Charles University, Prague,
Faculty of Mathematics and Physics

Doctoral Thesis

Mathematical Analysis of Fluids in Large Domains

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Study branch: M-3 Mathematical analysis

2008

Univerzita Karlova v Praze,
Matematicko–fyzikální fakulta

Dizertační práce

Matematická analýza tekutin na neomezených oblastech

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Studijní obor: M-3 Matematická analýza

2008

Acknowledgements

The author wishes to express his deep thanks to Eduard Feireisl for his kind support, fruitful ideas, and happy suggestions while progressing with the research related to the thesis.

Moreover, the results published in this thesis were supported by

- the research grant 6/2005/R of the Grant Agency of the Charles University, Prague;
- the Jindřich Nečas Center for Mathematical Modeling – a project number LC06052 financed by the Czech Ministry for Education, Youth and Sports;
- the research grant no. 201/05/0164 of the Grant Agency of the Czech Republic; and
- the grant number IAA100190606 and Premium Academiae of the Grant Agency of the Academy of Sciences of the Czech Republic in the framework of the general research program of the Academy of Sciences of the Czech Republic, Institutional Research Plan AV0Z10190503.

All these supports are gratefully acknowledged.

Statement of Honesty

I hereby claim that this thesis is solely my independent work if not explicitly stated otherwise.

Prague, June 22, 2008, RNDr. Lukáš Poul

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Abstrakt: Práce obsahuje soubor článků vztahujících se k proudění viskózní, stlačitelné a tepelně vodivé tekutiny na neomezených oblastech. První část práce se věnuje otázce existence slabých řešení na neomezených oblastech. Výsledky jsou uváděny tak, jak byly v průběhu času získány a začínají od prvotního rozšíření teorie na omezené oblasti s Lipschitzovsky spojitou hranicí a sahají až k obecnému modelu proudění tekutiny v obecné otevřené množině. Tyto výsledky existenčního charakteru jsou následně doplněny studiem existence slabých řešení v modelu s předepsanou nehomogenní okrajovou podmínkou pro teplotu a hustotu v nekonečnu. Na závěr je v práci věnována pozornost limitnímu přechodu stlačitelné tekutiny pro malá Machova čísla.
Klíčová slova: Navierův–Stokesův–Fourierův systém, stlačitelné tekutiny, neomezené oblasti, váhy, nehomogenní okrajové podmínky, low Mach number limit.

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Abstract: This thesis contains a set of articles concerned with flow of a viscous, compressible and heat conducting fluid in large domains. In the first part of the thesis, the existence of the weak solutions in unbounded domains is studied. The results follow each other in the way they were obtained through the time, and range from a simple extension to bounded domains with Lipschitz boundary up to the most general existence theorem for fluid flow in general open sets. The existence results are supplemented with the study of existence of weak solutions in the unbounded domain case with prescribed nonvanishing boundary conditions for density and temperature at infinity. The last contribution then concerns with the low Mach number limit in the compressible fluid flow.
Keywords: Navier–Stokes–Fourier system, compressible fluid flow, unbounded domains, weights, nonhomogeneous boundary conditions, low Mach number limit.

Chapter 1

Introduction

One of the most challenging problems of mathematics in recent decades concerns with the equations describing motion of a viscous, compressible and heat conducting fluid:

$$\left.\begin{array}{c} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \\ \partial_t(\rho \mathbf{u}_i) + \operatorname{div}(\rho \mathbf{u} \mathbf{u}_i) + \partial_{x_i} p = \sum_{j=1}^3 \partial_{x_j} \mathbb{S}_{ij} + \rho f_i, \quad i = 1,2,3 \\ \partial_t(\rho e) + \operatorname{div}(\rho e \mathbf{u}) + \operatorname{div} \mathbf{q} = \mathbb{S} : \nabla \mathbf{u} - p \operatorname{div} \mathbf{u} \end{array}\right\} \tag{1.0.1}$$

in $(0,T) \times \Omega$. The set of equations, even though looking a bit awkward, has a clear structure that reflects a set of physical conservation laws:

- The first equation expresses the conservation of mass, i.e. the fluid nowhere vanishes and nowhere emerges, the total amount of mass is a constant of motion.

- The second equation is in fact a triple of equations each expressing the Newton second law in each of the space coordinates: the linear momentum of each of the elements of the fluid can be changed only through an action of an external force (here 'external' means having its origin outside of the element).

- The third equation balances out the internal energy of the fluid and can be viewed as a mathematical formulation of the *first law of thermodynamics.*

Note that taking the scalar product of the linear momentum equation with $\mathbf{u}$, summing up with the balance of internal energy and integrating over the whole spatial domain Ω, one recovers the *total energy equality*

$$\int_\Omega (\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e)(t,x)\,\mathrm{d}\mathbf{x} = \int_\Omega (\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e)(0,x)\,\mathrm{d}\mathbf{x} + \int_0^t \int_\Omega \rho\mathbf{f}\mathbf{u}(t,x)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \quad (1.0.2)$$

provided

$$\mathbf{u}\cdot\mathbf{n} = 0, (\mathbb{S}\mathbf{n})\cdot\mathbf{u} = 0, \text{ and } \mathbf{q}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega. \quad (1.0.3)$$

The boundary conditions (1.0.3) express the fact that there is no heat flux between the fluid and the outer world, the boundary is impermeable and there is no energy input from friction between the fluid and the boundary. That is, the system is both thermally and mechanically isolated.

In order to satisfy the common sense of the balance of the total energy, the velocity should satisfy that

- the normal part of the velocity vanishes at the boundary, and
- the velocity is perpendicular to the term $\mathbb{S}\mathbf{n}$.

Note that the second condition is satisfied whenever the first condition holds and the tangential part of $\mathbb{S}\mathbf{n}$ vanishes, or the tangential component of $\mathbf{u}$ is zero.

One of the most often used boundary conditions satisfying the conditions described above is the *homogeneous Dirichlet* boundary condition i.e.

$$\mathbf{u}|_{\partial\Omega} = 0. \quad (1.0.4)$$

The condition (1.0.4) says the particles of the fluid stick to the boundary. Gibbs equation

Instead of a direct attempt to solve the system (1.0.1) (which can be found in [8]), one can make a step aside and ask for any more friendly-to-solve representations of the system. In the case of the so called *weak solutions* (defined later on in this thesis), it turns out that the energy balance equation can be transformed introducing a new internal variable *entropy* to the *entropy balance equation.* The entropy is expressed via the *entropy density* s. The entropy rate is binded with the specific internal energy rate e through the *Gibbs equation*

$$\vartheta Ds = De + pD\frac{1}{\rho}$$

By virtue of the Gibbs relationship between the entropy, internal energy, density and pressure, the internal energy balance transforms into the entropy balance provided the temperature is positive:

$$\partial_t(\rho s) + \operatorname{div}(\rho s \mathbf{u}) + \operatorname{div}\frac{\mathbf{q}}{\vartheta} = \frac{1}{\vartheta}\mathbb{S} : \nabla\mathbf{u} - \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2}.$$

The advantage of this approach lies in getting around the issue of estimating the term $p\operatorname{div}\mathbf{u}$ while trying to estimate the gradient of the velocity hidden in the expression $\mathbb{S} : \nabla\mathbf{u}$ on the right-hand side of the internal energy balance.

The formulation of the system (1.0.1) is by no means complete. The behaviour of the system's unknowns density, velocity and temperature (present through entropy) is determined also by the particular kind of interactions inside the fluid. This is desribed by so called *constitutive assumptions* on the structure of the viscous stress tensor $\mathbb{S}$, the pressure p, the external forces term $\mathbf{f}$, the heat flux term $\mathbf{q}$, and the entropy production term σ that are determined by the physical properties of the system.

There are two main approaches to finding an answer to the question of existence of solutions to the system (1.0.1): the way of *strong solutions*, and the route of *weak solutions*.

The former concept of strong solutions is closely related with the functional-analytic framework and the semigroup theory, as it usually relies on some kind of perturbation of the linearized problem in the spirit of the Duhamel formula. Since the main tool is the fixed point argument, one obtains relatively smooth solutions (in the sense of regularity of the underlying semigroup). The drawbacks are, however, fatal: the existence theory is limited either for small initial data, or only short time intervals. This approach has got many important contributions, among which it is worth quoting the initiating works by Matsumura and Nishida [31], as well as significant contributions by Hoff [22, 23] and others.

The latter route of weak solutions does not promise any significant regularity properties at all. However, being based on the construction using the Faedo–Galerkin approximation coupled with addition of vanishing mollifying terms into the system, it provides an existence result for arbitrary large initial data on large time intervals. Despite the lack of regularity properties, the notion of the weak solution is still strong enough to ensure that the physical principles motivating (1.0.1) are valid.

The idea of weak solutions, originating from Leray's ideas on incompressible flows [28], was successfully applied by P.-L. Lions to the framework of compressible barotropic flows. The original limitation to the pressure term of the form $p(\rho) \sim \rho^\gamma$ with $\gamma \geq \frac{9}{5}$ was weakened to the physically relevant constrain $\gamma > \frac{3}{2}$ by E. Feireisl (see [15]). Later on, E. Feireisl extended the existence result to the full system of equations, i.e. the system with density, velocity and temperature (see [8]), and, few years later, to the full system with a general pressure function (see [14]).

In parallel with Feireisl's approach, there is a recent result for the temperature dependent case by Bresch and Desjardins [1], and by Vaĭgant and Kazhikhov [43] for the barotropic case. In connection with the barotropic case, one should mention the recent result by Kukučka [26] where the existence result with the total energy inequality in the differential form is given.

Whereas the existence theory for the barotropic flows has existed for arbitrary open sets since the result on the domain-dependence of weak solutions by Feireisl, Novotný and Petzeltová [12], the existence issue for the full system of equations was solved only for domains with $C^{2+\nu}$ smooth boundary, where $\nu > 0$. The extension of the existence results from smooth domains to domains with rough boundary is the main topic of the first part of this thesis.

In general, the proof of existence of weak solutions relies in general on two basic steps:

(a) the construction of an approximating sequence, and

(b) the proof of stability of the approximating sequence, i.e. showing that the limit of the approximating sequence solves the limit system.

In the existence theory for solutions on smooth domains, the approximating sequence is a sequence of solutions to the modified system where the equations of the original problem are enriched with mollifying terms that ease the solvability of the problem and vanish in the limit. The existence theory for nonsmooth domains, developed in the first part of this thesis, benefits from the existence theory for smooth domains and its main core focuses on *domain dependence of weak solutions*. This means an introduction of a suitable kind of set topology that is strong enough to ensure the weak solutions form a converging sequence, and mild enough to enrich the existence theory with more general domains or sets.

The second part of the thesis concerns with the qualitative properties of weak solutions to the full system of equations. It focuses on the issue connected with the so called 'characteristic parameters' of the fluid (like speed of sound etc.): What happens if some of the parameters tend to zero? From the physical point of view the qualities of the fluid change — it may tend to behave like an incompressible fluid, for example. At least this is the result of rough computations done by engineers, e.g. But the mathematical reality is a bit different: The limiting of the characteristic parameters induces nonfriendly effects that have to be treated in a very precise way in order to verify the conclusions done by the physicists before.

This work's contribution can be recognized in two ways – the improvement of the existence theory for domains with non-smooth boundaries, and analysis of the singular limit in case of a compressible fluid when the Mach number tends to zero. The former part is contained in the larger part of this thesis as it reflects the route followed by the author while attempting to solve the problem of existence of weak solutions to a fluid flow on unbounded domains. It starts showing existence of weak solutions of the full Navier–Stokes–Fourier system on Lipschitz domains, where approximation of the target domain with smooth domains is done for the case where the pressure term p consists of the elastic pressure $p_e = p_e(\rho)$ and linear perturbation of the temperature in the form $\vartheta p_\vartheta(\rho)$.

The same structural assumption on the pressure term was used later on, together with an additional growth assumption on the viscosity terms in the spirit of the thermodynamical theory developed by Oxenius [35], to show the existence of a weak solution on an unbounded domain. The key ingredient here is the verification of the internal energy inequality for the weak solutions under the assumptions on the viscosity terms, and later derivation of the estimates on the velocity term.

The general concept of the pressure described in terms of a thermodynamical function for a monoatomic gas, i.e. $p = p(\rho, \vartheta) = \vartheta^{5/2} P(\rho \vartheta^{-3/2})$ is used in the next chapter where the existence result of weak solutions to the full Navier–Stokes–Fourier system on unbounded domains with in general rough boundary is yielded. Here the complications with integrability of the entropy term implied the necessity of use of the theory of Muckenhaupt's weights which resulted in several technical complications.

The series of works devoted to the question of existence is finaly concluded with the part dedicated to the full Navier–Stokes–Fourier system with general pressure function in case of an unbounded domain where the density and temperature attain prescribed positive values 'at infinity'. The result relies on the notion of the Helmholtz-like total energy, which enables us to introduce the nonvanishing boundary conditions to the system in a relatively harmless way. The result then follows from the estimates on the Helmoltz-like total energy and the techniques introduced in the previous chapter. The advantage, in comparison with the approach of the preceding chapter, is no necessity of the weights, and sufficiently strong apriori estimates, that facilitate the whole limit passage.

The last part of this thesis deals with the particular singular limit of a compressible flow on large domains where the so called Mach number is assumed to vanish. The result obtained extends the knowledge concerning the singular limit problems for bounded domains studied by E. Feireisl and A. Novotný (see, e.g. [16]), and the unbounded limit problem for the isentropic case dealt by Desjardins and Grenier [5] and Lions and Masmoudi [30].

The results published in this thesis are included in the articles written either solely by the author himself or in cooperation with E. Feireisl. The chapters with original scientific results correspond to the following articles:

(i) Chapter 2, *Existence of a weak solution on bounded domains with Lipschitz continuous boundary*, contains the article *Existence of weak solutions to the Navier–Stokes–Fourier system on Lipschitz domains* [36] published by the author in Discreet and Continuous Dynamical Systems,

(ii) Chapter 3, *Oxenius-like model of a fluid flow in an unbounded domain case* is covered by the article *On the Oxenius-like model of a fluid flow in the unbounded domain case* [37] published by the author in the Proceedings of the Week of Doctoral Students 2007, Part I – Mathematics and Computer Sciences.

(iii) Chapter 4, *Fluid flow in an unbounded domain with a rough boundary*, corresponds with the paper *On dynamics of fluids in astrophysics* [38] submitted by the author to Journal of Evolution Equations.

(iv) Chapter 5, *Existence of a weak solution on an unbounded domain with prescribed nonvanishing density and temperature at infinity*, is covered by the article *On dynamics of fluids in meteorology* [39] submitted by the author and accepted for publication in the Central European Journal of Mathematics.

(v) Chapter 6, *Low Mach number limit for a viscous compressible fluid* corresponds to the joint article with Eduard Feireisl which was sent to Mathematical Methods in Applied Sciences under the name *On compactness of the velocity field in the incompressible limit of the full Navier-Stokes-Fourier system on large domains* [19].

Chapter 2

Existence of a weak solution on bounded domains with Lipschitz continuous boundary

Corresponds to the article by Poul, L.: Existence of weak solutions to the Navier–Stokes–Fourier system on Lipschitz domains, *Discr. Cont. Dyn. Syst.*, (2007).

Abstract: We prove existence of a weak solution to the Navier–Stokes–Fourier system on a bounded Lipschitz domain in $\mathbb{R}^3$. The key tool is the existence theory for weak solutions developed by Feireisl for the case of bounded smooth domains. We prove our result by inserting an additional limit passage where smooth domains approximate the Lipschitz one. Results on sensitivity of solutions with respect to the convergence of spatial domains are shortly discussed at the end of the paper.
Key words: Navier–Stokes–Fourier system, weak solutions, existence, Lipschitz domains.

2.1 Introduction

The immediate state of a viscous, compressible, and heat conducting fluid can be described by a triple of functions $(\rho, \mathbf{u}, \vartheta)$. These functions represent physical quantities of the fluid: density ρ, velocity $\mathbf{u}$, and temperature ϑ. The time-evolution of the system can be caught up by a system of partial differential equations representing basic physical principles. They are: The continuity equation expressing the total balance of mass of the system

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0. \tag{2.1.1}$$

The second Newton's law in form of the linear momentum equation

$$\partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = \operatorname{div}\mathbb{S} + \rho\mathbf{f}, \tag{2.1.2}$$

where p denotes the pressure and $\mathbb{S}$ denotes the Cauchy stress tensor. The exact forms of p and $\mathbb{S}$ are given by constitutive relations. External forces are expressed by $\mathbf{f}$.

The first law of thermodynamics specifies internal energy e as a conserved quantity. It is equivalent with the entropy equation.

$$\partial_t(\rho s) + \operatorname{div}(\rho s\mathbf{u}) + \operatorname{div}\frac{\mathbf{q}}{\vartheta} = \sigma, \tag{2.1.3}$$

where $\mathbf{q}$ denotes the heat flux and σ stands for the entropy production.

If the state variables ρ, $\mathbf{u}$ and ϑ are smooth, the entropy production σ is equal to $\frac{1}{\vartheta}\mathbb{S} : \nabla\mathbf{u} + \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2}$. However, for nonsmooth motions only one inequality holds

$$\sigma \geq \frac{1}{\vartheta}\mathbb{S} : \nabla\mathbf{u} + \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2} \tag{2.1.4}$$

In this case, the system is supplemented by a requirement on the total energy (in)equality.

The constitutive relations describing quantities p, $\mathbb{S}$ and $\mathbf{q}$ are given as follows

$$p = p(\rho,\vartheta) = p_e(\rho) + \vartheta p_\vartheta(\rho) + \frac{d}{3}\vartheta^4 \tag{2.1.5}$$

$$\mathbb{S} = \mu(\vartheta)\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) + \lambda(\vartheta)\operatorname{div}\mathbf{u}\mathbb{I} \tag{2.1.6}$$

$$= \mu(\vartheta)\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T - \frac{2}{3}\operatorname{div}\mathbf{u}\mathbb{I}\right) + \zeta(\vartheta)\operatorname{div}\mathbf{u}\mathbb{I}$$

$$\mathbf{q} = -\kappa(\vartheta)\nabla\vartheta. \tag{2.1.7}$$

Quantities p, s, and e are interrelated by Gibb's equation $\vartheta Ds = De + pD(\frac{1}{\rho})$, where D represents the total derivative with respect to variables ρ and ϑ. Consequently, assuming moreover that the *specific heat at constant volume* c_v is constant, e and s have the form

$$e(\rho,\vartheta) = P_e(\rho) + d\frac{\vartheta^4}{\rho} + c_v\vartheta,$$

$$s(\rho,\vartheta) = \frac{4}{3}d\frac{\vartheta^3}{\rho} + c_v\log\vartheta - P_\vartheta(\rho)$$

where $P_e(z) = \int_1^z \frac{p_e(s)}{s^2}\,\mathrm{d}s$, and $P_\vartheta(z) = \int_1^z \frac{p_\vartheta(s)}{s^2}\,\mathrm{d}s$.

We assume that there is no slip on the boundary and the system is thermally isolated, i.e.

$$\mathbf{u}|_{\partial\Omega} = 0, \text{ and } (\nabla\vartheta \cdot \mathbf{n})|_{\partial\Omega} = 0.$$

Moreover, we assume that the following structural assumptions hold.

$$p_e(0) = 0, \quad p_e'(\rho) \geq a_1\rho^{\gamma-1} - c_1, \quad p_e(\rho) \leq a_2\rho^\gamma + c_2,$$
$$p_\vartheta(0) = 0, \quad p_\vartheta'(\rho) \geq 0, \quad p_\vartheta(\rho) \leq a_3\rho^\Gamma + c_3,$$

$$0 < \underline{\mu}(1+\vartheta^\alpha) \leq \mu(\vartheta) \leq \overline{\mu}(1+\vartheta)^\alpha,$$
$$0 < \underline{\zeta}\vartheta^\alpha \leq \zeta(\vartheta) \leq \overline{\zeta}(1+\vartheta)^\alpha,$$

$$0 < \underline{\kappa_G} \leq \kappa_G(\vartheta) \leq \overline{\kappa_G}(1+\vartheta^3), \quad \kappa_R(\vartheta) = \sigma\vartheta^3,$$

where $a_1 > 0, \gamma \geq 2, \gamma > \frac{4\Gamma}{3}$, and $\frac{1}{2} \leq \alpha \leq 1$.

The notion of a weak (or variational) solution can be seen as an approach where one replaces the pointwise values of physical quantities by their integral averages around the given point. This concept, being started by Leray [28] for the case of incompressible fluids, leads to the following definition.

Definition 2.1.1. *Let $(\rho, \mathbf{u}, \vartheta)$ be a triple of measurable functions, ρ being nonnegative. We say that $(\rho, \mathbf{u}, \vartheta)$ is a weak solution to the Navier–Stokes–Fourier system on the domain* $(0, T) \times \Omega$

- $\rho, \mathbf{u}$ *solve the* renormalized continuity equation

$$\partial_t b(\rho) + \operatorname{div}(b(\rho)\mathbf{u}) + (b'(\rho)\rho - b(\rho))\operatorname{div}\mathbf{u} = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3) \tag{2.1.1*}$$

  provided ρ and $\mathbf{u}$ are extended to be zero outside Ω.

- $\rho, \mathbf{u}, \vartheta$ *solve the linear momentum equation* (2.1.2) *in* $\mathcal{D}'((0,T)\times\Omega)$,

- $\rho, \mathbf{u}, \vartheta$ *solve the entropy inequality* (2.1.3) *in* $\mathcal{D}'((0,T)\times\overline{\Omega})$*, and*

- $\rho, \mathbf{u}, \vartheta$ *satisfy the total energy equality*

$$\int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e(\rho,\vartheta)\right)(t)\,\mathrm{d}\mathbf{x} = \int_\Omega \left(\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\rho_0} + \rho_0 e(\rho_0,\vartheta_0)\right)\mathrm{d}\mathbf{x} + \int_0^t \int_\Omega \rho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s$$

We introduce the following concept of convergence of domains.

Definition 2.1.2. *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz continuous boundary. We say, that the sequence of domains Ω_n converges to Ω if the following holds:*

- *for any ball $B \subset \mathbb{R}^3 \setminus \Omega$ there exists n_0 such that $B \subset \mathbb{R}^3 \setminus \Omega_n$ for all $n \geq n_0$, and*
- *for any compact $K \subset \Omega$ there exists n_0 such that $K \subset \Omega_n$ for all $n \geq n_0$.*
- $\mathrm{cap}_2(\Omega \setminus \Omega_n) \to 0$ *as n tends to infinity.*

Note that for any bounded set Ω with Lipschitz continuous boundary there exists a sequence of domains Ω_n with smooth boundary being uniformly Lipschitz continuous with respect to n, that converge to Ω in the sense of our definition. Moreover, one can take $\Omega \subset \Omega_n$. This can be seen, for example, by smoothing the boundary's graph via mollifiers, and moving it outwards the target domain. In what follows, we will consider the sequence Ω_n with these properties granted.

The following lemma is an easy consequence of Lipschitz regularity of the boundary, in particular, of existence of the trace operator for Lipschitz domains (cf. Stein [40]).

Lemma 2.1.3. *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz continuous boundary and let Ω_n be a sequence of domains that approximate Ω in the sense of Definition 2.1.2. Assume, that u_n is a sequence of functions from $W^{1,2}(\mathbb{R}^N)$ and $u_n \in W_0^{1,2}(\Omega_n)$ for each n. If u_n converge weakly in $W^{1,2}(\mathbb{R}^N)$ to u, then $u \in W_0^{1,2}(\Omega)$.*

Theorem 2.1.4 (Main Theorem)**.** *Let Ω be a bounded domain in $\mathbb{R}^3$ with Lipschitz continuous boundary. Moreover, let the assumptions on terms $p_e, p_\vartheta, \kappa, \lambda, \mu$ hold, and let $\mathbf{f} \in L^\infty((0,T) \times \Omega)$. Then for any initial conditions $\rho(0) = \rho_0 \geq 0$, $\rho_0 \in L^\gamma(\Omega)$, $(\rho\mathbf{u})(0) = \mathbf{m}_0 \in L^1(\Omega; \mathbb{R}^3)$, $\frac{|\mathbf{m}_0|^2}{\rho_0} \in L^1(\Omega)$, $\vartheta(0) = \vartheta_0 \in L^\infty(\Omega)$, $\frac{1}{\vartheta_0} \in L^\infty(\Omega)$, $\vartheta_0 > 0$, there exists a weak solution to the Navier–Stokes–Fourier system on Ω.*
Moreover, there exists a weak solution $(\rho, \mathbf{u}, \vartheta)$ satisfying the initial conditions above and enjoying the following properties: $\mathbf{u} \in L^r(0,T; W_0^{1,r}(\Omega)^3)$ for some $r > 1$; $\vartheta, \log\vartheta \in L^2(0,T; W^{1,2}(\Omega))$; $\rho \in C([0,T]; L^1(\Omega)) \cap L^\infty(0,T; L^\gamma(\Omega))$; $\rho\mathbf{u} \in C([0,T]; L_{weak}^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^3))$; the quantities $\rho\mathbf{u} \otimes \mathbf{u}$, $\mathbb{S} : \nabla\mathbf{u}$, p, $\rho\mathbf{f}$ are integrable on $(0,T) \times \Omega$.

2.2 Existence of δ-approximate solutions on Lipschitz domains

Following the method for proving existence of solutions developed by Ducomet and Feireisl in [6], one starts solving the modified system of equations namely the continuity equation with the artificial viscosity term, the linear momentum equation with artificial pressure term and equation for internal energy, which is equivalent to the entropy equation. The approximate solutions are constructed so that they satisfy the (approximate) total energy equality. Consider a domain with smooth, at least $C^{2+\nu}$, boundary. This regularity is necessary for construction of approximate solutions satisfying the continuity equation with the artificial viscosity term $\varepsilon\Delta\rho$ on the right-hand side, and corresponding parabolic estimates to hold. Then applying the vanishing-viscosity part of the proof in [6] we obtain solution $(\rho_n, \mathbf{u}_n, \vartheta_n)$ of the δ-approximated system of equations with $\delta > 0$ on the domain Ω_n:

$$\left.\begin{array}{rcll} \partial_t\rho_n + \operatorname{div}(\rho_n\mathbf{u}_n) &=& 0 & ,\ \text{in } \Omega_n \\ \rho_n(0) &=& \rho_{0,n} & ,\ \text{in } \Omega \end{array}\right\} \tag{2.2.1}$$

$$\left.\begin{array}{rcll} \partial_t(\rho_n\mathbf{u}_n) + \operatorname{div}(\rho_n\mathbf{u}_n\otimes\mathbf{u}_n) + \nabla p_\delta &=& \operatorname{div}\mathbb{S}_n + \rho_n\mathbf{f} & ,\ \text{in } \Omega_n \\ \mathbf{u}_n &=& 0 & ,\ \text{on } \partial\Omega_n \\ (\rho_n\mathbf{u}_n)(0) &=& \mathbf{m}_{0,n} & ,\ \text{in } \Omega_n \end{array}\right\} \tag{2.2.2}$$

$$\left.\begin{array}{rcll} \partial_t(\rho_n s_n) + \operatorname{div}(\rho_n s_n\mathbf{u}_n) - \operatorname{div}\frac{\kappa(\vartheta_n)\nabla\vartheta_n}{\vartheta_n} &=& \sigma_n & ,\ \text{on } \Omega_n \\ \nabla\vartheta_n\cdot\mathbf{n}_n &=& 0 & ,\ \text{on } \partial\Omega_n \\ \rho_n(0)s_n(0) &=& \rho_{0,n}s_{0,n} & ,\ \text{in } \Omega_n \end{array}\right\} \tag{2.2.3}$$

where $p_\delta = p_e(\rho_n) + \vartheta_n p_\vartheta(\rho_n) + \frac{d}{3}\vartheta_n^4 + \delta\rho_n^\beta$ represents the pressure term with artificial part $\delta\rho_n^\beta$ and σ_n stands for production of the entropy s_n. Using results of the corresponding part of the existence-proof by Ducomet and Feireisl [6], one can state the following lemma on boundedness of approximate solutions.

Lemma 2.2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{2+\nu}, \nu > 0$ smooth boundary. Moreover, consider that that the assumptions on constitutive terms hold. Then for any $\delta > 0$ there exists a triple $(\rho, \mathbf{u}, \vartheta)$ solving the problem* (2.2.1), (2.2.2) *and* (2.2.3) *in the sense of distributions. Moreover, there* *exists a solution satisfying the total energy equality*

$$-\int_0^T\int_\Omega \partial_t\xi\left(\frac{1}{2}\rho|\mathbf{u}|^2+\rho P_e(\rho)+\frac{\delta}{\beta-1}\rho^\beta+d\vartheta^4+c_v\rho\vartheta\right)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$=\int_\Omega \frac{|\mathbf{m}_0|^2}{2\rho_0}+\rho_0P_e(\rho_0)+\frac{\delta}{\beta-1}\rho_0^\beta+d\vartheta_0^4+c_v\rho_0\vartheta_0\,\mathrm{d}\mathbf{x}+\int_0^T\int_\Omega \xi\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \tag{2.2.4}$$

for any $\xi\in C^\infty[0,T],\xi(0)=1,\xi(T)=0$, *and enjoying the following estimates indepedently of the smoothness of the boundary:*

- $\rho\in L^\infty(0,T;L^\beta(\Omega))$, $\rho|\mathbf{u}|^2\in L^\infty(0,T;L^1(\Omega))$, $\rho\vartheta\in L^\infty(0,T;L^1(\Omega))$,
- $\vartheta\in L^\infty(0,T;L^4(\Omega))$, $\rho\log\vartheta\in L^\infty(0,T;L^1(\Omega))$, $\frac{\mathbb{S}:\nabla\mathbf{u}}{\vartheta}\in L^1((0,T)\times\Omega)$,
- $\nabla\log\vartheta\in L^2((0,T)\times\Omega)$, $\nabla\vartheta^{3/2}\in L^2((0,T)\times\Omega)$,
- $\mathbf{u}\in L^r(0,T;W_0^{1,r}(\Omega)), r=\frac{8}{5-\alpha}$.

Now we can benefit from the technique by Ducomet and Feireisl [6]. For domains Ω_n with smooth boundary, that converge to domain Ω with boundary being merely Lipschitz continuous, we obtain solutions $(\rho_n,\mathbf{u}_n,\vartheta_n)$ which satisfy estimates stated in the lemma above. Note that these estimates are independent of n.

First, we use the test function

$$\varphi_n(t,x)=\psi(t)\mathcal{B}\left[\rho_n(t,\cdot)-\fint_\Omega\rho_n(t)\right](x)$$

where to obtain $\rho_n|_\Omega\in L^{\beta+1}((0,T)\times\Omega)$. Here, $\psi\in\mathcal{D}(0,T)$ and $\mathcal{B}$ denotes so called Bogovskii operator on domain Ω (and we consider it is extended by zero outside Ω). It expresses certain kind of inverse to the div operator and its main properties are stated in the following lemma.

Lemma 2.2.2 (Bogovskii operator, paragraph 3.3 in [34])**.** *Let* $\Omega\subset\mathbb{R}^3$ *be a bounded domain with Lipschitz continuous boundary. Then there exists a bounded linear operator* $\mathcal{B}=(B_1,B_2,B_3)$ *satisfying the following properties:*

- $\mathcal{B}:\overline{L^p(\Omega)}:=\left\{f\in L^p(\Omega):\int_\Omega f\,\mathrm{d}\mathbf{x}=0\right\}\to W_0^{1,p}(\Omega;\mathbb{R}^3)$ *with*

  $$\|\mathcal{B}(f)\|_{W_0^{1,p}(\Omega;\mathbb{R}^3)}\le c(p)\|f\|_{L^p(\Omega)}$$

  for any $1<p<\infty$.

- *the function* $\mathbf{v} = \mathcal{B}[f]$ *solves the problem*

$$\operatorname{div} \mathbf{v} = f \text{ in } \Omega, \quad \mathbf{v}|_{\partial\Omega} = 0.$$

- *for any* $f \in L^p(\Omega)$ *such that there exists* $\mathbf{g} \in L^q(\Omega; \mathbb{R}^3)$ *satisfying* $f = \operatorname{div} \mathbf{g}$ *and* $\mathbf{g} \cdot \mathbf{n}|_{\partial\Omega} = 0$ *we have*

$$\|\mathcal{B}[f]\|_{L^r(\Omega;\mathbb{R}^3)} \leq c(r)\|\mathbf{g}\|_{L^r(\Omega;\mathbb{R}^3)}$$

for any $1 < r < \infty$.

More precisely, testing (2.2.2) with φ_n and employing the estimates given in Lemma 2.2.1 we obtain uniform bound in the form

$$\int_0^T \int_\Omega \rho_n^{\beta+1} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \leq c(\delta) \tag{2.2.5}$$

for any $\delta > 0$.

2.2.1 Strong compactness of the temperature

Up to this moment, we only have weak compactness of the temperature which follows from the estimates in Lemma 2.2.1. In order to strenghten the convergence to the strong one, we shall use the variational formulation of the entropy inequality

$$\partial_t \left(\frac{4d}{3}\vartheta_n^3 + c_v \rho_n \log \vartheta_n \right) + \operatorname{div} \left(\frac{4d}{3}\vartheta_n^3 + c_v \rho_n \log \vartheta_n)\mathbf{u}_n \right) - \operatorname{div} \left(\frac{\kappa_G(\vartheta_n) + \sigma\vartheta_n^3}{\vartheta_n} \nabla\vartheta_n \right) \geq -p_\vartheta(\rho_n) \operatorname{div} \mathbf{u}_n + \frac{\mathbb{S}_n : \nabla\mathbf{u}_n}{\vartheta_n} + \frac{\kappa_G(\vartheta_n) + \sigma\vartheta_n^3}{\vartheta_n^2} |\nabla\vartheta_n|^2 \text{ in } \mathcal{D}'([0,T) \times \overline{\Omega_n})$$

In order to show relative compactness of the sequence of functions bounded in Bochner spaces, one can utter the following version of the Aubin-Lions lemma (see Lemma 6.3, Chapter 6 by Feireisl [8]).

Lemma 2.2.3. *Let* $\Omega \subset \mathbb{R}^N$, $N \geq 2$ *be a bounded Lipschitz domain. Let* $\{v_n\}$ *be a sequence of functions bounded in*

$$L^2(0,T; L^q(\Omega)) \cap L^\infty(0,T; L^1(\Omega)), \quad q > \frac{2N}{N+2}.$$

Furthermore, suppose that $\partial_t v_n \geq g_n$ *in* $\mathcal{D}'((0,T) \times \Omega)$, *where distributions* g_n *are bounded in* $L^1(0,T; W^{-m,p}(\Omega))$ *for certain* $m \geq 1, p > 1$. *Then the sequence* $\{v_n\}$ *is relatively compact in the space* $L^2(0,T; W^{-1,2}(\Omega))$.

Applying Lemma 2.2.3 to the sequence $\left\{\frac{4d}{3}\vartheta_n^3 + c_v\rho_n \log\vartheta_n\right\}_n$ as in the part 5.4 and 6.2 of Ducomet and Feireisl [6] and using $\vartheta_n \rightharpoonup \vartheta$ in $L^2(0,T;W^{1,2}(\Omega))$ we obtain

$$\int_0^T \int_\Omega \left(\frac{4d}{3}\vartheta_n^3 + c_v\rho_n \log\vartheta_n\right)\vartheta_n \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ \to \int_0^T \int_\Omega \left(\frac{4d}{3}\overline{\vartheta^3} + c_v\rho_n\overline{\log\vartheta}\right)\vartheta \,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \quad (2.2.6)$$

This immediately yields $\vartheta_n \to \vartheta$ in $L^2((0,T)\times\Omega)$.

2.2.2 Propagation of density oscillations

Having proved pointwise convergence of the temperature, the next thing we have to show is convergence in the linear momentum equation. In order to pass, we need to show convergence of the nonlinear pressure term. This can be done by showing pointwise convergence of the density.

Similarly to the part 6.3 by Ducomet and Feireisl [6] we can show that

$$\rho_n\mathbf{u}_n\otimes\mathbf{u}_n \to \rho\mathbf{u}\otimes\mathbf{u} \text{ in } L^1((0,T)\times\Omega)^{3\times 3}$$

Growth assumptions on the pressure term and results of Lemma 2.2.1 yield

$$\partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\left(\overline{p_e(\rho)} + \vartheta\overline{p_\vartheta(\rho)} + \frac{d}{3}\vartheta^4 + \delta\overline{\rho^\beta}\right) = \\ \operatorname{div}\mathbb{S} + \rho\mathbf{f} \text{ in } \mathcal{D}'((0,T)\times\Omega)$$

By the Div-Curl lemma (see e.g. Lemma 6.1 by Feireisl [8]), the functions $\rho\in L^\infty(0,T;L^\beta(\Omega))$ and $\mathbf{u}\in L^2(0,T;W_0^{1,2}(\Omega)^3)$ solve the continuity equation in $\mathcal{D}'((0,T)\times\Omega)$ and it is easy to see that, provided we extend them by zero, the equation holds in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$. Moreover, we can take β sufficiently large to recover that ρ and $\mathbf{u}$ solve the renormalized continuity equation on $\mathbb{R}^3$

$$\partial_t b(\rho) + \operatorname{div}(b(\rho)\mathbf{u}) + (b'(\rho)\rho - b(\rho))\operatorname{div}\mathbf{u} = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3) \quad (2.1.1^*)$$

where the function b satisfies certain growth assumptions (for details, see e.g. Novotný and Straškraba [34], Chapter 6).

