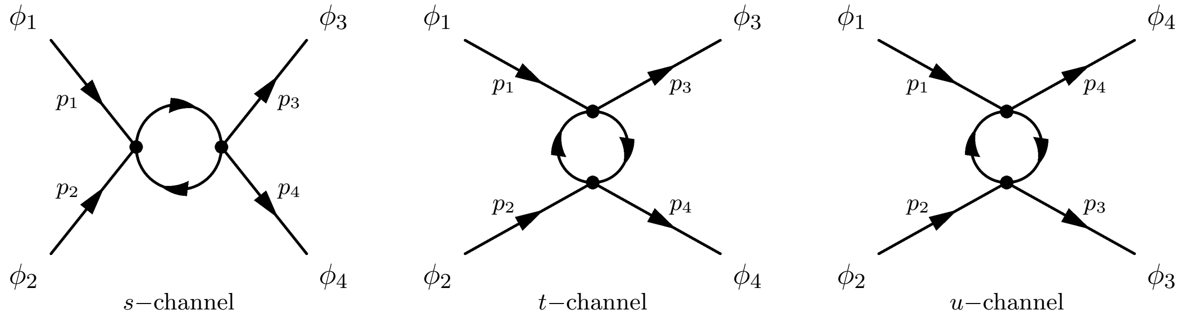

Figure 4.1: Contributions to the unitarity part of the amplitude at one loop.

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2 \tag{4.4}$$

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 = M^2. \tag{4.5}$$

We will limit ourselves to two particle intermediate states up to one loop. This condition is sufficient at one loop level in χPT regardless of the power counting and holds up to $O(p^6)$. At this order two loops have to be considered, but even in this case the following procedure can be generalized, in fact it was introduced as such already in [14]. At the $O(p^8)$ order three loop diagrams contribute and the two particle intermediate state limitation is generally not adequate.

There are of course other contributions, direct vertices and tadpoles, but these are analytical in nature and thus not interesting in our context. They will be tacitly assumed to be contained in the polynomial part of the amplitude.

Under these assumptions, graphs with only three basic types of topology contribute to the unitarity part of amplitude, conventionally named the s-, t- and u-channel. These are depicted in figure 4.1. The next step is to extend the amplitude $A(s,t,u)$ to the whole complex plane in the Mandelstam variables and identify the non-analyticities. We will assume the amplitude to be analytic except branch cuts generated by loops in fig.4.1.

It is most straightforward to analyze the s-channel contribution. As follows from the unitarity of the S-matrix, the amplitude has to start an imaginary part in s at the threshold of real particle production in the intermediate two particle states. Each threshold crossing generates a branching point. This produces cuts on the real axis for

$$s \in (m_s^2, \infty), \quad m_s^2 = \min_{(i,j)_s}(m_i + m_j)^2, \tag{4.6}$$

where $(i,j)_s$ run over all the possible intermediate virtual particle pairs in this channel.

The location of the branching points and associated cuts in the t- and u-channels is determined by crossing symmetry. From the general structure of relativistic QFT it follows that the amplitudes of the 'crossed' processes

$$\phi_1(p_1)\,\bar{\phi}_3(-p_3) \to \bar{\phi}_2(-p_2)\,\phi_4(p_4) \tag{4.7}$$

$$\phi_1(p_1)\,\bar{\phi}_4(-p_4) \to \phi_3(p_3)\,\bar{\phi}_2(-p_2) \tag{4.8}$$

must have the same form (up to a phase factor we omit)

$$A(s,t,u) \;=\; B(t,s,u) \;=\; C(u,t,s), \tag{4.9}$$

where amplitudes A, B and C correspond to the processes (4.1), (4.7) and (4.8) respectively. The analytic structure of the t-channel amplitude $A_t(s,t,u)$ in the t variable is thus the same as the s-channel one $B_s(s,t,u)$ in s and analogously for the u-channel. The location of the branch cuts therefore is

$$t \in (m_t^2, \infty), \quad m_t^2 = \min_{(i,j)_t}(m_i + m_j)^2 \tag{4.10}$$

$$u \in (m_u^2, \infty), \quad m_u^2 = \min_{(i,j)_u}(m_i + m_j)^2. \tag{4.11}$$

However, the Mandelstam variables are not independent but rather tied together by the condition (4.5). This transfers the cuts to additional locations. E.g., if we fix u at some point $u = u_0$ and express $t = M^2 - u_0 - s$, we will find extra cuts in s beyond (4.6) at

$$s \in (-\infty, M^2 - u_0 - m_t^2), \quad m_t^2 = \min_{(i,j)_t}(m_i + m_j)^2, \tag{4.12}$$

generated by the original discontinuity in t. The situation is similar for the rest of the combinations.

Now we can move on to demonstrate where lies the benefit of the analytic extension of the amplitude outside of the physical range of the parameters. For simplicity, we will derive the most elementary form of a dispersive relation for a hypothetical case of amplitude depending only on one variable instead of three. We will generalize this to the physical case in the last section of this chapter. Let us therefore assume the following analytic structure of a "scattering amplitude" $A(s)$:

- $A(s)$ has a branch cut along the real axis for $s > m^2$
- $A(s)$ is real along the real axis for $s < m^2$
- $A(s)$ is otherwise analytic in the whole complex plane

In other words, at this moment we are formally taking only the s-channel diagram discontinuity (4.6) into account.

The amplitude in any point s_0 can be calculated using Cauchy's theorem

$$A(s_0) \;=\; \frac{1}{2\pi i}\int_C \frac{A(s)}{s - s_0}\,\mathrm{d}s\,, \tag{4.13}$$

C is an arbitrary contour which encloses the point s_0. Therefore we can choose the integration path in the following way

$$A(s_0) \;=\; \frac{1}{2\pi i}\int_{m^2}^{\lambda^2} \frac{A(s+i\varepsilon)}{s - s_0}\,\mathrm{d}s \;+\; \frac{1}{2\pi i}\int_{\lambda^2}^{m^2} \frac{A(s-i\varepsilon)}{s - s_0}\,\mathrm{d}s \;+\; \frac{1}{2\pi i}\oint_{|s|=\lambda^2} \frac{A(s)}{s - s_0}\,\mathrm{d}s, \tag{4.14}$$

where $i\varepsilon$ denotes the infinitesimal amount by which the path is shifted above and under the real axis. After using Schwartz' reflection principle and a shift in the integration variable, the formula simplifies

$$A(s_0) \;=\; \frac{1}{\pi}\int_{m^2}^{\lambda^2} \frac{\mathrm{Im}A(s)}{s-s_0-i\varepsilon}\,\mathrm{d}s \;+\; \frac{1}{2\pi i}\oint_{|s|=\lambda^2} \frac{A(s)}{s-s_0}\,\mathrm{d}s\,. \tag{4.15}$$

The goal is to express the value of the amplitude in any given point s_0 using only its imaginary part. That would be achieved if

$$\lim_{\lambda^2\to\infty} \frac{1}{2\pi i}\oint_{|s|=\lambda^2} \frac{A(s)}{s-s_0}\,\mathrm{d}s \;=\; 0\,. \tag{4.16}$$

However, we did not assume that there are no poles at infinity and generally there might be some. The solution is to subtract them by using a different function $F(s)$, defined as the difference between the amplitude and its Taylor expansion

$$F(s) \;=\; \frac{1}{s^{n+1}}\sum_{k=0}^{n}[A(s) - A^{(k)}(0)\,\frac{s^k}{k!}\,]. \tag{4.17}$$

The expansion is done to the order n if the amplitude behaves at infinity as s^n. So when $F(s)$ is used in the preceding analysis, the resulting *dispersion relation* for the amplitude is

$$A(s_0) \;=\; P^{(n)}(s_0) + \frac{s_0^{n+1}}{\pi}\int_{m^2}^{\infty} \frac{\mathrm{d}s}{s^{n+1}}\frac{\mathrm{Im}A(s)}{s-s_0-i\varepsilon}\;, \tag{4.18}$$

where $P^{(n)}(s_0)$ is an n-th order *subtraction polynomial* in s. This relation recasts the assumptions about the analytical structure of the amplitude into a tactile expression. As we can see, if we were able to calculate the imaginary part, we could constrain its form up to a polynomial.

It is also useful to notice that if it is possible to factor a (real) polynomial $R^{(l)}$ out of the amplitude

$$A(s) \;=\; R^{(l)}(s)\bar{A}(s), \tag{4.19}$$

then less subtractions are needed by using the following function in the dispersive integral

$$\bar{F}(s) \;=\; \frac{1}{s^{n+1-l}}\sum_{k=0}^{n-l}[\bar{A}(s) - \bar{A}^{(k)}(0)\,\frac{s^k}{k!}\,] \tag{4.20}$$

and the integration simplifies in the final relation

$$A(s_0) \;=\; P^{(n)}(s_0) + \frac{s_0^{n+1-l}}{\pi}R^{(l)}(s_0)\int_{m^2}^{\infty} \frac{\mathrm{d}s}{s^{n+1-l}}\frac{\mathrm{Im}\bar{A}(s)}{s-s_0-i\varepsilon}\;. \tag{4.21}$$

4.2 Unitarity

After analyzing the analytic properties of the S-matrix, we will take its another basic attribute into account, the unitarity. The consequences will match up nicely with the outcome of the preceding section.

From the unitarity of the S-matrix

$$S^+S \;=\; 1 \tag{4.22}$$

follows for the transition matrix $iT = S - 1$

$$-i(T - T^+) \;=\; T^+T \;. \tag{4.23}$$

The amplitude A_{fi} can be introduced as

$$\langle\, f\,|T\,|\,i\,\rangle \;=\; (2\pi)^4 N_{P_f} N_{P_i}\, \delta^{(4)}(P_f - P_i)\, iA_{fi}\,, \tag{4.24}$$

where i and f denote the initial and final states, P_i and P_f the sum of momenta in these states. N_{P_i} and N_{P_f} are the products of the factors

$$N_p \;=\; \frac{1}{(2\pi)^{3/2}(2p_0)^{1/2}} \tag{4.25}$$

through all the momenta in the initial or final states

$$N_{P_i} \;=\; \prod_i N_{p_k}, \quad N_{P_f} \;=\; \prod_f N_{p_k}\,. \tag{4.26}$$

When inserting all possible intermediate states in (4.23), for the amplitude can be derived

$$-i(A_{fi} - A^*_{if}) \;=\; \sum_n (2\pi)^4 N_{P_n}\, \delta^{(4)}(P_n - P_i) A^*_{nf} A_{ni}\;. \tag{4.27}$$

P_n is the sum of momenta in the intermediate states, N_{P_n} is defined similarly to N_{P_i} and N_{P_f}. Because we are interested in strong interactions of mesons, we will assume time-reversal invariance, which leads to

$$2\,\mathrm{Im}\, A_{fi} \;=\; \sum_n\, (2\pi)^4 N_{P_n}\, \delta^{(4)}(P_n - P_i)\, A^*_{nf} A_{ni}\;. \tag{4.28}$$

This formula, the "Cutkosky rule", can be used to determine the imaginary part of the complete scattering amplitude directly from the simpler partial amplitudes involving the intermediate states.

The simplest and most often used instance of the rule is the two-to-two particle scattering with two particle intermediate states, as was introduced in the previous section. In this case it is convenient to use the partial wave decomposition

$$A_{fi}(s,\cos\theta_{fi}) \;=\; 32\pi\, N_{fi}\sum_{l}(2l+1)P_l(\cos\theta_{fi})A_l^{i\to f}(s)\,, \tag{4.29}$$

where $s = P_i^2 = P_f^2$ is the standard square of scattering energy in the center of mass, θ_{fi} the scattering angle and P_l the Legendre polynomials. N_{fi} is a normalization factor of the decomposition. The amplitudes involving the intermediate states can be treated in a similar way.

When the decomposition is inserted into the rule (4.28) and the right hand side is integrated over all possible values of the intermediate momenta, one can compare the coefficients standing in front of the Legendre polynomials. The result can be written as

$$\mathrm{Im}\, A_l^{i\to f}(s) \;=\; \sum_n \frac{2N_{ni}N_{fn}}{S}\,\frac{\lambda^{1/2}(s,m_{n_1}^2,m_{n_2}^2)}{s}\, A_l^{i\to n}(s)A_l^{n\to f}(s)^*. \tag{4.30}$$

The symmetry factor S is equal either to 2 or 1, depending on whether the intermediate states are indistinguishable or not. The masses standing in the standard triangle function $\lambda(s,m_{n_1}^2,m_{n_2}^2)$, defined by

$$\lambda(s,m_i^2,m_j^2) \;=\; s^2+m_i^4+m_j^4-2m_i^2s-2m_j^2s-2m_i^2m_j^2\;, \tag{4.31}$$

are the on-shell masses of the particles in the intermediate states.

4.3 Reconstruction theorem

The results of the last two sections can be directly generalized to the case of very low energy QCD. Because effective theories are expansions in terms of momenta, it's straightforward to determine how many subtractions has to be done in the dispersion relation (4.18). As the discussed approximation ceases validity at $O(p^6)$, the analytic, polynomial part of the amplitude can therefore be at maximum of second order in Mandelstam variables, which is equivalent to three subtractions needed generally.

As the first step of the generalization of the dispersion relation (4.18) to the case of three intertwined Mandelstam variables, one of them can be fixed. If one puts $u = u_0$ and expresses $t = M^2 - u_0 - s$, as already discussed in section 4.1, the amplitude will depend only on one variable s with the branch cuts placed at

$$s\in(m_s^2,\infty),\quad m_s^2=\min_{(i,j)_s}(m_i+m_j)^2 \tag{4.32}$$

$$s\in(-\infty,M^2-u_0-m_t^2),\quad m_t^2=\min_{(i,j)_t}(m_i+m_j)^2. \tag{4.33}$$

The dispersion relation then reads

$$A(s,t;u_0) \;=\; P_s^{(2)}(s,t;u_0)+\frac{s^3}{\pi}\int_{m_s^2}^{\infty}\frac{\mathrm{d}x}{x^3}\frac{\mathrm{Im}A(x,y;u_0)}{x-s}+\frac{t^3}{\pi}\int_{m_t^2}^{\infty}\frac{\mathrm{d}x}{x^3}\frac{\mathrm{Im}A(y,x;u_0)}{x-t}\,, \tag{4.34}$$

with $t = M^2 - u_0 - s$ and $y = M^2 - u_0 - x$.

It can be more convenient to use the crossing symmetry

$$A(s,t;u) \;=\; B(t,s;u)\,, \tag{4.35}$$

where $B(t,s;u)$ is the amplitude of the crossed process (4.7), to express the dispersion relation in the form

$$A(s,t;u_0) \;=\; P_s^{(2)}(s,t;u_0) + \frac{s^3}{\pi}\int_{m_s^2}^{\infty}\frac{\mathrm{d}x}{x^3}\frac{\mathrm{Im}A(x,y;u_0)}{x-s} + \frac{t^3}{\pi}\int_{m_t^2}^{\infty}\frac{\mathrm{d}x}{x^3}\frac{\mathrm{Im}B(x,y;u_0)}{x-t}\,. \tag{4.36}$$

The amplitude can be decomposed in terms of partial waves (4.29)

$$A(s,t;u) \;=\; 32\pi(A_0(s) + 3\cos\theta A_1(s) + \dots)\,. \tag{4.37}$$

It can be shown [51, 52] that in our case it is sufficient to consider the s and p waves only, as higher ones contribute to order $O(p^8)$ at best. The angle $cos\theta$ can be expressed using the Mandelstam variables as

$$\cos\theta \;=\; \frac{s(t-u) + \Delta_{ab}\Delta_{cd}}{\lambda_{ab}^{1/2}(s)\lambda_{cd}^{1/2}(s)}\,, \tag{4.38}$$

$$\Delta_{ij} = m_i^2 - m_j^2\,, \quad \lambda_{ij}(s) = \lambda(s, m_i^2, m_j^2)\,. \tag{4.39}$$

After inserting the decomposition of $A(s,t;u)$ and $B(s,t;u)$ into (4.36) and some manipulation resulting in the redefinition of the subtraction polynomial, one gets (see again [51, 52] for details)

$$\begin{aligned} A(s,t;u_0) \;=\;& P_s^{(2)}(s,t;u_0) + \\ &+ 32s^3\int_{m_s^2}^{\infty}\frac{\mathrm{d}x}{x^3}\frac{\mathrm{Im}A_0(x)}{x-s} + 96s^3(s(t-u)+\Delta_{ab}\Delta_{cd})\int_{m_s^2}^{\infty}\frac{\mathrm{d}x}{x^3}\frac{\mathrm{Im}A_1(x)}{(x-s)\lambda_{ab}^{1/2}(x)\lambda_{cd}^{1/2}(x)} \\ &+ 32t^3\int_{m_t^2}^{\infty}\frac{\mathrm{d}x}{x^3}\frac{\mathrm{Im}B_0(x)}{x-t} + 96t^3(t(s-u)+\Delta_{ac}\Delta_{bd})\int_{m_t^2}^{\infty}\frac{\mathrm{d}x}{x^3}\frac{\mathrm{Im}B_1(x)}{(x-t)\lambda_{ac}^{1/2}(x)\lambda_{bd}^{1/2}(x)}\,. \end{aligned} \tag{4.40}$$

As can be seen, the fixed variable u is now contained only in the polynomial part of the expression. One is now able to generalize the representation of the amplitude in order to obtain the correct crossing symmetry properties in all the Mandelstam variables. The result is a general dispersion relation in s, t and u, the Reconstruction theorem:

$$
\begin{aligned}
A(s,t;u) \;=\; & P^{(2)}(s,t;u) + \\
& + 32s^3 \int_{m_s^2}^{\infty} \frac{\mathrm{d}x}{x^3} \frac{\mathrm{Im}A_0(x)}{x-s} \;+\; 96s^3(s(t-u)+\Delta_{ab}\Delta_{cd}) \int_{m_s^2}^{\infty} \frac{\mathrm{d}x}{x^3} \frac{\mathrm{Im}A_1(x)}{(x-s)\lambda_{ab}^{1/2}(x)\lambda_{cd}^{1/2}(x)} \\
& + 32t^3 \int_{m_t^2}^{\infty} \frac{\mathrm{d}x}{x^3} \frac{\mathrm{Im}B_0(x)}{x-t} \;+\; 96t^3(t(s-u)+\Delta_{ac}\Delta_{bd}) \int_{m_t^2}^{\infty} \frac{\mathrm{d}x}{x^3} \frac{\mathrm{Im}B_1(x)}{(x-t)\lambda_{ac}^{1/2}(x)\lambda_{bd}^{1/2}(x)} \\
& + 32u^3 \int_{m_u^2}^{\infty} \frac{\mathrm{d}x}{x^3} \frac{\mathrm{Im}C_0(x)}{x-u} + 96u^3(u(t-s)+\Delta_{ad}\Delta_{bc}) \int_{m_u^2}^{\infty} \frac{\mathrm{d}x}{x^3} \frac{\mathrm{Im}C_1(x)}{(x-u)\lambda_{ad}^{1/2}(x)\lambda_{bc}^{1/2}(x)} \,. \quad (4.41)
\end{aligned}
$$

It is directly suited to use with the Cutkosky rule in the form (4.30).

Chapter 5

Resonance chiral theory

Efforts to incorporate heavier degrees of freedom into the effective theory are as old as χPT itself [6]. The energy range in which Chiral perturbation theory is valid is fairly limited and its plethora of low energy couplings forces theoreticians to search for ways how to estimate them. This can be achieved by extending the effective theory to the nearest heavier bound states beyond the Goldstone bosons, the low lying meson resonances.

We'll be using such an approach in a bit untraditional way in chapter 8. Here the classical basics will be introduced mainly by closely following the papers [53, 54].

5.1 Reaching beyond the χPT energy range

The limits of χPT as an effective theory are given by the lowest heavier resonant states not included explicitly in the effective Lagrangian. As an illustration, for $\pi^+\pi^-$ scattering the ρ^0 resonance appears at center of mass energy of about 770MeV. The S-matrix thus has to contain the corresponding pole, its effects are integrated out into the effective vertices expanded in momenta. Clearly, when approaching the limiting energy from below, higher and higher orders in the expansion have to be retained in order to keep the truncation error under control. Finally, at 770MeV the expansion in momenta necessarily fails. At higher energies the ρ meson can appear as a real, even if short lived particle and thus the χPT description is missing a bound states which leads to the violation of the S-matrix unitarity. The range of validity may depend on the specific process considered though, for example in the case of neutral pion scattering, the ρ meson as a vector particle cannot appear and thus the limit is pushed higher to the closest scalar resonance in this channel (apart from σ).

The range of validity of the effective theory can be enlarged by reconstructing the non-analyticities generated by some part of the heavier bound states. This means including them explicitly in the resonance Lagrangian $\mathcal{L}_{eff}^{R\chi T}$

$$e^{i\int \mathrm{d}^4x\ \mathcal{L}_{eff}^{\chi PT}(\phi)} = \int \mathcal{D}R\ e^{i\int \mathrm{d}^4x\ \mathcal{L}_{eff}^{R\chi T}(\phi,R)} = \int \mathcal{D}\Phi_H\, \mathcal{D}R\ e^{i\int \mathrm{d}^4x\ \mathcal{L}(\phi,R,\Phi_H)}\,. \tag{5.1}$$

The symmetry properties have to be the same in all cases, i.e. identical to QCD. As can be seen, by integrating out the resonances one has to obtain the original χPT Lagrangian. If we include the lowest multiplets of vector, axial vector, scalar and pseudoscalar resonances, their contribution to the Standard $O(p^4)$ LEC's can be written in the form

$$L_i^r(\mu) \;=\; \sum_{R=V,A,S,P} L_i^R + \hat{L}_i(\mu) \tag{5.2}$$

and similarly for higher orders, as the resonance discontinuities contribute to all orders of the derivative expansion. The RχT Lagrangian therefore consists of two parts, one contains the resonances and their interactions including the ones with the Goldstone bosons, the other has the same form as the χPT Lagrangian with the LEC's redefined according to (5.2)

$$\mathcal{L}_{R\chi T} \;=\; \hat{\mathcal{L}}_{\chi PT} + \mathcal{L}_R\,, \tag{5.3}$$

The original effective vertices are thus decomposed into the effect of heavy degrees of freedom still described effectively and the part that is made manifest now.

Apart from extending the theory to higher energies, another purpose of such a procedure might be the estimate of the original χPT LEC's. It is clear that a necessary condition for such an estimate to be feasible is the dominance of the included resonances in the decomposition (5.2), while the impact of the rest of the degrees of freedom has to be negligible. This assumption of *resonance saturation* is quite intuitive, one expects the closest poles to have the largest contribution, while that of the ones far away should be small. Nevertheless, it remains a conjecture to be verified.

As mentioned, the enlarged effective Lagrangian has to possess the same symmetry properties as the fundamental theory. This is a basic guide for its construction. However, in contrast to the Goldstone bosons, the transformational requirements constrain the form of the resonance interaction terms much more weakly and that in principle leads to an infinite number of possible contributions at each chiral order. Additional information is needed and this ingredient is provided by QCD in the limit of large number of colors.

5.2 Large N_c limit

The motivation behind introducing the large N_c limit [55, 56, 54] was the search for a weak coupling regime in QCD. Such a regime exists in the high energy region with asymptotic freedom and, as we now know, also at the lowest energies for the interactions of (nearly) massless Goldstone bosons. However, in the intermediate region of 1-2GeV, where the bulk of hadronic physics lies, no conventional perturbative expansion exists and the properties of hadrons thus remain lagely unexplained from the first principles. In this section we will recapitulate the basic qualitative results obtained by the approach.

The limit is performed by sending the number of colors to infinity while simultaneously suppressing the QCD coupling as $g = g_0/\sqrt{N_c}$, i.e. $g_0^2 = g^2 N_c$ is held constant. The number of flavors is also kept at the physical value. Considering now a generic perturbative QCD diagram in terms of quarks and gluons and looking at the QCD Lagrangian (1.1), we can see that any such diagram is suppressed in the limit as

- $g_0/\sqrt{N_c}$ for each quark-gluon vertex
- $g_0/\sqrt{N_c}$ for each three gluon vertex
- g_0^2/N_c for each quartic gluon vertex

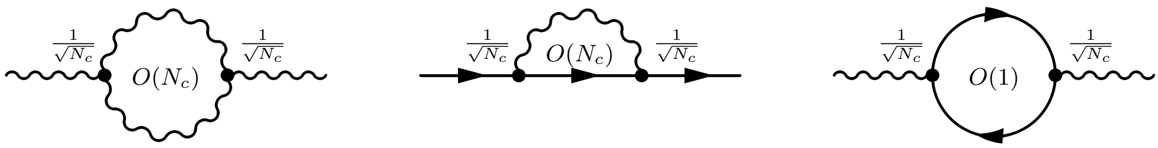

Figure 5.1: The gluon and mixed quark-gluon loops can survive the large N_c limit, pure quark loops are suppressed.

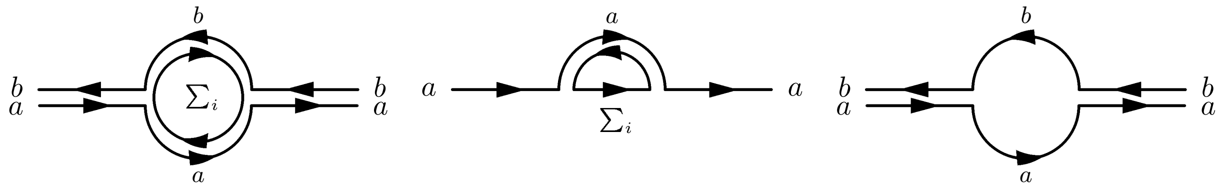

Figure 5.2: Schematically denoted flow of the color indices in diagrams from fig.5.1. While quarks carry a single color index, gluons a pair of them.

So when formally counting the four gluon one as a double vertex, the diagrams get suppressed as $1/\sqrt{N_c}$ for each vertex they contain.

On the other hand, the factor N_c can also appear in the numerator. This happens when the color index is contracted and the sum generates the number of colors. Color conservation in the vertices guarantees that there are only two such cases - either there is a sum over colors on the external lines, typically when one wants to have color singlets in the incoming and outgoing states, or when there is a loop with N_c possibilities of intermediate states. Two types of loops can have such an arrangement (fig.5.1):

- a gluonic loop
- a mixed quark-gluon loop

Pure quark loops do not contract the color index and thus they are suppressed in the limit. The flow of color is demonstrated in fig.5.2.

One also has to realize that not all gluonic or mixed quark-gluon loops lead to a summation over color states. If an internal gluon propagator exits the loop while crossing another propagator without creating a vertex, the color index escapes the loop and the contraction does not occur (fig.5.3). More precisely, either two or more loops then share a single color index loop or in some cases the color gets bound to an external line and does not contract at all. Such suppressed graphs cannot be drawn on a plane, their are *non-planar*, and thus the leading ones in $1/N_c$ are always planar.

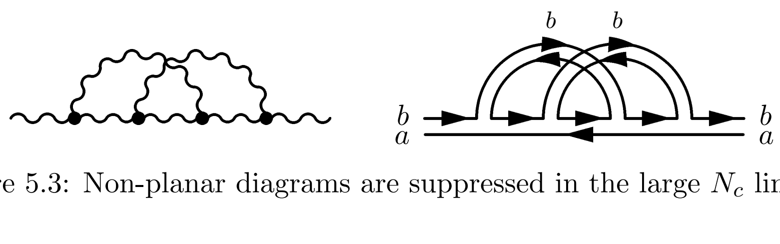

Figure 5.3: Non-planar diagrams are suppressed in the large N_c limit.

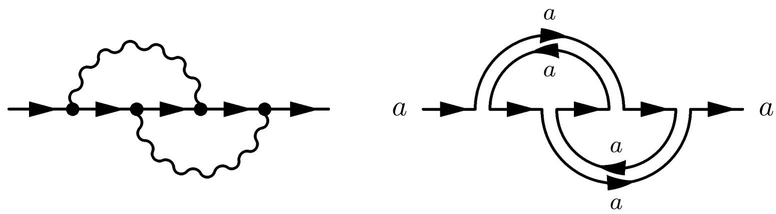

Figure 5.4: Inner fermion lines produce suppressed diagrams.

As a second possibility, demonstrated in fig.5.4, a similar situation occurs in a mixed quark-gluon loop when the quark line is interrupted by a vertex, where the gluon exits the loop while carrying away the color. One can notice that in this case the quark line cannot be drawn at the edge of the diagram, which will be convenient in the following.

Hence every color contraction leads to a factor N_c in the numerator, while every vertex to $1/\sqrt{N_c}$ in the denominator. It's easy to see that the only scenario surviving the limit, of order $O(1)$ in $1/N_c$, is the one when the number of color index loops is equal to twice the number of vertices. The leading contributions to both propagators possess this structure (fig.5.1). Because there is an infinite number of such diagrams, QCD cannot be considered a weakly coupled theory in the large N_c limit in terms of quarks and gluons. This is not such a problem, however. As the aim is to describe the world consisting of hadrons, one actually hopes to retain the confinement in the limit.

The assumption that the confinement is still present for $N_c \to \infty$, and that the number of colors is sufficiently large, leads to the basic qualitative predictions about the hadrons. We will concentrate on the mesons, baryons are infinitely heavy in the limit and decouple from both the mesons and glueballs [54].

Let's denote a generic quark bilinear (e.g. $\bar{q}q$, $\bar{q}\gamma^\mu q$ etc.) as $J(x)$. These composite operators are regarded as the interpolating fields for all the varieties of mesons in QCD (compare with (1.52)). Perturbatively, the quark current $J(k)$ creates a color singlet quark-antiquark pair from the vacuum.

We'll be interested in the analysis of the perturbative diagrams for the two point Green function $\langle 0 | J(x)J(y) | 0 \rangle$. As can be derived from the previous considerations, planar graphs with the valence quark and antiquark lines at the edges are leading in the large N_c limit in this case. There cannot be any internal quark loops, only gluon ones. It's easy to see that any intermediate state in such a diagram is a color singlet with a valence quark-antiquark pair and an arbitrary number of gluons. Based on the presumed quark confinement, the infinite sum of these perturbative intermediate states with the appropriate quantum numbers are interpreted as the mesons. No other states, like glueballs or multiquarks are possible. Thus one arrives to the spectral representation

$$\langle 0 | J(k)J(-k) | 0 \rangle = \sum_n \frac{a_n^2}{k^2 - M_n^2}, \quad a_n = \langle 0 | J(k) | n \rangle, \quad | n \rangle \dots \text{n-th meson}. \tag{5.4}$$

The number of meson states has to be infinite, there can't be a heaviest one otherwise the two point function would behave as $1/k^2$ for $k \to \infty$, which is in contradiction with what is expected in the region of asymptotic freedom. One can also see from (5.4) that the mesons are stable (they are infinitely narrow) and the masses converge at $N_c \to \infty$. The two point function diagram is of order $O(N_c)$, due to the sum over all colors in the external quark lines (to constitute a singlet), which effectively creates a color index loop. This also implies that $a_n = \langle\, 0\,|J(k)\,|\, n\,\rangle$ is of order $O(\sqrt{N_c})$.

One can now look at a general n-point function of the type $\langle\, 0\,|J_1(x_1)\dots J_n(x_n)\,|\,0\,\rangle$. Drawing a perturbative diagram, cutting it to search for all possible intermediate states and essentially repeating the logic from the simplest two point function case, one arrives to the following conclusions. Any intermediate states, if it contains a pole, is a single meson. The QCD graphs can be then redrawn in terms of effective meson lines only - the topology is given by the position of the meson poles in all possible intermediate states. The result is generally a sum of tree diagrams where meson lines are connected by effective meson vertices.

The large N_c order of any such n-point function is $O(N_c)$ for the same reason as in the simplest case. For the amplitude, we can use an effective LSZ formula and cut off the poles by multiplying the Green function with a factor $(k_i^2 - M_i^2)/a_i$ for each of the incoming and outgoing states. As a_i is of order $O(\sqrt{N_c})$, a scattering amplitude of n mesons is of order $N_c^{1-n/2}$. For example a three meson amplitude is of order $1/\sqrt{N_c}$. The counting is consistent, a two particle scattering can contain a four meson vertex of order $O(1/N_c)$ or two three meson ones connected with a meson propagator which is also $O(1/N_c) = O(1/\sqrt{N_c}\,.1/\sqrt{N_c})$.

The conclusion is that mesons are stable and noninteracting in the limit. One can repeat the previous steps with gluon composite operators to derive that glueballs decouple from the mesons as well. Hence we arrive to the desired weakly coupled theory at $N_c \to \infty$.

We can now summarize the phenomenological predictions following from the assumption of sufficiently large number of colours:

- existence of clearly defined multiplets of meson resonances organized according to their flavor quantum numbers
- suppression of the axial $U(1)_A$ anomaly (nonets rather than octets+singlet)
- suppression or decoupling of the exotic states such as multiquarks and glueballs
- unstable meson decays dominated by two particle final states, two particle scattering with single resonant intermediate states
- Zweig rule

The observed mesons obey these rules well and the large N_c limit is up to date the only explanation for many of their properties. Two of the multiplets, however, deserve further discussion - the scalars and light pseudoscalars. We already looked at the first case in chapter 3, the scalar mesons are not organized easily in a proper multiplet, some of them are suspected to be exotic states and they do not like the Zweig rule. The reason for this is unknown apart from the speculative conjecture that this might somehow be connected to the phase structure of QCD with varying number of light quark flavors.

In the case of the light pseudoscalars, Chiral perturbation theory in its traditional form is not organized according to a $1/N_c$ expansion. Even if the Goldstone bosons do form a nice multiplet, the theory itself does not suppress multiparticle interactions and the Zweig rule violating terms are democratically present at the same chiral order as the leading order ones in $1/N_c$. If the number of colors could be considered large, it should manifest itself only indirectly in the values of the low energy constants - the ones suppressed in large N_c should be negligible. It is possible to rebuild the theory explicitly in accord with the large N_c approximation though [8]. In fact, the question whether the large N_c limit is appropriate in the case pseudoscalars, in close connection to the scalar sector, is quite intimately tied to the main topic of this work.

Expansion in $1/N_c$ around the weakly coupled large N_c limit might not be an easy solution for all mesons, but as we will see in the following section, without a more direct knowledge about the resonance bound states of QCD, it is a necessary ingredient in constructing the extended effective Lagrangian. The basic framework of the theory is still chiral expansion, but the low energy LEC's are expanded in $1/N_c$, which allows a straightforward categorization of resonance contributions.

5.3 Resonance Lagrangian

We can now proceed to the construction of the extended effective Lagrangian. We will derive the most simple case, in the steps of [53]. The underlying logic is to find the most general form compatible with spontaneously and explicitly broken chiral symmetry $SU(3)_L \times SU(3)_R$.

Because additional fields will now be present in the Lagrangian, the Goldstone bosons have to be parametrized using the matrix field $u(x)$ rather than $U(x)$ (see [4])

$$u(x) \;=\; \exp \frac{i}{2F_0}\, \phi^a(x)\lambda^a, \quad U(x) = u(x)^2, \tag{5.5}$$

which transforms as

$$u'(x) \;=\; U_R u(x)\, h(\phi)^+ \;=\; h(\phi)u(x)\, U_L^+, \quad h \in SU(3)_V. \tag{5.6}$$

As for the resonance fields, we only know that they have to form $SU(3)_V$ multiplets in the chiral limit. This implies the following transformational properties for the octet and singlet representations [4]

$$R'(x) \;=\; h(\phi)R(x)\, h(\phi)^+, \quad R(x) \;=\; \frac{1}{\sqrt{2}}\, R^a(x)\lambda^a \tag{5.7}$$

$$R_1{}'(x) \;=\; R_1(x). \tag{5.8}$$

The local $SU(3)_L \times SU(3)_R$ group and consequently the electroweak interactions can be introduced through the covariant derivative

$$\nabla_\mu R \;=\; \partial_\mu R + [\,\Gamma_\mu, R\,] \tag{5.9}$$

$$\Gamma_\mu \;=\; \frac{1}{2}[u^+[\partial_\mu - i(v_\mu + a_\mu)]u + u[\partial_\mu - i(v_\mu - a_\mu)]u^+], \tag{5.10}$$

transforming as an octet

$$(\nabla_\mu R)' \;=\; h(\phi)\nabla_\mu R\, h(\phi)^+. \tag{5.11}$$

We can include the lightest multiplets of vector, axial vector, scalar and pseudoscalar resonances. For the description of the spin 1 particles, one can choose either a vector or a antisymmetric tensor formalism, which are generally not equivalent without adding additional contact terms [57]. We will use the tensor one, as only this directly contributes to the $O(p^4)$ chiral order. The vector 1^{--} multiplet fields are then collected in the matrix

$$V_{\mu\nu} = \begin{pmatrix} \frac{\rho^0}{\sqrt{2}} + \frac{\omega_8}{\sqrt{6}} & \rho^+ & K^{*+} \\ \rho^- & -\frac{\rho^0}{\sqrt{2}} + \frac{\omega_8}{\sqrt{6}} & K^{*0} \\ K^{*-} & \bar{K}^{*0} & -\frac{2\omega_8}{\sqrt{6}} \end{pmatrix}_{\mu\nu} \tag{5.12}$$

and similarly for the rest of the resonances. It can be noted that in the case of the scalars the identification of the states is more complicated as was discussed in chapter 3.

One can hardly proceed further in the construction because the transformational properties of the resonance fields (5.7-5.11) constrain the contributions only weakly. Additional information has to be input and this is provided by the large N_c limit. The resonance part can be composed in the form of a $1/N_c$ expansion with the assumption that the first orders dominate. In the following only the leading order in $1/N_c$ is kept, which means the result can contain purely tree graphs with the lowest possible number of resonances.

The Lagrangian of the effective theory

$$\mathcal{L}_{R\chi T} = \hat{\mathcal{L}}_{\chi PT} + \mathcal{L}_R, \tag{5.13}$$

where $\hat{\mathcal{L}}_{\chi PT}$ is the χPT Lagriangian with the LEC's redefined according to (5.2) and $\mathcal{L}_R$ is the part containing the resonances, then takes the form

$$\mathcal{L}_R = \sum_{R=V,A,S,P} [\mathcal{L}_{kin}(R) + \mathcal{L}_{int}(R)]. \tag{5.14}$$

The kinetic term can be written as

$$\mathcal{L}_{kin}(R) = -\frac{1}{2}\mathrm{Tr}[\nabla^\lambda R_{\lambda\mu}\nabla_\nu R^{\nu\mu} - \frac{1}{2}M_R^2 R_{\mu\nu}R^{\mu\nu}] - \frac{1}{2}[\partial^\lambda R_{1\lambda\mu}\partial_\nu R_1^{\nu\mu} - \frac{1}{2}M_{R_1}^2 R_{1\mu\nu}R_1^{\mu\nu}] \tag{5.15}$$

for the vector and axial vector resonances ($R = V, A$) and

$$\mathcal{L}_{kin}(R) = \frac{1}{2}\mathrm{Tr}[\nabla^\mu R\nabla_\mu R - M_R^2 R^2] + \frac{1}{2}[\nabla^\mu R_1\nabla_\mu R_1 - M_{R_1}^2 R_1^2] \tag{5.16}$$

for the scalar and pseudoscalar ones ($R = S, P$). M_R are the masses of the resonances in the chiral limit, usually chosen according to the lightest member of the multiplet.

