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Gibbs Particle Processes

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Abstract: We consider a recent result for the existence of infinite-volume marked Gibbs point processes and try to apply it to geometric models. At first, we reformulate a problematic assumption of the considered existence result and check that the theorem still holds. We use this result for the family of Gibbs facet processes (a special case of particle processes) and prove the existence for repulsive interactions. We find counterexamples for the process with attractive interactions and prove that the finite-volume Gibbs facet process in $\mathbb{R}^2$ does not exist in this case. We also study the class of Gibbs-Laguerre tessellations of $\mathbb{R}^2$. We cannot use the mentioned existence result in general, but we are able to prove the existence of an infinite-volume Gibbs-Laguerre process with particular energy function, under the assumption that we almost surely see a point.

Keywords: infinite-volume Gibbs measure, existence, Gibbs facet process, Gibbs-Laguerre tessellation

Contents

Basic notation and abbreviations

Here we present a list of basic notation and abbreviations used in our work. Some other notation concerning the locally finite measures can be found in Section 1.1.

$\mathbb{N}, \mathbb{Z}$	the set of all natural numbers and the set of all integers
$(\mathbb{R}^d, \lvert\cdot\rvert)$	the d-dimensional Euclidean space with the standard norm
$\langle \cdot, \cdot \rangle$	the standard dot product in $\mathbb{R}^d$
$\mathbb{Z}^d$	the set of all integer valued vectors in $\mathbb{R}^d$
$\mathcal{B}^d$, $\mathcal{B}^d_b$,	the Borel σ-algebra on $\mathbb{R}^d$ and its subset of bounded sets
λ, λ_Λ	the Lebesgue measure on $\mathbb{R}^d$ and its restriction to the set Λ
$B(x, r)$, $U(x, r)$	the closed and open ball in $\mathbb{R}^d$ with centre x and radius r
$\mathrm{int}(A)$, $\mathrm{clo}(A)$	the interior and closure of $A \subset \mathbb{R}^d$
$\mathrm{bd}(A)$, $\mathrm{conv}\{A\}$	the boundary and convex hull of $A \subset \mathbb{R}^d$
$A \oplus B$, A^c	Minkowski sum of sets $A, B \subset \mathbb{R}^d$ and the complement of A
$\lvert A\rvert$	the Lebesgue or counting (for A at most countable) measure of the set A
Λ_n	the set $[-n, n)^d$ for $n \in \mathbb{N}$, $d \in \mathbb{N}$
$\mathbb{H}^k$	the k-dimensional Hausdorff measure, $k \in \mathbb{N}$
$\mathbb{1}\{\cdot\}$, δ_x	the indicator function and Dirac measure, $x \in \mathbb{R}^d$
$\mathcal{M}(\mathcal{E})$	the set of all simple counting locally finite measures on $\mathcal{E}$
$(\mathcal{S}, \lVert \cdot \rVert)$	the mark space and its norm
ϑ_z	the shift operator on $\mathbb{R}^d$ defined as $\vartheta_z(x) = x + z$, $z \in \mathbb{R}^d$
π_μ, π^z_Λ	the distribution of a Poisson point process
$\sum^{\neq}$	sum over all pairwise different tuples
w. r. t.	with respect to
w. l. o. g.	without loss of generality
a. s., a. a.	almost surely, almost all

Introduction

It is a basic knowledge in the field of spatial modelling, that the Poisson point process is a model for complete spatial randomness, but if we wish to consider more general situation with interactions between the points, we need to consider more complicated processes.

A useful class of point processes are the Gibbs point processes (GPP). GPP consist of a broad family of models, which take into consideration various possibilities of interactions between the points. The effect of these interactions is explained through the notion of *an energy function*, with states with lower energy being more probable then the states with higher energy. This convention stems from the physical interpretation, as the notion of GPP was first introduced in statistical mechanics, see Ruelle [1969] for the standard reference. The family of GPP includes for example the well known *pairwise interaction process*, *Strauss's hard–core process* or *the Widow–Rowlinson process* but also more complicated geometric models such as the *quermass–interaction process* or models for *random tessellations*. Among others, Møller and Waagepetersen [2004], Chapter 6, and Dereudre [2019] provide a general introduction to the topic of GPP in the context of spatial modelling.

Gibbs point processes in a bounded window are defined using a density w. r. t. the distribution of a Poisson point process and they are characterized uniquely by the **DLR** equations. The problem with this approach is that the normalizing constant (also called *the partition function*) is often intractable. Although we do not consider this in our work, we note that this problem can be overcome by considering *conditional intensities* instead of densities and the **GNZ** equations instead of the **DLR** equations, as these two approaches are equivalent.

However, the situation gets much more complicated, once we start to consider processes in the whole $\mathbb{R}^d$. As we can no longer use the approach with a density w. r. t. a reference process, we can no longer define the distribution of an infinite–volume Gibbs process (also called the *infinite–volume Gibbs measure*) explicitly. Instead, we use the **DLR** equations, which prescribe the distribution of the process inside a bounded window conditionally on a fixed configuration outside of this window. The question, still relevant today, is whether and under what conditions such processes exist.

The standard approach on how to obtain an infinite–volume Gibbs measure is based on the topology of *local convergence* and the result from Georgii and Zessin [1993] for level sets of *a specific entropy*. Instead of an entropy tools, other approaches can be used, see Appendix B in Jansen [2019] for a proof based on the convergence of correlation functions and Janossy densities.

One of the standard assumptions for the energy function is the *finite–range* assumption which enforces that the range of interactions is uniformly bounded. In Dereudre [2009], it was proved that the quermass–interaction process with unbounded grains (i. e. unbounded interactions) exists. Using this article as an inspiration, an existence result for marked Gibbs point processes with unbounded interaction was proved in Rœlly and Zass [2020]. The aim of this work was to consider different models for marked Gibbs point processes with unbounded range of the interaction and use the existence result from Rœlly and Zass [2020] to show that the infinite–volume Gibbs processes exist.

The content of this thesis is separated into four chapters. In the first one, we summarize the theory of marked point processes and present the class of Gibbs processes and the set of tempered configurations as well as the entropy tools.

The second chapter is devoted to the proof of the existence theorem from Rœlly and Zass [2020]. We address the assumptions posed on the energy function and present the modified version of the range assumption, as we have a major objection to its formulation in the original work. Afterwards, we go through particular parts of the proof to see that it still holds.

In the third chapter we study Gibbs facet processes in $\mathbb{R}^d$, which present a special case of particle processes. A facet in $\mathbb{R}^d$ is a $(d-1)$-dimensional bounded set, which is obtained by intersecting a d-dimensional ball with $(d-1)$-dimensional linear subspace of $\mathbb{R}^d$. The energy is a function of the intersections of tuples of facets. We prove that the repulsive model (i. e. model with non–negative energy function) satisfies assumptions of the existence theorem and therefore the infinite–volume Gibbs facet process exists in this case. On the other hand, for the mixed and clustering models (i. e. models with possibly negative energy function) we find counterexamples in $\mathbb{R}^2$ and $\mathbb{R}^3$ for the stability assumption and we extend the counterexample in $\mathbb{R}^2$ to prove that the Gibbs facet processes in bounded windows do not exist.

In the last chapter, we consider a model for a random tessellation of $\mathbb{R}^2$. A tessellation is a locally finite partition of the space $\mathbb{R}^2$ into bounded cells C_i, which are convex polytopes. We consider *the Laguerre tessellation* $L(\gamma)$, which partitions $\mathbb{R}^2$ based on *the set of generators* $\gamma \subset \mathbb{R}^2 \times (0, \infty)$. Each generator $(x', x'') \in \gamma$ defines a circle $S(x', x'')$ and the space $\mathbb{R}^2$ is partitioned based on the power w. r. t. the generating circles. The random Laguerre tessellation can be modelled using a marked point process as its set of generators. The Poisson–Laguerre tessellation $L(\Psi)$, where Ψ is a stationary marked Poisson point process, has been studied in Lautensack [2007]. We consider more general situation with the random generator Ψ being a marked Gibbs point process with the energy function depending on the geometric properties of cells of $L(\Psi)$.

Gibbs point processes with geometry-dependent interactions (which include random tessellations) were considered in Dereudre et al. [2012]. Using the hypergraph structure, an existence result was derived for the unmarked case under a set of complicated assumptions. It was remarked in the same work that the same existence result would extend to the marked case, and based on this remark the existence of an infinite–volume Gibbs measure for several models of Gibbs–Laguerre tessellations of $\mathbb{R}^3$ was derived in Jahn and Seitl [2020]. However, the marks were considered bounded. Our intent was to use Rœlly and Zass [2020] and consider the unbounded case. Unfortunately, the range assumption from Rœlly and Zass [2020] turned out to be more restricting than initially expected.

However, noticing that we can still use several of the results from Rœlly and Zass [2020] for a non–negative energy function and after a careful analysis of the behaviour of the Laguerre diagram, we considered a model with energy given by the number of vertices in the tessellation. For this model we were able to prove new existence theorem, which states that under the condition that we almost surely see a point, there exists an infinite–volume Gibbs–Laguerre tessellation of $\mathbb{R}^2$ with energy given by the number of vertices. As a by-product of this proof, several useful observations arose, which could be helpful in the study of other Gibbs–Laguerre models.

To conclude we note that the original results in this work consist of Sections 2.1.3 and 2.3, Chapter 3 and Chapter 4 (without Section 4.1 up to Theorem 23).

1. Theory of Point Processes

In this chapter, we present the necessary theory for our work. In the first section, we briefly summarize the theory for marked point processes, for references see Rataj [2006] (in Czech) or Møller and Waagepetersen [2004] (in English). Afterwards, we summarize the theory for Gibbs point processes. The notation we use will mainly be chosen so that it is in accordance with the article Rœlly and Zass [2020], which is the core of this work.

1.1 Marked Point Processes

Let $(\mathcal{E}, \rho)$ be a complete separable metric space which satisfies that every closed and bounded subset of $\mathcal{E}$ is compact. The space $\mathcal{E}$ will be called *the state space.* Denote by $\mathcal{B}(\mathcal{E})$ the Borel σ-algebra on $\mathcal{E}$, by $\mathcal{B}_b(\mathcal{E})$ the set of all bounded Borel subsets of $\mathcal{E}$ and by $\mathcal{K}(\mathcal{E})$ the set of all compact subsets of $\mathcal{E}$.

Remark. For $\mathcal{E} = \mathbb{R}^d$, $d \geq 2$, we write shortly $\mathcal{B}^d = \mathcal{B}(\mathbb{R}^d)$, $\mathcal{B}_b^d = \mathcal{B}_b(\mathbb{R}^d)$ and $\mathcal{K}^d = \mathcal{K}(\mathbb{R}^d)$. The standard Euclidean norm on $\mathbb{R}^d$ will be denoted by $|x|, x \in \mathbb{R}^d$, and the Lebesgue measure on $\mathbb{R}^d$ will be denoted by λ.