Thus we can take $z\mapsto z\log z$ for b and write

$$\partial_t(\rho\log\rho) + \operatorname{div}(\rho\log\rho\mathbf{u}) + \rho\operatorname{div}\mathbf{u} = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3) \quad (2.2.7)$$

On the other hand, $\rho_n \in L^\infty(0,T;L^\beta(\Omega))$ and $\mathbf{u}_n \in L^2(0,T;W_0^{1,r}(\Omega)^3)$ satisfy the renormalized continuity equation with $b(z) = z\log z$, passing with n to infinity we get

$$\partial_t(\overline{\rho\log\rho}) + \mathrm{div}\,(\overline{\rho\log\rho}\mathbf{u}) + \overline{\rho\mathrm{div}\,\mathbf{u}} = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3) \tag{2.2.8}$$

Substracting (2.2.7) and (2.2.8), and integrating yields

$$\int_\Omega (\overline{\rho\log\rho} - \rho\log\rho)(\tau)\,\mathrm{d}\mathbf{x} = \int_0^\tau\int_\Omega \rho\mathrm{div}\,\mathbf{u} - \overline{\rho\mathrm{div}\,\mathbf{u}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \tag{2.2.9}$$

As the function $z \mapsto z\log z$ is strictly convex and continuous, we have that the term on the left-hand side is always non-negative and vanishes if and only if $\rho_n \to \rho$ strongly in $L^1((0,T)\times\Omega)$. Therefore our next step is to obtain suitable bounds on the right-hand side. In order to do this, we employ the strategy by Lions [29] to use a test function of the form

$$\varphi_n(t,x) := \psi(t)\eta(x)(\nabla\Delta^{-1})[\rho_n(t,\cdot)](x), \quad \psi\in\mathcal{D}(0,T), \eta\in\mathcal{D}(\Omega)$$

for problem on the set Ω_n. This yields

$$\begin{aligned}
&\int_0^T\int_\Omega \psi\eta\left(p_e(\rho_n) + \vartheta_n p_\vartheta(\rho_n) + \frac{d}{3}\vartheta_n^4 + \delta\rho_n^\beta - \lambda(\vartheta_n)\mathrm{div}\,\mathbf{u}_n\right)\rho_n\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad - 2\int_0^T\int_\Omega \psi\eta\mu(\vartheta_n)\nabla\mathbf{u}_n : (\nabla\Delta^{-1}\nabla)[\rho_n]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&= \int_0^T\int_\Omega \psi\left[\lambda(\vartheta_n)\mathrm{div}\,\mathbf{u}_n - \left(p_e(\rho_n) + \vartheta_n p_\vartheta(\rho_n) + \frac{d}{3}\vartheta_n^4 + \delta\rho_n^\beta\right)\right]\nabla\eta\times\\
&\quad\times(\nabla\Delta^{-1})[\rho_n]\\
&\quad + \int_0^T\int_\Omega \psi\left[\mu(\vartheta_n)(\nabla\mathbf{u}_n + \nabla\mathbf{u}_n^T) - \rho(\mathbf{u}_n\otimes\mathbf{u}_n)\right]\nabla\eta\cdot(\nabla\Delta^{-1})[\rho_n]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad - \int_0^T\int_\Omega \partial_t\psi\eta\mathbf{u}_n\cdot(\nabla\Delta^{-1})[\rho_n]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^T\int_{\Omega_n}\psi\eta\mathbf{f}\cdot(\nabla\Delta^{-1})[\rho_n]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad + \int_0^T\int_\Omega \psi\eta\mathbf{u}_n\cdot\big(\rho_n(\nabla\Delta^{-1}\mathrm{div}\,)[\rho_n\mathbf{u}_n] - (\nabla\Delta^{-1}\nabla)[\rho_n](\rho_n\mathbf{u}_n)\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t
\end{aligned}$$

where the terms $\nabla\Delta^{-1}\mathrm{div}$ and $\nabla\Delta^{-1}\nabla$ are defined in terms of the Fourier transformation and represent continuous linear operators from $L^p(\mathbb{R}^3)^3$ to $L^p(\mathbb{R}^3)^3$, $L^p(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)^{3\times3}$ respectively, with $1<p<\infty$ (see e.g. Stein [40]) for details).

Similarly, one can use the test function $\varphi(t,x) = \psi(t)\eta(x)(\nabla\Delta^{-1})[\rho]$, with $\psi\in\mathcal{D}(0,T)$, and $\eta\in\mathcal{D}(\Omega)$ in the limit version of the linear momentum equation. Substracting both equations and passing to the limit, results on the weak continuity of the bilinear forms of singular integrals (Lemma 3.4 in Feireisl, Novotný and Petzeltová [10]) can be used to obtain

$$\begin{aligned}
\lim_{n\to\infty} \Big(\int_0^T \int_{\Omega_n} & \psi\eta(p_e(\rho_n) + \vartheta_n p_\vartheta(\rho_n) + \delta\rho_n^\beta - \lambda(\vartheta_n)\operatorname{div}\mathbf{u}_n)\rho_n \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
& -2\int_0^T \int_{\Omega_n} \psi\eta\mu(\vartheta_n)\nabla\mathbf{u}_n : (\nabla\Delta^{-1}\nabla)[\rho_n]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \Big) \\
= \int_0^T \int_\Omega & \psi\eta(\overline{p_e(\rho)} + \vartheta\overline{p_\vartheta(\rho)} + \delta\overline{\rho^\beta} - \lambda(\vartheta)\operatorname{div}\mathbf{u})\rho\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
& - 2\int_0^T \int_\Omega \psi\eta\mu(\vartheta)\nabla\mathbf{u} : (\nabla\Delta^{-1}\nabla)[\rho]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.
\end{aligned}$$

Our next step is to simplify the integrals in the equation, more precisely, we wish to obtain

$$\begin{aligned}
\lim_{n\to\infty} & \int_0^T \int_\Omega \psi\eta\left(2\mu(\vartheta_n)\nabla\mathbf{u}_n : (\nabla\Delta^{-1}\nabla)[\rho_n] - 2\mu(\vartheta)\nabla\mathbf{u} : (\nabla\Delta^{-1}\nabla)[\rho]\right)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= \lim_{n\to\infty} \int_0^T \int_\Omega \psi\eta\left(2\mu(\vartheta_n)\operatorname{div}\mathbf{u}_n\rho_n - 2\mu(\vartheta)\operatorname{div}\mathbf{u}\rho\right)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.
\end{aligned}$$

To this end, we employ the commutator theory for singular integrals developed by Coifman and Meyer [4].

Lemma 2.2.4 (Commutator Lemma (Proposition 5.1 in [7]))**.** *Let* $\mathbf{v} : (0,T)\times\mathbb{R}^N \to \mathbb{R}^N$ *be a vector field and let* $g : (0,T)\times\mathbb{R}^N \to \mathbb{R}$ *be a scalar function such that*

$$\mathbf{v} \in L^r(0,T;L^r(\mathbb{R}^N;\mathbb{R}^N)), g \in L^p(0,T;W^{1,p}(\mathbb{R}^N)) \cap L^\infty(0,T;(L^q\cap L^1)(\mathbb{R}^N)),$$

where $\frac{Nr}{Nr+r-N} < p < N$, $\frac{1}{q} + \frac{1}{r} < 1$. *Furthermore, assume that*

$$\operatorname*{ess\,sup}_{t\in(0,T)} \Big(\|g(t)\|_{L^q(\mathbb{R}^N)} + \|g(t)\|_{L^1(\mathbb{R}^N)} \Big) \le K.$$

Then there exist constants $c = c(p,q,r,K) > 0$, $\omega = \omega(p,q,r) > 0$, *and* $s = s(p,q,r) > 1$ *such that*

$$\begin{aligned}
\big\|(\partial_i\Delta^{-1}\operatorname{div})[g\mathbf{v}] - g(\partial_i\Delta^{-1}\operatorname{div})[\mathbf{v}]\big\|_{L^s(0,T;W^{\omega,s}(\mathbb{R}^N))} \\
\le c\|g\|_{L^p(0,T;W^{1,p}(\mathbb{R}^N))}\|\mathbf{v}\|_{L^r(0,T;W^{1,r}(\mathbb{R}^N;\mathbb{R}^N))}, \ \textit{for } i = 1,\dots,N.
\end{aligned}$$

Taking $\mathbf{v} = \nabla u^i$ and $g = \eta\mu(\vartheta)$ in the preceding lemma and using the strong convergence of the temperature we see that the identity we claim holds.

We have shown, that

$$\begin{aligned}&\overline{p_e(\rho)\rho} + \vartheta\overline{p_\vartheta(\rho)\rho} + \delta\overline{\rho^{\beta+1}} - (2\mu(\vartheta) + \lambda(\vartheta))\overline{\rho \mathrm{div}\,\mathbf{u}} \\ &\quad = \overline{p_e(\rho)}\rho + \vartheta\overline{p_\vartheta(\rho)}\rho + \delta\overline{\rho^\beta}\rho - (2\mu(\vartheta) + \lambda(\vartheta))\rho \mathrm{div}\,\mathbf{u} \text{ in } \mathcal{D}((0,T)\times\Omega).\end{aligned}$$

This relation can be rewritten to the form

$$\rho \mathrm{div}\,\mathbf{u} - \overline{\rho \mathrm{div}\,\mathbf{u}} = \frac{1}{2\mu(\vartheta) + \lambda(\vartheta)}(Q_1 + Q_2 + Q_3)$$

where

$$Q_1 = \overline{p_e(\rho)}\rho - \overline{p_e(\rho)\rho}, \quad Q_2 = \vartheta\left(\overline{p_\vartheta(\rho)}\rho - \overline{p_\vartheta(\rho)\rho}\right), \quad Q_3 = \delta\left(\overline{\rho^\beta}\rho - \overline{\rho^{\beta+1}}\right).$$

As p_ϑ is non-decreasing, we have $Q_2 \leq 0$, and similarly $Q_3 \leq 0$. What remains is to estimate the term Q_1. We can use the pressure decomposition technique by Feireisl [8] in order to show that $Q_1 \leq \overline{p_b(\rho)}\rho - \overline{p_b(\rho)\rho}$, where the term p_b is a bounded part of the pressure $p_e = p_c + p_m + p_b$ with the convex part p_c and the monotone part p_m. Now we can estimate the difference of the bounded pressure parts as it was done by Feireisl [8], and employ (2.2.9) in order to obtain existence of $\Lambda < \infty$ such that

$$\int_\Omega \left(\overline{\rho\log\rho} - \rho\log\rho\right)(\tau)\,\mathrm{d}\mathbf{x} \leq \frac{\Lambda}{\underline{\mu}}\int_0^\tau\int_\Omega \left(\overline{\rho\log\rho} - \rho\log\rho\right)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$

for almost every $\tau \in [0,T]$. Consequently, the Gronwall lemma yields $\overline{\rho\log\rho} = \rho\log\rho$ which is equivalent to

$$\rho_n \to \rho \text{ in } L^1((0,T)\times\Omega).$$

2.2.3 Approximate entropy inequality and total energy equality

As we already know, the limit functions ρ and ϑ satisfy the continuity equation as well as the linear momentum equation, our next task is to verify that also the entropy inequality and energy equality are satisfied. In the previous parts, we have proved convergence of all the terms involved in the energy and entropy formulae except for $\rho P_e(\rho)$ and $\rho P_\vartheta(\rho)$, but this follows as ρ and $\mathbf{u}$ solve the renormalized continuity equation on $(0,T)\times\mathbb{R}^3$.

Passing to the limit in the entropy inequality, we see that the terms $\frac{\mathbb{S}_n:\nabla\mathbf{u}_n}{\vartheta_n}$, $\frac{\kappa_G(\vartheta_n)+\sigma\vartheta_n^3}{\vartheta_n}$ and $\frac{\kappa_G(\vartheta_n)+\sigma\vartheta_n^3}{\vartheta_n^2}|\nabla\vartheta_n|^2$ need some special care.

Lemma 2.2.5 (Lemma 5.4 in [6]). *Let* $\vartheta_n \to \vartheta$ *in* $L^2((0,T)\times\Omega)$, *and* $\log\vartheta_n \rightharpoonup \overline{\log\vartheta}$ *in* $L^2((0,T)\times\Omega)$. *Then* ϑ *is strictly positive a.e. on* $(0,T)\times\Omega$, *and* $\log\vartheta = \overline{\log\vartheta}$.

A direct consequence of the lemma above yields $\log\vartheta_n \to \log\vartheta$ in $L^2((0,T)\times\Omega)$. As $\nabla\log\vartheta_n = \frac{\nabla\vartheta_n}{\vartheta_n}$ is uniformly bounded in $L^2((0,T)\times\Omega)$, we obtain

$$\frac{\kappa_G(\vartheta_n)+\sigma\vartheta_n^3}{\vartheta_n}\nabla\vartheta_n \to \frac{\kappa_G(\vartheta)+\sigma\vartheta^3}{\vartheta}\nabla\vartheta \text{ in } \mathcal{D}'([0,T)\times\overline{\Omega})$$

Convergence in terms $\frac{1}{\vartheta}\mathbb{S}:\mathbf{u}$ and $\frac{\kappa_G(\vartheta)+\sigma\vartheta^3}{\vartheta^2}|\nabla\vartheta|^2$ follows by the weak lower semicontinuity of the norm and formulae

$$\frac{\mathbb{S}:\nabla\mathbf{u}}{\vartheta} = \left|\sqrt{\frac{\mu(\vartheta)}{\vartheta}}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T-\frac{1}{3}\mathrm{div}\,\mathbf{u}\mathbb{I}\right)\right|^2+\left|\sqrt{\frac{\zeta(\vartheta)}{\vartheta}}\mathrm{div}\,\mathbf{u}\right|^2,$$

$$\frac{\kappa_G(\vartheta)+\sigma\vartheta^3}{\vartheta^2}|\nabla\vartheta|^2 = |\nabla\mathcal{K}_{G,\sigma}(\vartheta)|^2, \text{ where } \mathcal{K}_{G,\sigma}(z) = \int_1^z \frac{\sqrt{\kappa_G(s)+\sigma s^3}}{s}\,\mathrm{d}s.$$

To complete our considerations, it is enough to write for any $\varphi \in \mathcal{D}([0,T)\times\mathbb{R}^3)$, $\varphi|_{(0,T)\times\overline{\Omega}} \geq 0$.

$$
\begin{aligned}
&\int_0^T \int_\Omega \partial_t \varphi \left(\frac{4d}{3} \vartheta_n^3 + c_v \rho_n \log \vartheta_n - \rho_n P_\vartheta(\rho_n) \right) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \left(\frac{4d}{3} \vartheta_n^3 + c_v \rho_n \log \vartheta_n - \rho_n P_\vartheta(\rho_n) \right) \mathbf{u}_n \cdot \nabla \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\
&\quad - \int_0^T \int_\Omega \frac{\kappa_G(\vartheta_n) + \sigma \vartheta_n^3}{\vartheta_n} \nabla \vartheta_n \cdot \nabla \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\
&\quad + \int_0^T \int_{\Omega_n \setminus \Omega} \partial_t \varphi \left(\frac{4d}{3} \vartheta_n^3 + c_v \rho_n \log \vartheta_n - \rho_n P_\vartheta(\rho_n) \right) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\
&\quad + \int_0^T \int_{\Omega_n \setminus \Omega} \left(\frac{4d}{3} \vartheta_n^3 + c_v \rho_n \log \vartheta_n - \rho_n P_\vartheta(\rho_n) \right) \mathbf{u}_n \cdot \nabla \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\
&\quad - \int_0^T \int_{\Omega_n \setminus \Omega} \frac{\kappa_G(\vartheta_n) + \sigma \vartheta_n^3}{\vartheta_n} \nabla \vartheta_n \cdot \nabla \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\
&\leq - \int_0^T \int_{\Omega_n} \varphi \left(\frac{\mathbb{S}_n : \nabla \mathbf{u}_n}{\vartheta_n} + \frac{\kappa_G(\vartheta_n) + \sigma \vartheta_n^3}{\vartheta_n^2} |\nabla \vartheta_n|^2 \right) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\
&\quad - \int_\Omega \varphi(0) \left(\frac{4d}{3} \vartheta_{0,n}^3 + c_v \rho_{0,n} + \log(\vartheta_{0,n}) - \rho_{0,n} P_\vartheta(\rho_{0,n}) \right) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\
&\quad - \int_{\Omega_n \setminus \Omega} \varphi(0) \left(\frac{4d}{3} \vartheta_{0,n}^3 + c_v \rho_{0,n} + \log(\vartheta_{0,n}) - \rho_{0,n} P_\vartheta(\rho_{0,n}) \right) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\
&\leq - \int_0^T \int_{\Omega} \varphi \left(\frac{\mathbb{S}_n : \nabla \mathbf{u}_n}{\vartheta_n} + \frac{\kappa_G(\vartheta_n) + \sigma \vartheta_n^3}{\vartheta_n^2} |\nabla \vartheta_n|^2 \right) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\
&\quad - \int_\Omega \varphi(0) \left(\frac{4d}{3} \vartheta_{0,n}^3 + c_v \rho_{0,n} + \log(\vartheta_{0,n}) - \rho_{0,n} P_\vartheta(\rho_{0,n}) \right) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\
&\quad - \int_{\Omega_n \setminus \Omega} \varphi(0) \left(\frac{4d}{3} \vartheta_{0,n}^3 + c_v \rho_{0,n} + \log(\vartheta_{0,n}) - \rho_{0,n} P_\vartheta(\rho_{0,n}) \right) \, \mathrm{d}\mathbf{x} \, \mathrm{d}t
\end{aligned}
$$

where we have taken for φ its nonnegative part, $\varphi(t,x) := (\varphi(t,x))^+$, which is possible by the density argument.

It is now easy to see that all the integrals over $\Omega_n \setminus \Omega$ vanish in the limit. As the weak lower semicontinuity of the first integral on the right-hand side preserves the inequality sign in the limit, we are done.

2.3 Vanishing artificial pressure

As we have proved existence of a solution to the δ-approximate problem on domain Ω with boundary being merely Lipschitz continuous, we are now able to employ the rest of procedures of the proof by Ducomet and Feireisl [6] and obtain solution to the Navier–Stokes–Fourier system on Ω.

2.4 Remarks on sensitivity with respect to the boundary

Throughout our proof we considered approximation of the Lipschitz domain Ω by smooth domains Ω_n, $\Omega \subset \Omega_n$. It was shown that only some reasonable property that for any ball $B \subset \mathbb{R}^3 \setminus \Omega$ there exists $n(B)$ such that if $n \geq n(B)$, then $B \subset \mathbb{R}^3 \setminus \Omega_n$, is needed. The question is what can one obtain in the case of approximation by "smaller" smooth domains, that is $\Omega_n \subset \Omega$. It turns out, that in addition to the rather natural requirement that any compact subset $K \subset \Omega$ is absorbed by Ω_n for all $n \geq n_0(K)$, we have to require even more – $\mathrm{cap}_2(\Omega \setminus \Omega_n) \to 0$ as $n \to \infty$.

Chapter 3

Oxenius-like model of a fluid flow in an unbounded domain case

Corresponds to the article by Poul, L.: On the Oxenius–like model of fluid flow in the unbounded domain case, *WDS'07 Proceedings of Contributed Papers: Part I - Mathematics and Computer Sciences (eds. J. Safrankova and J. Pavlu), Prague, Matfyzpress*, (2007).

Abstract This article is concerned with the study of existence of weak solutions to the Navier–Stokes–Fourier–Poisson system on an unbounded domain with minimally smooth boundary in $\mathbb{R}^3$. The tools used in this paper are deeply related to the ones used by Feireisl in the proof of existence of weak solutions on smooth domains.

3.1 Introduction

The immediate state of a flow at each point in the time-space can be described by three physical quantities – the density ρ, the velocity $\mathbf{u}$ and the temperature ϑ. These quantities evolve in time so that basic physical principles are satisfied. These principles can be expressed in terms of partial differential equations. They are:

The principle of balance of mass says that the total mass of the system is a conserved quantity, and is expressed in the way of the so called *continuity equation*:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{3.1.1}$$

Second Newton's law says that the linear momentum is a ballanced quantity and the *linear momentum equation* translates this into the language of partial differential equations:

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \operatorname{div} \mathbb{S} + \rho \nabla \Phi, \tag{3.1.2}$$

where p stands for the pressure and $\mathbb{S}$ for the Cauchy stress tensor. Φ denotes the gravitational potential.

The second principle of thermodynamics claims that the total entropy of the system is a nondecreasing quantity. In our framework it writes as follows:

$$\partial_t(\rho s) + \operatorname{div}(\rho s \mathbf{u}) + \operatorname{div} \frac{\mathbf{q}}{\vartheta} = \sigma, \tag{3.1.3}$$

where s denotes the entropy, $\mathbf{q}$ is the heat flux, and σ is the entropy production function.

Finally, the Poisson equation states that the gravitational potential of the fluid with the density ρ is a solution to the following elliptic equation on $\mathbb{R}^3$, where we suppose the right-hand side is extended by zero outside the domain Ω:

$$-\Delta \Phi = G\rho. \tag{3.1.4}$$

We suppose the flow sticks on the boundary and the fluid is thermally isolated. This yields the following boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{3.1.5}$$

The initial conditions are given as

$$\rho(0) = \rho_0, \quad \rho(0)\mathbf{u}(0) = \mathbf{m}_0, \quad \vartheta(0) = \vartheta_0. \tag{3.1.6}$$

The *total energy* of the system is given as

$$E(t) := \int_\Omega \left[\frac{1}{2} \rho |\mathbf{u}|^2(t) + \rho e(\rho, \vartheta)(t) - \frac{1}{2} \rho \Phi(t) \right] \mathrm{d}\mathbf{x}, \tag{3.1.7}$$

where the term e denotes the internal energy of the fluid and its evolution is caught by the *(internal) energy (in)equality*:

$$\partial_t(\rho e) + \operatorname{div}(\rho e \mathbf{u}) + \operatorname{div} \mathbf{q} = \mathbb{S} : \nabla \mathbf{u} - p \operatorname{div} \mathbf{u}. \tag{3.1.8}$$

The notion of the internal energy is closely related with the notion of the entropy and, in fact, one can use both (3.1.3) and (3.1.8) simultaneously, provided all the terms are well defined in appropriate spaces. In fact, if the motion is smooth and the temperature is strictly positive, one can switch between these two equations.

3.1.1 Constitutive Assumptions

In the equations governing the system, some terms were mentioned without explicit dependence on the state variables. The structure of these terms is given through the *consitutive assumptions* that reflect particular properties of the fluid.

We suppose the fluid is Newtonian and so its Cauchy stress tensor is given by

$$\mathbb{S} = \mu(\vartheta)\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T - \frac{2}{3}\mathrm{div}\,\mathbf{u}\mathbb{I}\right) + \zeta(\vartheta)\mathrm{div}\,\mathbf{u}\mathbb{I} \tag{3.1.9}$$

where μ and ζ are shear and bulk viscosity coefficients, respectively. The heat flux $\mathbf{q}$ obeys the Fourier law and so

$$\mathbf{q} = -\kappa(\vartheta)\nabla\vartheta \tag{3.1.10}$$

where κ stands for the heat conductivity coefficient and is supposed to consist of the thermal and radiative part

$$\kappa(\vartheta) = \kappa_G(\vartheta) + \sigma\vartheta^3 \tag{3.1.11}$$

The relation between thermodynamical quantities: pressure, energy, and entropy, is given by Gibb's equation

$$\vartheta Ds = De + pD\frac{1}{\rho}, \tag{3.1.12}$$

where D denotes the total derivative. We suppose that the pressure is composed from the interaction between particles the fluid consists of, and the radiation term due to the temperature. This means

$$p = p_G + p_R. \tag{3.1.13}$$

Furthermore, the interaction between the particles can be decomposed to the *elastic part* $p_e(\rho)$ and the *thermal part* $\vartheta p_\vartheta(\rho)$ so that

$$p_G(\rho,\vartheta) = p_e(\rho) + \vartheta p_\vartheta(\rho). \tag{3.1.14}$$

Consequently, using (3.1.12) we conclude the structure of the entropy and internal energy terms:

$$e(\rho,\vartheta) = P_e(\rho) + d\frac{\vartheta^2}{\rho} + Q(\vartheta), \quad s(\rho,\vartheta) = \frac{4}{3}d\frac{\vartheta^3}{\rho} + C_v(\vartheta) - P_\vartheta(\rho), \tag{3.1.15}$$

where $P_e(\rho) = \int_1^\rho \frac{p_e(z)}{z^2}\,\mathrm{d}z$ and similarly for P_ϑ, and $C_v(\vartheta) = \int_0^\vartheta \frac{Q'(z)}{z}\,\mathrm{d}z$.

If the motion is smooth, one can obtain the entropy production function σ equals $\frac{1}{\vartheta}\mathbb{S} : \nabla\mathbf{u} - \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2}$. However, in the weak solutions case one can have only one inequality, which is supported by the second principle of thermodynamics. More precisely,

$$\sigma \geq \frac{1}{\vartheta}\mathbb{S} : \nabla\mathbf{u} - \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2}. \tag{3.1.16}$$

3.1.2 Smoothness and Growth Assumptions

We suppose the pressure terms to satisfy the following growth and smoothness assumptions:

$$\left.\begin{array}{c} p_e, p_\vartheta \in C^1[0,\infty) \\ p_e(0) = 0, p_e'(\rho) \geq a_1\rho^{\gamma-1} - c_1, p_e(\rho) \leq a_2\rho^\gamma + c_2 \\ p_\vartheta(0) = 0, p_\vartheta'(\rho) \geq 0, p_\vartheta(\rho) \leq a_3\rho^\Gamma + c_3 \end{array}\right\} \tag{3.1.17}$$

where we suppose $\gamma \geq 2$ and $\gamma > 18\Gamma/7$. Due to Oxenius [35], the radiative interaction implies dependence of the viscosity terms also on temperature and the radiative parts of the shear and bulk viscosity grow like ϑ^4. Merging the radiative and standard part of the viscosity together, we come to the following structural hypotheses:

$$\left.\begin{array}{c} \mu \in C^1[0,\infty), 0 < \underline{\mu}(1+\vartheta^4) \leq \mu(\vartheta) \leq \overline{\mu}(1+\vartheta^4) \\ \zeta \in C^1[0,\infty), 0 < \underline{\zeta}\vartheta^4 \leq \zeta(\vartheta) \leq \overline{\zeta}(1+\vartheta^4) \end{array}\right\} \tag{3.1.18}$$

Finally, we suppose the specific heat at constant volume tends to zero while temperature is small. This implies that the entropy of the system stays bounded for low temperature. This means we assume:

$$Q \in BC^1[0,\infty), \quad \lim_{z\to 0}\frac{Q'(z)}{z^\alpha} < \infty \text{ for some } \alpha \in (0,1) \tag{3.1.19}$$

Finally, we introduce the notion of a weak solution:

Definition 3.1.1 (Weak solution)**.** *Let $(\rho, \mathbf{u}, \vartheta)$ be a triple of locally integrable functions, ρ being nonnegative. We say that $(\rho, \mathbf{u}, \vartheta)$ is a weak solution to the Navier–Stokes–Fourier system on $(0,T)\times\Omega$ if*

- $\rho, \mathbf{u}$ *solve the* renormalized continuity equation

$$\partial_t b(\rho) + \operatorname{div}(b(\rho)\mathbf{u}) + (b'(\rho)\rho - b(\rho))\operatorname{div}\mathbf{u} = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3) \tag{3.1.20}$$

  provided ρ and $\mathbf{u}$ are extended by zero outside Ω,

- $\rho, \mathbf{u}, \vartheta$ *solve the linear momentum equation* (3.1.2) *in* $\mathcal{D}'((0,T)\times\Omega)$,
- $\rho, \mathbf{u}, \vartheta$ *solve the entropy inequality* (3.1.3) *in* $\mathcal{D}'((0,T)\times\overline{\Omega})$, *and*
- $\rho, \mathbf{u}, \vartheta$ *satisfy the* total energy inequality*:*

$$E(\tau) \leq E_0 \text{ for } t \in (0,T) \tag{3.1.21}$$

Theorem 3.1.2 (Main Result)**.** *Let* Ω *be a domain with minimally smooth boundary in* $\mathbb{R}^3$. *Moreover, let the assumptions on* $p_e, p_\vartheta, \kappa, \lambda, \mu, \zeta$ *hold. Then for any initial conditions* $\rho(0) = \rho_0 \geq 0$, $\rho_0 \in L^1 \cap L^\gamma(\Omega)$, $(\rho\mathbf{u})(0) = \mathbf{m}_0 \in L^1(\Omega;\mathbb{R}^3)$, $\frac{|\mathbf{m}_0|^2}{\rho_0} \in L^1(\Omega)$, $\vartheta(0) = \vartheta_0 \in L^4 \cap L^\infty(\Omega)$, $\frac{1}{\vartheta_0} \in L^\infty_{loc}(\overline{\Omega}), \vartheta_0 > 0$ *there exists a weak solution* $(\rho, \mathbf{u}, \vartheta)$ *satisfying the initial conditions mentioned above and enjoying the following properties:* $\mathbf{u} \in L^2(0,T; D^{1,2}_0(\Omega;\mathbb{R}^3))$*;* $\vartheta, \vartheta^{3/2} \in L^2(0,T; W^{1,2}_{loc}(\overline{\Omega}))$*;* $\rho \in C([0,T]; L^1(\Omega)) \cap L^\infty(0,T; L^\gamma(\Omega))$*;* $\rho\mathbf{u} \in C([0,T]; L^{2\gamma/(\gamma+1)}_{weak}(\Omega;\mathbb{R}^3))$*; the quantities* $\rho\mathbf{u}\otimes\mathbf{u}$, $\mathbb{S} : \nabla\mathbf{u}$, p, $\rho\nabla\Delta^{-1}[\rho]$ *are integrable on* $(0,T)\times\Omega$.

For the definition of a domain with minimally smooth boundary, see e.g. Stein [40].

The approach benefits from the existence theory for weak solutions on bounded domains. However, the unbounded domain case includes some obstacles that avoid straightforward approach. The main differences are due to the following facts:

1. The existence theory is developped for bounded domains with Lipschitz continuous boundary. The approach to an unbounded domain with minimally smooth boundary requires analysis of the system's limit while underlying domains are beign blown-up.
2. The theory is developed for the case with viscosity coefficients growing as ϑ^α with $1/2 < \alpha \leq 1$. The structural assumptions on viscosity proposed by Oxenius [35] state, however, the viscosity coefficients have growth of order ϑ^4.

The latter point can be dealt just by checking the approximation schemes. The first difficulty is more delicate: in the unbounded-domain case the entropy is not (known to be) integrable, which yields lack of suitable estimates on the velocity field. However, the internal energy inequality can be applied to get this point over with. Unfortunately, the use of the energy inequality requires more restrictive assumptions on the growth of the thermal pressure term.

3.2 Existence Results for Bounded Domain Case

If Ω is a bounded domain with boundary of class $C^{2+\nu}$, there are existence results for weak solutions of the Navier–Stokes–Fourier–Poisson system by Ducomet and Feireisl [6]. This was later generalized in [36] for domains with Lipschitz continuous boundary. One can make the following statement:

Proposition 3.2.1. *Let Ω be a bounded domain with Lipschitz continuous boundary. Suppose that all the assumptions of Theorem 3.1.2 hold. Then there exists a weak solution $(\rho, \mathbf{u}, \vartheta)$ to the Navier–Stokes–Fourier–Poisson system on $(0,T) \times \Omega$ enjoying the same properties as the solution from Theorem 3.1.2 does.*

Our intention is to use the existence result for bounded domains, construct a sequence of bounded Lipschitz domains Ω_n that converges to Ω and show that the solutions of bounded-domain problems converge to a weak solution on the unbounded domain. In order to do this, we have to derive estimates that are independent of the measure of the domain.

3.3 Total Energy Estimates

For any bounded Lipschitz domain Ω_n we have existence of a weak solution by Proposition 3.2.1. This solution satisfies the total energy equality which yields the following set of estimates:

- ρ_n bounded uniformly in $L^\infty(0,T; L^1 \cap L^\gamma(\Omega_n))$,
- $\rho|\mathbf{u}|^2$ bounded uniformly $L^\infty(0,T; L^1(\Omega_n))$,
- ϑ bounded uniformly in $L^\infty(0,T; L^4(\Omega_n))$,

provided the initial total energy $E_n(0)$ is uniformly bounded with respect to n. We had to use an estimate on (nonpositive) term $\rho\Delta^{-1}\rho$ in the form

$$\|\rho\Delta^{-1}\rho\|_{L^1(\mathbb{R}^3)} \leq c\|\rho_n\|^{4/3}_{L^1(\Omega_n)}\|\rho\|^{2/3}_{L^2(\Omega_n)}. \tag{3.3.1}$$

3.4 Internal Energy Inequality

In the bounded-domain method of constructing weak solutions the internal energy (in)equality is used only at the very beginning while showing that the system of Galerkin approximations posseses a solution. After this, the internal energy equation is transformed into the entropy equation which enables us to reach better results on growth conditions of constitutive terms. However, in the unbounded-domain case the lack of estimates on the global integrability of the entropy requires us to verify that the internal energy inequality holds as well. This approach, however, requires more restrictive assumptions on the thermal pressure part growth, namely, one has to have $\gamma > 18\Gamma/7$. Checking the limit passages in the bounded-domain case, one can recover that besides the entropy inequality, also the internal energy inequality holds in the form:

$$\partial_t(d\vartheta^4 + \rho Q(\vartheta)) + \operatorname{div}((d\vartheta^4 + \rho Q(\vartheta))\mathbf{u}) - \operatorname{div}(\kappa(\vartheta)\nabla\vartheta) \geq \mathbb{S} : \nabla\mathbf{u} - \left(\vartheta p_\vartheta(\rho) + \frac{d}{3}\vartheta^4\right)\operatorname{div}\mathbf{u}. \quad (3.4.1)$$

3.5 Energy estimates

Integrating (3.4.1) over Ω_n yields

$$E_n(\tau) \geq \int_0^\tau \int_{\Omega_n} \mathbb{S}_n : \nabla\mathbf{u}_n \,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \int_0^t \int_{\Omega_n} \left(\vartheta_n p_\vartheta(\rho_n) + \frac{d}{3}\vartheta_n^4\right)\operatorname{div}\mathbf{u}_n \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$

Using the Young's inequality we can write:

$$\int_0^\tau \int_{\Omega_n} \vartheta_n^4 |\operatorname{div}\mathbf{u}_n| \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \leq C_\varepsilon \int_0^\tau \int_{\Omega_n} \vartheta_n^4 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \varepsilon\vartheta_n^4 |\operatorname{div}\mathbf{u}_n|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$

Similarly, for the thermal pressure part we obtain

$$\begin{aligned}
&\int_0^\tau \int_{\Omega_n} \vartheta_n p_\vartheta(\rho_n)|\operatorname{div}\mathbf{u}_n| \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \leq \\
&\leq C(\varepsilon, \tau, \|\rho_n\|_{L^\infty(0,\tau;L^\gamma(\Omega))}) + \varepsilon\tau\|\vartheta_n\|^2_{L^\infty(0,\tau;L^4(\Omega_n))} \\
&\quad + \varepsilon \int_0^\tau \int_{\Omega_n} \vartheta_n^4 |\operatorname{div}\mathbf{u}_n|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + c\varepsilon\|\nabla\mathbf{u}_n\|^2_{L^2((0,\tau)\times\Omega_n)}
\end{aligned}$$

Since $\mathbf{u}_n$ vanishes on the boundary of Ω_n, we can extend it by zero to $\mathbb{R}^3 \setminus \Omega_n$ and apply (3.1.18) together with Korn's inequality

$$\left\|\nabla\mathbf{v} + (\nabla\mathbf{v})^T - \frac{2}{3}\operatorname{div}\mathbf{v}\mathbb{I}\right\|_{L^p(\mathbb{R}^3;\mathbb{R}^{3\times 3})} \leq C_p\|\nabla\mathbf{v}\|_{L^p(\Omega;\mathbb{R}^3)}, \quad \text{for any } \mathbf{v} \in D_0^{1,p}(\mathbb{R}^3), 1 < p < \infty, \quad (3.5.1)$$

where $D^{1,p}(\mathbb{R}^3)$ denotes the homogeneous Sobolev space. Employing the Sobolev imbedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ we conclude

$$\mathbf{u}_n \text{ is uniformly bounded in } L^2(0,T;D_0^{1,2}(\mathbb{R}^3;\mathbb{R}^3)) \hookrightarrow L^2(0,T;L^6(\mathbb{R}^3;\mathbb{R}^3)). \tag{3.5.2}$$

3.6 Entropy estimates

Lack of global integrability estimates of the entropy suggests the use of a suitable weight function in place of a test function for (3.1.3). The particular type of weight we are going to use is

$$\varphi(x) = \exp(-\sqrt{1+|x|^2}) \tag{3.6.1}$$

Note that φ is smooth, positive, bounded, integrable with any power on $\mathbb{R}^3$ and $|\nabla\varphi(x)| \leq c\varphi(x)$.