Denoting some of the octet combinations

$$u_\mu = iu^+ D_\mu U u^+ = u_\mu^+ \tag{5.17}$$

$$u_{\mu\nu} = iu^+ D_\mu D_\nu U u^+ \tag{5.18}$$

$$\chi_\pm = u^+ \chi u^+ \pm u\chi^+ u \tag{5.19}$$

$$f_\pm^{\mu\nu} = uF_L^{\mu\nu} u^+ \pm u^+ F_R^{\mu\nu} u, \tag{5.20}$$

where

$$F_{R,L}^{\mu\nu} = \partial^\mu (v^\nu \pm a^\nu) - \partial^\nu (v^\mu \pm a^\mu) - i[v^\mu \pm a^\mu, v^\nu \pm a^\nu], \tag{5.21}$$

the leading order interaction part in the chiral expansion takes the form [53]

$$\mathcal{L}_{int}(V) = \frac{F_V}{2\sqrt{2}} \mathrm{Tr}[V_{\mu\nu} f_+^{\mu\nu}] + \frac{iG_V}{\sqrt{2}} \mathrm{Tr}[V_{\mu\nu} u^\mu u^\nu] \tag{5.22}$$

$$\mathcal{L}_{int}(A) = \frac{F_A}{2\sqrt{2}} \mathrm{Tr}[A_{\mu\nu} f_-^{\mu\nu}] \tag{5.23}$$

$$\mathcal{L}_{int}(S) = c_d \mathrm{Tr}[S u_\mu u^\mu] + c_m \mathrm{Tr}[S\chi_+] + \tilde{c}_d S_1 \mathrm{Tr}[u_\mu u^\mu] + \tilde{c}_m S_1 \mathrm{Tr}[\chi_+] \tag{5.24}$$

$$\mathcal{L}_{int}(P) = i d_m \mathrm{Tr}[P\chi_-] + i\tilde{d}_m P_1 \mathrm{Tr}[\chi_-]. \tag{5.25}$$

After the resonances are integrated out and considering that at the leading order in $1/N_c$

$$M_S = M_{S_1} \tag{5.26}$$

$$\tilde{c}_d = \pm \frac{1}{\sqrt{3}} c_d, \quad \tilde{c}_m = \pm \frac{1}{\sqrt{3}} c_m, \tag{5.27}$$

the following relations for the Standard $O(p^4)$ LEC's are obtained

$$2L_1^R = L_2^R = \frac{G_V^2}{4M_V^2}, \quad L_3^R = -\frac{3G_V^2}{4M_V^2} + \frac{c_d^2}{2M_S^2}, \quad L_5^R = \frac{c_d c_m}{M_S^2}, \quad L_7^R = \frac{d_m^2}{6M_P^2} - \frac{\tilde{d}_m^2}{2M_{P_1}^2} \tag{5.28}$$

$$L_8^R = \frac{c_m^2}{2M_S^2} - \frac{d_m^2}{2M_P^2}, \quad L_9^R = \frac{F_V G_V}{2M_V^2}. \quad L_{10}^R = -\frac{F_V^2}{4M_V^2} + \frac{F_A^2}{4M_A^2}\,. \tag{5.29}$$

As expected, there is no contribution to the large N_c suppressed LEC's L_4 and L_6. The situation is similar for L_7 as well, but as it is connected to the $U(1)_A$ anomaly and the description of the η' meson ($P_1 = \eta_1$), we keep it listed explicitly.

It can be noted that in this limit, owing to the dominance of tree graphs at $N_c \to \infty$, the correct scale dependence of the coupling constants is not reconstructed. Therefore there is a need to define *a saturation scale* at which the dominance of the resonance contributions to LEC's is presumed. It is usually taken at $\mu = M_\rho = 770$MeV.

Part II

Results

Chapter 6

$\eta \to \pi^0\pi^0\gamma\gamma$ decay in GχPT

The second part of the thesis is concerned by presenting the results of our original work. It is divided into three closely related topics, each connected to η meson physics in the low energy domain. The core of each chapter is a reprint of a published article or an article accepted for publication in its original form, with only minor changes required by the incorporation into the layout of the thesis. The articles are accompanied by a commentary in the beginning of the chapters, which intends to provide some background about the topic and a bit more of the context and motivations behind the study. Each article also contains its own list of references. Chapter 9 then summarized the thesis. The list of the articles and some of our additional results cited in the second part can be found in chapter 10.

The interest in the radiative rare decay $\eta \to \pi^0\pi^0\gamma\gamma$ was aroused by the anticipation of large number of η decays to be observed at various facilities [58]. Though η physics is indeed studied in several experiments, these expectations were generally not fulfilled in the sense that the resolution is still not sufficient up to date. Our motivation to look into this rare decay was the possibility that the validity of the assumptions of Standard χPT could be tested. The presented results [I] build on earlier work [IV,V], more details including kinematics and the structure of the amplitude can be found in these references.

Our first investigations [IV] showed a promising sensitivity of the partial decay width to the violation of the Standard assumption $r{\sim}25$ in a part of the kinematic region, see in what follows. However, as already discussed, experimental data [20] made such a scenario less probable, thus we concentrated on involving a second parameter Y, denoted in the following article as X_{GOR}. We stumbled on the uncertainty generated by the large number of GχPT LEC's at NLO, without a way to estimate them reliably, further progress in the Generalized framework turned out to be difficult. The conclusion therefore was that the decay seems to be sensitive to the violation of the Standard assumptions, but a way to get rid of the uncertainties have to be found in order to be able to identify a more modest deviation. Almost simultaneously, such an approach arose in the form of the Resummed framework.

The $\eta \to \pi^0\pi^0\gamma\gamma$ decay in Generalized χPT

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Abstract

Calculations of $\eta \to \pi^0\pi^0\gamma\gamma$ decay in Generalized chiral perturbation theory are presented. Tree level and next-to-leading corrections are involved. Sensitivity to violation of the Standard counting is discussed.

6.1 Introduction

The $\eta(p) \to \pi^0(p_1)\,\pi^0(p_2)\,\gamma(k)\,\gamma(k')$ process is a rare decay, which has been recently studied by several authors in context of Standard chiral perturbation theory (SχPT), namely at the lowest order by Knöchlein, Scherer and Drechsel [1] and to next-to-leading by Bellucci and Isidori [2] and Ametller et al. [3]. The experimental interest for such a process comes from the anticipation of large number of η's to be produced at various facilities.[1] The goal of our computations is to add the result for the next-to-leading order in Generalized chiral perturbation theory (GχPT). The motivation is that one of the important contributions involve the $\eta\,\pi \to \eta\,\pi$ off-shell vertex which is very sensitive to the violation of the Standard scheme and thus this decay provides a possibility of its eventual observation. We have completed the calculations at the tree level, added 1PI one loop corrections, corrections to the $\eta\,\pi \to \eta\,\pi$ vertex and phenomenological corrections to the resonant contribution. These preliminary results we would like to present in this paper.

6.2 Kinematics and parameters

The amplitude of the process can be defined

$$\langle \pi^0(p_1)\pi^0(p_2)\gamma(k,\epsilon)\gamma(k',\epsilon')_{\rm out}|\eta(p)_{\rm in}\rangle = i(2\pi)^4\delta^{(4)}(P_f - p)\mathcal{M}_{fi}. \tag{6.1}$$

In the square of the amplitude summed over the polarizations $\overline{|\mathcal{M}_{fi}|^2} = \sum_{\rm pol.} |\mathcal{M}_{fi}|^2$ we integrated out all of the independent Lorentz invariants except the diphoton energy square

$$s_{\gamma\gamma} = (\,k + k')^2, \quad 0 < \, s_{\gamma\gamma} \le \, (M_\eta - 2M_\pi)^2. \tag{6.2}$$

Our goal is to calculate the partial decay width dΓ of the η particle as the function of the diphoton energy square $s_{\gamma\gamma}$.

At the lowest order, the SχPT does not depend on any unknown free order parameters. In contrast, there are two free parameters controlling the violation of the Standard picture in the Generalized scheme. We have chosen them as

$$r \;=\; \frac{m_s}{\hat{m}}\,, \quad X_{GOR} \;=\; \frac{2B\hat{m}}{M_\pi^2} \tag{6.3}$$

[1] according to [4], at DAΦNE about 10^8 decays per year

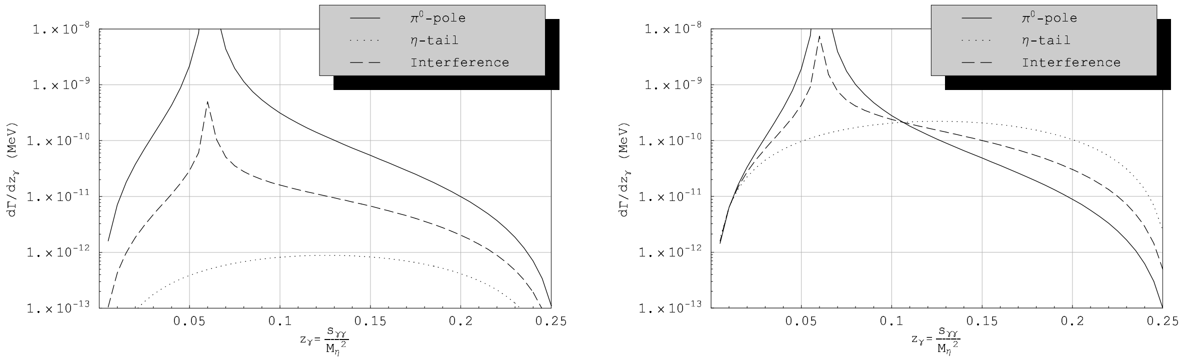

Figure 6.1: SχPT and GχPT tree level contributions to the partial decay rate $\mathrm{d}\Gamma/\mathrm{d}z_\gamma$

and their ranges are $r \sim r_1 - r_2 \sim 6 - 26\,,\ 0 \ \leq\ X_{\mathrm{GOR}} \ \leq\ 1$. We use abbreviations for $\hat{m} = (m_u + m_d)/2$, $r_1 = 2M_K/M_\pi - 1$ and $r_2 = 2M_K^2/M_\pi^2 - 1$. The Standard values of these parameters are $r = r_2$ and $X_{\mathrm{GOR}} = 1$.

6.3 Tree level

At the $O(p^4)$ tree level, the amplitude has two contributions, with a pion and an eta propagator. The first one is resonant, 'π^0-pole', the other is not, 'η-tail'.

The Standard values of the contributions to the partial decay rate and the maximum possible violation of the Standard counting ($r = r_1, X_{\mathrm{GOR}} = 0$) are represented in Fig. 6.1. The pole of the resonant contribution at $s_{\gamma\gamma} = M_\pi^2 \sim 0.06M_\eta^2$ is transparent. While in the Standard case it is fully dominant, in the Generalized scheme the η-tail could be determining in the whole area $s_{\gamma\gamma} > 0.11M_\eta^2$. The reason can be found in the $\eta\pi \to \eta\pi$ vertex. Its contribution in the Generalized amplitude can jump up to 16 times its Standard value.

The full decay width for the Standard ($r = r_2, X_{\mathrm{GOR}} = 1$) and Generalized case ($r =$

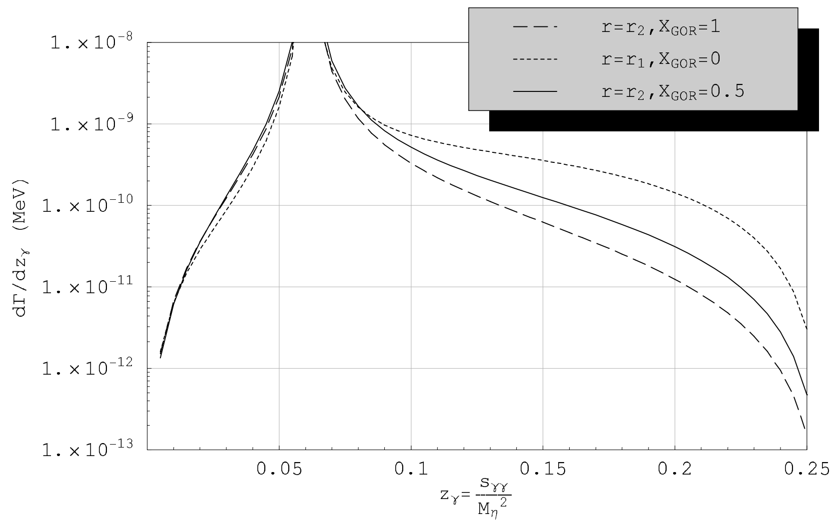

Figure 6.2: Full tree level decay width depending on the parameters r and X_{GOR}

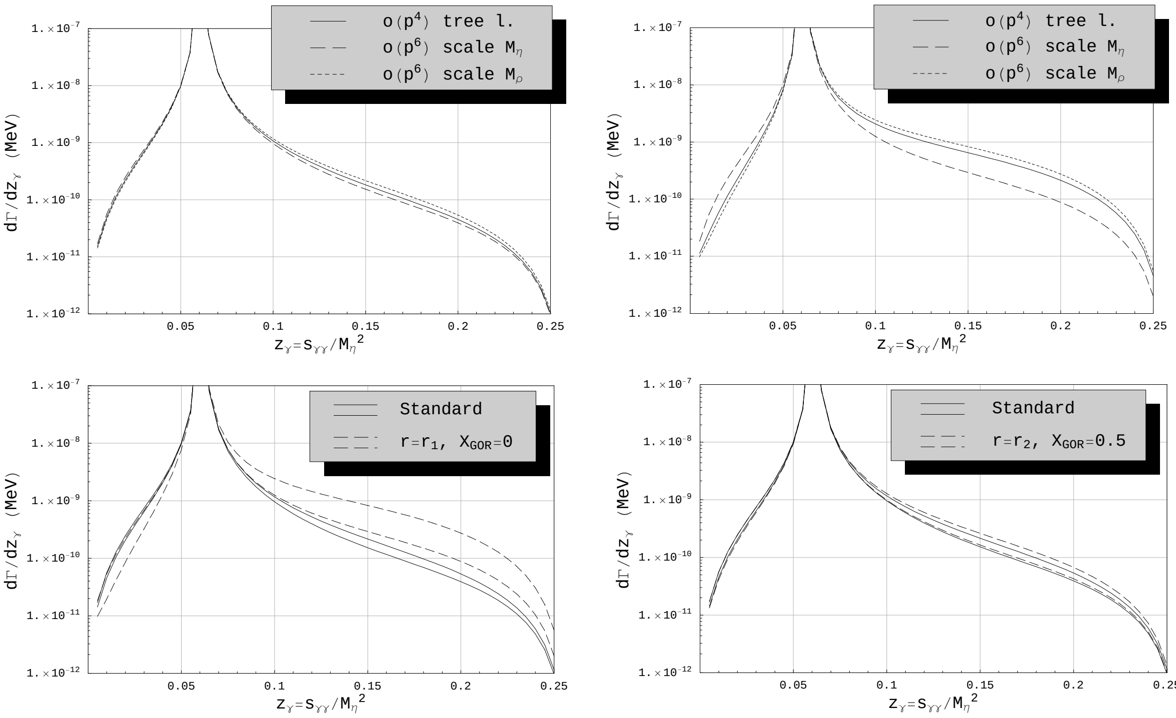

Figure 6.3: SχPT and GχPT tree level and one loop corrected full decay widths

r_2,$X_{\mathrm{GOR}} = 0.5$ and $r = r_1$,$X_{\mathrm{GOR}} = 0$) is displayed in Fig. 6.2. It can be seen, that even in the conservative intermediate case the change is quite interesting.

6.4 One loop corrections

There are four distinct contributions at the next-to-leading order: one loop corrections to the π^o-pole and the η-tail, one particle irreducible diagrams (1PI) and counterterms.

In the latter case we rely upon the results of [3]. Their estimate from vector meson dominated counterterms indicates, that it causes only a slight decrease of the full decay width. Because the estimate is the same for both schemes, for our purpose of studying the differences between them we can leave it for later investigation.

More important are the corrections to the η-tail diagram. We did take into account the corrections to the $\eta\pi \to \eta\pi$ vertex. These involve loop corrections and counterterms with many unknown higher order parameters. As a first approximation, we set these parameters equal to zero and estimated their effect through the remaining dependence on the renormalization scale. The scale was moved in the range from the mass of the η to the mass of ρ-meson.

We decided, similarly to [2], to correct the π^0-pole amplitude by a phenomenological parametrization of the $\eta \to 3\pi^0$ vertex and fix the parameters from experimental $\eta \to 3\pi^0$ data. We made an estimate of its phase by expanding the $\eta \to 3\pi^0$ one loop amplitude around the center of the Dalitz Plot.
In the 1PI amplitude, we neglected the suppressed kaon loops.

Fig. 6.3 represents the one loop corrected decay widths for the Standard and the maximum violation of the Standard scheme. The dependence on the renormalization scale is used to estimate the uncertainty in the unknown higher order coupling constants. We can see that the scale dependence is small in the Standard counting and not too terrible in the Generalized variant. In the case of the maximum violation of SχPT, the difference is big enough to not to be washed out by the uncertainty. However, in the conservative case $r = r_2$,$X_{\rm GOR} = 0.5$ this is not true and the promising results from the tree level are lost.

6.5 Conclusion

We have analyzed the $\eta \to \pi^0\pi^0\gamma\gamma$ decay to the next-to-leading order of chiral perturbation theory in its both variants. The tree level results are promising, the sensitivity to the change in parameters controlling the violation of the Stndard χPT is considerable.

At the one loop level, we tried to estimate the uncertainty in the higher order couplings constants in the crucial $\eta\pi \to \eta\pi$ vertex through their dependence on the renormalization scale. Although for big violation of the Standard case the difference is preserved, for the more realistic conservative case the output is not satisfactory. We would like to stress that these results are preliminary and there are several ways how to deal with the unknown order parameters. One of them is to take into account the vector mesons, similarly to the counterterm estimate in [3]. Other way is to treat the whole χPT expansion differently, with more caution, as developed in [5]. This approach, called 'resumed' χPT could provide results similar to the tree level case even if the one loop corrections are involved.

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Chapter 7

η decay constant in RχPT

In contrast to the pion and kaon decay constants, the eta one is much harder to determine experimentally. Phenomenological analyses vary in their results quite widely. The reason why it's not so easy to obtain a reliable value is the strong $\eta - \eta'$ mixing. Both states couple to the two relevant quark currents and therefore one needs to fit a set of four decay constants from data, not just one.

At the same time, the knowledge of the eta decay constant is important in χPT describing eta physics, as amplitudes are related to the corresponding Green functions collected in the generating functional through pseudoscalar decay constants and these then generally appear in the denominator. In view of the presumably large value of F_η, magnified by the possible suppression of F_0, the leading order relation $F_0 = F_\pi = F_\eta$ can't be considered satisfying. Thus replacing F_η in the denominator by its chiral expansion might be risky. Even in the Resummed framework, where one generally avoids dangerous manipulations, such replacement of F_η is inconvenient because of the additional uncertainty generated by the extra higher order remainder connected to the F_η chiral expansion.

The η' is not included in $SU(3)$ χPT explicitly, but it's contained effectively through low energy LEC's, more specifically L_7 at NLO. It is therefore possible to calculate F_η solely in the $SU(3)$ framework, which then appears as one of the four decay constants in the $U(3)$ theory, related to the coupling of η to the axial current A^8_μ. In the following article [II], we tried to use the Resummed approach in order to analyze the η decay constant chiral expansion, obtain a prediction and estimate the uncertainties. We compared the result to the latest phenomenological fit $F_\eta{\sim}1.38F_\pi$ [59] and investigated what consequences could be drawn for the parameters Y, r and the higher order remainders if such a higher value was confirmed. The effect of the remainders was also estimated by using the GχPT Lagrangian. Apart from the practical importance of F_η, this is also a very illustrative simplified case on which the Resummed procedure can be demonstrated.

THE η DECAY CONSTANT IN 'RESUMMED' CHIRAL PERTURBATION THEORY

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The recently developed 'Resummed' χPT is illustrated on the case of pseudoscalar meson decay constants. We try to get an estimate of the η decay constant, which is not well known from experiments, while using several ways including the Generalized χPT Lagrangian to gather information beyond Standard next-to-leading order. We compare the results to published χPT predictions, our own Standard χPT calculations and available phenomenological estimates.

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Keywords: chiral perturbation theory, pseudoscalar decay constants, η meson

7.1 Introduction

As was discussed recently [1, 2, 3, 4, 5], chiral perturbation theory [6, 7], the low energy effective theory of QCD with N_f light quark flavors, could posses different behavior for N_f=2 and N_f=3. As a consequence of vacuum fluctuations of the growing number of light quark flavors, the most important order parameters of spontaneous chiral symmetry breaking ($SB\chi S$), namely the pseudoscalar decay constant and the quark condensate in the chiral limit, obey paramagnetic inequalities $F_0(N_f+1) < F_0(N_f)$ and $\Sigma(N_f+1) < \Sigma(N_f)$ [3]. In particular, the fluctuations of the sea $\bar{s}s$-pairs need not be suppressed due to the relatively small value of the s-quark mass $m_s \lesssim \Lambda_{QCD}$ and could bring about a possibly significant suppression of $F_0(3)$ and $\Sigma(3)$ w.r.t. N_f=2. This should manifest itself through the OZI rule violation in the scalar sector as can be seen from

$$F_0(2)^2 \; = \; F_0(3)^2 + 16 m_s B_0 L_4^r - 2\bar{\mu}_K + \mathcal{O}(m_s^2) \tag{7.1}$$

$$\Sigma(2) \; = \; \Sigma(3)(1 + \frac{32 m_s B_0}{F_0^2} L_6^r - 2\bar{\mu}_K - \frac{1}{3}\bar{\mu}_\eta) + \mathcal{O}(m_s^2), \tag{7.2}$$

where $B_0^2 = \Sigma(3)/F_0(3)^2$, $\bar{\mu}_P = \mu_P|_{m_{u,d}\to 0}$. Indeed, L_4 and L_6 are the $1/N_c$ suppressed LEC's (connected to the scalar mesons), traditionally considered negligible. Predictions for L_4 and L_6 derived from sum rules involving scalars [10, 11, 5, 2], calculations on the lattice [12, 13] and NNLO SχPT [14] produce numbers significantly different from traditional expectations [7]. Convenient parameters relating the order parameters to physical quantities can be introduced: $Z(N_f) = F_0(N_f)^2/F_\pi^2$ and $X(N_f) = 2\hat{m}\Sigma(N_f)/F_\pi^2 M_\pi^2$, with $\hat{m}$=$(m_u+m_d)/2$. The large $\bar{s}s$ vacuum fluctuations could lead to $Z(3) \ll Z(2)$, $X(3) \ll X(2)$. Analysis [8] of the K_{e_4} decay experimental results for the $\pi\pi$ s-wave scattering length [9] lead to values $X(2)$=0.81 ± 0.07, $Z(2)$=0.89 ± 0.03. $\pi\pi$ and πK scattering data constrain the three flavor parameters much less strictly [1, 4], $X(3)$<0.8, $Z(3)$∼0.2-0.9, Y=$X(3)/Z(3)$<1.1, r=$m_s/\hat{m}$>15. Sum rule approaches [5, 2] yield approximately $X(2)$,$Z(2)$∼0.9 and $X(3)$,$Z(3)$∼0.5 at r=25.

year	cit.	model input	F_8	ϑ_8	F_η^8
2005	[18]		$(1.51 \pm 0.05)F_\pi$	$(-24 \pm 1.6)^o$	$1.38F_\pi$
2000	[19]	sum rules	$1.44F_\pi$	-8.4^o	$1.42F_\pi$
1999	[20]	VMD			$1.27F_\pi$
1998	[21]		$1.26F_\pi$	-21.2^o	$1.17F_\pi$

Table 7.1: Recent two angle η-η' analyses leading to a value of F_η^8

Small $X(3)$ and $Z(3)$ would lead to irregularities of the chiral expansion connected to numerical competition of the LO and NLO terms, which could be consequently seen as unusually large higher order corrections. An alternative approach, dubbed 'Resummed' χPT (RχPT), has been introduced recently [1]. It takes this possible scenario into account and is based on the effective resummation of the vacuum fluctuation discussed above. The goal of the article is to illustrate the 'Resummed' approach on the sector of decay constants and to try to use it for theoretical predictions of the η decay constant and related parameters, including uncertainty estimates.

The $SU(3)_L \times SU(3)_R$ η decay constant in the isospin limit

$$ip_\mu \, F_\eta \;=\; \langle\, 0 \,|\, A_\mu^8 \,|\, \eta(p)\, \rangle, \tag{7.3}$$

where A^i_μ are the QCD axial vector currents, can be calculated in $SU(3)_L \times SU(3)_R$ χPT without the introduction of the η' meson. In the usually investigated η-η' mixing sector, the following definitions are used

$$ip_\mu \, F^{8,0}_{\eta,\eta'} \;=\; \langle\, 0 \,|\, A_\mu^{8,0} \,|\, \eta, \eta' \,\rangle. \tag{7.4}$$

As can be seen, the $SU(3)_L \times SU(3)_R$ constant F_η is defined identically to F_η^8 in the $U(3)_L \times U(3)_R$ framework. A general two angle mixing scheme [15, 16]

$$F_\eta^8 \;=\; F_8 \cos\vartheta_8, \;\; F_{\eta'}^8 \;=\; F_8 \sin\vartheta_8, \;\; F_\eta^0 \;=\; -F_0 \sin\vartheta_0, \;\; F_{\eta'}^0 \;=\; F_0 \cos\vartheta_0 \tag{7.5}$$

has been shown to provide better agreement with experimental data and χPT predictions [16, 17, 18] than a single mixing angle scenario. Table 7.1 collects some recent two angle phenomenological analyses leading to a value of F_η^8. Older one angle mixing scheme results generally provided much lower numbers $F_\eta^8 \sim F_\pi$.

Several recent χPT results can be cited. Standard χPT to $\mathcal{O}(p^6)$ [22] gives $F_\eta/F_\pi = 1 + 0.242 + 0.066 = 1.308$. Large N_c χPT to NNLO [23] leads to $F_8 = 1.34F_\pi$, $\vartheta = -22^o$ and thus $F_\eta^8 = 1.24F_\pi$. We build on the 'Resummed' χPT result [1] $F_\eta^2 = F_\pi^2(1.651 + 0.036Y)$ (at r=24, remainders neglected).

7.2 Decay constants in ‘Resummed’ χPT

‘Resummed’ χPT [1] starts from the same form of the effective Lagrangian as the Standard variant (SχPT) [7]. The difference is in the treatment of the chiral series, RχPT assumes possible irregularities. Overall convergence to all orders is taken for granted, but only for expansions directly obtained from the generating functional. These ‘bare’ expansions are then dealt with additional caution.

The first step is to derive a strict chiral expansion fully expressed in terms of the original parameters of the effective Lagrangian. In our case we have

$$F_\pi^2 \;=\; F_0^2(1-4\mu_\pi-2\mu_K)+16B_0\hat{m}(L_4(r+2)+L_5)+\Delta_{F_\pi}^{(4)} \tag{7.6}$$

$$F_K^2 \;=\; F_0^2(1-\frac{3}{2}\mu_\pi-3\mu_K-\frac{3}{2}\mu_\eta)+16B_0\hat{m}(L_4(r+2)+\frac{1}{2}L_5(r+1))+\Delta_{F_K}^{(4)} \tag{7.7}$$

$$F_\eta^2 \;=\; F_0^2(1-6\mu_K)+16B_0\hat{m}(L_4(r+2)+\frac{1}{3}L_5(2r+1))+\Delta_{F_\eta}^{(4)}\;. \tag{7.8}$$

The expansions for the squares of the decay constants are used, as they are directly related to two point Green functions obtained from the generating functional. At this point, the chiral logs $\mu_P = m_P^2/32\pi^2F_0^2\ln(m_P^2/\mu^2)$ contain non-physical $\mathcal{O}(p^2)$ masses $m_\pi^2 = 2B_0\hat{m}$, $m_K^2 = B_0\hat{m}(1+r)$, $m_\eta^2 = 2/3B_0\hat{m}(1+2r)$. $\Delta_{F_P}^{(4)}$ denote the higher order remainders, not neglected in this approach.

The second step is the definition of the bare expansion, which usually involves changes to the strict form in order to incorporate additional requirements, such as physically correct analytical structure. In our case this narrows down to a question, whether to replace the original leading order masses inside the chiral logarithms with physical ones. In some cases (see [28]) this is a nontrivial question, so we will keep both options and evaluate them.

The next stage is the reparametrization of the unknown LEC's in terms of physical observables. In RχPT the leading order ones are left free, only re-expressed in terms of more convenient parameters r, Z and X resp. Y. Two NLO LEC's are present in our formulae, the equations for F_π and F_K (7.6,7.7) can be used for the reparametrization. Note that this is done in a pure algebraic way, no additional expansion is made. The final formula for the η decay constant [1] is then obtained by insertion into (7.8)

$$F_\eta^2 = \frac{1}{3}\Big[4F_K^2 - F_\pi^2 + \frac{M_\pi^2 Y}{16\pi^2}(\ln\frac{m_\pi^2}{m_K^2} + (2r+1)\ln\frac{m_\eta^2}{m_K^2}) + 3\Delta_{F_\eta}^{(4)} - 4\Delta_{F_K}^{(4)} + \Delta_{F_\pi}^{(4)}\Big]. \tag{7.9}$$

The expression is valid to all orders, it's only divided into an explicitly calculated part and the unknown higher order remainders $\Delta_{F_P}^{(4)}$.

The last step consists of the treatment of the remainders. We will use three ways to estimate them. The first relies on an assumption about the convergence of the chiral series [1, 4] and assumes the typical size of the NNLO remainders is $|\Delta_{F_P}^{(4)}| \sim 0.1F_P^2$. These are added in squares to obtain the final uncertainty. The result is a prediction in the sense that a value significantly outside of the resulting variance is not compatible with such an assumption about a reasonably quick convergence of the chiral expansion.

expansion	$O(p^4)\, L_5,\ r{=}r_2$	$O(p^6)\, L_5,\ r{=}r_2$	$O(p^4)\, L_5,\ r{=}\tilde{r}_2$	$O(p^6)\, L_5,\ r{=}\tilde{r}_2$
F_η	**1.31±0.07**	1.21±0.02	1.29±0.07	1.19±0.02
$\sqrt{F_\eta^2}$	1.27±0.06	1.19±0.01	1.25±0.05	**1.17±0.01**

Table 7.2: Various NLO SχPT results for the η decay constant in F_π units.

Then we try to use information outside core χPT to get a feeling about remainder magnitudes. As can be seen, the RχPT framework is very suitable for incorporating such additional sources of information. We collect various published estimates for L_5^r and use them to check the remainder differences $\Delta_{F_K}^{(4)}$-$\Delta_{F_\pi}^{(4)}$ and $\Delta_{F_\eta}^{(4)}$-$\Delta_{F_\pi}^{(4)}$. We also use the Generalized χPT Lagrangian [24, 25] to get a sense of the magnitude of the higher order corrections

$$\Delta_{F_P}^{(4)} \;=\; (F_P^{(2)})_{G\chi PT} - F_P^{(2)} + \Delta_{F_P}^{(G\chi PT)}. \tag{7.10}$$

More details about this procedure will be published elsewhere [28].

7.3 Numerical results

For the numerical results we use the physical values M_π=135MeV, M_K=496MeV, M_η=548MeV, M_ρ=770MeV, F_π=92.4MeV and F_K=113MeV. At first, let us investigate the NLO Standard χPT. There are several differences compared to the procedure outlined in the previous section. One can use the quadratic form of the expansion obtained from the two point Green function or a linearized form, as is more usual. For the LEC reparametrization inverted expansions for F_0^2 and $2B_0\hat{m}$ are used, while r is fixed at $r = r_2 = 2M_K^2/M_\pi^2 - 1$ or $r = \tilde{r}_2 = 3M_\eta^2/2M_\pi^2 - 1/2$. One then obtains the following formulae

$$\frac{F_\eta}{F_\pi} = 1 + 2\mu_\pi - 2\mu_K + \frac{8M_\pi^2(r-1)}{3F_\pi^2}L_5^r, \quad \frac{F_\eta^2}{F_\pi^2} = 1 + 4\mu_\pi - 4\mu_K + \frac{16M_\pi^2(r-1)}{3F_\pi^2}L_5^r \tag{7.11}$$

where the chiral logs contain physical masses only $\mu_P = M_P^2/32\pi^2F_\pi^2\ln(M_P^2/\mu^2)$.

As for L_5^r, we opted to use the published values $L_5^r(M_\rho) = (1.4 \pm 0.5).10^{-3}$ ($\mathcal{O}(p^4)$ fit [7, 26]) and $L_5^r(M_\rho) = (0.65 \pm 0.12).10^{-3}$ ($\mathcal{O}(p^6)$ fit [27]).

All these possibilities differ merely in redefinitions of the usually neglected remainders. Table 7.2 shows that it might be worth to spend the additional effort to bring the higher order uncertainties explicitly under control. Numerically, the sensitivity to the change in L_5^r is in the range $\Delta F_\eta/F_\pi = (0.11 - 0.14)\Delta L_5.10^3$.

Proceeding to RχPT, we generally investigate a standard and a low r scenario $r \sim$15-25 and vary Y in the range 0-1.6. Keep in mind, though, that the $\pi\pi$ and πK scattering analyses [1, 4] suggest $Y < 1.1$.

Let us first neglect the remainders and have a look on the dependence of the explicitly calculated part on the free parameters Y, r and the treatment of chiral logs. As can be seen from Fig.1, the dependence on both is very small. For physical masses inside the logs one gets $F_\eta^2/F_\pi^2 = 1.661 - 0.011Y + 0.002Yr$. The decay constant sector might thus be insensitive to the particular scenario of $SB\chi S$ and more information is needed to extract the values of the parameters.

Neglecting these weak dependencies one gets the sensitivity on the remainders as $\Delta F_\eta/F_\pi = 1.5 \cdot 10^{-5}\sqrt{((3\Delta^{(4)}_{F_\eta})^2 + (4\Delta^{(4)}_{F_K})^2 + (\Delta^{(4)}_{F_\pi})^2)}$.

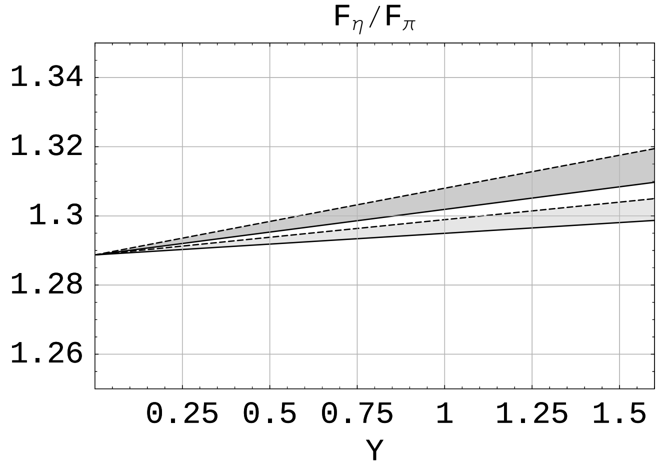

Figure 1: F_η in RχPT, remainders neglected. Chiral logs: solid - physical masses, dashed - $\mathcal{O}(p^2)$ masses. Dark: $r = 25$, light: $r = 15$

Applying the 10% uncertainty remainder estimate, the following all order approximation is obtained

$$F_\eta \;=\; (1.3\pm0.1)F_\pi. \quad (7.12)$$

All phenomenological and theoretical results cited in the introduction fall in or very close to this range and are thus compatible with a reasonable convergence of chiral series. However, the mentioned one mixing angle scheme results are significantly outside.

The difference F_K^2-F_π^2 (7.6,7.7) depends only on L_5^r. This yields an order estimate on the remainder difference $\Delta^{(4)}_{F_K}$-$\Delta^{(4)}_{F_\pi}$ if independent information on L_5^r can be gathered. Several estimates for L_5^r beyond $\mathcal{O}(p^4)$ χPT are available:

- SχPT $\mathcal{O}(p^6)$ fit: $L_5^r(M_\rho) \sim (0.5 - 1.0) \cdot 10^{-3}$ [27, 22]
- Resonance saturation: $L_5^r \sim (1.6 - 2.1) \cdot 10^{-3}$ [11]
- QCD sum rules: $L_5^r(M_\rho) > 1.0 \cdot 10^{-3}$ [2, 5]
- χPT on lattice: $L_5^r \sim 1.8 - 2.2 \cdot 10^{-3}$ [12, 13]

The result of varying $L_5^r(M_\rho)$ in the range $(0.5 - 2).10^{-3}$ can be seen in Fig.2. $\mathcal{O}(p^2)$ masses were kept inside logarithms, physical ones make the remainder estimate somewhat larger. The estimate is compatible with small remainders.

We can also utilize the information about the difference F_η^2-F_π^2 (7.6,7.8). If we use the latest phenomenological result $F_\eta^8 \sim 1.38F_\pi$ [18] as an input, an estimate of $\Delta^{(4)}_{F_\eta}$-$\Delta^{(4)}_{F_\pi}$ is obtained. It should be stressed that older results produced lower values, so this should be taken as a preliminary look on the possible consequences if such a higher value of F_η was confirmed. We don't make a full statistical analysis, only provide some first feelings where it could lead to. Keep in mind $|\Delta^{(4)}_{F_P}| \sim 0.1F_P^2$ and $Y < 1.1$ as suggestions following from [1, 4]. These assumptions hint the following consequences, demonstrated in Fig.3

- $r \sim 15$ and $\Delta^{(4)}_{F_\eta}$-$\Delta^{(4)}_{F_\pi}$$<0.2F_\eta^2$ implies $Y>1$ or $L_5^r(M_\rho)>2.10^{-3}$.
- $r \sim 25$ and $\Delta^{(4)}_{F_\eta}$-$\Delta^{(4)}_{F_\pi}$$<0.2F_\eta^2$ implies $Y>0.5$ or $L_5^r(M_\rho)>2.10^{-3}$.
- $L_5^r(M_\rho)<1.10^{-3}$ and $r \sim 25$ implies $Y>1.2$, $\Delta^{(4)}_{F_\eta}$-$\Delta^{(4)}_{F_\pi}$$>0.2F_\eta^2$.

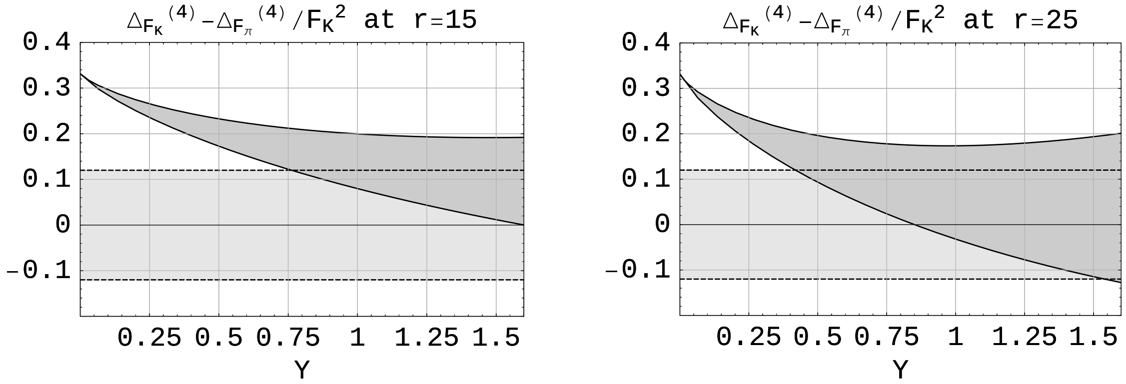

Figure 2: $\Delta^{(4)}_{F_K}$-$\Delta^{(4)}_{F_\pi}$ estimate for $L^r_5(M_\rho) \sim (0.5-2).10^{-3}$ (dark band). Upper bound corresponds to low values of L^r_5. Left: r=15, right: r=25.
Light: expected uncertainty $\pm 0.12 F_K^2$ from the 10% uncertainty estimate.

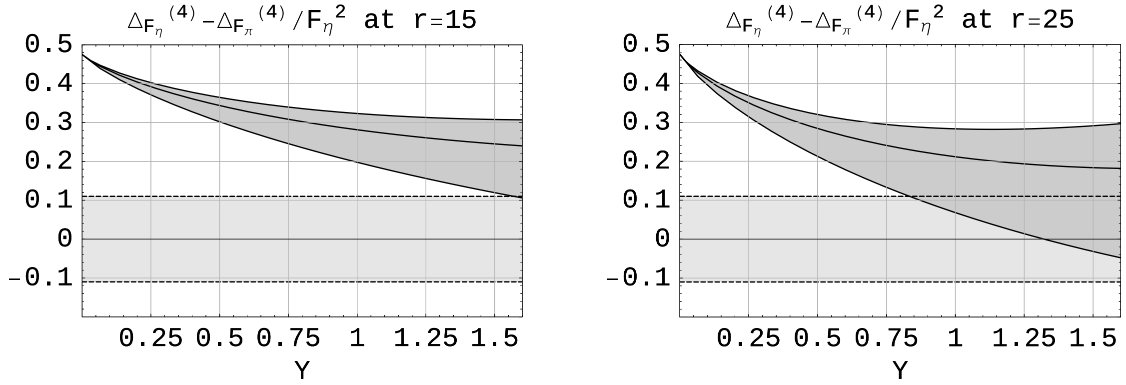

Figure 3: $\Delta^{(4)}_{F_\eta}$-$\Delta^{(4)}_{F_\pi}$ estimate for $L^r_5(M_\rho) \sim (0.5-2).10^{-3}$ (dark band). Upper bound corresponds to low values of L^r_5. Left: r=15, right: r=25.
Light: expected uncertainty $\pm 0.11 F_\eta^2$ from the 10% uncertainty estimate.