Definition 1. *A Borel measure ν on $\mathcal{E}$ is called **locally finite**, if it holds that $\nu(K) < \infty$, $\forall K \in \mathcal{K}(\mathcal{E})$.*

Thanks to our assumptions on the space $\mathcal{E}$, every locally finite measure ν is also finite on all bounded Borel subsets of $\mathcal{E}$.

Definition 2. *We say that a locally finite measure ν on $\mathcal{E}$ is*

- *a **counting** measure, if it holds that $\nu(B) \in \mathbb{N} \cup \{0, \infty\}$, $\forall B \in \mathcal{B}(\mathcal{E})$.*
- ***simple**, if it is a counting measure such that $\nu(\{x\}) \leq 1, \forall x \in \mathcal{E}$.*

Let us denote by $\mathcal{N}(\mathcal{E})$ the set of all locally finite measures on the space $\mathcal{E}$, by $\mathcal{N}^*(\mathcal{E})$ the set of all counting locally finite measures on $\mathcal{E}$ and by $\mathcal{M}^*(\mathcal{E})$ the set of all simple counting locally finite measures on $\mathcal{E}$. Particularly we have that $\mathcal{M}^*(\mathcal{E}) \subset \mathcal{N}^*(\mathcal{E}) \subset \mathcal{N}(\mathcal{E})$.

As is usual, we endow the set $\mathcal{N}(\mathcal{E})$ with σ-algebra $\mathfrak{N}(\mathcal{E})$, where $\mathfrak{N}(\mathcal{E})$ is the smallest σ-algebra on $\mathcal{N}(\mathcal{E})$ such that the projections $p_B : \mathcal{N}(\mathcal{E}) \to \mathbb{R}$, where $p_B(\nu) = \nu(B)$, are measurable $\forall B \in \mathcal{B}(\mathcal{E})$. We then endow the set $\mathcal{N}^*(\mathcal{E})$ with the σ-algebra $\mathfrak{N}^*(\mathcal{E})$ defined as the trace of the σ-algebra $\mathfrak{N}(\mathcal{E})$ on $\mathcal{N}^*(\mathcal{E})$ and the set $\mathcal{M}^*(\mathcal{E})$ is analogously endowed with the σ-algebra $\mathfrak{M}^*(\mathcal{E})$ defined as the trace of $\mathfrak{N}(\mathcal{E})$ on $\mathcal{M}^*(\mathcal{E})$.

Now we can define a point process in $\mathcal{E}$.

Definition 3. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Any measurable mapping $\Phi : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathcal{N}^*(\mathcal{E}), \mathfrak{N}^*(\mathcal{E}))$ is called a **point process** in $\mathcal{E}$. We say that a point process Φ is **simple**, if $\mathbb{P}(\Phi \in \mathcal{M}^*(\mathcal{E})) = 1$.*

The *distribution* of a point process Φ will be denoted by P_Φ, i. e. for every $U \in \mathfrak{N}^*(\mathcal{E})$ we have $\mathsf{P}_\Phi(U) = \mathbb{P}(\Phi \in U)$. Let us remark that for a point process Φ

and a set $B \in \mathcal{B}_b(\mathcal{E})$, the number of points of Φ in the set B is a random variable denoted by $\Phi(B)$.

The essential definition for the theory of point processes is the definition of a Poisson point process.

Definition 4. *Let $\mu \in \mathcal{N}(\mathcal{E})$ be a locally finite measure. A point process Φ is called a* ***Poisson point process*** *with intensity measure μ, if it satisfies the following two conditions:*

i) $\Phi(B)$ *has a Poisson distribution with parameter* $\mu(B)$, $\forall B \in \mathcal{B}_b(\mathcal{E})$,

ii) $\Phi(B_1), \dots, \Phi(B_n)$ *are independent random variables* $\forall B_1, \dots, B_n \in \mathcal{B}_b(\mathcal{E})$ *pairwise disjoint,* $\forall n \in \mathbb{N}$.

In this work, we will consider only a special type of point processes, so-called marked point processes. Take state space in the product form $\mathcal{E} = \mathbb{R}^d \times \mathcal{S}$, where $d \geq 2$ and the so-called *mark space* $(\mathcal{S}, \|\cdot\|)$ is a normed space.[1] Each point $(x, m) \in \mathbb{R}^d \times \mathcal{S}$ consists of two parts, the location part x and the mark m. Marked point processes are defined as a special class of point processes in the product space $\mathbb{R}^d \times \mathcal{S}$.

Definition 5. *Let $\mathcal{N}^*_m(\mathcal{E}) = \{\nu \in \mathcal{N}^*(\mathcal{E}) : \nu(\cdot \times \mathcal{S}) \in \mathcal{N}^*(\mathbb{R}^d)\}$. We say that a point process Φ on $\mathcal{E}$ is a* ***marked point process****, if $\mathbb{P}(\Phi \in \mathcal{N}^*_m(\mathcal{E})) = 1$. Furthermore, let $\mathcal{M}^*_m(\mathcal{E}) = \{\nu \in \mathcal{N}^*(\mathcal{E}) : \nu(\cdot \times \mathcal{S}) \in \mathcal{M}^*(\mathbb{R}^d)\}$. We say that a marked point process Φ is* ***simple****, if $\mathbb{P}(\Phi \in \mathcal{M}^*_m(\mathcal{E})) = 1$.*

Notice that not every point process in $\mathbb{R}^d \times \mathcal{S}$ is a marked point process – it is required that the so-called *ground process* $\Phi'(\cdot) = \Phi(\cdot \cap \mathcal{S})$ is a. s. a point process in $\mathbb{R}^d$. Also, not every simple point process in $\mathbb{R}^d \times \mathcal{S}$ is a simple marked point process – for the marked point process to be simple, we require that the ground process is simple. In the following text, we will only consider simple marked point processes. For simplicity, we will use $\mathcal{M}(\mathcal{E})$ instead of $\mathcal{M}^*_m(\mathcal{E})$ and we will denote by $\mathfrak{M}(\mathcal{E})$ the usual σ-algebra on $\mathcal{M}(\mathcal{E})$.

Before we proceed further, we state several useful remarks considering the notation for $\gamma \in \mathcal{M}(\mathcal{E})$ and define some special subsets of $\mathcal{M}(\mathcal{E})$.

1. Each $\gamma \in \mathcal{M}(\mathcal{E})$ can be written as

$$\gamma = \sum_{i=1}^{N} \delta_{(x_i, m_i)},$$

where $(x_i, m_i) \in \mathbb{R}^d \times \mathcal{S}$ are pairwise different points and $N \in \mathbb{N} \cup \{0, \infty\}$. Therefore we can identify γ with its support

$$\gamma \equiv \mathbf{supp}\ \gamma = \{(x_1, m_1), (x_2, m_2), \dots\} \subset \mathcal{E}.$$

The *zero measure* $\bar{o}$ is identified with $\emptyset$. Throughout this text, we will write $(x, m) \in \gamma$, instead of $(x, m) \in \mathbf{supp}\ \gamma$ and use this remark when it will be convenient to regard $\gamma \in \mathcal{M}(\mathcal{E})$ as a (locally finite) subset of $\mathcal{E}$ instead of a simple counting locally finite measure. The marked point $\boldsymbol{x} = (x, m) \in \gamma$ is called an *atom* of γ.

[1] In general, it is possible to consider a metric space as the mark space, however, for the purposes of this work, we require the existence of a norm.

2. Denote for $\gamma \in \mathcal{M}(\mathcal{E})$, $z \in \mathbb{R}^d$ and $\Lambda \in \mathcal{B}^d$

$$\begin{aligned}
\gamma_\Lambda &= \sum_{i: x_i \in \Lambda} \delta_{(x_i, m_i)} && \text{... the restriction of } \gamma \text{ to } \Lambda \times \mathcal{S}, \\
|\gamma| &= \gamma(\mathcal{E}) && \text{... the number of atoms of } \gamma, \\
\mathsf{m}(\gamma) &= \sup_{(x,m) \in \gamma} \|m\| && \text{... the supremum of norms of all marks in } \gamma, \\
\gamma + z &= \sum_{(x,m) \in \gamma} \delta_{(x+z, m)} && \text{... the measure } \gamma \text{ shifted by the vector } z.
\end{aligned}$$

3. For $\gamma, \xi \in \mathcal{M}(\mathcal{E})$ we denote by

$$\gamma\,\xi = \sum_{(x,m) \in \gamma} \delta_{(x,m)} + \sum_{(y,n) \in \xi} \delta_{(y,n)}$$

the sum of measures γ and ξ. Notice that $\gamma\,\xi$ always belongs to $\in \mathcal{N}_m^*(\mathcal{E})$, but it does not necessarily lie in $\mathcal{M}(\mathcal{E})$. However, it does hold that $\gamma_A\,\xi_B$ belongs to $\mathcal{M}(\mathcal{E})$ for $A, B \in \mathcal{B}^d$, $A \cap B = \emptyset$.

4. For $\gamma \in \mathcal{M}(\mathcal{E})$, $a > 0$ and $\Lambda \in \mathcal{B}^d$ define special subsets of $\mathcal{M}$:

$$\mathcal{M}_\Lambda(\mathcal{E}) = \{\gamma \in \mathcal{M}(\mathcal{E}) : \gamma = \gamma_\Lambda\}$$

is the set of all measures whose atoms lie in $\Lambda \times \mathcal{S}$,

$$\mathcal{M}_f(\mathcal{E}) = \{\gamma \in \mathcal{M}(\mathcal{E}) : |\gamma| < \infty\}$$

is the set of all finite measures $\gamma \in \mathcal{M}(\mathcal{E})$ and

$$\mathcal{M}_a(\mathcal{E}) = \{\gamma \in \mathcal{M}(\mathcal{E}) : \mathsf{m}(\gamma) \le a\}$$

is the set of measures whose marks have norm at most a.

5. Let $\gamma \in \mathcal{M}(\mathcal{E})$ and let $f : \mathcal{E} \to \mathbb{R}$ be a measurable γ-integrable function. Then we will write

$$\langle \gamma, f \rangle = \int f(\boldsymbol{x}) \gamma(\mathrm{d}\boldsymbol{x}) = \sum_{\boldsymbol{x} \in \gamma} f(\boldsymbol{x}).$$

6. We will often use the term *configuration* for $\gamma \in \mathcal{M}(\mathcal{E})$. If the state space $\mathcal{E}$ is clear from the context, we will write $\mathcal{M}$ instead of $\mathcal{M}(\mathcal{E})$.