Merging the entropy inequality and testing with the weight function φ given by (3.6.1), we obtain

$$\begin{aligned}\int_{\Omega_n} \rho_n s_n(t)\varphi \,\mathrm{d}\mathbf{x} - \int_{\Omega_n} \rho_{0,n}s_{0,n}\varphi \geq & \int_0^t \int_{\Omega_n} \rho_n s_n \mathbf{u}_n \cdot \nabla\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ & + \int_0^t \int_{\Omega_n} \frac{\kappa(\vartheta_n)\nabla\vartheta_n}{\vartheta_n}\nabla\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ & + \int_0^t \int_{\Omega_n} \left(\frac{1}{\vartheta_n}\mathbb{S}_n : \nabla\mathbf{u}_n + \frac{1}{\vartheta_n^2}\kappa(\vartheta_n)|\nabla\vartheta_n|^2\right)\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\end{aligned}$$

We wish to use the dissipation estimates that follow from $\kappa(\vartheta_n)\frac{|\nabla\vartheta_n|^2}{\vartheta_n^2}$. In order to do this, we need to show that the remaining terms in the inequality are integrable. First of all, let us use the energy estimates to show boundedness of the left-hand side. Taking into account the entropy structure (3.1.15), we can write

$$\begin{aligned}\int_{\Omega_n} \rho_n|s(\rho_n,\vartheta_n)|(t)\varphi \,\mathrm{d}\mathbf{x} \leq & \int_{\Omega_n} \left(\frac{4d}{3}\vartheta_n^3 + \rho_n P_\vartheta(\rho_n) + \rho_n C_v(\vartheta_n)\right)\varphi \,\mathrm{d}\mathbf{x} \\ \leq & C\int_{\Omega_n} \vartheta_n^3\varphi + 1 + \rho_n^\Gamma\varphi + \rho_n\vartheta_n^\alpha\varphi \,\mathrm{d}\mathbf{x} \leq C(E_{0,n}, M_{0,n}, \varphi) < \infty\end{aligned}$$

Estimates of the terms without given sign can be written as follows:

$$\left|\int_0^t\int_{\Omega_n}\rho_n s_n \mathbf{u}_n\cdot\nabla\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right| \le \int_0^t \|\rho_n s_n\varphi\|_{L^{6/5}(\Omega_n)}\|\mathbf{u}_n\|_{L^6(\Omega_n)}\,\mathrm{d}t$$
$$\le \|\rho_n s_n\varphi\|_{L^\infty(0,T;L^{5/6}(\Omega_n))}\|\mathbf{u}_n\|_{L^2(0,T;L^6(\Omega_n))}$$
$$\le C(E_{0,n},M_{0,n},\varphi)<\infty,\ \text{and}$$

$$\left|\int_0^t\int_{\Omega_n}\kappa(\vartheta_n)\frac{\nabla\vartheta_n}{\vartheta_n}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right| \le C\int_0^t\int_{\Omega_n}|\vartheta_n^2\nabla\vartheta_n||\nabla\varphi_n| + |\frac{\nabla\vartheta_n}{\vartheta_n}||\nabla\varphi_n|\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$\le C\int_0^t \|\vartheta_n\|^2_{L^4(\Omega_n)}\|\nabla\vartheta_n\varphi\|_{L^2(\Omega_n)} + \|\nabla\log\vartheta_n\varphi\|_{L^1(\Omega_n)}$$
$$\le C_\varepsilon(\varphi)\left[\|\vartheta_n\|^4_{L^\infty(0,T;L^4(\Omega_n))}+1\right]$$
$$+\,\varepsilon\left[\|\nabla\vartheta_n\sqrt{\varphi}\|^2_{L^2(0,T;L^2(\Omega_n))}+\|\nabla\log\vartheta_n\sqrt{\varphi}\|^2_{L^2(0,T;L^2(\Omega_n))}\right].$$

Merging all the estimates above together, we bound the temperature term $\kappa(\vartheta_n)\frac{|\nabla\vartheta_n|^2}{\vartheta_n^2}$ and the disipation term $\frac{1}{\vartheta_n}\mathbb{S}_n : \nabla\mathbf{u}_n$. Growth assumptions on coefficients κ and μ yield

$$\frac{1}{C}\left(\|\nabla\log\vartheta_n\sqrt{\varphi}\|^2_{L^2(0,T;L^2(\Omega_n))}+\|\nabla\vartheta_n^{3/2}\sqrt{\varphi}\|^2_{L^2(0,T;L^2(\Omega_n))}\right)$$
$$\le\int_0^T\int_{\Omega_n}\left(\frac{1}{\vartheta_n}\mathbb{S}_n:\nabla\mathbf{u}_n+\kappa(\vartheta_n)\frac{|\nabla\vartheta_n|^2}{\vartheta_n^2}\right)\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$\le\int_{\Omega_n}\rho s_n(t)\varphi\,\mathrm{d}\mathbf{x}-\int_{\Omega_n}\rho_{0,n}s_{0,n}\varphi\,\mathrm{d}\mathbf{x}$$
$$+\int_0^T\int_{\Omega_n}|\rho s_n||\mathbf{u}_n||\nabla\varphi|\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_0^T\int_{\Omega_n}\kappa(\vartheta_n)\frac{|\nabla\vartheta_n|}{\vartheta_n}|\nabla\varphi_n|\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$\le C(E_{0,n},M_{0,n},\varphi)+\varepsilon\left(\|\nabla\vartheta_n\sqrt{\varphi}\|^2_{L^2(0,T;L^2(\Omega_n))}+\|\nabla\log\vartheta_n\sqrt{\varphi}\|^2_{L^2(0,T;L^2(\Omega_n))}\right)$$

This procedure yields

$$\sqrt{\varphi}\nabla\log\vartheta_n,\ \sqrt{\varphi}\nabla\vartheta_n^{3/2}\ \text{bounded in}\ L^2(0,T;L^2(\Omega_n)). \tag{3.6.2}$$

This implies $\vartheta_n^{3/2},\vartheta_n\in L^2(0,T;W^{1,2}(\Omega_n,\sqrt{\varphi}))$. In particular, $\vartheta_n\in L^3(0,T;L^9_{loc}(\overline{\Omega_n}))$.

3.7 Local estimates on density

Taking the Bogovskii operator of ρ_n^ν for the test function of the linear momentum equation (for details on Bogovskii operator, see Novotný and Straškraba [34]), we can apply similar procedure to the one used by Ducomet and Feireisl [6], and conclude

$$\rho_n \in L^{\gamma+\nu}_{loc}([0,T] \times \mathbb{R}^3) \text{ uniformly with respect to } n, \text{ for some } \nu > 0 \text{ small.} \tag{3.7.1}$$

3.8 The Limit Process

At this point, we have all the estimates we need to show that the weak limit of a sequence $(\rho_n, \mathbf{u}_n, \vartheta_n)$ is a weak solution to the Navier–Stokes–Fourier system on Ω. First of all, we construct a sequence of approximating Lipschitz domains $\Omega_n \subset \Omega$ with the following property: given $R > 0$ there exists $n_0 \geq 0$ such that if $n \geq n_0$, then $\Omega \cap B_R = \Omega_n \cap B_R$. Due to [36], there exists a weak solution $(\rho_n, \mathbf{u}_n, \vartheta_n)$ of the Navier–Stokes–Fourier–Poison system on $(0, T) \times \Omega_n$. We can apply the results on continuous dependence of weak solutions developed by Feireisl (see, e.g. [6]) to show that the limit $(\rho, \mathbf{u}, \vartheta)$ solves the renormalized continuity equation, the linear momentum equation, and the entropy inequality in the weak sense. What remains to be shown is the total energy inequality. However, by [12], Lemma 5.2, we have

$$\liminf_n \int_0^T \int_{\mathbb{R}^3} \psi \rho_n |\mathbf{u}_n|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \geq \int_0^T \int_{\mathbb{R}^3} \psi \rho |\mathbf{u}|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$

for any $\psi \geq 0, \psi \in \mathcal{D}(0,T)$. Similarly, one can use the [12], Lemma 5.2 to show that the negative part of the elastic pressure potential converges, namely

$$P_e^-(\rho_n)(t) \to P_e^-(\rho) \text{ in } L^1(\mathbb{R}^3) \text{ for any } t \in [0,T].$$

Convergence in the gravitational potential $\rho_n \Delta^{-1}[\rho_n]$ follows from equiintegrability of the density (see the proof of Lemma 4.4 in [12]). Indeed, we can argue similarly as for (3.3.1) and estimate the size of the potential term outside large balls:

$$\begin{aligned} \left| \int_{\mathbb{R}^3 \setminus B_{R_\varepsilon}} \rho_n \Delta^{-1}[\rho_n] \,\mathrm{d}\mathbf{x} \right| &\leq \|\rho_n\|_{L^{4/3}(\mathbb{R}^3 \setminus B_R)} \|\Delta^{-1}[\rho_n]\|_{L^4(\mathbb{R}^3)} \\ &\leq c \|\rho_n\|^{(3\gamma-4)/(4\gamma-4)}_{L^1(\mathbb{R}^3 \setminus B_R)} C(M_{0,n}, E_{0,n}) < \varepsilon. \end{aligned}$$

The remaining terms appearing in the total energy (3.1.7) are nonnegative. Their convergence on bounded set is clear and passing then to the unbounded domain we get the total energy inequality. Hence, $(\rho, \mathbf{u}, \vartheta)$ is a weak solution to the Navier–Stokes–Fourier–Poisson system.

Chapter 4

Fluid flow in an unbounded domain with a rough boundary

Corresponds to the article by Poul, L.: On dynamics of fluids in astrophysics. Submitted to Journal of Evolution Equations.

Abstract: The object of this article is existence of weak solutions to the Navier–Stokes–Fourier–Poisson system on (in general) unbounded domains. The topic is a natural continuation of the author's results on existence of weak solutions of the problem on Lipschitz domains and of the Oxenius system on unbounded domains. Technique of the proof is based on the tools developed in series of works by Feireisl and others during the recent years. The weak solution's sensitivity to a change of the domain is discussed as well.
Keywords: compressible Navier–Stokes–Fourier system, weak solutions, unbounded domains, weights.
Acknowledgment: The project was supported by the Jindrich Necas Center for Mathematical Modeling, project LC06052 financed by MSMT.

4.1 Introduction

We study existence of weak solutions to equations of a fluid flow in the case of unbounded domains, which has particular applications in models used in astrophysics. The model proposed in this work is inspired by thermodynamical properties of a monoatomic gas successfully used e.g. in [13].

The contribution of this paper is the treatment of the unbounded-domain case, and a detailed analysis of the essential weak continuous dependence of weak solutions on convergence of the underlying spatial domain, which pushes forward a shortnote published in [36] for the case of bounded domains with Lipschitz continuous boundary.

This paper is a continuation of a series dedicated to the question of existence of weak solutions of the Navier–Stokes–Fourier system on domains with nonsmooth boundary. It benefits, like the author's previous works [36] and [37], from the existence theory for the system on bounded domain with boundary of class $C^{2+\nu}, \nu > 0$, developed by Feireisl and others, see e.g. [8], [6], [13]. It generalizes the results on domain dependence of weak solutions for the barotropic flow (i.e. the flow where the pressure is a function of density and the temperature terms are not included) known from the paper by Feireisl, Novotný and Petzeltová [12] to the full system of equations.

The paper is organized as follows: In the first section, we introduce the most important introductory issues for futher work. In section two, the estimates independent of the size of the domain are obtained: first, we get bounds from the total energy inequality and, consequently, we apply the weighted theory in order to get the estimates based on the disipation terms from the entropy inequality. Finally, we get the local-in-space estimates on the pressure due to the Bogovskii lemma. The concluding third part is devoted to the limit process where we prove that the weak limit of the constructed approximating sequence is the desired solution of the problem on an unbounded domain Ω. At the very end, we briefly discuss the question of strict positivity of the temperature.

4.1.1 Equations governing the system

A fluid flow can be described as a time-evolution of the fluid's immediate state. This state is characterized by a triple of physical quantities: the density ρ, the velocity $\mathbf{u}$, and the temperature ϑ. The fluid flow evolves in time so that a set of basic physical principles is satisfied. These principles are translated into the language of partial differential equations as follows.

The total balance of mass in the system, described in terms of the *continuity equation*

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0; \tag{4.1.1}$$

Newton's second law, saying that the linear momentum is a balanced quantity, captured by the *linear momentum equation*,

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \operatorname{div} \mathbb{S} + \rho(\nabla \Phi + \mathbf{f}), \tag{4.1.2}$$

where p denotes the pressure, $\mathbb{S}$ is the so called Cauchy stress tensor, $\mathbf{f}$ stands for external forces with the origin out of the fluid, and Φ is the gravitational potential of the fluid itself; the first law of thermodynamics which says the internal energy e is a conserved quantity, which is equivalent to the *entropy* *production equation*:

$$\partial_t(\rho s) + \operatorname{div}(\rho s \mathbf{u}) + \operatorname{div}\frac{\mathbf{q}}{\vartheta} = \Sigma, \tag{4.1.3}$$

where s is the entropy, $\mathbf{q}$ is the heat flux and Σ stands for the entropy production rate – a nonnegative quantity (possibly a measure). If the motion is smooth, Σ is represented by a nonnegative function and $\Sigma = \frac{1}{\vartheta}\mathbb{S} : \nabla\mathbf{u} + \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2}$. For non-smooth motion, however, only the inequality

$$\Sigma \geq \frac{1}{\vartheta}\mathbb{S} : \nabla\mathbf{u} - \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2}$$

holds. Moreover, the second law of thermodynamics implies $\Sigma \geq 0$ since $\frac{1}{\vartheta}\mathbb{S} : \nabla\varphi + \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2} \geq 0$. The gravitational potential Φ is given in terms of the *Poisson equation*

$$-\Delta\Phi = G\rho, \tag{4.1.4}$$

with $G > 0$, considered on the whole $\mathbb{R}^3$ provided the density ρ is considered to equal zero outside Ω.

The relation between entropy, pressure, and internal energy terms is given through *Gibbs' equation*

$$\vartheta Ds = De + pD\frac{1}{\rho} \tag{4.1.5}$$

where D denotes the total derivative.

The pressure is supposed to be composed from the interaction between particles the fluid consists of, and the radiation term due to the temperature. This means

$$p = p_G + p_R, \tag{4.1.6}$$

where the radiation part is given by

$$p_R = p_R(\vartheta) = \frac{1}{3}d\vartheta^4, \tag{4.1.7}$$

and $d > 0$ is the Stefan–Boltzmann constant.

Similarly, the decomposition of the entropy and the energy yields

$$s = s_G + s_R,\ s_R(\rho,\vartheta) = \frac{4d}{3}\frac{\vartheta^3}{\rho}, \quad e = e_G + e_R,\ e_R(\rho,\vartheta) = d\frac{\vartheta^4}{\rho}.$$

Furthermore, in a monoatomic gas, there is a relation between pressure, density and energy:

$$p_G = \frac{2}{3}\rho e_G. \tag{4.1.8}$$

Following the analysis by Feireisl and Novotný [11], (4.1.5) and (4.1.8) yield the following formulae for functions p_G and s_G:

$$\left.\begin{array}{rcl} p_G &=& p_G(\rho,\vartheta) = \vartheta^{5/2} P\left(\frac{\rho}{\vartheta^{3/2}}\right), \\ s_G &=& s_G(\rho,\vartheta) = S\left(\frac{\rho}{\vartheta^{3/2}}\right), \end{array}\right\} \tag{4.1.9}$$

where P is a function from $C^1[0,\infty)$ which choice will be restricted later on so that the thermodynamic principles hold. S is related with P through

$$S'(Y) = -\frac{3}{2}Y^{-2}\left(\frac{5}{3}P(Y) - P'(Y)Y\right), \quad Y > 0. \tag{4.1.10}$$

Which means S is determined by P up to an additive constant. Throughout this paper the function S is supposed to satisfy

$$\lim_{Y\to\infty} S(Y) = 0. \tag{4.1.11}$$

The fluid under consideration is assumed to be Newtonian. This means the stress tensor depends linearly on the velocity's gradient and so the Cauchy stress tensor $\mathbb{S}$ is given by

$$\mathbb{S} = \mu(\vartheta)\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T - \frac{2}{3}\mathrm{div}\,\mathbf{u}\mathbb{I}\right) + \zeta(\vartheta)\mathrm{div}\,\mathbf{u}\mathbb{I}, \tag{4.1.12}$$

where μ and ϑ are viscosity coefficients. The heat flux $\mathbf{q}$ obeys the Fourier law and so

$$\mathbf{q} = -\kappa(\vartheta)\nabla\vartheta, \tag{4.1.13}$$

where κ stands for the heat conductivity coefficient.

We suppose the flow sticks on the boundary and the system is thermally isolated. This yields the boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{4.1.14}$$

The *total energy* of the system is given as

$$E := \int_\Omega \left[\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e(\rho,\vartheta) - \frac{1}{2}\rho\Phi\right] \mathrm{d}\mathbf{x}. \tag{4.1.15}$$

Finaly, the system is supplemented with the initial conditions

$$\rho(0) = \rho_0, \quad (\rho\mathbf{u})(0) = \mathbf{m}_0, \quad \vartheta(0) = \vartheta_0. \tag{4.1.16}$$

4.1.2 Some mathematical tools and notation

Since the underlying spatial domain is unbounded, the imbeddings of L^p spaces fail and, moreover, boundedness of derivatives is not sufficient for boundedness of the original function in the same space. To be more precise, in section 2 we show that the temperature is bounded in $L^\infty(0,T;L^4(\Omega))$ with the bound independent of the size of Ω. This is, however, insufficient to get the total entropy bounded in $L^1(\Omega)$ since it contains the term ϑ^3. This is why weights are used in this work. By *weight* we understand a positive measurable function. Moreover, we will restrict ourselves to a special type of radial weights given by $w_\beta(x) = (1+|x|^2)^{\beta/2}$. Appearance of weights forces us to state here a few basic comments concerning them and their role in weighted spaces.

Given a weight w, we define the *weighted Lebesgue space* $L^p(\Omega;w)$, or $L^p_w(\Omega)$, $1 \le p < \infty$ as

$$\left\{ f \in L^1_{loc}(\Omega) : \|f\|^p_{L^p_w(\Omega)} := \int_\Omega |f|^p w < \infty \right\}.$$

The *weighted homogeneous Sobolev space with zero trace*

$$D^{1,p}_{0,w}(\Omega) \equiv D^{1,p}_0(\Omega;w)$$

is a closure of the set $\mathcal{D}(\Omega)$ in the topology given by the norm

$$\|f\|_{D^{1,p}_{0,w}(\Omega)} := \|\nabla f\|_{L^p_w(\Omega)}.$$

The *weighted Sobolev space* $W^{1,p}(\Omega;w_1,w_2)$ is a set of functions from $W^{1,1}_{loc}(\Omega)$ such that

$$\|f\|_{W^{1,p}(\Omega;w_1,w_2)} := \|f\|_{L^p(\Omega;w_1)} + \|\nabla f\|_{L^p(\Omega;w_2)} < \infty.$$

For the sake of clarity what do we mean by some rather obscure notation used in this work, we present here several examples: $L^p_{loc}(\overline{\Omega})$ denotes the set of functions $f \in L^1_{loc}(\Omega)$ such that for any compact subset of $\overline{\Omega}$ $f \in L^p(K)$. Clearly, if Ω is bounded, one has $L^p_{loc}(\overline{\Omega}) = L^p(\Omega)$. However, having the case of Ω unbounded the situation changes since we don't care about integrability in distant regions of Ω. The space $W^{1,p}_{loc}(\overline{\Omega})$ may be interpreted in a similar way. The space $W^{-1,p}_{loc}(\overline{\Omega})$ is a set of functionals $\psi \in \mathcal{D}'(\Omega)$ such that for any open bounded subdomain $\tilde{\Omega} \subset \Omega$ it holds $\psi \in W^{-1,p}(\tilde{\Omega})$.

Furthermore, we fix the notation on exponents related to duality in the Lebesgue spaces as well as to the Sobolev imbeddings:

$$p' = \frac{p}{p-1}, \quad p^* = \frac{np}{n-p},$$

where n denotes the dimension. Throughout this work, we distinguish between different types of convergence by the following notation:

1. $\rightarrow$ means the standard norm-convergence,

2. $\rightharpoonup$ stands for the weak convergence, and

3. $\overset{*}{\rightharpoonup}$ denotes the weak* convergece.

Last but not least, to get around merely local-in-space estimates, there is a beautiful virtue of the invading domains lemma:

Proposition 4.1.1 (Invading domains lemma, Lemma 6.6 in [34]). *Let* $\{f_n\}$, $f_n \in L^p(0,T;L^q_{loc}(\mathbb{R}^3))$ *with* $1 < p, q \leq \infty$, *a sequence such that*

$$\|f_n\|_{L^p(0,T;L^q(B_M))} \leq K(M) \text{ for } M = M_0, M_0+1, M_0+2, \ldots.$$

Then there exists a subsequence $\{n'\} \subset \{n\}$ *such that* $f_{n'} \rightarrow f$ *weakly-*$*$ *in* $L^p(0,T;L^q(B_R))$ *for any* $R > 0$.

Finally, throughout this work we denote B_R the open ball of diameter $R > 0$ centered at the origin.

4.1.3 Weak solutions

The idea of weak solutions, originating from Leray's work on incompressible flows [28], and further developed for the compressible case by P.-L. Lions [29] and Feireisl [8], is a fruitful way with no limitations on the time of existence or size of the initial data. On the other hand, weak solutions promise no regularity properties. Moreover, as Feireisl has mentioned in [14], the theory of compressible fluid is more likely to rely on the concept of 'genuinely weak' solutions incorporating various types of discontinuities and other irregular phenomena. However, even though the weak solutions fail to have regularity properties, their mathematical formulation as a set of integral identities is much closer to the original formulation of the balance laws. Moreover, the notion of a weak solution is strong enough to ensure the 'classical' interpretation of balance laws as it is discussed in [14], for discussion on initial conditions attained by the weak solutions of the Navier–Stokes–Fourier problem one can see [13].

We take over the notion of the weak solution to the Navier–Stokes–Fourier system in a similar way as it was introduced in the works of P.-L. Lions, E. Feireisl and others.

Definition 4.1.2 (Weak solution). *Let Ω be an open subset of $\mathbb{R}^3$ and let $\mathbf{u} \in L^2((0,T); D^{1,p}_{0,loc}(\overline{\Omega}))$ for some $p > 1$, $\rho \in L^\infty((0,T); L^1(\Omega) \cap L^{5/3}(\Omega))$, and $\vartheta \in L^2((0,T); W^{1,2}_{loc}(\overline{\Omega}))$ be such that $\sqrt{\rho}\mathbf{u} \in L^\infty(0,T; L^2(\Omega))$. We say that the triple $(\rho, \mathbf{u}, \vartheta)$ is a weak solution to the Navier–Stokes–Fourier system on the domain Ω, if the following holds.*

(i) The continuity equation is satisfied in the sense of distributions in the renormalized form, i.e.

$$\partial_t b(\rho) + \operatorname{div}(b(\rho)\mathbf{u}) + (b'(\rho)\rho - b(\rho))\operatorname{div}\mathbf{u} = 0 \text{ in } \mathcal{D}'((0,T) \times \mathbb{R}^3 \quad (4.1.1^*)$$

for all $b \in BC^1[0,\infty)$, where we suppose ρ and $\mathbf{u}$ are extended to be zero outside Ω,

(ii) the linear momentum equation (4.1.2) holds in the sense of distributions, i.e. in $\mathcal{D}'((0,T) \times \Omega)$,

(iii) the entropy production inequality (4.1.3) holds in $\mathcal{D}'((0,T) \times \overline{\Omega})$,

(iv) the total energy (in)equality holds, i.e.

$$E(t) \le E_0 + \int_0^T \int_\Omega \rho\mathbf{u} \cdot \mathbf{f} \,\mathrm{d}\mathbf{x} \,\mathrm{d}t, \quad t \ge 0, \quad (4.1.17)$$

where E is defined by (4.1.15),

(v) and the density and temperature are non-negative on $(0,T) \times \Omega$.

4.1.4 Assumptions & the main results

Partly due to the requirements of thermodynamics, partly because of technical reasons, we impose here assumptions on particular terms in the system of equations:

The thermodynamical function P defining the pressure as well as the entropy term, is $C^1([0,\infty))$. Moreover,

$$\left. \begin{array}{l} P(0) = 0, P'(z) > 0 \text{ for all } z \ge 0, \lim_{z\to\infty} \frac{P(z)}{z^{5/3}} = P_\infty > 0, \text{ and} \\ 0 < \frac{5}{3}P(z) - P'(z)z \le cz^r, \text{ for some } 0 < r < 1, \text{ for all } z > 0. \end{array} \right\} \quad (4.1.18)$$

The viscosity coefficients μ and ζ satisfy:

$$\left. \begin{array}{l} 0 < \underline{\mu}(1 + \vartheta^\alpha) \le \mu(\vartheta), \quad |\mu'(\vartheta)| \le \overline{\mu}(1 + \vartheta^{\alpha-1}) \\ \zeta(\vartheta) \ge 0, \quad \underline{\zeta}\vartheta^{\alpha-1} - 1 \le \zeta(\vartheta), \quad |\zeta'(\vartheta)| \le \overline{\zeta}\vartheta^\alpha, \end{array} \right\} \quad (4.1.19)$$

where $\alpha \in (2/5, 1]$, $\underline{\mu}, \overline{\mu}, \overline{\zeta}, \underline{\zeta}$ are positive constants.

The heat conductivity coefficient κ consists of, similarly to the pressure and entropy terms, the heat conductivity between particles of the fluid, and the heat transfer due to radiation: $\kappa(\rho, \vartheta) = \kappa_G(\vartheta) + \kappa_R(\vartheta)$, where κ_G and κ_R are continuously differentiable functions with growth conditions

$$0 < \underline{\kappa} \leq \kappa_G(\vartheta) \leq \overline{\kappa}(1 + \vartheta^3), \quad \kappa_R(\vartheta) = \sigma\vartheta^3 \tag{4.1.20}$$

for some positive constants $\underline{\kappa}$, $\overline{\kappa}$, and σ.

Theorem 4.1.3 (Main Result I)**.** *Let $\Omega \subset \mathbb{R}^3$ be an open set in $\mathbb{R}^3$. Suppose that the initial state of a fluid is given by the initial density $\rho_0 \in L^1(\Omega) \cap L^{5/3}(\Omega)$ such that $\rho_0 \log \rho_0 \in L^1(\Omega)$, linear momentum $\mathbf{m}_0 \in L^1(\Omega) \cap L^{5/4}(\Omega)$, and temperature $\vartheta_0 \in L^4(\Omega) \cap L^3(\Omega)$, $\vartheta_0 > 0$, $\vartheta \in L^\infty(\Omega)$. Then for any time $T > 0$ there exists a weak solution to the Navier–Stokes–Fourier–Poisson system on $(0,T) \times \Omega$. Moreover, there exists a weak solution $(\rho, \mathbf{u}, \vartheta)$ on $(0,T) \times \Omega$ such that the following holds:*

- *$\rho \in C([0,T]; L^1(\Omega)) \cap L^\infty(0,T; L^{5/3}(\Omega))$, $\rho|\mathbf{u}|^2 \in L^\infty(0,T; L^1(\Omega))$, $\vartheta \in L^\infty(0,T; L^4(\Omega))$;*
- *$\mathbf{u} \in L^2(0,T; D_0^{1,p}(\Omega; w)), \vartheta \in L^2(0,T; W^{1,2}_{loc}(\Omega))$ for some weight w and $p > 1$;*
- *$\mathbb{S} \in L^q_{loc}(\overline{(0,T) \times \Omega})$ for some $q > 1$.*

Futhermore, the temperature ϑ is positive in the sense that $\frac{1}{\vartheta}\mathbb{S} : \nabla\mathbf{u}$ is locally integrable in $\overline{(0,T) \times \Omega}$ and, moreover, there exists time $t_0 > 0$ such that for all $0 < t < t_0$ the temperature is strictly positive in the sense that $\log \vartheta(t, \cdot)$ is locally integrable on $\overline{\Omega}$.

Remark 4.1.4. It is worth discussing what boundary conditions does the particular solution addressed in the theorem satisfy at infinity. Let us assume that Ω is unbounded and nongenerate, that is $|\Omega| = \infty$. Then one obtains that $\rho(t,x), \vartheta(t,x), \rho\mathbf{u}(t,x) \to 0$ as $|x| \to \infty$ because they belong to some L^p space. Since the weight w, given by (4.2.1), does not belong to $L^1(\Omega)$, we obtain that $\nabla\mathbf{u}(t,x), \nabla\vartheta(t,x) \to 0$ for large $|x|$. The statement of a similar property for the velocity $\mathbf{u}$ follows from the estimate

$$\begin{aligned} \|\mathbf{u} w_{\beta/2}\|_{L^2(0,T;L^{24/7}(\mathbb{R}^3))} &\leq \|\nabla\mathbf{u} w_{\beta/2}\|_{L^2(0,T;L^{8/5}(\mathbb{R}^3))} \\ &\leq \|\vartheta^{1/2}\|_{L^\infty(0,T;L^8(\mathbb{R}^3))} \|\frac{1}{\vartheta^{1/2}}\nabla\mathbf{u} w_{\beta/2}\|_{L^2(0,T;L^2(\mathbb{R}^3))}. \end{aligned} \tag{4.1.21}$$

Since for $\beta \in [-21/24, -1/2)$ we have $w_{\beta/2} \notin L^{24/7}(\mathbb{R}^3)$. The result on $\mathbf{u} \to 0$ for large $|x|$ follows.

Remark 4.1.5. Similarly to the boundary conditions, one can ask as well in which sense the initial condition (4.1.16) is satisfied. It is easy to use the regularization procedure to show that $\rho \in C([0,T]; L^1(\mathbb{R}^3) \cap L^{5/3}_{loc,weak}(\overline{\Omega}))$ and $\rho\mathbf{u} \in C([0,T]; L^1_{weak}(\Omega) \cap L^{5/4}_{loc,weak}(\overline{\Omega}))$ and so the initial conditions for ρ and $\rho\mathbf{u}$ are attained in an approriate weak sense. Concerning the initial condition for the temperature, it is clear that

$$\liminf_{t\to 0+} \int_\Omega (\rho s(\rho,\vartheta))(t)\varphi \,\mathrm{d}\mathbf{x} \geq \int_\Omega \rho_0 s(\rho_0,\vartheta_0)\varphi \,\mathrm{d}\mathbf{x}, \quad \varphi \in \mathcal{D}(\mathbb{R}^3), \varphi \geq 0, \text{ and}$$

$$\limsup_{t\to 0+} \int_\Omega (\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e(\rho,\vartheta))(t) \,\mathrm{d}\mathbf{x} \leq \int_\Omega (\frac{1}{2}\rho_0|\mathbf{u}_0|^2 + \rho_0 e(\rho_0,\vartheta_0)) \,\mathrm{d}\mathbf{x}.$$

The stronger result is, however, much more delicate: Let us assume the initial temperature ϑ_0 is positive and smooth enough ($W^{1,\infty}(\Omega) \cap W^{1,1}(\Omega)$). Using its product with $\varphi \geq 0, \varphi \in \mathcal{D}(\Omega)$ as a test function in the entropy inequality and subtracking it from the total energy inequality yields

$$\begin{aligned}
&\int_\Omega \frac{1}{2}(\rho|\mathbf{u}|^2(t) - \rho_0|\mathbf{u}_0|^2) \,\mathrm{d}\mathbf{x} \\
&+ \int_\Omega (\rho e_G(\rho,\vartheta)(t) - \rho_0 e_G(\rho_0,\vartheta_0))(1-\varphi) \,\mathrm{d}\mathbf{x} + d\int_\Omega (\vartheta^4(t) - \vartheta_0)(1-\varphi) \,\mathrm{d}\mathbf{x} \\
&+ \int_\Omega [(\rho e_G(\rho,\vartheta) - \vartheta_0 \rho s_G(\rho,\vartheta))(t) - (\rho_0 e_G(\rho_0,\vartheta_0) - \vartheta_0\rho_0 s(\rho_0,\vartheta_0))] \,\mathrm{d}\mathbf{x} \\
&+ \frac{d}{3}\int_\Omega (3\vartheta^4 - 4\vartheta_0\vartheta^3(t) + \vartheta_0^4)\varphi \,\mathrm{d}\mathbf{x} + \int_0^t \int_\Omega (\rho s(\rho,\vartheta)\mathbf{u} - \kappa(\vartheta)\frac{\nabla\vartheta}{\vartheta}) \cdot \nabla(\vartheta_0\varphi) \,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\
&+ \Sigma(\vartheta_0\varphi 1_{[0,t]\times\overline{\Omega}}) \leq \int_0^t \int_\Omega \rho\mathbf{f}\cdot\mathbf{u} \,\mathrm{d}\mathbf{x}\,\mathrm{d}s
\end{aligned} \tag{4.1.22}$$

Passing with t to zero one can observe, similarly as in [13], that

$$\operatorname{ess}\liminf_{t\to 0+} \int_\Omega \rho|\mathbf{u}|^2(t) - \rho_0|\mathbf{u}_0|^2 \,\mathrm{d}\mathbf{x} \geq 0.$$

Then the properties of the function

$$H_{\vartheta_0}(\rho,\vartheta) = \rho e(\rho,\vartheta) - \vartheta_0 \rho s(\rho,\vartheta)$$

(the function $(\rho,\vartheta) \mapsto H_{\vartheta_0}(\rho,\vartheta) - H_{\vartheta_0}(\rho_0,\vartheta_0) - \frac{\partial H_{\vartheta_0}}{\partial\rho}(\rho_0,\vartheta_0)(\rho-\rho_0)$ is strictly convex and attains its minimum at (ρ_0,ϑ_0), see [13]) transfer the problem to the investigation of the limit

$$\lim_{t\to 0+} \int_\Omega (\rho-\rho_0)\frac{\partial H_{\vartheta_0}}{\partial\rho}(\rho_0,\vartheta_0)\varphi \,\mathrm{d}\mathbf{x}$$

which turns out to be zero if $\rho_0, \log\rho_0, \vartheta_0$ belong to $L^\infty_{loc}(\overline{\Omega})$.

Thus we get

$$\int_\Omega \frac{d}{3}(\vartheta-\vartheta_0)^2(3\vartheta+2\vartheta_0\vartheta+\vartheta_0^2)(t)\varphi\,\mathrm{d}\mathbf{x}$$
$$-\int_\Omega(\rho_0 e_G(\rho_0,\vartheta_0)+d\vartheta_0^4)(1-\varphi)\,\mathrm{d}\mathbf{x} \le h(t,\varphi)\to 0 \text{ for } t\to 0+ \quad (4.1.23)$$

From integrability of $\rho_0 e_G(\rho_0,\vartheta_0), \vartheta_0^4$ and continuity of the integral it follows that for any $\varepsilon>0$ there exists a compact set K_ε such that

$$\int_{K_\varepsilon}(\rho_0 e_G(\rho_0,\vartheta_0)+d\vartheta_0^4)\,\mathrm{d}\mathbf{x}<\varepsilon.$$

Now consider the test function $\varphi_\varepsilon\in\mathcal{D}(\mathbb{R}^3)$ such that $0\le\varphi_\varepsilon\le 1$ and $\varphi_\varepsilon|_{K_\varepsilon}\equiv 1$. This yields for any bounded set $B\subset\Omega$

$$\operatorname*{ess\,lim}_{t\to 0+}\int_B\|\vartheta-\vartheta_0\|^2(t)\,\mathrm{d}\mathbf{x}\le\varepsilon C(B) \text{ for any } \varepsilon>0. \quad (4.1.24)$$

Thus we conclude that the initial condition for the temperature is satisfied in the sense

$$\operatorname*{ess\,lim}_{t\to 0+}\|\vartheta(t)-\vartheta_0\|_{L^2(B)}=0, \quad B \text{ bounded } \subset\Omega \quad (4.1.25)$$

As a certain kind of a by-product to the analysis connected with Theorem 4.1.3, we recover the following result on the relative compactness of solutions to the Navier–Stokes–Fourier system with respect to the domain convergence, in particular, this implies the essential weak continuity of the solutions with respect to the convergence of domains contained in Definition 4.3.1.

Theorem 4.1.6 (Main Result II). *Let Ω_n be a sequence of domains in $\mathbb{R}^3$. Suppose that for any compact $K\subset\Omega$ there exists n_0 such that for $n\ge n_0$ one has $K\subset\Omega_n$, and $\Omega_n\setminus\Omega$ is bounded and $\mathrm{cap}_2(\overline{(\Omega_n\setminus\Omega)})\to 0$ as $n\to\infty$. Moreover, assume that $\rho_{0,n}$, $\rho_{0,n}\log\rho_{0,n}$, $\mathbf{m}_{0,n}$, and $\vartheta_{0,n}$ converge to ρ_0, $\rho_0\log\rho_0$, $\mathbf{m}_0$, and ϑ_0 respectively so that the convergence is strong in the appropriate function spaces stated in Theorem 4.1.3. Then there exists a subsequence $(\rho_n,\mathbf{u}_n,\vartheta_n)$ converging to a triple $(\rho,\mathbf{u},\vartheta)$ which is a solution to the problem on Ω with initial conditions $(\rho_0,\mathbf{u}_0,\vartheta_0)$.*

4.2 Estimates for weak solutions

The basic strategy of constructing a weak solution on a given unbounded domain is to approximate the domain with a sequence of smooth bounded domains and pass to an essential limit with corresponding approximate solutions. Once we succeed in getting estimates on the approximate solutions independently of the size of the domain, we can apply the limiting procedure well known from works by Feireisl (see, e.g, [8]) and obtain the wanted triple of functions.

4.2.1 Weak solutions on smooth bounded domains

Consider a bounded domain Ω with $C^{2+\nu}, \nu > 0$, boundary. The existence result by Feireisl, Petzeltová and Trivisa [13] gives us a weak solution $(\rho, \mathbf{u}, \vartheta)$ to the Navier–Stokes–Poisson–Fourier system on Ω. Given a sequence of bounded smooth domains Ω_n, our aim is to obtain estimates on corresponding solutions $(\rho_n, \mathbf{u}_n, \vartheta_n)$, independent of the size of Ω_n.