The remainder estimate using the Generalized χPT Lagrangian [28] provides

$$\begin{aligned} 3\Delta^{(4)}_{F_\eta} - 4\Delta^{(4)}_{F_K} + \Delta^{(4)}_{F_\pi} \;&=\; 2A^r_2 F^2_\pi \hat{m}^2 (r-1)^2 + 8A^r_3 F^2_\pi \hat{m}^2 (r^2+1) - 4B^r_2(\mu) F^2_\pi \hat{m}^2 (r-1)^2 \\ &\quad - 8C^{P^r}_1(\mu) F^2_\pi \hat{m}^2 (r-1)^2 - \frac{M^2_\pi (X-1)}{16\pi^2} \ln[\frac{M^2_\pi}{\mu^2}] \\ &\quad - \frac{4M^2_K - 2M^2_\pi (r+1) X}{16\pi^2} \ln[\frac{M^2_K}{\mu^2}] + \frac{3M^2_\eta - M^2_\pi (2r+1) X}{16\pi^2} \ln[\frac{M^2_\eta}{\mu^2}] + \Delta^{(5)}_{G\chi PT}. \end{aligned} \tag{7.13}$$

We use two ways to estimate the unknown GχPT LEC's. The first is the usual simple variation of scale, the constants are set to zero at two different scales and the sensitivity is checked. The second assumes a probabilistic distribution of possible values depending on scale variation

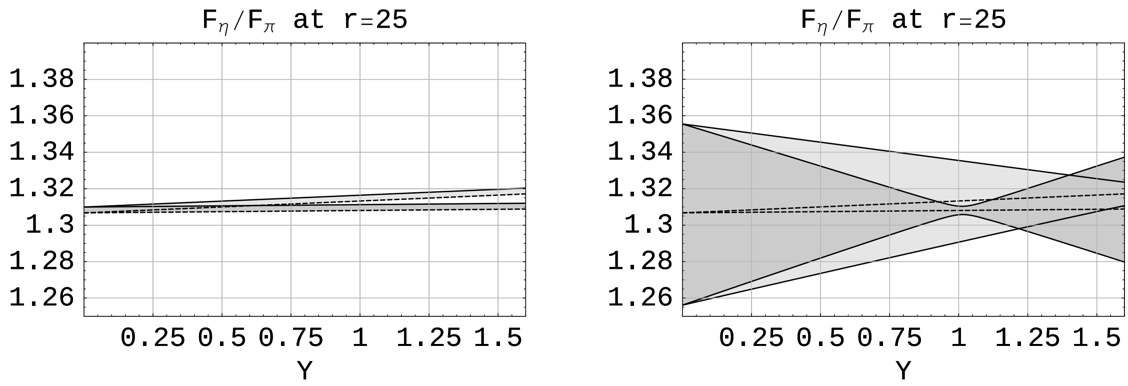

Figure 4: GχPT remainder estimate for r=25. Dark: Z=0.9, light: Z=0.5. Left: simple variation of scale, solid: $\mu = 1$GeV, dashed: $\mu = M_\rho$.
Right: the LEC estimate described in the text, solid: error bars $\mu = 1\text{GeV}/M_\rho$, dashed: central values $\mu = M_\rho$.

$$B_2^r(M_\rho) = 0 \pm \frac{Z_0^S + Z_0^P}{4\pi^2 F_\pi^2} \ln[\frac{1\text{GeV}}{M_\rho}], \quad C_1^{Pr}(M_\rho) = 0 \pm \frac{A_0 - Z_0^S}{16\pi^2 F_\pi^2} \ln[\frac{1\text{GeV}}{M_\rho}]. \tag{7.14}$$

These are then added in squares. Note that the insensitivity in the first case assures independence on where the central value is chosen in the latter one.

The results for r=25 can be seen in Fig.4, low values of r do not change the overall picture. However, of the four GχPT LEC's present in our case only two depend on scale, which is hardly a good statistical ensemble. There is no indication of large higher order corrections nevertheless.

7.4 Summary

We have studied the case of pseudoscalar decay constants in the 'Resummed' χPT framework and tried to obtain an estimate for the η decay constant and related parameters. We used several ways to get a feeling about the effect of higher order remainders.

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Chapter 8

$\pi\eta$ scattering in Resummed χPT

After analyzing the $\eta \to \pi^0\pi^0\gamma\gamma$ decay [I,IV,V] we have seen that the off-shell $\eta\pi \to \eta^*\pi$ vertex contribution is the source of the sensitivity of the result to the violation of the Standard assumptions. This was confirmed by our $\pi\eta$ calculations in GχPT [VI]. We decided to look into $\pi\eta$ scattering in more detail and apply the Resummed approach to this case first.

The downside is that there are no experimental data available in the low energy domain and thus the effects can be observed only indirectly. One application is of course the $\eta \to \pi^0\pi^0\gamma\gamma$ decay, but at the moment the experimental situation have not turned to meet the first expectations neither. Another possibility is the $\eta \to 3\pi^0$ decay, where the off-shell $2\pi2\eta$ vertex effectively contributes through $\pi - \eta$ mixing. Data are readily available here and so this case might prove itself to be an interesting opportunity to try to test the Standard assumptions using the Resummed framework.

Moreover, the $\eta\pi$ scattering can be considered as a theoretical laboratory by itself. That certain kind of behavior is present in χPT at all can have important consequences, not only theoretical but also practical. In the following article [III] we have put Standard and Resummed χPT under theoretical scrutiny, we have looked at the convergence properties in the NLO Standard case and tried to identify the uncertainties related to the Standard treatment of the chiral expansion. We have also analyzed the Resummed procedure in a detailed way, starting with the assumptions behind the definition of the bare expansion, through checking the consistence of the two approaches in the Standard assumption domain, and finally investigating the influence of the higher order remainders while also trying to estimate them in novel ways - using resonance saturation and the Generalized χPT Lagrangian.

$\pi\eta$ scattering and the resummation of vacuum fluctuation in three-flavour χPT

Marian Kolesár and Jiří Novotný[1]

Abstract

We discuss various aspects of resummed chiral perturbation theory, which was developed recently in order to consistently include the possibility of large vacuum fluctuations of the $\bar{s}s$-pairs and the scenario with smaller value of the $\bar{q}q$ condensate for $N_f = 3$. The subtleties of this approach are illustrated using a concrete example of observables connected with $\pi\eta$ scattering. This process seems to be a suitable theoretical laboratory for this purpose due to its sensitivity to the values of the $O(p^4)$ LEC's, namely to the values of the fluctuation parameters L_4 and L_6. We discuss several issues in detail, namely the choice of "good" observables and properties of their bare expansions, the "safe" reparametrization in terms of physical observables, the implementation of exact perturbative unitarity and exact renormalization scale independence, the role of higher order remainders and their estimates. We make a detailed comparison with standard chiral perturbation theory and use generalized χPT as well as resonance chiral theory to estimate the higher order remainders.

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8.1 Introduction

As it is well known, at the energy scales $E \ll \Lambda_H \sim 1\mathrm{GeV}$ the physics of QCD is nonperturbative and governed by chiral symmetry (χS) $SU(N_f)_L{\times}SU(N_f)_R$. This global symmetry is present on the classical level within the QCD with N_f massless quarks (in the chiral limit of QCD) and on the quantum level there exist strong theoretical (for $N_f \geq 3$) and phenomenological arguments for spontaneous symmetry breakdown (SSB) of χS according to the pattern $SU(N_f)_L{\times}SU(N_f)_R \rightarrow SU(N_f)_V$. Due to the confinement, quark and gluon fields do not represent appropriate low energy degrees of freedom within the above mentioned energy range; the relevant degrees of freedom correspond to the lightest colourless hadrons in the QCD spectrum. As far as the Green functions of quark currents are concerned, it is possible to obtain a general solution of the chiral Ward identities in terms of the low energy expansion. This expansion can be organized most efficiently using the methods of effective field theory corresponding to the low-energy limit of QCD with N_f light quark flavours which is known as chiral perturbation theory (χPT) [1, 2, 3]. χPT describes the low energy QCD dynamics in terms of the lightest (N_f^2-1)-plet of the pseudoscalar mesons identified with the Goldstone bosons (GB) of the spontaneously broken chiral symmetry which appear in the particle spectrum of the theory as a consequence of the Goldstone theorem. In the chiral limit these pseudoscalars are massless and dominate the low energy dynamics of QCD. They interact weekly at low energies $E \ll \Lambda_H$, where $\Lambda_H \sim 1\mathrm{GeV}$ is the hadronic scale corresponding to the masses of the lightest nongoldstone hadrons. This feature of the GB dynamics enables systematic perturbative treatment with the expansion parameter (E/Λ_H). Within the real QCD the quark mass term $\mathcal{L}^{QCD}_{f,mass}$ breaks χS explicitly and the Goldstone bosons become pseudogoldstone bosons (PGB) with nonzero masses. Though $m_f \neq 0$, for $m_f \ll \Lambda_H$ the mass term $\mathcal{L}^{QCD}_{f,mass}$ can be treated as a perturbation. As a consequence, PGB correspond to the lightest hadrons in the QCD spectrum[2] (identified with $\pi^0, \pi^\pm$ for $N_f = 2$ and $\pi^0, \pi^\pm, K^0, \overline{K^0}, K^\pm, \eta$ for $N_f = 3$) and the interaction of PGB at the energy scale $E \ll \Lambda_H$ continues to be weak. Because $M_P < \Lambda_H$, the QCD dynamics at $E \ll \Lambda_H$ is still dominated by these particles and the effective theory provides us with a simultaneous expansion in powers of (E/Λ_H) and (m_f/Λ_H). The Lagrangian of χPT can be constructed on the basis of symmetry arguments only; the unknown information about the nonperturbative properties of QCD are hidden in the parameters known as low energy constants (LEC)[2, 3]. These are related to the (generally nonlocal) order parameters of the SSB of χS, the most prominent of them are the Goldstone boson decay constant F_0 and the chiral condensate[3] $B_0 = \Sigma/F_0^2$ where $\Sigma = -\langle \overline{u}u \rangle_0$.

To be more precise, N_f-flavour χPT is in fact an expansion in m_i, around the $SU(N_f)_L \times SU(N_f)_R$ chiral limit $m_i = 0$, $i \leq N_f$, while keeping all the other quark masses for $i > N_f$ at their physical values. Because $m_{u,d}$ are much smaller not only in comparison with the hadronic scale Λ_H, but also in comparison with the intrinsic QCD scale Λ_{QCD}, the two-flavour χPT is expected to produce well-behaved expansion corresponding to small corrections to the $SU(2)_L \times SU(2)_R$ chiral limit.

The strange quark mass on the other hand, though still small enough with respect to Λ_H

[2]The PGB masses M_P can be expanded in the powers (and logarithms) of the quark masses starting from the linear term and therefore vanish in the chiral limit.

[3]The parameter F_0 is however more fundamental in the sense that $F_0 \neq 0$ is both necessary and sufficient condition for SSB, while $\langle \overline{q_f} q_f \rangle_0 \neq 0$ corresponds to the sufficient condition only. (The lower index zero means here the chiral limit.)

to be treated as an expansion parameter within the three-flavour χPT (relating real QCD with its $SU(3)_L \times SU(3)_R$ chiral limit), is of comparable size with respect to Λ_{QCD}. This fact, besides the expected worse convergence of the three-flavour χPT, might also have interesting consequences for the possible difference between the $N_f = 2$ and $N_f = 3$ chiral dynamics. As discussed intensively in a series of papers [4, 5, 6, 7, 8, 9], $m_s \lesssim \Lambda_{QCD}$ suggest, that the loop effects of the vacuum $\overline{s}s$ pairs are not suppressed as strongly as it is for the heavy quarks and might enhance the magnitude of the $N_f = 2$ chiral order parameters relatively to their $N_f = 3$ chiral limits. This applies mainly to $F_0(N_f)$ and $\Sigma(N_f) = F_0^2(N_f)B_0(N_f)$, which should satisfy paramagnetic inequalities [4]

$$\begin{aligned} \Sigma(2) &> \Sigma(3) = \lim_{m_s \to 0} \Sigma(2) \\ F_0(2) &> F_0(3) = \lim_{m_s \to 0} F_0(2). \end{aligned} \tag{8.1}$$

The leading order difference between the two-flavour and three-flavour values is proportional to m_s, with coefficients measuring the violation of the OZI rule in the 0^{++} channel, *e.g.*

$$\Sigma(2) = \Sigma(3) + m_s \overline{Z}_1^s + \ldots \tag{8.2}$$

(see [4] for details) where

$$\overline{Z}_1^s = \lim_{m_s \to 0} \int \mathrm{d}^4 x \langle \overline{u}u(x) \overline{s}s(0) \rangle_c \tag{8.3}$$

and analogously for F_0. The fluctuation parameter $\overline{Z}_1^s$ is related to the LEC $L_6^r(\mu)$ (and $L_4^r(\mu)$ for F_0) of the three-flavour χPT . As discussed in [4, 5, 6, 7, 8, 9], these parameters might be larger than their estimate based on the large N_c expansion, provided $N_f = 3$ is close to the critical number of light quark flavours N_f^{crit}, for which the chiral symmetry is restored. Available estimates vary widely, some indicate a larger number $N_f^{crit} \sim 10 - 12$ for $N_c = 3$ [10, 11, 12], while other approaches [13, 14] and lattice calculations [15, 16, 17, 18, 19] discuss a possibly much lower value $N_f^{crit} \leq 6$. Provided the scenario of large vacuum fluctuations takes place, the second term in (8.2) (called the induced condensate in [5, 8, 20]) can be numerically comparable with the first term and the three-flavour condensate $\Sigma(3)$ could be substantially smaller than the two-flavour one, the value of which is experimentally accessible in the recent experiments. Analogous reasonings apply to the relationship of $F_0(2)$ and $F_0(3)$.

These effects could possibly have strong consequences for the organization of the chiral expansion in the $N_f = 3$ case [4, 6, 7, 8]. Let us remind that the general form of the Lagrangian of χPT is

$$\mathcal{L} = \sum_{m,n} \mathcal{L}^{(m,n)} \tag{8.4}$$

where

$$\mathcal{L}^{(m,n)} = \sum_k C_k^{(m,n)} O_k^{(m,n)}. \tag{8.5}$$

with LEC's $C_k^{(m,n)}$ and independent set of the operators $O_k^{(m,n)} = O(\partial^m m_f^n)$.

In order to be able to treat the double expansion consistently, it is necessary to assign a single integer parameter called *chiral order* to each term $\mathcal{L}^{(m,n)} = O(\partial^m m_f^n)$ of the effective Lagrangian. The terms $\mathcal{L}_k$ with chiral order k are then called $O(p^k)$ terms. Obviously, $\partial = O(p)$. The matter of discussion might be, however, the question concerning the chiral power of m_f. This question is intimately connected to the scenario according which the SSB of χS is realized.

The standard scenario [2, 3] corresponds to the assumption, that the SSB order parameters $\Sigma(N_f)$ and $F_0(N_f)$ are large in the sense, that the ratios

$$X(N_f) = \frac{2\widehat{m}\Sigma(N_f)}{F_\pi^2 M_\pi^2} \tag{8.6}$$

(where and $\widehat{m} = (m_u + m_d)/2$) and

$$Z(N_f) = \frac{F_0^2(N_f)}{F_\pi^2} \tag{8.7}$$

are close to one. Because $M_\pi^2 = O(p^2)$, it is then natural to take $m_f = O(p^2)$, i.e. $k = m + 2n$. This results in the *standard* χPT ($S\chi PT$ in what follows). This scenario seems to be experimentally confirmed [21] for $N_f = 2$; the recent analysis of the data yields [22]

$$X(2) = 0.81 \pm 0.07, \quad Z(2) = 0.89 \pm 0.03. \tag{8.8}$$

The $O(p^2)$ Lagrangian [2, 3]

$$\mathcal{L}_2 = \frac{F_0^2}{4}\left(\langle\partial_\mu U^+\partial^\mu U\rangle + 2B_0\langle U^+\mathcal{M} + \mathcal{M}^+U\rangle\right) \tag{8.9}$$

gives $\Sigma(2)_{LO} = \Sigma(3) = B_0F_0^2$ at the leading order, thus postponing the difference $\Sigma(2)-\Sigma(3)$ to higher orders. The same is true for the parameters $F_0(N_f)$. Let us also note that the quark mass ratio $r = m_s/\widehat{m}$ is not a free parameter here[4], at the leading order one has

$$r = 2\frac{M_K^2}{M_\pi^2} - 1. \tag{8.10}$$

An alternative way of chiral power counting for $N_f = 3$ is the *generalized* χPT ($G\chi PT$) [24, 25, 26, 27, 28, 29], originally designed to treat the scenario with small quark condensate $X(3) \ll 1$ and to take the quark mass ratio r as a free parameter. In the case $X(3) \ll 1$ it is natural to take $m_f = O(p)$ and $B_0 = O(p)$, this means $k = m + n$. In contrast to $S\chi PT$, there are also odd chiral orders and the $O(p^2)$ Lagrangian contains additional terms which are $O(p^4)$ within the standard chiral counting[5](see *e.g.* [24, 25, 28]):

$$\begin{aligned}\mathcal{L}_2 \;=\; & \frac{F_0^2}{4}\big(\langle\partial_\mu U^+\partial^\mu U\rangle + 2B_0\langle U^+\mathcal{M} + \mathcal{M}^+U\rangle + A_0\langle(U^+\mathcal{M})^2 + (\mathcal{M}^+U)^2\rangle \\ & + Z_0^P\langle U^+\mathcal{M} - \mathcal{M}^+U\rangle^2 + Z_0^S\langle U^+\mathcal{M} + \mathcal{M}^+U\rangle^2\big)\,.\end{aligned} \tag{8.11}$$

For the condensate $\Sigma = -\langle\overline{u}u\rangle$ we get at the leading order for $N_f = 3$

$$\Sigma_{LO} = B_0F_0^2 + Z_0^S(2\widehat{m} + m_s) = \Sigma(3) + Z_0^S(2\widehat{m} + m_s) \tag{8.12}$$

and therefore

$$\Sigma_{LO}(2) = \Sigma(3) + Z_0^S m_s. \tag{8.13}$$

[4]On the contrary, the value of r is usually taken as an input in standard $O(p^6)$ fits, see *e.g.* [23] and references therein.

[5]Effectively the generalized chiral power-counting means partial resummation of these terms.

This allows the difference $\Sigma(2)-$ $\Sigma(3)$ to appear already at the leading order, consistently with the small $\Sigma(3)$ scenario. The next-to-leading order Lagrangian $O(p^3)$

$$\mathcal{L}_3 = \frac{F_0^2}{4}\left(\xi\langle\partial_\mu U^+\partial^\mu U U^+\mathcal{M} + \mathcal{M}^+U\rangle + \widetilde{\xi}\langle\partial_\mu U^+\partial^\mu U\rangle\langle U^+\mathcal{M} + \mathcal{M}^+U\rangle + \ldots\right) \tag{8.14}$$

(where the ellipses stand for the additional terms which are of the order $O(p^6)$ in $S\chi PT$) gives rise to the $N_f = 3$ relation

$$F^2_{\pi,NLO} = F_0^2(3)(1 + 2\widetilde{\xi}(m_s + 2\widehat{m}) + 2\widehat{m}\xi), \tag{8.15}$$

which implies that the difference $F_0^2(2) - F_0^2(3)$ is treated as an effect of the next-to-leading order

$$F_0^2(2) = F_0^2(3)(1 + 2\widetilde{\xi}m_s). \tag{8.16}$$

Therefore, neither $S\chi PT$ nor $G\chi PT$ can accumulate the case of large fluctuation parameter $\widetilde{\xi}$ and the ratio $Z(3) \ll 1$ at the leading order.

Quite recently, a consistent method of handling the case $X(3)$, $Z(3) \ll 1$ was proposed [4, 6, 7, 8, 9]. Instead of changing the chiral power counting, it is based on a more careful manipulations with the chiral expansion. As it was discussed in the above references, the case $X(3)$, $Z(3) \ll 1$ could significantly influence the properties of the chiral expansion inducing instabilities of the perturbative series corresponding to the observables, which cannot be linearly related to the QCD correlators (such as the *ratios* like PGB masses, scattering amplitudes etc.). For such quantities, one should not perform a perturbative chiral expansion of the denominators but rather keep the ratios in a nonperturbative "resummed" form. The possibly large vacuum $\bar{s}s$ pair fluctuations are then parameterized in terms of $X(3)$, $Z(3)$ and r and treated as free parameters. We return to the detailed formulation of this recipe in the next section.

The aim of this paper is to illustrate the "resummed" form of the chiral expansion with special attention to its formal properties and to the details and subtleties of the general procedure. Motivated by our preliminary results on $\pi\eta$ scattering within the $G\chi PT$ [34], we have chosen the observables connected with this process as a concrete example which seems to be sensitive to the deviations from the standard assumption $X(3)$, $Z(3) \sim 1$ (note that some recent phenomenological studies suggest a possibility of $X(3) \sim 0.5$, cf. [30, 31, 32] and [6, 33]). Also, from the phenomenological point of view, the off-shell $\pi\eta\pi\eta^*$ vertex is a necessary building block for the non-resonant part of the amplitude for the rare decay $\eta \to \pi^0\pi^0\gamma\gamma$. Preliminary estimates within $G\chi PT$ [35, 36] suggest, that the effect of deviation of this off-shell vertex from the standard case might be at least in principle observed. The details will be presented elsewhere [37].

The amplitude of $\pi\eta$ scattering was already calculated within $S\chi PT$ to $O(p^4)$ (and within the extended $S\chi PT$ with explicit resonance fields) in the paper [38], where the authors presented prediction for the scattering lengths and phase shifts of the S, P and D partial waves. We quote here their $O(p^4)$ results for the S- and P-wave scattering lengths (in the units of the pion Compton wavelength): $a_0^{S\chi PT} = 7.2 \times 10^{-3}$ and $a_1^{S\chi PT} = -5.2 \times 10^{-4}$.

The paper is organized as follows. In Section 2 we recapitulate the motivation for the resummed version of χPT and the construction of the bare expansion of "good" observables. We make a detailed general discussion, connected with the four-meson amplitude, of the strict chiral expansion, the dispersive representation and matching of both approaches with stress to the reconciling of exact perturbative unitarity and exact renormalization scale independence in Section 3. Section 4 is devoted to the general properties of the $\pi\eta$ scattering amplitude. We discuss the kinematics, the definition of suitable "good" observables, the dispersion representation of the amplitude and the construction of the bare expansion. Various possibilities of its reparametrization are described in a detailed way in Section 5. The numerical illustration of the particular variants is made in Section 6, where we also numerically illustrate the subtleties of the construction of the bare expansion. We recapitulate the results of the standard variant of χPT and compare them with the resummed approach. We concentrate on the dependence on the LEC's as well as on the sensitivity to the higher order reminders and make an attempt to estimate their values using a matching with $G\chi PT$ and a simple version of resonance chiral theory. Section 8.7 contains the summary and conclusions. Some technical details are postponed to the appendices.

8.2 Motivation and basic notation

As we have mentioned in the Introduction, the potentially large vacuum fluctuations of the $\bar{s}s$ pairs might result in the instabilities of the chiral expansion, which originate in the possibility that for some observables the next-to-leading order correction could be numerically comparable with the leading order one. As discussed in [8, 9], this could generally cause problems with the convergence of the formal chiral expansion. Nevertheless, at least for some carefully defined "good" observables, it is natural to presume some sort of satisfactory convergence properties. Such "good" observables are assumed to be those which can be obtained directly from the low-energy correlation functions in the domain of their analyticity far away from singularities and which are related to the corresponding correlator *linearly* [8, 9]. Typical examples are the squares of the PGB decay constants F_P^2 , the products $F_P^2 M_P^2$ where M_P are the PGB masses and also the subthreshold parameters which can be derived from the products $A\prod_{i=1}^4 F_{P_i}$ where A is the PGB scattering amplitude $1+2 \to 3+4$. Let us write the expansion of such a "good" observable G in the form of a (carefully defined) *bare expansion* [8] as[6]

$$G = G^{(2)} + G^{(4)} + G\delta_G, \tag{8.17}$$

where $G^{(2)} = g^{(2)}(F_0, B_0, m_q)$ and $G^{(4)} = g^{(4)}(F_0, B_0, m_q, L_i, M_P^2)$ correspond to the sum of the leading and next-to-leading order terms respectively and the renormalization scale independent quantity δ_G accommodates the higher order remainders.

As a terminological note, in what follows we use the term *strict* chiral expansion for an unmodified expansion in terms of the LEC's strictly respecting the chiral orders. The *bare* expansion, though still expressed in terms of LEC's, accumulates some modifications dictated by physical requirements. It is the bare expansion which is assumed to be globally convergent.

For a "good" observable it is then assumed

$$|\delta_G| \ll 1 \tag{8.18}$$

as a natural assumption. This property of the bare chiral expansion (8.17) is called global convergence in [8, 9]. Note however, that the validity of the inequality (8.18) might depend on the definition of the reminder δ_G which is not fixed unambiguously and might differ according to the calculation scheme in use. We will comment on this point later on.

[6] Here we tacitly assume the standard chiral power counting. Analogous expansion could be written also for the generalized case.

The above mentioned possible instability in (8.17) appears when $G^{(2)} \sim G^{(4)}$, *i.e.* $X_G \lesssim 1$, where

$$X_G = \frac{G^{(2)}}{G}. \tag{8.19}$$

Such an instability manifests itself in the expansion of the observables depending on G non-linearly [8]. For instance, for a ratio of two "good" observables G and $G^{'}$ formally expanded in the form (8.17)

$$\frac{G}{G'} = \left(\frac{G^{(2)}}{G'^{(2)}}\right) + \left(\frac{G^{(2)}}{G'^{(2)}}\right)\left(\frac{G^{(4)}}{G^{(2)}} - \frac{G'^{(4)}}{G'^{(2)}}\right) + \frac{G}{G'}\delta_{G/G'}, \tag{8.20}$$

we get for the remainder $\delta_{G/G'}$

$$\delta_{G/G'} = \frac{(1 - X_{G'})(X_G - X_{G'})}{X_G'^2} + \frac{\delta_G}{X_{G'}} - \frac{X_G \delta_{G'}}{X_G'^2}. \tag{8.21}$$

For $X_{G'} \lesssim 1$ this might be numerically large even if both $|\delta_G|$, $|\delta_{G'}|$ were reasonably small. In this sense, a ratio of two globally convergent observables need not to be necessarily globally convergent too. It should be therefore much safer not to expand such "dangerous" observables and rather write the ratio in the "resummed" form

$$\frac{G}{G'} = \frac{G^{(2)} + G^{(4)}}{G'^{(2)} + G'^{(4)}} + \frac{G}{G'}\widetilde{\delta}_{G/G'}. \tag{8.22}$$

The relation (8.22) is an exact algebraic identity provided we keep explicitly the remainder

$$\widetilde{\delta}_{G/G'} = \frac{\delta_G - \delta_{G'}}{1 - \delta_{G'}}. \tag{8.23}$$

In this case $\widetilde{\delta}_{G/G'}$ remains for $|\delta_G|, |\delta_{G'}| \ll 1$ under numerical control.

Of course, only the fact that the bare expansion of some observable is not globally convergent does not necessarily correspond to the collapse of the convergence, because the next-to-next-to-leading order $G^{(6)}$ can saturate the series in such a way that the next-to-next-to-leading remainder

$$G\delta_G^{NNLO} = G - G^{(2)} - G^{(4)} - G^{(6)} \tag{8.24}$$

is reasonably small. Namely this is the usual assumption behind the $O(p^6)$ calculations. Violation of the global convergence property here means merely that the $O(p^6)$ contribution have unnatural size, *i.e.* $G^{(6)} \lesssim G^{(2)} + G^{(4)}$. This could, however, destabilize the $O(p^6)$ chiral expansion of ratios in the way similar to that discussed above.

Provided we allow the expansion of the "good" observables only, we are also pressed to modify the next step leading from the bare expansion to the usual output of the χPT, consisting of a reparametrization of the expansion by expressing some of the LEC's in terms of the physical observables such as masses and PGB decay constants. This step converts the series into an expansion in powers and logs of the (squared) PGB masses instead of quark masses. To achieve this, it is either necessary to invert a bare chiral expansion of some observable (in the case of the $O(p^2)$ LEC's) or to use an observable which might be generally a "dangerous" one. Let us briefly discuss the first case. Schematically, suppose that some

$O(p^2)$ LEC G_0 (*e.g.* F_0^2) just corresponds to the leading term $G^{(2)}$ of the expansion of the observable G. Then we can write an algebraic identity

$$G_0 = G - G^{(4)}(G_0) - G\delta_G, \tag{8.25}$$

where we explicitly point out the dependence of the next-to-leading term on G_0. To convert this expansion and express G_0 by means of the series in G one substitutes G for G_0 on the right hand side. This defines a new remainder δ_{G_0}

$$G_0 = G - G^{(4)}(G) + G_0\delta_{G_0}, \tag{8.26}$$

for which we get

$$\delta_{G_0} = -\frac{1 - X_G}{X_G} + \frac{1}{X_G}\frac{G^{(4)}(G)}{G}. \tag{8.27}$$

This could cause an instability of the converted expansion for G_0 in terms of G for $X_G \lesssim 1$ even if the relative size of the next-to-leading order $G^{(4)}(G)/G$ is reasonably small, irrespective of the condition for global convergence $|\delta_G| \ll 1$.

On the other hand, suppose that some $O(p^4)$ constant G_1 coincides with the next-to-leading term $G^{(4)}$. In this case we have an algebraic identity for G_1

$$G_1 = G - G^{(2)} - G\delta_G \tag{8.28}$$

and the remainder here is perfectly under control, provided G has a globally convergent bare expansion and we *do not* re-express $G^{(2)}$ in terms of physical observables (i.e. provided we treat the $O(p^2)$ LEC's as free parameters).

From the above simple considerations follows that in order to avoid potential problems with the instabilities of the chiral expansion, which might be present in the three-flavor χPT in the case of small $X(3)$ and $Z(3)$ (cf. (8.6, 8.7)), we should [8, 9]

- carefully define the bare expansion
- confine ourselves (as far as the bare chiral expansion is concerned) to the linear space of "good" observables and keep the "dangerous" observables in the nonperturbative "resummed" form
- use rather $\Sigma(3)$, $F_0(3)$ (or $X(3)$ and $Z(3)$) and $r = 2m_s/(m_u + m_d)$ as free parameters[7] instead of expressing them in the form of the series in PGB masses and decay constants
- eliminate the $O(p^4)$ LEC's algebraically, using bare expansions of "good" observables such as F_P^2, $F_P^2 M_P^2$ [8].

In the next section we shall illustrate the possible subtleties of the first step of this general recipe on the concrete example[9] of the PGB scattering amplitude $P_1P_2 \to P_3P_4$

[7]Note, that r is related to the "dangerous" observable

$$2\frac{F_K^2 M_K^2}{F_\pi^2 M_\pi^2} - 1 = r + \ldots$$

[8]We will do it for $L_4 - L_8$ but leave $L_1 - L_3$ free, also L_7 is a special case, see in what follows.

[9]We shall tacitly assume the case of three light flavours in what follows.

8.3 Bare expansion for the scattering $P_1P_2 \to P_3P_4$

8.3.1 Chiral expansion of the "good" observable

Let us assume a scattering of pseudoscalar mesons $P_1P_2 \to P_3P_4$ with masses M_{P_i}. The amplitude $S(s,t;u)$ is defined as

$$\langle P_3(k_3)P_4(k_4)_{out}|P_1(k_1)P_2(k_2)_{in}\rangle = \mathrm{i}(2\pi)^4\delta^{(4)}(k_3+k_4-k_1-k_2)S(s,t;u), \tag{8.29}$$

where s, t and u are the usual Mandelstam variables. The amplitude is related to the "good" observable[10]

$$G(s,t;u) = \prod_{i=1}^{4} F_{P_i} S(s,t;u), \tag{8.30}$$

(where F_{P_i} are the decay constants) which can be directly obtained from the (cut) four-point function of the axial currents. Let us write for $G(s,t;u)$ the following *strict* chiral expansion in terms of the low energy constants

$$G = G^{(2)} + G^{(4)}_{ct} + G^{(4)}_{tad} + G^{(4)}_{unit} + G\delta_G. \tag{8.31}$$

$G\delta_G$ accommodates the higher order remainders. Using the functional method, G can be obtained from the generating functional

$$\begin{aligned} F_0^4 Z[U,v,p,a,s] &= F_0^4 \int d^4x \left(\mathcal{L}^{(2)}(U,v,p,a,s) + \mathcal{L}^{(4)}(U,v,p,a,s)\right) \\ &\quad + F_0^4 Z^{(4)}_{loop}[U,v,p,a,s] + \ldots \end{aligned} \tag{8.32}$$

by setting $v = s = p = 0$, $s = 2B_0\mathcal{M}$ and expanding in the fields Φ where $U = \exp(i\Phi/F_0)$. Following the notation in [3], we have

$$\begin{aligned} Z^{(4)}_{loop}[U,v,p,a,s] &= Z^{(4)}_{tad}[U,v,p,a,s] + Z^{(4)}_{unit}[U,v,p,a,s] \\ &= \frac{i}{2}\ln\det D_0 + \frac{i}{4}\mathrm{Tr}(D_0^{-1}\delta) - \frac{i}{4}\mathrm{Tr}(D_0^{-1}\delta D_0^{-1}\delta) + \ldots . \end{aligned} \tag{8.33}$$

In the above formulae,

$$D_0^{ab} = \delta^{ab}\Box + \frac{1}{2}B_0 tr(\{\lambda^a,\lambda^b\}\mathcal{M}) \tag{8.34}$$

and $\mathcal{M}$ is the quark mass matrix. Note this representation of $Z^{(4)}_{loop}$ assumes that the masses running in the loops are the $O(p^2)$ masses rather than the physical masses. Or, in more detail, provided we start with the chiral expansion of the squared product of the masses and decay constants

$$F_P^2 M_P^2 = (F_P^2 M_P^2)^{(2)} + (F_P^2 M_P^2)^{(4)} + F_P^2 M_P^2 \delta_{FM_P}, \tag{8.35}$$

the masses in the loops are defined as

$$\overset{o}{M}{}^2_P = \frac{(F_P^2 M_P^2)^{(2)}}{F_0^2}. \tag{8.36}$$

[10]Strictly speaking, the "good" observables correspond to the subthreshold parameters derived form $G(t,s;u)$ in an unphysical point away from singularities.

Note, however, that this is the first term in a potentially "dangerous" expansion of the ratio

$$M_P^2 = \frac{F_P^2 M_P^2}{F_P^2} = \frac{(F_P^2 M_P^2)^{(2)} + (F_P^2 M_P^2)^{(4)} + F_P^2 M_P^2 \delta_P}{F_0^2 + (F_P^2)^{(4)} + F_P^2 \delta_{F_P}} = \overset{o}{M}{}_P^2 + \ldots \tag{8.37}$$

From this definition of $Z_{loop}^{(4)}$ we obtain $G^{(4)} = G_{ct}^{(4)} + G_{tad}^{(4)} + G_{unit}^{(4)}$ which is exactly renormalization scale independent even for the external momenta off-shell. This meets the requirement of the renormalization scale independence of the remainder δ_G.

The first two terms of the above strict chiral expansion for $G(s,t;u)$ have a serious drawback in the sense that the singularities in the complex stu planes required by unitarity are not placed at the physical thresholds but rather at points given by the leading order terms $\overset{o}{M}_P$ of the chiral expansion of the PGB masses. Straightforward substitution $\overset{o}{M}_P \to M_P$ in the propagators of the loops, which apparently means merely a redefinition of the remainder δ_G, could, however, in general spoil its exact renormalization scale independence. It is therefore desirable to use the freedom in the definition of the remainder more carefully in order to reconcile both scale independence of $G^{(4)}$ and unitarity. For this purpose, a useful tool is the matching with a dispersive representation [8] of the amplitude $S(s,t;u)$ based on the reconstruction theorem [25, 29].

8.3.2 Dispersive representation for $G(s,t;u)$

The above mentioned reconstruction theorem for the PGB scattering amplitude is based on the basic properties of unitarity, analyticity and crossing symmetry and provides us with the most general form of the PGB scattering amplitude up to the order $O(p^6)$ in terms of dispersive integrals with known discontinuities. It was first proved for the case of $\pi\pi$ scattering in [25, 29] and for πK scattering in [39, 32] and since then it has been intensively used in various contexts. Here we use the general form of the theorem, more detailed discussion of which will be presented elsewhere [40].

For the scattering of pseudoscalar mesons $P_1 P_2 \to P_3 P_4$, let us denote the $s-$, $t-$ and $u-$channel amplitudes as $S(s,t;u)$, $T(s,t;u)$ and $U(s,t;u)$ and write their partial wave expansion as

$$A(s,t;u) = 32\pi \sum_{l=0}^{\infty} (2l+1) A_l(s) P_l(\cos\theta_A), \tag{8.38}$$

where $A = S$, T, U and

$$\cos\theta_A = \frac{s(t-u) + \Delta_{A_i}\Delta_{A_f}}{\lambda_{A_i}^{1/2}(s)\lambda_{A_f}^{1/2}(s)}. \tag{8.39}$$

Here $A_l(s)$ are the partial waves,

$$\lambda_{A_{i,f}}(s) = (s - (M_{P_j} + M_{P_k})^2)(s - (M_{P_j} - M_{P_k})^2) \tag{8.40}$$

is the triangle function which corresponds to the initial/final state $A_{i,f}$ (consisting of the pseudoscalars $P_j P_k$) of the process in the channel A and

$$\Delta_{A_{i,f}} = M_{P_j}^2 - M_{P_k}^2. \tag{8.41}$$

According to the theorem, we get the following representation for the amplitude $S(s,t;u)$

$$S(s,t;u) = \mathcal{S}(s,t;u) + \mathcal{S}_{unit}(s,t;u) + O(p^8), \tag{8.42}$$

where $\mathcal{S}(s,t;u)$ is a third order polynomial with the same symmetries as the whole amplitude $S(s,t;u)$. The nontrivial analytical properties are incorporated in the unitarity part $\mathcal{S}_{unit}(s,t;u)$, which can be expressed as

$$\begin{aligned}\mathcal{S}_{unit}(s,t;u) &= \Phi^S(s)+\Phi^T(t)+\Phi^U(u)\\ &+[s(t-u)+\Delta_{12}\Delta_{34}]\Psi^S(s)\\ &+[t(s-u)+\Delta_{13}\Delta_{24}]\Psi^T(t)\\ &+[u(t-s)+\Delta_{14}\Delta_{23}]\Psi^U(u).\end{aligned} \tag{8.43}$$

In the last expression, $\Delta_{ij}=M_{P_i}^2-M_{P_j}^2$. The functions $\Phi^A(s)$ and $\Psi^A(s)$ with $A=S,T,U$ are analytic in the cut complex plane with the right hand cut from $\tau_A=\min_{i,j}(M_{P_i}+M_{P_j})^2$ (where P_iP_j are the possible intermediate states in the given channel A) to infinity with discontinuities given by the formulae

$$\mathrm{disc}\,\Phi^A(s) = 32\pi\theta(s-\tau_A)\mathrm{disc}\,A_0(s) \tag{8.44}$$

$$\mathrm{disc}\,\Psi^A(s) = 96\pi\theta(s-\tau_A)\mathrm{disc}\frac{A_1(s)}{\lambda_{A_i}^{1/2}(s)\lambda_{A_f}^{1/2}(s)}. \tag{8.45}$$

Here $A_0(s)$, $A_1(s)$ are the corresponding $l=0,1$ partial waves.

Consequently, once the right hand sides of (8.44, 8.45) are known, the unitarity part $\mathcal{S}_{unit}(s,t;u)$ of the amplitude can be uniquely reconstructed to $O(p^6)$ up to the polynomial, which encompass subtraction polynomials for the dispersion integrals.

Let us now assume the chiral expansion of the amplitudes in the form

$$A(s,t;u) = A^{(2)}(s,t;u)+A^{(4)}(s,t;u)+A\delta_A, \tag{8.46}$$

$$A^{(n)}(s,t;u) = 32\pi\sum_{l=0}^{\infty}(2l+1)A_l^{(n)}(s)P_l(\cos\theta_s(t)). \tag{8.47}$$

Starting from the $O(p^2)$ amplitudes, we can use the two particle partial wave unitarity to get the discontinuity of the partial waves $A_l^{(4)}(s)$ along the right hand cut[11]

$$\mathrm{disc}A_l^{(4)}(s)=\sum_{ij}\frac{2}{z_{ij}}\frac{\lambda_{ij}^{1/2}(s)}{s}A_l^{(2)ij\to A_f}(s)A_l^{(2)ij\to A_i}(s)^*+O(p^6) \tag{8.48}$$

Here $z_{ij}=1,2$ is a symmetry factor taking into account the possibility of identical particles in the intermediate state ij. Inserting this into the dispersive integrals we easily[12] get a minimal form for the $O(p^4)$ unitarity corrections in terms of the functions $\Phi^{(4)A}(s)$ and $\Psi^{(4)A}(s)$ reconstructed from the $O(p^2)$ amplitudes

$$\Phi^{(4)A}(s) = (32\pi)^2\sum_{ij}\frac{1}{z_{ij}}\overline{\overline{J}}_{ij}(s)A_0^{(2)A_i\to ij}(s)A_0^{(2)ij\to A_f}(s)^* \tag{8.49}$$

$$\Psi^{(4)A}(s) = \frac{(96\pi)^2}{3}\sum_{ij}\frac{1}{z_{ij}}\overline{\overline{J}}_{ij}(s)\frac{A_1^{(2)A_i\to ij}(s)A_1^{(2)ij\to A_f}(s)^*}{\lambda_{A_i}^{1/2}(s)\lambda_{A_f}^{1/2}(s)}. \tag{8.50}$$

[11] It can be shown that more than two particle intermediate states yield contribution of the order $O(p^8)$ and higher.