Let us now recall the standard definition of a point process with density in the setting of marked point processes.

1.1.1 Processes with density

Let π_μ be the distribution of a simple marked Poisson point process in $\mathcal{E}$ with finite intensity measure μ. Then it is a finite process (i. e. $\pi_\mu(\mathcal{M}_f) = 1$) and for every measurable set $U \in \mathfrak{M}(\mathcal{E})$ we can write

$$\begin{aligned}
\pi_\mu(U) =& e^{-\mu(\mathcal{E})} \cdot \mathbb{1}\{\bar{o} \in U\} \\
&+ e^{-\mu(\mathcal{E})} \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathcal{E}} \cdots \int_{\mathcal{E}} \mathbb{1}\left\{\sum_{i=1}^{k} \delta_{\boldsymbol{x}_i} \in U\right\} \mu(\mathrm{d}\boldsymbol{x}_1) \cdots \mu(\mathrm{d}\boldsymbol{x}_k).
\end{aligned} \tag{1.1}$$

It is well known that the points of the Poisson point process do not interact with each other. To define more complicated models with interactions between the points, the following definition is often useful. We consider π_μ–integrable non–negative function, which specifies our new model, and define the new (finite) point process as the process with absolutely continuous distribution w. r. t. π_μ. As we will see in the next section, this is how we define the finite–volume Gibbs process.

Definition 6. *Let $p : \mathcal{M}_f \to [0, \infty)$ be a measurable function, which satisfies $\int p(\gamma)\, \pi_\mu(\mathrm{d}\gamma) = 1$. Then we define* ***a point process with density*** *p* ***w. r. t.*** *π_μ, as the marked point process Φ with distribution*

$$\mathsf{P}_\Phi(\mathrm{d}\gamma) = p(\gamma)\, \pi_\mu(\mathrm{d}\gamma). \tag{1.2}$$

Particularly, it easy to see, using (1.1), that for a measurable set $U \in \mathfrak{M}(\mathcal{E})$ we can write

$$\begin{aligned}\mathsf{P}_\Phi(U) &= e^{-\mu(\mathcal{E})} \cdot \mathbb{1}\{\bar{o} \in U\} \cdot p(\bar{o})\\ &+ e^{-\mu(\mathcal{E})} \cdot \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathcal{E}} \cdots \int_{\mathcal{E}} \mathbb{1}\left\{\sum_{i=1}^{k} \delta_{\boldsymbol{x_i}} \in U\right\} p\left(\sum_{i=1}^{k} \delta_{\boldsymbol{x_i}}\right) \mu(\mathrm{d}\boldsymbol{x_1}) \cdots \mu(\mathrm{d}\boldsymbol{x_k}).\end{aligned}$$

Clearly from (1.2) the distribution P_Φ favours configurations with higher values of p.

1.2 Tempered Configurations

From now on we fix $\delta > 0$. Before we dive into the theory of Gibbs processes, we will define a special set $\mathcal{M}^{temp} \subset \mathcal{M}$ of so-called *tempered configurations* (for reference see Section 2.2 in Rœlly and Zass [2020]). As we will see later, in Section 2.2.3, the infinite–volume Gibbs measure is concentrated on the set of tempered configurations and therefore many of the assumptions and theoretical results can be stated only for the set $\mathcal{M}^{temp}$.

Remark. For $x \in \mathbb{R}^d$ and $r > 0$ we denote by $U(x, r)$ the open ball with centre x and radius r and by $B(x, r)$ the closed ball with centre x and radius r. The complement of a set $A \subset \mathbb{R}^d$ will be denoted by A^c.

Definition 7. *Define the set of* ***tempered configurations***

$$\mathcal{M}^{temp} = \bigcup_{t \in \mathbb{N}} \mathcal{M}^t,$$

where $\mathcal{M}^t = \left\{\gamma \in \mathcal{M} : \left\langle \gamma_{U(0,l)}, (1 + \|m\|^{d+\delta}) \right\rangle \leq t \cdot l^d \text{ holds } \forall l \in \mathbb{N}\right\}$.

Clearly, for $t < s$, we have that $\mathcal{M}^t \subset \mathcal{M}^s$. Tempered configurations have the following important properties (see Lemmas 1 and 2 from Rœlly and Zass [2020]) which we present together with their proofs.

Lemma 1. *Let $\gamma \in \mathcal{M}^t$, $t \geq 1$, then it holds that*

$$\lim_{l \to \infty} \frac{1}{l} \mathsf{m}(\gamma_{U(0,l)}) = 0. \tag{1.3}$$

Furthermore there exists $l(t)$ such that $\forall l \geq l(t)$ the following implication holds

$$(x, m) \in \gamma_{U(0,2l+1)^c} \implies B(x, \|m\|) \cap U(0, l) = \emptyset. \tag{1.4}$$

Proof. Fix $t \in \mathbb{N}$ and take $\gamma \in \mathcal{M}^t$. Let us prove (1.3). From the definition of the set $\mathcal{M}^t$, we can write for all $l \in \mathbb{N}$ that

$$\mathsf{m}(\gamma_{U(0,l)}) \leq \left(tl^d\right)^{\frac{1}{d+\delta}} \implies \frac{\mathsf{m}(\gamma_{U(0,l)})}{l} \leq \frac{\left(tl^d\right)^{\frac{1}{d+\delta}}}{l} = t^{\frac{1}{d+\delta}} \cdot l^{\frac{-\delta}{d+\delta}} \underset{l\to\infty}{\longrightarrow} 0. \tag{1.5}$$

Now consider (1.4). At first, define

$$l_1\left(t, \frac{1}{2}\right) = \left(\frac{t}{(1/2)^{d+\delta}}\right)^{\frac{1}{\delta}}.$$

Then for all $l \geq l_1\left(t, \frac{1}{2}\right)$ we can write, using (1.5)

$$\frac{\mathsf{m}(\gamma_{U(0,l)})}{l} \leq \frac{\left(tl^d\right)^{\frac{1}{d+\delta}}}{l} = t^{\frac{1}{d+\delta}} \cdot l^{\frac{-\delta}{d+\delta}} \leq t^{\frac{1}{d+\delta}} \cdot \left(l_1\left(t, \frac{1}{2}\right)\right)^{\frac{-\delta}{d+\delta}} = \frac{1}{2}. \tag{1.6}$$

Let $l(t) = \frac{1}{2}l_1\left(t, \frac{1}{2}\right)$ and take $l \geq l(t)$. Denote for $a > 0 : \lceil a \rceil = \lfloor a \rfloor + 1$, where $\lfloor a \rfloor$ is the integer part of a (i. e. the largest natural number less or equal to a).

Then for any point $(x, m) \in \gamma_{U(0,2l+1)^c}$ we have that $x \in U(0, \lceil |x| \rceil)$ and $|x| \geq 2l + 1$. Therefore $\lceil |x| \rceil \geq l_1\left(t, \frac{1}{2}\right)$ and using (1.6) for $\lceil |x| \rceil$, we can write

$$|x| - \|m\| \geq |x| - \frac{1}{2}\lceil |x| \rceil \geq \frac{1}{2}|x| - \frac{1}{2} \geq l, \tag{1.7}$$

which completes the proof, since if there existed $z \in B(x, \|m\|) \cap U(0, l)$, then we would have $|x| \leq |x - z| + |z| < \|m\| + l$, which is a contradiction with (1.7). □

Notice, that $l(t)$ depends only on t, i. e. the implication (1.4) holds $\forall \gamma \in \mathcal{M}^t$. We also need to define the following increasing sequence of subsets of $\mathcal{M}^{temp}$, whose definition is inspired by Lemma 1.

Definition 8. *Take $l \in \mathbb{N}$ and define*

$$\underline{\mathcal{M}}^l = \Big\{\gamma \in \mathcal{M}^{temp} : \forall k \in \mathbb{N},\, k \geq l,\, \forall (x, m) \in \gamma_{U(0,2k+1)^c} \\ B(x, \|m\|) \cap U(0, k) = \emptyset\Big\}.$$

We can see from Lemma 1 that $\forall t \geq 1$ we have $\mathcal{M}^t \subset \underline{\mathcal{M}}^{\lceil l(t) \rceil}$. We can also see that

$$\mathcal{M}^{temp} = \bigcup_{l \in \mathbb{N}} \underline{\mathcal{M}}^l.$$

For simplicity, we will write $\underline{\mathcal{M}}^{l(t)}$ instead of $\underline{\mathcal{M}}^{\lceil l(t) \rceil}$ in the following text.

Remark. The sets $\mathcal{M}^t$, $t \geq 1$ and $\underline{\mathcal{M}}^l$, $l \geq 1$, and consequently also $\mathcal{M}^{temp}$, are measurable. It also holds that whenever $\xi \in \mathcal{M}^t$ then $\forall B \subset \mathbb{R}^d$ also $\xi_{B^c} \in \mathcal{M}^t$ and analogously for $\underline{\mathcal{M}}^l$: $\xi \in \underline{\mathcal{M}}^l \implies \xi_{B^c} \in \underline{\mathcal{M}}^l, \forall B \subset \mathbb{R}^d$.

1.3 Gibbs Measures and Processes

In this section, we summarize the theory for marked Gibbs point processes. As a foundation we take the theory presented in Rœlly and Zass [2020] and enlarge it to give broader introduction to the theory of Gibbs point processes, with Dereudre [2019] as our reference.

1.3.1 Finite–volume Gibbs measures

As we have stated before, the finite–volume Gibbs point process is defined as a point process with density (see Section 1.1.1). For us, the *reference distribution* π_μ will be the distribution of an independently marked Poisson point process.

The density of Gibbs process depends on energy of a configuration.

Definition 9. ***An energy function*** *is a mapping* $H : \mathcal{M}_f \to \mathbb{R} \cup \{+\infty\}$ *which is measurable, translation invariant and satisfies*[2] $H(\bar{o}) = 0$.

Consider a probability measure Q on the mark space $(\mathcal{S}, \|\cdot\|)$, which will serve as our *reference mark distribution.* Random variable with values in $\mathcal{S}$ distributed according to Q will be called a *typical mark.* Take $\Lambda \in \mathcal{B}_b^d$ and $z > 0$ and denote by π_Λ^z the distribution of the marked Poisson point process in $\mathcal{E}$ with intensity measure $z\lambda_\Lambda(\mathrm{d}x) \bigotimes \mathsf{Q}(\mathrm{d}m)$, where $\lambda_\Lambda(\mathrm{d}x)$ is the restriction of the Lebesgue measure λ on Λ and $\cdot \bigotimes \cdot$ denotes the standard product of measures. Now we define Gibbs processes.