First of all, show that for any weak solution the total mass is a balanced quantity.

Proposition 4.2.1. *Let $(\rho, \mathbf{u}, \vartheta)$ be a weak solution according to Definition 4.1.2. Then the total mass of the system is a conserved quantity, i.e.:*

$$\int_\Omega \rho(t)\,\mathrm{d}\mathbf{x} = \int_\Omega \rho_0\,\mathrm{d}\mathbf{x} \text{ for any } t \in [0,T].$$

In particular, the total mass of the system at every time $t \in [0,T]$ is bounded in terms of the total mass at the begining.

Proof. First of all, the regularization procedure by DiPerna–Lions shows $\rho \in C([0,T]; L^1_{loc}(\overline{\Omega}))$. Thus the instantaneous value of ρ is well defined at every time $t \in [0,T]$. Taking test functions φ_k such that $|\nabla\varphi_k| \leq \frac{1}{k}$ and $\operatorname{supp}\varphi_k \subset B(0,2k)$ yields

$$\int_\Omega \rho(t)\varphi_k\,\mathrm{d}\mathbf{x} = \int_\Omega \rho_0\varphi_k\,\mathrm{d}\mathbf{x} + \int_0^t \int_\Omega \rho\mathbf{u}\cdot\nabla\varphi_k\,\mathrm{d}\mathbf{x}\,\mathrm{d}s.$$

Hölder inequality and $\rho \in L^\infty(0,T;L^1(\Omega)$, and $\sqrt{\rho}\mathbf{u} \in L^\infty(0,T;L^2(\Omega))$ yield $\rho\mathbf{u} \in L^\infty(0,T;L^1(\Omega)$, the claim follows then passing with k to infinity. □

4.2.2 Energy estimates

The energy inequality yields boundedness of the term $\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e_G(\rho,\vartheta) + d\vartheta^4 - \frac{1}{2}\rho\Phi$ in $L^\infty(0,T;L^1(\Omega))$. As all the terms except the last one are non-negative, we can proceed if we manage to bound it in $L^\infty(0,T;L^1(\Omega))$. However, the potential Φ is a solution to the Laplace equation $-\Delta\Phi =$

$G\rho$ on $\mathbb{R}^3$ (provided we have extended the function ρ to be zero outside Ω) which is given in terms of a weakly singular kernel $\mathcal{E} = -\frac{1}{4\pi}\frac{1}{|x|}$ as $\Phi = \mathcal{E} * (-G\rho)$. Finally, as ρ is bounded in $L^\infty(0,T;L^1 \cap L^{5/3}(\Omega))$, Φ is bounded in $L^\infty(0,T;L^p(\Omega))$ for any $3 < p < \infty$. Now the Hölder inequality yields

$$\|\rho\Phi\|_{L^1(\Omega)} \leq \|\rho\|_{L^{r'}(\Omega)}\|\Phi\|_{L^r(\Omega)} \leq c\|\rho\|_{L^{r/(r-1)}(\Omega)}\|\rho\|_{L^{3r/(3+2r)}(\Omega)}, r > 3$$

By interpolation,

$$\|\rho\|_{L^{r'}(\Omega)} \leq \|\rho\|_{L^1(\Omega)}^{1-\frac{5}{2r}}\|\rho\|_{L^{5/3}(\Omega)}^{5/2r}, \text{ and } \|\rho\|_{L^{3r/(3+2r)}(\Omega)} \leq \|\rho\|_{L^1(\Omega)}^{\frac{1}{6}-\frac{5}{2r}}\|\rho\|_{L^{5/3}(\Omega)}^{\frac{5}{6}-\frac{5}{2r}}.$$

Thus,

$$\|\rho\Phi\|_{L^1(\Omega)} \leq \|\rho\|_{L^1(\Omega)}^{7/6}\|\rho\|_{L^{5/3}(\Omega)}^{5/6}.$$

By hypothesis, $\mathbf{f} \in L^\infty(0,T;L^\infty(\Omega))$ and so we can write the total energy (in)equality in the form

$$\begin{aligned}\int_\Omega \frac{1}{2}\rho|\mathbf{u}|^2(t) + \rho e_G(\rho,\vartheta) + d\vartheta^4 + \frac{1}{2}\rho\Phi(\rho)\,\mathrm{d}\mathbf{x} &\leq E_0 + \int_0^t\int_\Omega \rho\mathbf{u}\cdot\mathbf{f}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\ &\leq E_0 + \int_0^t M^{1/2}\|\mathbf{f}\|_{L^\infty(0,T;L^\infty(\Omega))}\left(\int_\Omega \rho|\mathbf{u}|^2\,\mathrm{d}\mathbf{x}\right)^{1/2}\mathrm{d}s \\ &\leq C(1+t) + \int_0^t\int_\Omega \frac{1}{2}\rho|\mathbf{u}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}s\end{aligned}$$

Consequently, the Gronwall lemma yields the following bounds:

- $\sqrt{\rho}\mathbf{u}$ bounded in $L^\infty(0,T;L^2(\Omega))$,
- ρ bounded in $L^\infty(0,T;L^1 \cap L^{5/3}(\Omega))$, and
- ϑ bounded in $L^\infty(0,T;L^4(\Omega))$

independently of the size of Ω.

Remark 4.2.2. One can allow more general classes of the external forces term $\mathbf{f}$, which can be treated similarly. However, in our scope where $\mathbf{f}$ represents forces originating far away from the fluid, it is reasonable to assume the boundedness of $\mathbf{f}$ in the space-time and avoid further technical difficulties.

4.2.3 Entropy estimates with weights

From total energy estimates we know that the temperature is integrable with the fourth power. However, since the measure of the underlying domain may be infinite, one cannot state anything concerning the integrability of temperature with the third power which appears in the entropy term. In order to deal with this difficulty, we are forced to employ weights. The problems connected with the use of weights are the following:

- the terms $\operatorname{div}(\rho s\mathbf{u})$ and $\operatorname{div}(\kappa(\vartheta)\frac{\nabla\vartheta}{\vartheta})$ don't vanish while integrated over the whole domain Ω and have to be estimated by virtue of the dissipation terms $\frac{1}{\vartheta}\mathbb{S} : \nabla\mathbf{u}$ and $\kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}$, and the energy estimates from the previous section.

- the estimate of the term $\operatorname{div}(\rho s\mathbf{u})$ needs to know certain bounds on the velocity. However, from the dissipation term $\frac{1}{\vartheta}\mathbb{S} : \nabla\mathbf{u}$ we obtain only bounds on the traceless symmetric gradient of the velocity in the weighted space. This requires us to combine weighted Korn and Poincaré inequalities.

We use the results of weighted integral operator theory connected with the Muckenhoupt weights which definition is as follows:

Definition 4.2.3 (Muckenhoupt weights)**.** *Let $w : \mathbb{R}^N \to \mathbb{R}$ be a measurable, nonnegative and locally integrable function. Then we say that w satisfies the A_p condition (or, belongs to the Muckenhoupt class $\mathcal{A}_p$) if and only if there exists a constant $C > 0$ such that for any cube $Q \subset \mathbb{R}^3$ one has*

$$\left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\right)^{p-1} \leq C.$$

Lemma 4.2.4. *Consider function $w_\alpha : \mathbb{R}^3 \to \mathbb{R}$ defined as follows:*

$$w_\alpha(x) = (1 + |x|^2)^\alpha. \tag{4.2.1}$$

Then w_α is smooth, bounded, strictly positive, $|\nabla w_\alpha(x)| \leq C(w_\alpha(x))^{1-1/2\alpha}$ and for $\alpha < 0$ w_α belongs to any $L^p(\mathbb{R}^3), p \geq 1$ with $p > \frac{-3}{2\alpha}$. Moreover, for cylinder $C(c, r) := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1 - c| < r, x_2^2 + x_3^2 < r^2\}$ one has the following asymptotics:

$$w_\alpha[C(c,R)] \sim \begin{cases} (1+c)^{2\alpha}R^3 & \text{, for } \frac{R}{c} \text{ small,} \\ c_{\alpha,1} + c_{\alpha,2}(1+R)^{2\alpha+3} & \text{, for } \frac{R}{c} \text{ large, } c \text{ large,} \\ (1+R)^{2\alpha+3} & \text{, for } R \sim c \text{ large}, 2\alpha+3 > 0, \\ c_{\alpha,3} & \text{, for } R \sim c \text{ large}, 2\alpha+3 < 0 \end{cases}$$

provided $\alpha \neq -\frac{1}{2}, -1, -\frac{3}{2}$.

Proof. The only thing to show is the asymptotics on cylinders. Direct computation recovers that for $R < c$ one has

$$w_\alpha[C(c,R)] \sim \frac{-2\pi}{(2\alpha+1)(2\alpha+2)(2\alpha+3)}$$
$$\big[(1+c+2R)^{2\alpha+3} - (1+c)^{2\alpha+3} - (1+c+R)^{2\alpha+3} + (1+c-R)^{2\alpha+3}\big]$$
$$+ \frac{2\pi R}{(2\alpha+1)(2\alpha+2)}\big[(1+c+2R)^{2\alpha+2} - (1+c)^{2\alpha+2}\big],$$

and for $c < R$ it holds

$$w_\alpha[C(c,R)] \sim 2\pi\Bigg\{\frac{-1}{(2\alpha+1)(2\alpha+2)(2\alpha+3)}$$
$$\big[(1+2R+c)^{2\alpha+3}+(1+2R-c)^{2\alpha+3}-2(1+R)^{2\alpha+3}-(1+R+c)^{2\alpha+3}-(1+R-c)^{2\alpha+3}$$
$$+2\big]$$
$$+\frac{R}{(2\alpha+1)(2\alpha+2)}\big[(1+2R+c)^{2\alpha+2}+(1+2R-c)^{2\alpha+2}-2(1+R)^{2\alpha+2}\big]\Bigg\}.$$

The rest follows by straightforward analysis. □

Corollary 4.2.5. *Consider the function w_β given by* (4.2.1)*. Then w_β is a doubling weight, and for $-\frac{3}{2} < \beta < \frac{3}{2}(p-1)$ it satisfies Muckenhoupt's $\mathcal{A}_p$ condition.*

Weighted Poincaré and Korn Inequalities

The bounds on the dissipation term $\frac{1}{\vartheta}\mathbb{S} : \nabla\mathbf{u}$ give us estimates on the traceless symmetric gradient of the velocity in the weighted space. Since the velocity at the boundary vanishes, one can ask for some generalizations of the classical Korn and Poincaré–Sobolev inequalities into the framework of weighted spaces.

Lemma 4.2.6 (Weighted Korn's inequality)**.** *Let $w : \mathbb{R}^3 \to \mathbb{R}$ belong to the Muckenhoupt $\mathcal{A}_p$ class. Then there exists a constant $C > 0$ such that for any $v \in D^{1,p}_{0,w}(\mathbb{R}^3;\mathbb{R}^3)$ one has*

$$\|v\|_{D^{1,p}_{0,w}(\mathbb{R}^3;\mathbb{R}^3)} := \|\nabla v\|_{L^p_w(\mathbb{R}^3)} \leq C\| < \nabla v > \|_{L^p_w(\mathbb{R}^3)}$$

Proof. Suppose that $\mathbf{u}$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^3)$. Then the Fourier transform is well defined for $\mathbf{u}$.

Denote

$$F(\mathbf{u}) =< \nabla \mathbf{u} >= \nabla \mathbf{u} + \nabla \mathbf{u}^T - \frac{2}{3}\mathrm{div}\,\mathbf{u}\mathbb{I}.$$

Applying the div operator to $F(\mathbf{u})$ we obtain

$$\Delta \mathbf{u} + \frac{1}{3}\nabla \mathrm{div}\,\mathbf{u} = \mathrm{div}\,F(\mathbf{u})$$

Applying the Δ^{-1} operator we obtain

$$(\mathbb{I} + \frac{1}{3}\nabla \Delta^{-1}\mathrm{div}\,)\mathbf{u} = \Delta^{-1}\mathrm{div}\,F(\mathbf{u})$$

Since the inverse to the operator $(\mathbb{I} + \frac{1}{3}\nabla \mathrm{div}\,)$ can be expressed in terms of a matrix of multipliers $\mathbb{M}(\xi)$:

$$\mathbb{M}(\xi) := \frac{1}{4}\begin{pmatrix} \frac{3\xi_1^2+4\xi_3^2+4\xi_2^2}{\xi_1^2+\xi_2^2+\xi_3^2} & -\frac{\xi_1\,\xi_2}{\xi_1^2+\xi_2^2+\xi_3^2} & \frac{\xi_1\,\xi_3}{\xi_1^2+\xi_2^2+\xi_3^2} \\ -\frac{\xi_1\,\xi_2}{\xi_1^2+\xi_2^2+\xi_3^2} & \frac{4\xi_1^2+4\xi_3^2+3\xi_2^2}{\xi_1^2+\xi_2^2+\xi_3^2} & -\frac{\xi_2\,\xi_3}{\xi_1^2+\xi_2^2+\xi_3^2} \\ -\frac{\xi_1\,\xi_3}{\xi_1^2+\xi_2^2+\xi_3{}^2} & -\frac{\xi_2\,\xi_3}{\xi_1^2+\xi_2^2+\xi_3^2} & \frac{4\xi_1^2+3\xi_3^2+4\xi_2^2}{\xi_1^2+\xi_2^2+\xi_3^2} \end{pmatrix},$$

we can write

$$\nabla \mathbf{u} = (\nabla T_{\mathbb{M}} \Delta^{-1}\mathrm{div}\,)[F(\mathbf{u})].$$

Clearly, $\mathbb{M}$ is a multiplier on $L^p(\mathbb{R}^3; w), 1 < p < \infty$, for any weight $w \in \mathcal{A}_p$ because it is a linear combination of compositions of Riesz transforms. Therefore the operator $\nabla \Delta^{-1/2} T_{\mathbb{M}} \Delta^{-1/2}\mathrm{div}$ is bounded on $L^p_w(\mathbb{R}^3)$. Consequently,

$$\|\nabla \mathbf{u}\|_{L^p_w(\mathbb{R}^3)} \leq C\| < \nabla \mathbf{u} > \|_{L^p_w(\mathbb{R}^3)}$$

for any $w \in \mathcal{A}_p$. □

In the second part, we report and apply the result concerning imbeddings of weighted Sobolev spaces by Gurka and Opic [21].

Proposition 4.2.7. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with boundary $\partial\Omega$ Lipschitz continuous. Then for $\frac{\alpha}{2q} - \frac{\beta}{2p} + \frac{N}{q} + \frac{N}{p} + 1 \leq 0$ and $\frac{N}{q} - \frac{N}{p} + 1 \geq 0$ one has*

$$\|v\|_{L^q_{w_\alpha}(\Omega)} \leq \|\nabla v\|_{L^p_{w_\beta}(\Omega)}, \text{ for all } v \in D^{1,p}_{0,w_\beta}(\Omega). \quad (4.2.2)$$

As a direct consequence, we obtain the following corollary.

Corollary 4.2.8. *Let w_β be given by (4.2.1). Furthermore, suppose that $\vartheta \in L^\infty(0,T;L^4(\Omega))$, $\mathbf{u}$ vanishes at the boundary, $\int_0^T \int_\Omega \frac{1}{\vartheta}\mathbb{S} : \nabla \mathbf{u}\, w_\beta \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$ is finite and the constitutive assumptions (4.1.12) on $\mathbb{S}$ hold. Then*

$$\|\mathbf{u}\|^2_{L^2(0,T;L^{p^*}_{w_{\beta p^*/2}}(\Omega))} \leq \; C\int_0^T \int_\Omega \frac{1}{\vartheta}\mathbb{S} : \nabla \mathbf{u}\, w_\beta \,\mathrm{d}\mathbf{x}\,\mathrm{d}t.$$

In particular, $\mathbf{u} \in L^{p^*}(\Omega; w_{\beta p^*/2})$

Proof. It suffices to write

$$\begin{aligned}
&\int_0^T \left(\int_\Omega |\mathbf{u}|^{p^*} w_{\beta p^*/2}\,\mathrm{d}\mathbf{x}\right)^{2/p^*} \mathrm{d}t \\
&\leq C\int_0^T \left(\int_\Omega |\nabla \mathbf{u}|^p w_{\beta p/2}\,\mathrm{d}\mathbf{x}\right)^{2/p} \mathrm{d}t \\
&\leq C\int_0^T \left(\int_\Omega | < \nabla \mathbf{u} > |^p w_{\beta p/2}\,\mathrm{d}\mathbf{x}\right)^{2/p} \mathrm{d}t \\
&\leq \int_0^T \left(\int_\Omega \vartheta^{(1-\alpha)p/2}\vartheta^{(\alpha-1)p/2}|\nabla \mathbf{u}|^p w_{\beta p/2}\,\mathrm{d}\mathbf{x}\right)^{2/p} \mathrm{d}t \\
&\leq C\|\vartheta\|^{(1-\alpha)}_{L^\infty(0,T;L^4(\Omega))}\int_0^T \int_\Omega \vartheta^{\alpha-1}| < \nabla \mathbf{u} > |^2 w_\beta \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\leq C\|\vartheta\|^{(1-\alpha)}_{L^\infty(0,T;L^4(\Omega))}\int_0^T \int_\Omega \frac{1}{\vartheta}\mathbb{S} : \nabla \mathbf{u}\, w_\beta \,\mathrm{d}\mathbf{x}\,\mathrm{d}t
\end{aligned} \tag{4.2.3}$$

with $p = \frac{8}{5-\alpha}$. □

At this point, we are ready to obtain estimates from the dissipation terms in the entropy inequality. Testing (4.1.3) with the weight w_β we get:

$$\int_\Omega \rho s(t) w_\beta \,\mathrm{d}\mathbf{x} - \langle S_0, w_\beta\rangle \geq \int_0^t \int_\Omega \rho s \mathbf{u}\cdot \nabla w_\beta \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$+ \int_0^t \int_\Omega \frac{\kappa(\vartheta)\nabla\vartheta}{\vartheta}\nabla w_\beta \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^t \int_\Omega \left(\frac{1}{\vartheta}\mathbb{S} : \nabla \mathbf{u} + \frac{1}{\vartheta^2}\kappa(\vartheta)|\nabla\vartheta|^2\right) w_\beta \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$

The structural and constitutive assumptions (4.1.10), (4.1.11) and (4.1.18) yield bounds for the entropy function S:

$$\rho S(\rho\vartheta^{-3/2}) \leq \begin{cases} C\rho|\log\rho| + S(1)\rho, & \rho < \vartheta^{3/2} \leq 1 \\ C\rho|\log\rho| + \rho(\vartheta - 1) + S(1)\rho, & \rho < \vartheta^{3/2} \text{ and } \vartheta > 1 \\ S(1)\rho, & \rho \geq \vartheta^{3/2}. \end{cases} \tag{4.2.4}$$

The energy estimates and (4.2.4) yield bounds on the entropy term on the left-hand side.

$$
\begin{aligned}
&\int_\Omega \rho s(\rho,\vartheta)(t) w_\beta \,\mathrm{d}\mathbf{x} \le \int_\Omega \rho S(\rho/\vartheta^{3/2})(t) w_\beta + \frac{4d}{3}\vartheta^3(t) w_\beta \,\mathrm{d}\mathbf{x} \\
\le& C \int_{\Omega\cap\{\rho\le\vartheta^{-3/2}\}} \rho|\log(\rho)|(t) w_\beta \,\mathrm{d}\mathbf{x} + C\int_{\Omega\cap\{\rho>\vartheta^{-3/2}\}} \rho(t) w_\beta \,\mathrm{d}\mathbf{x} \\
&+ \int_\Omega \rho\vartheta(t) w_\beta \,\mathrm{d}\mathbf{x} + \int_\Omega \frac{4d}{3}\vartheta^3(t) w_\beta \,\mathrm{d}\mathbf{x} \\
\le& C \int_\Omega \left(\rho|\log\rho| + \rho\vartheta + \rho + \frac{4d}{3}\vartheta^3\right)(t) w_\beta \,\mathrm{d}\mathbf{x} \le C(M, E_0, \mathbf{f}, \beta) < \infty.
\end{aligned}
$$

Similarly, the entropy term $\operatorname{div}(\rho s\mathbf{u})$ on the right-hand side can be estimated as follows:

$$
\begin{aligned}
&\left|\int_0^t \int_\Omega \rho s\mathbf{u}\cdot\nabla w_\beta \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right| \le C\int_0^t \|\rho s w_{\beta/2-1/2}\|_{L^{p^*\prime}(\Omega)} \|\mathbf{u} w_{\beta/2}\|_{L^{p^*}(\Omega)} \,\mathrm{d}t \\
&\quad\le C_\varepsilon(\beta,t)\left(\|\vartheta\|^6_{L^\infty(0,T;L^4(\Omega))} + \|\rho S(\rho\vartheta^{-3/2}) w_{\beta/2-1/2}\|^2_{L^\infty(0,T;L^{p^*\prime}(\Omega))}\right) \\
&\qquad + \varepsilon\|\mathbf{u} w_{\beta/2}\|^2_{L^2(0,T;L^{p^*}(\Omega))} \\
&\quad\le C_\varepsilon(\beta, E_0, \mathbf{f}, t) + \varepsilon\|\nabla\mathbf{u}\|^2_{L^2(0,T;L^p(\Omega;w_{\beta p/2}))},
\end{aligned}
$$

where we have used (4.2.4) to write

$$
\begin{aligned}
&\|\rho S(\rho\vartheta^{-3/2}) w_{\beta/2-1/2}\|_{L^2(0,T;L^{p^*\prime}(\Omega))} \\
&\le C(\beta,t)\left(\|\rho\log\rho\|_{L^\infty(0,T;L^{p^*\prime}(\Omega))} + \|\rho\|_{L^\infty(0,T;L^{p^*\prime}(\Omega))} + \|\vartheta\|^{5/2}_{L^\infty(0,T;L^4(\Omega))}\right).
\end{aligned}
$$

The estimate of the heat flux term reads as follows:

$$
\begin{aligned}
&\left|\int_0^t \int_\Omega \kappa(\vartheta)\frac{\nabla\vartheta}{\vartheta}\cdot\nabla w_\beta \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right| \\
\le& C\int_0^t \int_\Omega |\vartheta^2\nabla\vartheta||\nabla w_\beta| + |\frac{\nabla\vartheta}{\vartheta}||\nabla w_\beta| \,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\
\le& C\int_0^t \|\vartheta\|^2_{L^4(\Omega)}\|\nabla\vartheta w_{\beta-1/2}\|_{L^2(\Omega)} + \|\nabla\log\vartheta w_{\beta-1/2}\|_{L^1(\Omega)} \,\mathrm{d}s \\
\le& C_{\varepsilon,\beta,t}\left[\|\vartheta\|^4_{L^\infty(0,T;L^4(\Omega))} + 1\right] + \varepsilon\left[\|\nabla\vartheta\|^2_{L^2(0,T;L^2_{w_\beta}(\Omega))} + \|\nabla\log\vartheta\|^2_{L^2(0,T;L^2_{w_\beta}(\Omega))}\right]
\end{aligned}
$$

for any $\varepsilon > 0$ provided $\beta < -1/2$.

Merging all the estimates above together, we bound the temperature term $\kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}$ and the disipation term $\frac{1}{\vartheta}\mathbb{S}:\nabla\mathbf{u}$. Growth assumptions on coefficients κ and μ yield the following

$$
\begin{aligned}
&\frac{1}{C}\left(\|\nabla\mathbf{u}\|^2_{L^2(0,T;L^p_{w_{\beta p/2}}(\Omega))} + \|\nabla\log\vartheta\|^2_{L^2(0,T;L^2_{w_\beta}(\Omega))} + \|\nabla\vartheta^{3/2}\|^2_{L^2(0,T;L^2_{w_\beta}(\Omega))}\right)\\
&\leq \int_0^T\int_\Omega\left(\frac{1}{\vartheta}\mathbb{S}:\nabla\mathbf{u} + \kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}\right)\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\leq \int_\Omega \rho s(T) w_\beta\,\mathrm{d}\mathbf{x} - \langle S_0, w_\beta\rangle\\
&\quad + \int_0^T\int_\Omega |\rho s||\mathbf{u}||\nabla w_\beta|\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^T\int_\Omega \kappa(\vartheta)\frac{|\nabla\vartheta|}{\vartheta}|\nabla w_\beta|\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\leq\varepsilon\left(\|\mathbf{u}\|^2_{L^2(0,T;L^{p^*}_{w_{\beta p/2}}(\Omega))} + \|\nabla\vartheta\|^2_{L^2(0,T;L^2_{w_\beta}(\Omega))} + \|\nabla\log\vartheta\|^2_{L^2(0,T;L^2_{w_\beta}(\Omega))}\right)\\
&\quad + C_{\varepsilon,\beta,T}(E_0, M, \mathbf{f}).
\end{aligned}
$$

By virtue of Corollary 4.2.8, we obtain the following estimates:

- $\nabla\mathbf{u}$ is bounded in $L^2(0,T;L^p(\Omega;\mathbb{R}^{3\times 3}, w_{\beta p/2}))$,

- $\nabla\vartheta^{3/2}$, $\frac{\nabla\vartheta}{\vartheta}$ are bounded in $L^2(0,T;L^2(\Omega;\mathbb{R}^3, w_\beta))$

provided $-\frac{3}{2} < \beta < -\frac{1}{2}$, with the bound independent of the size of Ω. By virtue of the energy estimates, $\vartheta^{3/2}$ in $L^2(0,T;W^{1,2}_{loc}(\overline{\Omega}))$ and, by the Sobolev imbedding,

$$\vartheta \in L^3(0,T;L^9_{loc}(\overline{\Omega})). \tag{4.2.5}$$

Taking advantage of (4.2.5) we finally obtain

$$\nabla\mathbf{u} \in L^{\frac{6}{4-\alpha}}(0,T;L^{\frac{18}{10-\alpha}}_{loc}(\overline{\Omega})) \tag{4.2.6}$$

which means, by the Sobolev imbedding,

$$\mathbf{u} \in L^{\frac{6}{4-\alpha}}(0,T;L^{\frac{18}{4-\alpha}}_{loc}(\overline{\Omega})) \cap L^2(0,T;L^{\frac{24}{7-3\alpha}}_{loc}(\overline{\Omega})).$$

As a consequence, we get estimate of

$$\rho\mathbf{u}\otimes\mathbf{u} \in L^{\frac{6}{4-\alpha}}(0,T;L^{\frac{90}{92-5\alpha}}_{loc}(\overline{\Omega}))$$

provided $\alpha > \frac{2}{5}$. Similarly, writing

$$\mathbb{S} = (\vartheta\mu(\vartheta))^{1/2}\left(\frac{\mu(\vartheta)}{\vartheta}\right)^{1/2} < \nabla\mathbf{u} > + (\vartheta\zeta(\vartheta))^{1/2}\left(\frac{\zeta(\vartheta)}{\vartheta}\right)^{1/2} \operatorname{div}\mathbf{u}\mathbb{I}$$

we bound $\mathbb{S}$ by virtue of bounds on $\frac{1}{\vartheta}\mathbb{S} : \nabla\mathbf{u}$ and ϑ so that

$$\mathbb{S} \in L^{34/29}(0,T; L^{34/29}_{loc}(\overline{\Omega})). \tag{4.2.7}$$

4.2.4 Refined pressure estimates

In this part, we improve the estimates on integrability of the density. This has to be done in order to get bounds on the pressure term in $L^p_{loc}(\overline{\Omega})$ with $p > 1$. We use the procedure known from the works by Feireisl and others, see e.g. [9], [13] etc. This method is based on using a special test function on the linear momentum equation based on the so called *Bogovskii operator*. The Bogovskii operator has a meaning of an 'inverse' to the divergence operator; it was first introduced in the paper by Bogovskii [3] and can be characterized, for example, in terms of the following statement:

Proposition 4.2.9 (Bogovskii operator ([34], section 3.3)). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Then there exists a linear operator $\mathcal{B}_\Omega = (\mathcal{B}^1_\Omega, \ldots, \mathcal{B}^N_\Omega)$ with the following properties:*

(i) $\mathcal{B}_\Omega : L^p(\Omega) \to W^{1,p}_0(\Omega), 1 < p < \infty,$

(ii) $\operatorname{div}\mathcal{B}_\Omega[f] = f - \frac{1}{|\Omega|}\int_\Omega f\,\mathrm{d}\mathbf{x}$ *a.e. in* $\Omega, f \in L^p(\Omega),$

(iii) $\|\nabla\mathcal{B}_\Omega[f]\|_{L^p(\Omega)} \leq c(p,\Omega)\|f - \frac{1}{|\Omega|}\int_\Omega f\|_{L^p(\Omega)}, 1 < p < \infty,$

(iv) if $f = \operatorname{div}\mathbf{g}$*, where* $\mathbf{g}, \operatorname{div}\mathbf{g} \in L^q(\Omega), \mathbf{g}\cdot\mathbf{n}|_{\partial\Omega} = 0$ *for some* $1 < q < \infty$*, then* $\|\mathcal{B}_\Omega[f]\|_{L^q(\Omega)} \leq c(q,\Omega)\|\mathbf{g}\|_{L^q(\Omega)}$.

Let Ω_R be a bounded domain with Lipschitz continuous boundary. The result on local-in-space integrability of the density function is given through testing the momentum equation (4.1.2) with a function

$$\varphi(t,x) := \psi(t)\mathcal{B}_{\Omega_R}[\eta\rho(t,\cdot)](x)$$

where $\psi \in D(0,T)$, $\eta \in \mathcal{D}(\mathbb{R}^3)$ and the Bogovskii solution is considered to be extended by zero outside Ω_R.

$$
\begin{aligned}
\int_0^T \int_{\Omega_R} \psi p(\rho,\vartheta)\rho^\nu \eta \,\mathrm{d}\mathbf{x}\,\mathrm{d}t &= \int_0^T \int_{\Omega_R} \psi p(\rho,\vartheta) \left(\frac{1}{|\Omega_R|} \int_{\Omega_R} \eta\rho^\nu \,\mathrm{d}\mathbf{y} \right) \mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&- \int_0^T \int_{\Omega_R} \partial_t \psi \rho \mathbf{u} \cdot \mathcal{B}_{\Omega_R}[\eta\rho^\nu] - \psi\rho\mathbf{u} \cdot \mathcal{B}_{\Omega_R}[\mathrm{div}\,(\eta\rho^\nu)] + \\
&+ \psi\rho\mathbf{u}\cdot \mathcal{B}_{\Omega_R}[\rho^\nu \mathbf{u}\cdot\nabla\eta] \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+ \int_0^T \int_{\Omega_R} (\nu-1)\psi\rho\mathbf{u}\cdot\mathcal{B}_{\Omega_R}[\eta\rho^\nu \mathrm{div}\,\mathbf{u}] - \rho(\mathbf{u}\otimes\mathbf{u}) : \nabla\mathcal{B}_{\Omega_R}[\eta\rho^\nu] \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+ \int_0^T \int_{\Omega_R} \psi\mathbb{S} : \nabla\mathcal{B}_{\Omega_R}[\eta\rho^\nu] - \psi\rho(\mathbf{f}+\nabla\Phi)\cdot\mathcal{B}_{\Omega_R}[\eta\rho^\nu] \,\mathrm{d}\mathbf{x}\,\mathrm{d}t
\end{aligned}
$$

This would yield bounds on the density in $L^q((0,T)\times\Omega_R)$ for some $q > 5/3$ provided we succeed in estimating the terms on the left-hand side. Clearly, $\mathcal{B}_{\Omega_R}[\eta\rho^\nu] \in L^\infty(0,T;W_0^{1,\frac{5}{3\nu}}(\Omega_R))$, $\mathcal{B}_{\Omega_R}[\rho^\nu\mathbf{u}\cdot\nabla\eta] \in L^2(0,T;W_0^{1,\frac{15p}{15+9\nu p-5p}}(\Omega_R))$, $\mathcal{B}_{\Omega_R}[\eta\rho^\nu\mathrm{div}\,\mathbf{u}] \in L^2(0,T;W_0^{1,\frac{5p}{5+3\nu p}}(\Omega_R))$. In view of estimates on $\rho\mathbf{u}$, $\rho\mathbf{u}\otimes\mathbf{u}$, $\mathbb{S}$, and $\nabla\Phi$, we conclude that for $\nu > 0$ small enough the right-hand side is bounded. The structural assumptions (4.1.18) yield

$$
P_\infty \rho^{5/3} \le p_G(\rho,\vartheta) \le P(1)(\rho^{5/3} + \vartheta^{5/2}),
$$

which implies

$$
\rho \in L^{5/3+\nu}((0,T)\times\Omega_R), \quad p_G \in L^q((0,T)\times\Omega_R) \tag{4.2.8}
$$

for some $q > 1$.

4.3 The Limit

In the previous parts, the apriori estimates independent of the size of the domain (or local in the domain) have been recovered. The second step is the construction of a solution to the system (4.1.1 − 4.1.4) on a given unbounded domain. As the reader could expect, the theory of weak solutions on bounded domains will be used to construct an approximating sequence.

4.3.1 Approximation scheme for domains

Analyzing the weak formulation of the problem in Definition 4.1.2, we come to the following definition on convergence of domains:

Definition 4.3.1 (Convergence of domains). *Let Ω be an open set in $\mathbb{R}^3$. We say that open sets $\Omega_n \subset \mathbb{R}^3$ converge to the domain Ω, if:*

- *for any compact $K \subset \Omega$ there exists n_0 such that for any $n \geq n_0$ $K \subset \Omega_n$.*
- *the set $\Omega_n \setminus \Omega$ is bounded, and, moreover,*

$$\mathrm{cap}_2(\overline{\Omega_n \setminus \Omega}) \to 0 \text{ for } n \to \infty. \tag{4.3.1}$$

Where the exterior capacity cap_2 is defined as

$$\mathrm{cap}_2(M) = \inf \left\{ \int_{\mathbb{R}^3} |\nabla v|^2 \, \mathrm{d}\mathbf{x} : v \in \mathcal{D}(\mathbb{R}^3), v|_M \geq 1 \right\}$$

for any compact $M \subset \mathbb{R}^3$.

The condition on the capacity of the target emerges from the requirement on the transfer of the boundary conditions for the velocity field $\mathbf{u}$. Indeed, we state here the following lemma on the Mosco–type convergence inspired by Lemma 3.1 from [12].

Proposition 4.3.2. *Let $\Omega_n \to \Omega$ in the sense of Definition 4.3.1. Moverover, let $v_n \in D_0^{1,p}(\Omega_n; w) \subset D^{1,p}(\mathbb{R}^3; w)$ be an arbitrary sequence such that $v_n \rightharpoonup v$ in $D^{1,p}(\mathbb{R}^3; w)$, where w is given by (4.2.1). Then $v \in D_0^{1,p}(\Omega; w)$.*

Remark 4.3.3. It is worth noting that the capacitary condition in the definition 4.3.1 can be much relaxed. For some ideas, one can look into [12], Part 6. Another condition emerges from the procedure of domain-approximation in [36] and is supported by the concept of locally Lipschitz convergence of the graphs of boundaries $\partial\Omega_n$, i.e.:

Definition 4.3.4 (A domain with locally Lipschitz boundary). *Let Ω be an opet set in $\mathbb{R}^N$. We say that the boundary $\partial\mathcal{D}$ is locally Lipschitz if there exists $R_0 > 0$ such that for any $R \geq R_0$ there exists a domain Ω_R enjoying the following properties:*

1. *Ω_R is a domain with Lipschitz continuous boundary, and*
2. *$\Omega \cap B_R \subset \Omega_R \subset \Omega \cap B_{2R}$.*

Definition 4.3.5 (Domain convergence for domains with locally Lipschitz boundary). *A sequence of domains Ω_n converges to Ω if the parameters of locally Lipschitz covering are uniformly bounded with respect to n, and*

- *for any compact* $K \subset \Omega$ *there exists* n_0 *such that for any* $n \geq n_0$ $K \subset \Omega_n$.
- *for any compact* $K \subset \mathbb{R}^3$ *one has* $|(\Omega_n \setminus \Omega) \cap K| \to 0$.

It is a staightforward consequence of the above definition that the conclusion of Proposition 4.3.2 remains valid even for this type of converging domains.

The existence of an approximating sequence of bounded smooth domains Ω_n that converge to any open set Ω in the sense of Definition 4.3.1 is then granted by virtue of the following proposition:

Proposition 4.3.6 ([12], Lemma 7.1)**.** *Let* $\Omega \subset \mathbb{R}^3$ *be a non-void open set. Then there exists a sequence of open sets* Ω_n *such that* $\Omega_n \subset \Omega$, $\Omega_n \to \Omega$ *in the sense of Definition 4.3.1, and*

$$\Omega_n = \bigcup_{k=1}^{k(n)} \Omega_{n,k} \text{ with } k(n) \text{ finite}$$

where $\Omega_{n,k}$ *are bounded domains with the boundary of the class* $\mathcal{C}^\infty$ *and such that* $\overline{\Omega}_{n,i} \cap \overline{\Omega}_{n,j}$ *for* $i \neq j$.

4.3.2 Convergence in continuity and linear momentum equations

Having all the necessary estimates, we can pass with $(\rho_n, \mathbf{u}_n, \vartheta_n)$ to its weak limit $(\rho, \mathbf{u}, \vartheta)$. We have to show that the weak limit is a variational solution to the problem on $(0,T) \times \Omega$. This means we have to verify that $(\rho, \mathbf{u}, \vartheta)$ solves the continuity equation in the renormalized sense in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$, the linear momentum equation holds in $\mathcal{D}'((0,T)\times\Omega)$, the entropy inequality is satisfied in $\mathcal{D}'((0,T)\times\overline{\Omega})$, and finally that the total energy inequality holds.