[12] Note that $A^{(2)}(s,t;u)$ are real polynomials of the first order in s, t and u.

$\overline{\overline{J}}_{ij}(s) = J^r_{ij}(s) - J^r_{ij}(0) - sJ^{r\prime}_{ij}(s)$ corresponds to the twice subtracted scalar bubble with internal line masses M_{P_i,P_j}. Provided $\Delta_{A_i} = 0$ or $\Delta_{A_f} = 0$, which will be our case, it can be shown that we only need one subtraction, $\overline{J}_{ij}(s) = J^r_{ij}(s) - J^r_{ij}(0)$ instead of $\overline{\overline{J}}_{ij}(s)$. The explicit form of the function $\overline{J}_{ij}(s)$ is given in the Appendix 8.11.

The above formulae can be used to write a dispersive representation of the "good" observable $G(s,t;u) = \prod_{i=1}^4 F_{P_i} S(s,t;u)$ to the next-to-leading order in the form

$$G(s,t;u) = \mathcal{G}(s,t;u) + \mathcal{G}_{unit}(s,t;u), \tag{8.51}$$

where $\mathcal{G}(s,t;u)$ is the polynomial part and the unitarity corrections up to $O(p^6)$ are included in

$$\begin{aligned}\mathcal{G}_{unit}(s,t;u) &= \phi^S(s) + \phi^T(t) + \phi^U(u)\\ &\quad +[s(t-u) + \Delta_{12}\Delta_{23}]\psi^S(s)\\ &\quad +[t(s-u) + \Delta_{13}\Delta_{24}]\psi^T(t)\\ &\quad +[u(t-s) + \Delta_{14}\Delta_{23}]\psi^U(u).\end{aligned} \tag{8.52}$$

Our goal is to write down a representation of $\phi^{(4)A}$ and $\psi^{(4)A}$, which, notice, are distinct quantities from $\Phi^{(4)A}$ and $\Psi^{(4)A}$, analogous to (8.49, 8.50). Note, however, that while the relation of $G(s,t;u)$ and $S(s,t;u)$ is unambiguously fixed to all orders by (8.30), the amplitude can be defined order by order in various ways. For example, for the "good" observable G, the leading order piece $G^{(2)}$ of its strict chiral expansion is fixed by the lowest order Lagrangian $\mathcal{L}^{(2)}$, but the corresponding $O(p^2)$ piece of the amplitude S can be related in various ways. Similarly, the same is true order by order, where the amplitude at the given order can be defined up to higher order corrections.

The most straightforward way is to write a safe expansion for $S(s,t;u)$ in the form

$$S(s,t;u) = \left(\prod_{i=1}^4 F_{P_i}\right)^{-1} \left(G^{(2)}(s,t;u) + G^{(4)}(s,t;u) + G\delta_G\right) \tag{8.53}$$

with physical values of F_{P_i}, thus satisfying the relation (8.30) order by order

$$S^{(n)}(s,t;u) = \left(\prod_{i=1}^4 F_{P_i}\right)^{-1} G^{(n)}(s,t;u). \tag{8.54}$$

As we'll see, the minimal modification of the form derived from the generating functional is obtained by using an alternative, potentially "dangerous" expansion

$$\begin{aligned}S(s,t;u) &= \left(\prod_{i=1}^4 F_{P_i}\right)^{-1} G(s,t;u)\\ &= \left(\prod_{i=1}^4 F_0\Big(1 + \frac{1}{2}\frac{(F^2_{P_i})^{(4)}}{F_0^2} + \ldots\Big)\right)^{-1} \left(G^{(2)}(s,t;u) + G^{(4)}(s,t;u) + \ldots\right)\\ &= F_0^{-4} G^{(2)}(s,t;u) - \frac{1}{2}F_0^{-6}G^{(2)}(s,t;u)\sum_{i=1}^4 (F^2_{P_i})^{(4)} + F_0^{-4}G^{(4)}(s,t;u) + \ldots\end{aligned} \tag{8.55}$$

which defines

$$\widetilde{S}^{(2)}(s,t;u) = F_0^{-4}G^{(2)}(s,t;u) \tag{8.56}$$

$$\widetilde{S}^{(4)}(s,t;u) = F_0^{-4}G^{(4)}(s,t;u) - \frac{1}{2}F_0^{-6}G^{(2)}(s,t;u)\sum_{i=1}^{4}(F_{P_i}^2)^{(4)}. \tag{8.57}$$

The representation of $\phi^{(4)A}$ and $\psi^{(4)A}$ is therefore not unique. According to our definitions of the amplitude we get either (we assume partial wave expansion of $G(s,t;u)$ analogous to (8.38))

$$\phi^{(4)A}(s) = (32\pi)^2\sum_{ij}\frac{1}{z_{ij}}\frac{\overline{\overline{J}}_{ij}(s)}{F_{P_i}^2F_{P_j}^2}G_0^{(2)A_i\to ij}(s)G_0^{(2)ij\to A_f}(s)^* \tag{8.58}$$

$$\psi^{(4)A}(s) = \frac{(96\pi)^2}{3}\sum_{ij}\frac{1}{z_{ij}}\frac{\overline{\overline{J}}_{ij}(s)}{F_{P_i}^2F_{P_j}^2}\frac{G_1^{(2)A_i\to ij}(s)G_1^{(2)ij\to A_f}(s)^*}{\lambda_{A_i}^{1/2}(s)\lambda_{A_f}^{1/2}(s)}, \tag{8.59}$$

corresponding to the definition (8.54) or

$$\widetilde{\phi}^{(4)A}(s) = (32\pi)^2F_0^{-4}\sum_{ij}\frac{1}{z_{ij}}\overline{\overline{J}}_{ij}(s)G_0^{(2)A_i\to ij}(s)G_0^{(2)ij\to A_f}(s)^* \tag{8.60}$$

$$\widetilde{\psi}^{(4)A}(s) = \frac{(96\pi)^2}{3}F_0^{-4}\sum_{ij}\frac{1}{z_{ij}}\overline{\overline{J}}_{ij}(s)\frac{G_1^{(2)A_i\to ij}(s)G_1^{(2)ij\to A_f}(s)^*}{\lambda_{A_i}^{1/2}(s)\lambda_{A_f}^{1/2}(s)}, \tag{8.61}$$

when reconstructing the bare expansion of G from the "dangerous" expansion (8.55) and using the definitions (8.56, 8.57) for the $O(p^2)$ and $O(p^4)$ amplitudes.

8.3.3 Matching the strict chiral expansion to the dispersive representation

The dispersive representation (8.51) can be now matched to the formula (8.31). As we have mentioned above, the positions of the cuts in the formulas (8.31) and (8.51) are not the same; in the former case they correspond to the $O(p^2)$ masses (8.36), which ensures the renormalization scale independence, while in the latter they are determined by the physical ones, as required by the unitarity conditions. In order to reconcile both these requirements, one can proceed as follows (c.f. also [8]).

In (8.31), the nonanalytic terms are generally of the form $P(s)J_{ij}^r(s)$, where $J_{ij}^r(s)$ is the renormalized scalar bubble defined in Appendix 8.11 and $P(s)$ is some second order polynomial. As the first step, one rewrites these expressions in terms of $\overline{\overline{J}}_{ij}(s)$ writing $J_{ij}^r(s) = J_{ij}^r(0) + s\overline{J}_{ij}'(0) + \overline{\overline{J}}_{ij}(s)$. This adjustment allows us to split G uniquely into a polynomial part G_{pol} and a nonanalytic part G_{cut} which accumulates the unitarity cuts

$$G(s,t;u) = G_{pol}(s,t;u) + G_{cut}(s,t;u) + G\delta_G, \tag{8.62}$$

where

$$G_{pol}(s,t;u) = (G(s,t;u) - G\delta_G)\,|_{\overline{J}_{ij}=\overline{\overline{J}}_{ij}=0}. \tag{8.63}$$

Both parts are now renormalization scale independent.

As a second step, we replace the $G_{cut}(s,t;u)$ with $\mathcal{G}_{unit}(s,t;u)$ from (8.51). This means we write

$$G(s,t;u) = G_{pol}(s,t;u) + \mathcal{G}_{unit}(s,t;u) + G\delta_G^{'} \tag{8.64}$$

where $\delta_G^{'}$ is a new remainder defined by this equation. According to the naive chiral power counting, $G_{cut}(s,t;u) - \mathcal{G}_{unit}(s,t;u) = O(p^6)$.

The third step, not necessary from the point of view of preserving unitarity and renormalization scale invariance, consists of a further modification of $G_{pol}(s,t;u)$ by means of replacement of the $O(p^2)$ masses $\overset{o}{M}{}_P^2$ in $J_{ij}^r(0)$ with the physical masses M_P^2. This replacement does not spoil the renormalization scale independence of the $G_{pol}(s,t;u)$ and corresponds to the convention introduced in [8, 9]. This again means a redefinition of the remainders $\delta_G^{'}$, *i.e.* re-shuffling of the terms of the next-to-next-to-leading order.

Note that the origin of $J_{ij}^r(0)$'s in one loop generating functional (8.33) is twofold: they can stem either from the tadpole part $Z_{tad}^{(4)}$ or from the unitarity corrections $Z_{unit}^{(4)}$. It was argued in [9] that in the former case the above mentioned replacement does not necessarily modify the numerical value of the remainders much. The reason should be that the chiral logs appear only in the combination $\mu_P \propto \overset{o}{M}{}_P^2 \ln(\overset{o}{M}{}_P^2/\mu^2)$. The replacement here means

$$\overset{o}{M}{}_P^2 \ln(\overset{o}{M}{}_P^2/\mu^2) \to \overset{o}{M}{}_P^2 \ln(M_P^2/\mu^2). \tag{8.65}$$

Because $\overset{o}{M}{}_P^2 \propto Y = X/Z$, the difference should therefore either be small for $Y \sim 1$ (where $\overset{o}{M}{}_P^2 \sim M_P^2$) or the contribution of μ_P itself is tiny for $Y \to 0$.

On the other hand, the logs from $Z_{unit}^{(4)}$ do not generally come with such a prefactor. Therefore, with a replacement $\overset{o}{M}{}_P^2 \to M_P^2$ inside $J_{ij}^r(0)$, one might create large differences between the "old" and "new" remainders due to the enhancement of the contributions of chiral logs for small Y. However, without the replacement inside the chiral logs of this type we could expect an unphysical increase (and irregularities) of the observables for $Y \to 0$. Also, here the replacement is natural physically, remember that the matching with the dispersive representation consist essentially of an analogous replacement within the unitary corrections. Let us also note that the splitting of the generating functional into the tadpole and unitarity part is not unique (it depends *e.g.* on the parametrization of the fluctuations around the classical solution of the $O(p^2)$ field equations in the functional integral), though the sum must be independent on this and therefore it is more consistent to use the same rule for both[13]. Nevertheless, it could be of some worth to test the differences between various treatments of the chiral logs numerically (see Subsection 8.6.2).

The resulting *bare* expansion (8.64) now not only meets the requirement of the exact scale independence of the remainder $\delta_G^{'}$, it has also correct physical location of the unitarity cuts. Of course, we could achieve the last property simply by inserting physical masses into the functions $\overline{\overline{J}}_{ij}(s)$ in $G_{cut}(s,t;u)$. The replacement $G_{cut}(s,t;u) \to \mathcal{G}_{unit}(s,t;u)$ has

[13] Also notice that the offending Y dependence of the chiral logs with $O(p^2)$ masses inside comes always in the combination Y/μ^2 where μ is the renormalization scale. Provided we were able to reparametrize the bare expansion in such a way that all the running $O(p^4)$ constants were completely expressed in terms of the physical observables, the explicit independence on μ would at the same time guarantee an elimination of the irregularities for $Y \to 0$. Such a treatment has to include the reparametrization of L_1-L_3, which is, however, beyond the scope of our paper.

however another advantage. Namely, using the prescription (8.58), (8.59), the corresponding amplitude, written in the form (without any expansion of the denominator)

$$S(s,t;u) = \frac{G(s,t;u)}{\prod_{i=1}^{4} F_{P_i}} \tag{8.66}$$

satisfies the relations of perturbative unitarity (with $S^{(2)}$ and $S^{(4)}$ given by (8.54))

$$\mathrm{disc} S_l^{(4)}(s) = \sum_{ij} \frac{2}{z_{ij}} \frac{\lambda_{ij}^{1/2}(s)}{s} S_l^{(2)ij\to A_f}(s) S_l^{(2)ij\to A_i}(s)^* \tag{8.67}$$

exactly (*i.e.* not only modulo the next-to-next-to-leading correction), which can be sometimes technically useful (*e.g.* for the unitarization by means of the inverse amplitude method, cf. [41]). The same is true using the prescription (8.60), (8.61) with $\widetilde{S}^{(2)}$ and $\widetilde{S}^{(4)}$ given by (8.56, 8.57). As we shall see in what follows, the latter prescription gives a minimal modification of the strict expansion (8.31) compatible with exact perturbative unitarity.

8.4 General properties of $\pi\eta$ scattering amplitude

8.4.1 Basic notation

Let us denote the $s-$and $u-$ channel amplitude in the isospin conservation limit as

$$\langle \pi^b(p^b)\eta(q)_{out}|\pi^a(p^a)\eta(p)_{in}\rangle = \mathrm{i}(2\pi)^4\delta(P_f - P_i)\delta^{ab}S(s,t;u) \tag{8.68}$$

and the crossed amplitude in the $t-$ channel as

$$\langle \eta(p)\eta(q)_{out}|\pi^a(p^a)\pi^b(p^b)_{in}\rangle = \mathrm{i}(2\pi)^4\delta(P_f - P_i)\delta^{ab}T(s,t;u). \tag{8.69}$$

Crossing and Bose symmetries then yield

$$\begin{aligned} T(s,t;u) &= S(t,s;u) \\ S(s,t;u) &= S(u,t;s) \\ T(s,t;u) &= T(s,u;t). \end{aligned} \tag{8.70}$$

Writing the partial wave expansion as

$$\begin{aligned} S(s,t;u) &= 32\pi\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta_s)S_l(s), \\ \cos\theta_s &= \frac{(t-u)s+\Delta_{\eta\pi}^2}{\lambda_{\eta\pi}(s)}, \end{aligned} \tag{8.71}$$

the scattering lengths a_l and phase shifts $\delta_l(s)$ are given by the formulae

$$\begin{aligned} \mathrm{Re}S_l(s) &= \frac{\sqrt{s}}{4}P^{2l}(a_l + O(P^2)) \quad \text{for} \quad P\to 0\,,\ s\to(M_\eta+M_\pi)^2 \\ \delta_l(s) &= \arctan\left(\frac{4P}{\sqrt{s}}\mathrm{Re}S_l(s)\right), \end{aligned} \tag{8.72}$$

where $P = \lambda_{\eta\pi}^{1/2}(s)/2\sqrt{s}$ is the CMS momentum. I.e., in the units of (pion Compton wavelength)$^{2l+1}$

$$a_l = M_\pi^{2l+1} \lim_{P\to 0} \frac{4}{\sqrt{s}P^{2l}} \mathrm{Re} A_l(s). \tag{8.73}$$

Let us also define the subthreshold parameters c_{ij} in terms of the expansion of the amplitude in the point of analyticity $t = 0$, $s = u = \Sigma_{\eta\pi} = M_\eta^2 + M_\pi^2$

$$S(s,t;u) = \sum_{i,j} c_{ij} t^i \nu^{2j} \tag{8.74}$$

where

$$\nu = \frac{s-u}{4M_\eta} = \frac{2s + t - 2\Sigma_{\eta\pi}}{4M_\eta}. \tag{8.75}$$

The dimension c_{ij} is $\dim[c_{ij}] = mass^{-2i-2j}$, in what follows we will refer to the dimensionless numbers $c_{ij}M_\pi^{2i+2j}$. Let us note that in the limit $m_u = m_d = 0$ we have two Adler zeros at $p^a = 0$ and $p^b = 0$, which implies the following $SU(2)_L \times SU(2)_R$ theorem

$$\lim_{m_u = m_d \to 0} c_{00} = 0. \tag{8.76}$$

We can also quote the low-energy current algebra result [42]

$$S(s,t;u) = \frac{M_\pi^2}{3F_\eta^2}, \tag{8.77}$$

which is in agreement with (8.76).

8.4.2 Dispersive representation

As a result of the symmetry properties of the amplitudes, the dispersive representation to the next-to-leading order (8.52) for $G_{\pi\eta}(s,t;u) = F_\pi^2 F_\eta^2 S(s,t;u)$ simplifies, namely $\phi^S = \phi^U \equiv \phi$ and $\psi^S = \psi^U \equiv \psi$. The intermediate states in (8.58, 8.59) are[14] $\pi\eta$ and $\overline{K}K$ in the s and u channels and $\pi\pi$, $\eta\eta$ and $\overline{K}K$ in the $t-$channel. This implies $\psi(s) = O(p^6)$, because the $P-$waves in the $s-$channel start at $O(p^4)$ due to the low-energy theorem (8.77) for the $\pi\eta \to \pi\eta$ amplitude and as a result of charge conjugation invariance of the $\pi\eta \to \overline{K}K$ amplitude. Moreover, $\psi^T = 0$, because the partial wave decomposition of the $t-$channel amplitude $T(s,t;u)$ contains only even partial waves due to Bose symmetry and charge conjugation. We therefore get

$$G_{\pi\eta}(s,t;u) = G_{\pi\eta,\,pol}(s,t;u) + \mathcal{G}_{\pi\eta,unit}(s,t;u) + O(p^6), \tag{8.78}$$

where the polynomial part has the following general form

$$G_{\pi\eta,\,pol}(s,t;u) = \alpha + \beta t + \gamma t^2 + \omega(s-u)^2. \tag{8.79}$$

Note that the parameters $\alpha, \ldots, \omega$ are related to the expansion of the Green function $G_{\pi\eta}(s,t;u)$ at the point of analyticity $t = 0$, $s = u = \Sigma_{\eta\pi} = M_\eta^2 + M_\pi^2$ and therefore they represent "good observables" according to our classification.

[14] Here we assume isospin conservation.

The dispersive part is

$$\mathcal{G}_{\pi\eta,unit}(s,t;u) = \phi^T(t) + \phi(s) + \phi(u), \tag{8.80}$$

where $\phi^T(t)$ and $\phi(s)$ are given by the formula (8.58). A complete list of relevant leading order contributions $G_{0,1}^{(2)12\to ij}$ and $G_{0,1}^{(2)ij\to 34}$ can be found in Appendix 8.10, here we give the resulting expressions (transcription to the convention (8.60, 8.61) is straightforward)

$$\begin{aligned}
\phi(s) &= F_0^4\left\{\frac{1}{9}\overset{o}{M}{}_\pi^4\frac{\overline{J}_{\pi\eta}(s)}{F_\pi^2F_\eta^2}\right.\\
&+\frac{3}{8}[(s-\frac{1}{3}M_\eta^2-\frac{1}{3}M_\pi^2-\frac{2}{3}M_K^2)\\
&\left.-\frac{1}{3}(2\overset{o}{M}{}_K^2-\overset{o}{M}{}_\pi^2-\overset{o}{M}{}_\eta^2)]^2\frac{\overline{J}_{KK}(s)}{F_K^4}\right\},\\
\phi^T(s) &= F_0^4\left\{\frac{1}{3}\overset{o}{M}{}_\pi^2[(s-\frac{4}{3}M_\pi^2)+\frac{5}{6}\overset{o}{M}{}_\pi^2]\frac{\overline{J}_{\pi\pi}(s)}{F_\pi^4}\right.\\
&-\frac{1}{18}\overset{o}{M}{}_\pi^2\left(\overset{o}{M}{}_\pi^2-4\overset{o}{M}{}_\eta^2\right)\frac{\overline{J}_{\eta\eta}(s)}{F_\eta^4}\\
&+\frac{1}{8}[(s-\frac{2}{3}M_\pi^2-\frac{2}{3}M_K^2)+\frac{2}{3}(\overset{o}{M}{}_K^2+\overset{o}{M}{}_\pi^2)]\\
&\left.\times[(3s-2M_K^2-2M_\eta^2)+(2\overset{o}{M}{}_\eta^2-\frac{2}{3}\overset{o}{M}{}_K^2)]\frac{\overline{J}_{KK}(s)}{F_K^4}\right\}.
\end{aligned}\tag{8.81}$$

In terms of these functions, we have (notice that $\phi^T(0)=0$)

$$\begin{aligned}
a_0 &= \frac{1}{8\pi F_\eta^2F_\pi^2}\frac{M_\pi}{(M_\pi+M_\eta)}\left(\alpha+16\omega M_\eta^2M_\pi^2+\phi((M_\pi+M_\eta)^2)+\phi((M_\eta-M_\pi)^2)\right)\\
a_1 &= \frac{1}{12\pi F_\eta^2F_\pi^2}\frac{M_\pi^3}{(M_\pi+M_\eta)}\left(\beta+8\omega M_\eta M_\pi+\phi^{T'}(0)-\phi^{'}((M_\eta-M_\pi)^2)\right)
\end{aligned}\tag{8.82}$$

and

$$\begin{aligned}
c_{00} &= \frac{1}{F_\eta^2F_\pi^2}\left(\alpha+2\phi(\Sigma_{\eta\pi})\right)\\
c_{10} &= \frac{1}{F_\eta^2F_\pi^2}\left(\beta+\phi^{T'}(0)-\phi^{'}(\Sigma_{\eta\pi})\right)\\
c_{20} &= \frac{1}{F_\eta^2F_\pi^2}\left(\gamma+\frac{1}{2}\phi^{T''}(0)+\frac{1}{4}\phi^{''}(\Sigma_{\eta\pi})\right)\\
c_{01} &= \frac{16M_\eta^2}{F_\eta^2F_\pi^2}\left(\omega+\frac{1}{4}\phi^{''}(\Sigma_{\eta\pi})\right).
\end{aligned}\tag{8.83}$$

While the scattering lengths, being related to the value of the amplitude at the threshold, are not candidates for "good observables", the situation is a little bit more subtle in the case of the subthreshold parameters. Provided the η decay constant was known from experiments as accurately as F_π, then (similarly to α, $\beta,\ldots$) also the c_{ij} could be treated as "good observables". However, this is not the case, and we should rather use a chiral expansion of F_η in the above formulae. Therefore, the subthreshold parameters are typical examples of the dangerous ratios, which should be treated with care.

8.4.3 Bare expansion for $G(s,t;u)$

For the strict expansion in terms of LEC's (i.e. without any reparametrization in terms of physical observables) derived from (8.32, 8.33), we have confirmed the results of the article [38] by independent calculation. The $O(p^4)$ expansion can be written in the form

$$G_{\pi\eta} = G^{(2)} + G^{(4)}_{ct} + G^{(4)}_{tad} + G^{(4)}_{unit} + G\delta_G, \tag{8.84}$$

where

$$\begin{aligned}
G^{(2)}(s,t;u) &= \frac{F_0^2}{3}\mathring{M}_\pi^2 \\
G^{(4)}_{ct}(s,t;u) &= 8(L_1^r(\mu) + \frac{1}{6}L_3^r(\mu))(t-2M_\pi^2)(t-2M_\eta^2) \\
&\quad +4(L_2^r(\mu) + \frac{1}{3}L_3^r(\mu))[(s-M_\pi^2-M_\eta^2)^2 + (u-M_\pi^2-M_\eta^2)^2] \\
&\quad +8L_4^r(\mu)[(t-2M_\pi^2)\mathring{M}_\eta^2 + (t-2M_\eta^2)\mathring{M}_\pi^2] \\
&\quad -\frac{8}{3}L_5^r(\mu)(M_\pi^2+M_\eta^2)\mathring{M}_\pi^2 + 8L_6^r(\mu)\mathring{M}_\pi^2(\mathring{M}_\pi^2 + 5\mathring{M}_\eta^2) \\
&\quad +32L_7^r(\mu)(\mathring{M}_\pi^2 - \mathring{M}_\eta^2)\mathring{M}_\pi^2 + \frac{64}{3}L_8^r(\mu)\mathring{M}_\pi^4 \\
G^{(4)}_{tad}(s,t;u) &= -\frac{F_0^2}{3}\mathring{M}_\pi^2\left(3\mu_\pi + 2\mu_K + \frac{1}{3}\mu_\eta\right) \\
G^{(4)}_{unit}(s,t;u) &= \frac{1}{9}\mathring{M}_\pi^4[J^r_{\pi\eta}(s) + J^r_{\pi\eta}(u)] \\
&\quad +\frac{3}{8}[s-M_\pi^2-M_\eta^2+\frac{2}{3}\mathring{M}_\pi^2]^2 J^r_{KK}(s) + \frac{3}{8}[u-M_\pi^2-M_\eta^2+\frac{2}{3}\mathring{M}_\pi^2]^2 J^r_{KK}(u) \\
&\quad +\frac{1}{3}\mathring{M}_\pi^2[t-2M_\pi^2+\frac{3}{2}\mathring{M}_\pi^2]J^r_{\pi\pi}(t) \\
&\quad +\frac{2}{9}\mathring{M}_\pi^2(\mathring{M}_\eta^2 - \frac{1}{4}\mathring{M}_\pi^2)J^r_{\eta\eta}(t) \\
&\quad +\frac{1}{8}[t-2M_\pi^2+2\mathring{M}_\pi^2][3t-6M_\eta^2+4\mathring{M}_\eta^2-\frac{2}{3}\mathring{M}_\pi^2]J^r_{KK}(t)
\end{aligned} \tag{8.85}$$

are the $O(p^2)$, counterterm, tadpole and unitarity contributions respectively. In the above formulae, the masses within the loop functions $J^r_{PQ}(t)$ are the $O(p^2)$ masses

$$\mathring{M}_\pi^2 = 2B_0\widehat{m}, \quad \mathring{M}_K^2 = B_0\widehat{m}\,(r+1)\,, \quad \mathring{M}_\eta^2 = \frac{2}{3}B_0\widehat{m}\,(2r+1)\,. \tag{8.86}$$

The chiral logs μ_P can be expressed using $J^r_{PP}(0)$

$$\mu_P = \frac{\mathring{M}_P^2}{32\pi^2F_0^2}\ln\frac{\mathring{M}_P^2}{\mu^2} = -\frac{\mathring{M}_P^2}{2F_0^2}\left(J^r_{PP}(0) + \frac{1}{16\pi^2}\right). \tag{8.87}$$

Written in such a form, the sum $G^{(4)}_{ct} + G^{(4)}_{tad} + G^{(4)}_{unit}$ is *exactly* renormalization scheme independent by construction. Let us now proceed as described in the previous section and write the bare expansion of $G(s,t;u)$ in the form

$$G_{\pi\eta}(s,t;u) = G_{\pi\eta,\,pol}(s,t;u) + \mathcal{G}_{\pi\eta,unit}(s,t;u) + G\delta_G^{'}. \tag{8.88}$$

Writing $J^r_{ij}(s) = J^r_{ij}(0) + \overline{J}_{ij}(s)$ in (8.85), we get the renormalization scale independent polynomial part

$$\begin{aligned} G_{\pi\eta,\,pol}(s,t;u) &= G^{(2)}(s,t;u) + G^{(4)}_{ct}(s,t;u) + G^{(4)}_{tad}(s,t;u) \\ &+\frac{1}{3}\overset{o}{M}{}^2_\pi\,[t-2M_\pi^2+\frac{3}{2}\overset{o}{M}{}^2_\pi]J^r_{\pi\pi}(0) \\ &+\frac{2}{9}\overset{o}{M}{}^4_\pi\,J^r_{\pi\eta}(0)+\frac{2}{9}\overset{o}{M}{}^2_\pi\,(\overset{o}{M}{}^2_\eta-\frac{1}{4}\overset{o}{M}{}^2_\pi)J^r_{\eta\eta}(0) \\ &\frac{3}{8}\{[s-M_\pi^2-M_\eta^2+\frac{2}{3}\overset{o}{M}{}^2_\pi]^2+[u-M_\pi^2-M_\eta^2+\frac{2}{3}\overset{o}{M}{}^2_\pi]^2 \\ &+[t-2M_\pi^2+2\overset{o}{M}{}^2_\pi][t-2M_\eta^2+\frac{4}{3}\overset{o}{M}{}^2_\eta-\frac{2}{9}\overset{o}{M}{}^2_\pi]\}J^r_{KK}(0). \end{aligned} \tag{8.89}$$

Comparing this with the general form (8.79) of $G_{\pi\eta,\,pol}(s,t;u)$, we get for the bare expansions of the parameters $\alpha - \omega$ the following manifestly renormalization scale independent form

$$\begin{aligned} \alpha &= \frac{1}{3}F_0^2\,\overset{o}{M}{}^2_\pi+\frac{1}{96\pi^2}\overset{o}{M}{}^2_\pi\left(\frac{7}{2}\overset{o}{M}{}^2_\pi+\frac{11}{6}\overset{o}{M}{}^2_\eta\right) \\ &+4[8(L_1^r(\mu)+\frac{1}{6}L_3^r(\mu))+\frac{3}{8}J^r_{KK}(0)]M_\pi^2M_\eta^2 \\ &-[16L_4^r(\mu)+J^r_{KK}(0)]M_\pi^2\,\overset{o}{M}{}^2_\eta-[16L_4^r(\mu)+\frac{8}{3}L_5^r(\mu)+\frac{3}{2}J^r_{KK}(0)]M_\eta^2\,\overset{o}{M}{}^2_\pi \\ &-[\frac{8}{3}L_5^r(\mu)-\frac{1}{6}J^r_{KK}(0)+\frac{2}{3}J^r_{\pi\pi}(0)]M_\pi^2\,\overset{o}{M}{}^2_\pi \\ &+[40L_6^r(\mu)+\frac{5}{18}J^r_{\eta\eta}(0)+\frac{5}{4}J^r_{KK}(0)]\,\overset{o}{M}{}^2_\pi\overset{o}{M}{}^2_\eta \\ &+32L_7^r(\mu)\,\overset{o}{M}{}^2_\pi\,(\overset{o}{M}{}^2_\pi-\overset{o}{M}{}^2_\eta) \\ &+[8L_6^r(\mu)+\frac{64}{3}L_8^r(\mu)+J^r_{\pi\pi}(0)+\frac{2}{9}J^r_{\pi\eta}(0)-\frac{1}{18}J^r_{\eta\eta}(0)+\frac{1}{4}J^r_{KK}(0)]\,\overset{o}{M}{}^4_\pi+\frac{1}{3}F_\pi^2M_\pi^2\delta_\alpha \end{aligned} \tag{8.90}$$

$$\begin{aligned} \beta &= -2\Sigma_{\eta\pi}[8(L_1^r(\mu)+\frac{1}{6}L_3^r(\mu))+\frac{3}{8}J^r_{KK}(0)] \\ &+[8L_4^r(\mu)(\overset{o}{M}{}^2_\eta+\overset{o}{M}{}^2_\pi)+\frac{1}{2}J^r_{KK}(0)\,\overset{o}{M}{}^2_\eta+\frac{1}{3}(J^r_{\pi\pi}(0)+\frac{1}{2}J^r_{KK}(0))\,\overset{o}{M}{}^2_\pi]+\beta\delta_\beta \end{aligned} \tag{8.91}$$

$$\gamma = [8(L_1^r(\mu)+\frac{1}{6}L_3^r(\mu))+\frac{3}{8}J^r_{KK}(0)]+[2(L_2^r(\mu)+\frac{1}{3}L_3^r(\mu))+\frac{3}{16}J^r_{KK}(0)]+\gamma\delta_\gamma \tag{8.92}$$

$$\omega = [2(L_2^r(\mu)+\frac{1}{3}L_3^r(\mu))+\frac{3}{16}J^r_{KK}(0)]+\omega\delta_\omega. \tag{8.93}$$

8.5 Reparametrization of the bare expansion

Let us now discuss the various possibilities of the reparametrization of the bare expansion.

8.5.1 $\pi\eta$ scattering within the standard chiral perturbation theory to $O(p^4)$

The standard way of dealing with the chiral expansion consists of two "dangerous" steps. The first one involves using the inverted expansions of the type (8.26) in order to express the amplitude in terms of the masses and decay constants instead of the parameters $B_0\widehat{m}$, F_0 and $r = m_s/\widehat{m}$ of the $O(p^2)$ chiral Lagrangian. Here one encounters an ambiguity connected with different possibilities how to choose the observable G in (8.26), the chiral expansion of which starts with the desired $O(p^2)$ parameter G_0.

Let us fix this ambiguity by using the expansions of F_π^2, M_π^2 and M_K^2, inverting of which leads to[15]

$$
\begin{aligned}
F_0^2 &= F_\pi^2\,(1+4\mu_\pi+2\mu_K) - 8M_\pi^2\,(L_4^r(\mu)(2+r)+L_5^r(\mu)) &(8.94)\\
2B_0\widehat{m} &= M_\pi^2\Big[1-\mu_\pi+\frac{1}{3}\mu_\eta \\
&\quad -\frac{8M_\pi^2}{F_\pi^2}\Big(2L_8^r(\mu)+2(2+r)L_6^r(\mu)-L_5^r(\mu)-(2+r)L_4^r(\mu)\Big)\Big] &(8.95)\\
r &= r_2 = \frac{2M_K^2}{M_\pi^2}-1+O(p^2). &(8.96)
\end{aligned}
$$

Inserting the inverted expansions (8.94–8.96) into (8.90) and (8.91) and keeping terms up to the order $O(p^4)$ we get

$$
\begin{aligned}
\alpha &= \frac{1}{3}F_\pi^2M_\pi^2+\frac{16}{3}M_\pi^2M_\eta^2L_3^r(\mu)-\frac{64}{3}L_7^r(\mu)M_\pi^4(r_2-1)\\
&\quad +M_\pi^2M_\eta^2[32L_1^r(\mu)-16L_4^r(\mu)-\frac{8}{3}L_5^r(\mu)]\\
&\quad +\frac{1}{3}(2r_2+1)M_\pi^4[32L_6^r(\mu)-16L_4^r(\mu)+\frac{2}{9}J_{\eta\eta}^r(0)]\\
&\quad +M_\pi^4[-\frac{8}{3}L_5^r(\mu)+16L_8^r(\mu)-\frac{1}{6}J_{\pi\pi}^r(0)+\frac{2}{9}J_{\pi\eta}^r(0)-\frac{1}{18}J_{\eta\eta}^r(0)+\frac{1}{3}J_{KK}^r(0)]+\alpha\delta_\alpha^{st}
\end{aligned}
\tag{8.97}
$$

$$
\begin{aligned}
\beta &= -2\Sigma_{\eta\pi}[8(L_1^r(\mu)+\frac{1}{6}L_3^r(\mu))+\frac{3}{8}J_{KK}^r(0)]\\
&\quad +\frac{1}{3}M_\pi^2[16L_4^r(\mu)(r_2+2)+J_{KK}^r(0)(r_2+1)+J_{\pi\pi}^r(0)]+\beta\delta_\beta^{st},
\end{aligned}
\tag{8.98}
$$

with new remainders δ_α^{st} and δ_β^{st}, which might be, however, out of control as we have already discussed. In fact, this first step involves three "unsafe" manipulations from the point of

[15] Instead of M_K^2 we could use the chiral expansion of M_η^2 to obtain

$$r = \widetilde{r}_2 = \frac{3}{2}\left(\frac{M_\eta^2}{M_\pi^2}-\frac{1}{3}\right)$$

or even $F_K^2M_K^2$ to get

$$r = r_2^* = 2\frac{F_K^2M_K^2}{F_\pi^2M_\pi^2}-1.$$

The latter choice, formally as good as the previous two, could also involve the redefinition of the loop masses to $\overset{o}{M}{}_P^2 = F_P^2M_P^2/F_\pi^2$ instead of the simple $\overset{o}{M}{}_P^2 = M_P^2$ as in the case of the other standard reparametrizations. Even then, however, it suffers from numerically large $O(p^4)$ corrections which could produce instabilities of the reparametrization based on this observable.

view of resummed χPT: using "dangerous" expansions for the masses as a starting point, the inversion and finally the negligence of all higher order terms generated by this procedure after the insertion.

It's understood to use physical masses inside the chiral logarithms. Higher order LEC's are then fitted by using additional experimental input, no parameters are therefore left free. Also note that (8.96) effectively implements the classical Gell-Mann-Okubo formula

$$3M_\eta^2 - 4M_K^2 + M_\pi^2 = 0. \tag{8.99}$$

This insures renormalization scale independence. We, however, leave M_η at its physical value in cases when it was produced by on-shell mass on outer legs or inside chiral logarithms, which is compatible with the requirement of scale independence.

The second step is connected to the fact that the amplitude is used in standard χPT rather than $G(s,t;u)$. As was shown in Section 3.2, the expansion of the amplitude can be organized in various ways, of which only (8.54) is considered safe in the resummed approach. On the other hand, from the standard point of view it often seems more advantageous to use (8.56, 8.57), as together with (8.94) it leads to only the experimentally very well known pion decay constant being present in the formulae. This can be seen on the case of F_η, which is experimentally poorly known due to $\eta - \eta'$ mixing [43] and thus if it's kept at its physical value as was done in [38], a significant uncertainty is introduced into the results. As the normalization (8.56, 8.57) is used more often in NLO $S\chi PT$, we will adhere to this view and perform this second step by expanding the kaon and eta decay constants from the denominators and subsequently cutting off the higher orders.

Using therefore the prescription (8.60, 8.61), the dispersive part of the $O(p^4)$ amplitude (8.81) simplifies using the reparametrization recipe described above

$$\begin{aligned}
\phi(s) &= \frac{1}{9}M_\pi^4 \overline{J}_{\pi\eta}(s) \\
&\quad + \frac{3}{8}[(s - \frac{1}{3}M_\eta^2 - \frac{1}{3}M_\pi^2 - \frac{2}{3}M_K^2) - \frac{1}{9}M_\pi^2(r_2-1)]^2 \overline{J}_{KK}(s), \\
\phi^T(s) &= \frac{1}{3}M_\pi^2(s - \frac{1}{2}M_\pi^2)\overline{J}_{\pi\pi}(s) + \frac{1}{54}M_\pi^4\,(8r_2+1)\,\overline{J}_{\eta\eta}(s) \\
&\quad + \frac{1}{8}[(s - \frac{2}{3}M_\pi^2 - \frac{2}{3}M_K^2) + \frac{1}{3}M_\pi^2(r_2+3)] \\
&\quad \times [(3s - 2M_K^2 - 2M_\eta^2) + \frac{1}{3}M_\pi^2(3r_2+1)]\overline{J}_{KK}(s).
\end{aligned} \tag{8.100}$$

The second step also propagates itself to the case of the subthreshold parameters c_{ij} and the scattering lengths a_i, where it consists of the expansion of F_η^2 in the denominator of the formulae (8.83) and (8.82). This step could in principle produce uncontrollable contribution to the remainders as well.

8.5.2 Resummation of the vacuum fluctuation

In order to preserve the global convergence, as was discussed, in the context of resummed χPT we are not allowed to perform "dangerous" inverted expansions and thus to express the $O(p^2)$ masses $\overset{o}{M}_P$ and the decay constant F_0 in terms of the physical ones in the way it is common within the standard χPT calculations sketched above. Instead of this, the $O(p^2)$

LEC's are left free, or more precisely, rewritten using parameters directly related to the order parameters of the chiral symmetry breaking[16]

$$r=\frac{m_s}{\widehat{m}},\quad X=\frac{\overset{o}{M}{}_\pi^2 F_0^2}{M_\pi^2F_\pi^2},\quad Z=\frac{F_0^2}{F_\pi^2}. \tag{8.101}$$

The bare expansions for masses $F_P^2M_P^2$ and decay constants F_P^2 are used the reparametrize the NLO LEC's L_4-L_8. As the dependence is linear, it can be done in a purely non-perturbative algebraic way by introduction of an unknown higher order remainder to each observable used. The relevant formulae for L_4-L_8 can be found in Appendix 8.12.

As masses and decay constants do not depend on L_1, L_2, L_3, bare expansions of some additional, experimentally well known observables is needed for these LEC's. This is, however, even if highly desirable, out of the scope of our article and we make a shortcut and use the standard tabular values for these constants. We will make an analysis of the sensitivity of our results to a change in the value of L_1-L_3 in the next section devoted to numerical results.