Definition 10. *Take* $\Lambda \in \mathcal{B}_b^d$, $z > 0$ *and energy function* H. *Then* ***finite–volume Gibbs process*** *in* Λ *with energy function* H, *activity* z *and with free boundary condition is the point process with density* p *w.r.t.* π_Λ^z, *where*

$$p(\gamma) = \frac{1}{Z_\Lambda} \cdot e^{-H(\gamma_\Lambda)}.$$

Here, Z_Λ *is called the* ***partition function***, $Z_\Lambda = \int e^{-H(\gamma_\Lambda)} \pi_\Lambda^z(\mathrm{d}\gamma)$.

Notice that Gibbs processes favour configurations with small energy compared to those with high energy, which is in accordance with the physical interpretation. It is possible for a configuration γ to have infinite energy and such configurations are called *forbidden*, since they occur with probability 0.

Throughout this text we mostly work with the distributions of Gibbs processes, which are called Gibbs measures.

Definition 11. *Take* $\Lambda \in \mathcal{B}_b^d$, $z > 0$ *and energy function* H. *We then define* ***the finite–volume Gibbs measure*** *in* Λ *with energy function* H, *activity* z *and with free boundary condition as the distribution* P_Λ *of the corresponding Gibbs process, i. e.*

$$\mathsf{P}_\Lambda(\mathrm{d}\gamma) = \frac{1}{Z_\Lambda} \cdot e^{-H(\gamma_\Lambda)}\, \pi_\Lambda^z(\mathrm{d}\gamma).$$

[2]The assumption that $H(\bar{o}) = 0$ is in fact not restricting. Consider function H such that $H(\bar{o}) = a$, $a \in \mathbb{R}$, $a \neq 0$. Then we can take $\tilde{H} = H - a$.

Clearly, for the finite–volume Gibbs measure and Gibbs process to be well defined, we need $0 < Z_\Lambda < \infty$. This does not hold in general for every energy function H. Let us briefly comment on some standard assumptions for the energy function of a Gibbs point process. Our set of assumptions on H and Q, under which the inequalities $0 < Z_\Lambda < \infty$ hold, will be specified later, in Section 2.1.2.

At first consider the *hereditarity* assumption:

$$\forall \gamma \in \mathcal{M}_f, (x,m) \in \gamma : \; H(\gamma) < \infty \implies H(\gamma - \delta_{(x,m)}) < \infty.$$

It means that removing a point from a configuration, which is not forbidden, cannot lead to a forbidden configuration[3].

Another standard assumption is the *stability* assumption:

$$\exists C \in \mathbb{R} : H(\gamma) \geq C \cdot |\gamma|, \forall \gamma \in \mathcal{M}_f.$$

Notice that this assumption ensures, that the partition function is finite.

To state some examples, we present the following two energy functions (see Example 2 in Rœlly and Zass [2020]), which are based on interactions between pairs of points. Some other examples can be found in Dereudre [2019], Section 5.2.2. We will treat more complicated models in the following chapters.

Example (Pairwise interaction models)*.* Let $\mathcal{E} = \mathbb{R}^d \times \mathcal{S}$ and consider

$$\begin{aligned} H_1(\gamma) &= \sum_{(x,m),\,(y,n)\in\gamma}^{\neq} \phi(x,y) \cdot \mathbb{1}\{|x-y| \leq \|m\| + \|n\|\} \\ H_2(\gamma) &= \sum_{(x,m),\,(y,n)\in\gamma}^{\neq} (+\infty) \cdot \mathbb{1}\{|x-y| \leq \|m\| + \|n\|\} \end{aligned} \tag{1.8}$$

where $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a non–negative measurable function, called the *pair potential.* We use the convention $+\infty \cdot 0 = 0$.

One of the key properties of a finite–volume Gibbs measure is that it satisfies the **DLR** equations, named after Dobrushin, Lanford and Ruelle. These equations prescribe the conditional distributions for a configuration inside some bounded window Δ given a fixed configuration outside this window (i. e. given a fixed boundary condition).

Proposition 2. *Let* $\Lambda, \Delta \in \mathcal{B}_b^d$*,* $\Delta \subset \Lambda$*. Then it holds for any bounded measurable function* $F : \mathcal{M} \to \mathbb{R}$ *that*

$$\int_{\mathcal{M}_\Lambda} F(\gamma)\, \mathsf{P}_\Lambda(\mathrm{d}\gamma) = \int_{\mathcal{M}_\Lambda} \int_{\mathcal{M}_\Delta} F(\gamma_\Delta \xi_{\Delta^c}) \frac{1}{Z_\Lambda^\Delta(\xi)} e^{-(H(\gamma_\Delta\xi_{\Delta^c}) - H(\xi_{\Delta^c}))} \pi_\Delta^z(\mathrm{d}\gamma)\, \mathsf{P}_\Lambda(\mathrm{d}\xi),$$

where $Z_\Lambda^\Delta(\xi) = \int_{\mathcal{M}_\Delta} e^{-(H(\gamma_\Delta\xi_{\Delta^c}) - H(\xi_{\Delta^c}))} \pi_\Delta^z(\mathrm{d}\gamma)$ *is the normalizing constant.*

Proof. See Proposition 5.3 in Dereudre [2019]. □

[3]While Rœlly and Zass [2020] do not state this assumption explicitly, we note that without hereditarity we could get the conditional energy (see Definition 12) equal to $-\infty$, which would lead to contradiction with the local stability assumption (see Section 2.1.2).

1.3.2 Infinite–volume Gibbs measures

Although there is a natural generalization of the measures π_Λ^z to π^z, where π^z is the distribution of a marked Poisson point process with intensity measure $z\lambda(\mathrm{d}x) \otimes \mathsf{Q}(\mathrm{d}m)$, we cannot generalize the definition of a finite–volume Gibbs measure for an infinite–volume Gibbs measure using density w. r. t. the measure π^z. One reason is that we would need a suitable generalization for energy of an infinite configuration (as it could be $+\infty$ simply due to the infinite number of points even for a configuration which should not be forbidden).

Therefore we need to use a different approach, based on the conditional energy of a configuration γ in a bounded window Λ.

Definition 12. *For energy function H and $\Lambda \in \mathcal{B}_b^d$ we define* ***the conditional energy of*** $\gamma \in \mathcal{M}$ ***in*** Λ *given its environment as*

$$H_\Lambda(\gamma) = \lim_{n\to\infty} H(\gamma_{\Lambda_n}) - H(\gamma_{\Lambda_n\setminus\Lambda}), \tag{1.9}$$

where $\Lambda_n = [-n, n)^d$.

Again, it is not clear whether the conditional energy is well defined or not. Indeed, we will need to pose some assumptions on H for this to be true. We will in fact work with such energy functions, for which the limit (1.9) is attained in finitely many steps.

Let us note that the basic assumption for the conditional energy is the *finite–range* assumption:

$$\exists R > 0 \text{ such that } \forall\gamma \in \mathcal{M}, \forall\Lambda \in \mathcal{B}_b^d : H_\Lambda(\gamma) = H(\gamma_{\Lambda\oplus B(0,R)}) - H(\gamma_{\Lambda\oplus B(0,R)\setminus\Lambda}),$$

where $\Lambda \oplus B(0, R) = \{x \in \mathbb{R}^d : \exists y \in \Lambda,\ |x - y| \le R\}$. This means that the range of interactions between the points is uniformly bounded over all configurations. On the contrary, Rœlly and Zass [2020] deals with a situation, where the range is finite, but unbounded, i. e. the finite–range assumption is not satisfied. We will address our range assumption in Section 2.1.3.

To define an infinite–volume Gibbs measure, we need the following definition.

Definition 13. *Let $\Lambda \in \mathcal{B}_b^d$. Function $F : \mathcal{M} \to \mathbb{R}$ is called* Λ***-local*** *if it satisfies $F(\gamma) = F(\gamma_\Lambda)$ for all $\gamma \in \mathcal{M}$. Function $F : \mathcal{M} \to \mathbb{R}$ is called* ***local****, if there exists $\Lambda \in \mathcal{B}_b^d$ such that it is Λ-local.*

In other words, the value of a local function depends only on the configuration in a bounded window. We also need to define a Gibbs probability kernel.

Definition 14. *For $\Lambda \in \mathcal{B}_b^d$, $z > 0$, energy function H and $\xi \in \mathcal{M}$, define* ***the Gibbs probability kernel*** *associated to H as*

$$\Xi_\Lambda(\xi, \mathrm{d}\gamma) = \frac{e^{-H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c})}}{Z_\Lambda(\xi)} \pi_\Lambda^z(\mathrm{d}\gamma),$$

where $Z_\Lambda(\xi) = \int e^{-H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c})} \pi_\Lambda^z(\mathrm{d}\gamma)$ is the normalizing constant.

Again, for $\Xi_\Lambda(\xi, \mathrm{d}\gamma)$ to be well defined, we need $0 < Z_\Lambda(\xi) < \infty$. We will show that under the assumptions given in Section 2.1.2, this will be true for $\xi \in \mathcal{M}^{temp}$. Notice that $Z_\Lambda(\xi) = Z_\Lambda(\xi_{\Lambda^c})$ for any $\xi \in \mathcal{M}$.

Now we can define an infinite–volume Gibbs measure as the probability measure on $\mathcal{M}$, which satisfies the **DLR** equations. This definition follows naturally from the fact that the finite–volume Gibbs measure also satisfies **DLR**.

Definition 15. *A probability measure* P *on* $\mathcal{M}$ *is called* ***an infinite–volume Gibbs measure*** *with energy function* H *and activity* z*, if for all* $\Lambda \in \mathcal{B}_b^d$ *and for all measurable bounded local functions* $F : \mathcal{M} \to \mathbb{R}$ *the* $\boldsymbol{DLR}_\Lambda$ *equation holds:*

$$\int_{\mathcal{M}} F(\gamma)\,\mathsf{P}(\mathrm{d}\gamma) = \int_{\mathcal{M}} \int_{\mathcal{M}_\Lambda} F(\gamma_\Lambda \xi_{\Lambda^c})\,\Xi_\Lambda(\xi, \mathrm{d}\gamma)\,\mathsf{P}(\mathrm{d}\xi).$$

For completion, we also define an infinite–volume Gibbs process.

Definition 16. *A marked point process* Φ *on* $\mathcal{E}$ *is called an* ***infinite–volume marked Gibbs point process*** *with energy function* H *and activity* z*, if its distribution is an infinite volume Gibbs measure.*

Now the questions are, whether and under what assumptions such measure exists and whether it is uniquely defined by the **DLR** equations. The existence problem will be addressed in the next chapter, where we prove that under some variations of the stability and range assumptions on the energy function H, an infinite–volume Gibbs measure exists. Many existence results for different models have been published over the years.