The Div–Curl lemma is a valuable tool and we report it shortly here in the form as it can be found in [14]:

Proposition 4.3.7. *Let* $Q \subset \mathbb{R}^M$ *be a bounded domain. Assumne* $\{\mathbf{U}_n\}_n$ *and* $\{\mathbf{V}_n\}_n$ *are vector fields such that*

$$\mathbf{U}_n \rightharpoonup \mathbf{U} \text{ in } L^p(Q;\mathbb{R}^M), \quad \mathbf{V}_n \rightharpoonup \mathbf{V} \text{ in } L^q(Q;\mathbb{R}^M),$$

where $1 < p, q < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1$. *Furthermore, let* $\{\operatorname{div} \mathbf{U}_n\}, \{\operatorname{curl} \mathbf{V}_n\}$ *be precompact in* $W^{-1,s}(Q), W^{-1,s}(Q;\mathbb{R}^{M\times M})$ *respectively, for certain* $s > 1$.

Then

$$\mathbf{U}_n \cdot \mathbf{V}_n \rightharpoonup \mathbf{U} \cdot \mathbf{V} \text{ in } L^r(Q).$$

The Div–Curl lemma can be applied in the same way as is was shown in [14] to verify that

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3),$$

$$\partial_t \overline{b(\rho)} + \operatorname{div}(\overline{b(\rho)}\mathbf{u}) + \overline{(\rho b'(\rho) - b(\rho))\operatorname{div}\mathbf{u}} = 0 \text{ in } \mathcal{D}'((0,T)\times\Omega)$$

for $b \in BC^1[0,\infty)$, and

$$\partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\overline{p(\rho,\vartheta)} = \operatorname{div}\overline{\mathbb{S}} + \rho(\nabla\Phi + \mathbf{f}) \text{ in } \mathcal{D}'((0,T)\times\Omega),$$

where $\overline{p(\rho,\vartheta)}$ denotes the weak limit of the sequence $p(\rho_n,\vartheta_n)$ and by analogy for $\overline{b(\rho)}$ and other terms. The question whether $\overline{p(\rho,\vartheta)} = p(\rho,\vartheta)$, $\overline{b(\rho)} = b(\rho)$ etc. will be answered (affirmatively) later on.

4.3.3 Convergence of the temperature

We report here a version of the Aubin–Lions lemma (for further details, see Lemma 6.3 by Feireisl [8]).

Proposition 4.3.8. *Let $D \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let $\{v_n\}_n$ be a sequence of functions bounded in*

$$L^2(0,T;L^q(D)) \cap L^\infty(0,T;L^1(D))$$

with $q > \frac{6}{5}$.

Furthermore, assume that

$$\partial_t v_n \geq g_n \text{ in } \mathcal{D}'((0,T)\times D),$$

where the sequence of distributions g_n is bounded in $L^1(0,T;W^{-m,p}(D))$ for certain $m \geq 1, p > 1$.

Then it holds

$$v_n \to v \text{ in } L^2(0,T;W^{-1,2}(D))$$

passing to a subsequence as the case may be.

Applying Proposition 4.3.8 to the variational formulation of the entropy inequality together with the estimates from section 2 yields

$$\rho_n s_G(\rho_n,\vartheta_n) + \frac{4d}{3}\vartheta_n^3 \text{ is relatively compact in } L^2(0,T;W^{-1,2}_{loc}(\Omega)),$$

and

$$\frac{4d}{3}\vartheta_n^3 + \rho_n s_G(\rho_n,\vartheta_n) \to \frac{4d}{3}\overline{\vartheta^3} + \overline{\rho s_G(\rho,\vartheta)} \text{ in } L^2(0,T;W^{-1,2}_{loc}(\Omega)).$$

As ϑ_n is uniformly bounded in $L^2(0,T;W^{1,2}_{loc}(\overline{\Omega}))$, we conclude

$$\frac{4d}{3}\overline{\vartheta^4} + \overline{\rho s_G(\rho,\vartheta)\vartheta} = \frac{4d}{3}\overline{\vartheta^3}\vartheta + \overline{\rho s_G(\rho,\vartheta)}\vartheta \text{ in } L^1_{loc}([0,T]\times\Omega).$$

Finally, the nonlinearity of the radiative part of the entropy, together with the arguments similar to the ones in [14] implies:

$$\vartheta_n \to \vartheta \text{ in } L^4_{loc}([0,T]\times\Omega),$$

where we have used

$$\begin{aligned}&\rho_n s_G(\rho_n,\vartheta_n)\vartheta_n - \rho_n s_G(\rho_n,\vartheta_n)\vartheta \\ &\quad = (\rho_n s_G(\rho_n,\vartheta_n) - \rho_n s_G(\rho_n,\vartheta))(\vartheta_n - \vartheta) + s_G(\rho_n,\vartheta)(\vartheta_n-\vartheta). \quad (4.3.2)\end{aligned}$$

Since $z \mapsto \rho_n s_G(\rho_n, z)$ is monotone, the first term is nonnegative while the second one tends to zero by virtue of the Div–Curl Lemma.

As a direct consequence of the pointwise convergence of the temperature we get $\overline{\mathbb{S}} = \mathbb{S}$.

4.3.4 Convergence of the density revisited

The main aim of this part is to introduce two major results: (a) quantities $\rho, \mathbf{u}$ solve the renormalized continuity equation, and (b) ρ_n converges to ρ strongly in $L^1((0,T)\times\Omega)$.

From the previous parts, we already know that $\rho, \mathbf{u}$ solve the continuity equation in $\mathcal{D}'((0,T)\times\Omega)$. The question whether ρ and $\mathbf{u}$ solve also the renormalized continuity equation can be answered affirmatively (see Corollary 4.1 by Feireisl [8]) under the condition $\rho \in L^2(0,T;L^2_{loc}(\Omega))$. However, this is not satisfied since we merely have $\rho \in L^\infty(0,T;L^{5/3}(\Omega))$. In order to bypass this obstacle, we have to introduce Feireisl's notion of the *oscillations defect measure*

$$\operatorname{osc}_p[\rho_n \to \rho](Q) := \sup_{k\geq 1}\left(\limsup_{n\to\infty}\int_Q |T_k(\rho_n) - T_k(\rho)|^p \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right), \qquad (4.3.3)$$

where T_k are the cut-off functions,

$$T_k(z) = kT\left(\frac{z}{k}\right), k \geq 1, \qquad (4.3.4)$$

with $T \in C^\infty(\mathbb{R})$, $T(-z) = -T(z)$ for all z in $\mathbb{R}$, T concave on $(0,\infty)$, and

$$T(z) = \begin{cases} z, & 0 \leq z \leq 1 \\ 2, & z \geq 3. \end{cases}$$

The following proposition is a modification of Lemma 5.3 in [14] so that it fits our case, it says that if $\rho_n, \mathbf{u}_n$ are solutions of the renormalized continuity equation (4.1.1∗) in $\mathcal{D}'((0,T)\times\Omega)$, and the oscillations defect measure is bounded, then the weak limits $\rho, \mathbf{u}$ solve the renormalized continuity equation as well:

Proposition 4.3.9. *Let $\Omega \subset \mathbb{R}^N, N \geq 2$ be an arbitrary domain. Let $\{\rho_n\}_n$ be a sequence of non-negative functions such that*

$$\begin{aligned} \rho_n &\rightharpoonup \rho \ \textit{in} \ L^1_{loc}(\overline{(0,T)\times\Omega}), \\ \mathbf{u}_n &\rightharpoonup \mathbf{u} \ \textit{in} \ L^1_{loc}(\overline{(0,T)\times\Omega}), \\ \nabla\mathbf{u}_n &\rightharpoonup \nabla\mathbf{u} \ \textit{in} \ L^p_{loc}(\overline{(0,T)\times\Omega}), p>1, \end{aligned}$$

and

$$\mathrm{osc}_q[\rho_n \to \rho](Q) \leq c(Q) \ \textit{for some } q \textit{ such that } \frac{1}{p}+\frac{1}{q}<1$$

for any bounded $Q \subset (0,T)\times\Omega$. Let $\rho_n, \mathbf{u}_n$ solve the renormalized continuity equation on $(0,T)\times\Omega$ in $\mathcal{D}'((0,T)\times\Omega)$. Then $\rho, \mathbf{u}$ is a renormalized continuity equation on $(0,T)\times\Omega$ in $\mathcal{D}'((0,T)\times\Omega)$.

Thus showing boundedness of the oscillations defect measure we obtain the claim (a). In order to yield this we may combine the way shown in the work by Feireisl, Petzeltová and Trivisa [13] (with obvious modifications due to the fact that we need to work only on bounded subdomains of Ω) with the arguments stated by Feireisl, Novotný and Petzeltová in [12] and obtain

$$\mathrm{osc}_q[\rho_n \to \rho](Q) \leq c(Q), \text{ for any } Q \text{ bounded } \subset (0,T)\times\mathbb{R}^3.$$

for certain $q > \frac{8}{3+\alpha}$.

Having claim (a) in mind, we can use it in order to show claim (b). Since $\rho, \mathbf{u}$ solve the renormalized continuity equation in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ and $\rho_n, \mathbf{u}_n$ are solutions to the renormalized continuity equation in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ as well, we can write

$$\begin{aligned} \partial_t L_k(\rho) + \mathrm{div}\,(L_k(\rho)\mathbf{u}) + T_k(\rho)\mathrm{div}\,\mathbf{u} &= 0 \\ \partial_t \overline{L_k(\rho)} + \mathrm{div}\,(\overline{L_k(\rho)}\mathbf{u}) + \overline{T_k(\rho)\mathrm{div}\,\mathbf{u}} &= 0, \end{aligned}$$

where the line over terms in the second equation denotes the terms' weak limits. The functions L_k are defined as solutions to the differential equation $L_k'(z)z - L_k(z) = T_k(z), L_k(0)=0$ and are given by the following formula

$$L_k(z) = z\int_1^z \frac{T_k(s)}{s^2}\,\mathrm{d}s.$$

Subtracking the renormalized equations and testing the result with a function $\varphi \in \mathcal{D}(\mathbb{R}^3)$ yields

$$\int_{\mathbb{R}^3} (\overline{L_k(\rho)} - L_k(\rho))(\tau)\varphi \,\mathrm{d}\mathbf{x} - \int_0^\tau \int_{\mathbb{R}^3} (\overline{L_k(\rho)} - L_k(\rho))\mathbf{u}\cdot\nabla\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^\tau \int_{\mathbb{R}^3} (\overline{T_k(\rho)\mathrm{div}\,\mathbf{u}} - \overline{T_k(\rho)}\mathrm{div}\,\mathbf{u})\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_0^\tau \int_{\mathbb{R}^3} ((T_k(\rho) - \overline{T_k(\rho)})\mathrm{div}\,\mathbf{u})\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{\mathbb{R}^3} (\overline{L_k(\rho)} - L_k(\rho))(0)\varphi \,\mathrm{d}\mathbf{x} \tag{4.3.5}$$

Since we can write

$$(\frac{4}{3}\mu(\vartheta) + \zeta(\vartheta))(\overline{T_k(\rho)\mathrm{div}\,\mathbf{u}} - \overline{T_k(\rho)}\mathrm{div}\,\mathbf{u}) = \overline{p_G(\rho,\vartheta)T_k(\rho)} - \overline{p_G(\rho,\vartheta)}\;\overline{T_k(\rho)},$$

and we have already shown the pointwise convergence of the temperature in $L^1_{loc}(\overline{(0,T)\times\Omega})$, and p_G is non-decreasing in ρ, we have

$$\overline{T_k(\rho)\mathrm{div}\,\mathbf{u}} - \overline{T_k(\rho)}\mathrm{div}\,\mathbf{u} \geq 0 \text{ in } \mathcal{D}'(\overline{(0,T)\times\Omega}).$$

Next, we deal with the first integral on the right-hand side. We use boundedness of $\varphi\mathrm{div}\,\mathbf{u}$ in $L^p(\mathbb{R}^3)$ (because of the compactness of φ's support). To show the term $\overline{T_k(\rho)} - T_k(\rho) \to 0$ in L^{p^*}, we use boundedness of the oscillations defect measure (4.3.3) in some L^q, $q > \frac{8}{3+\alpha} = p^*$ together with the interpolation inequalities and the fact that $T_k(\rho) - \overline{T_k(\rho)} \to 0$ in $L^1(\mathbb{R}^3)$. So, passing with k to infinity yields

$$\int_{\mathbb{R}^3} (\overline{\rho\log\rho} - \rho\log\rho)(\tau)\varphi \,\mathrm{d}\mathbf{x} \leq \int_0^\tau \int_{\mathbb{R}^3} (\overline{\rho\log\rho} - \rho\log\rho)\mathbf{u}\cdot\nabla\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{\mathbb{R}^3} (\overline{\rho\log\rho}(0) - \rho_0\log\rho_0)\varphi \,\mathrm{d}\mathbf{x}$$

Taking a sequence of test functions φ_k such that $|\nabla\varphi_k| \leq \frac{1}{k}$ and $\mathrm{supp}\,\varphi \subset B(0,2k)$ one recovers that

$$|\nabla\varphi_k| \leq C w_{\beta/2}$$

with $\beta = -1$ and w_β given by (4.2.1).

Writing $|\nabla\varphi_k| \leq C w_{-3/8}|\nabla\varphi_k|^{1/4}$ and estimating $\rho_n\log\rho_n$ in L^{p^*} in terms of $\|\rho\|_{L^1(\mathbb{R}^3)\cap L^{5/3}(\mathbb{R}^3)}$ yields the final estimate

$$\int_{\mathbb{R}^3} (\overline{\rho\log\rho} - \rho\log\rho)(\tau)\varphi_k \,\mathrm{d}\mathbf{x} \leq \int_{\mathbb{R}^3} (\overline{\rho\log\rho}(0) - \rho_0\log\rho_0)\varphi_k \,\mathrm{d}\mathbf{x} + C_\tau \frac{1}{k^{1/4}}, \tag{4.3.6}$$

which means that the density converges pointwise provided it did so at the initial time.

4.3.5 The entropy and the total energy inequality

The total energy inequality follows immediately since for any K compact subset of Ω and n large enough so that $K \subset \Omega_n$ one has

$$\begin{aligned}
\int_K \frac{1}{2}\rho_n|\mathbf{u}_n|^2(t) + d\vartheta_n^4(t) + \rho_n e(\rho_n,\vartheta_n)(t)\,\mathrm{d}\mathbf{x} - \int_{B_R\cup(\mathbb{R}^3\setminus B_R)} \frac{1}{2}\rho_n\Phi_n(t)\,\mathrm{d}\mathbf{x} \\
\leq E_n(t) = \int_{\Omega_n} \frac{1}{2}\rho_n|\mathbf{u}_n|^2(t) + d\vartheta_n^4(t) + \rho_n e(\rho_n,\vartheta_n)(t) - \frac{1}{2}\rho_n\Phi_n\,\mathrm{d}\mathbf{x} \\
= E_{0,n} + \int_0^t\int_{\Omega_n} \rho_n\mathbf{f}_n\cdot\mathbf{u}_n\,\mathrm{d}\mathbf{x}\,\mathrm{d}s.
\end{aligned}$$

By the equi-integrability of ρ_n and the estimates of Φ_n we show the integral over $\mathbb{R}^3 \setminus B_R$ is less than arbitrary $\varepsilon > 0$ provided the diameter $R = R(\varepsilon)$ is chosen large enough. Passing with n to infinity yields

$$\begin{aligned}
\int_K \frac{1}{2}\rho|\mathbf{u}|^2(t) + d\vartheta^4(t) + \rho e(\rho,\vartheta)(t)\,\mathrm{d}\mathbf{x} - \int_{B_R} \frac{1}{2}\rho\Phi(t)\,\mathrm{d}\mathbf{x} - \varepsilon \\
\leq E_0 + \int_0^t\int_\Omega \rho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s,
\end{aligned}$$

Taking supremum over all $R > 0$ and K compact subset of Ω finaly verifies the total energy inequality since $\varepsilon > 0$ was arbitrary.

The entropy inequality, however, does not possess so fast approach. As it was already mentioned in concluding remarks of [36], the convergence of the temperature term is deeply related to the convergence of the underlying domain's boundary, as described in Definition 4.3.1. For any solution $(\rho_n, \mathbf{u}_n, \vartheta_n)$ corresponding to the spatial domain Ω_n the weak formulation for the entropy inequality reads:

$$\begin{aligned}
\int_0^T\int_{\Omega_n} \rho_n s(\rho_n,\vartheta_n)\partial_t\varphi + \rho_n s(\rho_n,\vartheta_n)\mathbf{u}_n\cdot\nabla\varphi - \kappa(\vartheta_n)\frac{\nabla\vartheta_n}{\vartheta_n}\cdot\nabla\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
\leq -\int_0^T\int_{\Omega_n}\left(\frac{1}{\vartheta_n}\mathbb{S}_n : \nabla\mathbf{u}_n + \kappa(\vartheta_n)\frac{|\nabla\vartheta_n|^2}{\vartheta_n^2}\right)\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \langle S_{0,n}, \varphi(0,\cdot)\rangle,
\end{aligned} \tag{4.3.7}$$

for all $\varphi \in \mathcal{D}([0,T)\times\mathbb{R}^3)$, $\varphi \geq 0$.

The inequality for the limit problem would read:

$$\int_0^T \int_\Omega \rho s(\rho,\vartheta)\partial_t\varphi + \rho s(\rho,\vartheta)\mathbf{u}\cdot\nabla\varphi - \kappa(\vartheta)\frac{\nabla\vartheta}{\vartheta}\cdot\nabla\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ \leq -\int_0^T \int_\Omega \left(\frac{1}{\vartheta}\mathbb{S}:\nabla\mathbf{u} + \kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}\right)\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \langle S_0, \varphi(0,\cdot)\rangle, \quad (4.3.8)$$

for all $\varphi \in \mathcal{D}([0,T)\times\mathbb{R}^3)$, $\varphi \geq 0$.

For any compact set $K \subset \Omega$ there exists n_0 such that if $n \geq n_0$, then $K \subset \Omega_n$. For this compact set we may apply the results on strong convergence of the density and temperature, along with the weak convergence of $\frac{\nabla\vartheta_n}{\vartheta_n}$ and $\mathbf{u}$ in order to rewrite (4.3.7) to

$$\int_0^T \int_\Omega \rho s(\rho,\vartheta)\partial_t\varphi + \rho s(\rho,\vartheta)\mathbf{u}\cdot\nabla\varphi - \kappa(\vartheta)\frac{\nabla\vartheta}{\vartheta}\cdot\nabla\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ - \int_0^T \int_{\Omega\setminus K} \rho s(\rho,\vartheta)\partial_t\varphi + \rho s(\rho,\vartheta)\mathbf{u}\cdot\nabla\varphi - \kappa(\vartheta)\frac{\nabla\vartheta}{\vartheta}\cdot\nabla\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ + \lim_{n\to\infty}\int_0^T \int_{\mathbb{R}^3\setminus K} 1_{\Omega_n}\left[\rho_n s(\rho_n,\vartheta_n)(\partial_t\varphi + \mathbf{u}_n\cdot\nabla\varphi) - \kappa(\vartheta_n)\frac{\nabla\vartheta_n}{\vartheta_n}\cdot\nabla\varphi\right]\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ \leq -\liminf_{n\to\infty}\left[\int_0^T\int_{\Omega_n}\left(\frac{1}{\vartheta_n}\mathbb{S}_n:\nabla\mathbf{u}_n + \kappa(\vartheta_n)\frac{|\nabla\vartheta_n|^2}{\vartheta_n^2}\right)\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \right. \\ \left. - \langle S_{0,n}, \varphi(0,\cdot)\rangle\right]. \quad (4.3.9)$$

Since $\operatorname{supp}\varphi \subset [0,T)\times\mathbb{R}^3$ is compact, Definition 4.3.1 implies that the measure of $[0,T)\times(\Omega_n\setminus K)\cap\operatorname{supp}\varphi$ and $[0,T)\times(\Omega\setminus K)\cap\operatorname{supp}\varphi$ vanishes as n goes to infinity. Consequently, the L^p estimates on $\rho_n s(\rho_n,\vartheta_n)$ and $\rho_n s(\rho_n,\vartheta_n)\mathbf{u}_n$ imply, by virtue of the Hölder inequality, convergence of these parts. The discussion on the term $\kappa(\vartheta_n)\frac{\nabla\vartheta_n\cdot\nabla\varphi}{\vartheta_n}$, as well as the limit terms in the first integral, is similar. The integral terms on the right-hand side are dealt with in the following way: by Definition 4.3.1, for any $K \subset \Omega$ there exists n_0 such that for any $n \geq n_0$ $K \subset \Omega_n$. So, taking n large enough, the integral is estimated from above by

$$-\int_0^T\int_K\left(\frac{1}{\vartheta+\varepsilon}\mathbb{S}_n:\nabla\mathbf{u}_n + \frac{\kappa(\vartheta_n)}{\vartheta_n^2+\varepsilon(1+\vartheta^3)}|\nabla\vartheta_n|^2\right)\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \text{ for any } \varepsilon > 0.$$

Passing with n to infinity yields finally

$$\int_0^T \int_\Omega \rho s(\rho,\vartheta)\partial_t \varphi + \rho s(\rho,\vartheta)\mathbf{u}\cdot\nabla\varphi - \kappa(\vartheta)\frac{\nabla\vartheta\cdot\nabla\varphi}{\vartheta}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$\leq -\int_0^T \int_K \left(\frac{1}{\vartheta+\varepsilon}\mathbb{S}:\nabla\mathbf{u} + \kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2+\varepsilon(1+\vartheta^3)}\right)\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \langle S_0, \varphi(0,\cdot)\rangle.$$

As ε and K were arbitrary, we have

$$\int_0^T \int_\Omega \rho s(\rho,\vartheta)\partial_t \varphi + \rho s(\rho,\vartheta)\mathbf{u}\cdot\nabla\varphi - \kappa(\vartheta)\frac{\nabla\vartheta\cdot\nabla\varphi}{\vartheta}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$\leq -\int_0^T \int_\Omega \left(\frac{1}{\vartheta}\mathbb{S}:\nabla\mathbf{u} + \kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}\right)\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \langle S_0, \varphi(0,\cdot)\rangle,$$

for any $\varphi \in \mathcal{D}([0,T)\times\mathbb{R}^3)$. In other words, the entropy inequality is verified as well.

4.3.6 Strict positivity of the temperature

Lemma 4.3.10. *Suppose the assumptions of Theorem 4.1.3 hold and let $(\rho,\mathbf{u},\vartheta)$ be the solution to the Navier–Stokes–Fourier system on $(0,T)\times\Omega$ discussed throughout this paper. Then there exists time t_0 such that for $t \in (0,t_0)$ the temperature ϑ is strictly positive.*

Proof. Take weight w_α, $w_\alpha(x) = (1+|x|^2)^\alpha$ and test the entropy inequality. We obtain:

$$\langle S_0, w_\alpha\rangle - S(N)\int_\Omega \rho(t) w_\alpha\,\mathrm{d}\mathbf{x}$$
$$\leq \int_\Omega \frac{4d}{3}\vartheta^3(t)w_\alpha + (NS(1)+c)\vartheta^{3/2}(t)w_\alpha\,\mathrm{d}\mathbf{x} + \int_0^t\int_\Omega \vartheta^{3/2}|\mathbf{u}\sqrt{w_\alpha}|w_\alpha^{1/2-1/2\alpha}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s$$
$$+\int_0^t\int_\Omega \frac{4d}{3}\vartheta^3|\mathbf{u}\sqrt{w_\alpha}|w_\alpha^{1/2-1/2\alpha}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s + \int_0^t\int_\Omega \kappa(\vartheta)|\nabla\log\vartheta|\sqrt{w_\alpha}w_\alpha^{1/2-1/2\alpha}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s$$
$$-\int_0^t\int_\Omega \frac{1}{\vartheta}\mathbb{S}:\nabla\mathbf{u}w_\alpha + \kappa(\vartheta)|\nabla\log\vartheta|^2 w_\alpha\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \quad (4.3.10)$$

In the terms where the integration in time appears, we use the Hölder and the Young inequality to get

$$
\begin{aligned}
&\langle S_0, w_\alpha\rangle - S(N)\int_\Omega \rho(t) w_\alpha \,\mathrm{d}\mathbf{x} \\
&\quad\le \int_\Omega \frac{4d}{3}\vartheta^3(t) w_\alpha + (NS(1)+c)\vartheta^{3/2}(t) w_\alpha \,\mathrm{d}\mathbf{x} + \varepsilon \|\mathbf{u}\sqrt{w_\alpha}\|^2_{L^2(0,t;L^{p^*}(\Omega))} \\
&\quad + C_\varepsilon t^{3/2}\|\vartheta\|^3_{L^\infty(0,t;L^4(\Omega))}\|w_\alpha^{1/2-1/2\alpha}\|^2_{L^{24p/(23p-24)}(\Omega)} + \varepsilon\|\mathbf{u}\sqrt{w_\alpha}\|^2_{L^2(0,t;L^{p^*}(\Omega))} \\
&\quad + C_\varepsilon t^3 \|\vartheta\|^6_{L^\infty(0,t;L^4(\Omega))}\|w_\alpha^{1/2-1/2\alpha}\|^2_{L^{12p/(7p-12)}(\Omega)} + \varepsilon\|\nabla\log\vartheta\sqrt{w_\alpha}\|^2_{L^2(0,t;L^2(\Omega))} \\
&\quad + C_\varepsilon t\|w_\alpha^{1-1/\alpha}\|_{L^1(\Omega)} + C_\varepsilon t^2\|\vartheta\|^6_{L^\infty(0,t;L^4(\Omega))}\|w_\alpha^{1/2-1/2\alpha}\|^2_{L^4(\Omega)} \\
&\quad - \int_0^t\int_\Omega \frac{1}{\vartheta}\mathbb{S}:\nabla\mathbf{u} w_\alpha + \kappa(\vartheta)|\nabla\log\vartheta|^2 w_\alpha \,\mathrm{d}\mathbf{x}\,\mathrm{d}s \qquad (4.3.11)
\end{aligned}
$$

For w_α belonging to the Muckenhoupt class $\mathcal{A}_p$ (in our case it means $\alpha p > -3$) we can use the imbedding of weighted Sobolev spaces and the weighted Korn's inequality in order to write

$$
\int_0^t\int_\Omega \frac{1}{\vartheta}\mathbb{S}:\nabla\mathbf{u} w_\alpha \,\mathrm{d}\mathbf{x}\,\mathrm{d}s \ge \frac{1}{C}\|\mathbf{u}\sqrt{w_\alpha}\|^2_{L^2(0,t;L^{p^*}(\Omega))}.
$$

Moreover, one can use the growth estimates on the heat conductivity coefficient κ to obtain

$$
\begin{aligned}
&\int_0^t\int_\Omega \kappa(\vartheta)|\nabla\log\vartheta|^2 w_\alpha \,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\
&\qquad \ge \underline{\kappa}\|\nabla\log\vartheta\sqrt{w_\alpha}\|^2_{L^2(0,t;L^2(\Omega))} + \underline{\kappa}\|\nabla\vartheta^{3/2}\sqrt{w_\alpha}\|^2_{L^2(0,t;L^2(\Omega))}.
\end{aligned}
$$

We collect the estimates just obtained above, take $\varepsilon > 0, t > 0$ small enough and N large enough in order to obtain

$$
\begin{aligned}
0 < c_0 &\le \int_\Omega \vartheta^3(t) w_\alpha + \vartheta^{3/2}(t) w_\alpha \,\mathrm{d}\mathbf{x} \\
&\le \int_{\Omega\cap\{\vartheta>\delta\}} (\vartheta^3(t) + \vartheta^{3/2}(t)) w_\alpha \,\mathrm{d}\mathbf{x} + (\delta^3 + \delta^{3/2}) w_\alpha(\Omega) \qquad (4.3.12)
\end{aligned}
$$

Consequently, the Hölder inequality yields

$$
\begin{aligned}
&0 < c_0 - w_\alpha(\Omega)(\delta^3 + \delta^{3/2}) \\
&\quad \le (w_\alpha^4[\{\vartheta>\delta\}])^{1/4}\|\vartheta(t)\|^3_{L^4(\Omega)} + (w_\alpha^{8/5}[\{\vartheta>\delta\}])^{5/8}\|\vartheta(t)\|^{3/2}_{L^4(\Omega)} \qquad (4.3.13)
\end{aligned}
$$

for $\delta > 0$ small enough. Therefore for any $t > 0$ small enough there is a set $M(t)$ of positive measure where $\vartheta(t,\cdot)$ is greater that δ and so $\log(\vartheta(t,\cdot))$ is integrable. □

Chapter 5

Existence of a weak solution on an unbounded domain with prescribed nonvanishing density and temperature at infinity

Corresponds to the article by Poul, L.: On dynamics of fluids in meteorology. Accepted for publication in Central European Journal of Mathematics.

Abstract: We consider the full Navier–Stokes–Fourier system of equations on an unbounded domain with prescribed nonvanishing boundary conditions for the density and temperature at infinity. The topic of this article continues author's previous works on existence of the Navier–Stokes–Fourier system on nonsmooth domains. The procedure deeply relies on the techniques developed by Feireisl and others in the series of works on compressible, viscous and heat conducting fluids.
Keywords: unbounded domains, Navier-Stokes-Fourier system, compressible fluid flow, weak solutions.

5.1 Introduction

Many models arising in meteorology can be regarded as a flow of a viscous, compressible and heat conducting fluid in an unbounded spatial domain with prescribed nonvanishing density and temperature 'at infinity'. The fluid flow is governed by a set of physical principles expressed in the way of partial differential equations: The total balance of mass in the system, described in terms of the *continuity equation*

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0; \tag{5.1.1}$$

Newton's second law, saying that the linear momentum is a balanced quantity, captured by the *linear momentum equation*,

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \operatorname{div} \mathbb{S} + \rho \mathbf{f}, \tag{5.1.2}$$

where p denotes the pressure, $\mathbb{S}$ is the so called viscous stress tensor, and $\mathbf{f} = \nabla F$ stands for external forces of potential type; the first law of thermodynamics which says the internal energy e is a balanced quantity, which is equivalent to the *entropy production equation*:

$$\partial_t(\rho s) + \operatorname{div}(\rho s \mathbf{u}) + \operatorname{div} \frac{\mathbf{q}}{\vartheta} = \Sigma, \tag{5.1.3}$$

where s is the specific entropy, $\mathbf{q}$ is the heat flux and Σ stands for the entropy production rate – a nonnegative quantity. *Gibbs' equation*

$$\vartheta D s = D e + p D \frac{1}{\rho}, \tag{5.1.4}$$

implies $\Sigma = \frac{1}{\vartheta} \mathbb{S} : \nabla \mathbf{u} + \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta^2}$, provided the motion is smooth.

We suppose the flow sticks on the boundary and the system is thermally isolated. This yields the boundary conditions on $\partial\Omega$

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{5.1.5}$$

and we require our solutions to satisfy certain boundary conditions 'at infinity', i.e.:

$$\lim_{\substack{|x| \to \infty \\ x \in \Omega}} \mathbf{u}(t,x) = 0, \lim_{\substack{|x| \to \infty \\ x \in \Omega}} \rho(t,x) = \rho_\infty, \lim_{\substack{|x| \to \infty \\ x \in \Omega}} \vartheta(t,x) = \vartheta_\infty \tag{5.1.6}$$

Finaly, the system is supplemented with the initial conditions

$$\rho(0) = \rho_0, \quad (\rho \mathbf{u})(0) = \mathbf{m}_0, \quad \vartheta(0) = \vartheta_0. \tag{5.1.7}$$

The *total energy* of the system is given as

$$E(t) := \int_\Omega \left[\frac{1}{2} \rho |\mathbf{u}|^2 + \rho e(\rho, \vartheta) + (\rho - \rho_\infty) F \right] (t) \, \mathrm{d}\mathbf{x}. \tag{5.1.8}$$

However, in our case the total energy is unbounded since ρ and ϑ are expected to stay far away from zero on a domain of infinite measure. Thus we will omit the classical total energy of the system and instead of it we shall introduce the *Helmholtz-like energy* that is much more convenient for dealing with nonvanishing density and temperature at infinity. Denote

$$H_{\vartheta_\infty}(\rho,\vartheta) := \rho e(\rho,\vartheta) - \vartheta_\infty \rho s(\rho,\vartheta). \tag{5.1.9}$$

Then we consider the total Helmholtz-like energy in the system given by the form

$$\begin{aligned}\mathcal{H}(t) := \int_\Omega \bigg[\frac{1}{2}\rho|\mathbf{u}|^2 + H_{\vartheta_\infty}(\rho,\vartheta) \\ - H_{\vartheta_\infty}(\rho_\infty,\vartheta_\infty) - \frac{\partial H_{\vartheta_\infty}}{\partial \rho}(\rho_\infty,\vartheta_\infty)(\rho-\rho_\infty) + (\rho-\rho_\infty)F\bigg](t)\,\mathrm{d}\mathbf{x}\end{aligned} \tag{5.1.10}$$

Note that, in the bounded-domain case (5.1.10) corresponds with the standard total energy minus entropy of the system minus the energy minus entropy at the state $(\rho_\infty, \vartheta_\infty)$ provided ρ_∞ is the integral mean of ρ, that is:

$$\int_\Omega (\rho - \rho_\infty)\,\mathrm{d}\mathbf{x} = 0.$$

It is easy to see that in the bounded-domain case the integral mean of the density is a preserved quantity as long as ρ and $\mathbf{u}$ lie in a suitable Lebesgue (or Sobolev) space.

This paper is a continuation of a series dedicated to the question of existence of weak solutions of the Navier–Stokes–Fourier system on domains with nonsmooth boundary. It benefits, like the author's previous works [36], [37], and [38] from the existence theory for the system on a bounded domain with boundary of class $C^{2+\nu}, \nu > 0$, developed by Feireisl and others, see e.g. [8], [6], [13]. As a continuation of [38], the paper fills the open gap answering the question about existence of weak solutions on unbounded domains with prescribed nonzero values for density and temperature 'at infinity'.

The paper is organized as follows: In the first section, we present some introductory material concerning the system and state the main result on existence of solutions. In section 2, the estimates necessary for the weak relative compactness are obtained. The concluding section 3 is devoted then to the analysis of the limit system and recovering the main theorem.

5.1.1 Constitutive Assumptions

The pressure is supposed to be composed from the interaction between particles the fluid consists of, and the radiation term due to the temperature. This means

$$p = p_G + p_R, \tag{5.1.11}$$

where the radiation part is given by

$$p_R = p_R(\vartheta) = \frac{1}{3} d\vartheta^4, \tag{5.1.12}$$

and $d > 0$ is the Stefan–Boltzmann constant.

Similarly, the decomposition of the entropy and the internal energy yields

$$s = s_G + s_R,\ s_R(\rho,\vartheta) = \frac{4d}{3}\frac{\vartheta^3}{\rho}, \quad e = e_G + e_R,\ e_R(\rho,\vartheta) = d\frac{\vartheta^4}{\rho}.$$

Furthermore, in a monoatomic gas, there is a relation between pressure, density and energy:

$$p_G = \frac{2}{3}\rho e_G. \tag{5.1.13}$$

Following the analysis by Feireisl and Novotný [11], (5.1.4) and (5.1.13) yield the following formulae for functions p_G and s_G:

$$\left.\begin{array}{rcl} p_G & = & p_G(\rho,\vartheta) = \vartheta^{5/2} P\left(\frac{\rho}{\vartheta^{3/2}}\right), \\ s_G & = & s_G(\rho,\vartheta) = S\left(\frac{\rho}{\vartheta^{3/2}}\right), \end{array}\right\} \tag{5.1.14}$$

where P is a function from $C^1[0,\infty)$ which choice will be restricted later on so that the thermodynamic principles hold. S is related with P through

$$S'(Y) = -\frac{3}{2}Y^{-2}\left(\frac{5}{3}P(Y) - P'(Y)Y\right), \quad Y > 0. \tag{5.1.15}$$

This means S is determined by P up to an additive constant. Throughout this paper the function S is supposed to satisfy the *third law of thermodynamics*

$$\lim_{Y\to\infty} S(Y) = 0, \tag{5.1.16}$$

that is, the entropy vanishes for degenerate states of high density and/or low temperature.

The fluid under consideration is assumed to be Newtonian. This means the viscous stress tensor depends linearly on the velocity's gradient and so it is given by

$$\mathbb{S} = \mu(\vartheta)\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T - \frac{2}{3}\mathrm{div}\,\mathbf{u}\mathbb{I}\right) + \zeta(\vartheta)\mathrm{div}\,\mathbf{u}\mathbb{I}, \tag{5.1.17}$$

where μ and ϑ are viscosity coefficients. The heat flux $\mathbf{q}$ obeys the Fourier law and so

$$\mathbf{q} = -\kappa(\vartheta)\nabla\vartheta, \tag{5.1.18}$$

where κ stands for the heat conductivity coefficient.

5.1.2 Some mathematical tools and notation

We introduce the definition of homogeneous Sobolev spaces $D_0^{1,p}(\Omega)$ as a closure of $\mathcal{D}(\Omega)$ with respect to the norm $\|\cdot\|_{D_0^{1,p}(\Omega)} := \|\nabla\cdot\|_{L^p(\Omega)}$.

We fix the notation on exponents related to the duality in the Lebesgue spaces as well as to the Sobolev imbeddings:

$$p' = \frac{p}{p-1}, \text{ and } \quad p^* = \frac{np}{n-p} \text{ for } 1 \leq p < n,$$

where n denotes the dimension. Throughout this work, we distinguish between different types of convergence by the following notation:

1. $\rightarrow$ means the standard norm-convergence,
2. $\rightharpoonup$ stands for the weak convergence, and
3. $\overset{*}{\rightharpoonup}$ denotes the weak* convergece.