For the resulting expression for the parameters α and β, we use the following abbreviation for some repeatedly occurring combinations

$$\begin{aligned}
r_2^* &= 2\frac{F_K^2M_K^2}{F_\pi^2M_\pi^2}-1 && (8.102)\\
\varepsilon(r) &= 2\frac{r_2^*-r}{r^2-1} && (8.103)\\
\eta(r) &= \frac{2}{r-1}\left(\frac{F_K^2}{F_\pi^2}-1\right) && (8.104)\\
\Delta_{GMO} &= \frac{3F_\eta^2M_\eta^2+F_\pi^2M_\pi^2-4F_K^2M_K^2}{F_\pi^2M_\pi^2}, && (8.105)
\end{aligned}$$

in terms of which we get

$$\begin{aligned}
\alpha \;=\;& \frac{1}{3}XF_\pi^2M_\pi^2+\frac{1}{3}F_\pi^2M_\pi^2(1-X)\frac{5r+4}{r+2}+\frac{1}{3}F_\pi^2M_\pi^2\varepsilon(r)r\frac{2r+1}{r+2}-\frac{2}{3}\frac{F_\pi^2M_\pi^2}{r-1}\Delta_{GMO}\\
&+2\frac{F_\pi^2}{r+2}(Z-1)\left(\frac{1}{3}M_\pi^2(2r+1)+M_\eta^2\right)+\frac{F_\pi^2}{r+2}\eta(r)\left(rM_\pi^2-\frac{1}{3}(r-4)M_\eta^2\right)\\
&+\frac{1}{96\pi^2}\frac{X}{Z}M_\pi^4\,(4r+5)+\frac{3}{32\pi^2}\frac{X}{Z}M_\pi^2M_\eta^2-\frac{1}{864\pi^2}\left(\frac{X}{Z}\right)^2M_\pi^4(44r+67)\\
&-\frac{M_\pi^4}{2(r+2)(r-1)}\frac{X}{Z}[J_{\eta\eta}^r(0)(2r+1)+2J_{KK}^r(0)r-J_{\pi\pi}^r(0)(4r+1)]r\\
&+\frac{M_\pi^2M_\eta^2}{6(r+2)(r-1)}\frac{X}{Z}[J_{\eta\eta}^r(0)(2r+1)(r-4)+J_{\pi\pi}^r(0)(19r-4)-2J_{KK}^r(0)(r^2+6r-4)]\\
&+\frac{M_\pi^4}{18(r+2)(r-1)}\left(\frac{X}{Z}\right)^2[J_{\eta\eta}^r(0)(5r^2-10r-4)+6J_{KK}^r(0)(3r^2-2r-4)\\
&+4J_{\pi\eta}^r(0)(r^2+r-2)-9J_{\pi\pi}^r(0)(3r^2-2r-4)]\\
&+4[8(L_1^r(\mu)+\frac{1}{6}L_3^r(\mu))+\frac{3}{8}J_{KK}^r(0)]M_\pi^2M_\eta^2+\frac{1}{3}F_\pi^2M_\pi^2\delta_\alpha',
\end{aligned}$$

[16] Here we omit the explicit dependence of X, Y and Z on N_f keeping in mind that $N_f=3$ in what follows.

$$\beta = \frac{2}{3}F_\pi^2(1-Z-\eta(r)) + \frac{1}{3}\frac{M_\pi^2}{r-1}\frac{X}{Z}[J^r_{\eta\eta}(0)(2r+1)+J^r_{KK}(0)(r+1)-J^r_{\pi\pi}(0)(3r+2)-\frac{1}{16\pi^2}(r+2)(r-1)] -2\Sigma_{\eta\pi}[8(L_1^r(\mu)+\frac{1}{6}L_3^r(\mu))+\frac{3}{8}J^r_{KK}(0)]+\beta\delta'_\beta. \tag{8.106, 8.107}$$

The new primed remainders are the following functions of the original remainders entering the game

$$\begin{aligned}\delta'_\alpha &= \delta_\alpha - \frac{5r+4}{r+2}\delta_{F_\pi M_\pi} + \frac{2}{r+2}\left((2r+1)+\frac{3M_\eta^2}{M_\pi^2}\right)\delta_{F_\pi} \\ &\quad - \frac{2}{(r+2)(r-1)}\left(3r-(r-4)\frac{M_\eta^2}{M_\pi^2}\right)\left(\frac{F_K^2}{F_\pi^2}\delta_{F_K}-\delta_{F_\pi}\right) \\ &\quad - \frac{2r(2r+1)}{(r+2)(r^2-1)}\left(2\frac{F_K^2M_K^2}{F_\pi^2M_\pi^2}\delta_{F_KM_K}-(r+1)\delta_{F_\pi M_\pi}\right) \\ &\quad + \frac{2}{r-1}\left(3\frac{F_\eta^2M_\eta^2}{F_\pi^2M_\pi^2}\delta_{F_\eta M_\eta}+\delta_{F_\pi M_\pi}-4\frac{F_K^2M_K^2}{F_\pi^2M_\pi^2}\delta_{F_KM_K}\right) \qquad (8.108)\\ \delta'_\beta &= \delta_\beta + \frac{2}{3}\frac{2F_K^2\delta_{F_K}-(r+1)F_\pi^2\delta_{F_\pi}}{\beta(r-1)}. \qquad (8.109)\end{aligned}$$

This is an alternative to the first step used in the standard approach to χPT. Because the value of F_η is not very well known, we make in a sense a parallel to the second step as well, i.e. using the chiral expansion for F_η^2 in the denominator of (8.81), (8.82) and (8.83). It involves the reparametrization in terms of X, Z, r (see Appendix 8.9 for details), but, contrary to the standard case, the denominator is not further expanded and the result is given in a nonperturbative resummed form of a ratio of two "safe" expansions.

8.5.3 $\pi\eta$ scattering within generalized χPT to $O(p^4)$ - the bare expansion

In analogy with (8.85), the strict chiral expansion for $G(s,t;u)$ within the generalized χPT can be straightforwardly obtained by using the Lagrangian summarized in the Appendix 8.13, where we use the traditional notation for the LEC's. The result has the following structure

$$G_{\pi\eta} = \widetilde{G}^{(2)} + \widetilde{G}^{(3)} + \widetilde{G}^{(4)}_{ct} + \widetilde{G}^{(4)}_{tad} + \widetilde{G}^{(4)}_{unit} + G\delta_G^{G\chi PT}, \tag{8.110}$$

where[17]

$$\begin{aligned}\widetilde{G}^{(2)}(s,t;u) &= \frac{1}{3}F_0^2[\widetilde{M_\pi^2}+4\widehat{m}^2(3A_0-4(r-1)Z_0^P+2(2r+1)Z_0^S)] \\ \widetilde{G}^{(3)}(s,t;u) &= \frac{1}{3}F_0^2\Big[-2\,\widehat{m}\,\left(6\,M_\eta^2+M_\pi^2\,(2+4\,r)-2\,(2+r)\,t\right)\tilde{\xi}-2\widehat{m}\,\Sigma_{\pi\eta}\,\xi \\ &\quad + 81\,\widehat{m}^3\,\rho_1+\widehat{m}^3\,\rho_2+\left(80-64\,r-16\,r^2\right)\widehat{m}^3\,\rho_3\end{aligned}$$

[17] All the LEC's in the following formulae are the renormalized LEC's at scale μ. We have omitted explicit notation of this in order to simplify the expressions.

$$+\left(100+64\,r+34\,r^2\right)\widehat{m}^3\,\rho_4+\left(2\,+\,r^2\right)\widehat{m}^3\,\rho_5$$
$$+\,(96-96\,r)\,\widehat{m}^3\,\rho_6+\left(144\,+288\,r+108\,r^2\right)\widehat{m}^3\,\rho_7\big]$$

$$\begin{aligned}
\widetilde{G}^{(4)}_{ct}(s,t;u) \;=\;& 8(L_1+\frac{1}{6}L_3)(t-2M_\pi^2)(t-2M_\eta^2)\\
&+4(L_2+\frac{1}{3}L_3)[(s-M_\pi^2-M_\eta^2)^2+(u-M_\pi^2-M_\eta^2)^2]\\
&+\frac{8}{3}\widehat{m}^2F_0^2\left\{-(B_1-B_2)\Sigma_{\pi\eta}+2D^PM_\pi^2(r-1)-2C_1^PM_\eta^2(r-1)\right.\\
&+C_1^S(2r+1)t-D^S[\frac{1}{2}\Sigma_{\pi\eta}(5r+4)-(2r+1)t]\\
&\left.-\,2B_4[3M_\eta^2+M_\pi^2(2r^2+1)-(r^2+2)t]\right\}\\
&+\frac{1}{3}\widehat{m}^4F_0^2\left[256E_1+16E_2+F_1^P(256-256r^2)+F_4^S(32+16r^2)\right.\\
&+F_1^S(256+320r^2)+F_5^{SP}(192-320r+160r^2-32r^3)\\
&+F_2^P(240-216r-24r^3)+F_6^{SP}(32-32r+16r^2-16r^3)\\
&+F_3^P(16-8r-8r^3)+F_3^S(16+10r+10r^3)\\
&+F_6^{SS}(32+40r+16r^2+20r^3)+F_7^{SP}(384-160r-256r^2+32r^3)\\
&\left.+F_2^S(400+234r+74r^3)+F_5^{SS}(576+720r+480r^2+168r^3)\right]
\end{aligned}$$

$$\begin{aligned}
\widetilde{G}^{(4)}_{tad}(s,t;u) \;=\;& -\frac{1}{9}F_0^2\left[2\widehat{m}B_0(\mu_\eta+6\mu_K+9\mu_\pi)+8A_0\widehat{m}^2(8\mu_\eta+3\mu_K(r+8)+48\mu_\pi)\right.\\
&+4Z_0^S\widehat{m}^2(\mu_\eta(16+41r)+\mu_K(48+90r)+\mu_\pi(96+45r))\\
&\left.-\,16Z_0^P\widehat{m}^2(2\mu_\eta(5r-2)+3\mu_K(6r-4)+3\mu_\pi(3r-8)\right]
\end{aligned}$$

$$\begin{aligned}
\widetilde{G}^{(4)}_{unit}(s,t;u) \;=\;& \frac{1}{9}[\widetilde{M}_\pi^2+4\widehat{m}^2(3A_0-4(r-1)Z_0^P+2(2r+1)Z_0^S)]^2[J^r_{\pi\eta}(s)+J^r_{\pi\eta}(u)]\\
&+\frac{3}{8}[s-M_\pi^2-M_\eta^2+\frac{2}{3}\widetilde{M}_\pi^2-\frac{8}{3}(r-1)\widehat{m}^2(A_0+2Z_0^P)]^2J^r_{KK}(s)\\
&+\frac{3}{8}[u-M_\pi^2-M_\eta^2+\frac{2}{3}\widetilde{M}_\pi^2-\frac{8}{3}(r-1)\widehat{m}^2(A_0+2Z_0^P)]^2J^r_{KK}(u)\\
&+\frac{1}{3}[\widetilde{M}_\pi^2+4\widehat{m}^2(3A_0-4(r-1)Z_0^P+2(2r+1)Z_0^S)]\\
&\times[t-2M_\pi^2+\frac{3}{2}\widetilde{M}_\pi^2+10\widehat{m}^2(A_0+2Z_0^S)]J^r_{\pi\pi}(t)\\
&+\frac{2}{9}[\widetilde{M}_\pi^2+4\widehat{m}^2(3A_0-4(r-1)Z_0^P+2(2r+1)Z_0^S)]\\
&\times[\widetilde{M}_\eta^2-\frac{1}{4}\widetilde{M}_\pi^2+\widehat{m}^2((8r^2+1)A_0+8r(r-1)Z_0^P+2(2r+1)^2Z_0^S)]J^r_{\eta\eta}(t)\\
&+\frac{1}{8}[t-2M_\pi^2+2\widetilde{M}_\pi^2+8(r+1)\widehat{m}^2(A_0+2Z_0^S))]\\
&\times[3t-6M_\eta^2+6\widetilde{M}_\eta^2-\frac{8}{3}\widetilde{M}_K^2\\
&+\frac{8}{3}(r+1)\widehat{m}^2(3rA_0+2(r-1)Z_0^P+2(2r+1)Z_0^S]J^r_{KK}(t).
\end{aligned}\qquad(8.111)$$

In the above formulae, the generalized $O(p^2)$ masses (also present implicitly in the chiral logs μ_P and the loop functions $J^r_{PQ}(s)$) are

$$\begin{aligned}
\widetilde{M}^2_\pi &= 2[B_0 + 2\widehat{m}(r+2)Z^S_0]\widehat{m} + 4A_0\widehat{m}^2 \\
\widetilde{M}^2_K &= [B_0 + 2\widehat{m}(r+2)Z^S_0]\widehat{m}\,(r+1) + A_0\widehat{m}^2\,(r+1)^2 \\
\widetilde{M}^2_\eta &= \frac{2}{3}[B_0 + 2\widehat{m}(r+2)Z^S_0]\widehat{m}\,(2r+1) + \frac{4}{3}A_0\widehat{m}^2\left(2r^2+1\right) + \frac{8}{3}Z^P_0\widehat{m}^2(r-1)^2
\end{aligned} \tag{8.112}$$

The unitarity part can be further split into the polynomial and dispersive part

$$\widetilde{G}^{(4)}_{unit} = \widetilde{G}^{(4)}_{unit,\,pol} + \widetilde{G}^{(4)}_{unit,\,disp} = \widetilde{G}^{(4)}_{unit}|_{J^r\to J(0)} + \widetilde{G}^{(4)}_{unit}|_{J^r\to\bar{J}}. \tag{8.113}$$

According to the general prescription, the dispersive part can be replaced with that of the dispersive representation which has the general form

$$\widetilde{\mathcal{G}}^{(4)}_{unit} = \phi^T(t) + \phi(s) + \phi(u), \tag{8.114}$$

where now

$$\begin{aligned}
\phi(s) &= F_0^4\left\{\left(\frac{1}{3}\alpha_{\pi\eta}\widetilde{M}^2_\pi\right)^2\frac{\overline{J}_{\pi\eta}(s)}{F_\pi^2F_\eta^2}\right. \\
&\left. +\frac{3}{8}[s - \frac{1}{3}M^2_\eta - \frac{1}{3}M^2_\pi - \frac{2}{3}M^2_K - \frac{1}{3}(2\widetilde{M}^2_K - \widetilde{M}^2_\pi - \widetilde{M}^2_\eta + \alpha_{\pi\eta K}\widetilde{M}^2_\pi)]^2\frac{\overline{J}_{KK}(s)}{F_K^4}\right\}
\end{aligned} \tag{8.115}$$

$$\begin{aligned}
\phi^T(s) &= F_0^4\left\{\frac{1}{3}\alpha_{\pi\eta}\widetilde{M}^2_\pi[s - \frac{4}{3}M^2_\pi + \frac{5}{6}\alpha_{\pi\pi}\widetilde{M}^2_\pi]\frac{\overline{J}_{\pi\pi}(s)}{F_\pi^4}\right. \\
&-\frac{1}{18}\alpha_{\eta\eta}\alpha_{\pi\eta}\widetilde{M}^2_\pi\left(\widetilde{M}^2_\pi - 4\widetilde{M}^2_\eta\right)\frac{\overline{J}_{\eta\eta}(s)}{F_\eta^4} \\
&+\frac{1}{8}[s - \frac{2}{3}M^2_\pi - \frac{2}{3}M^2_K + \frac{2}{3}((\widetilde{M}_K - \widetilde{M}_\pi)^2 + 2\alpha_{\pi K}\widetilde{M}_K\widetilde{M}_\pi] \\
&\left.\times[3s - 2M^2_K - 2M^2_\eta + \alpha_{\eta K}(2\widetilde{M}^2_\eta - \frac{2}{3}\widetilde{M}^2_K)]\frac{\overline{J}_{KK}(s)}{F_K^4}\right\}
\end{aligned} \tag{8.116}$$

for (8.58, 8.59) and analogously for (8.60, 8.61). The coefficients $\alpha_{\pi\eta}\ \ldots$ parametrize the difference between the standard and the generalized cases, within the standard $O(p^4)$ chiral expansion their values are either one or zero. The dependence of these constants on the LEC's are given in the Appendix 8.14.

8.5.4 Observables within $G\chi PT$ - the reparametrization

As it can be easily seen from the above formulae (in fact it is a consequence of the construction of $G\chi PT$), after identifying the parameters of the Lagrangians,

$$\begin{aligned}
\frac{B_0\widehat{m}}{F_0^2}L^r_4(\mu) &\to \frac{1}{8}\widehat{m}\widetilde{\xi} \\
\frac{B_0\widehat{m}}{F_0^2}L^r_5(\mu) &\to \frac{1}{8}\widehat{m}\xi^r
\end{aligned}$$

$$\begin{aligned}
\frac{B_0^2\widehat{m}^2}{F_0^2}L_6^r(\mu) &\rightarrow \frac{1}{16}\widehat{m}^2 Z_0^S \\
\frac{B_0^2\widehat{m}^2}{F_0^2}L_7^r(\mu) &\rightarrow \frac{1}{16}\widehat{m}^2 Z_0^P \\
\frac{B_0^2\widehat{m}^2}{F_0^2}L_8^r(\mu) &\rightarrow \frac{1}{16}\widehat{m}^2 A_0
\end{aligned} \tag{8.117}$$

and defining the remainders using the *physical* masses inside the chiral logarithms and the loop functions J^r_{PQ}, the generalized *bare* chiral expansions contain all the terms of the standard one. More precisely, the generalized $O(p^4)$ bare expansions include extra $O(p^4)$ terms, which are counted as $O(p^6)$ and $O(p^8)$ within the standard chiral power counting scheme. As a consequence, after writing the generalized bare chiral expansion of a generic "good" observable g in the form

$$g = g^{(2),\,G\chi PT} + g^{(3),\,G\chi PT} + g^{(4),\,G\chi PT} + g\delta_g^{G\chi PT} \tag{8.118}$$

and then collecting the "standard" terms together, this expansion can be formally rewritten as

$$g = g^{(2),\,std} + g^{(4),\,std} + g\delta_g, \quad g\delta_g = g\delta_g^{(G)} + g\delta_g^{G\chi PT}, \tag{8.119}$$

where the identification (8.117) is assumed. The extra $O(p^4)$ terms mentioned above are now accumulated in $g\delta_g^{(G)}$. In the case of the polynomial parameters $\alpha\ldots\omega$ (8.90–8.93), the two versions of the chiral expansion coincide for γ and ω[18]

$$\delta_\gamma = \delta_\gamma^{G\chi PT} \qquad \delta_\omega = \delta_\omega^{G\chi PT}, \tag{8.120}$$

while the "standard" remainders δ_α, δ_β can be split into an explicitly known part, which includes the extra "nonstandard" terms, and the unknown remainders inherent to $G\chi PT$

$$\delta_\alpha = \delta_\alpha{}^{loops}(\mu) + 3\frac{\widehat{m}^2F_0^2}{F_\pi^2 M_\pi^2}\delta_\alpha^{CT}(\mu) + \delta_\alpha^{G\chi PT} \tag{8.121}$$

$$\beta\delta_\beta = \beta\delta_\beta{}^{loops}(\mu) + \widehat{m}^2F_0^2\delta_\beta^{CT}(\mu) + \beta\delta_\beta^{G\chi PT}. \tag{8.122}$$

Here the first terms correspond to the new loops and the second to the new counterterm contributions. The explicit expressions for them can be easily extracted from the formulae of the previous subsection, the results are however rather lengthy and we postpone them to Appendix 8.15.

Let us note that both $\delta^{loop}_{\alpha,\beta}(\mu)$ and $\delta^{CT}_{\alpha,\beta}(\mu)$ are generally renormalization scale dependent. However, due to the running of the $G\chi PT$ LEC's A_0, Z_0^S, Z_0^P, ξ and $\widetilde{\xi}$,which, after identification (8.117), is the same as in the standard case, the "standard" remainders δ_α and δ_β given by (8.121, 8.122) are μ-independent. Of course, the "true $G\chi PT$" remainders $\delta_\alpha^{G\chi PT},\ldots,\delta_\omega^{G\chi PT}$ are scale independent by construction. That means the sum of the loop and counterterm contributions to the "standard" remainders is μ-independent too.

The usual way how to handle the reparametrization of the $G\chi PT$ bare expansions is quite similar to the standard one. The difference is that as there are three additional $O(p^2)$ LEC's, not all of them can be reparametrized using inverted mass and decay constant expansions. The solution is to leave two of them free (*e.g.* r and $\zeta = Z_0^S/A_0$). Consequently, the expansion

[18] The reason is that they stem from the terms quadratic in the Mandelstam variables.

is performed according to the generalized power counting scheme and the terms of order higher than $O(p^4)$ are discarded.

We will, however, not use this approach, but rather exploit the relation (8.119), *i.e.* sew the standard and generalized bare expansions together. The reparametrization is then an extension of the resummed one (Appendix 8.12), where all the remainders of mass and decay constant bare expansions are split according to (8.119). We can use all the resummed formulae as they are exact algebraic identities, valid independently on the version of χPT. The generalized contributions to the remainders can be found in Appendices 8.9 and 8.16. The outcome for the parameters α and β is then obtained by simply inserting all the generalized results for the remainders (Appendices 8.9, 8.15, 8.16) into the expression for δ_α' and δ_β' (8.108, 8.109).

After this procedure, the generalized LEC's are present only in the formulae for the standard remainders. Also note that $\delta_\alpha{}^{loops}(\mu)$, $\delta_\beta{}^{loops}(\mu)$ as well as the generalized loop contributions to mass and decay constant remainders depend explicitly on the $O(p^2)$ LEC's $B_0 = XM_\pi^2/2\widehat{m}$, $F_0^2 = ZF_\pi^2$, A_0, Z_0^S and Z_0^P.[19] So as the last step of the reparametrization, the remaining dependence of the generic "loop" remainders $\delta_\alpha{}^{loops}(\mu), \ldots, etc.$ on the $O(p^2)$ LEC's F_0, A_0, Z_0^S and Z_0^P can be removed up to the order $O(p^4)$ using the leading order expressions

$$\begin{aligned}
F_0^2 = F_K^2 = F_\eta^2 \quad &\to \quad F_\pi^2 \\
\widehat{m}^2 F_0^2 Z_0^S \quad &\to \quad \frac{1}{4}\frac{F_\pi^2 M_\pi^2}{r+2}(1 - X - \varepsilon(r)) \\
\widehat{m}^2 F_0^2 Z_0^P \quad &\to \quad -\frac{1}{8}F_\pi^2 M_\pi^2\left(\varepsilon(r) - \frac{\Delta_{GMO}}{(r-1)^2}\right) \\
\widehat{m}^2 F_0^2 A_0 \quad &\to \quad \frac{1}{4}F_\pi^2 M_\pi^2 \varepsilon(r).
\end{aligned} \tag{8.123}$$

As a summery, our handling of the generalized bare expansion can be viewed in two ways - either as a partial estimate of the standard remainders present in the resummed approach or as a special treatment within the generalized framework, where the $O(p^2)$ (and partly $O(p^3)$) LEC's are reparametrized algebraically at the leading order, while treated perturbatively at the $O(p^4)$ one. The numerical results including a simple estimate of the remaining NLO and NNLO LEC's are presented in Subsection 6.6, also Appendix 8.9 contains an illustrative example of applying this procedure on F_η.

8.6 Numerical results

In this section we shall present the numerical analysis of the observables connected to the $\pi\eta-$ scattering amplitude and the results which illustrate the subtleties of the various versions of the chiral expansions described above. In the numerical estimates we use $M_\pi = 135$MeV, $M_\eta = 548$MeV, $M_K = 496$MeV, $\mu = M_\rho = 770$MeV, $F_\pi = 92.4$MeV and $F_K = 113$MeV. For the calculation within the standard χPT, the $O(p^4)$ LEC's are taken from ref. [44] and [45]. In the alternative reparametrization schemes, where only L_1, L_2 and L_3 remain among the free parameters and the other $O(p^4)$ LEC's are expressed in terms of physical masses, decay

[19] More precisely, the loops depend on the "true $O(p^2)$ LEC's"$\overline{A}_0$, $\overline{Z}_0^{S,P}$ (cf. Appendix 8.13), the difference is however of the higher order in the generalized power counting.

L_i	α/α^{CA}	$10^3\beta/M_\eta^2$	$10^3\gamma$	$10^4\omega$
[44]	1.68	0.90	−1.52	2.24
[45]	1.91	−0.68	−0.23	−5.03
Δ	2.48	7.49	3.31	9.48

Table 8.1: Standard $O(p^4)$ values of the polynomial parameters for the two sets of LEC's taken form references [44] and [45]. In the last row, the sensitivity on the LEC's is (over)estimated by adding the uncertainties associated with the LEC's [44] in quadrature (This is of course only a rough estimate, because in fact not all the uncertainties of the L_i's are independent).

constants and the indirect remainders, we again keep (though rather non-systematically) the values of L_1, L_2 and L_3 from the same references. The sensitivity on this LEC's might be then estimated by means of the variation around these central values. In this chapter we insert the physical masses into the functions $J^r_{PQ}(0)$ unless stated otherwise.

8.6.1 The standard chiral perturbation theory

This subsection discusses the predictions of the standard chiral expansion to the order $O(p^4)$, which are summarized in Table 8.1 and 8.2. Let us start with the parameter α of the polynomial part of the amplitude. The relevant formulae from the Subsection 8.5.1 and the LEC's taken from [44][20] result in the following value

$$\alpha = \frac{1}{3}\left(1 + 0.683 + \delta^{st}_\alpha\ \right) F_\pi^2 M_\pi^2 \tag{8.124}$$

In this expression, the first term corresponds to the current algebra result $\alpha^{CA} = F_\pi^2 M_\pi^2/3$, while the second one represents the $O(p^4)$ correction. The third term is the standard remainder, which might be out of control when $X, Z \ll 1$ and r far from r_2, even if the bare expansion of α were globally convergent (let us remind that α is a "good" observable) as we have discussed in Section 8.2. Let us also notice the unusually large next-to-leading correction, which could also indirectly indicate the numerical importance of the remainder in this scheme.

The actual numerical value of the NLO correction is very sensitive to a shift in the $O(p^4)$ LEC's. The corresponding variation $\Delta\alpha$ is numerically

$$\frac{\Delta\alpha}{\alpha^{CA}} = (3.38\Delta L_1 + 0.56\Delta L_3 - 3.50\Delta L_4 - 0.30\Delta L_5 + 3.62\Delta L_6 - 3.42\Delta L_7 + 0.10\Delta L_8) \times 10^3. \tag{8.125}$$

For example, using the $O(p^6)$ analysis based LEC's from [45] instead of those from [44], we get (cf. Table 8.1)[21]

$$\frac{\Delta\alpha}{\alpha^{CA}} = 0.23. \tag{8.126}$$

Note that the large coefficients in front of the L_4 and L_6 contributions indicate sensitivity of this observable to the vacuum fluctuations of $\bar{s}s$ pairs as mentioned in the Introduction.

[20] This set of LEC's is used in numerical estimates unless stated otherwise.

[21] Because the values of the LEC's L_i based on the $O(p^6)$ fit include implicitly parts of the $O(p^6)$ corrections, the large variation can be interpreted as a signal of the importance of the NNLO contributions to the parameter α. The same is true for other observables from the Table 8.1.

L_i	c_{00}/c_{00}^{CA}	$10^3 c_{10}$	$10^3 c_{20}$	$10^3 c_{01}$	a_0/a_0^{CA}	$10^3 a_1$
[44]	1.06	0.91	−1.23	8.27	1.96	0.59
[45]	1.51	−0.67	0.07	−3.36	1.18	−0.60
Δ	2.49	7.49	3.31	15.16	3.21	2.80

Table 8.2: Standard $O(p^4)$ values of the subthreshold and threshold parameters as in Tab.8.1. The c_{ij} parameters are given in their natural units described in the main text. Analogously to Tab.8.1, Δ is the sensitivity on the LEC's (over)estimated by adding the uncertainties associated with the LEC's [44] in quadrature

Let us compare this case with the related “dangerous“ observable, namely the subthreshold parameter c_{00}. From (8.83) we get

$$c_{00} = \frac{1}{3}\left(1 + 0.683 - 0.625 + 0.006\right)\frac{M_\pi^2}{F_\pi^2} = 1.064 c_{00}^{CA}, \tag{8.127}$$

where the individual terms are the leading order contribution $c_{00}^{CA} = M_\pi^2/3F_\pi^2$, the next-to-leading correction to the parameter α, the next-to-leading correction to F_η^2 induced by the expansion of the denominator and the contribution stemming from the unitarity correction $\phi(s)$ respectively. The first two large corrections accidentally cancel here, this, however, does not automatically imply similar cancellation of the potentially large remainders (we have not written them down explicitly here). Also, the strong sensitivity of α to the variation of the LEC's propagates here giving

$$\frac{\Delta c_{00}}{c_{00}^{CA}} = \frac{\Delta\alpha}{\alpha^{CA}} - 0.28\Delta L_5 \times 10^3 \tag{8.128}$$

and furthermore increases the uncertainty of the $O(p^4)$ correction. This strong sensitivity supports the possibility that the standard remainders for c_{00} might be numerically larger than the next-to-leading correction. Namely using the LEC's from the $O(p^6)$ fit [45], which generates part of the $O(p^6)$ corrections to the reparametrized expansion of c_{00}, we get

$$\frac{\Delta c_{00}}{c_{00}^{CA}} = 0.45. \tag{8.129}$$

We can also check the sensitivity of the next-to-leading order contributions to the way we rewrite them in terms of the physical masses and decay constants (*i.e.* how we use the $O(p^2)$ relations generating here a difference of the order $O(p^6)$). Provided we insert the alternative $O(p^2)$ expressions for r into the chiral expansions of α and c_{00}

$$\widetilde{r}_2 = \frac{1}{2}\left(\frac{3M_\eta^2}{M_\pi^2} - 1\right) = 24.2 \tag{8.130}$$

$$r_2^* = 2\frac{F_K^2 M_K^2}{F_\pi^2 M_\pi^2} - 1 = 39.4, \tag{8.131}$$

instead of the standard $O(p^2)$ value $r = r_2 = 2M_K^2/M_\pi^2 - 1 = 25.9$, we get as a result

$$\widetilde{\alpha} = \frac{1}{3}(1 + 0.601)\, F_\pi^2 M_\pi^2 \tag{8.132}$$

$$\widetilde{c}_{00} = \frac{1}{3}(1 + 0.031)\frac{M_\pi^2}{F_\pi^2} \tag{8.133}$$

and

$$\begin{aligned} \alpha^* &= \frac{1}{3}(1+1.297)\, F_\pi^2 M_\pi^2 && (8.134)\\ c_{00}^* &= \frac{1}{3}(1+0.325)\frac{M_\pi^2}{F_\pi^2}. && (8.135) \end{aligned}$$

The remaining parameters of the polynomial part start at $O(p^4)$ and we get them from (8.98), (8.92) and (8.93). Their numerical values and the related subthreshold parameters c_{ij} in natural units (chosen in such a way to make the comparison with the polynomial parameters easy, *i.e.* we take c_{10} and c_{01} in units of M_η^2/F_π^2 and c_{20} in units of F_π^{-4}, cf. (8.83)) are shown in Tables 8.1 and 8.2 for the two sets of $O(p^4)$ LEC's.

All the considered parameters are strongly sensitive to the variations of the LEC's. For instance the parameter β varies with the L_i's as

$$\Delta\beta = (17.0\Delta L_1 - 2.8\Delta L_3 + 9.1\Delta L_4)M_\eta^2. \tag{8.136}$$

For the LEC's [44] we get $\beta = 0.90 \times 10^{-3} M_\eta^2$. Using the set [45] we get a drastic change

$$\Delta\beta = -1.58 \times 10^{-3} M_\eta^2. \tag{8.137}$$

Let us turn to the "doubly dangerous" observables represented by the scattering lengths now. For the s-wave we obtain from (8.82) and the LEC's [44]

$$\begin{aligned} a_0 &= \frac{1}{24\pi F_\pi^2}\frac{M_\pi^3}{M_\eta + M_\pi}(1 + 0.683 + 0.378 - 0.625 + 0.527)\\ &= \frac{1}{24\pi F_\pi^2}\frac{M_\pi^3}{M_\eta + M_\pi}\,(1 + 0.963) = 11.0 \times 10^{-3}. && (8.138) \end{aligned}$$

Here the individual terms in the first line represent the current algebra result, the correction stemming from the $O(p^4)$ contributions to the parameters α and ω, the next-to-leading correction to F_η^2 induced by the expansion of the denominator and the correction induced by the dispersive part of the amplitude $\phi(s)$, in this order. This result confirms the expectations about a bad convergence of the chiral expansion for the observables which are connected to the threshold values of the amplitude - even if the polynomial NLO corrections were small, which are not, the dispersive part would be still as large as 50% of the leading order term.

The sensitivity to the $O(p^4)$ LEC's is illustrated in Table 8.2. The p-wave scattering length then starts at $O(p^4)$, we get the values in the last column of the table from (8.82).

When comparing our standard χPT results for the scattering lengths (first row of Table 8.2)

$$a_0 = 11.0 \times 10^{-3} \qquad a_1 = 5.9 \times 10^{-4} \tag{8.139}$$

with those of [38], quoted in the Introduction

$$a_0^{BKM} = 7.2 \times 10^{-3} \qquad a_1^{BKM} = -5.2 \times 10^{-4}, \tag{8.140}$$

we can see a seemingly large discrepancy. The difference is produced by a different set of $O(p^4)$ LEC's, the alternative treatment of the F_η in the denominator and by another form of the unitarity corrections - the authors do not use a matching with a dispersive representation.

Taken these distinctions into account, we get more consistent numbers (with our inputs for the masses and decay constants):

$$a_0 = 7.0 \times 10^{-3} \qquad a_1 = -5.0 \times 10^{-4}. \tag{8.141}$$

As we can see, a slightly different treatment of the standard chiral expansion may lead to a significant shift in the results. This does not necessarily mean that the standard counting is not consistent, though. As follows from from Table 8.2, the nominal uncertainty associated with $O(p^4)$ LEC error bars encompasses the difference

$$\Delta a_0 = 18.0 \times 10^{-3} \qquad \Delta a_1 = 28.0 \times 10^{-4}. \tag{8.142}$$

What can be concluded is that the standard approach has a large theoretical uncertainty attached, which is hard to estimate. The sensitivity to the L_i^r values also leads to a considerable difference when one uses the $O(p^6)$ fit (second row in Table 8.2). As the two fits effectively differ only in a rearrangement of the expansion, both cannot have small higher order corrections at the same time.

8.6.2 Resummation of vacuum fluctuations - basic properties

In the resummed case, the free parameters are X, Z, and r together with the remaining LEC's L_1, L_2 and L_3 and the direct and indirect remainders $\delta_\alpha \dots \delta_\omega$ and δ_{F_P}, $\delta_{F_P M_P}$. Because F_η is experimentally not known with enough accuracy, we also have to fix how to treat the observable Δ_{GMO} which was introduced to eliminate the LEC L_7 using the bare expansion for $F_\eta^2 M_\eta^2$. Let us remind that our definition of Δ_{GMO} follows the ref. [8], where it is based on the good observables $F_P^2 M_P^2$ instead of M_P^2 and differs from that originally defined in [3]. One possibility is to treat Δ_{GMO} as an additional independent parameter. The another, similarly to the treatment of F_η in the denominators of (8.81), (8.82) and (8.83) discussed earlier, is to use a (resummed) chiral expansion of F_η^2 inserted to Δ_{GMO} for the numerical estimates, *i.e.* to insert the following exact algebraic identity into (8.106) (cf. Appendix 8.9 for details)

$$\begin{aligned} F_\pi^2 M_\pi^2 \Delta_{GMO} &= F_\pi^2 M_\pi^2 - 4M_K^2 F_K^2 + M_\eta^2 (1-\delta_{F_\eta})^{-1} \Bigg(4F_K^2 (1-\delta_{F_K}) - F_\pi^2 (1-\delta_{F_\pi}) \\ &\quad - M_\pi^2 \left(\frac{X}{Z}\right) \left(J_{\pi\pi}^r(0) - 2(r+1) J_{KK}^r(0) + (2r+1) J_{\eta\eta}^r(0) \right) \Bigg). \end{aligned} \tag{8.143}$$

This generates the indirect remainder δ_{F_η} in a nonlinear way.

Before doing a more detailed analysis, let us first illustrate the numerical sensitivity connected with the subtleties of the definition of the bare expansion. As we have discussed in Section 3, there is still some freedom how to define the amplitudes entering the dispersive part of the $G_{\pi\eta}$ (cf. (8.58, 8.59) and (8.60, 8.61) and also how to treat the masses inside the chiral logs. Based on general considerations it was argued [9], that in the latter case the different prescriptions should not make much difference. Nevertheless, it might be interesting to test this assumption numerically in our concrete case and also to check what is the numerical influence of the varying amplitude definition.

In Figure 8.1 we plot the comparison of various definitions of the dispersive part of the amplitude using the scattering length a_0 as an example, *i.e* we illustrate its sensitivity on the various versions of the unitarity corrections. The cusps on the full line, which uses the *strict* chiral expansion with the unphysical choice of the $O(p^2)$ masses in all J_{ij}^r, originate in the

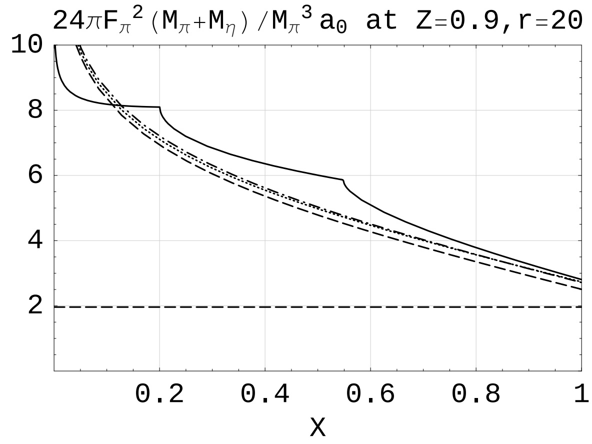

Figure 8.1: *Comparison of the numerical impact of the various forms of the dispersive part on the scattering length a_0. The full line represents the strict chiral expansion, dotted, dashed and dash-dotted lines the "minimal" modification, (8.58, 8.59) and (8.60, 8.61) respectively. The horizontal line shows our standard NLO prediction.*

conflict of the physical masses used for the on-shell outer legs and the unphysical location of the thresholds. This illustrates the fact that the original strict chiral expansion is unsuitable for realistic physical predictions and its redefinition into a bare one is necessary. The dotted line shows the "minimal" physical modification of the strict expansion by means of insertion physical masses into all J^r_{ij}. While the "minimal" version and the unitary choice (8.60, 8.61) give numerically almost the same result, the difference between the these two and the third possibility (8.58, 8.59) is up to $\sim 0.3\ a_0^{CA}$.

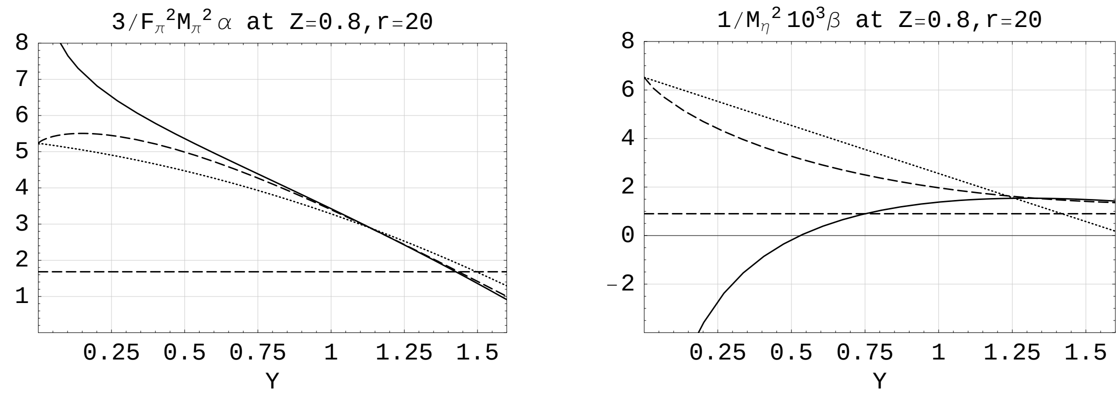

Figure 8.2: *In this figure we illustrate the sensitivity of the "good" variables α and β to the treating of the chiral logs. The full line corresponds to the $O(p^2)$ masses in all $J^r_{ij}(0)$'s, while doted and dashed lines to the physical masses either in all $J^r_{ij}(0)$'s or only in $J^r_{ij}(0)$'s originating from the unitarity corrections. Horizontal lines are standard NLO predictions.*

Figure 8.2 shows the dependence of the polynomial parameters α and β on $Y = X/Z$ for $Z = 0.8$ and $r = 20$, using the the various possible treatments of chiral logarithms in the bare expansion. The results demonstrate that the difference might be numerically important in some range of Y. For α the various possibilities do not differ drastically in comparison with the value of α itself, on the other hand the differences become comparable with α^{CA} at $Y \sim 0.5$. As we have discussed in Section 2, in the region of small Y the case with $O(p^2)$ masses in the tadpoles only should not differ drastically from the case when all the masses are physical. However, the convergence to the common value at $Y = 0$ is rather slow and in the intermediate region of Y the difference of this two cases for α is $\sim 0.5\ \alpha^{CA}$ in a relatively wide interval. Keeping $O(p^2)$ masses also in the unitarity corrections produce instabilities for $Y \to 0$ as expected. The parameter β (which starts at $O(p^4)$) is even much more sensitive.