The uniqueness problem is, as far as we have seen, a much harder problem, which is not always addressed. It is believed, however, that for small activity z and (in some sense) low energy function, the Gibbs measure is unique. For more information and references on this problem as well as proofs for a uniqueness and a non–uniqueness result, see Dereudre [2019], Sections 5.3.7 and 5.3.8. For an example of a sufficient condition for uniqueness of the Gibbs measure with non–negative pairwise potential see Jansen [2019].

1.3.3 Topology of local convergence

The standard method used to obtain an infinite–volume Gibbs measure is based on the result from Georgii and Zessin [1993]. At first, we have to define a suitable topology on the space $\mathcal{P}(\mathcal{M})$ of probability measures on $\mathcal{M}$.

Definition 17. *A function* F *on* $\mathcal{M}$ *is called* ***tame****, if there exists* $a > 0$ *such that* $|F(\gamma)| \leq a\left(1 + \left\langle \gamma, 1 + \|m\|^{d+\delta} \right\rangle\right)$.

Recall that function F is local, if for some $\Lambda \in \mathcal{B}_b^d$ we have $F(\gamma) = F(\gamma_\Lambda)$ for all configurations γ.

Definition 18. *Denote by* $\mathcal{L}$ *the set of all tame local functions* $F : \mathcal{M} \to \mathbb{R}$*. We define* ***the topology*** $\tau_{\mathcal{L}}$ ***of local convergence*** *on* $\mathcal{P}(\mathcal{M})$ *as the smallest topology such that* $\forall F \in \mathcal{L}$ *the mapping* $\mathsf{P} \to \int F \mathrm{d}\mathsf{P}$ *is continuous.*

Finally we define a relative and specific entropy for two probability measures, which are important tools in existence proofs. We remark that $\mathsf{P} \ll \mathsf{P}'$ denotes that P is absolutely continuous w. r. t. P' and the corresponding density is denoted by $\frac{d\mathsf{P}}{d\mathsf{P}'}$.

Definition 19. *Let* $\Lambda \in \mathcal{B}_b^d$ *and take two probability measures* P, P' *on* $\mathcal{M}$. *Then we define the* ***relative entropy of*** P ***with respect to*** P' ***on*** Λ *as*

$$I_\Lambda(\mathsf{P}|\mathsf{P}') = \int \log f \, \mathrm{d}\mathsf{P}_\Lambda \quad \text{if } \mathsf{P}_\Lambda \ll \mathsf{P}'_\Lambda \text{ and } f = \frac{d\mathsf{P}_\Lambda}{d\mathsf{P}'_\Lambda},$$
$$I_\Lambda(\mathsf{P}|\mathsf{P}') = +\infty \qquad \text{otherwise,}$$

where P_Λ *denotes the image of* P *under the mapping* $\gamma \to \gamma_\Lambda$.

We then define ***specific entropy of*** P ***with respect to*** P' *as*

$$\mathcal{I}(\mathsf{P}|\mathsf{P}') = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} I_{\Lambda_n}(\mathsf{P}|\mathsf{P}').$$

For us the reference measure P' is the distribution of a marked Poisson point process with intensity measure $z\lambda(\mathrm{d}x) \bigotimes \mathsf{Q}(\mathrm{d}m)$, $\mathsf{P}' = \pi^z$. Let $z \in \mathbb{R}^d$ and recall that $\mathsf{P} \in \mathcal{P}(\mathcal{M})$ is invariant under translation ϑ_z,

$$\vartheta_z : \mathcal{M} \to \mathcal{M}, \, \vartheta_z(\gamma) = \gamma + z,$$

if $\mathsf{P} = \mathsf{P} \circ \vartheta_z^{-1}$, where $\mathsf{P} \circ \vartheta_z^{-1}$ is the image of measure P under ϑ_z. Then we have the key property.

For any $a > 0$ *the level sets*

$$\mathcal{P}(\mathcal{M})_a = \{\mathsf{P} \in \mathcal{P}(\mathcal{M}) : \mathsf{P} \text{ invariant under } \vartheta_\kappa, \, \kappa \in \mathbb{Z}^d, \, \mathcal{I}(\mathsf{P}|\pi^z) \le a\} \qquad (1.10)$$

are relatively compact in the $\tau_\mathcal{L}$ *topology.*

Specially, any sequence $(\mathsf{P}_k)_{k\in\mathbb{N}} \subset \mathcal{P}(\mathcal{M})_a$ has a subsequence with limit $\mathsf{P} \in \mathcal{P}(\mathcal{M})$ in the $\tau_\mathcal{L}$ topology. It holds that this limit is also invariant under translations by $\kappa \in \mathbb{Z}^d$. As a standard reference, we state[4] Proposition 2.6 from Georgii and Zessin [1993].

[4] However, we add that this article works in the setting of stationary probability measures (i. e. invariant under translation by all $z \in \mathbb{R}^d$). In our work, we have measures invariant under translations by $z \in \mathbb{Z}^d$. We appeal to Dereudre [2009] and Rœlly and Zass [2020] for a reference for this case.

2. Existence of Marked Gibbs Point Processes with Unbounded Interactions

In this section, we present the proof of the existence of an infinite–volume Gibbs measure from Section 3 in Rœlly and Zass [2020]. At first, we present the assumptions on the energy function and reference mark distribution. As we state in Section 2.1.3, we have some objections to the formulation of the range assumption and its use in the proof. We propose a modified version of this assumption, which overcomes the particular problems, and afterwards go through specific parts of the proof to see that it still holds.

2.1 Assumptions

To be able to prove the existence of an infinite–volume Gibbs measure, we will need the following four assumptions: the moment assumption $\mathcal{H}_m$, the stability assumption $\mathcal{H}_s$, the local stability assumption $\mathcal{H}_l$ and the range assumption $\mathcal{H}_r$.

2.1.1 The moment assumption

Recall that we have fixed $\delta > 0$ in Section 1.2 and we have chosen a reference mark distribution Q in Section 1.3.1. We need to assume that Q satisfies

$$\mathcal{H}_m : \int_{\mathcal{S}} \exp(\|m\|^{d+2\delta})\mathsf{Q}(\mathrm{d}m) < \infty.$$

This means that the distribution of the norm of the typical mark $\|M\|$ has super-exponential moment.

2.1.2 Stability assumptions

We now pose a version of the stability assumption for the energy function H.

$$\mathcal{H}_s : \text{There exists } c \geq 0 \text{ such that } \forall \gamma \in \mathcal{M}_f : H(\gamma) \geq -c \left\langle \gamma, 1 + \|m\|^{d+\delta} \right\rangle.$$

Notice that this assumption is weaker than the standard stability assumption (see Section 1.3.1) thanks to the additional term $\|m\|^{d+\delta}$. It is clear that if H is non-negative, then $\mathcal{H}_s$ holds trivially for $c = 0$.

As we have promised, we will now show that under assumptions $\mathcal{H}_m$ and $\mathcal{H}_s$, the partition function is finite and therefore the finite–volume Gibbs measures are well defined.

Lemma 3. *Under the assumptions $\mathcal{H}_s$ and $\mathcal{H}_m$ we have $0 < Z_\Lambda < \infty$, $\forall \Lambda \in \mathcal{B}_b^d$.*

Proof. We have that

$$Z_\Lambda = \int e^{-H(\gamma_\Lambda)} \pi_\Lambda^z(\mathrm{d}\gamma) \geq e^{-H(\bar{o})} \pi_\Lambda^z(\{\bar{o}\}) = e^{-z|\Lambda|} > 0.$$

On the other hand we can write

$$Z_\Lambda = \int e^{-H(\gamma_\Lambda)}\pi^z_\Lambda(\mathrm{d}\gamma) \overset{\mathcal{H}_s}{\leq} \int e^{c\left\langle \gamma, 1+\|m\|^{d+\delta}\right\rangle}\pi^z_\Lambda(\mathrm{d}\gamma)$$
$$= \exp\{-z\,|\Lambda|\}\cdot\exp\{e^c z\,|\Lambda|\int_{\mathcal{S}}\exp(c\|m\|^{d+\delta})\mathsf{Q}(\mathrm{d}m)\} \overset{\mathcal{H}_m}{<} \infty.$$

The second equality holds thanks to the formula (1.1) and the Levi formula for the non–negative function $e^{c\left\langle \gamma, 1+\|m\|^{d+\delta}\right\rangle}$. □

For the infinite volume Gibbs measure to be well defined, we need an analogue of the stability assumption for the conditional energy – so-called *local stability assumption.*

$\mathcal{H}_l$: For all $\Lambda \in \mathcal{B}^d_b$ and all $t \in \mathbb{N}$ there exists $c(\Lambda, t) \geq 0$ such that $\forall \xi \in \mathcal{M}^t$ the following inequality holds for any $\gamma_\Lambda \in \mathcal{M}_\Lambda$:

$$H_\Lambda(\gamma_\Lambda\,\xi_{\Lambda^c}) \geq -c(\Lambda, t)\left\langle \gamma_\Lambda, 1+\|m\|^{d+\delta}\right\rangle.$$

Let us emphasize that the lower bound for the conditional energy must hold uniformly over $\mathcal{M}^t$.

Contrary to the stability assumption, local stability is not automatically satisfied for non-negative energy functions. However, it will often be the case. We state the following claim.

Claim 4. *Assume that the energy function H satisfies $H(\gamma_A) - H(\gamma_B) \geq 0$, for all $\gamma \in \mathcal{M}_f$ whenever $B \subset A$, $\forall A, B \in \mathcal{B}^d_b$. Then the conditional energy is non-negative and the local stability assumption $\mathcal{H}_l$ holds.*

Proof. Let $\gamma \in \mathcal{M}$ and $\Lambda \in \mathcal{B}^d_b$. Then $\exists K \in \mathbb{N}$ and points $\boldsymbol{x_1}, \ldots, \boldsymbol{x_K} \in \mathbb{R}^d \times \mathcal{S}$ such that $\gamma_\Lambda = \sum_{i=1}^K \delta_{\boldsymbol{x_i}}$ and we can write

$$H_\Lambda(\gamma) = \lim_{n\to\infty} H(\gamma_{\Lambda_n}) - H(\gamma_{\Lambda_n\setminus\Lambda})$$
$$= \lim_{n\to\infty}\sum_{i=1}^K H(\gamma_{\Lambda_n\setminus\{x_1,\ldots,x_{i-1}\}}) - H(\gamma_{\Lambda_n\setminus\{x_1,\ldots,x_i\}}) \geq 0.$$

□

In the same way the stability ensures that the partition function is finite, the local stability condition $\mathcal{H}_l$ ensures that the normalizing constants of Gibbs probability kernels are positive and finite for tempered configurations.