Last but not least, we get around merely local-in-space estimates by virtue of the invading domains lemma:

Proposition 5.1.1 (Invading domains lemma, Lemma 6.6 in [34]). *Let* $\{f_n\}$, $f_n \in L^p(0,T;L^q_{loc}(\mathbb{R}^3))$ *with* $1 < p, q \leq \infty$, *a sequence such that*

$$\|f_n\|_{L^p(0,T;L^q(B_M))} \leq K(M) \textit{ for } M = M_0, M_0+1, M_0+2, \ldots.$$

Then there exists a subsequence $\{n'\} \subset \{n\}$ *such that* $f_{n'} \rightarrow f$ *weakly-∗ in* $L^p(0,T;L^q(B_R))$ *for any* $R > 0$.

5.1.3 Weak solutions

We deal with the problem (5.1.1) – (5.1.3) through the concept of so called *weak solutions*, introduced by Leray [28] for the incompressible case and further developed for the compressible case by P.-L. Lions [29] and Feireisl [8].

We propose the following definition of the weak solution to the Navier–Stokes–Fourier system in a similar way as it was introduced in the works of E. Feireisl and others.

Definition 5.1.2 (Weak solution). *Let* Ω *be an open subset of* $\mathbb{R}^3$ *and let* $\mathbf{u} \in L^2((0,T);D_0^{1,p}(\Omega))$ *for some* $p > 1$, $\rho \in L^\infty((0,T);L^1_{loc}(\overline{\Omega}) \cap L^{5/3}_{loc}(\overline{\Omega}))$, *and* $\vartheta \in L^2((0,T);W^{1,2}_{loc}(\overline{\Omega}))$ *such that* $\log\vartheta \in L^2(0,T;W^{1,2}_{loc}(\overline{\Omega}))$. *We say that the triple* $(\rho, \mathbf{u}, \vartheta)$ *is a weak solution to the Navier–Stokes–Fourier system on the domain* Ω *with nonvanishing boundary conditions at infinity* $\rho = \rho_\infty$, $\vartheta = \vartheta_\infty$, *if the following holds.*

(i) The continuity equation is satisfied in the sense of distributions in the renormalized form, i.e.

$$\partial_t b(\rho)+\operatorname{div}(b(\rho)\mathbf{u})+(b'(\rho)\rho-b(\rho))\operatorname{div}\mathbf{u}=0 \ \textit{in}\ \mathcal{D}'((0,T)\times\mathbb{R}^3 \quad (5.1.1^*)$$

for all $b \in BC^1[0,\infty)$, where we suppose ρ and $\mathbf{u}$ are extended to be zero outside Ω,

(ii) the linear momentum equation (5.1.2) *holds in the sense of distributions, i.e. in $\mathcal{D}'((0,T)\times\Omega)$,*

(iii) the entropy production inequality (5.1.3) *holds in the sense of integral inequalities:*

$$\int_0^T\int_\Omega \rho s\partial_t\varphi+\rho s\mathbf{u}\nabla\varphi+\frac{\mathbf{q}}{\vartheta}\nabla\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \leq \\ -\int_0^T\int_\Omega\left(\frac{1}{\vartheta}\mathbb{S}:\nabla\mathbf{u}+\kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t-\int_\Omega s(0)\varphi(0)\,\mathrm{d}\mathbf{x} \quad (5.1.19)$$

for any $\varphi\geq 0, \varphi\in\mathcal{D}([0,T)\times\mathbb{R}^3)$.

(iv) the total Helmholtz-like energy inequality holds, i.e.

$$\mathcal{H}(t)+\vartheta_\infty\Sigma([0,t]\times\overline{\Omega})\leq\mathcal{H}(0),\quad t\geq 0 \qquad (5.1.20)$$

where $\mathcal{H}$ is defined by (5.1.10)*, and Σ is the entropy production rate given in* (5.1.3)*,*

(v) and the density is non-negative on $(0,T)\times\Omega$.

The definition delineated above deserves several comments: First, the appearance of the renormalized solutions to the continuity equation is much stronger than the 'usual' distributional solution. Indeed, the notion of a renormalized solution in our scope preserves some kind of regularity, since any 'classical' solution is the renormalized one, though not every distributional solution satisfies the renormalized equation. The second point for discussion concerns the violation of the equality sign in the weak formulation of the entropy inequality (5.1.19). This is gap is balanced out by supplementing the Helmholtz-like energy balance (5.1.20).

5.1.4 Assumptions & the main result

Partly due to the requirements of thermodynamics, partly because of technical reasons, we impose assumptions on particular terms in the system of equations:

The thermodynamical function P defining the pressure term is $C^1([0,\infty))$. Moreover,

$$\left.\begin{array}{c} P(0)=0, P'(z)>0 \text{ for all } z\geq 0, \lim_{z\to\infty}\frac{P(z)}{z^{5/3}}=P_\infty>0, \text{ and} \\ 0<\underline{c_v}z\leq \frac{5}{3}P(z)-P'(z)z\leq \overline{c_v}z \text{ for all } z>0. \end{array}\right\} \quad (5.1.21)$$

The viscosity coefficients μ and ζ satisfy:

$$\left.\begin{array}{r} 0<\underline{\mu}(1+\vartheta)\leq\mu(\vartheta), \quad |\mu'(\vartheta)|\leq\overline{\mu} \\ \zeta(\vartheta)\geq 0, \quad \underline{\zeta}\vartheta-1\leq\zeta(\vartheta), \quad |\zeta'(\vartheta)|\leq\overline{\zeta}, \end{array}\right\} \quad (5.1.22)$$

where $\underline{\mu},\overline{\mu},\overline{\zeta},\underline{\zeta}$ are positive constants.

The heat conductivity coefficient κ consists of, similarly to the pressure and entropy terms, the heat conductivity between particles of the fluid, and the heat transfer due to radiation: $\kappa(\rho,\vartheta)=\kappa_G(\vartheta)+\kappa_R(\vartheta)$, where κ_G and κ_R are continuously differentiable functions with growth conditions

$$0<\underline{\kappa}\leq\kappa_G(\vartheta)\leq\overline{\kappa}(1+\vartheta^3), \quad \kappa_R(\vartheta)=\sigma\vartheta^3 \quad (5.1.23)$$

for some positive constants $\underline{\kappa}$, $\overline{\kappa}$, and σ.

Theorem 5.1.3 (Main Result)**.** *Let $\Omega\subset\mathbb{R}^3$ be an unbounded domain of infinite measure with boundary locally Lipschitz continuous. Suppose that the initial state of the fluid is given by the initial density $\rho_0>0$, the initial linear momentum $\mathbf{m}_0$ and the initial temperature $\vartheta_0>0$. Moreover, let the integral average of $\rho_0-\rho_\infty>0$ vanish and suppose $\vartheta_\infty>0$ and $\vartheta_0-\vartheta_\infty\in W^{1,1}\cap W^{1,\infty}(\Omega)$. Futhermore, let $\rho_0,\vartheta_0>0$, and $\rho_0|\mathbf{u}_0|^2\in L^1(\Omega)$. Suppose that $\mathbf{f}\in L^2\cap L^{5/2}(\Omega)$, $\mathbf{m}_0\in L^1\cap L^2(\Omega)$. Then for any time $T>0$ there exists a weak solution to the Navier–Stokes–Fourier system on $(0,T)\times\Omega$. Moreover, there exists a weak solution $(\rho,\mathbf{u},\vartheta)$ on $(0,T)\times\Omega$ such that the following holds:*

- $\rho-\rho_\infty\in L^\infty(0,T;L^2\cap L^\infty(\Omega)+L^1\cap L^{5/3}(\Omega))$, $\rho|\mathbf{u}|^2\in L^\infty(0,T;L^1(\Omega))$, $\vartheta-\vartheta_\infty\in L^\infty(0,T;L^2\cap L^\infty(\Omega)+L^1\cap L^4(\Omega))$;
- $\mathbf{u}\in L^2(0,T;W_0^{1,2}(\Omega))$, $\vartheta-\vartheta_\infty\in L^2(0,T;W^{1,2}(\Omega))$;
- $\mathbb{S}\in L^2(0,T;L^4_{loc}(\overline{\Omega}))$.

Futhermore, the temperature ϑ is strictly positive in the sense that

$$\log\vartheta - \log\vartheta_\infty \in L^2(0,T;W^{1,2}(\Omega)).$$

Remark 5.1.4. The boundary conditions for $\mathbf{u}$ at infinity are formally satisfied since $\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$. Boundary conditions for the density and temperature are satisfied in the sense that $\rho - \rho_\infty \in L^\infty(0,T;L^{5/3} + L^2(\Omega))$ and $\vartheta - \vartheta_\infty \in L^2(0,T;W^{1,2}(\Omega))$.

5.2 Estimates on weak solutions

The basic strategy of constructing a weak solution on a given unbounded domain is to approximate the domain with a sequence of smooth bounded domains and pass to an essential limit with corresponding approximate solutions. Once we succeed in getting estimates on the approximate solutions independently of the size of the domain, we can apply the limiting procedure well known from works by Feireisl (see, e.g, [8]) and obtain the desired triple of functions.

5.2.1 Weak solutions on smooth bounded domains

Consider a bounded domain Ω with $C^{2+\nu}, \nu > 0$, boundary. The existence result by Feireisl, Petzeltová and Trivisa [13] gives us a weak solution $(\rho, \mathbf{u}, \vartheta)$ to the Navier–Stokes–Poisson–Fourier system on Ω. Given a sequence of bounded smooth domains Ω_n, our aim is to obtain estimates on corresponding solutions $(\rho_n, \mathbf{u}_n, \vartheta_n)$, independent of the size of Ω_n.

5.2.2 Estimates from the Helmholtz-like energy balance

In order to simplify the forthcoming formulae, we introduce the following notation: Let $\mathcal{M}_{\rm ess} = \{(\rho,\vartheta) : \rho_\infty/2 \leq \rho \leq 2\rho_\infty, \text{ and } \vartheta_\infty/2 \leq \vartheta 2\vartheta_\infty\}$ denote the "essential set", whereas $\mathcal{M}_{\rm res} = [0,\infty) \times [0,\infty) \setminus \mathcal{M}_{\rm ess}$ denote the set of "residual" values of ρ, ϑ. Similarly, for any function h defined on $(0,T) \times \Omega$ let $[h]_{\rm ess} := 1_{\{(t,x):(\rho(t,x),\vartheta(t,x))\in\mathcal{M}_{\rm ess}\}} h$ and $[h]_{\rm res} := 1_{\{(t,x):(\rho(t,x),\vartheta(t,x))\in\mathcal{M}_{\rm res}\}} h$.

Let us state the following proposition:

Proposition 5.2.1 (Properties of the function H)**.** *Let H be given by* (5.1.9)*, then there exist positive constants $c_i, i = 1,..,4,$ depending solely on $\rho_\infty, \vartheta_\infty$ such that*

(i)

$$c_1(|\rho-\rho_\infty|^2+|\vartheta-\vartheta_\infty|^2) \le H(\rho,\vartheta)-H(\rho_\infty,\vartheta_\infty)-(\rho-\rho_\infty)\frac{\partial H}{\partial \rho}(\rho_\infty,\vartheta_\infty)$$

$$\le c_2(|\rho-\rho_\infty|^2+|\vartheta-\vartheta_\infty|^2) \text{ for all } (\rho,\vartheta)\in\mathcal{M}_{\mathrm{ess}}, \quad (5.2.1)$$

(ii)

$$c_1|\rho-\rho_\infty|^2 \le H(\rho,\vartheta)-H(\rho_\infty,\vartheta_\infty)-(\rho-\rho_\infty)\frac{\partial H}{\partial \rho}(\rho_\infty,\vartheta_\infty)$$

$$\text{for all } \rho_\infty/2 \le \rho \le 2\rho_\infty, \quad (5.2.2)$$

(iii)

$$H(\rho,\vartheta)-H(\rho_\infty,\vartheta_\infty)-(\rho-\rho_\infty)\frac{\partial H}{\partial \rho}(\rho_\infty,\vartheta_\infty)$$

$$\ge \inf_{(r,\Theta)\in\partial\mathcal{M}_{\mathrm{ess}}} \Big\{H(r,\Theta)-H(\rho_\infty,\vartheta_\infty)-(r-\rho_\infty)\frac{\partial H}{\partial \rho}(\rho_\infty,\vartheta_\infty)\Big\} = c_3 > 0, \quad (5.2.3)$$

for all $(\rho,\vartheta)\in\mathcal{M}_{\mathrm{res}}$,

(iv)

$$H(\rho,\vartheta)-H(\rho_\infty,\vartheta_\infty)-(\rho-\rho_\infty)\frac{\partial H}{\partial \rho}(\rho_\infty,\vartheta_\infty) \ge c_4(\rho e(\rho,\vartheta)+\rho|s(\rho,\vartheta)|) \quad (5.2.4)$$

for all $(\rho,\vartheta)\in\mathcal{M}_{\mathrm{res}}$.

Proof. Parts $(i), (iii)$ and (iv) are contained in Lemma 6.1 by Feireisl and Novotný [16]. The proof of part (ii) proceeds with similar arguments since the function $\vartheta\mapsto H(\rho,\vartheta)$ has its strict minimum at ϑ_∞. □

Since $P(Y)Y^{-5/3}$ is decreasing, we immediately obtain $\rho e(\rho,\vartheta) \ge c\rho^{5/3}$. Consequently, the Helmholtz-like energy balance and Proposition 5.2.1 yield

$$\int_\Omega \Big\{\frac{1}{2}\rho|\mathbf{u}|^2 + c_1(|[\rho-\rho_\infty]_{\mathrm{ess}}|^2+|[\vartheta-\vartheta_\infty]_{\mathrm{ess}}|^2)$$

$$+ c_2(|[\rho-\rho_\infty]_{\mathrm{res}}|^{5/3}+|[\vartheta-\vartheta_\infty]_{\mathrm{res}}|^4)\Big\}(t)\,\mathrm{d}\mathbf{x} + \vartheta_\infty\Sigma([0,t)\times\overline{\Omega})$$

$$\le C + \int_\Omega (\rho-\rho_\infty)F\Big\}(t)\,\mathrm{d}\mathbf{x}. \quad (5.2.5)$$

By virtue of the Hölder and Young inequality we obtain

$$\begin{aligned}
&\int_\Omega \Big\{ \frac{1}{2}\rho|\mathbf{u}|^2 + c_1(|\left[\rho-\rho_\infty\right]_{\rm ess}|^2 + |\left[\vartheta-\vartheta_\infty\right]_{\rm ess}|^2) \\
&\qquad + c_2(|\left[\rho-\rho_\infty\right]_{\rm res}|^{5/3} + |\left[\vartheta-\vartheta_\infty\right]_{\rm res}|^4) \Big\}(t)\,{\rm d}\mathbf{x} \\
&\qquad + \vartheta_\infty \int_0^t \int_\Omega \frac{1}{\vartheta}\mathbb{S} : \nabla\mathbf{u} + \kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}\,{\rm d}\mathbf{x}\,{\rm d}s \\
&\quad \le C + \eta\|\left[\rho-\rho_\infty\right]_{\rm ess}\|^2_{L^\infty(0,t;L^2(\Omega))} + C_{\eta,T}\|F\|^2_{L^2(\Omega)} \\
&\qquad + \eta\|\left[\rho-\rho_\infty\right]_{\rm res}\|^{5/3}_{L^\infty(0,t;L^{5/3}(\Omega))} + C_{\eta,T}\|F\|^{5/2}_{L^{5/2}(\Omega)}. \qquad (5.2.6)
\end{aligned}$$

Since $\eta > 0$ can be taken arbitrary small, we conclude with the following bounds:

(i) $\sqrt{\rho}\mathbf{u}$ is bounded in $L^\infty(0,T;L^2(\Omega))$,

(ii) $[\rho-\rho_\infty]_{\rm ess}$, $(\rho-\rho_\infty)1_{\{\rho_\infty/2\le\rho\le 2\rho_\infty\}}$, and $[\vartheta-\vartheta_\infty]_{\rm ess}$ are bounded in $L^\infty(0,T;L^2(\Omega))$,

(iii) $[\rho-\rho_\infty]_{\rm res}$ is bounded in $L^\infty(0,T;L^{5/3}(\Omega))$,

(iv) $[\vartheta-\vartheta_\infty]_{\rm res}$ is bounded in $L^\infty(0,T;L^4(\Omega))$,

(v) $\nabla\mathbf{u}$, $\nabla\vartheta$, $\nabla\vartheta^{3/2}$, and $\frac{\nabla\vartheta}{\vartheta}$ are bounded in $L^2((0,T)\times\Omega)$,

(vi) $[1]_{\rm res}$ is bounded in $L^\infty(0,T;L^1(\Omega))$.

provided $F \in L^2 \cap L^{5/2}(\Omega)$ and

$$H(\rho_0,\vartheta_0) - H(\rho_\infty,\vartheta_\infty) - \frac{\partial H}{\partial\rho}(\rho_\infty,\vartheta_\infty)(\rho_0-\rho_\infty) + \rho_0|\mathbf{u}_0|^2 + (\rho_0-\rho_\infty)F \in L^1(\Omega)$$

independent of the size of Ω.

Note that in view of the last estimate, the measure of $\Omega_{\rm res} := \{(t,x) : (\rho,\vartheta) \in \mathcal{M}_{\rm res}\}$ is bounded independent of the size of Ω. As a direct consequence, one has

$$[\rho-\rho_\infty]_{\rm res} \text{ bounded in } L^\infty(0,T;L^p(\Omega)) \text{ for any } 1 \le p \le 5/3, \qquad (5.2.7)$$

and

$$[\vartheta-\vartheta_\infty]_{\rm res} \text{ bounded in } L^\infty(0,T;L^q(\Omega)) \text{ for any } 1 \le q \le 4. \qquad (5.2.8)$$

Moreover, since

$$\begin{aligned}\int_0^T \int_\Omega |\mathbf{u}|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t &\\ &\le \frac{2}{\rho_\infty} \int_0^T \int_{\Omega\cap\{\rho_\infty/2\le\rho\le2\rho_\infty\}} \rho|\mathbf{u}|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &+ C\int_0^T \int_{\Omega\cap\{\rho<\rho_\infty/2 \text{ or } \rho>2\rho_\infty\}} |\rho-\rho_\infty|^{10/9}|\mathbf{u}|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\le C_T \left(1 + \|[\rho-\rho_\infty]_{\mathrm{res}}\|^{10/9}_{L^\infty(0,T;L^{5/3}(\Omega))} \|\nabla\mathbf{u}\|^2_{L^2(0,T;L^2(\Omega))}\right), \quad (5.2.9)\end{aligned}$$

we conclude $\mathbf{u} \in L^2(0,T;L^2(\Omega))$ and consequently,

$$\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega)). \quad (5.2.10)$$

The estimate for $\vartheta - \vartheta_\infty$ is more straightforward, since $[\vartheta-\vartheta_\infty]_{\mathrm{ess}} \in L^\infty(0,T;L^2(\Omega))$, and $[\vartheta-\vartheta_\infty]_{\mathrm{res}} \in L^\infty(0,T;L^2(\Omega))$ by (5.2.8). Thus

$$\vartheta - \vartheta_\infty \in L^2(0,T;W^{1,2}(\Omega)). \quad (5.2.11)$$

For the logarithm of the temperature we get

$$\log\vartheta - \log\vartheta_\infty \in L^2((0,T;W^{1,2}(\Omega)), \quad (5.2.12)$$

where we have used the following lemma:

Lemma 5.2.2. *Let Ω be a domain in $\mathbb{R}^3$ and consider a set $M \subset \Omega$ of finite measure such that $|M| < |\Omega|$. Then for any $1 < p < \infty$ there exists a constant $C = C(|\Omega|, |K|, p)$ such that*

$$\|f\|_{W^{1,p}(\Omega)} \le C(\|\nabla f\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega\setminus M)}. \quad (5.2.13)$$

provided the right-hand side is finite.

Proof. There exists a covering $\{\Omega_k\}$ of Ω such that $|M| \le 2|\Omega_k|$ for every k and $|\{j : |\Omega_k\cap\Omega_j| \ne \emptyset\}|$ is bounded uniformly for k. Thus we may decompose the L^p norm of f over Ω writing

$$\begin{aligned}\|f\|^p_{L^p(\Omega)} \le \sum_k \|f\|^p_{L^p(\Omega_k)} &\\ &\le \sum_k C\left(\|\nabla f\|^p_{L^p(\Omega_k)} + \|f\|^p_{L^p(\Omega_k\setminus M)}\right) \\ &\le C\left(\|\nabla f\|^p_{L^p(\Omega)} + \|f\|^p_{L^p(\Omega\setminus M)}\right),\end{aligned}$$

where we have used the Poincaré-type inequality from Proposition 9.6 in [16]. □

5.2.3 Estimates on the entropy-related terms

Taking into account the structural assumptions, the entropy term is bounded as

$$\rho|s(\rho,\vartheta)| \leq C(1 + \vartheta^3 + \rho|\log\rho| + \rho|\log\vartheta - \log\vartheta_\infty|). \tag{5.2.14}$$

Clearly, the essential part of the entropy is bounded in L^∞ and so we have

$$\left.\begin{array}{r} [\rho s(\rho,\vartheta)]_{\rm ess} \in L^\infty(0,T;L^\infty(\Omega)) \\ [\rho s(\rho,\vartheta)]_{\rm ess}\mathbf{u} \in L^2(0,T;L^2 \cap L^6(\Omega)) \end{array}\right\} \tag{5.2.15}$$

and for the heat flux term we get

$$\left[\kappa(\vartheta)\frac{\nabla\vartheta}{\vartheta}\right]_{\rm ess} \in L^2(0,T;L^2(\Omega)). \tag{5.2.16}$$

The estimates of the residual part of the entropy require a bit more technical manipulations: Since

$$\Big\|\Big[\frac{\vartheta^3}{\varepsilon^{3/2}}\Big]_{\rm res}\Big\|_{L^\infty(0,T;L^{4/3}(\Omega))} = \Big\|\Big[\frac{\vartheta}{\varepsilon^{1/2}}\Big]_{\rm res}\Big\|^3_{L^\infty(0,T;L^4(\Omega))} \leq C, \tag{5.2.17}$$

$$\Big\|\big[\rho\log\rho\big]_{\rm res}\Big\|_{L^\infty(0,T;L^q(\Omega))} \leq C_q \text{ for } \quad 1 \leq q < 5/3, \tag{5.2.18}$$

and

$$\|[\rho(\log\vartheta - \log\vartheta_\infty)]_{\rm res}\|_{L^2(0,T;L^p_{loc}(\overline{\Omega}))} \leq C_p \text{ for } 1 \leq p \leq 30/23, \tag{5.2.19}$$

we get a bound on the residual part of the entropy in the form

$$\|[\rho s(\rho,\vartheta)]_{\rm res}\|_{L^2(0,T;L^q(\Omega))} \leq C_q \tag{5.2.20}$$

with $1 \leq q \leq 30/23$.

Similarly, we can write

$$\|[\rho(\log\vartheta - \log\vartheta_\infty)\mathbf{u}]_{\rm res}\|_{L^2(0,T;L^{30/29}(\Omega)} \leq C, \tag{5.2.21}$$

and

$$\Big\|\big[\kappa(\vartheta)\frac{\nabla\vartheta}{\vartheta}\big]_{\rm res}\Big\|_{L^2(0,T;L^{8/7}(\Omega)} \leq C, \tag{5.2.22}$$

where we have used the structural assumptions on κ to write

$$\kappa(\vartheta)\frac{|\nabla\vartheta|}{\vartheta} \leq \overline{\kappa}\left(\frac{|\nabla\vartheta|}{\vartheta} + \vartheta^{3/2}|\nabla\vartheta^{3/2}|\right).$$

5.2.4 Refined pressure estimates

Next, we concentrate on bounds on the pressure term in $L^p_{loc}(\overline{\Omega})$ for certain $p > 1$. This is done through bounds on the density in $L^{5/3+\nu}_{loc}(\overline{\Omega})$ for certain $\nu > 0$ and the procedure relies on the method known from the works by Feireisl and others, see e.g. [9], [13] etc. The method is based on using a special test function on the linear momentum equation based on the so called *Bogovskii operator*. The Bogovskii operator has a meaning of an 'inverse' to the divergence operator; it was first introduced in the paper by Bogovskii [3] and can be characterized, for example, in terms of the following statement:

Proposition 5.2.3 (Bogovskii operator ([34], section 3.3)). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Then there exists a linear operator $\mathcal{B}_\Omega = (\mathcal{B}^1_\Omega, \dots, \mathcal{B}^N_\Omega)$ with the following properties:*

(i) $\mathcal{B}_\Omega : L^p(\Omega) \to W^{1,p}_0(\Omega), 1 < p < \infty,$

(ii) $\operatorname{div} \mathcal{B}_\Omega[f] = f - \frac{1}{|\Omega|}\int_\Omega f \,\mathrm{d}\mathbf{x}$ *a.e. in* $\Omega, f \in L^p(\Omega),$

(iii) $\|\nabla \mathcal{B}_\Omega[f]\|_{L^p(\Omega)} \le c(p,\Omega)\|f - \frac{1}{|\Omega|}\int_\Omega f\|_{L^p(\Omega)}, 1 < p < \infty,$

(iv) if $f = \operatorname{div}\mathbf{g}$, *where* $\mathbf{g}, \operatorname{div}\mathbf{g} \in L^q(\Omega), \mathbf{g}\cdot\mathbf{n}|_{\partial\Omega} = 0$ *for some* $1 < q < \infty$, *then* $\|\mathcal{B}_\Omega[f]\|_{L^q(\Omega)} \le c(q,\Omega)\|\mathbf{g}\|_{L^q(\Omega)}$.

Let Ω_R be a bounded domain with Lipschitz continuous boundary. The result on local-in-space integrability of the density function is given through testing the momentum equation (5.1.2) with a function

$$\varphi(t,x) := \psi(t)\mathcal{B}_{\Omega_R}[\eta\rho(t,\cdot)](x)$$

where $\psi \in D(0,T)$, $\eta \in \mathcal{D}(\mathbb{R}^3)$ and the Bogovskii solution is considered to be extended by zero outside Ω_R.

$$\begin{aligned}
\int_0^T \int_{\Omega_R} \psi p(\rho,\vartheta)\rho^\nu \eta \,\mathrm{d}\mathbf{x}\,\mathrm{d}t &= \int_0^T \int_{\Omega_R} \psi p(\rho,\vartheta)\left(\frac{1}{|\Omega_R|}\int_{\Omega_R} \eta\rho^\nu \,\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&- \int_0^T \int_{\Omega_R} \partial_t\psi\rho\mathbf{u}\cdot\mathcal{B}_{\Omega_R}[\eta\rho^\nu] - \psi\rho\mathbf{u}\cdot\mathcal{B}_{\Omega_R}[\operatorname{div}(\eta\rho^\nu)] + \\
&\quad + \psi\rho\mathbf{u}\cdot\mathcal{B}_{\Omega_R}[\rho^\nu\mathbf{u}\cdot\nabla\eta]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+ \int_0^T \int_{\Omega_R} (\nu-1)\psi\rho\mathbf{u}\cdot\mathcal{B}_{\Omega_R}[\eta\rho^\nu \operatorname{div}\mathbf{u}] - \rho(\mathbf{u}\otimes\mathbf{u}) : \nabla\mathcal{B}_{\Omega_R}[\eta\rho^\nu]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+ \int_0^T \int_{\Omega_R} \psi\mathbb{S} : \nabla\mathcal{B}_{\Omega_R}[\eta\rho^\nu] - \psi\rho\mathbf{f}\cdot\mathcal{B}_{\Omega_R}[\eta\rho^\nu]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t
\end{aligned}$$

This will yield bounds on the density in $L^q((0,T)\times\Omega_R)$ for some $q>5/3$ provided we succeed in estimating the terms on the left-hand side. Clearly, $\mathcal{B}_{\Omega_R}[\eta\rho^\nu]\in L^\infty(0,T;W_0^{1,\frac{5}{3\nu}}(\Omega_R))$, $\mathcal{B}_{\Omega_R}[\rho^\nu\mathbf{u}\cdot\nabla\eta]\in L^2(0,T;W_0^{1,\frac{15p}{15+9\nu p-5p}}(\Omega_R))$, $\mathcal{B}_{\Omega_R}[\eta\rho^\nu\operatorname{div}\mathbf{u}]\in L^2(0,T;W_0^{1,\frac{5p}{5+3\nu p}}(\Omega_R))$. In view of estimates on $\rho\mathbf{u}$, $\rho\mathbf{u}\otimes\mathbf{u}$, and $\mathbb{S}$, we conclude that for $\nu>0$ small enough the right-hand side is bounded. The structural assumptions (5.1.21) yield

$$P_\infty\rho^{5/3}\le p_G(\rho,\vartheta)\le P(1)(\rho^{5/3}+\vartheta^{5/2}),$$

which implies

$$\rho\in L^{5/3+\nu}((0,T)\times\Omega_R),\quad p_G\in L^q((0,T)\times\Omega_R) \tag{5.2.23}$$

for some $q>1$.

5.3 The Limit

In the previous parts, the apriori estimates independent of the size of the domain (or local in the domain) have been recovered. The second step is the construction of a solution to the system $(5.1.1-5.1.3)$ on a given unbounded domain. As the reader could expect, the theory of weak solutions on bounded domains will be used to construct an approximating sequence.

5.3.1 Approximation scheme for domains

There are several different approaches to the technique of the domain approximation that differ in the smoothness of the target (approximated) domain. In this paper, we propose the concept of unbounded domains with locally Lipschitz boundary as described in the following definition.

Definition 5.3.1 (A domain with locally Lipschitz boundary)**.** *Let Ω be an opet set in $\mathbb{R}^N$. We say that the boundary $\partial\Omega$ is locally Lipschitz if there exists $R_0>0$ such that for any $R\ge R_0$ there exists a domain Ω_R enjoying the following properties:*

1. *Ω_R is a domain with Lipschitz continuous boundary, and*
2. *$\Omega\cap B_R\subset\Omega_R\subset\Omega\cap B_{2R}$.*

Definition 5.3.2 (Domain convergence for domains with locally Lipschitz boundary)**.** *A sequence of domains Ω_n converges to Ω if the parameters of locally Lipschitz covering are uniformly bounded with respect to n, and*

- *for any compact $K \subset \Omega$ there exists n_0 such that for any $n \geq n_0$ $K \subset \Omega_n$.*
- *for any compact $K \subset \mathbb{R}^3$ one has $|(\Omega_n \setminus \Omega) \cap K| \to 0$.*

One of the most important parts the concept of domain convergence should capture is the convergence of the velocity: provided the velocity field $\mathbf{u}_n$ vanishes on the boundary of each Ω_n, we would like to see the trace of $\mathbf{u}$ on $\partial\Omega$ be zero as well. In view of Definition 5.3.2 and the trace and extension theorems (see e.g. Stein [40]), one can state the following proposition:

Proposition 5.3.3. *Let $\Omega_n \to \Omega$ in the sense of Definition 5.3.2. Moverover, let $v_n \in D_0^{1,p}(\Omega_n) \subset D^{1,p}(\mathbb{R}^3)$ be an arbitrary sequence such that $v_n \rightharpoonup v$ in $D^{1,p}(\mathbb{R}^3)$. Then $v \in D_0^{1,p}(\Omega)$.*

The construction of the approximating sequence is straightforward: We take the Lipschitz subdomains $\Omega_R \subset \Omega$ which existence follows from the definition, and mollify the graph of the boundary so that the mollifying parameter tends to zero for large R. Moreover, we suppose that the initial conditions are approximated so that

- $(\rho_{0,n} - \rho_\infty) \to (\rho_0 - \rho_\infty)$ in $L^1(\Omega) \cap L^2(\Omega)$, and
$$\int_{\mathbb{R}^3} (\rho_{0,n} - \rho_\infty)\,\mathrm{d}\mathbf{x} = 0.$$
- $1_{\Omega_n}(\vartheta_{0,n} - \vartheta_\infty) \to 1_\Omega(\vartheta_0 - \vartheta_\infty)$ in $L^3 \cap L^\infty(\mathbb{R}^3)$,
- $1_{\Omega_n}(\log\vartheta_{0,n} - \log\vartheta_\infty) \to 1_\Omega(\log\vartheta_{0,n} - \log\vartheta_\infty)$ in $L^2(\mathbb{R}^3)$,
- $\mathbf{m}_{0,n} \to \mathbf{m}_0$ in $L^1 \cap L^2(\Omega)$.

5.3.2 Convergence in the continuity and the linear momentum equation

Having all the necessary estimates, we can pass with $(\rho_n, \mathbf{u}_n, \vartheta_n)$ to its weak limit $(\rho, \mathbf{u}, \vartheta)$. We have to show that the weak limit is a variational solution to the problem on $(0,T) \times \Omega$. This means we have to verify that $(\rho, \mathbf{u}, \vartheta)$ solves the continuity equation in the renormalized sense in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$, the linear momentum equation holds in $\mathcal{D}'((0,T)\times\Omega)$, the entropy inequality holds in $\mathcal{D}'([0,T) \times \overline{\Omega})$, and the total energy inequality holds.

The Div–Curl lemma, discovered by Murat [32] and Tartar [42] is a valuable tool and we report it shortly here in the form as it can be found in [14]:

Proposition 5.3.4. *Let $Q \subset \mathbb{R}^M$ be a bounded domain. Assumne $\{\mathbf{U}_n\}_n$ and $\{\mathbf{V}_n\}_n$ are vector fields such that*

$$\mathbf{U}_n \rightharpoonup \mathbf{U} \text{ in } L^p(Q;\mathbb{R}^M), \quad \mathbf{V}_n \rightharpoonup \mathbf{V} \text{ in } L^q(Q;\mathbb{R}^M),$$

where $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1$. Furthermore, let $\{\mathrm{div}\, \mathbf{U}_n\}, \{\mathrm{curl}\, \mathbf{V}_n\}$ be precompact in $W^{-1,s}(Q), W^{-1,s}(Q;\mathbb{R}^{M\times M})$ respectively, for certain $s > 1$.

Then

$$\mathbf{U}_n \cdot \mathbf{V}_n \rightharpoonup \mathbf{U} \cdot \mathbf{V} \text{ in } L^r(Q).$$

The Div–Curl lemma can be applied in the same way as is was shown in [14] to verify that

$$\partial_t \rho + \mathrm{div}\, (\rho \mathbf{u}) = 0 \text{ in } \mathcal{D}'((0,T) \times \mathbb{R}^3),$$

$$\partial_t \overline{b(\rho)} + \mathrm{div}\, (\overline{b(\rho)}\mathbf{u}) + \overline{(\rho b'(\rho) - b(\rho))\mathrm{div}\, \mathbf{u}} = 0 \text{ in } \mathcal{D}'((0,T) \times \Omega)$$

for $b \in BC^1[0,\infty)$, and

$$\partial_t(\rho\mathbf{u}) + \mathrm{div}\, (\rho\mathbf{u} \otimes \mathbf{u}) + \nabla \overline{p(\rho,\vartheta)} = \mathrm{div}\, \overline{\mathbb{S}} + \rho \mathbf{f} \text{ in } \mathcal{D}'((0,T) \times \Omega),$$

where the line over a particular term denotes the weak limit.

5.3.3 Convergence of the temperature

We report here a version of the Aubin–Lions lemma (for further details, see Lemma 6.3 by Feireisl [8]).

Proposition 5.3.5. *Let $D \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let $\{v_n\}_n$ be a sequence of functions bounded in*

$$L^2(0,T;L^q(D)) \cap L^\infty(0,T;L^1(D))$$

with $q > \frac{6}{5}$.

Furthermore, assume that

$$\partial_t v_n \geq g_n \text{ in } \mathcal{D}'((0,T) \times D),$$

where the sequence of distributions g_n is bounded in $L^1(0,T;W^{-m,p}(D))$ for certain $m \geq 1, p > 1$.

Then it holds

$$v_n \to v \text{ in } L^2(0,T;W^{-1,2}(D))$$

passing to a subsequence as the case may be.

Applying Proposition 5.3.5 to the variational formulation of the entropy inequality together with the estimates from section 2 yields

$$\rho_n s_G(\rho_n, \vartheta_n) + \frac{4d}{3}\vartheta_n^3 \text{ is relatively compact in } L^2(0,T; W^{-1,2}_{loc}(\Omega)),$$

and

$$\frac{4d}{3}\vartheta_n^3 + \rho_n s_G(\rho_n, \vartheta_n) \to \frac{4d}{3}\overline{\vartheta^3} + \overline{\rho s_G(\rho,\vartheta)} \text{ in } L^2(0,T; W^{-1,2}_{loc}(\Omega)).$$

As ϑ_n is uniformly bounded in $L^2(0,T; W^{1,2}_{loc}(\overline{\Omega}))$, we conclude

$$\frac{4d}{3}\overline{\vartheta^4} + \overline{\rho s_G(\rho,\vartheta)\vartheta} = \frac{4d}{3}\overline{\vartheta^3}\vartheta + \overline{\rho s_G(\rho,\vartheta)}\vartheta \text{ in } L^1_{loc}([0,T]\times\Omega).$$

Finally, the nonlinearity of the radiative part of the entropy, together with the arguments similar to the ones in [14] implies:

$$\vartheta_n \to \vartheta \text{ in } L^4_{loc}([0,T]\times\Omega), \tag{5.3.1}$$

where we have used

$$\begin{aligned}\rho_n s_G(\rho_n, \vartheta_n)\vartheta_n &- \rho_n s_G(\rho_n, \vartheta_n)\vartheta \\ &= (\rho_n s_G(\rho_n,\vartheta_n) - \rho_n s_G(\rho_n,\vartheta))(\vartheta_n - \vartheta) + s_G(\rho_n,\vartheta)(\vartheta_n - \vartheta).\end{aligned} \tag{5.3.2}$$

Since $z \mapsto \rho_n s_G(\rho_n, z)$ is monotone, the first term is nonnegative while the second one tends to zero by virtue of the Div–Curl Lemma.