In the following numerical analysis we take a pragmatic point of view and fix the bare expansion in such a way that the comparison of the resummed and standard reparametrizations remains as simple as possible, *i.e.* we insert physical masses into $J^r_{ij}(0)$ and define the amplitude according to (8.56, 8.57) and (8.60, 8.61).

8.6.3 Numerical comparison of the resummed and standard reparametrization

Within the standard χPT we have an $O(p^4)$ prediction for X, Z, r and Δ_{GMO} based on the standard formal $O(p^4)$ chiral expansion (8.160) and (8.162), cf. Appendix 8.8. Using the LEC's from [44] and [45], we get numerically the following central values, which should confirm the self-consistency of the standard chiral expansion scheme

L_i set	X^{std}	Z^{std}	r^{std}	r^{*std}	Δ^{std}_{GMO}
[44]	0.902	0.865	25.2	26.7	6.41
[45]	0.726	0.734	25.9	31.7	3.31

As we can see, while expectations are fulfilled in the first case, there is a considerable shift when using the $O(p^6)$ fitted constants. These numbers, moreover, should be taken with some caution, because they originate in the expansions of the "dangerous" observables and can be therefore plagued with large $O(p^6)$ remainders as well as with strong sensitivity to the $O(p^4)$ LEC's[22]. In the above table r^{std} stems from the chiral expansion of r_2 while r^{*std} uses expansion of r_2^*. [23]

Let us now illustrate the relationship of the resummed and standard approach using the observables from Subsection 8.6.1.

For the "good" observables α and β we can expect that the numerical values of X^{std}, Z^{std}, r^{std} and Δ^{std}_{GMO}, with L_1, L_2 and L_3 taken from [44] for definiteness, should produce numbers consistent with the first row of Table 8.1 when inserted into (8.106, 8.107). The results for the various possibilities how to approach the standard predictions for α and β (which is independent on Δ_{GMO}) within the resummed version of χPT are summarized[24] in the Table 8.3. The last row corresponds to the resummed treating of Δ_{GMO} explained above. The dependence of the central values of α and β on the parameters r, X and Z in the broader

[22] As it was analyzed in detail in [8], the actual values of X^{std} and Z^{std} are strongly sensitive to the values of the LEC's L_6 and L_4 connected with the vacuum fluctuation of the $\bar{s}s$ pairs, the same is true for the sensitivity of r^{std} and Δ^{std}_{GMO} to L_8 and L_7. This causes large error bars to be attached to these values. Nevertheless, in the following we take these central values as a reference point for an illustrative numerical comparison of the two versions of the chiral expansion.

[23] Though the difference between the values of r^{std} and r^{*std} is within the standardly expected accuracy of the $O(p^4)$ approximation, note, however, that for r^{*std} the $O(p^4)$ correction is much larger than in the first alternative ($r_2 = 25.9$ while $r_2^* = 39.4$).

[24] In this and the following tables in this subsection we ignore the uncertainty stemming from the remainders and L_i, $i = 1, 2, 3$ and give only the central values (assuming the central values of the remainders to be zero).

X	Z	r	Δ_{GMO}	α/α^{CA}	$10^3\beta$	c_{00}/c_{00}^{CA}	$10^3 c_{10}$	a_0/a_0^{CA}	$10^3 a_1$
X^{std}	Z^{std}	r^{std}	Δ_{GMO}^{std}	1.88	0.69	1.11	0.41	1.64	0.30
X^{std}	Z^{std}	r^{*std}	Δ_{GMO}^{std}	1.61	0.55	0.95	0.33	1.47	0.26
X^{std}	Z^{std}	r_2	Δ_{GMO}^{std}	1.74	0.62	1.03	0.37	1.55	0.28
Z^{std}	Z^{std}	r^{*std}	Δ_{GMO}^{std}	1.76	0.75	1.04	0.45	1.57	0.31
Z^{std}	Z^{std}	r^{std}	Δ_{GMO}^{std}	2.02	0.89	1.20	0.53	1.72	0.35
X^{std}	Z^{std}	r^{std}	Δ_{GMO}	2.07	0.69	1.22	0.41	1.75	0.30
X^{std}	Z^{std}	r^{*std}	Δ_{GMO}	1.78	0.55	1.05	0.33	1.57	0.26

Table 8.3: *The values of the the polynomial parameters α and β and the related subthreshold and threshold parameters near the standard reference point. For Δ_{GMO} we take either the standard value (Δ_{GMO}^{std}) or the resummed prediction described (Δ_{GMO}) in the main text.*

vicinity of their standard values is illustrated in Fig.8.3. These results can be interpreted as a consistency of both variants of reparametrization for good observables near the standard reference point X^{std}, Z^{std} and r^{std}, where the predictions of the resummed version almost coincide with the standard results[25]. This coincidence together with the working hypothesis about the controllable remainders of good observables within the resummed reparametrization scheme confirms again a self-consistency of the standard expansion based on the assumption $X \sim 1$, $Z \sim 1$ and $r \sim r_2$. Away from the standard reference point, however, the standard reparametrization might be dangerous in the sense that the difference between the standard and the resummed prediction diverges rapidly and the importance of the standard $O(p^6)$ remainders might therefore increase considerably.

X	Z	r	$10^4 c_{20}$	$10^3 c_{01}$
X^{std}	Z^{std}	r^{std}	-7.10	4.86
X^{std}	Z^{std}	r^{*std}	-6.97	4.79
X^{std}	Z^{std}	r_2	-7.04	4.83

Table 8.4: *The values of the subthreshold parameters c_{20} and c_{01} related to the polynomial parameters γ and ω at the standard reference point.*

For the "dangerous" observables like c_{ij} we cannot *a priori* expect coincidence of both expansion even near the standard values of X, Z, and r due to the different treatment of the denominators, which contain large $O(p^4)$ corrections and are not expanded in the resummed case. Comparison of both approaches is illustrated in Table 8.3 (with the same treatments of Δ_{GMO} as above), Table 8.4 and in Fig. 8.4. For the dispersive part we use the prescription (8.60, 8.61), which differs from the corresponding standard contributions of the unitarity corrections to c_{ij} for $X = Z = 1$ and $r = r_2$ by a factor $F_\pi^2/F_\eta^2 \approx 0.6$. This is reflected by the values of those c_{ij} that start at $O(p^4)$ (cf. Table 8.3, 8.4). Namely in this case the contribution of the polynomial part is reduced near the reference point roughly by the same factor with respect to the standard value (which includes only the first term of the expansion of the denominator). On the other hand, c_{00} is compatible with the standard value, because

[25] As a rule, the point X^{std}, Z^{std} and r^{std} cannot give the best coincidence with the standard values in all cases, the reason can be understood *e.g.* by having a closer look on the resummed reparametrization of β (cf. (8.107)). In order to reproduce the dependence of β on L_4 satisfactorily, we need $Z = Z^{std}$ and $r = r^{std}$, on the other hand to reproduce the chiral logs we need rather $X/Z = 1$ and $r = r_2$. This can explain why β approaches the standard value best for $X = Z = Z^{std}$.

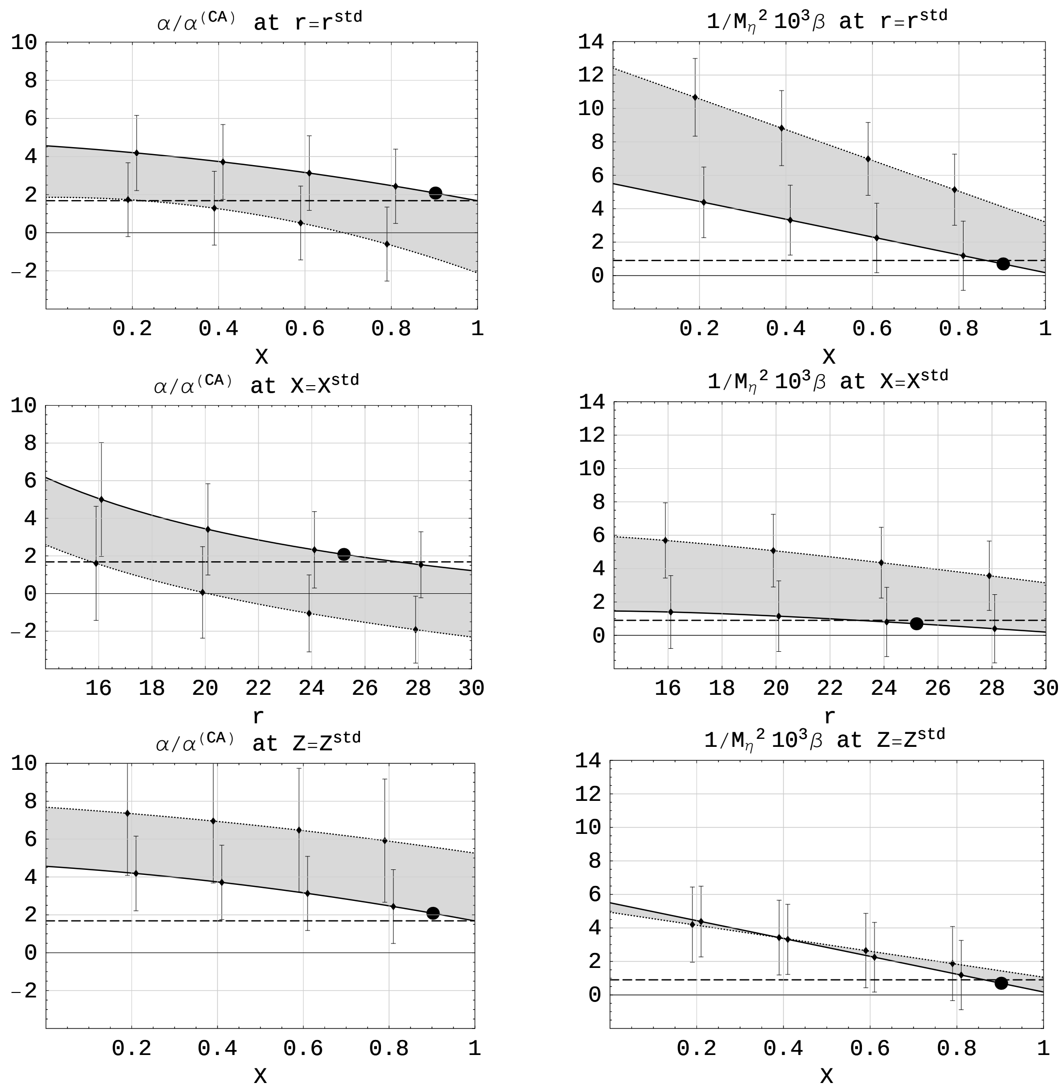

Figure 8.3: *The dependence of the parameters α and β on r, X and Z is plotted, one of the parameters being fixed at its standard reference value in each figure. The dashed horizontal line shows the standard values from the first row of the Table 8.1, the full circle depicts the corresponding resummed value at the standard reference point $[r^{std}, X^{std}, Z^{std}]$. The error bars represent the 10% uncertainties from the direct and indirect remainders added in quadrature. In the first row, r is fixed at r^{std}, the filled are highlight the dependence on Z between $Z = Z^{std}$ (solid line) and $Z = 0.5$ (dotted one). Similarly, in the second row $X = X^{std}$, the filled area shows the dependence on Z again. Z is fixed at Z^{std} in the last row, the solid line shows the case with $r = r^{std}$ while the dotted the one with a lower value $r = 15$.*

of the large $O(p^2)$ contribution, tiny dispersive contribution and the fact that within the standard reparametrization of the bare expansion also the second term from the expansion of the denominator is taken into account.

Let us now proceed to the "doubly dangerous" observables a_0 and a_1. These are related to the values of the amplitude at the threshold and receive therefore large contribution from the dispersive part of the amplitude. While a_0 is reproduced well at the standard reference

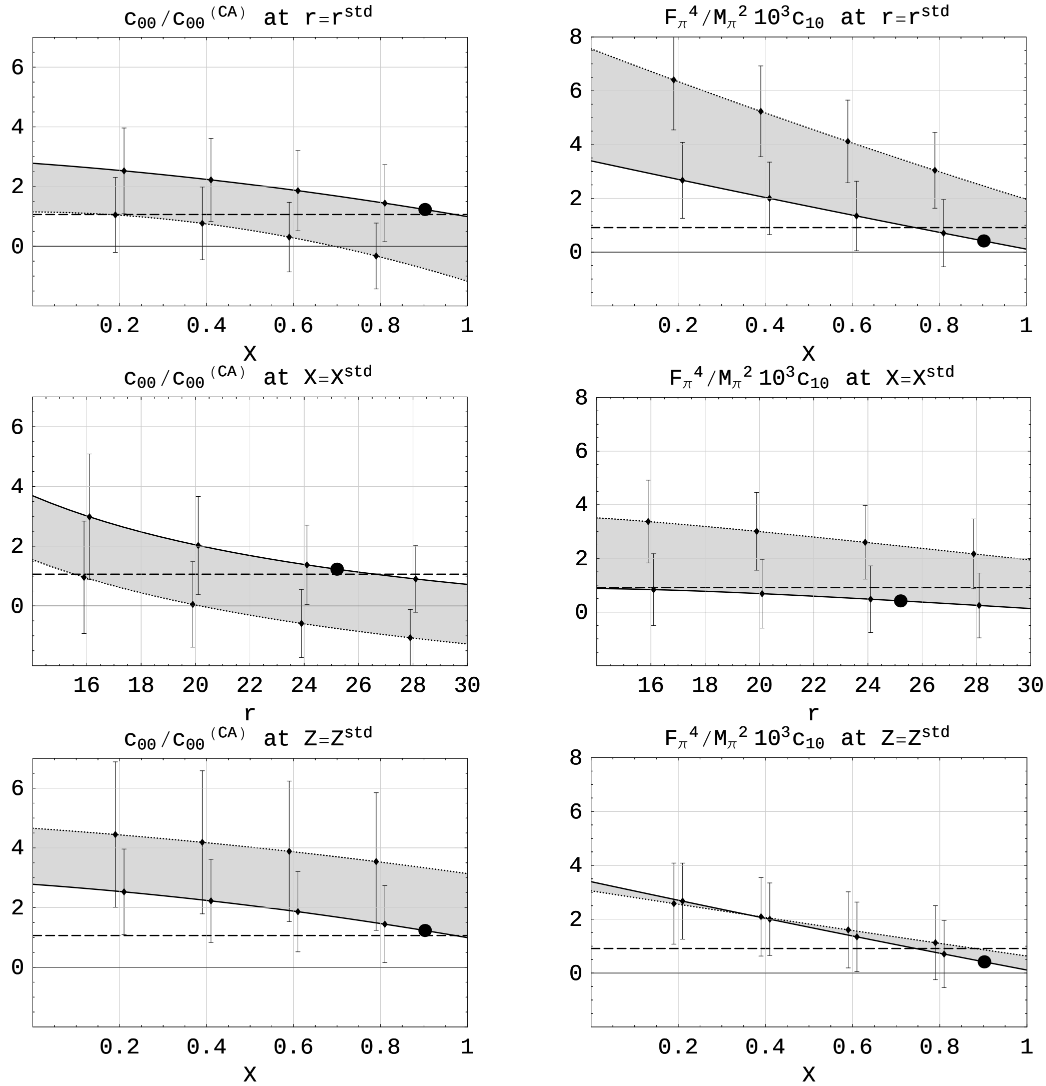

Figure 8.4: *Dependence of the subthreshold parameters* c_{00} *and* c_{10} *related to the polynomial parameters* α *and* β*. The figures are in one-to one correspondence to those in Figure 8.3.*

point, a_1 (which starts at the NLO) is off the standard value roughly by a factor 0.6 from the same reasons as for the c_{ij} parameters. The dependence of this observables on X, Z and r is depicted in Fig. 8.5.

8.6.4 The role of the remainders

Up to now we have not discussed the uncertainties of the observables calculated within the resummed scheme. They are connected with the direct and indirect remainders as well as with the LEC's L_i, $i = 1, 2, 3$. As the first illustration, we have added the error bars stemming from the remainders to the central values of the various observables depicted in the figures 8.3-8.5. These illustrate the rough estimate of the remainders $\delta \sim (30\%)^2 \sim 0.1$ as suggested in [8] and adding the uncertainties in quadrature.

In more detail, at the standard reference point $X = X^{std}$, $Z = Z^{std}$ and $r = r^{std}$, using

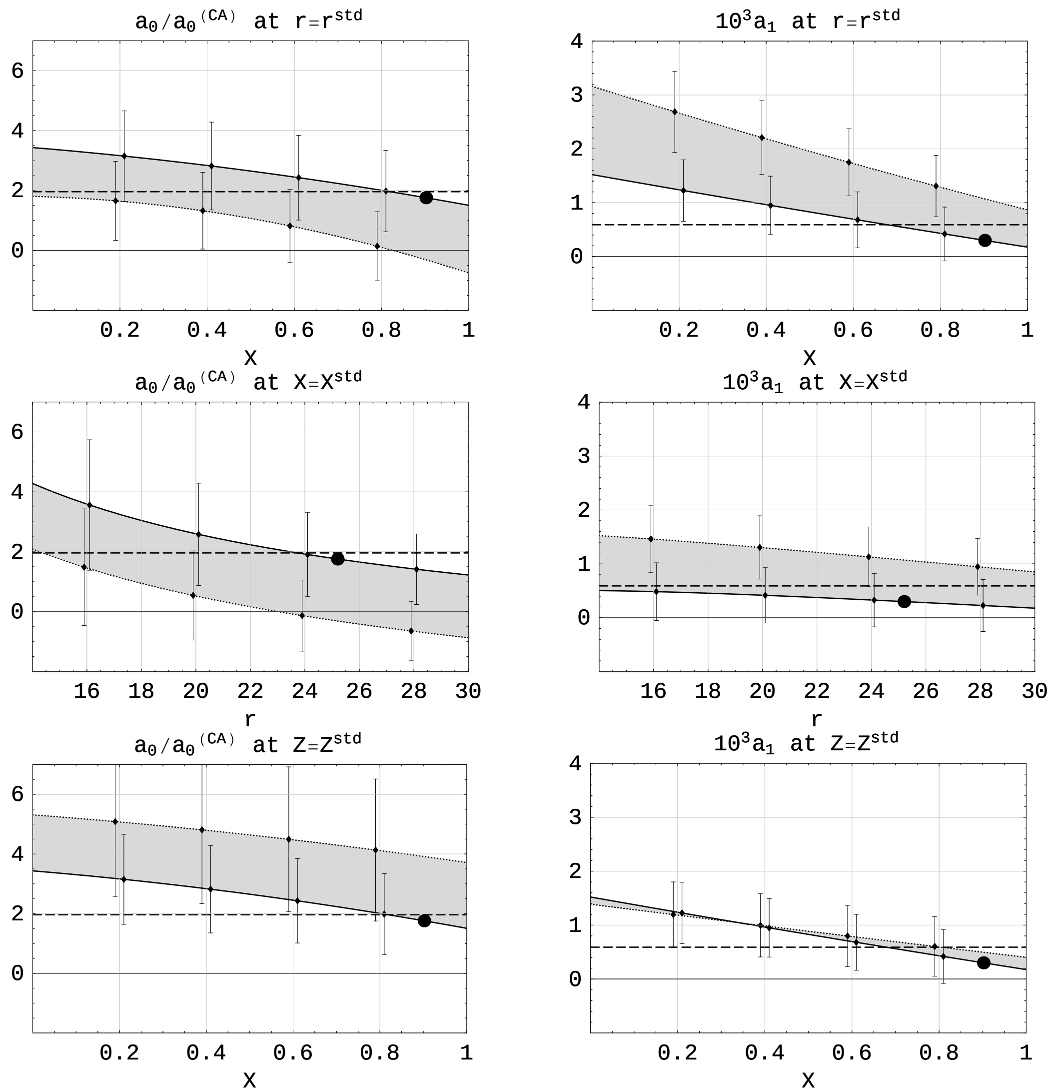

Figure 8.5: *Dependence of the scattering lengths a_0 and a_1 related to the polynomial parameters α and β. The figures are in one-to one correspondence to those in Figure 8.3.*

(8.108, 8.109) and (8.125, 8.136), we numerically get for the corresponding variations (to the first order in the remainders)

$$
\begin{aligned}
\frac{\Delta\alpha}{\alpha^{CA}} &= (\delta_\alpha + 6.92\delta_{F_\eta M_\eta} - 12.71\delta_{F_K M_K} - 0.76\delta_{F_\pi M_\pi} + 9.37\delta_{F_K} + 5.22\delta_{F_\pi} - 6.92\delta_{F_\eta}) \\
&\quad +(3.38\Delta L_1 + 0.56\Delta L_3)\times 10^3 \qquad (8.144)\\
\Delta\beta &= [(0.69\delta_\beta + 2.34\delta_{F_K} - 20.52\delta_{F_\pi}) + (17.0\Delta L_1 - 2.8\Delta L_3)\times 10^3]\times 10^{-3}M_\eta^2 \qquad (8.145)
\end{aligned}
$$

This reveals strong sensitivity on both δs and LEC's. Assuming again the typical size of the remainders to be $\delta \sim 0.1$ and adding all the uncertainties in quadrature (for ΔL_i we take the error bars form [44]) we obtain rough (over)estimates

$$
\left|\frac{\overline{\Delta\alpha}}{\alpha^{CA}}\right| = \sqrt{1.93^2 + 1.19^2} = 2.27 \qquad (8.146)
$$

$$\left|\overline{\Delta\beta}\right| = \sqrt{2.06^2 + 5.96^2} \times 10^{-3} M_\eta^2 = 6.31 \times 10^{-3} M_\eta^2, \tag{8.147}$$

where the first number under the square root represents the contribution of the remainders while the second accumulates the uncertainty from $L_{1,3}$. Though these numbers are a little bit more optimistic than those in the last row of Table 8.1 (note that the latter originated purely from the uncertainties of ΔL_i and did not include any estimates of the higher order corrections to α and β), it is clear that, without more restrictive information on the remainders (and L_i, $i = 1, 2, 3$)[26], the predictive power of χPT is reduced considerably in the case of $\pi\eta$ scattering even for "good" observables. In other words, small remainders are not a guarantee of an equivalently small final uncertainty. In what follows, we therefore try to gain some additional information outside the (resummed) χPT expansion to get further estimates of the size of the remainders.

The sources of the remainders are twofold: on one hand there are the unknown terms of the pure derivative expansion, on the other hand the contributions coming from the expansion in the quark masses. We try to get estimates for both of them from different sources, namely using the resonance estimate for the first type as well as independent information from generalized χPT for the second.

8.6.5 Resonance estimate of the direct remainders

In order to partially estimate the derivative part of the higher order corrections to the chiral expansion, we use the assumption, that the process under consideration is saturated by the exchange of the lowest laying resonances, the interactions of which can be described by the Lagrangian of the resonance chiral theory ($R\chi T$). The leading order Lagrangian of $R\chi T$ was originally formulated in the seminal paper [46] and applied to $\pi\eta$ scattering in [38]. To this process, only scalar resonances as well as $\eta_8 - \eta_0$ mixing contribute. Our result for the amplitude agree with [38] (cf. Appendix 8.17), which we can rewrite in terms of the resonance contribution $G^R_{\pi\eta}$ to $G_{\pi\eta}$ in the form

$$G^R_{\pi\eta}(s,t;u) = \alpha_R^{(4)} + \beta_R^{(4)} t + \gamma_R^{(4)} t^2 + \omega_R^{(4)} (s-u)^2 + \Delta G^R_{\pi\eta}(s,t;u). \tag{8.148}$$

The polynomial part with the coefficients (in what follows, M_S and M_{S_1} are the octet and singlet scalar mass respectively, c_d, c_m, $\widetilde{c}_d$, $\widetilde{c}_m$and $\widetilde{d}_m$ are the couplings defined in [46])

$$\begin{aligned}
\alpha_R^{(4)} = {} & 8M_\pi^2 M_\eta^2 \left(-\frac{c_d^2}{3M_S^2} + \frac{2\widetilde{c}_d^2}{M_{S_1}^2} \right) + 16 M_\pi^2 \overset{o}{M}{}_\eta^2 \left(\frac{c_d c_m}{3M_S^2} - \frac{\widetilde{c}_d \widetilde{c}_m}{M_{S_1}^2} \right) \\
& + 8 M_\eta^2 \overset{o}{M}{}_\pi^2 \left(\frac{c_d c_m}{3M_S^2} - \frac{2\widetilde{c}_d \widetilde{c}_m}{M_{S_1}^2} \right) - 16 \frac{\widetilde{d}_m^2}{M_{\eta_1}^2} \overset{o}{M}{}_\pi^2 \left(\overset{o}{M}{}_\pi^2 - \overset{o}{M}{}_\eta^2 \right) \\
& + 20 \overset{o}{M}{}_\pi^2 \overset{o}{M}{}_\eta^2 \left(\frac{\widetilde{c}_m^2}{M_{S_1}^2} - \frac{c_m^2}{3M_S^2} \right) - M_\pi^2 \overset{o}{M}{}_\pi^2 \frac{8 c_d c_m}{3M_S^2} + 4 \overset{o}{M}{}_\pi^4 \left(\frac{7c_m^2}{M_S^2} + \frac{\widetilde{c}_m^2}{M_{S_1}^2} \right)
\end{aligned} \tag{8.149}$$

$$\beta_R^{(4)} = -\frac{8\widetilde{c}_d \widetilde{c}_d}{M_{S_1}^2} \Sigma_{\eta\pi} + \frac{4}{3} \frac{c_d^2}{M_S^2} \Sigma_{\eta\pi} + 8 \left(\frac{\widetilde{c}_d \widetilde{c}_m}{M_{S_1}^2} - \frac{c_d c_m}{3M_S^2} \right) \left(\overset{o}{M}{}_\pi^2 + \overset{o}{M}{}_\eta^2 \right) \tag{8.150}$$

[26] As already discussed, the explicit dependence on these constants could be eliminated by means of reparametrization similar to those for L_i, $i = 4, \ldots, 6$ using further experimental input *e.g.* form K_{e4} decay. The price to pay is to introduce additional remainders connected with observables used for such a reparametrization.

$$\gamma_R^{(4)} = \frac{4\widetilde{c}_d^2}{M_{S_1}^2} - \frac{c_d^2}{3M_S^2} \tag{8.151}$$

$$\omega_R^{(4)} = \frac{c_d^2}{3M_S^2} \tag{8.152}$$

gathers the complete $O(p^4)$ resonance contribution (here we can recognize the resonance saturation of the LEC's in (8.90)-(8.93)). This part of the amplitude is already included in our resummed version of χPT, either explicitly through the LEC's $L_1 \ldots L_3$ or implicitly using the reparametrization in terms of the masses, decay constants and parameters r, X and Z. On the other hand, $\Delta G_{\pi\eta,R}(s,t;u)$ can be formally understood as an infinite sum of the higher order corrections in the (purely) derivative expansion, summed up to

$$\begin{aligned}
\Delta G_{\pi\eta,R}(s,t;u) &= \frac{4t}{M_{S_1}^2(M_{S_1}^2-t)}\left(\widetilde{c}_d(t-2M_\pi^2)+2\widetilde{c}_m \overset{o}{M}{}_\pi^2\right)\left(\widetilde{c}_d(t-2M_\eta^2)+2\widetilde{c}_m \overset{o}{M}{}_\eta^2\right) \\
&+\frac{2}{3}\frac{s}{M_S^2(M_S^2-s)}\left(c_d(s-M_\pi^2-M_\eta^2)+2c_m \overset{o}{M}{}_\pi^2\right)^2 \\
&+\frac{2}{3}\frac{u}{M_S^2(M_S^2-u)}\left(c_d(u-M_\pi^2-M_\eta^2)+2c_m \overset{o}{M}{}_\pi^2\right)^2 \\
&-\frac{2}{3}\frac{t}{M_S^2(M_S^2-t)}\left(c_d(t-2M_\pi^2)+2c_m \overset{o}{M}{}_\pi^2\right) \\
&\times\left(c_d(t-2M_\eta^2)+2c_m(2\overset{o}{M}{}_\eta^2-\overset{o}{M}{}_\pi^2)\right) \\
&+16\frac{\widetilde{d}_m^2 M_\eta^2}{M_{\eta_1}^2(M_{\eta_1}^2-M_\eta^2)}\overset{o}{M}{}_\pi^2\left(\overset{o}{M}{}_\eta^2-\overset{o}{M}{}_\pi^2\right).
\end{aligned} \tag{8.153}$$

Of course, this does not exhaust all of the possible higher order corrections (note *e.g.* that the resonance Lagrangian we use contains only the leading order interaction terms with one resonance field and chiral building blocks of the order $O(p^2)$), nevertheless we can use it at least as a rough estimate of the effect of higher orders of the derivative expansion. This is in some sense a procedure opposite to the usual resonance saturation; instead of LEC's we "saturate" the remainders by means of sewing together the resummed chiral expansion $G_{\pi\eta}^{\chi PT}$ (without remainders) with the resonance chiral theory writing the full $R\chi T$ amplitude as

$$G_{\pi\eta}^{R\chi T}(s,t;u) = G_{\pi\eta}^{\chi PT}(s,t;u) + \Delta G_{\pi\eta}^R(s,t;u). \tag{8.154}$$

and identifying $G_{\pi\eta}^{R\chi T}$ with the full χPT amplitude, the remainder being $\Delta G_{\pi\eta}^R$. Under this assumption, we can derive the following higher order contributions to the direct remainders from $\Delta G_{\pi\eta}^R$

$$\Delta\alpha_R = \frac{1}{3}F_\pi^2 M_\pi^2 \delta_\alpha^R = \frac{16}{3}c_m^2 \overset{o}{M}{}_\pi^4 \frac{\Sigma_{\eta\pi}}{M_S^2(M_S^2-\Sigma_{\eta\pi})} + 16\frac{\widetilde{d}_m^2 M_\eta^2}{M_{\eta_1}^2(M_{\eta_1}^2-M_\eta^2)}\overset{o}{M}{}_\pi^2\left(\overset{o}{M}{}_\eta^2-\overset{o}{M}{}_\pi^2\right) \tag{8.155}$$

$$\Delta\beta_R = \beta\delta_\beta^R = \frac{16}{M_{S_1}^4}\left(\widetilde{c}_d M_\pi^2 - \widetilde{c}_m \overset{o}{M}{}_\pi^2\right)\left(\widetilde{c}_d M_\eta^2 - \widetilde{c}_m \overset{o}{M}{}_\eta^2\right)$$

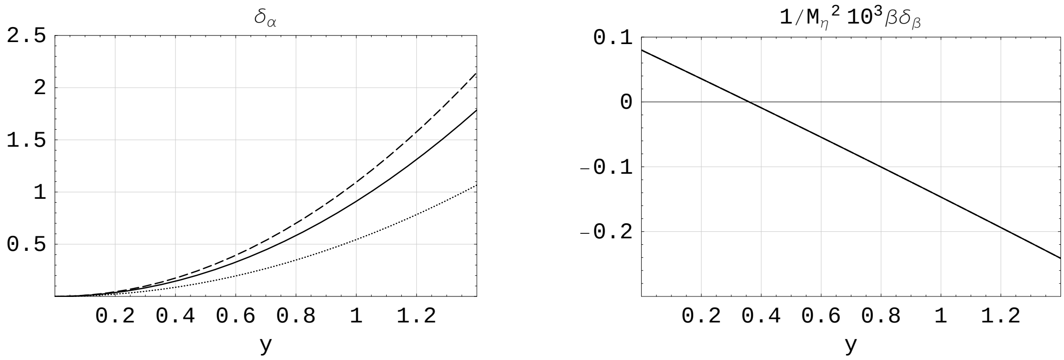

Figure 8.6: *Dependence of the resonance estimates of the direct remainders on* $Y = X/Z$ *for* $r = 15$ *(dots),* r_2 *(solid) and* 30 *(dashed). Note that for* $M_{S_1} = M_S$ *and* $\widetilde{c}_{m,d} = c_{m,d}/\sqrt{3}$*, the remainder* δ_β^R *is exactly independent on* r*.*

$$+\frac{8}{3}\frac{1}{M_S^4}\left(c_d M_\pi^2 - c_m \overset{o}{M}{}_\pi^2\right)\left(c_m(2\overset{o}{M}{}_\eta^2 - \overset{o}{M}{}_\pi^2) - c_d M_\eta^2\right)$$
$$-\frac{8}{3}\frac{c_m^2 \overset{o}{M}{}_\pi^4}{M_S^2(M_S^2-\Sigma_{\eta\pi})} - \frac{8}{3}\frac{c_d c_m \overset{o}{M}{}_\pi^2 \Sigma_{\eta\pi}}{M_S^2(M_S^2-\Sigma_{\eta\pi})} - \frac{8}{3}\frac{c_m^2 \overset{o}{M}{}_\pi^4 \Sigma_{\eta\pi}}{M_S^2(M_S^2-\Sigma_{\eta\pi})^2} \qquad (8.156)$$

and similarly for δ_γ^R and δ_ω^R (see Appendix 8.17). Note that the dependence on X and Z is exclusively through the ratio $Y = X/Z$ here.

One can notice that there are two distinct features of this procedure as compared to the usual LEC saturation. First, there is no need to fix a saturation scale, which is the result of "saturating" the renormalization scale independent remainder instead of the scale dependent LEC's. And second, as the resonance contribution are resummed to all chiral orders, the resonance poles are explicitly present in our result, as can be seen in (8.153), (8.155) and (8.156) as well as the formulae for δ_γ^R and δ_ω^R in the Appendix 8.17.

For the rough numerical estimates we use M_S, M_{S_1}, M_{η_1} and the couplings c_d, c_m, $\widetilde{c}_d$, $\widetilde{c}_m$, $\widetilde{d}_m$ from [46]. This gives at the standard reference point $X = X^{std}$, $Z = Z^{std}$ and $r = r^{std}$

$$\delta_\alpha^R = 1.00 \qquad (8.157)$$
$$\beta\delta_\beta^R = -0.15 \times 10^{-3} M_\eta^2, \qquad (8.158)$$

which represents roughly 55% and 20% correction to values in the first row of the Table 8.3 respectively. The dependence of δ_α^R and δ_β^R on $Y = X/Z$ and r is depicted in Figure 8.6. The effect of the resonance remainder estimate on the parameters α and β in a wider range of the X, r and Z is illustrated in the first column of Figure 8.8, analogous plots for γ and ω are in Figure 8.7. As can be seen, these results suggests the conclusion that the derivative part of the expansion could in some cases produce higher order remainders with much bigger value than 10%.

8.6.6 Generalized χPT

In the previous subsection we have tried to estimate the contributions to the remainders generated by the derivative expansion. The resulting expressions (8.155) and (8.156) could,

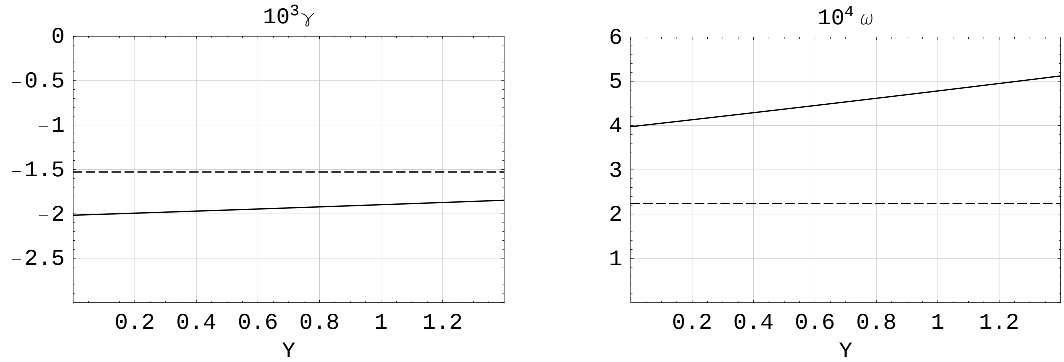

Figure 8.7: *Polynomial parameters γ and ω depending on Y. Horizontal dashed line: standard $O(p^4)$ and central $R\chi PT$ value. The result with resonance remainder estimates is shown by the solid line.*

however, gather only terms of at most the second order[27] in the quark mass expansion due to the lowest order resonance Lagrangian used. Also, the indirect remainders have not been included in this way as there is no contribution to them in this simplest approach. For the appraisal of the importance of the missing terms we therefore need additional information. One possibility might be to use a resonance Lagrangian with additional terms of higher chiral order suited for saturation of the $O(p^6)$ LEC's [47] and/or to go to the next-to-next-to leading order in the chiral expansion, this is, however, beyond the scope of our paper.

Instead we try to get some flavour of the size of the effect by means of comparison of our previous results with generalized χPT, which was originally designed to handle the badly convergent quark mass expansion in the case $X \ll 1$ and therefore also includes terms which correspond to higher orders in the standard chiral power counting.

In Subsection 8.5.4, we have already rewritten the generalized expansion of the parameters α and β (as well as that of the masses and decay constants in the Appendices 8.9, 8.16) in the "resummed" form (8.106–8.109) by means of splitting the "standard" remainders into the "nonstandard" extra terms $\delta^{loops}(\mu)$ and $\delta^{CT}(\mu)$ originating in $G\chi PT$ and the unknown part $\delta^{G\chi PT}$. Therefore, neglecting the latter, the sum $\delta^{loops}(\mu) + \delta^{CT}(\mu)$ could be in a sense interpreted as the rough estimate of the contribution to the standard remainders stemming from the higher orders of the quark mass expansion.

While $\delta^{loops}(\mu)$ are known, $\delta^{CT}(\mu)$ depend on the unknown LEC's of the $G\chi PT$ Lagrangian (cf. Appendix 8.13). We therefore set $\delta^{CT}(\mu) = 0$ at a fixed scale μ and by varying this scale in $\delta^{loops}(\mu)$ from $\mu = M_\eta$ to $\mu = M_\rho$ we can get some information on the contribution of the unknown LEC's (note that $\delta^{loops}(\mu) + \delta^{CT}(\mu)$ is renormalization scale independent). We apply this procedure both to the direct and indirect remainders.

The usual way of handling the generalized χPT expansion is to neglect the unknown remainders $\delta^{G\chi PT}$. We can repeat the considerations from the previous subsection and partially appreciate them using the resonance estimate. In order to avoid double counting, we have to further modify the resonance contribution to the remainder (8.153) subtracting terms of the order $O(p^4)$ within the generalized power counting in the same way as it was done in the

[27] Note that the physical masses in (8.155), (8.156) originate in the derivative expansion.