Lemma 5. *Let $\Lambda \in \mathcal{B}^d_b$ and $\xi \in \mathcal{M}^{temp}$. Under assumptions $\mathcal{H}_l$ and $\mathcal{H}_m$ we have that $0 < Z_\Lambda(\xi) < \infty$.*

Proof. Let $\xi \in \mathcal{M}^t$ and $\Lambda \in \mathcal{B}^d_b$. Then we can write

$$Z_\Lambda(\xi) = \int e^{-H_\Lambda(\gamma_\Lambda\xi_{\Lambda^c})}\pi^z_\Lambda(\mathrm{d}\gamma) \geq e^{-H_\Lambda(\xi_{\Lambda^c})}\pi^z_\Lambda(\{\bar{o}\}) = e^{-z|\Lambda|} > 0,$$

since $\forall \xi \in \mathcal{M}^t$ we have that $H_\Lambda(\xi_{\Lambda^c}) = 0$. On the other hand, in the same way as in the proof of Lemma 3:

$$Z_\Lambda(\xi) = \int e^{-H_\Lambda(\gamma_\Lambda\xi_{\Lambda^c})}\pi^z_\Lambda(\mathrm{d}\gamma) \overset{\mathcal{H}_l}{\leq} \int e^{c(\Lambda,t)\left\langle \gamma_\Lambda, 1+\|m\|^{d+\delta}\right\rangle}\pi^z_\Lambda(\mathrm{d}\gamma)$$
$$= \exp\{-z\,|\Lambda|\}\cdot\exp\{e^{c(\Lambda,t)} z\,|\Lambda|\int_{\mathcal{S}}\exp(c(\Lambda,t)\|m\|^{d+\delta})\mathsf{Q}(\mathrm{d}m)\} \overset{\mathcal{H}_m}{<} \infty.$$

□

2.1.3 The range assumption

The last assumption (and perhaps the most crucial one as $\mathcal{H}_s$ and $\mathcal{H}_l$ are often satisfied thanks to H being non-negative), considers the range of the interactions among the points. Contrary to the usual assumption of finite range, the existence result in Rœlly and Zass [2020] allows for the range to be finite, but unbounded in the sense that it can depend on the whole configuration. In other words, the range of the interaction is an unbounded random variable.

Let us at first state the range assumption from Rœlly and Zass [2020].

$\tilde{\mathcal{H}}_r$: Fix $\Lambda \in \mathcal{B}_b^d$. For any $\gamma \in \mathcal{M}^t$, $t \geq 1$, there exists $\tau(\gamma, \Lambda) > 0$ such that
$$H_\Lambda(\gamma) = H(\gamma_{\Lambda \oplus B(0,\tau(\gamma,\Lambda))}) - H(\gamma_{\Lambda \oplus B(0,\tau(\gamma,\Lambda))\setminus\Lambda}).$$

It is noted that the choice of $\tau(\gamma, \Lambda)$ can be

$$\tau(\gamma, \Lambda) = 2l(t) + 2\mathsf{m}(\gamma_\Lambda) + 1, \tag{2.1}$$

and this choice is used in the proof of the existence theorem. However, we have two comments regarding this choice.

1. Contrary to the claims in Rœlly and Zass [2020], this choice of $\tau(\gamma, \Lambda)$ does not work for the presented examples of the energy function. For a counterexample supporting our claim, see Lemma 6. It is not hard to see, using the proof of this lemma, that this choice of τ will not work for such models, where two points (x, m) and (y, n) interact with each other if $B(x, \|m\|) \cap B(y, \|n\|) \neq \emptyset$.

2. The choice of the range (2.1) assumes a very specific dependence of the range on the configuration both inside and outside of Λ. Particularly, the range depends on γ_{Λ^c} only through $l(t)$, which is later used in the proof to find a certain uniform estimate over $\mathcal{M}^t$. Therefore, the proof cannot be directly modified for general $\tau(\gamma, \Lambda)$.

In the following lemma, we present the counterexample, which shows that the choice of range (2.1) is not suitable for energy functions (1.8).

Lemma 6. *Let $\mathcal{E} = \mathbb{R}^2 \times \mathbb{R}$ and take energy function*

$$H(\gamma) = \sum_{(x,m),(y,n)\in\gamma}^{\neq} \phi(x, y) \cdot \mathbb{1}\{|x - y| \leq \|m\| + \|n\|\}.$$

Then $\forall \delta > 0$ there exist $\Lambda \in \mathcal{B}_b^2$ and a set $\mathcal{M}_C \subset \mathcal{M}^1$ such that $\forall \gamma \in \mathcal{M}_C$

$$\lim_{n\to\infty} H(\gamma_{\Lambda_n}) - H(\gamma_{\Lambda_n\setminus\Lambda}) = H_\Lambda(\gamma) \neq H(\gamma_{\Lambda\oplus B(0,\tau)}) - H(\gamma_{\Lambda\oplus B(0,\tau)\setminus\Lambda})$$

if we choose $\tau = 2l(1) + 2\mathsf{m}(\gamma_\Lambda) + 1$.

Proof. We will at first consider for simplicity $\delta = \frac{1}{2}$ and afterwards modify the example for general $\delta > 0$.

Step 1) Let $\delta = \frac{1}{2}$. It holds that (according to the proof od Lemma 1)

$$l(t) = \frac{1}{2} l_1\left(t, \frac{1}{2}\right) = \frac{1}{2} \cdot t^{\frac{1}{\delta}} \cdot 2^{\frac{2+\delta}{\delta}} = t^2 \cdot 2^4.$$

Therefore for $t = 1$ we get that $l(t) = 2^4 = 16$. Take points $(x, m), (y, n) \in \mathbb{R}^2 \times \mathbb{R}$, where $x = (120, 120)$, $m = 1$, $y = (150, 150)$ and $n = 43$. Let $\Lambda = B(x, \varepsilon)$, where $\varepsilon \in [0, 1]$ and set $\gamma = \delta_{(x,m)} + \delta_{(y,n)}$. Then it holds that

i) $\gamma \in \mathcal{M}^1$,

ii) $B(x,m) \cap B(y,n) \neq \emptyset$,

iii) $(y,n) \notin \gamma_{\Lambda \oplus B(0,\tau)}$ for $\tau = 2l(1) + 2\mathsf{m}(\gamma_\Lambda) + 1 = 35$.

Part i) can be easily shown by checking the definition, ii) follows from the simple computation $|x-y| = \sqrt{(150-120)^2 + (150-120)^2} = \sqrt{2} \cdot 30 < 43$ and iii) follows from $|x-y| = \sqrt{2} \cdot 30 > 36$.

We get that

$$\lim_{n\to\infty} H(\gamma_{\Lambda_n}) - H(\gamma_{\Lambda_n \setminus \Lambda}) = \phi(x,y),$$
$$H(\gamma_{\Lambda \oplus B(0,\tau)}) - H(\gamma_{\Lambda \oplus B(0,\tau) \setminus \Lambda}) = 0 - 0 = 0.$$

Choose $k \in \mathbb{N}$ such that $k \geq l(1)$ and $\Lambda \oplus B(0,1) \subset U(0,k)$ and define the set $\mathcal{M}_C = \{\gamma\, \xi_{U(0,2k+1)^c} : \xi \in \mathcal{M}^1, (x,m) \in \xi, (y,n) \in \xi\}$. Then we get that also $\forall \gamma \in \mathcal{M}_C$

$$\lim_{n\to\infty} H(\gamma_{\Lambda_n}) - H(\gamma_{\Lambda_n \setminus \Lambda}) = \phi(x,y),$$
$$H(\gamma_{\Lambda \oplus B(0,\tau)}) - H(\gamma_{\Lambda \oplus B(0,\tau) \setminus \Lambda}) = 0 - 0 = 0,$$

and $\mathcal{M}_C \subset \mathcal{M}^1$.

Step 2) Let $\delta > 0$. Then we can choose (x,m) and (y,n) in the following way:

1. $m = 1$ and $x = (x', 0)$ where x' is large enough so that
$$4 + 2 \cdot 2^{\frac{2}{\delta}} \leq \frac{1}{2} + ((4 + 2 \cdot 2^{\frac{2}{\delta}} + x')^2 - 3)^{\frac{1}{2+\delta}} \text{ and } x' > 1.$$

2. $y = (y', 0)$, where $y' = 4 + 2 \cdot 2^{\frac{2}{\delta}} + x'$ and $n = ((y')^2 - 3)^{\frac{1}{2+\delta}}$

Set $\gamma = \delta_{(x,m)} + \delta_{(y,n)}$ and choose $\Lambda = B(x, \varepsilon)$, where $\varepsilon \in \left[0, \frac{1}{2}\right]$. We again get that

i) $\gamma \in \mathcal{M}^1$

ii) $B(x,m) \cap B(y,n) \neq \emptyset$

iii) $(y,n) \notin \gamma_{\Lambda \oplus B(0,\tau)}$ for $\tau = 2l(1) + 2\mathsf{m}(\gamma_\Lambda) + 1 = 2 \cdot 2^{\frac{2}{\delta}} + 2 + 1$.

and the choice of $\mathcal{M}_C$ proceeds in the same way as in the first step. □

Particularly, we have found a counterexample to the claim that for any configuration $\gamma_\Lambda \in \mathcal{M}_\Lambda$ and any $\xi \in \mathcal{M}^t$ the equality $H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c}) = H_\Lambda(\gamma_\Lambda \xi_{\Delta \setminus \Lambda})$ holds as soon as $\Lambda \oplus B(0, 2l(t) + 2\mathsf{m}(\gamma_\Lambda) + 1) \subset \Delta$.

We have given the counterexample for $d = 2$, $t = 1$ and pairwise–interaction model, however it should be clear that it would be possible to find counterexamples in the same way for $d \geq 3$, $t \in \mathbb{N}$ and other energy functions, for which the interactions between two points is given strictly by the intersection of their respective balls.

Considering the two comments above, we propose the following modification of the range assumption.

$\mathcal{H}_r$: Fix $\Lambda \in \mathcal{B}_b^d$ and $l \in \mathbb{N}$. Then for all $\gamma \in \mathcal{M}^{temp}$ such that $\gamma_{\Lambda^c} \in \underline{\mathcal{M}}^l$ there exists $\tau = \tau(\mathsf{m}(\gamma_\Lambda), l, \Lambda) > 0$ such that

$$H_\Lambda(\gamma) = H(\gamma_{\Lambda \oplus B(0,\tau)}) - H(\gamma_{\Lambda \oplus B(0,\tau) \setminus \Lambda}),$$

holds and $\tau(\mathsf{m}(\gamma_\Lambda), l, \Lambda)$ is a non-decreasing function of $\mathsf{m}(\gamma_\Lambda)$. Particularly, τ depends on γ_{Λ^c} only through l.

In the next section, we will show that the proof of the existence theorem from Rœlly and Zass [2020] does work for the choice of the range assumed in $\mathcal{H}_r$ instead of (2.1).