Note that, by virtue of (5.3.1) we get $\overline{\mathbb{S}} = \mathbb{S}$ in the limit version of the linear momentum equation.

5.3.4 Convergence of the density revisited

The main aim of this part is to introduce two major results: (a) quantities $\rho, \mathbf{u}$ solve the renormalized continuity equation, and (b) ρ_n converges to ρ strongly in $L^1_{loc}([0,T]\times\Omega)$.

From the previous parts, we already know that $\rho, \mathbf{u}$ solve the continuity equation in $\mathcal{D}'((0,T)\times\Omega)$. The question whether ρ and $\mathbf{u}$ solve also the renormalized continuity equation can be answered affirmatively (see Corollary 4.1 by Feireisl [8]) under the condition $\rho \in L^2(0,T; L^2_{loc}(\Omega))$. However, this is not satisfied since we merely have $\rho \in L^\infty(0,T; L^{5/3}(\Omega))$. In order to bypass this obstacle, we have to introduce a notion of the so called *oscillations defect measure*

$$\operatorname{osc}_p[\rho_n \to \rho](Q) := \sup_{k\ge 1}\left(\limsup_{n\to\infty}\int_Q |T_k(\rho_n) - T_k(\rho)|^p \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right), \tag{5.3.3}$$

where T_k are the cut-off functions,

$$T_k(z) = kT\left(\frac{z}{k}\right), k \geq 1, \tag{5.3.4}$$

with $T \in C^\infty(\mathbb{R})$, $T(-z) = -T(z)$ for all z in $\mathbb{R}$, T concave on $(0, \infty)$, and

$$T(z) = \begin{cases} z, & 0 \leq z \leq 1 \\ 2, & z \geq 3. \end{cases}$$

The following proposition is a modification of Lemma 5.3 in [14] so that it fits our case. It says, if $\rho_n, \mathbf{u}_n$ are solutions of the renormalized continuity equation (5.1.1$*$) in $\mathcal{D}'((0,T) \times \Omega)$, and the oscillations defect measure is bounded, then the weak limits $\rho, \mathbf{u}$ solve the renormalized continuity equation as well:

Proposition 5.3.6. *Let $\Omega \subset \mathbb{R}^N, N \geq 2$ be an arbitrary domain. Let $\{\rho_n\}_n$ be a sequence of non-negative functions such that*

$$\rho_n \rightharpoonup \rho \text{ in } L^1_{loc}(\overline{(0,T) \times \Omega}),$$
$$\mathbf{u}_n \rightharpoonup \mathbf{u} \text{ in } L^1_{loc}(\overline{(0,T) \times \Omega}),$$
$$\nabla \mathbf{u}_n \rightharpoonup \nabla \mathbf{u} \text{ in } L^2(\overline{(0,T) \times \Omega}),$$

and

$$\operatorname{osc}_q[\rho_n \to \rho](Q) \leq c(Q) \text{ for some } q > 2$$

for any bounded $Q \subset (0,T) \times \Omega$. Let $\rho_n, \mathbf{u}_n$ solve the renormalized continuity equation on $(0,T) \times \Omega$ in $\mathcal{D}'((0,T) \times \Omega)$. Then $\rho, \mathbf{u}$ is a renormalized continuity equation on $(0,T) \times \Omega$ in $\mathcal{D}'((0,T) \times \Omega)$.

Thus showing boundedness of the oscillations defect measure we obtain the claim (a). In order to get this we may combine the way shown in the work by Feireisl, Petzeltová and Trivisa [13] (with obvious modifications due to the fact that we need to work only on bounded subdomains of Ω) with the arguments stated by Feireisl, Novotný and Petzeltová in [12] and obtain

$$\operatorname{osc}_q[\rho_n \to \rho](Q) \leq c(Q), \text{ for any } Q \text{ bounded } \subset (0,T) \times \mathbb{R}^3.$$

for certain $q > 2$.

Having claim (a) in mind, we can use it in order to show claim (b). Since $\rho, \mathbf{u}$ solve the renormalized continuity equation in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$ and $\rho_n, \mathbf{u}_n$ are solutions to the renormalized continuity equation in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$ as well, we can write

$$\partial_t L_k(\rho) + \operatorname{div}(L_k(\rho)\mathbf{u}) + T_k(\rho)\operatorname{div}\mathbf{u} = 0$$
$$\partial_t \overline{L_k(\rho)} + \operatorname{div}(\overline{L_k(\rho)}\mathbf{u}) + \overline{T_k(\rho)\operatorname{div}\mathbf{u}} = 0,$$

where the line over terms in the second equations denotes the weak limits. The functions L_k are defined as solutions to the differential equation $L_k'(z)z - L_k(z) = T_k(z), L_k(0) = 0$ and are given by the following formula

$$L_k(z) = z\int_1^z \frac{T_k(s)}{s^2}\,\mathrm{d}s.$$

Subtracking the renormalized equations and testing the result with a function $\varphi \in \mathcal{D}(\mathbb{R}^3)$ yields

$$\begin{aligned}\int_{\mathbb{R}^3}(\overline{L_k(\rho)} - L_k(\rho))(\tau)\varphi\,\mathrm{d}\mathbf{x} - \int_0^\tau\int_{\mathbb{R}^3}(\overline{L_k(\rho)} - L_k(\rho))\mathbf{u}\cdot\nabla\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ + \int_0^\tau\int_{\mathbb{R}^3}(\overline{T_k(\rho)\mathrm{div}\,\mathbf{u}} - \overline{T_k(\rho)}\mathrm{div}\,\mathbf{u})\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ = \int_0^\tau\int_{\mathbb{R}^3}((T_k(\rho) - \overline{T_k(\rho)})\mathrm{div}\,\mathbf{u})\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{\mathbb{R}^3}(\overline{L_k(\rho)} - L_k(\rho))(0)\varphi\,\mathrm{d}\mathbf{x}\end{aligned} \tag{5.3.5}$$

Since we can write

$$(\frac{4}{3}\mu(\vartheta) + \zeta(\vartheta))(\overline{T_k(\rho)\mathrm{div}\,\mathbf{u}} - \overline{T_k(\rho)}\mathrm{div}\,\mathbf{u}) = \overline{p_G(\rho,\vartheta)T_k(\rho)} - \overline{p_G(\rho,\vartheta)}\;\overline{T_k(\rho)},$$

and we have already shown the pointwise convergence of the temperature in $L^1_{loc}(\overline{(0,T)\times\Omega})$, and p_G is non-decreasing in ρ, we have

$$\overline{T_k(\rho)\mathrm{div}\,\mathbf{u}} - \overline{T_k(\rho)}\mathrm{div}\,\mathbf{u} \geq 0 \text{ in } \mathcal{D}'(\overline{(0,T)\times\Omega}).$$

Next, we deal with the first integral on the right-hand side of (5.3.5). We use boundedness of $\varphi\mathrm{div}\,\mathbf{u}$ in $L^2(\mathbb{R}^3)$ to show the term $\overline{T_k(\rho)} - T_k(\rho) \to 0$ in L^2, we use boundedness of the oscillations defect measure (5.3.3) in L^q, for certain $q > 2$ together with the interpolation inequalities and the fact that $T_k(\rho) - \overline{T_k(\rho)} \to 0$ in $L^1_{loc}(\mathbb{R}^3)$. So, passing with k to infinity yields

$$\begin{aligned}&\int_{\mathbb{R}^3}(\overline{\rho\log\rho} - \rho\log\rho)(\tau)\varphi\,\mathrm{d}\mathbf{x} \\ &\leq \int_0^\tau\int_{\mathbb{R}^3}(\overline{\rho\log\rho} - \rho\log\rho)\mathbf{u}\cdot\nabla\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{\mathbb{R}^3}(\overline{\rho\log\rho}(0) - \rho_0\log\rho_0)\varphi\,\mathrm{d}\mathbf{x}\end{aligned}$$

Taking a sequence of test functions φ_k such that $|\nabla\varphi_k| \leq \frac{1}{k}$ and $\mathrm{supp}\,\varphi \subset$

$B(0,2k)$, we can write

$$\int_{\mathbb{R}^3} (\overline{\rho\log\rho} - \rho\log\rho)(\tau)\varphi_k \,\mathrm{d}\mathbf{x}$$
$$\leq \frac{1}{k}\int_0^\tau \int_{\mathbb{R}^3} |\overline{\rho\log\rho} - \rho_\infty\log\rho_\infty - \rho\log\rho + \rho_\infty\log\rho_\infty||\mathbf{u}| \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$+ \int_{\mathbb{R}^3} (\overline{\rho\log\rho}(0) - \rho_0\log\rho_0)\varphi_k \,\mathrm{d}\mathbf{x}$$

Decomposing the integral to the residual set of finite measure where ρ belongs to $L^{5/3}$ and the essential set where we know the estimate of $\rho - \rho_\infty$ in L^2 yields

$$\int_{\mathbb{R}^3} (\overline{\rho\log\rho} - \rho\log\rho)(\tau)\varphi_k \,\mathrm{d}\mathbf{x}$$
$$\leq \frac{1}{k}C_\tau \left(\|[\rho-\rho_\infty]_{\mathrm{ess}}\|_{L^\infty(0,T;L^2(\Omega))} + \|[\rho]_{\mathrm{res}}\|_{L^\infty(0,T;L^{5/3}(\Omega))}\right)$$
$$\times \|\mathbf{u}\|_{L^2(0,T;W^{1,2}(\Omega))}$$
$$+ \int_{\mathbb{R}^3} (\overline{\rho\log\rho}(0) - \rho_0\log\rho_0)\varphi_k \,\mathrm{d}\mathbf{x}.$$

And so we get the final estimate

$$\int_{\mathbb{R}^3} (\overline{\rho\log\rho} - \rho\log\rho)(\tau)\varphi_k \,\mathrm{d}\mathbf{x}$$
$$\leq \int_{\mathbb{R}^3} (\overline{\rho\log\rho}(0) - \rho_0\log\rho_0)\varphi_k \,\mathrm{d}\mathbf{x} + C_T\frac{1}{k}, 0 \leq \tau \leq T \quad (5.3.6)$$

which means that the density converges pointwise provided it did so at the initial time.

5.3.5 The entropy and the total energy inequality

The Helmholtz-like total energy inequality follows immediately since for any K compact subset of Ω and n large enough so that $K \subset \Omega_n$ one has

$$\int_K \frac{1}{2}\rho_n|\mathbf{u}_n|^2 + H(\rho_n,\vartheta_n) - H(\rho_\infty,\vartheta_\infty) - \frac{\partial H}{\partial\rho}(\rho_\infty,\vartheta_\infty)(\rho_n-\rho_\infty)\,\mathrm{d}\mathbf{x}$$
$$\leq \mathcal{H}_n(t) = \int_{\Omega_n} \frac{1}{2}\rho_n|\mathbf{u}_n|^2 + H(\rho,\vartheta) - H(\rho_\infty,\vartheta_\infty) - \frac{\partial H}{\partial\rho}(\rho_\infty,\vartheta_\infty)(\rho-\rho_\infty)\,\mathrm{d}\mathbf{x}$$
$$\leq \mathcal{H}_{0,n} - \vartheta_\infty\Sigma_n([0,t]\times K) + \int_0^t \int_{\Omega_n} \rho_n\mathbf{f}_n\cdot\mathbf{u}_n \,\mathrm{d}\mathbf{x}\,\mathrm{d}s.$$

Passing with n to infinity yields

$$\int_K \frac{1}{2}\rho|\mathbf{u}|^2(t) + H(\rho,\vartheta) - H(\rho_\infty,\vartheta_\infty) - \frac{\partial H}{\partial \rho}(\rho_\infty,\vartheta_\infty)(\rho-\rho_\infty)\,\mathrm{d}\mathbf{x} + \vartheta_\infty \Sigma([0,t]\times K) \leq \mathcal{H}_{0,n} + \int_0^t \int_\Omega \rho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s.$$

Thus taking supremum over all compact subsets of Ω finaly verifies the Helmholtz-like total energy inequality.

The entropy inequality does not possess so fast approach. For any solution $(\rho_n, \mathbf{u}_n, \vartheta_n)$ corresponding to the spatial domain Ω_n the weak formulation for the entropy inequality reads:

$$\int_0^T \int_{\Omega_n} \rho_n s(\rho_n,\vartheta_n)\partial_t\varphi + \rho_n s(\rho_n,\vartheta_n)\mathbf{u}_n\cdot\nabla\varphi - \kappa(\vartheta_n)\frac{\nabla\vartheta_n}{\vartheta_n}\cdot\nabla\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \leq -\int_0^T \int_{\Omega_n} \left(\frac{1}{\vartheta_n}\mathbb{S}_n : \nabla\mathbf{u}_n + \kappa(\vartheta_n)\frac{|\nabla\vartheta_n|^2}{\vartheta_n^2}\right)\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \langle S_{0,n},\varphi(0,\cdot)\rangle, \tag{5.3.7}$$

for all $\varphi \in \mathcal{D}([0,T)\times\mathbb{R}^3)$, $\varphi \geq 0$.

The inequality for the limit problem would read:

$$\int_0^T \int_{\Omega} \rho s(\rho,\vartheta)\partial_t\varphi + \rho s(\rho,\vartheta)\mathbf{u}\cdot\nabla\varphi - \kappa(\vartheta)\frac{\nabla\vartheta}{\vartheta}\cdot\nabla\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \leq -\int_0^T \int_{\Omega} \left(\frac{1}{\vartheta}\mathbb{S} : \nabla\mathbf{u} + \kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}\right)\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \langle S_0,\varphi(0,\cdot)\rangle, \tag{5.3.8}$$

for all $\varphi \in \mathcal{D}([0,T)\times\mathbb{R}^3)$, $\varphi \geq 0$.

For any compact set $K \subset \Omega$ there exists n_0 such that if $n \geq n_0$, then $K \subset \Omega_n$. For this compact set we may apply the results on strong convergence of the density and temperature, along with the weak convergence of $\frac{\nabla\vartheta_n}{\vartheta_n}$ and

$\mathbf{u}_n$ in order to rewrite (5.3.7) to

$$\int_0^T \int_\Omega \rho s(\rho,\vartheta)\partial_t\varphi + \rho s(\rho,\vartheta)\mathbf{u}\cdot\nabla\varphi - \kappa(\vartheta)\frac{\nabla\vartheta}{\vartheta}\cdot\nabla\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$- \int_0^T \int_{\Omega\setminus K} \rho s(\rho,\vartheta)\partial_t\varphi + \rho s(\rho,\vartheta)\mathbf{u}\cdot\nabla\varphi - \kappa(\vartheta)\frac{\nabla\vartheta}{\vartheta}\cdot\nabla\varphi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$+ \lim_{n\to\infty}\int_0^T \int_{\mathbb{R}^3\setminus K} 1_{\Omega_n}\left[\rho_n s(\rho_n,\vartheta_n)(\partial_t\varphi + \mathbf{u}_n\cdot\nabla\varphi) - \kappa(\vartheta_n)\frac{\nabla\vartheta_n}{\vartheta_n}\cdot\nabla\varphi\right]\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$\leq -\liminf_{n\to\infty}\left[\int_0^T\int_{\Omega_n}\left(\frac{1}{\vartheta_n}\mathbb{S}_n : \nabla\mathbf{u}_n + \kappa(\vartheta_n)\frac{|\nabla\vartheta_n|^2}{\vartheta_n^2}\right)\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \langle S_{0,n},\varphi(0,\cdot)\rangle\right]. \quad (5.3.9)$$

Since $\operatorname{supp}\varphi \subset [0,T)\times\mathbb{R}^3$ is compact, Definition 5.3.2 implies that the measure of $[0,T)\times(\Omega_n\setminus K)\cap\operatorname{supp}\varphi$ vanishes as n goes to infinity, while $[0,T)\times(\Omega\setminus K)\cap\operatorname{supp}\varphi$ can be made arbitrary small. Consequently, the L^p estimates on $\rho_n s(\rho_n,\vartheta_n)$ and $\rho_n s(\rho_n,\vartheta_n)\mathbf{u}_n$ imply, by virtue of the Hölder inequality, convergence of these parts. The discussion on the term $\kappa(\vartheta_n)\frac{\nabla\vartheta_n\cdot\nabla\varphi}{\vartheta_n}$, as well as the limit terms in the first integral, is similar. The integral terms on the right-hand side are dealt with in the following way: by Definition 5.3.2, for any compact $K\subset\Omega$ there exists n_0 such that for any $n\geq n_0$ $K\subset\Omega_n$. So, taking n large enough, the integral is estimated from above by

$$-\int_0^T\int_K\left(\frac{1}{\vartheta+\varepsilon}\mathbb{S}_n : \nabla\mathbf{u}_n + \frac{\kappa(\vartheta_n)}{\vartheta_n^2+\varepsilon(1+\vartheta^3)}|\nabla\vartheta_n|^2\right)\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \text{ for any } \varepsilon>0.$$

Passing with n to infinity yields finally

$$\int_0^T \int_\Omega \rho s(\rho,\vartheta)\partial_t\varphi + \rho s(\rho,\vartheta)\mathbf{u}\cdot\nabla\varphi - \kappa(\vartheta)\frac{\nabla\vartheta\cdot\nabla\varphi}{\vartheta}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$\leq -\int_0^T\int_K\left(\frac{1}{\vartheta+\varepsilon}\mathbb{S} : \nabla\mathbf{u} + \kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2+\varepsilon(1+\vartheta^3)}\right)\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \langle S_0,\varphi(0,\cdot)\rangle.$$

As ε and K were arbitrary, we have

$$\int_0^T \int_\Omega \rho s(\rho,\vartheta)\partial_t\varphi + \rho s(\rho,\vartheta)\mathbf{u}\cdot\nabla\varphi - \kappa(\vartheta)\frac{\nabla\vartheta\cdot\nabla\varphi}{\vartheta}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$\leq -\int_0^T\int_\Omega\left(\frac{1}{\vartheta}\mathbb{S} : \nabla\mathbf{u} + \kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}\right)\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \langle S_0,\varphi(0,\cdot)\rangle,$$

for any $\varphi \in \mathcal{D}([0, T) \times \mathbb{R}^3)$. In other words, the entropy inequality is verified as well.

Acknowledgement The author wishes to express his gratitude to Prof. E. Fereisl for his support in terms of Premium Academiae without which this paper would hardly be conceivable.

Chapter 6

Low Mach number limit for a viscous compressible fluid

Corresponds to the article by Feireisl, E., Poul, L.: Low Mach number limit in large domains. Submitted to Mathematical Methods in Applied Sciences.

6.1 Introduction

Many problems in continuum fluid mechanics are considered on *unbounded* spatial domains, in particular on the whole space R^3. Although it seems intuitively clear that any observable physical space is necessarily bounded, the concept of unbounded domain offers a useful approximation when the influence of the boundary on the motion is negligible. For instance, the presence of *acoustic waves* is usually neglected in meteorological models, where the underlying physical domain is large and the speed of sound dominates the characteristic speed of the fluid (see Klein [24]). Under these circumstances, a relevant mathematical description can be obtained through a suitable scaling of the primitive equations typically represented by the complete Navier-Stokes-Fourier system.

We examine the situation when the characteristic velocity of the fluid $\mathbf{u}_{\text{char}} = \varepsilon$, the characteristic time $t_{\text{char}} = 1/\varepsilon$ as well as the characteristic viscosity $\mu_{\text{char}} = \varepsilon$ are given in terms of a small parameter $\varepsilon > 0$. The motion of the fluid is governed by the standard Navier-Stokes-Fourier system in the dimensionless form:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \tag{6.1.1}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p = \mathrm{div}_x\mathbb{S}, \tag{6.1.2}$$

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s\mathbf{u}) + \mathrm{div}_x\Big(\frac{\mathbf{q}}{\vartheta}\Big) = \sigma, \tag{6.1.3}$$

where $\varrho = \varrho(t,x)$ denotes the density, $\mathbf{u} = \mathbf{u}(t,x)$ is the velocity field, and $\vartheta = \vartheta(t,x)$ is the absolute temperature. The pressure $p = p(\varrho,\vartheta)$ and the specific entropy $s = s(\varrho,\vartheta)$ are given functions of the state variables ϱ, ϑ. The symbol $\mathbb{S}$ denotes the viscous stress tensor assumed to satisfy the standard Newton's rheological law

$$\mathbb{S} = \mu\Big(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\Big), \tag{6.1.4}$$

while $\mathbf{q}$ denotes the heat flux obeying Fourier's law

$$\mathbf{q} = -\kappa\nabla_x\vartheta. \tag{6.1.5}$$

Finally, the entropy production σ satisfies

$$\sigma = \frac{1}{\vartheta}\Big(\varepsilon^2\mathbb{S} : \nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\Big). \tag{6.1.6}$$

The singular coefficient in the pressure term in (6.1.3) corresponds to the Mach number proportional to ε (Klein et al. [25]).

System (6.1.1 - 6.1.6) will be considered on a spatial domain Ω_ε large enough in order to eliminate the effect of the boundary on propagation of the acoustic waves. Seeing that the speed of sound in (6.1.1 - 6.1.6) is proportional to $1/\varepsilon$ we shall assume that the family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ enjoys the following property:

Property (L)

For any $x \in R^3$, there is $\varepsilon_0 = \varepsilon_0(x)$ such that $x \in \Omega_\varepsilon$ for all $0 < \varepsilon < \varepsilon_0$. Moreover, there exists a function h, $\lim_{z\to\infty} h(z)/z = \infty$ such that

$$\mathrm{dist}[x, \partial\Omega_\varepsilon] > h(1/\varepsilon) \text{ for all } 0 < \varepsilon < \varepsilon_0. \tag{6.1.7}$$

In addition to (6.1.7), we suppose that the initial distribution of the density and the temperature are close to a spatially homogeneous state. More specifically,

$$\varrho(0,\cdot) = \overline{\varrho} + \varepsilon\varrho^{(1)}_{0,\varepsilon}, \tag{6.1.8}$$

$$\vartheta(0,\cdot) = \overline{\vartheta} + \varepsilon\vartheta^{(1)}_{0,\varepsilon}, \tag{6.1.9}$$

where $\overline{\varrho}$, $\overline{\vartheta}$ are positive constants and

$$\int_{\Omega_\varepsilon} \left(|\varrho_{0,\varepsilon}^{(1)}|^2 + |\vartheta_{0,\varepsilon}^{(1)}|^2 \right) \mathrm{d}x \leq c \tag{6.1.10}$$

uniformly for $\varepsilon \to 0$.

Consider a family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ of (weak) solutions to problem (6.1.1 - 6.1.6) on a compact time interval $(0, T)$ emanating from the initial state satisfying (6.1.8), (6.1.9), (6.1.10). The main goal of the present paper is to show that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } L^2(0,T;L^2(B;R^3)) \text{ for any bounded ball } B \subset R^3, \tag{6.1.11}$$

at least for a suitable subsequence $\varepsilon \to 0$, where the limit velocity field complies with the standard incompressibility constraint

$$\mathrm{div}_x \mathbf{u} = 0. \tag{6.1.12}$$

As already pointed out, the result should be independent of the behavior of $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ "far away" from the set B, in particular we do not impose any specific boundary conditions. On the other hand, certain restrictions have to be made in order to prevent the energy to be "pumped" into the system at infinity. Specifically, the following hypotheses specified below are required:

- The total mass of the fluid contained in Ω_ε is a constant of motion.
- The system dissipates energy, specifically, the total energy of the fluid contained in Ω_ε is non-increasing in time.
- The system produces entropy, in particular, the total entropy of the system is non-decreasing in time.

Apart from the general stipulations stated above, we assume that the quantities $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ solve (6.1.1 - 6.1.5) in the sense of distributions while (6.1.6) is replaced by an inequality

$$\sigma \geq \frac{1}{\vartheta}\Big(\varepsilon^2 \mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\Big) \tag{6.1.13}$$

in the spirit of the existence theory developed in [14].

Our technique is based on uniform estimates of the family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ resulting from the dissipation inequality deduced in a similar way as in [18] (see Section 6.2). The time evolution of the acoustic waves is governed by a wave equation (acoustic equation) derived in Section 6.3. At this stage, the finite speed of propagation of the waves is used in order to reduce the problem to a bounded spatial domain (Section 6.4). Finally, we use the dispersive estimates for the acoustic equation in order to obtain the desired conclusion stated in (6.1.11) (see Section 6.5). The paper is concluded by a rigorous formulation of the main result stated in Section 6.6.

A similar problem for the Navier-Stokes system in the isentropic regime posed on the whole space R^3 was addressed by Desjardins and Grenier [5]. In contrast to their work, the acoustic equation for the complete system contains the contribution of "thermal" waves including the entropy production rate σ being merely a positive measure. In order to handle this additional difficulty, a regularization and "time lifting" technique is used in combination with the standard L^1-dispersive estimates for the acoustic equation (see Section 6.5).

6.2 Uniform estimates

6.2.1 Estimates based on the hypothesis of thermodynamics stability

In accordance with the principle of thermodynamics stability, we shall assume that

$$\frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho,\vartheta)}{\partial \vartheta} > 0, \tag{6.2.1}$$

where $e = e(\varrho,\vartheta)$ is the specific energy interrelated to p and s through Gibbs' equation

$$\vartheta D s(\varrho,\vartheta) = De(\varrho,\vartheta) + p(\varrho,\vartheta) D\Big(\frac{1}{\varrho}\Big). \tag{6.2.2}$$

The former condition in (6.2.1) asserts that the compressibility of the fluid is always positive while the latter says that the specific heat at constant volume is positive (see Gallavotti [20]).

In accordance with the general principles delineated in the previous section, we shall assume that the total mass is a conserved quantity, specifically,

$$\int_{\Omega_\varepsilon} \Big(\varrho_\varepsilon(t,\cdot) - \overline{\varrho}\Big) \ \mathrm{d}x = 0 \text{ for a.a. } t \in (0,T), \tag{6.2.3}$$

in particular, we have to take

$$\int_{\Omega_\varepsilon} \varrho_{0,\varepsilon}^{(1)} \ \mathrm{d}x = 0 \tag{6.2.4}$$

in (6.1.8).

Similarly, the total energy is a non-decreasing function in time, meaning

$$\int_{\Omega_\varepsilon} \Big[\frac{\varepsilon^2}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2(t) + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon)(t) - \frac{\varepsilon^2}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2(0) - \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon)(0) \Big] \ \mathrm{d}x \leq 0 \tag{6.2.5}$$

while the entropy is being produced:

$$\int_{\Omega_\varepsilon} \Big[\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)(t) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)(0) \Big] \ \mathrm{d}x = \sigma_\varepsilon[[0,t] \times \overline{\Omega}_\varepsilon] \tag{6.2.6}$$

for a.a. $t \in (0,T)$, where the entropy production rate σ_ε is a non-negative measure satisfying

$$\sigma_\varepsilon \geq \frac{1}{\vartheta_\varepsilon} \Big(\varepsilon^2 \frac{\mu}{2} \Big| \nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3} \mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I} \Big|^2 + \frac{\kappa |\nabla_x \vartheta_\varepsilon|^2}{\vartheta_\varepsilon} \Big). \tag{6.2.7}$$

Combining (6.2.3) with (6.2.5), (6.2.6) we get, first formally, the so-called dissipation inequality

$$\int_{\Omega_\varepsilon} \Big[\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2} \Big(H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})(\varrho_\varepsilon - \overline{\varrho}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \Big) \Big](t) \ \mathrm{d}x \tag{6.2.8}$$

$$+ \frac{\overline{\vartheta}}{\varepsilon^2} \sigma_\varepsilon[[0,t] \times \overline{\Omega}_\varepsilon]$$

$$\leq \int_{\Omega_\varepsilon} \Big[\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2} \Big(H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})(\varrho_\varepsilon - \overline{\varrho}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \Big) \Big](0) \ \mathrm{d}x$$

for a.a. $t \in [0,T]$, where we have introduced

$$H_{\overline{\vartheta}}(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \overline{\vartheta} \varrho s(\varrho, \vartheta). \tag{6.2.9}$$

Since, by virtue of Gibbs' relation (6.2.2),

$$\frac{\partial^2 H_{\overline{\vartheta}}(\varrho, \overline{\vartheta})}{\partial \varrho^2} = \frac{1}{\varrho} \frac{\partial p(\varrho, \overline{\vartheta})}{\partial \varrho}, \ \ \frac{\partial H_{\overline{\vartheta}}(\varrho, \vartheta)}{\partial \vartheta} = \frac{\varrho}{\vartheta} (\vartheta - \overline{\vartheta}) \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta},$$

the thermodynamics stability hypothesis (6.2.1) implies that

$$\varrho \mapsto H_{\overline{\vartheta}}(\varrho, \overline{\vartheta}) \text{ is strictly convex on } (0, \infty),$$

and

$$\vartheta \mapsto H_{\overline{\vartheta}}(\varrho, \vartheta) \text{ is decreasing for } \vartheta < \overline{\vartheta} \text{ and increasing for } \vartheta > \overline{\vartheta}.$$

Introducing the essential and residual set of values as follows

$$\mathcal{M}_{\rm ess} = \{(\varrho,\vartheta) \mid \overline{\varrho}/2 < \varrho < 2\overline{\varrho},\ \overline{\vartheta}/2 < \vartheta < 2\overline{\vartheta}\},\ \mathcal{M}_{\rm res} = [0,\infty)^2 \setminus \mathcal{M}_{\rm ess}$$

we report the following estimates (see Lemma 2.1 in [17]):

$$c_1\Big(|\varrho-\overline{\varrho}|^2 + |\vartheta-\overline{\vartheta}|^2\Big) \tag{6.2.10}$$

$$\leq H_{\overline{\vartheta}}(\varrho,\vartheta) - (\varrho-\overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})}{\partial\varrho} - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})$$

$$\leq c_2\Big(|\varrho-\overline{\varrho}|^2 + |\vartheta-\overline{\vartheta}|^2\Big) \text{ for all } (\varrho,\vartheta)\in\mathcal{M}_{\rm ess},$$

$$H_{\overline{\vartheta}}(\varrho,\vartheta) - (\varrho-\overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})}{\partial\varrho} - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta}) \tag{6.2.11}$$

$$\geq \inf_{(r,\Theta)\in\partial\mathcal{M}_{\rm ess}} \Big\{H_{\overline{\vartheta}}(r,\Theta) - (r-\overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})}{\partial\varrho} - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})\Big\} > 0 \text{ for all } (\varrho,\vartheta)\in\mathcal{M}_{\rm res},$$

and

$$H_{\overline{\vartheta}}(\varrho,\vartheta) - (\varrho-\overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})}{\partial\varrho} - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta}) \tag{6.2.12}$$

$$\geq c\Big(\varrho e(\varrho,\vartheta) + \varrho|s(\varrho,\vartheta)|\Big) \text{ for all } (\varrho,\vartheta)\in\mathcal{M}_{\rm res}.$$

It follows from (6.2.10) that the integral on the right-hand side of the dissipation inequality (6.2.8) is bounded uniformly with respect to $\varepsilon\to 0$ as soon as the initial data satisfy (6.1.8), (6.1.9), together with

$$\mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon}, \tag{6.2.13}$$

where

$$\|\varrho_{0,\varepsilon}^{(1)}\|_{L^1\cap L^\infty(\Omega_\varepsilon)} + \|\vartheta_{0,\varepsilon}^{(1)}\|_{L^1\cap L^\infty(\Omega_\varepsilon)} + \|\mathbf{u}_{0,\varepsilon}\|_{L^2\cap L^\infty(\Omega_\varepsilon;R^3)} \leq c, \tag{6.2.14}$$

with c independent of ε.

Thus relation (6.2.8), together with the structural properties of the function $H_{\overline{\vartheta}}$ listed in (6.2.10 - 6.2.12), can be used to deduce uniform bounds on the family $\{\varrho_\varepsilon,\mathbf{u}_\varepsilon,\vartheta_\varepsilon\}_{\varepsilon>0}$. To this end, it is convenient to associate to a family $\{h_\varepsilon\}_{\varepsilon>0}$ its *essential* and *residual* part as follows:

$$[h_\varepsilon]_{\rm ess} = h_\varepsilon 1_{\{(t,x)\ \mid\ (\varrho_\varepsilon,\vartheta_\varepsilon)(t,x)\in\mathcal{M}_{\rm ess}\}},\ [h_\varepsilon]_{\rm res} = h_\varepsilon 1_{\{(t,x)\ \mid\ (\varrho_\varepsilon,\vartheta_\varepsilon)(t,x)\in\mathcal{M}_{\rm res}\}}.$$

The dissipation inequality (6.2.8) gives rise to the following uniform estimates:

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon;R^3)} \leq c, \tag{6.2.15}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right]_{\rm ess} \right\|_{L^2(\Omega_\varepsilon)} \leq c, \tag{6.2.16}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\right]_{\rm ess} \right\|_{L^2(\Omega_\varepsilon)} \leq c, \tag{6.2.17}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \| \, [\varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\rm res} \|_{L^1(\Omega_\varepsilon)} \leq \varepsilon^2 c, \tag{6.2.18}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \| \, [\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)]_{\rm res} \|_{L^1(\Omega_\varepsilon)} \leq \varepsilon^2 c, \tag{6.2.19}$$

and

$$\|\sigma_\varepsilon\|_{\mathcal{M}^+([0,T]\times\overline{\Omega}_\varepsilon} \leq \varepsilon^2 c. \tag{6.2.20}$$

Moreover, the measure of the "residual" set is small, specifically,

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|[1]_{\rm res}\|_{L^1(\Omega_\varepsilon)} \leq \varepsilon^2 c. \tag{6.2.21}$$

Finally, combining (6.2.7), (6.2.20) we conclude that

$$\int_0^T \int_{\Omega_\varepsilon} \frac{\mu}{\vartheta} \left| \nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3} {\rm div}_x \mathbf{u}_\varepsilon \mathbb{I} \right|^2 \, {\rm d}x \ \, {\rm d}t \leq c, \tag{6.2.22}$$

and

$$\int_0^T \int_{\Omega_\varepsilon} \frac{\kappa}{\vartheta^2} |\nabla_x \vartheta_\varepsilon|^2 \, {\rm d}x \ \, {\rm d}t \leq \varepsilon^2 c. \tag{6.2.23}$$

Note that all bounds established in (6.2.15 - 6.2.23) have been established assuming only the thermodynamics stability hypothesis (6.2.1), the uniform bound on the data (6.2.14), and the general physical principles (6.2.2), (6.2.3), (6.2.5), and (6.2.6). In particular, these bounds are independent of the specific form of the constitutive relations.

6.2.2 Estimates based on constitutive relations

Unlike the uniform bounds established in the previous part, the following estimates are derived under certain restrictions imposed on the material properties of the fluid. The purpose of these estimates is to control the residual part of the quantities appearing in the acoustic equation introduced in Section 6.3 below. Note that all restrictions introduced here are technical and by no means optimal.

Motivated by the existence theory developed in [14], we consider the state equation for the pressure in the form

$$p(\varrho,\vartheta) = \underbrace{p_M(\varrho,\vartheta)}_{\text{molecular pressure}} + \underbrace{p_R(\vartheta)}_{\text{radiation pressure}},\ p_M = \vartheta^{\frac{5}{2}} P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big),\ p_R = \frac{a}{3}\vartheta^4,\ a > 0, \tag{6.2.24}$$

while the integral energy reads

$$e(\varrho,\vartheta) = e_M(\varrho,\vartheta) + e_R(\varrho,\vartheta),\ e_M = \frac{3}{2}\frac{\vartheta^{\frac{5}{2}}}{\varrho} P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big),\ e_R = a\frac{\vartheta^4}{\varrho}, \tag{6.2.25}$$

and, in accordance with Gibbs' relation (6.2.2),

$$s(\varrho,\vartheta) = s_M(\varrho,\vartheta) + s_R(\varrho,\vartheta),\ s_M(\varrho,\vartheta) = S\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big),\ s_R = \frac{4}{3}a\frac{\vartheta^3}{\varrho}, \tag{6.2.26}$$

where

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} \text{ for all } Z > 0. \tag{6.2.27}$$

The thermodynamics stability hypothesis (6.2.1) reformulated in terms of the structural properties of P reads

$$P \in C^1[0,\infty) \cap C^2(0,\infty),\ P(0) = 0,\ P'(Z) > 0 \text{ for all } Z \geq 0, \tag{6.2.28}$$

$$0 < \frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} \leq \sup_{z>0} \frac{\frac{5}{3}P(z) - zP'(z)}{z} < \infty. \tag{6.2.29}$$

Furthermore, it follows from (6.2.29) that $P(Z)/Z^{5/3}$ is a decreasing function of Z, and we assume that

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0. \tag{6.2.30}$$

The transport coefficients μ and κ are continuously differentiable functions of the temperature ϑ satisfying the growth restrictions

$$\left\{ \begin{array}{r} 0 < \underline{\mu}(1+\vartheta) \leq \mu(\vartheta) \leq \overline{\mu}(1+\vartheta), \\ \\ 0 < \underline{\kappa}(1+\vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1+\vartheta^3) \text{ for all } \vartheta \geq 0, \end{array} \right\} \tag{6.2.31}$$

where $\underline{\mu}$, $\overline{\mu}$, $\underline{\kappa}$, and $\overline{\kappa}$ are positive constants.