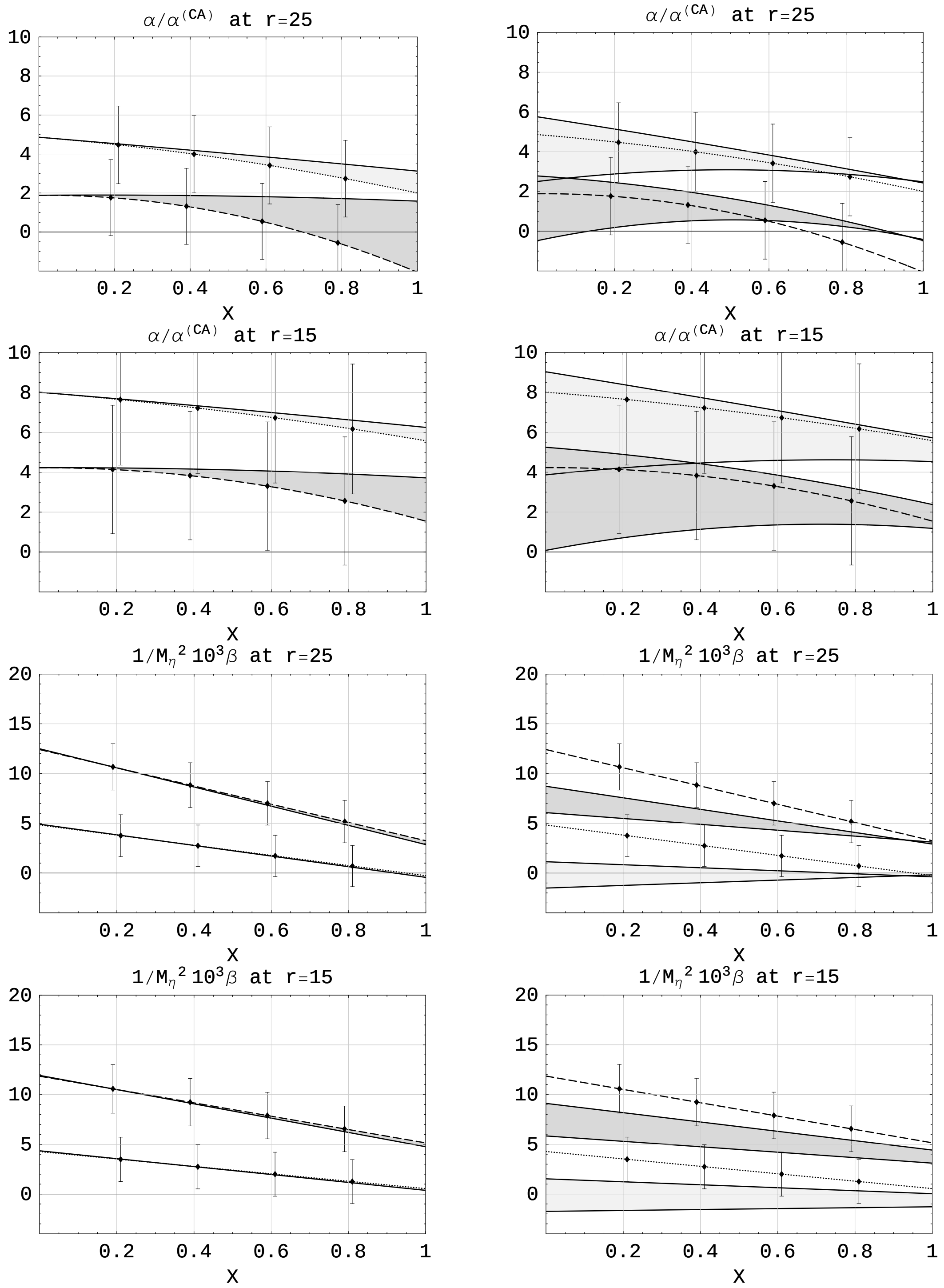

Figure 8.8: *Polynomial parameters α and β depending on X and Z for traditional and low value of r. The dotted line shows the central value for $Z = 0.9$, the dashed one is the same for $Z = 0.5$. The error bars correspond to the 10% estimates of the remainders. Left column: resonance estimate, filled areas highlight the $O(p^6)$ and higher corrections to the amplitude generated by resonances (lighter for $Z = 0.9$, darker for $Z = 0.5$). Right column: results with combined resonance and $G\chi PT$ estimate of remainders. Filled areas show the scale dependence ($\mu \sim M_\eta - M_\rho$), lighter for $Z = 0.9$ and darker for $Z = 0.5$.*

previous subsection (c.f. (8.148))

$$\Delta G^{G\chi PT}_{\pi\eta,R}(s,t;u) = \Delta G_{\pi\eta,R}(s,t;u) - \frac{16t}{M^4_{S_1}}\widetilde{c}^2_m \overset{o}{M}{}^2_\pi \overset{o}{M}{}^2_\eta - \frac{16}{M^4_S} c^2_m \overset{o}{M}{}^2_\pi (\overset{o}{M}{}^2_\pi (M^2_\eta + M^2_\pi) - \overset{o}{M}{}^2_\eta t)$$
$$- \frac{16}{M^4_{\eta_1}} \widetilde{d}^2_m \overset{o}{M}{}^2_\pi \left(\overset{o}{M}{}^2_\eta - \overset{o}{M}{}^2_\pi \right) M^2_\eta. \quad (8.159)$$

This combined $G\chi PT$ and resonance estimate of the remainders is illustrated in Figure 8.8, the right column shows the result in the case of the polynomial parameters α and β. The effect of the unknown GχPT LEC's is estimated by their scale dependence. The lines closer to the central $R\chi$PT results with neglected remainders are the ones at the scale $\mu = M_\rho$, *i.e.* the constants are set to zero at the usually chosen scale. The filled grey areas then show the change when the LEC's are set to the difference when moving from the scale $\mu = M_\rho$ to M_η. Admittedly, this assigns quite arbitrary numbers to the LEC's, so the uncertainty should be viewed as a rough estimate which can go both ways. The result can be interpreted as being quite consistent with the 10% estimate of the remainders, though clearly exceeding it for some range of the free parameters X, Z and r.

As for the parameters γ and ω, because their contribution in the polynomial expansion is quadratic in the Mandelstam variables, the GχPT estimate does not contribute here.

8.7 Summary and conclusions

In this paper we have studied the properties of various variants of the chiral expansion, namely the recently introduced resummed χPT as compared to the standard and partly generalized versions, on the concrete example of $\pi\eta$ scattering. Our calculations payed special attention to the possible reparametrization in terms of the physical observables. We have tried to illustrate several issues in detail, specifically:

- the necessity of carefully choosing a class of "good" observables for which the condition of global convergence is believed to be satisfied in the sense that the $O(p^6)$ and higher remainders are small and under control.
- the necessity to carefully define the bare expansion of "good" observables. Here we have concentrated on the requirements dictated by the exact renormalization scale independence as well as the exact perturbative unitarity. As we have shown, both these requirements can be met by means of sewing together the strict chiral expansion in terms of the LEC's with the dispersive representation for the corresponding Green function. Nevertheless, the resulting bare expansion is not yet defined uniquely; one has to fix the way how to treat the chiral logs and also the $O(p^2)$ amplitudes entering the dispersive integrals. Though the difference is formally of the same order as the remainder itself, we have found that, it might be numerically significant in some region of the free parameters.
- the properties of the standard chiral expansion, based on the potentially "dangerous" reparametrization of the bare expansion implicitly assuming $X, Z \sim 1$ and $r \sim r_2$. In this case we have established strong sensitivity of the observables for $\pi\eta$ scattering on the $O(p^4)$ LEC's; this plagues the standard prediction with large uncertainty. In the case of L_4 and L_6 this also means strong sensitivity to the vacuum fluctuations of the

$\overline{s}s$ pairs and therefore to the deviation from the standard scenario with X, $Z \sim 1$. The unusually large absolute values of the NLO corrections as well as large variations achieved for most of the observables (including the "good" ones) when moving from the $O(p^4)$ fit of the LEC's L_i [44] to the $O(p^6)$ based fit [45] might be interpreted as a signal of the importance of the NNLO corrections within the standard chiral expansion. This seems to be also supported by the sensitivity of the NLO contributions to their form when expressed in terms of the physical masses and decay constants (*i.e.* how the $O(p^2)$ relations like *e.g.* the Gell-Mann-Okubo formula are used).

- properties of the "safe" reparametrization and resummation of the vacuum fluctuations. We have confirmed that, for the "good" observables, the resummed and standard values coincide near the standard reference point $[X^{std}, Z^{std}, r^{std}]$. Under our working hypothesis, which assumes the "good" observables to be accompanied with small and controllable remainders, this can be interpreted as a consistence of the standard $O(p^4)$ chiral expansion of "good" observables in the sense that the potentially large higher order remainders are in fact small. On the other hand, in most cases of the "dangerous" observables the standard and resummed values do not meet at $[X^{std}, Z^{std}, r^{std}]$ (typically for those of them that start at $O(p^4)$). Though this might indicate that the standard expansion is convergent less satisfactorily in this case and the higher order remainders might be important here, the difference between the standard and resummed values lies within the estimated uncertainty of the resummed prediction. Away from the standard reference point, however, we have established that the central values diverge substantially from those of the standard approach even for the "good" observables. This is a signal that, unless X, $Z \sim 1$, the higher order remainders of the standard chiral expansion might be huge in comparison with LO+NLO value. Though this feature does not exclude the possibility that in this case the standard remainders might be saturated by the NNLO corrections, it could be nevertheless interpreted as an indication of the instability of the standard chiral expansion.

- the role of the remainders within the resummed approach. We have found a strong sensitivity of the observables connected to $\pi\eta$ scattering to the higher order remainders. This might reduce the predictive power of this approach, unless additional information on the actual size of the remainders is available. We have tried to make an independent estimate of the remainders using the simplest version of the resonance chiral Lagrangian as well as making a comparison with $G\chi PT$. Both these estimates seem to be in accord with the rough expectation $\delta \sim 10\%$ for the remainders only in some range of the parameters. For some observables and some corners of the parameter space, they can be substantially larger. Of course, the convergence properties of the bare expansion deserves further investigation by means of going to the NNLO, which is, however, beyond the scope of our article.

Let us add some final remarks concerning the interpretation of the above results from a practical point of view. The resummed version of the χPT expansion not only seems to be a suitable framework for taking the effect of large $s\overline{s}$-pairs vacuum fluctuations into account, but by keeping the remainders as explicit parameters it effectively includes all orders of the chiral expansion and thus it opens a space for incorporating further improvements of the predictions using additional information from various sources. As our analysis shows, $\pi\eta$ scattering allows to test the plausibility of the standard assumption $X, Z \sim 1$, $r \sim 25$ due to the sensitivity of the corresponding observables to the deviation of X, Z and r from these values. Provided experimental data were available, this could be done purely in the resummed framework using statistical methods similar to the ones used in the cases of $\pi\pi$ and πK scattering [8, 9].

On the other hand, to resolve a direct disagreement between the standard and resummed predictions is more delicate. At first sight, even though the $S\chi PT$ corrections at the NNLO are still not available, the possible experimental data which were in conflict with the standard $O(p^4)$ prediction but still compatible with that of resumed χPT might indicate problems with the standard chiral expansion based on the assumption $X, Z \sim 1$, $r \sim 25$. This might show itself either as unusually large $O(p^6)$ corrections or as $O(p^6)$ corrections too small to saturate the standard remainders. However, as we have illustrated in Subsection 6.1, the central values of the standard $O(p^4)$ predictions are plagued with large uncertainties even for the "good" observables. This feature together with the lack of information concerning the size of the standard $O(p^6)$ corrections would most likely prevents us from making a decisive conclusion concerning the possible deviations of the resummed χPT from the standard chiral expansion. In the light of our results, this is expected - bad convergence of the standard chiral expansion does not necessarily manifest itself as a direct conflict with experimental data at NLO, but rather in large uncontrollable uncertainties attached to its predictions.

Because of the current lack of low energy $\pi\eta$ scattering data, the comparison with experiment can only be done indirectly. As we have mentioned in the Introduction, the promising process here is the rare decay $\eta \to \pi^0\pi^0\gamma\gamma$, where the off shell $\pi\eta\pi\eta^*$ vertex enters the non-resonant part of the amplitude. As the preliminary studies [35, 36] using $G\chi PT$ show, the effect of the $s\bar{s}$-pairs vacuum fluctuations parametrized by X, Z away form their standard values might give large deviations from the prediction of $S\chi PT$ [48, 49, 50], resulting in the increase of the $\eta-$tail of the diphoton spectrum which can be in principle observed. Based on the above results, the more careful analysis using resummed version of χPT expansion is expected to yield qualitatively the same effect [37].

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Appendix

8.8 Standard chiral expansion of parameters X, Z and r

Here we summarize the formulae leading to the standard values of X, Z, Y and r used in Subsection 8.6.3. Using the standard reparametrization rules explained in Subsection 8.5.1, we get up to the NLO order in terms of the $O(p^4)$ LEC's

$$\begin{aligned}
X^{std} &= 1 - \frac{M_\pi^2}{2F_\pi^2}\left(32(L_6^r(r_2+2) + L_8^r)\right. \\
&\quad \left. +3J_{\pi\pi}^r(0) + (r_2+1)J_{KK}^r(0) + \frac{1}{9}(2r_2+1)J_{\eta\eta}^r(0) + \frac{11r_2+37}{144\pi^2}\right) \\
Z^{std} &= 1 - \frac{M_\pi^2}{2F_\pi^2}\left(16\,(L_4^r(r_2+2) + L_5^r) + (r_2+1)J_{KK}^r(0) + 4J_{\pi\pi}^r(0) + \frac{r_2+5}{16\pi^2}\right)
\end{aligned}$$

$$r^{std} = r_2 - \frac{M_\pi^2(r_2+1)}{2F_\pi^2}\left(8(2L_8^r - L_5)(r_2-1) - \frac{1}{3}(2r_2+1)J^r_{\eta\eta}(0) + J^r_{\pi\pi}(0) - \frac{(r_2-1)}{24\pi^2}\right). \quad (8.160)$$

For r^{std} we can also use an alternative expression based on the chiral expansion of r_2^*

$$\begin{aligned} r^{*std} &= r_2^* - \frac{M_\pi^2(r_2+1)}{2F_\pi^2}\left(16L_8^r(r_2-1)\right. \\ &\left.+\frac{1}{6}(2r_2+1)J^r_{\eta\eta}(0) + \frac{1}{2}(r_2+1)J^r_{KK}(0) - \frac{3}{2}J^r_{\pi\pi}(0) + \frac{5(r_2-1)}{96\pi^2}\right). \end{aligned} \quad (8.161)$$

Δ_{GMO} has the following standard chiral expansion

$$\begin{aligned} \Delta^{std}_{GMO} &= \frac{M_\pi^2(r_2-1)}{2F_\pi^2}\left(32(2L_7^r + L_8^r)(r_2-1)\right. \\ &\left.+\frac{1}{3}(2r_2+1)J^r_{\eta\eta}(0) + (r_2+1)J^r_{KK}(0) - 3J^r_{\pi\pi}(0) + \frac{5(r_2-1)}{48\pi^2}\right). \end{aligned} \quad (8.162)$$

8.9 Chiral expansion of the η decay constant

For the bare expansion of the "good" observables F_P^2 we rewrite the standard formulae in the form

$$\begin{aligned} F_\pi^2 &= F_0^2\left[1 + \frac{B_0\widehat{m}}{F_0^2}\left(16L_4^r(\mu)(r+2) + 16L_5^r(\mu) + (r+1)J^r_{KK}(0) + 4J^r_{\pi\pi}(0) + \frac{1}{16\pi^2}(r+5)\right)\right] \\ &+F_\pi^2\delta_{F_\pi}, \end{aligned} \quad (8.163)$$

$$\begin{aligned} F_K^2 &= F_0^2\left[1 + \frac{B_0\widehat{m}}{F_0^2}\left(16L_4^r(\mu)(r+2) + 8L_5^r(\mu)(r+1) + \frac{3}{2}(r+1)\left(J^r_{KK}(0) + \frac{1}{16\pi^2}\right)\right.\right. \\ &\left.\left.+\frac{3}{2}\left(J^r_{\pi\pi}(0) + \frac{1}{16\pi^2}\right) + \frac{1}{2}(2r+1)\left(J^r_{\eta\eta}(0) + \frac{1}{16\pi^2}\right)\right)\right] \\ &+F_K^2\delta_{F_K} \end{aligned} \quad (8.164)$$

$$\begin{aligned} F_\eta^2 &= F_0^2\left[1 + \frac{B_0\widehat{m}}{F_0^2}\left(16L_4^r(\mu)(r+2) + \frac{16}{3}L_5^r(\mu)(2r+1) + 3(r+1)\left(J^r_{KK}(0) + \frac{1}{16\pi^2}\right)\right)\right] \\ &+F_\eta^2\delta_{F_\eta}. \end{aligned} \quad (8.165)$$

Within the standard χPT, the $O(p^2)$ parameters B_0, F_0 and r are expressed using inverted expansions of the observables F_P^2, M_P^2, as explained in subsection 8.5.1. This yields the standard formula for F_η^2

$$\begin{aligned} F_\eta^2 &= F_\pi^2\left[1 + \frac{M_\pi^2}{F_\pi^2}\left(\frac{16}{3}L_5^r(\mu)(r_2-1) + (r_2+1)J^r_{KK}(0) - 2J^r_{\pi\pi}(0) + \frac{1}{16\pi^2}(r_2-1)\right)\right] \\ &+F_\eta^2\delta^{st}_{F_\eta}. \end{aligned} \quad (8.166)$$

with a potentially large remainder $\delta^{st}_{F_\eta}$. Numerically, with $L_5^r(M_\rho)$ taken from [44] we get

$$F_\eta^2 = 1.625F_\pi^2. \quad (8.167)$$

On the other hand, the "safe" reparametrization in terms of r, X and Z gives

$$\begin{aligned} F_\eta^2 &= F_\pi^2\left[1+\frac{2}{3}(r-1)\eta(r)-\frac{1}{3}\frac{M_\pi^2}{F_\pi^2}\left(\frac{X}{Z}\right)\left(J_{\pi\pi}^r(0)-2(r+1)J_{KK}^r(0)+(2r+1)J_{\eta\eta}^r(0)\right)\right] \\ &+\frac{1}{3}\left(3F_\eta^2\delta_{F_\eta}+F_\pi^2\delta_{F_\pi}-4F_K^2\delta_{F_K}\right), \end{aligned} \tag{8.168}$$

which is valid as an exact algebraic identity[28].

Following the $G\chi PT$ procedure outlined in Section 5.4, after identifying the corresponding LEC's in both approaches

$$\begin{aligned} \frac{B_0\widehat{m}}{F_0^2}L_4^r(\mu) &\to \frac{1}{8}\widehat{m}\widetilde{\xi} \\ \frac{B_0\widehat{m}}{F_0^2}L_5^r(\mu) &\to \frac{1}{8}\widehat{m}\xi^r, \end{aligned} \tag{8.169}$$

and defining the remainders using the physical masses inside the chiral logs, we can use the exact formula (8.165) and write the remainder δ_{F_η} within $G\chi PT$ as

$$F_\eta^2\delta_{F_\eta}=F_\eta^2\delta_{F_\eta}^{loop}(\mu)+F_0^2\delta_{F_\eta}^{(4)\ CT}(\mu)+F_\eta^2\delta_{F_\eta}^{G\chi PT}.$$

Here

$$F_\eta^2\delta_{F_\eta}^{loop}(\mu)=3\widehat{m}^2\,(r+1)\left(A_0\,(r+1)+2(r+2)Z_0^S\right)\left(J_{KK}^r(0)+\frac{1}{16\pi^2}\right) \tag{8.170}$$

is the extra loop contribution and $\delta_{F_\eta}^{(4)\,CT}(\mu)$ the contribution of the counterterms from the $O(p^4)$ $G\chi PT$ Lagrangian renormalized at scale μ

$$\begin{aligned} \delta_{F_\eta}^{(4)\,CT}(\mu) &= \frac{2}{3}\widehat{m}^2\left[\frac{1}{2}(2A_1+A_2+4A_3+2B_1-2B_2)(1+2r^2)\right. \\ &\left. +3(A_4+2B_4)(1+\frac{1}{2}r^2)-4\,C_1^P(r-1)^2+2D^S\,(r+2)\,(2r+1)\right]. \end{aligned} \tag{8.171}$$

$\delta_{F_\eta}^{G\chi PT}$ is a new remainder, which is exactly independent on the renormalization scale.

Analogously, for the $G\chi PT$ formula for F_π^2 we have, besides the substitution (8.169) to (8.163), to insert

$$F_\pi^2\delta_{F_\pi}=F_\pi^2\delta_{F_\pi}^{loop}(\mu)+F_0^2\delta_{F_\pi}^{(4)\ CT}(\mu)+F_\pi^2\delta_{F_\pi}^{G\chi PT}, \tag{8.172}$$

---

[28] This identity can be also rewritten as

$$4F_K^2(1-\delta_{F_K})-F_\pi^2(1-\delta_{F_\pi})-3F_\eta^2(1-\delta_{F_\eta})=\left(\frac{X}{Z}\right)M_\pi^2\left(J_{\pi\pi}^r(0)-2(r+1)J_{KK}^r(0)+(2r+1)J_{\eta\eta}^r(0)\right).$$

Within the standard approach, the parameters on the r.h.s. of this identity can be expressed to the order $O(p^4)$ in terms of the physical observables and it is interpreted as a $O(p^4)$ sum rule

$$4F_K^2-F_\pi^2-3F_\eta^2=M_\pi^2\left(J_{\pi\pi}^r(0)-2(r_2+1)J_{KK}^r(0)+(2r_2+1)J_{\eta\eta}^r(0)\right).$$

This gives

$$F_\eta^2=1.697F_\pi^2.$$

where the loop and counterterm contribution are now

$$\begin{aligned}
F_\pi^2 \delta_{F_\pi}^{loop}(\mu) &= 8\widehat{m}^2(A_0 + (r+2)Z_0^S)) \left(J_{\pi\pi}^r(0) + \frac{1}{16\pi^2} \right) \\
&\quad + \widehat{m}^2 (r+1) (A_0 (r+1) + 2(r+2)Z_0^S) \left(J_{KK}^r(0) + \frac{1}{16\pi^2} \right) \\
\delta_{F_\pi}^{(4)\,CT}(\mu) &= 2\widehat{m}^2 \Big[A_1 + \frac{1}{2}A_2 + 2A_3 + (A_4 + 2B_4)(1 + \frac{1}{2}r^2) \\
&\quad + B_1 - B_2 + 2D^S (r+2) \Big] . \qquad (8.173)
\end{aligned}$$

Finally, we have the expression for F_K^2, where the remainder is replaced with

$$F_K^2 \delta_{F_K} = F_K^2 \delta_{F_K}^{loop}(\mu) + F_0^2 \delta_{F_K}^{(4)\,CT}(\mu) + F_K^2 \delta_{F_K}^{G\chi PT} \qquad (8.174)$$

and the loops and counterterms contribute as

$$\begin{aligned}
F_K^2 \delta_{F_K}^{loop}(\mu) &= 3\widehat{m}^2(A_0 + (r+2)Z_0^S)) \left(J_{\pi\pi}^r(0) + \frac{1}{16\pi^2} \right) \\
&\quad + \frac{3}{2}\widehat{m}^2 (r+1) (A_0 (r+1) + 2(r+2)Z_0^S) \left(J_{KK}^r(0) + \frac{1}{16\pi^2} \right) \\
&\quad + \widehat{m}^2(A_0 \left(2r^2+1\right) + 2(r-1)^2 Z_0^P + (r+2)(2r+1)Z_0^S) \left(J_{\eta\eta}^r(0) + \frac{1}{16\pi^2} \right) \\
\delta_{F_K}^{(4)\,CT}(\mu) &= \widehat{m}^2 \Big[(A_1 + B_1)(r^2+1) + (A_2 - 2B_2)r + 2(A_4 + 2B_4)(1 + \frac{1}{2}r^2) \\
&\quad + 2D^S (r+2)(r+1) \Big] . \qquad (8.175)
\end{aligned}$$

In order to reparametrize the $G\chi PT$ bare expansion in terms of the masses and decay constants, we can proceed as follows. Because the exact identity (8.168) is valid independently of the version of χPT, we can also use it in the generalized case, provided we rewrite the remainders according to (8.171, 8.172) and (8.174). This step eliminates the LEC's ξ and $\widetilde{\xi}$. Collecting the chiral logs together we have

$$\begin{aligned}
F_\eta^2 &= F_\pi^2 \Bigg[1 + \frac{2}{3}(r-1)\eta(r) - \frac{1}{3F_\pi^2} \Bigg(\widetilde{M}_\pi^2 \left(J_{\pi\pi}^r(0) + \frac{1}{16\pi^2} \right) \\
&\quad - 4\widetilde{M}_K^2 \left(J_{KK}^r(0) + \frac{1}{16\pi^2} \right) + 3\widetilde{M}_\eta^2 \left(J_{\eta\eta}^r(0) + \frac{1}{16\pi^2} \right) \Bigg) \Bigg] + F_\eta^2 \Delta_{F_\eta}^{G\chi PT}(\mu) \qquad (8.176)
\end{aligned}$$

where the $O(p^2)$ masses are given by (8.112) and

$$F_\eta^2 \Delta_{F_\eta}^{G\chi PT}(\mu) = \frac{1}{3}F_0^2 \left(3\delta_{F_\eta}^{(4)\,CT} + \delta_{F_\pi}^{(4)\,CT} - 4\delta_{F_K}^{(4)\,CT} \right) + \frac{1}{3} \left(3F_\eta^2 \delta_{F_\eta}^{G\chi PT} + F_\pi^2 \delta_{F_\pi}^{G\chi PT} - 4F_K^2 \delta_{F_K}^{G\chi PT} \right) . \qquad (8.177)$$

The last step consists of replacing the LEC's F_0, A_0, Z_0^S and Z_0^P with the first term of their expansion in terms of the masses and decay constants as described in Subsection 8.5.4. This corresponds to a further redefinitions of the generalized remainders.

8.10 Dispersion representation of the $\pi\eta$ amplitude

For the dispersive representation of the amplitude we need the $S-$ and $T-$channel discontinuities at $O(p^4)$. In the following subsections we give a list of the relevant $O(p^2)$ amplitudes $G^{(2)A_i\to ij}$ and $G^{(2)ij\to A_f}$ and $O(p^4)$ discontinuities discG_0^{ij} corresponding to the different intermediate states ij.

8.10.1 $S-$channel discontinuities at $O(p^4)$

- $\pi\eta$ intermediate state

$$\begin{aligned} G^{(2)\pi\eta\to\pi\eta} &= \frac{1}{3}F_0^2\,\mathring{M}_\pi^2 \\ \mathrm{disc}G_0^{\pi\eta}(s) &= 2\frac{\lambda^{1/2}(s,M_\pi^2,M_\eta^2)}{s}\left(\frac{1}{32\pi}\frac{1}{3}\,\mathring{M}_\pi^2\right)^2\frac{F_0^4}{F_\pi^2F_\eta^2} \end{aligned} \tag{8.178}$$

- $\overline{K}K$ intermediate state

$$\begin{aligned} G^{(2)\pi\eta\to\overline{K^0}K^0(K^+K^-)} &= -\frac{\sqrt{3}}{4}F_0^2(s-\frac{1}{3}M_\eta^2-\frac{1}{3}M_\pi^2-\frac{2}{3}M_K^2) \\ &\quad +\frac{1}{4\sqrt{3}}F_0^2(2\,\mathring{M}_K^2-\mathring{M}_\pi^2-\mathring{M}_\eta^2) \\ G^{(2)\overline{K^0}K^0(K^+K^-)\to\pi\eta} &= -\frac{\sqrt{3}}{4}F_0^2(s-\frac{1}{3}M_\eta^2-\frac{1}{3}M_\pi^2-\frac{2}{3}M_K^2) \\ &\quad +\frac{1}{4\sqrt{3}}F_0^2(2\,\mathring{M}_K^2-\mathring{M}_\pi^2-\mathring{M}_\eta^2) \\ \mathrm{disc}G_0^{\overline{K^0}K^0(K^+K^-)}(s) &= 2\sqrt{1-\frac{4M_K^2}{s}}\left(\frac{1}{32\pi}\right)^2\frac{3}{16}[(s-\frac{1}{3}M_\eta^2-\frac{1}{3}M_\pi^2-\frac{2}{3}M_K^2) \\ &\quad -\frac{1}{3}(2\,\mathring{M}_K^2-\mathring{M}_\pi^2-\mathring{M}_\eta^2)]^2\frac{F_0^4}{F_K^4} \end{aligned} \tag{8.179}$$

8.10.2 $T-$channel discontinuities at $O(p^4)$

- $\pi\pi$ intermediate state

$$\begin{aligned} G^{(2)\pi\pi\to\pi\pi,\,I=0} &= F_0^2[(s-\frac{4}{3}M_\pi^2)+\frac{5}{6}\,\mathring{M}_\pi^2] \\ \mathrm{disc}G_0^{\pi\pi,\,I=0}(s) &= 2\sigma(s)\left(\frac{1}{32\pi}\right)^2\frac{1}{3}\,\mathring{M}_\pi^2\,[(s-\frac{4}{3}M_\pi^2)+\frac{5}{6}\,\mathring{M}_\pi^2]\frac{F_0^4}{F_\pi^4} \end{aligned} \tag{8.180}$$

- $\eta\eta$ intermediate state

$$\begin{aligned} G^{(2)\eta\eta\to\eta\eta} &= -\frac{1}{3}F_0^2\left(\mathring{M}_\pi^2-4\,\mathring{M}_\eta^2\right) \\ \mathrm{disc}G_0^{\eta\eta}(s) &= -2\frac{1}{2}\sqrt{1-\frac{4M_\eta^2}{s}}\left(\frac{1}{32\pi}\right)^2\frac{1}{9}\,\mathring{M}_\pi^2\left(\mathring{M}_\pi^2-4\,\mathring{M}_\eta^2\right)\frac{F_0^4}{F_\eta^4} \end{aligned} \tag{8.181}$$

- $\overline{K}K$ intermediate state[29]

$$
\begin{aligned}
G^{(2)\pi\pi\to\overline{K^0}K^0(K^+K^-),\,I=0} &= \mp\frac{\sqrt{3}}{4}F_0^2[(s-\frac{2}{3}M_\pi^2-\frac{2}{3}M_K^2)\\
&\quad+\frac{2}{3}(\mathring{M}_K^2+\mathring{M}_\pi^2)]\\
G^{(2)\overline{K^0}K^0(K^+K^-)\to\eta\eta,\,I=0} &= \pm\frac{1}{4}F_0^2[(3s-2M_K^2-2M_\eta^2)\\
&\quad+(2\mathring{M}_\eta^2-\frac{2}{3}\mathring{M}_K^2)]\\
\mathrm{disc}G_0^{\overline{K^0}K^0(K^+K^-),\,I=0}(s) &= \sqrt{1-\frac{4M_K^2}{s}}\left(\frac{1}{32\pi}\right)^2\frac{1}{16}[(s-\frac{2}{3}M_\pi^2-\frac{2}{3}M_K^2)\\
&\quad+\frac{2}{3}(\mathring{M}_K^2+\mathring{M}_\pi^2)]\times[(3s-2M_K^2-2M_\eta^2)\\
&\quad+(2\mathring{M}_\eta^2-\frac{2}{3}\mathring{M}_K^2)]]\frac{F_0^4}{F_K^4} \qquad (8.182)
\end{aligned}
$$

8.11 The scalar bubble

In this appendix we summarize the formulae for the scalar bubble, defined as

$$
\begin{aligned}
J_{PQ}(q^2) &= -\mathrm{i}\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-M_P^2+\mathrm{i}0)((k-q)^2-M_Q^2+\mathrm{i}0)}\\
&= -2\lambda_\infty+J_{PQ}^r(q^2). \qquad (8.183)
\end{aligned}
$$

Here, as usual

$$
\lambda_\infty = \frac{\mu^{d-4}}{16\pi^2}\left(\frac{1}{d-4}-\frac{1}{2}(\ln 4\pi+\Gamma'(1)+1)\right) \qquad (8.184)
$$

and $J_{PQ}^r(s)=J_{PQ}^r(0)+\overline{J}_{PQ}(s)$, where

$$
J_{PQ}^r(0)=-\frac{1}{16\pi^2}\frac{M_P^2\ln(M_P^2/\mu^2)-M_Q^2\ln(M_Q^2/\mu^2)}{M_P^2-M_Q^2} \qquad (8.185)
$$

and $\overline{J}_{PQ}(s)$, sometimes called Chew-Mandelstam function, can be expressed by means of once subtracted dispersion relation as

$$
\overline{J}_{PQ}(s)=\frac{s}{16\pi^2}\int_{(M_P+M_Q)^2}^{\infty}\frac{\mathrm{d}x}{x}\frac{\lambda^{1/2}(x,M_P^2,M_Q^2)}{x}\frac{1}{x-s}. \qquad (8.186)
$$

The explicit form of $\overline{J}_{PQ}(s)$ reads

$$
\overline{J}_{PQ}(s)=\frac{1}{32\pi^2}\left(2+\frac{\Delta_{PQ}}{s}\ln\frac{M_Q^2}{M_P^2}-\frac{\Sigma_{PQ}}{\Delta_{PQ}}\ln\frac{M_Q^2}{M_P^2}+2\frac{(s-(M_P-M_Q)^2)}{s}\sigma_{PQ}(s)\ln\frac{\sigma_{PQ}(s)-1}{\sigma_{PQ}(s)+1}\right), \qquad (8.187)
$$

[29]Let us note

$$
G^{I=0}(s,t;u)=-\frac{1}{\sqrt{3}}\delta^{ab}G^{ab}(s,t;u)=-\sqrt{3}G(s,t;u)
$$

where

$$\begin{aligned}
\Delta_{PQ} &= M_P^2 - M_Q^2 \\
\Sigma_{PQ} &= M_P^2 + M_Q^2 \\
\sigma_{PQ}(t) &= \sqrt{\frac{s-(M_P+M_Q)^2}{s-(M_P-M_Q)^2}} = \sqrt{1-\frac{4M_PM_Q}{s-(M_P-M_Q)^2}}\,. \qquad (8.188)
\end{aligned}$$

In the limit $M_P \to M_Q$ we get

$$\begin{aligned}
J^r_{PP}(0) &= -\frac{1}{16\pi^2}\left(\ln\frac{M_P^2}{\mu^2}+1\right) \\
\overline{J}_{PP}(s) &= \frac{1}{16\pi^2}\left(2+\sigma_{PP}(s)\ln\frac{\sigma_{PP}(s)-1}{\sigma_{PP}(s)+1}\right). \qquad (8.189)
\end{aligned}$$

8.12 L_4-L_8 in terms of masses and decay constants

In this Appendix we summarize the formulae used in the text for the reparametrization of bare expansions of "good" observables. We use the abbreviated notation (8.102–8.105). From the bare expansion of "good" variables F_π^2 and F_K^2 we obtain

$$\begin{aligned}
4\overset{o}{M}{}_\pi^2\, L_4^r(\mu) &= \frac{1}{2}(1-Z-\eta(r))\frac{F_\pi^2}{r+2} \\
&-\frac{M_\pi^2}{4(r+2)(r-1)}\frac{X}{Z}[(4r+1)J^r_{\pi\pi}(0)+(r-2)(r+1)J^r_{KK}(0) \\
&-(2r+1)J^r_{\eta\eta}(0)+\frac{(r+2)(r-1)}{16\pi^2}] \\
&+\frac{2F_K^2\delta_{F_K}-(r+1)F_\pi^2\delta_{F_\pi}}{2(r+2)(r-1)}, \qquad (8.190) \\
4\overset{o}{M}{}_\pi^2\, L_5^r(\mu) &= \frac{1}{2}F_\pi^2\eta(r) \\
&+\frac{M_\pi^2}{4(r-1)}\frac{X}{Z}[5J^r_{\pi\pi}(0)-(r+1)J^r_{KK}(0)-(2r+1)J^r_{\eta\eta}(0)-\frac{3(r-1)}{16\pi^2}] \\
&-\frac{F_K^2\delta_{F_K}-F_\pi^2\delta_{F_\pi}}{(r-1)}. \qquad (8.191)
\end{aligned}$$

In the same way, from the expansion of $F_P^2M_P^2$ we get

$$\begin{aligned}
4\overset{o}{M}{}_\pi^4\, L_6^r(\mu) &= \frac{1}{4}\frac{F_\pi^2M_\pi^2}{r+2}(1-X-\varepsilon(r)) \\
&-\frac{M_\pi^4}{72(r-1)(r+2)}\left(\frac{X}{Z}\right)^2[27rJ^r_{\pi\pi}(0)+9(r+1)(r-2)J^r_{KK}(0) \\
&+(2r+1)(r-4)J^r_{\eta\eta}(0)+\frac{11(r-1)(r+2)}{16\pi^2}] \qquad (8.192) \\
&-\frac{F_\pi^2M_\pi^2\delta_{F_\pi M_\pi}[(r+1)^2]-4F_K^2M_K^2\delta_{F_KM_K}}{4(r^2-1)(r+2)}
\end{aligned}$$

$$4\overset{o}{M}{}_{\pi}^{4}\, L_7^r(\mu) = -\frac{1}{8}F_\pi^2 M_\pi^2\left(\varepsilon(r) - \frac{\Delta_{GMO}}{(r-1)^2}\right) - \frac{3(1+r)F_\eta^2 M_\eta^2 \delta_{F_\eta M_\eta} + (2r^2+r-1)F_\pi^2 M_\pi^2 \delta_{F_\pi M_\pi} - 8rF_K^2 M_K^2 \delta_{F_K M_K}}{8(r-1)^2(r+1)} \tag{8.193}$$

$$\begin{aligned}4\overset{o}{M}{}_{\pi}^{4}\, L_8^r(\mu) &= \frac{1}{4}F_\pi^2 M_\pi^2 \varepsilon(r) \\ &+\frac{M_\pi^4}{24(r-1)}\left(\frac{X}{Z}\right)^2 [9J_{\pi\pi}^r(0) - 3(r+1)J_{KK}^r(0) - (2r+1)J_{\eta\eta}^r(0) - \frac{5(r-1)}{16\pi^2}] \\ &- \frac{2F_K^2 M_K^2 \delta_{F_K M_K} - (r+1)F_\pi^2 M_\pi^2 \delta_{F_\pi M_\pi}}{2(r^2-1)}.\end{aligned} \tag{8.194}$$

8.13 Lagrangian of $G\chi PT$ to $O(p^4)$

Here we give the traditional form of the $G\chi PT$ Lagrangian. In the following formulae

$$\begin{aligned}\chi &= \mathcal{M} + s + \mathrm{i}p \\ \nabla U &= \partial U - \mathrm{i}(v+a)U + \mathrm{i}U(v-a) \\ \chi &= \partial\chi - \mathrm{i}(v+a)\chi + \mathrm{i}\chi(v-a).\end{aligned} \tag{8.195}$$

Up to the order $O(p^4)$, the Lagrangian can be split into the $O(p^2)$, $O(p^3)$ and $O(p^4)$ parts

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4, \tag{8.196}$$

where

$$\mathcal{L}_n = \sum_{i+j+k=n} \mathcal{L}^{(i,j,k)} \tag{8.197}$$

and (i,j,k) indicates the number of derivatives, χ sources and powers of B_0 respectively. Then for $O(p^2)$ we get

$$\begin{aligned}\mathcal{L}^{(2,0,0)} &= \frac{F_0^2}{4}\langle\nabla_\mu U^+ \nabla^\mu U\rangle \\ \mathcal{L}^{(0,1,1)} &= \frac{F_0^2}{2}B_0\langle U^+\chi + \chi^+ U\rangle \\ \mathcal{L}^{(0,2,0)} &= \frac{F_0^2}{4}\Big(\overline{A}_0\langle(U^+\chi)^2 + (\chi^+U)^2\rangle \\ &+ \overline{Z}_0^S\langle U^+\chi + \chi^+U\rangle^2 + \overline{Z}_0^P\langle U^+\chi - \chi^+U\rangle^2\Big).\end{aligned} \tag{8.198}$$

At the order $O(p^3)$ one has

$$\begin{aligned}\mathcal{L}^{(2,1,0)} &= \frac{F_0^2}{4}(\overline{\xi}\langle\nabla_\mu U^+\nabla^\mu U(\chi^+U + U^+\chi)\rangle + \overline{\overline{\xi}}\langle\nabla_\mu U^+\nabla^\mu U\rangle\langle\chi^+U + U^+\chi\rangle) \\ \mathcal{L}^{(0,3,0)} &= \frac{F_0^2}{4}(\overline{\rho}_1\langle(\chi^+U)^3 + (U^+\chi)^3\rangle + \overline{\rho}_2\langle(\chi^+U + U^+\chi)\chi^+\chi\rangle \\ &+\overline{\rho}_3\langle(\chi^+U)^2 - (U^+\chi)^2\rangle\langle\chi^+U - U^+\chi\rangle \\ &+\overline{\rho}_4\langle(\chi^+U)^2 + (U^+\chi)^2\rangle\langle\chi^+U + U^+\chi \\ &+\overline{\rho}_5\langle\chi^+U + U^+\chi\rangle\langle\chi^+\chi\rangle + \overline{\rho}_6\langle\chi^+U - U^+\chi\rangle^2\langle\chi^+U + U^+\chi\rangle\end{aligned}$$

$$
\begin{aligned}
& \quad +\overline{\rho}_7\langle\chi^+U+U^+\chi\rangle^3) \\
\mathcal{L}^{(2,0,1)} &= \frac{F_0^2B_0}{4}\delta_d^{(1)}\langle\nabla_\mu U^+\nabla^\mu U\rangle \\
\mathcal{L}^{(0,2,1)} &= \frac{F_0^2B_0}{4}\Big(\delta^{(1)}\overline{A}_0\langle(U^+\chi)^2+(\chi^+U)^2 \\
& \quad +\delta^{(1)}\overline{Z}_0^S\langle U^+\chi+\chi^+U\rangle^2+\delta^{(1)}\overline{Z}_0^P\langle U^+\chi-\chi^+U\rangle^2\Big) \\
\mathcal{L}^{(0,1,2)} &= \frac{F_0^2}{2}B_0^2\delta_\chi^{(1)}\langle U^+\chi+\chi^+U\rangle.
\end{aligned}
\tag{8.199}
$$