2.2 The Proof of the Existence Result

From now on we assume that the energy function H satisfies $\mathcal{H}_l$, $\mathcal{H}_s$, $\mathcal{H}_m$ and our modified range assumption $\mathcal{H}_r$. The proof consists of the following four steps (corresponding to Sections 3.1 – 3.4 in Rœlly and Zass [2020]):

1. Introducing a stationarised sequence of probability measures $(\bar{\mathsf{P}}_n)_{n\in\mathbb{N}}$ with the help of the finite–volume Gibbs measures in windows $\Lambda_n \uparrow \mathbb{R}^d$.
2. Proving the existence of a limit measure $\bar{\mathsf{P}}$ for a subsequence of $(\bar{\mathsf{P}}_n)_{n\in\mathbb{N}}$ in the topology of local convergence.
3. Showing that the measures $\bar{\mathsf{P}}_n$ and $\bar{\mathsf{P}}$ are concentrated on the set of tempered configurations.
4. Introducing sequence $(\hat{\mathsf{P}}_n)_{n\in\mathbb{N}}$ of measures, that satisfy **DLR** and which has the same asymptotic behaviour as $(\bar{\mathsf{P}}_n)_{n\in\mathbb{N}}$. Using this sequence, we show that also $\bar{\mathsf{P}}$ satisfies **DLR** equations and is therefore an infinite–volume Gibbs measure.

Since steps 1,2 and 3 of the proof do not use the assumption $\mathcal{H}_r$, we will only state the necessary definitions and partial results from these sections and refer to the article Rœlly and Zass [2020] for the complete proofs. The last step will be presented in more detail.

2.2.1 Stationary measures $\bar{\mathsf{P}}_n$

Denote by $\mathsf{P}_n = \mathsf{P}_{\Lambda_n}$ the finite–volume Gibbs measure in Λ_n, where $\Lambda_n = [-n, n)^d$, $n \in \mathbb{N}$. For $n \in \mathbb{N}$ and $\kappa \in \mathbb{Z}^d$ set $\Lambda_n^\kappa = \Lambda_n + 2n\kappa$. Then $\{\Lambda_n^\kappa\}_{\kappa\in\mathbb{Z}^d}$ is a disjoint partition of the space $\mathbb{R}^d$.

For all $n \in \mathbb{N}$ let $\tilde{\mathsf{P}}_n$ be the probability measure on $\mathcal{M}$ satisfying that the marginal distributions of a configuration in disjoint sets Λ_n^κ are independent and identically distributed according to the finite volume Gibbs measure P_n, i.e.

$$\tilde{\mathsf{P}}_n = \bigotimes_{\kappa\in\mathbb{Z}^d} \mathsf{P}_n \circ \vartheta_{2n\kappa}^{-1}, \tag{2.2}$$

where $\mathsf{P} \circ \vartheta_\kappa^{-1}$ denotes the image of the measure P under the translation ϑ_κ, $\vartheta_\kappa : \mathcal{M} \to \mathcal{M}$, $\vartheta_\kappa(\gamma) = \gamma + z$. Then we can define the stationary sequence.

Definition 20. *For $n \in \mathbb{N}$ we define the empirical field associated to the probability measure $\tilde{\mathsf{P}}_n$ as a probability measure $\bar{\mathsf{P}}_n$:*

$$\bar{\mathsf{P}}_n = \frac{1}{(2n)^d} \sum_{\kappa \in \Lambda_n \cap \mathbb{Z}^d} \tilde{\mathsf{P}}_n \circ \vartheta_\kappa^{-1}.$$

The probability measures $\bar{\mathsf{P}}_n$, $n \in \mathbb{N}$ are invariant under $\vartheta_\kappa, \kappa \in \mathbb{Z}^d$ and the following upper bounds hold.

Lemma 7. *There exists a constant a_1 such that*

$$\forall n \in \mathbb{N} \qquad J_n = \int_{\mathcal{M}} \langle \gamma, 1 + \|m\|^{d+\delta} \rangle \mathsf{P}_n(\mathrm{d}\gamma) \leq a_1 \left| \Lambda_n \right|. \tag{2.3}$$

There also exists a constant a_2 such that

$$\forall n \in \mathbb{N} \qquad K_n = \int_{\mathcal{M}} \langle \gamma_{\Lambda_n}, 1 + \|m\|^{d+\delta} \rangle \bar{\mathsf{P}}_n(\mathrm{d}\gamma) \leq a_2 \left| \Lambda_n \right|. \tag{2.4}$$

Proof. For the proof of (2.3) see Lemma 5 in Rœlly and Zass [2020]. The inequalities (2.4) (stated as an observation without proof in Rœlly and Zass [2020]) can be proved using the following observation.

There exists $a^d \in \mathbb{N}$ and vectors $b_1, \ldots, b_{a^d} \in \mathbb{Z}^d$ such that $\forall n \in \mathbb{N}$ and $\forall \kappa \in \Lambda_n \cap \mathbb{Z}^d$: $(\Lambda_n - \kappa) \cap \Lambda_n^b = \emptyset$ whenever $b \neq b_i$ for all i. Therefore we can write $\forall n \in \mathbb{N}$:

$$\begin{aligned}
K_n &= \frac{1}{(2n)^d} \sum_{\kappa \in \Lambda_n \cap \mathbb{Z}^d} \int \langle \gamma_{\Lambda_n}, 1 + \|m\|^{d+\delta} \rangle \tilde{\mathsf{P}}_n \circ \vartheta_\kappa^{-1}(\mathrm{d}\gamma) \\
&= \frac{1}{(2n)^d} \sum_{\kappa \in \Lambda_n \cap \mathbb{Z}^d} \int \langle \gamma_{\Lambda_n - \kappa}, 1 + \|m\|^{d+\delta} \rangle \tilde{\mathsf{P}}_n(\mathrm{d}\gamma) \\
&= \frac{1}{(2n)^d} \sum_{\kappa \in \Lambda_n \cap \mathbb{Z}^d} \sum_{i=1}^{a^d} \int_{\mathcal{M}_{\Lambda_n^{b_i}}} \langle \gamma_{(\Lambda_n - \kappa) \cap \Lambda_n^{b_i}}, 1 + \|m\|^{d+\delta} \rangle \mathsf{P}_n \circ \vartheta_{2nb_i}^{-1}(\mathrm{d}\gamma) \\
&\leq \frac{1}{(2n)^d} \sum_{\kappa \in \Lambda_n \cap \mathbb{Z}^d} \sum_{i=1}^{a^d} \int_{\mathcal{M}_{\Lambda_n^{b_i}}} \langle \gamma_{\Lambda_n^{b_i}}, 1 + \|m\|^{d+\delta} \rangle \mathsf{P}_n \circ \vartheta_{2nb_i}^{-1}(\mathrm{d}\gamma) \\
&= \frac{1}{(2n)^d} \sum_{\kappa \in \Lambda_n \cap \mathbb{Z}^d} \sum_{i=1}^{a^d} \int_{\mathcal{M}_{\Lambda_n}} \langle \gamma, 1 + \|m\|^{d+\delta} \rangle \mathsf{P}_n(\mathrm{d}\gamma) \\
&\leq a_1 \cdot a^d \cdot \left| \Lambda_n \right|.
\end{aligned}$$

□

2.2.2 The existence of the limit measure $\bar{\mathsf{P}}$

Recall the definition of the topology of local convergence $\tau_{\mathcal{L}}$ from Section 1.3.3 as well as the definitions of the relative and the specific entropy. We will use the property (1.10) for level sets of the specific entropy.

Lemma 8. *Let $(\bar{\mathsf{P}}_n)_{n \in \mathbb{N}}$ be the stationarised sequence defined in Definition 20. Then there exists constant $a_3 > 0$ such that*

$$\forall n \in \mathbb{N} \qquad \mathcal{I}(\bar{\mathsf{P}}_n | \pi^z) \leq a_3.$$

Proof. See Proposition 1 in Rœlly and Zass [2020]. ☐

Therefore there exists a subsequence $(\bar{\mathsf{P}}_{n_k})_{k\in\mathbb{N}}$ such that $\bar{\mathsf{P}}_{n_k} \xrightarrow{\tau_\mathcal{L}} \bar{\mathsf{P}}$, where $\bar{\mathsf{P}}$ is a probability measure on $\mathcal{M}$ invariant under translations by $\kappa \in \mathbb{Z}^d$.

In the following text, we for simplicity denote the converging subsequence by $(\bar{\mathsf{P}}_n)_{n\in\mathbb{N}}$ instead of $(\bar{\mathsf{P}}_{n_k})_{k\in\mathbb{N}}$. Now our task is to show that the limit measure $\bar{\mathsf{P}}$ is the infinite–volume Gibbs measure.

2.2.3 Supports of measures $\bar{\mathsf{P}}_n$ and $\bar{\mathsf{P}}$

As we have stated earlier, the set of tempered configurations has the important property of containing the support of the infinite–volume Gibbs measure.

Lemma 9. *The measures $\bar{\mathsf{P}}_n$ and $\bar{\mathsf{P}}$ satisfy*

$$\forall n \in \mathbb{N} \qquad \bar{\mathsf{P}}_n(\mathcal{M}^{temp}) = 1 \qquad \text{and} \qquad \bar{\mathsf{P}}(\mathcal{M}^{temp}) = 1.$$

Proof. See Proposition 2 in Rœlly and Zass [2020]. ☐

Since $\mathcal{M}^{temp} = \bigcup_{t\in\mathbb{N}} \mathcal{M}^t$ and $\mathcal{M}^t \subset \mathcal{M}^{t+1}$, we can find $\forall n \in \mathbb{N}$ and $\forall \varepsilon > 0$ large enough t so that

$$\bar{\mathsf{P}}_n(\mathcal{M}^t) \geq 1 - \varepsilon.$$

However, in the next step of the proof, we need this t to be determined uniformly for all $n \in \mathbb{N}$. From the remark in Rœlly and Zass [2020], we know that this does not work for the sequence $\mathcal{M}^t$, $t \in \mathbb{N}$ but it is possible if we consider the sequence $\underline{\mathcal{M}}^l$, $l \in \mathbb{N}$.

Lemma 10. *Let $\varepsilon > 0$, then there exists $l \in \mathbb{N}$ such that*

$$\forall n \in \mathbb{N} \qquad \bar{\mathsf{P}}_n(\underline{\mathcal{M}}^l) \geq 1 - \varepsilon.$$

Proof. See Proposition 3 in Rœlly and Zass [2020]. ☐

2.2.4 $\bar{\mathsf{P}}$ is a Gibbs measure

In this final section, we will prove that the limit measure $\bar{\mathsf{P}}$ is indeed an infinite volume Gibbs measure with energy function H. At first, let us note the following observation concerning the Gibbs kernel Ξ_Λ (recall Definition 14).