By virtue of (6.2.31), the uniform estimate (6.2.22) yields

$$\int_0^T \| \nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3}\mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I} \|^2_{L^2(\Omega_\varepsilon;R^{3\times 3})} \ \mathrm{d}t \leq c, \tag{6.2.32}$$

with c independent of $\varepsilon \to 0$.

In order to get more information, we need the following version of Korn's inequality proved in [13, Proposition 6.1].

Proposition 6.2.1. *Let $\Omega \subset R^3$ be a bounded Lipschitz domain. Let $r \geq 0$ be a function such that*

$$0 < m = \int_\Omega r \ \mathrm{d}x, \ \int_\Omega r^\gamma \ \mathrm{d}x < K \text{ for a certain } \gamma > 6/5.$$

Then

$$\|\mathbf{v}\|^2_{W^{1,2}(\Omega;R^3)} \leq c(m,k,\Omega)\Big(\| \nabla_x \mathbf{v} + \nabla_x^t \mathbf{v} - \frac{2}{3}\mathrm{div}_x \mathbf{v} \mathbb{I} \|^2_{L^2(\Omega_\varepsilon;R^{3\times 3})} + \int_\Omega r|\mathbf{v}|^2 \ \mathrm{d}x \Big)$$

for any $\mathbf{v} \in W^{1,2}(\Omega;R^3)$.

Taking $r = [\varrho_\varepsilon]_{\mathrm{ess}}$, $\mathbf{v} = \mathbf{u}_\varepsilon$ we can cover the domains Ω_ε by a finite number of cubes and apply Proposition 6.2.1 in order to conclude that

$$\int_0^T \|\mathbf{u}_\varepsilon\|^2_{W^{1,2}(\Omega_\varepsilon;R^3)} \ \mathrm{d}t \leq c \text{ uniformly for } \varepsilon \to 0, \tag{6.2.33}$$

where we have used the uniform estimates (6.2.15), (6.2.32), together with the "smallness" of the residual set established in (6.2.21).

Similarly, we can use estimates (6.2.17), (6.2.23) in order to obtain

$$\int_0^T \|\vartheta_\varepsilon - \overline{\vartheta}\|^2_{W^{1,2}(\Omega_\varepsilon)} \ \mathrm{d}t + \int_0^T \| \log(\vartheta_\varepsilon) - \log(\overline{\vartheta})\|^2_{W^{1,2}(\Omega_\varepsilon)} \ \mathrm{d}t \leq \varepsilon^2 c. \tag{6.2.34}$$

Finally, a combination of (6.2.18), (6.2.30) yields

$$\text{ess} \sup_{t\in(0,T)} \int_{\Omega_\varepsilon} [\varrho_\varepsilon]_{\text{res}}^{5/3} \, \mathrm{d}x \le \varepsilon^2 c. \tag{6.2.35}$$

6.3 Acoustic equation

Acoustic equation is a wave equation governing the time evolution of the acoustic waves. It can be viewed as a linearization of system (6.1.1 - 6.1.3) around the static state $\{\overline{\varrho}, 0, \overline{\vartheta}\}$. If $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ satisfy (6.1.1 - 6.1.3) in the sense of distributions, we get

$$\int_0^T \int_{\Omega_\varepsilon} \Big[\varepsilon\Big(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big)\partial_t\varphi + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \varphi\Big] \, \mathrm{d}x \ \mathrm{d}t = 0 \tag{6.3.1}$$

for any test function $\varphi \in \mathcal{D}((0,T) \times \Omega_\varepsilon)$;

$$\int_0^T \int_{\Omega_\varepsilon} \varepsilon\varrho_\varepsilon\Big(\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big)\partial_t\varphi \, \mathrm{d}x \ \mathrm{d}t \tag{6.3.2}$$

$$= \int_0^T \int_{\Omega_\varepsilon} \varepsilon\varrho_\varepsilon\Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big)\mathbf{u}_\varepsilon \cdot \nabla_x\varphi \, \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^T \int_{\Omega_\varepsilon} \frac{\kappa\nabla_x\vartheta_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x\varphi \, \mathrm{d}x \ \mathrm{d}t - < \sigma_\varepsilon, \varphi >$$

for any test function $\varphi \in \mathcal{D}((0,T) \times \Omega_\varepsilon)$; and

$$\int_0^T \int_{\Omega_\varepsilon} \Big[\varepsilon(\varrho_\varepsilon\mathbf{u}_\varepsilon)\cdot\partial_t\varphi + \Big(\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big)\mathrm{div}_x\varphi\Big] \, \mathrm{d}x \ \mathrm{d}t \tag{6.3.3}$$

$$= \int_0^T \int_{\Omega_\varepsilon} \varepsilon\Big(\mathbb{S}_\varepsilon - \varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon\Big) : \nabla_x\varphi \, \mathrm{d}x \ \mathrm{d}t$$

for any test function $\varphi \in \mathcal{D}((0,T) \times \Omega_\varepsilon; R^3)$.

Thus, after a simple manipulation,

$$\int_0^T \int_{\Omega_\varepsilon} \Big[\varepsilon r_\varepsilon \partial_t\varphi + A(\varrho_\varepsilon\mathbf{u}_\varepsilon)\cdot\nabla_x\varphi\Big] \, \mathrm{d}x \ \mathrm{d}t \tag{6.3.4}$$

$$= B\int_0^T \int_{\Omega_\varepsilon} \varepsilon\varrho_\varepsilon\Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big)\mathbf{u}_\varepsilon \cdot \nabla_x\varphi \, \mathrm{d}x \ \mathrm{d}t$$

$$+B\int_0^T\int_{\Omega_\varepsilon}\frac{\kappa\nabla_x\vartheta_\varepsilon}{\vartheta_\varepsilon}\cdot\nabla_x\varphi\ \mathrm{d}x\ \ \mathrm{d}t-B<\sigma_\varepsilon,\varphi>$$

for all $\varphi\in\mathcal{D}((0,T)\times\Omega_\varepsilon)$, and

$$\int_0^T\int_{\Omega_\varepsilon}\Big[\varepsilon(\varrho_\varepsilon\mathbf{u}_\varepsilon)\cdot\partial_t\varphi+r_\varepsilon\mathrm{div}_x\varphi\Big]\ \mathrm{d}x\ \ \mathrm{d}t \tag{6.3.5}$$

$$=\int_0^T\int_{\Omega_\varepsilon}\Big[r_\varepsilon-\Big(\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon)-p(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big)\Big]\mathrm{div}_x\varphi\ \mathrm{d}x\ \ \mathrm{d}t$$

$$+\int_0^T\int_{\Omega_\varepsilon}\varepsilon\Big(\mathbb{S}_\varepsilon-\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon\Big):\nabla_x\varphi\ \mathrm{d}x\ \ \mathrm{d}t$$

for any test function $\varphi\in\mathcal{D}((0,T)\times\Omega_\varepsilon;R^3)$, where we have set

$$r_\varepsilon=A\Big(\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\Big)+B\varrho_\varepsilon\Big(\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big), \tag{6.3.6}$$

with A, B determined through

$$B\overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}=\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta},\ A+B\overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}=\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}. \tag{6.3.7}$$

As a direct consequence of Gibbs' relation (6.2.2), we have

$$\frac{\partial s}{\partial\varrho}=-\frac{1}{\varrho^2}\frac{\partial p}{\partial\vartheta},$$

in particular, $A>0$ as soon as e, p comply with the thermodynamics stability hypotheses (6.2.1).

Finally, introducing the "time lifting" Σ_ε of the measure σ_ε as

$$\Sigma_\varepsilon\in L^\infty(0,T;\mathcal{M}^+(\overline{\Omega}_\varepsilon)),\ <\Sigma_\varepsilon,\psi>=<\sigma_\varepsilon,\Psi>,\ \Psi(t,x)=\int_0^t\psi(s,x)\ \mathrm{d}s \tag{6.3.8}$$

we can rewrite system (6.3.4), (6.3.5) in a concise form

$$\int_0^T\int_{\Omega_\varepsilon}\Big[\varepsilon Z_\varepsilon\partial_t\varphi+A\mathbf{V}_\varepsilon\cdot\nabla_x\varphi\Big]\ \mathrm{d}x\ \ \mathrm{d}t=\int_0^T\int_{\Omega_\varepsilon}\varepsilon\mathbf{F}^1_\varepsilon\cdot\nabla_x\varphi\ \mathrm{d}x\ \ \mathrm{d}t \tag{6.3.9}$$

for all $\varphi\in\mathcal{D}((0,T)\times\Omega_\varepsilon)$,

$$\int_0^T\int_{\Omega_\varepsilon}\Big[\varepsilon\mathbf{V}_\varepsilon\cdot\partial_t\varphi+Z_\varepsilon\mathrm{div}_x\varphi\Big]\ \mathrm{d}x\ \mathrm{d}t=\int_0^T\int_{\Omega_\varepsilon}\Big(\varepsilon\mathbb{F}^2_\varepsilon:\nabla_x\varphi+\varepsilon F^3_\varepsilon\mathrm{div}_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{6.3.10}$$

for all $\varphi\in\mathcal{D}((0,T)\times\Omega_\varepsilon;R^3)$, where we have set

$$Z_\varepsilon = A\Big(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big) + B\varrho_\varepsilon\Big(\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big) + \frac{B}{\varepsilon}\Sigma_\varepsilon,\ \mathbf{V}_\varepsilon = \varrho_\varepsilon\mathbf{u}_\varepsilon, \quad (6.3.11)$$

$$\mathbf{F}^1_\varepsilon = B\varrho_\varepsilon\Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big)\mathbf{u}_\varepsilon + B\frac{\kappa\nabla_x\vartheta_\varepsilon}{\varepsilon\vartheta_\varepsilon}, \quad (6.3.12)$$

$$\mathbb{F}^2_\varepsilon = \mathbb{S}_\varepsilon - \varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon, \quad (6.3.13)$$

and

$$F^3_\varepsilon = \frac{B}{\varepsilon^2}\Sigma_\varepsilon + A\Big(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon^2}\Big) + B\varrho_\varepsilon\Big(\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\Big) - \Big(\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\Big). \quad (6.3.14)$$

6.4 Regularization and extension to R^3

6.4.1 Uniform estimates

To begin, we establish uniform estimates for all terms appearing on the right-hand side of acoustic equation (6.3.9), (6.3.10).

Writing

$$\varrho_\varepsilon\Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big)$$

$$= [\varrho_\varepsilon]_{\rm ess}\Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big) + [\varrho_\varepsilon]_{\rm res}\Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big),$$

we can use the uniform bounds (6.2.16), (6.2.17) in order to obtain

$$\operatorname*{ess\ sup}_{t\in(0,T)} \Big\| [\varrho_\varepsilon]_{\rm ess}\Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big)\Big\|_{L^2(\Omega_\varepsilon)} \le c. \quad (6.4.1)$$

Furthermore, estimate (6.4.1) combined with (6.2.33) yields

$$\int_0^T \Big\| [\varrho_\varepsilon]_{\rm ess}\Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big)\mathbf{u}_\varepsilon\Big\|^2_{L^1(\Omega_\varepsilon;R^3)} \le c, \quad (6.4.2)$$

where both estimates are uniform for $\varepsilon \to 0$.

On the other hand, in accordance with (6.2.18), (6.2.21, and (6.2.35),

$$\operatorname*{ess\ sup}_{t\in(0,T)} \Big\| [\varrho_\varepsilon]_{\rm res}\Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big)\Big\|_{L^1(\Omega_\varepsilon)} \le \varepsilon c, \quad (6.4.3)$$

Next it follows from the structural hypotheses (6.2.27 - 6.2.29) that

$$|\varrho s_M(\varrho, \vartheta)| \leq c(1 + \varrho|\log(\varrho)| + \varrho|\log(\vartheta)|) \text{ for all positive } \varrho,\ \vartheta.$$

In particular, we deduce from (6.2.21), (6.2.35) that

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \frac{[\varrho_\varepsilon]_{\rm res}|\log(\varrho_\varepsilon)|}{\varepsilon} \right\|_{L^{6/5}(\Omega_\varepsilon)} \leq c, \tag{6.4.4}$$

which, together with (6.2.33) and the Sobolev embedding relation $W^{1,2}(R^3) \hookrightarrow L^2 \cap L^6(R^3)$, gives rise to the uniform bound

$$\int_0^T \left\| \frac{[\varrho_\varepsilon]_{\rm res}|\log(\varrho_\varepsilon)|}{\varepsilon} \mathbf{u}_\varepsilon \right\|^2_{L^1(\Omega_\varepsilon)} \ {\rm d}t \leq c. \tag{6.4.5}$$

Similarly, we can write

$$\left| \frac{[\varrho_\varepsilon]_{\rm res}|\log(\vartheta_\varepsilon)|\mathbf{u}_\varepsilon}{\varepsilon} \right|$$

$$\leq \sqrt{[\varrho_\varepsilon]_{\rm res}} \frac{|\log(\vartheta_\varepsilon) - \log(\overline{\vartheta})|}{\varepsilon} \sqrt{[\varrho_\varepsilon]_{\rm res}}\ |\mathbf{u}_\varepsilon| + \frac{[\varrho_\varepsilon]_{\rm res}}{\varepsilon} |\mathbf{u}_\varepsilon|\ |\log(\overline{\vartheta})|$$

and use the uniform estimates (6.2.15), (6.2.21), (6.2.34), and (6.2.35) in order to conclude that

$$\int_0^T \left\| \frac{[\varrho_\varepsilon]_{\rm res}|\log(\vartheta_\varepsilon)|}{\varepsilon} \mathbf{u}_\varepsilon \right\|^2_{L^1(\Omega_\varepsilon)} \ {\rm d}t \leq c. \tag{6.4.6}$$

Since exactly the same estimates can be deduced also for the radiation component $\varrho_\varepsilon s_R(\varrho_\varepsilon, \vartheta_\varepsilon) \approx \vartheta_\varepsilon^3$, we infer that

$$\int_0^T \left\| [\varrho_\varepsilon]_{\rm res} \Big(\frac{s(\overline{\varrho}, \overline{\vartheta}) - s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \Big) \mathbf{u}_\varepsilon \right\|^2_{L^1(\Omega_\varepsilon; R^3)} \leq c, \tag{6.4.7}$$

Using estimates (6.2.22), (6.2.23), we get

$$\int_0^T \left(\| \ [\ \mathbb{S}_\varepsilon]_{\rm ess} \|^2_{L^2(\Omega_\varepsilon; R^{3\times 3})} + \left\| \ [\kappa]_{\rm ess} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon \vartheta_\varepsilon} \right\|^2_{L^2(\Omega_\varepsilon; R^3)} \right) \ {\rm d}t \leq c. \tag{6.4.8}$$

Finally, the contribution of the radiation energy in (6.2.18) gives rise to a bound

$$\operatorname*{ess\ sup}_{t\in(0,T)} \int_{\Omega_\varepsilon} [\vartheta_\varepsilon]^4_{\rm res} \ {\rm d}x \leq \varepsilon^2 c \tag{6.4.9}$$

which can be used in combination with (6.2.22), (6.2.23) in order to infer that

$$\int_0^T \left(\| [\mathbb{S}_\varepsilon]_{\rm res} \|^2_{L^2(\Omega_\varepsilon;R^{3\times 3})} + \left\| [\kappa]_{\rm res} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon \vartheta_\varepsilon} \right\|^2_{L^1(\Omega_\varepsilon;R^3)} \right) \, {\rm d}t \leq c. \tag{6.4.10}$$

As a matter of fact, it can be shown that the presence of radiation terms is not necessary, however we would have to content ourselves with a weaker bound

$$\int_0^T \left(\| [\mathbb{S}_\varepsilon]_{\rm res} \|_{L^2(\Omega_\varepsilon;R^{3\times 3})} + \left\| [\kappa]_{\rm res} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon \vartheta_\varepsilon} \right\|_{L^1(\Omega_\varepsilon;R^3)} \right) \, {\rm d}t \leq c.$$

Having established all the preliminary estimates we are ready to deduce uniform bounds on all quantities appearing in the acoustic equation (6.3.9), (6.3.10).

To begin, it follows from (6.2.16), (6.2.20), (6.2.21), (6.4.1), and (6.4.3) that

$$Z_\varepsilon = Z^1_\varepsilon + Z^2_\varepsilon + Z^3_\varepsilon, \tag{6.4.11}$$

with

$$\left\{ \begin{array}{c} \{Z^1_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;L^2(\Omega_\varepsilon)), \\ \\ \{Z^2_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;L^1(\Omega_\varepsilon)), \\ \\ \{Z^3_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;\mathcal{M}^+(\Omega_\varepsilon)). \end{array} \right\} \tag{6.4.12}$$

Similarly, using (6.2.15), (6.2.21) together with (6.2.35), we obtain

$$\mathbf{V}_\varepsilon = \mathbf{V}^1_\varepsilon + \mathbf{V}^2_\varepsilon, \tag{6.4.13}$$

where

$$\left\{ \begin{array}{c} \{\mathbf{V}^1_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T;L^2(\Omega_\varepsilon;R^3)), \\ \\ \{\mathbf{V}^2_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T;L^1(\Omega_\varepsilon;R^3)). \end{array} \right\} \tag{6.4.14}$$

Furthermore, in accordance with (6.4.2), (6.4.7 - 6.4.10),

$$\mathbf{F}^1_\varepsilon = \mathbf{F}^{1,1}_\varepsilon + \mathbf{F}^{1,2}_\varepsilon, \tag{6.4.15}$$

with

$$\left\{ \begin{array}{c} \{\mathbf{F}^{1,1}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;L^2(\Omega_\varepsilon;R^3)), \\ \\ \{\mathbf{F}^{1,2}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;L^1(\Omega_\varepsilon;R^3)). \end{array} \right\} \tag{6.4.16}$$

By the same token, 6.4.10),

$$\mathbb{F}^2_\varepsilon = \mathbb{F}^{2,1}_\varepsilon + \mathbb{F}^{2,2}_\varepsilon, \tag{6.4.17}$$

where

$$\left\{ \begin{array}{c} \{\mathbb{F}^{2,1}_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;L^2(\Omega_\varepsilon;R^{3\times 3})), \\ \\ \{\mathbb{F}^{2,2}_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;L^1(\Omega_\varepsilon;R^{3\times 3})). \end{array} \right\} \tag{6.4.18}$$

Finally, by virtue of our choice of the parameters A, B in (6.3.7), we conclude, by help of (6.2.16 - 6.2.21), that

$$F^3_\varepsilon = F^{3,1}_\varepsilon + F^{3,2}_\varepsilon, \tag{6.4.19}$$

with

$$\left\{ \begin{array}{c} \{F^{3,1}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;L^1(\Omega_\varepsilon)), \\ \\ \{F^{3,2}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;\mathcal{M}^+(\Omega_\varepsilon)). \end{array} \right\} \tag{6.4.20}$$

6.4.2 Regularization

Our final goal is to show strong convergence of the velocity fields claimed in (6.1.11). By virtue of the uniform estimates (6.2.33), we already have

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T;W^{1,2}(B;R^3)) \text{ for any bounded domain } B \subset R^3 \tag{6.4.21}$$

passing to a suitable subsequence (independent of B) as the case may be.

Let

$$[\mathbf{v}]_\delta(t,x) = \int_{R^3} \eta_\delta(x-y)\mathbf{v}(t,y)\ \mathrm{d}y$$

denote the smoothing operator associated to a family $\{\eta_\delta\}_{\delta>0}$ or smooth regularizing kernels $\mathrm{supp}[\eta_\delta] \subset \{|y| < \delta\}$. We claim that the desired relation (6.1.11) follows as soon as we are able to show

$$\begin{array}{c} [\varrho_\varepsilon \mathbf{u}_\varepsilon]_\delta \to \overline{\varrho}[\mathbf{u}]_\delta \text{ in } L^2(0,T;L^2(B;R^3)) \ \text{ as } \varepsilon \to 0 \\ \\ \text{for any bounded domain } B \subset R^3, \text{ and any fixed } \delta > 0. \end{array} \tag{6.4.22}$$

Indeed relation (6.4.22) implies

$$[\overline{\varrho}\mathbf{u}_\varepsilon]_\delta = \varepsilon[\frac{\overline{\varrho} - \varrho_\varepsilon}{\varepsilon}\mathbf{u}_\varepsilon]_\delta + [\varrho_\varepsilon \mathbf{u}_\varepsilon]_\delta \to \overline{\varrho}[\mathbf{u}]_\delta,$$

meaning

$$[\mathbf{u}_\varepsilon]_\delta \to [\mathbf{u}]_\delta \text{ in } L^2(0,T;L^2(B;R^3)) \text{ for any bounded } B \subset R^3;$$

whence the desired conclusion follows from compactness of the Sobolev embedding $W^{1,2}(B;R^3) \hookrightarrow\hookrightarrow L^2(B;R^3))$.

In order to see (6.4.22), we regularize the acoustic equation, that means, we take $\varphi_x(t,y) = \psi(t)\eta_\delta(x-y)$, $\psi \in \mathcal{D}(0,T)$, as test function in (6.3.9), (6.3.10). The resulting equation reads

$$\left\{ \begin{array}{c} \varepsilon\partial_t[Z_\varepsilon]_\delta + A\mathrm{div}_x[\mathbf{V}_\varepsilon]_\delta = \varepsilon\mathrm{div}_x\Big(\mathbf{G}^1_{\varepsilon,\delta} + \mathbf{G}^2_{\varepsilon,\delta}\Big) \\ \\ \varepsilon\partial_t[\mathbf{V}_\varepsilon]_\delta + \nabla_x[Z_\varepsilon]_\delta = \varepsilon\mathrm{div}_x\Big(\mathbb{H}^1_{\varepsilon,\delta} + \mathbb{H}^2_{\varepsilon,\delta}\Big), \end{array} \right\} \text{ a.a. in } (0,T)\times\Omega_\varepsilon, \tag{6.4.23}$$

where, by virtue of the uniform estimates (6.4.16), (6.4.18), and (6.4.20)

$$\begin{aligned} &\{\mathbf{G}^1_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;W^{k,1}(\Omega_\varepsilon;R^3)), \\ &\{\mathbf{G}^2_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;W^{k,2}(\Omega_\varepsilon;R^3)), \\ &\{\mathbb{H}^1_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;W^{k,1}(\Omega_\varepsilon;R^{3\times3})), \\ &\{\mathbb{H}^2_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;W^{k,2}(\Omega_\varepsilon;R^{3\times3})). \end{aligned} \tag{6.4.24}$$

Moreover,

$$[Z_\varepsilon]_\delta = Z^1_{\varepsilon,\delta} + Z^2_{\varepsilon,\delta},\ [\mathbf{V}_\varepsilon]_\delta = [\varrho_\varepsilon\mathbf{u}_\varepsilon]_\delta,$$

with

$$\begin{aligned} &\{Z^1_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;W^{k,1}(\Omega_\varepsilon)) \\ &\{Z^2_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;W^{k,2}(\Omega_\varepsilon)) \end{aligned} \tag{6.4.25}$$

for any $k = 0,1,\dots,$ where all bounds depend on k and δ but they are uniform for $\varepsilon \to 0$.

6.4.3 Extension to the whole space R^3

The acoustic equation (6.4.23) admits a finite speed of propagation proportional to ε^{-1}. Indeed multiplying the left-hand side of (6.4.23) on $[Z_\varepsilon, A\mathbf{V}_\varepsilon]$ we get the expression

$$\partial_t(|Z_\varepsilon|^2 + A|\mathbf{V}_\varepsilon|^2) + \frac{2A}{\varepsilon}\mathrm{div}_x(Z_\varepsilon \mathbf{V}_\varepsilon);$$

whence the desired result follows by integration over an appropriate space-time cone.

From now on we fix a bounded ball $B \subset R^3$. Since our goal is to show strong convergence of $\{[\varrho_\varepsilon \mathbf{u}_\varepsilon]_\delta\}_{\varepsilon>0}$ on B as claimed in (6.4.22), the family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ enjoys Property L formulated in Section 6.1, meaning the boundaries $\partial\Omega_\varepsilon$ are "far away" from B, and equation (6.4.23) admits the finite speed of propagation, we can extend all quantities in (6.4.23) onto the whole space R^3 in such a way that

- the acoustic equation (6.4.23) is satisfied a.a. in the set $(0,T) \times R^3$;
- the uniform bounds established in (6.4.24 - 6.4.25) hold with Ω_ε replaced by R^3;
- $$\{[\mathbf{V}_\varepsilon]_\delta(0,\cdot)\}_{\varepsilon>0} \text{ is bounded in } W^{k,1}(R^3;R^3) \text{ for any } k = 0,1,\ldots \tag{6.4.26}$$
  (see (6.2.14));
- $$\int_{R^3} [Z_\varepsilon]_\delta(0,x) \ \mathrm{d}\mathbf{x} = 0; \tag{6.4.27}$$
- all quantities appearing in (6.4.23) have compact support in R^3, the radius of which depends on ε.

6.5 Dispersion estimates and time-decay of the acoustic waves

The problem being reduced to the situation described in Section 6.4, the proof of the desired relation (6.4.22) will follow from the standard dispersive estimates for the acoustic equation (6.4.23).

Integrating the first equation in (6.4.23) and using (6.4.27) we observe that

$$\int_{R^3} [Z_\varepsilon]_\delta(t,x)\ \mathrm{d}\mathbf{x} = 0 \text{ for all } t \in [0,T). \tag{6.5.1}$$

Let us introduce the Helmholtz decomposition on R^3,

$$\mathbf{v} = \mathbf{H}[\mathbf{v}] + \mathbf{H}^\perp[\mathbf{v}],$$

where $\mathbf{H}^\perp \approx \nabla_x \Delta^{-1} \mathrm{div}_x$ can be determined in terms of the Fourier symbols as

$$\mathbf{H}^\perp[\mathbf{v}] = \mathcal{F}^{-1}_{\xi\to x}\Big[\frac{\xi\otimes\xi}{|\xi|^2}\mathcal{F}_{x\to\xi}[\mathbf{v}]\Big],$$

where $\mathcal{F}$ denotes the Fourier transform in the $x-$variable.

Applying $\mathbf{H}$ to the second equation in (6.4.23) we deduce easily that

$$\mathbf{H}[\ [\varrho_\varepsilon \mathbf{u}_\varepsilon]_\delta] \to \mathbf{H}[\ [\overline{\varrho}\mathbf{u}]_\delta] = \overline{\varrho}[\mathbf{u}]_\delta \text{ in } L^2(0,T;L^2(B;R^3)) \tag{6.5.2}$$

for any fixed $\delta > 0$. Consequently, in order to complete the proof of (6.4.22), it is enough to handle the gradient component

$$\mathbf{H}^\perp[\ [\varrho_\varepsilon \mathbf{u}_\varepsilon]_\delta] = \mathbf{H}^\perp[\ [\mathbf{V}_\varepsilon]_\delta\ ] = \nabla_x \Delta^{-1}\mathrm{div}_x[\mathbf{V}_\varepsilon]_\delta.$$

For

$$\Psi_\varepsilon \equiv \Delta^{-1}\mathrm{div}_x[\mathbf{V}_\varepsilon]_\delta,\ z_\varepsilon = -[Z_\varepsilon]_\delta$$

we have a "classical" wave equation

$$\varepsilon\partial_t z_\varepsilon - A\Delta\Psi_\varepsilon = \varepsilon(g^1_\varepsilon + g^2_\varepsilon) \tag{6.5.3}$$

$$\varepsilon\partial_t \Psi_\varepsilon - z_\varepsilon = \varepsilon(h^1_\varepsilon + h^2_\varepsilon), \tag{6.5.4}$$

supplemented with the initial conditions

$$\Psi_\varepsilon(0,\cdot) = \Psi_{0,\varepsilon},\ z_\varepsilon(0,\cdot) = z_{0,\varepsilon}, \tag{6.5.5}$$

where, in accordance with (6.4.24 - 6.4.26),

$$\{\Psi_{0,\varepsilon}\}_{\varepsilon>0},\ \{z_{0,\varepsilon}\}_{\varepsilon>0} \text{ are bounded in } W^{k,2}(R^3), \tag{6.5.6}$$

$$\int_{R^3} g^i_\varepsilon\ \mathrm{d}\mathbf{x} = \int_{R^3} h^i_\varepsilon\ \mathrm{d}\mathbf{x} = 0,\ i = 1,2, \tag{6.5.7}$$

$$\{g^1_\varepsilon\}_{\varepsilon>0},\ \{h^1_\varepsilon\}_{\varepsilon>0} \text{ are bounded in } L^2(0,T;W^{k,1}(R^3)), \tag{6.5.8}$$

and

$$\{g^2_\varepsilon\}_{\varepsilon>0},\ \{h^2_\varepsilon\}_{\varepsilon>0} \text{ are bounded in } L^2(0,T;W^{k,2}(R^3)) \tag{6.5.9}$$

for any $k = 0, 1, \ldots$

Since $[\mathbf{V}_\varepsilon]_\delta$ coincides with $[\varrho_\varepsilon \mathbf{u}_\varepsilon]_\delta$ on the set $(0,T)\times B$, and since we have already shown (6.5.2), relation (6.4.22) follows as soon as we are able to verify that

$$\Psi_\varepsilon \to 0 \text{ in } L^2(0,T;W^{1,2}(B)). \tag{6.5.10}$$

Any solution of (6.5.3 - 6.5.5) can be expressed by means of Duhamel's formula

$$\left[\begin{array}{c} z_\varepsilon \\ \Psi_\varepsilon \end{array}\right](t) = S\Big(\frac{t}{\varepsilon}\Big)\left[\begin{array}{c} z_{0,\varepsilon} \\ \Psi_{0,\varepsilon} \end{array}\right] + \int_0^t S\Big(\frac{t-s}{\varepsilon}\Big)\left[\begin{array}{c} (g^1_\varepsilon + g^2_\varepsilon)(s) \\ (h^1_\varepsilon + h^2_\varepsilon)(s) \end{array}\right] \mathrm{d}s, \tag{6.5.11}$$

where

$$S(t)\left[\begin{array}{c} z_0 \\ \Psi_0 \end{array}\right] = \left[\begin{array}{c} z(t) \\ \Psi(t) \end{array}\right] \tag{6.5.12}$$

is the unique solutions of the homogeneous problem

$$\partial_t z - A\Delta\Psi = 0,\ \partial_t \Psi - z = 0,\ z(0) = z_0,\ \Psi(0) = \Psi_0. \tag{6.5.13}$$

As we need only a local bound, the component

$$\int_0^t S\Big(\frac{t-s}{\varepsilon}\Big)\left[\begin{array}{c} g^1_\varepsilon(s) \\ h^1_\varepsilon(s) \end{array}\right] \mathrm{d}s$$

is easily controlled by means of the classical $L^1 - L^\infty$ dispersive estimates for the wave equation (see Strauss [41, Chapter 1]). In order to handle the L^2-terms, we use the following result by Burq [2, Theorem 3] (see also Metcalfe [27, Lemma 4.1]).

Proposition 6.5.1. *For any function $\chi \in \mathcal{D}(R^3)$, there is a constant $c = c(\chi)$ such that*

$$\int_{-\infty}^{\infty} \left\| \chi\ S(t)\left[\begin{array}{c} z_0 \\ \Psi_0 \end{array}\right]\right\|^2_{L^2(R^3)\times D^{1,2}_0(R^3)} \mathrm{d}t \le c \left\|\left[\begin{array}{c} z_0 \\ \Psi_0 \end{array}\right]\right\|^2_{L^2(R^3)\times D^{1,2}_0(R^3)}, \tag{6.5.14}$$

where $D^{1,2}_0(R^3)$ is the so called homogeneous Sobolev space*, i.e. a completion of functions from $\mathcal{D}(R^3)$ with respect to the gradient norm $\|\nabla_x \cdot \|_{L^2(R^3)}$.*

Rescaling (6.5.14) in t we get

$$\int_{-\infty}^{\infty} \left\| S\Big(\frac{t}{\varepsilon}\Big) \left[\begin{array}{c} z_{0,\varepsilon} \\ \Psi_{0,\varepsilon} \end{array} \right] \right\|^2_{L^2(B)\times D_0^{1,2}(B)} \, \mathrm{d}t \leq \varepsilon c \left\| \left[\begin{array}{c} z_{0,\varepsilon} \\ \Psi_{0,\varepsilon} \end{array} \right] \right\|^2_{L^2(R^3)\times D_0^{1,2}(R^3)} . \tag{6.5.15}$$

Finally, by the same token

$$\int_0^T \left\| \int_0^t S\Big(\frac{t-s}{\varepsilon}\Big) \left[\begin{array}{c} g_\varepsilon^2(s) \\ h_\varepsilon^2(s) \end{array} \right] \mathrm{d}s \right\|^2_{L^2(B)\times D_0^{1,2}(B)} \, \mathrm{d}t \tag{6.5.16}$$

$$\leq c(T) \int_0^T \int_{-\infty}^{\infty} \left\| S\Big(\frac{t}{\varepsilon}\Big) S\Big(\frac{-s}{\varepsilon}\Big) \left[\begin{array}{c} g_\varepsilon^2(s) \\ h_\varepsilon^2(s) \end{array} \right] \right\|^2_{L^2(B)\times D_0^{1,2}(B)} \, \mathrm{d}t \, \mathrm{d}s$$

$$\leq \varepsilon c(T) \int_0^T \left\| S\Big(\frac{-s}{\varepsilon}\Big) \left[\begin{array}{c} g_\varepsilon^2(s) \\ h_\varepsilon^2(s) \end{array} \right] \right\|^2_{L^2(R^3)\times D_0^{1,2}(R^3)} \, \mathrm{d}s$$

$$= \varepsilon c(T) \int_0^T \left\| \left[\begin{array}{c} g_\varepsilon^2(s) \\ h_\varepsilon^2(s) \end{array} \right] \right\|^2_{L^2(R^3)\times D_0^{1,2}(R^3)} \, \mathrm{d}s,$$

where we have used the fact that $(S(t))_{t\in R}$ is a group of isometries on $L^2(R^3) \times D_0^{1,2}(R^3)$.

Combining (6.5.9), (6.5.15), (6.5.16) we obtain (6.5.10).

6.6 Conclusion - main result

We have proved the following result.

Theorem 6.6.1. *Let* $\{\Omega_\varepsilon\}_{\varepsilon>0}$ *be a family of domains in* R^3 *enjoying Property L introduced in Section 6.1. Assume that the thermodynamics functions* p, e, s *as well as the transport coefficients* μ, κ *satisfy the structural hypotheses (6.2.24 - 6.2.31). Let* $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ *be a distributional solution of the Navier-Stokes-Fourier system (6.1.1 - 6.1.5) in* $(0,T) \times \Omega_\varepsilon$ *satisfying (6.2.3 - 6.2.7) and emanating from the initial data (6.1.8), (6.1.9), (6.2.13) satisfying (6.2.14).*
Then, at least for a suitable subsequence,

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } L^2(0,T; L^2(B; R^3)) \text{ for any bounded ball } B \subset R^3,$$

where $\text{div}_x \mathbf{u} = 0$.

The presence of the radiation terms in the system is not necessary. The same result can be obtained if $a = 0$ in (6.2.24).

Chapter 7

Conclusion

In the previous chapters, an approach to the problems of compressible fluid flow on unbounded domains was shown. In the first part, the existence of weak solutions was studied and the results were reported as they occured through the time. In the second part, i.e. in the last chapter, the question of qualitative behavior of the weak solutions in the process of low Mach number limit was studied.

The first contribution concerned with generalization of the existence result of the weak solutions to the compressible full system of equations to bounded Lipschitz domains. The main tool was existing theory and the strategy of the proof lied in inserting an additional approximating sequence inside the limit process. The restriction on Lipschitz domains and the type of convergence of these domains emerged from estimates for the limit passage in the entropy inequality and has been weakened by the later results (published in this thesis).

The second result dealt with the unbounded domain case. The model proposed reflects the additional radiative viscosity term due to Oxenius [35]. This radiative term, which presence is influential in the case of high temperature regimes, makes the viscosity to be proportional to the fourth power of the temperature. Thanks to the radiative viscosity phenomenon, one could employ the internal energy inequality in order to get estimates on the gradient of the velocity. The drawback of this approach was an additional constrain on the growth of the pressure terms.

The generalization to the unbounded domains reached its top in the third result. The model uses the general pressure function in the constitutive assumptions and the existence result given concerns with an arbitrary open set. Furthermore, the problem of attaining the initial data and so called *formal compatibility* are studied as well. The extent of these results required general approach: Unknown bounds on the temperature in L^3 required the use of the Muckenhoupt theory of weights, which required to pay a special attention to all the steps in order get through several narrow integrability constrains.

In the next result, the focus moved to the question of the full system of equations in the case of unbounded domains with nonhomogeneous boundary conditions for the density and temperature at infinity. The basic idea was similar to the previous approach – benefit from the existing theory for bounded domains and approximate the unbounded domains with the bounded ones. The presence of the nonvanishing density and temperature would, however, imply nonintegrability of the internal (and so the total) energy term. This obstacle was bypassed by the introduction of the linearized energy term in the form of the so called *Helmholtz-like total energy*. This approach facilitated the process of the integrability estimates. On the other hand, the price paid was a lack of the formal compatibility which seems to be disrupted in this approach.

The study of the existence question was closed at this point. In the following chapter, the attention was given to the qualitative behavior of the compressible flow on an unbounded domain in the low Mach number limit. The problem reduced onto the question of strong convergence of the velocity field, i.e. one had to focus on showing strong relative compactness of the velocity sequence. The compactness of the velocity sequence was recovered through the analysis of the related *accoustic equation* and the dispersive estimates.

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