For the $O(p^4)$ Lagrangian, the building blocks are

$$
\begin{aligned}
\mathcal{L}^{(4,0,0)} &= L_1\langle\nabla_\mu U^+\nabla^\mu U\rangle^2+L_2\langle\nabla_\mu U^+\nabla_\nu U\rangle\langle\nabla^\mu U^+\nabla^\nu U\rangle \\
& \quad +L_3\langle\nabla_\mu U^+\nabla^\mu U\nabla_\nu U^+\nabla^\nu U\rangle \\
& \quad -iL_9\left\langle F^R_{\mu\nu}\nabla^\mu U\nabla^\nu U^++F^L_{\mu\nu}\nabla^\mu U^+\nabla^\nu U\right\rangle \\
& \quad +L_{10}\left\langle U^+F^R_{\mu\nu}UF^L_{\mu\nu}\right\rangle+H_1\left\langle F^R_{\mu\nu}F^{R\mu\nu}F^L_{\alpha\beta}F^{L\alpha\beta}\right\rangle \\
\mathcal{L}^{(2,1,1)} &= \frac{F_0^2B_0}{4}(\delta^{(1)}\overline{\xi}\langle\nabla_\mu U^+\nabla^\mu U(\chi^+U+U^+\chi)\rangle+\delta^{(1)}\overline{\overline{\xi}}\langle\nabla_\mu U^+\nabla^\mu U\rangle\langle\chi^+U+U^+\chi\rangle) \\
\mathcal{L}^{(2,0,2)} &= \frac{F_0^2B_0^2}{4}\delta_d^{(2)}\langle\nabla_\mu U^+\nabla^\mu U\rangle \\
\mathcal{L}^{(2,2,0)} &= \frac{F_0^2}{4}\{A_1\langle\nabla_\mu U^+\nabla^\mu U(\chi^+\chi+U^+\chi\chi^+U)\rangle \\
& \quad +A_2\langle(\nabla_\mu U^+)U\chi^+(\nabla^\mu U)U^+\chi\rangle \\
& \quad +A_3\left\langle\nabla_\mu U^+U(\chi^+\nabla^\mu\chi-\nabla^\mu\chi^+\chi)+\nabla_\mu UU^+(\chi\nabla^\mu\chi^+-\nabla^\mu\chi\chi^+)\right\rangle \\
& \quad +A_4\langle\nabla_\mu U^+\nabla^\mu U\rangle\langle\chi^+\chi\rangle \\
& \quad +B_1\langle\nabla_\mu U^+\nabla^\mu U(\chi^+U\chi^+U+U^+\chi U^+\chi)\rangle \\
& \quad +B_2\langle\nabla_\mu U^+\chi\nabla^\mu U^+\chi+\chi^+\nabla^\mu U\chi^+\nabla_\mu U\rangle \\
& \quad +B_4\langle\nabla_\mu U^+\nabla^\mu U\rangle\langle\chi^+U\chi^+U+U^+\chi U^+\chi\rangle \\
& \quad +C_1^S\langle\nabla_\mu U\chi^++\chi\nabla_\mu U^+\rangle\langle\nabla^\mu U\chi^++\chi\nabla^\mu U^+\rangle \\
& \quad +C_2^S\langle\nabla_\mu\chi^+U+U^+\nabla_\mu\chi\rangle\langle\nabla^\mu\chi^+U+U^+\nabla^\mu\chi\rangle \\
& \quad +C_3^S\langle\nabla_\mu\chi^+U+U^+\nabla_\mu\chi\rangle\langle\nabla^\mu U^+\chi+\chi^+\nabla^\mu U\rangle \\
& \quad +C_1^P\langle\nabla_\mu U\chi^+-\chi\nabla_\mu U^+\rangle\langle\nabla^\mu U\chi^+-\chi\nabla^\mu U^+\rangle \\
& \quad +C_2^P\langle\nabla_\mu\chi^+U-U^+\nabla_\mu\chi\rangle\langle\nabla^\mu\chi^+U-U^+\nabla^\mu\chi\rangle \\
& \quad +C_3^P\langle\nabla_\mu\chi^+U-U^+\nabla_\mu\chi\rangle\langle\nabla^\mu U^+\chi-\chi^+\nabla^\mu U\rangle \\
& \quad +D^S\langle\nabla_\mu U^+\nabla^\mu U(\chi^+U+U^+\chi)\rangle\langle\chi^+U+U^+\chi\rangle \\
& \quad +D^P\langle\nabla_\mu U^+\nabla^\mu U(\chi^+U-U^+\chi)\rangle\langle\chi^+U-U^+\chi\rangle\} \\
& \quad +H_2\left\langle\nabla_\mu\chi\nabla^\mu\chi^+\right\rangle \\
\mathcal{L}^{(0,4,0)} &= \frac{F_0^2}{4}\{E_1\langle(\chi^+U)^4+(U^+\chi)^4\rangle \\
& \quad +E_2\langle\chi^+\chi(\chi^+U\chi^+U+U^+\chi U^+\chi)\rangle \\
& \quad +E_3\langle\chi^+\chi U^+\chi\chi^+U\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+F_1^S\langle\chi^+U\chi^+U+U^+\chi U^+\chi\rangle^2\\
&+F_2^S\langle(\chi^+U)^3+(U^+\chi)^3\rangle\langle\chi^+U+U^+\chi\rangle\\
&+F_3^S\langle\chi^+\chi(\chi^+U+U^+\chi)\rangle\langle\chi^+U+U^+\chi\rangle\\
&+F_4^S\langle(\chi^+U)^2+(U^+\chi)^2\rangle\langle\chi^+\chi\rangle\\
&+F_1^P\langle\chi^+U\chi^+U-U^+\chi U^+\chi\rangle^2\\
&+F_2^P\langle(\chi^+U)^3+(U^+\chi)^3\rangle\langle\chi^+U+U^+\chi\rangle\\
&+F_3^P\langle\chi^+\chi(\chi^+U-U^+\chi)\rangle\langle\chi^+U-U^+\chi\rangle\\
&+F_5^{SS}\langle(\chi^+U)^2+(U^+\chi)^2\rangle\langle\chi^+U+U^+\chi\rangle^2\\
&+F_6^{SS}\langle\chi^+\chi\rangle\langle\chi^+U+U^+\chi\rangle^2\\
&+F_5^{SP}\langle(\chi^+U)^2+(U^+\chi)^2\rangle\langle\chi^+U-U^+\chi\rangle\\
&+F_6^{SP}\langle\chi^+\chi\rangle\langle\chi^+U-U^+\chi\rangle^2\\
&+F_7^{SP}\langle(\chi^+U)^2-(U^+\chi)^2\rangle\\
&\times\langle\chi^+U-U^+\chi\rangle\langle\chi^+U+U^+\chi\rangle\\
&+H_3\left\langle\chi\chi^+\chi\chi^+\right\rangle+H_4\left\langle\chi\chi^+\right\rangle^2
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}^{(0,3,1)} &= \frac{F_0^2B_0}{4}(\delta^{(1)}\overline{\rho}_1\langle(\chi^+U)^3+(U^+\chi)^3\rangle+\delta^{(1)}\overline{\rho}_2\langle(\chi^+U+U^+\chi)\chi^+\chi\rangle\\
&+\delta^{(1)}\overline{\rho}_3\langle(\chi^+U)^2-(U^+\chi)^2\rangle\langle\chi^+U-U^+\chi\rangle\\
&+\delta^{(1)}\overline{\rho}_4\langle(\chi^+U)^2+(U^+\chi)^2\rangle\langle\chi^+U+U^+\chi\\
&+\delta^{(1)}\overline{\rho}_5\langle\chi^+U+U^+\chi\rangle\langle\chi^+\chi\rangle+\delta^{(1)}\overline{\rho}_6\langle\chi^+U-U^+\chi\rangle^2\langle\chi^+U+U^+\chi\rangle\\
&+\delta^{(1)}\overline{\rho}_7\langle\chi^+U+U^+\chi\rangle^3)\\
\mathcal{L}^{(0,2,2)} &= \frac{F_0^2B_0^2}{4}\Big(\delta^{(2)}\overline{A}_0\langle(U^+\chi)^2+(\chi^+U)^2\rangle\\
&+\delta^{(2)}\overline{Z}_0^S\langle U^+\chi+\chi^+U\rangle^2+\delta^{(2)}\overline{Z}_0^P\langle U^+\chi-\chi^+U\rangle^2\Big)\\
\mathcal{L}^{(0,1,3)} &= \frac{F_0^2}{2}B_0^3\delta_\chi^{(2)}\langle U^+\chi+\chi^+U\rangle.
\end{aligned}
\tag{8.200}
$$

In fact, identifying F_0 with the Goldstone boson decay constant and $B_0=\Sigma/F_0^2$ where $\Sigma=-\langle\overline{u}u\rangle_0$ (in the chiral limit), we have

$$\delta_d^{(i)}=\delta_\chi^{(i)}=0. \tag{8.201}$$

As usual, we can also resume the powers of B_0 already at the Lagrangian level and write

$$\mathcal{L}_n=\sum_{i+j=n}\mathcal{L}^{(i,j)} \tag{8.202}$$

with

$$\mathcal{L}^{(i,j)}=\sum_k\mathcal{L}^{(i,j,k)} \tag{8.203}$$

and denote

$$\begin{aligned}
A_0 &= \overline{A}_0 + B_0\delta^{(1)}\overline{A}_0 + B_0^2\delta^{(2)}\overline{A}_0 + \ldots \\
Z_0^{S,P} &= \overline{Z}_0^{S,P} + B_0\delta^{(1)}\overline{Z}_0^{S,P} + B_0^2\delta^{(2)}\overline{Z}_0^{S,P} + \ldots \\
\xi &= \overline{\xi} + B_0\delta^{(1)}\overline{\xi} + \ldots \\
\widetilde{\xi} &= \overline{\widetilde{\xi}} + B_0\delta^{(1)}\overline{\widetilde{\xi}} + \ldots \\
\rho_i &= \overline{\rho}_i + B_0\delta^{(1)}\overline{\rho}_i + \ldots,
\end{aligned} \tag{8.204}$$

etc.. These LEC's without the bars are then used in the main text. Note that while the $O(p^2)$ parameters $\overline{A}_0$, $\overline{Z}_0^{S,P}$ and the $O(p^3)$ LEC's $\overline{\widetilde{\xi}}$,$\overline{\xi}$ are renormalization scale independent, the renormalized resumed parameters $Z_0^{S,P,r}$, A_0^r and $\widetilde{\xi}^r$, ξ^r run with μ in the same way as $16(B_0/F_0)^2L_{6-8}^r$ and $8B_0/F_0^2L_{4,5}$ within the standard χPT.

8.14 Coefficients of the dispersive part of $G\chi PT$ amplitude

In these formulae as well as in the following two appendices, the masses $\widetilde{M_P^2}$ are the generalized $O(p^2)$ masses given by (8.112).

$$\begin{aligned}
\alpha_{\pi\eta}\widetilde{M_\pi^2} &= 2[\widehat{m}B_0 + 8\widehat{m}^2A_0 + 2\widehat{m}^2Z_0^S(5r+4) - 8\widehat{m}^2Z_0^P(r-1)] \\
\alpha_{\pi\eta K}\widetilde{M_\pi^2} &= 4\widehat{m}^2(r^2-1)(A_0+2Z_0^P) \\
\alpha_{\pi\pi}\widetilde{M_\pi^2} &= 2\widehat{m}B_0 + 16\widehat{m}^2A_0 + 4\widehat{m}^2Z_0^S(r+8) \\
\alpha_{\eta\eta}(4\widetilde{M_\eta^2} - \widetilde{M_\pi^2}) &= \frac{2}{3}\widehat{m}B_0(1+8r) + \frac{16}{3}\widehat{m}^2A_0(1+8r^2) \\
&\quad + \frac{4}{3}\widehat{m}^2Z_0^S(8+41r+32r^2) + \frac{32}{3}\widehat{m}^2Z_0^P(r-1)(4r-1) \\
(\alpha_{\pi K}-1)\widetilde{M}_K\widetilde{M}_\pi &= 6(A_0+2Z_0^S)\widehat{m}^2(r+1) \\
\alpha_{\eta K}(2\widetilde{M_\eta^2} - \frac{2}{3}\widetilde{M_K^2}) &= \frac{2}{3}[\widehat{m}B_0(1+3r) + \widehat{m}^2A_0(3+10r+19r^2) \\
&\quad + 2\widehat{m}^2Z_0^S(6+19r+11r^2) + 16\widehat{m}^2Z_0^P r(r-1)]
\end{aligned} \tag{8.205}$$

8.15 Parameters $\alpha - \omega$ within the generalized χPT

Here we summarize the formulae in terms of the decomposition of the remainders. For the parameter α we write

$$\delta_\alpha = \delta_\alpha^{loop} + 3\frac{\widehat{m}^2F_0^2}{F_\pi^2M_\pi^2}\delta_\alpha^{CT}(\mu) + \delta_\alpha^{G\chi PT}. \tag{8.206}$$

For the counterterm contribution we get

$$\begin{aligned}
\delta_\alpha^{CT}(\mu) &= \frac{1}{3}\widehat{m}\left[81\ \rho_1 + \rho_2 + \left(80 - 64\ r - 16\ r^2\right)\ \rho_3\right. \\
&\quad + \left(100 + 64\ r + 34\ r^2\right)\ \rho_4 + \left(2\ +\ r^2\right)\ \rho_5 \\
&\quad \left. + \left(96 - 96\,r\right)\ \rho_6 + \left(144\ + 288\ r + 108\ r^2\right)\ \rho_7\right] \\
&\quad + \frac{8}{3}\left[-(B_1-B_2)\Sigma_{\pi\eta} + 2D^PM_\pi^2(r-1) - 2C_1^PM_\eta^2(r-1) - \frac{1}{2}D^S[\Sigma_{\pi\eta}(5r+4)]\right.
\end{aligned}$$

$$
\begin{aligned}
&- 2B_4[3M_\eta^2 + M_\pi^2(2r^2+1)]] \\
&+\frac{1}{3}\widehat{m}^2\left[256E_1 + 16E_2 + F_1^P(256-256r^2) + F_4^S(32+16r^2)\right. \\
&+ F_1^S(256+320r^2) + F_5^{SP}(192-320r+160r^2-32r^3) \\
&+ F_2^P(240-216r-24r^3) + F_6^{SP}(32-32r+16r^2-16r^3) \\
&+ F_3^P(16-8r-8r^3) + F_3^S(16+10r+10r^3) \\
&+ F_6^{SS}(32+40r+16r^2+20r^3) + F_7^{SP}(384-160r-256r^2+32r^3) \\
&\left.+ F_2^S(400+234r+74r^3) + F_5^{SS}(576+720r+480r^2+168r^3)\right] \qquad (8.207)
\end{aligned}
$$

and the loops contribute as

$$
\begin{aligned}
\frac{1}{3}F_\pi^2 M_\pi^2 \delta_\alpha^{loop} \;=\;& \frac{1}{3}\left\{[\widetilde{M}_\pi^2\left(3B_0\widehat{m} + 64A_0\widehat{m}^2 + 2Z_0^S\widehat{m}^2(15r+32) - 8Z_0^P\widehat{m}^2(3r-8)\right)]\right. \\
&-6B_0^2\widehat{m}^2\} \left(J^r_{\pi\pi}(0) + \frac{1}{16\pi^2}\right) \\
&+\frac{2}{3}\left\{[\widetilde{M}_K^2\left(B_0\widehat{m} + 2A_0\widehat{m}^2(r+8) + 2Z_0^S\widehat{m}^2(15r+8) - 8Z_0^P\widehat{m}^2(3r-2)\right)]\right. \\
&-B_0^2\widehat{m}^2(r+1)\} \left(J^r_{KK}(0) + \frac{1}{16\pi^2}\right) \\
&\frac{1}{9}\left\{[\widetilde{M}_\eta^2(B_0\widehat{m} + 32A_0\widehat{m}^2 - 16Z_0^P\widehat{m}^2(5r-2) + 2Z_0^S\widehat{m}^2(41r+16)]\right. \\
&\left.-\frac{2}{3}B_0^2\widehat{m}^2(2r+1)\right\} \left(J^r_{\eta\eta}(0) + \frac{1}{16\pi^2}\right) \\
&+\frac{2}{9}\left\{[\widetilde{M}_\pi^2 + 4\widehat{m}^2(3A_0 - 4(r-1)Z_0^P + 2(2r+1)Z_0^S)]^2 - 4B_0^2\widehat{m}^2\right\} J^r_{\pi\eta}(0) \\
&+\frac{3}{4}\left\{[\frac{2}{3}\widetilde{M}_\pi^2 - \frac{8}{3}(r-1)\widehat{m}^2(A_0+2Z_0^P)]^2 - \frac{16}{9}B_0^2\widehat{m}^2\right\} J^r_{KK}(0) \\
&+\frac{1}{3}\left\{[\widetilde{M}_\pi^2 + 4\widehat{m}^2(3A_0 - 4(r-1)Z_0^P + 2(2r+1)Z_0^S)]\right. \\
&\left.\times[-2M_\pi^2 + \frac{3}{2}\widetilde{M}_\pi^2 + 10\widehat{m}^2(A_0+2Z_0^S)] - 2B_0\widehat{m}(3B_0\widehat{m} - 2M_\pi^2)\right\} J^r_{\pi\pi}(0) \\
&+\frac{2}{9}\left\{[\widetilde{M}_\pi^2 + 4\widehat{m}^2(3A_0 - 4(r-1)Z_0^P + 2(2r+1)Z_0^S)]\right. \\
&\times[\widetilde{M}_\eta^2 - \frac{1}{4}\widetilde{M}_\pi^2 + \widehat{m}^2((8r^2+1)A_0 + 8r(r-1)Z_0^P + 2(2r+1)^2Z_0^S)] \\
&\left.-\frac{1}{3}B_0^2\widehat{m}^2(8r+1)\right\} J^r_{\eta\eta}(0) \\
&+\frac{1}{8}\left\{[-2M_\pi^2 + 2\widetilde{M}_\pi^2 + 8(r+1)\widehat{m}^2(A_0+2Z_0^S))]\right. \\
&\times[-6M_\eta^2 + 6\widetilde{M}_\eta^2 - \frac{8}{3}\widetilde{M}_K^2 + \frac{8}{3}(r+1)\widehat{m}^2(3rA_0 + 2(r-1)Z_0^P + 2(2r+1)Z_0^S] \\
&\left.-2(2B_0\widehat{m} - M_\pi^2)(\frac{4}{3}B_0\widehat{m}(4r+1) - 6M_\eta^2)\right\} J^r_{KK}(0). \qquad (8.208)
\end{aligned}
$$

In the same way we have for β

$$\beta\delta_\beta = \beta\delta_\beta^{loop} + \widehat{m}^2 F_0^2 \delta_\beta^{CT}(\mu) + \beta\delta_\beta^{G\chi PT}, \tag{8.209}$$

where

$$\begin{aligned}
\delta_\beta^{CT}(\mu) &= \frac{8}{3}[(C_1^S + D^S)(2r+1) + 2B_4(r^2+2)] && (8.210)\\
\beta\delta_\beta^{loop} &= -\frac{3}{4}\left\{[\frac{2}{3}\widetilde{M}_\pi^2 - \frac{8}{3}(r-1)\widehat{m}^2(A_0 + 2Z_0^P)] - \frac{4}{3}B_0\widehat{m}\right\} J_{KK}^r(0)\\
&+\frac{1}{3}\left\{[\widetilde{M}_\pi^2 + 4\widehat{m}^2(3A_0 - 4(r-1)Z_0^P + 2(2r+1)Z_0^S)] - 2B_0\widehat{m}\right\} J_{\pi\pi}^r(0)\\
&+\frac{1}{8}\left\{[6(\widetilde{M}_\eta^2 - M_\eta^2 + \widetilde{M}_\pi^2 - M_\pi^2) - \frac{8}{3}\widetilde{M}_K^2\right.\\
&+\frac{8}{3}(r+1)\widehat{m}^2(3A_0(r+3) + 4Z_0^S(r+5) + 2(r-1)Z_0^P)]\\
&\left.-[\frac{8}{3}B_0\widehat{m}(2r+5) - 6M_\eta^2 - 6M_\pi^2]\right\} J_{KK}^r(0). && (8.211)
\end{aligned}$$

For the remaining two parameters the corresponding decomposition of the remainders

$$\begin{aligned}
\gamma\delta_\beta(\mu) &= \gamma\delta_\gamma{}^{loops}(\mu) + \widehat{m}^2 F_0^2 \delta_\gamma^{CT}(\mu) + \gamma\delta_\gamma^{G\chi PT} && (8.212)\\
\omega\delta_\omega(\mu) &= \omega\delta_\omega{}^{loops}(\mu) + \widehat{m}^2 F_0^2 \delta_\omega^{CT}(\mu) + \omega\delta_\omega^{G\chi PT} && (8.213)
\end{aligned}$$

is trivial, *i.e.*

$$\delta_\gamma{}^{loops}(\mu) = \delta_\gamma^{CT}(\mu) = \delta_\omega{}^{loops}(\mu) = \delta_\omega^{CT}(\mu) = 0. \tag{8.214}$$

8.16 GχPT bare expansion remainders for the masses

The expressions for ξ, $\widetilde{\xi}$ can be obtained from the exact algebraic identities (8.191) after identification (8.117) and using the representation (8.172) and (8.173) for the remainder of F_π^2 and (8.174) and (8.175) for the remainder of F_K^2. In the same spirit, A_0, Z_0^S and Z_0^P can be expressed using identities (8.194) and the following remainders

$$\begin{aligned}
F_\pi^2 M_\pi^2 \delta_{F_\pi M_\pi} &= F_\pi^2 M_\pi^2 \delta_{F_\pi M_\pi}^{loop}(\mu) + F_0^2\widehat{m}^2\delta_{F_\pi M_\pi}^{CT}(\mu) + F_\pi^2 M_\pi^2 \delta_{F_\pi M_\pi}^{G\chi PT}\\
F_K^2 M_K^2 \delta_{F_K M_K} &= F_K^2 M_K^2 \delta_{F_K M_K}^{loop}(\mu) + F_0^2\widehat{m}^2\delta_{F_K M_K}^{CT}(\mu) + F_K^2 M_K^2 \delta_{F_K M_K}^{G\chi PT}\\
F_\eta^2 M_\eta^2 \delta_{F_\eta M_\eta} &= F_\eta^2 M_\eta^2 \delta_{F_\eta M_\eta}^{loop}(\mu) + F_0^2\widehat{m}^2\delta_{F_\eta M_\eta}^{CT}(\mu) + F_\eta^2 M_\eta^2 \delta_{F_\eta M_\eta}^{G\chi PT}, && (8.215)
\end{aligned}$$

where

$$
\begin{aligned}
F_\pi^2 M_\pi^2 \delta^{loop}_{F_\pi M_\pi}(\mu) &= \left[\widetilde{M}_\pi^2\left(3B_0\widehat{m}+16A_0\widehat{m}^2+2Z_0^S\widehat{m}^2(3r+16)\right)-6B_0^2\widehat{m}^2\right] \\
&\times\left(J^r_{\pi\pi}(0)+\frac{1}{16\pi^2}\right) \\
&+2\left[\widetilde{M}_K^2\left(B_0\widehat{m}+2A_0\widehat{m}^2(r+2)+2Z_0^S\widehat{m}^2(3r+4)\right)-B_0^2\widehat{m}^2(r+1)\right] \\
&\times\left(J^r_{KK}(0)+\frac{1}{16\pi^2}\right) \\
&+\frac{1}{3}\left[\widetilde{M}_\eta^2(B_0\widehat{m}+8A_0\widehat{m}^2-8Z_0^P\widehat{m}^2(r-1)+2Z_0^S\widehat{m}^2(5r+4)\right. \\
&\left.-\frac{2}{3}B_0^2\widehat{m}^2(2r+1)\right]\left(J^r_{\eta\eta}(0)+\frac{1}{16\pi^2}\right) \qquad (8.216) \\
\delta^{CT}_{F_\pi M_\pi}(\mu) &= \widehat{m}[9\rho_1+\rho_2+2\rho_4(10+4r+r^2)+\rho_5(2+r^2)+12\rho_7(4+4r+r^2)] \\
&+2\widehat{m}^2[8E_1+2E_2+8F_1^S\left(2+r^2\right)+F_2^S\left(9r+r^3+20\right) \\
&+F_3^S\left(4+r+r^3\right)+2F_4^S\left(2+r^2\right)+4F_5^{SS}\left(r+2\right)\left(r^2+2r+6\right) \\
&+2F_6^{SS}\left(r+2\right)\left(2+r^2\right)], \qquad (8.217)
\end{aligned}
$$

then

$$
\begin{aligned}
F_K^2 M_K^2 \delta^{loop}_{F_K M_K}(\mu) &= \left\{\frac{3}{4}\left[\widetilde{M}_\pi^2\left(B_0\widehat{m}+A_0\widehat{m}^2(r+5)+2Z_0^S\widehat{m}^2(r+6)\right)-2B_0^2\widehat{m}^2\right]\right. \\
&\times\left(J^r_{\pi\pi}(0)+\frac{1}{16\pi^2}\right) \\
&+\frac{3}{2}\left[\widetilde{M}_K^2\left(B_0\widehat{m}+3A_0\widehat{m}^2(r+1)+2Z_0^S\widehat{m}^2(3r+4)\right)-B_0^2\widehat{m}^2(r+1)\right] \\
&\times\left(J^r_{KK}(0)+\frac{1}{16\pi^2}\right) \\
&+\frac{1}{12}\left[\widetilde{M}_\eta^2\left(5B_0\widehat{m}+A_0\widehat{m}^2(17r+5)+8Z_0^P\widehat{m}^2(r-1)+2Z_0^S\widehat{m}^2(13r+14)\right)\right. \\
&\left.\left.-\frac{10}{3}B_0^2\widehat{m}^2\left(2r+1\right)\right]\left(J^r_{\eta\eta}(0)+\frac{1}{16\pi^2}\right)\right\}(r+1) \qquad (8.218) \\
\delta^{CT}_{F_K M_K}(\mu) &= \frac{1}{2}\widehat{m}[3\rho_1(1+r)(1+r+r^2)+\rho_2(1+r^3)+6\rho_4\left(r+1\right)\left(2+2r+r^2\right) \\
&+\rho_5\left(r+1\right)\left(2+r^2\right)+12\rho_7\left(r+1\right)\left(r+2\right)^2] \\
&+\widehat{m}^2[2E_1\left(1+r\right)^2\left(1+r^2\right)+E_2\left(1+r\right)^2\left(1-r+r^2\right) \\
&+\frac{1}{2}E_3\left(r^2-1\right)^2 \\
&+4F_1^S\left(1+r\right)^2\left(2+r^2\right)+F_2^S\left(1+r\right)\left(8+9r+9r^2+4r^3\right) \\
&-F_3^S\left(1+r\right)\left(4-r+r^2+2r^3\right)+F_4^S\left(1+r\right)^2\left(2+r^2\right) \\
&+4F_5^{SS}\left(1+r\right)\left(2+r\right)\left(4+3r+2r^2\right) \\
&+2F_6^{SS}\left(1+r\right)\left(2+r\right)\left(2+r^2\right)] \qquad (8.219)
\end{aligned}
$$

and

$$\begin{aligned}
F_\eta^2 M_\eta^2 \delta^{loop}_{F_\eta M_\eta}(\mu) &= \Big[\widetilde{M}_\pi^2 \left(B_0\widehat{m} + 8\,A_0\,\widehat{m}^2 - 8\,\widehat{m}^2\,Z_0^P\,(r-1) + 2\,\widehat{m}^2\,Z_0^S\,(4+5\,r)\right) - 2B_0^2\widehat{m}^2\Big] \\
&\times \left(J^r_{\pi\pi}(0) + \frac{1}{16\pi^2}\right) \\
&+\frac{2}{3}\,\Big[\widetilde{M}_K^2\,\left(B_0\widehat{m}\,(1+4\,r) + 2\,A_0\,\widehat{m}^2\,\left(2+r+8\,r^2\right) + 8\,\widehat{m}^2\,Z_0^P\,(r-1)\,(2\,r-1)\right. \\
&\left.+2\,\widehat{m}^2\,Z_0^S\,\left(4+15\,r+8\,r^2\right)\right) - B_0^2\widehat{m}^2\,(4\,r+1)\,(r+1)\Big] \\
&\times \left(J^r_{KK}(0) + \frac{1}{16\pi^2}\right) \\
&+\frac{1}{9}\,\Big[\widetilde{M}_\eta^2\left(B_0\widehat{m}\,(1+8\,r) + 8A_0\,\widehat{m}^2\,\left(1+8\,r^2\right) + 16\,\widehat{m}^2\,Z_0^P\,(r-1)\,(4\,r-1)\right. \\
&\left.+2\,\widehat{m}^2\,Z_0^S\,\left(8+41\,r+32\,r^2\right)\right) - \frac{2}{3}B_0^2\widehat{m}^2\,(8\,r+1)\,(2r+1)\Big] \\
&\times \left(J^r_{\eta\eta}(0) + \frac{1}{16\pi^2}\right)
\end{aligned} \tag{8.220}$$

$$\begin{aligned}
\delta^{CT}_{F_\eta M_\eta}(\mu) &= \frac{1}{3}\widehat{m}[9\rho_1\left(1+2r^3\right) + \rho_2\left(1+2r^3\right) + 16\rho_3\,(r-1)^2\,(1+r) \\
&+2\rho_4\left(10+8r+17r^2+10r^3\right) \\
&+\rho_5\,(1+2r)\left(2+r^2\right) + 16\rho_6\left(2-3r+r^3\right) + 12\rho_7\,(2+r)^2\,(1+2r)] \\
&+\frac{2}{3}\widehat{m}^2[8E_1\left(1+2r^4\right) + 2E_2\left(1+2r^4\right) \\
&+8F_1^S\left(r^2+2\right)\left(2r^2+1\right) + 16F_1^P\left(r^2-1\right)^2 \\
&+F_2^S\left(20+13r+37r^3+20r^4\right) + 12F_2^P\left(r^2+r+1\right)(r-1)^2 \\
&+F_3^S\left(4+5r+5r^3+4r^4\right) + 4F_3^P\left(r^2+r+1\right)(r-1)^2 \\
&+2F_4^S\left(r^2+2\right)\left(2r^2+1\right) \\
&+12F_5^{SS}\,(r+2)\left(2r^3+3r^2+2r+2\right) + 8F_5^{SP}\left(r^2+2\right)(r-1)^2 \\
&+2F_6^{SS}\,(2r+1)\,(r+2)\left(r^2+2\right) + 4F_6^{SP}\left(r^2+2\right)(r-1)^2 \\
&+16F_7^{SP}\,(r+2)\,(r+1)\,(r-1)^2].
\end{aligned} \tag{8.221}$$

8.17 Resonance amplitude and remainders estimates

Here we give the contribution to the amplitude [38] related to the resonance exchange, derived from the leading order Lagrangian of $R\chi T$ (we have confirmed this expression by independent calculation)

$$\begin{aligned}
G_R(s,t;u) &= 4\frac{1}{M_{S_1}^2 - t}\left(\widetilde{c}_d(t-2M_\pi^2) + 2\widetilde{c}_m\,\overset{o}{M}{}_\pi^{\,2}\right)\left(\widetilde{c}_d(t-2M_\eta^2) + 2\widetilde{c}_m\,\overset{o}{M}{}_\eta^{\,2}\right) \\
&+4\frac{\widetilde{c}_m^2}{M_{S_1}^2}\,\overset{o}{M}{}_\pi^{\,2}\left(\overset{o}{M}{}_\pi^{\,2} + \overset{o}{M}{}_\eta^{\,2}\right) + 4\frac{c_m^2}{3M_S^2}\,\overset{o}{M}{}_\pi^{\,2}\left(\overset{o}{M}{}_\pi^{\,2} - \overset{o}{M}{}_\eta^{\,2}\right) \\
&+\frac{2}{3}\frac{1}{M_S^2 - s}\left(c_d(s-M_\pi^2-M_\eta^2) + 2c_m\,\overset{o}{M}{}_\pi^{\,2}\right)^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{3}\frac{1}{M_S^2-u}\left(c_d(u-M_\pi^2-M_\eta^2)+2c_m\overset{o}{M}{}_\pi^2\right)^2\\
&-\frac{2}{3}\frac{1}{M_S^2-t}\left(c_d(t-2M_\pi^2)+2c_m\overset{o}{M}{}_\pi^2\right)\left(c_d(t-2M_\eta^2)+2c_m(2\overset{o}{M}{}_\eta^2-\overset{o}{M}{}_\pi^2)\right)\\
&-16\frac{\widetilde{d}_m^2}{M_{\eta_1}^2}\overset{o}{M}{}_\pi^2\left(\overset{o}{M}{}_\pi^2-\overset{o}{M}{}_\eta^2\right).
\end{aligned}
\tag{8.222}
$$

The resonance estimate of the remainders δ_γ^R and δ_ω^R are

$$
\begin{aligned}
\gamma\delta_\gamma^R \;=\; & -\frac{8}{3M_S^6}\left(c_dM_\pi^2-c_m\overset{o}{M}{}_\pi^2\right)\left(c_dM_\eta^2-c_m(2\overset{o}{M}{}_\eta^2-\overset{o}{M}{}_\pi^2)\right)\\
&+\frac{4}{3}\frac{c_d}{M_S^4}\left(c_dM_\eta^2-c_m(2\overset{o}{M}{}_\eta^2-\overset{o}{M}{}_\pi^2)\right)+\frac{1}{3}\frac{c_d^2\Sigma_{\pi\eta}}{M_S^2(M_S^2-\Sigma_{\pi\eta})}\\
&+\frac{4}{3}\frac{c_dc_m}{(M_S^2-\Sigma_{\pi\eta})^2}\overset{o}{M}{}_\pi^2+\frac{4}{3}\frac{c_m^2}{(M_S^2-\Sigma_{\pi\eta})^3}\overset{o}{M}{}_\pi^4\\
&+\frac{16}{M_{S_1}^6}\left(\widetilde{c}_dM_\pi^2-\widetilde{c}_m\overset{o}{M}{}_\pi^2\right)\left(\widetilde{c}_dM_\eta^2-\widetilde{c}_m\overset{o}{M}{}_\eta^2\right)\\
&-\frac{8\widetilde{c}_d}{M_{S_1}^4}\left(\widetilde{c}_d\Sigma_{\pi\eta}-\widetilde{c}_m(\overset{o}{M}{}_\pi^2+\overset{o}{M}{}_\eta^2)\right)
\end{aligned}
\tag{8.223}
$$

$$
\omega\delta_\omega^R \;=\; -\frac{1}{3}\frac{c_d^2\Sigma_{\pi\eta}}{M_S^2(M_S^2-\Sigma_{\pi\eta})}+\frac{4}{3}\frac{c_m^2\overset{o}{M}{}_\pi^4}{(M_S^2-\Sigma_{\pi\eta})^3}+\frac{4}{3}\frac{c_dc_m}{(M_S^2-\Sigma_{\pi\eta})^2}\overset{o}{M}{}_\pi^2\,. \tag{8.224}
$$

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Chapter 9

Summary and conclusions

This work was devoted to the study of Chiral perturbation theory in its several versions according to the assumptions concerning the character of spontaneous breaking of chiral symmetry. Three interrelated topics connected to eta meson interactions were investigated from the theoretical point of view.

Chapter 6 summed up our calculations of the $\eta \to \pi^0\pi^0\gamma\gamma$ decay in Generalized χPT. We found a part of the kinematic region to be very sensitive to the violation of the Standard χPT assumptions, particularly at the tree level. One loop corrections brought the large number of GχPT LEC's, which could not be neglected based on the scale dependence analysis. The uncertainty in their value turned out to be too big to allow an unambiguous result for a more modest deviation. We concluded that while the Generalized counting might improve the convergence properties of the chiral expansion, its practical usefulness is limited already at the next-to-leading order due to the unknown values of several tens of unknown LEC's.

Future goals include applying the alternative Resummed χPT to this case. This approach might not suffer from such large uncertainties and allows several ways of estimating them. One can only hope future experiments improve the resolution to allow the practical study of this rare decay as was originally anticipated.

As a second topic, we have applied the Resummed χPT to the case of eta decay constant in chapter 7, a simpler illustrative case with quite some practical interest. We were able to derive a clear prediction, a value significantly outside $F_\eta = (1.3 \pm 0.1)F_\pi$ would be in contradiction with a satisfying convergence of the chiral series. This includes very low values of F_η obtained by older single mixing angle scheme fits, newer two angle phenomenological results generally fall in or close to this band.

Motivated by the latest fit based on a variety of experimental data (F_η=1.38F_π), we investigated the consequences of confirmation of such a higher value of the eta decay constant. In comparison with various L_5^r estimates outside NLO χPT and assuming reasonably small higher order remainders, we found several possibly interesting conclusions. Very small values of Y, for which no lower bound exists up to date, were generally disfavored. Very low values of $L_5^r(M_\rho)$<1.10^{-3}, obtained by some NNLO Standard χPT fits, imply Y>1.2, which seems to be in contradiction with the result Y<1.1 obtained from $\pi\pi$ and πK scattering. Large higher order remainders are hard to avoid in this case too, which does not play well with the Standard treatment of the chiral expansion.

We also used the Generalized χPT Lagrangian in a untraditional way to check the influence of higher order remainders, it was found reasonably small. Future work could continue in the direction of an in-depth statistical analysis in order to try to obtain more definite constraints on the parameters controlling the character of spontaneous breaking of chiral symmetry.

In chapter 8, the last subject, we have used a set of observables connected to $\eta\pi$ scattering, the source of sensitivity to the deviation from the Standard assumptions in the $\eta \to \pi^0\pi^0\gamma\gamma$ decay calculations, to make a detailed comparison of Standard and Resummed χPT. We found that the Standard case suffers from large hidden uncertainties in the form of a strong sensitivity to the value of $O(p^4)$ LEC's and the definition of the expansion. Next-to-leading order correction to the examined observables were found very large in the majority of cases.

In the Resummed framework, we introduced the dispersive representation as a systematical method to deal with the unphysical analytical structure in the strict chiral expansion of the four point Green function related to the amplitude. We numerically compared two possible constructions of the representation and a simple analyticity fix by hand. The differences were not large in absolute terms, but compared to the leading order terms they could be substantial in some part of the parameter space. We also checked the assumption that one can plant physical masses inside the chiral logarithms by hand, without large numerical significance of such a redefinition of the bare expansion. Our result is that it is not true in our case. We have also shown that a part of the logarithms have to be treated otherwise an unphysical divergence for $Y \to 0$ occurs.

We have checked whether the Resummed results reproduce the Standard ones when the parameters X, Z, r are sent to their Standard values. This is fairly true for the "good" observables, as one might expect, but for the "bad" ones there are indeed significant differences. Away from the Standard reference point we have found a strong sensitivity on the parameters and, as expected from our earlier results, the Resummed values deviate from the Standard one in many cases quickly.

The sensitivity on the unknown higher order remainders were found quite big. We have tried to estimate them using information outside the Standard $O(p^4)$ theory - resonance saturation and the Generalized χPT Lagrangian. We tried to get a feel about the magnitude of higher order corrections in the derivative expansion using Resonance chiral theory in an untraditional way. Instead of the usual LEC saturation we sewed the two theories together to all orders, thus getting an estimate of the remainders. In this way we also avoided the need to fix a saturation scale and the resonance poles appeared explicitly in the result. The higher order terms in the expansion in quark masses were estimated by including additional contributions from the Generalized Lagrangian. Altogether, the joint estimate of higher order remainders were found to be outside of the usually presumed 10% band in some parts of the parameter space.

Further theoretical improvements should include the reparametrization treatment of $L_{1,2,3}$ in a similar way to the rest of NLO LEC's. Resonances could be described by a more sophisticated up-to-date Lagrangian involving higher orders. The estimate of the remainders might also contain two loop χPT contributions or resonances to one loop. More practically, the off-shell $2\eta 2\pi$ vertex could be implemented into $\eta \to 3\pi^0$ and $\eta \to \pi^0\pi^0\gamma\gamma$ calculations, which is necessary in order to reach closer to experimental impact.

Chapter 10

List of original work

[I] M. Kolesar, J. Novotny, *The $\eta \to \pi^0\pi^0\gamma\gamma$ decay in GχPT*, based on talk presented at Int.Conf. "Advanced Studies Institute - Symmetries and Spin" (SPIN-Praha-2003), July 12 - 19, 2003, Prague, Czech Republic, Czech.J.Phys. 54 (2004) B63-B66, [arXiv:hep-ph/0402239]

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