Lemma 11. *For $\Lambda \in \mathcal{B}_b^d$ and $F : \mathcal{M} \to \mathbb{R}$ measurable function also the mapping $\xi \to \int_{\mathcal{M}_\Lambda} F(\gamma)\, \Xi_\Lambda(\xi, \mathrm{d}\gamma)$ defined on $\mathcal{M}^{temp}$ is measurable.*

Proof. See Lemma 7. in Rœlly and Zass [2020]. ☐

To prove that $\bar{\mathsf{P}}$ is a Gibbs measure, we will need the following estimate of the Gibbs probability kernel Ξ_Λ, which only considers bounded marks inside Λ and the outside environment only in some bounded set $\Delta \supset \Lambda$.

Definition 21. *For $\Delta \in \mathcal{B}_b^d$ such that $\Delta \supset \Lambda$ and for $m_0 > 0$ we define the (Δ, m_0)–**cut off** Ξ_Λ^{Δ,m_0} of the Gibbs kernel Ξ_Λ as the probability kernel*

$$\Xi_\Lambda^{\Delta,m_0}(\xi, \mathrm{d}\gamma) = \frac{\mathbb{1}\{\mathsf{m}(\gamma) \le m_0\} \cdot e^{-H_\Lambda(\gamma_\Lambda \xi_{\Delta\setminus\Lambda})}}{Z_\Lambda^{\Delta,m_0}(\xi_{\Delta\setminus\Lambda})} \pi_\Lambda^z(\mathrm{d}\gamma),$$

where $Z_\Lambda^{\Delta,m_0}(\xi_{\Delta\setminus\Lambda}) = \int \mathbb{1}\{\mathsf{m}(\gamma) \le m_0\} \cdot e^{-H_\Lambda(\gamma_\Lambda \xi_{\Delta\setminus\Lambda})} \pi_\Lambda^z(\mathrm{d}\gamma)$ is the normalizing constant.

We have the following remarks considering this definition.

i) The cut off Ξ_Λ^{Δ,m_0} is well defined, since $Z_\Lambda^{\Delta,m_0}(\xi_{\Delta\setminus\Lambda})$ is finite and positive.

ii) For any bounded measurable Λ-local function G the mapping

$$\xi \to \int_{\mathcal{M}_\Lambda} G(\gamma)\, \Xi_\Lambda^{\Delta,m_0}(\xi, \mathrm{d}\gamma)$$

is local (Δ-local), measurable and bounded. Particularly it belongs to $\mathcal{L}$.

The proof of i) would be the same as the proof of Lemma 5 and ii) is clear from the definition. The usefulness of this definition arises in the next lemma, where we prove that the cut–off Ξ_Λ^{Δ,m_0} is a uniform estimate of the Gibbs kernel Ξ_Λ over the set $\mathcal{M}^t$.

Lemma 12. *Let $\Lambda \in \mathcal{B}_b^d$ be a bounded set and take a measurable, bounded and Λ-local function $F : \mathcal{M} \to \mathbb{R}$. For any $\varepsilon > 0$ and any $t \in \mathbb{N}$ there exists $\underline{m_0} > 0$ and $\underline{\Delta} \supset \Lambda$ such that*

$$\sup_{\xi \in \mathcal{M}^t} \left| \int_{\mathcal{M}_\Lambda} F(\gamma)\, \Xi_\Lambda^{\Delta,m_0}(\xi, \mathrm{d}\gamma) - \int_{\mathcal{M}_\Lambda} F(\gamma)\, \Xi_\Lambda(\xi, \mathrm{d}\gamma) \right| \le \varepsilon, \tag{2.5}$$

whenever $\underline{\Delta} \subset \Delta$ and $\underline{m_0} \le m_0$.

Proof. Take $\gamma_\Lambda \in \mathcal{M}_\Lambda$ and $\xi \in \mathcal{M}^t$. Then from the assumption $\mathcal{H}_r$ we know that

$$H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c}) = H(\gamma_\Lambda \xi_{\Lambda \oplus B(0,\tau)\setminus\Lambda}) - H(\xi_{\Lambda \oplus B(0,\tau)\setminus\Lambda}),$$

where $\tau = \tau(\mathsf{m}(\gamma_\Lambda), l(t), \Lambda)$. Therefore for any $\xi \in \mathcal{M}^t$ we have that

$$\Lambda \oplus B(0, \tau(\mathsf{m}(\gamma_\Lambda), l(t), \Lambda)) \subset \Delta \implies H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c}) = H_\Lambda(\gamma_\Lambda \xi_{\Delta\setminus\Lambda}).$$

Denote for $\Delta \in \mathcal{B}_b^d$ and $m_0 > 0$

$$\mathbb{1}_{\Delta,m_0}(\gamma, l(t)) := \mathbb{1}\{\mathsf{m}(\gamma_\Lambda) > m_0 \text{ or } \Lambda \oplus B(0, \tau(\mathsf{m}(\gamma_\Lambda), l(t), \Lambda)) \not\subset \Delta\},$$

then we can write for any $\Delta \in \mathcal{B}_b^d$ and $m_0 > 0$

$$\begin{aligned} \left| Z_\Lambda^{\Delta,m_0}(\xi_{\Delta\setminus\Lambda}) - Z_\Lambda(\xi_{\Lambda^c}) \right| &= \left| \int (\mathbb{1}\{\mathsf{m}(\gamma_\Lambda) \le m_0\} e^{-H_\Lambda(\gamma_\Lambda \xi_{\Delta\setminus\Lambda})} - e^{-H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c})}) \pi_\Lambda^z(\mathrm{d}\gamma) \right| \\ &\le \int \mathbb{1}_{\Delta,m_0}(\gamma, l(t)) \cdot (e^{-H_\Lambda(\gamma_\Lambda \xi_{\Delta\setminus\Lambda})} + e^{-H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c})}) \pi_\Lambda^z(\mathrm{d}\gamma) \\ &\overset{\mathcal{H}_l}{\le} \int \mathbb{1}_{\Delta,m_0}(\gamma, l(t)) \cdot 2 \cdot e^{c(\Lambda,t)\langle \gamma_\Lambda, 1 + \|m\|^{d+\delta} \rangle} \pi_\Lambda^z(\mathrm{d}\gamma). \end{aligned}$$

This upper bound does not depend on ξ and therefore

$$\sup_{\xi\in\mathcal{M}^t}\left|Z_\Lambda^{\Delta,m_0}(\xi_{\Delta\setminus\Lambda})-Z_\Lambda(\xi_{\Lambda^c})\right| \\ \le \int \mathbb{1}_{\Delta,m_0}(\gamma,l(t))\cdot 2\cdot e^{c(\Lambda,t)\langle\gamma_\Lambda,1+\|m\|^{d+\delta}\rangle}\pi_\Lambda^z(\mathrm{d}\gamma).$$

Taking limit in $m_0\uparrow\infty$ and $\Delta\uparrow\mathbb{R}^d$, the right side goes to 0 (we can exchange limit and integral thanks to the assumption $\mathcal{H}_m$ and the dominated convergence theorem) and therefore for given ε and t there exist $\underline{m_0}(\varepsilon,t)>0$ and $\underline{\Delta}(\varepsilon,t)\supset\Lambda$ such that $\forall m_0\ge\underline{m_0}$ and $\forall\Delta\supset\underline{\Delta}$ we get that

$$\sup_{\xi\in\mathcal{M}^t}\left|Z_\Lambda^{\Delta,m_0}(\xi_{\Delta\setminus\Lambda})-Z_\Lambda(\xi_{\Lambda^c})\right|\le\varepsilon.$$

Since F is assumed to be bounded, the formula (2.5) can be proven analogously. □

We will also need to define *conditional Gibbs kernels*. Take $m_0>0$ and let

$$\begin{aligned}\Xi_\Lambda(\xi,\mathrm{d}\gamma|\{\nu:\mathsf{m}(\nu)\le m_0\})&=\frac{\mathbb{1}\{\gamma:\mathsf{m}(\gamma)\le m_0\}}{\Xi_\Lambda(\xi,\{\nu:\mathsf{m}(\nu)\le m_0\})}\frac{e^{-H_\Lambda(\gamma_\Lambda\xi_{\Lambda^c})}}{Z_\Lambda(\xi)}\pi_\Lambda^z(\mathrm{d}\gamma),\\ \Xi_\Lambda(\xi,\mathrm{d}\gamma|\{\nu:\mathsf{m}(\nu)>m_0\})&=\frac{\mathbb{1}\{\gamma:\mathsf{m}(\gamma)>m_0\}}{\Xi_\Lambda(\xi,\{\nu:\mathsf{m}(\nu)>m_0\})}\frac{e^{-H_\Lambda(\gamma_\Lambda\xi_{\Lambda^c})}}{Z_\Lambda(\xi)}\pi_\Lambda^z(\mathrm{d}\gamma).\end{aligned}\tag{2.6}$$

Now we are ready to prove that the probability measure $\bar{\mathsf{P}}$ satisfies the $\mathbf{DLR}_\Lambda$ equations for a given function F.

Lemma 13. *Let $\Lambda\in\mathcal{B}_b^d$ be a bounded set and take a measurable, bounded and Λ-local function $F:\mathcal{M}\to\mathbb{R}$ .Then the probability measure $\bar{\mathsf{P}}$ from Section 2.2.2 satisfies*

$$\int_{\mathcal{M}^{temp}}F(\gamma)\,\bar{\mathsf{P}}(\mathrm{d}\gamma)=\int_{\mathcal{M}^{temp}}\int_{\mathcal{M}_\Lambda}F(\gamma_\Lambda)\,\Xi_\Lambda(\xi,\mathrm{d}\gamma)\,\bar{\mathsf{P}}(\mathrm{d}\xi).$$

Proof. At first, denote by $i_0\in\mathbb{N}$ the smallest n such that $\Lambda\subset\Lambda_n$. Since $\bar{\mathsf{P}}_n$ do not satisfy $\mathbf{DLR}_\Lambda$, we need to at first define a sequence of measures $(\hat{\mathsf{P}}_n)_{n\in\mathbb{N}}$ which is asymptotically equivalent to $(\bar{\mathsf{P}}_n)_{n\in\mathbb{N}}$, but unlike $\bar{\mathsf{P}}_n$, satisfies $\mathbf{DLR}_\Lambda$, at least for $n\ge i_0$ (hence in the following we only consider $n\ge i_0$).

The estimating sequence is defined as follows. Take $n\in\mathbb{N}$, then

$$\hat{\mathsf{P}}_n=\frac{1}{|\Lambda_n|}\sum_{\kappa\in\mathbb{Z}^d\cap\Lambda_n:\,\Lambda\subset\vartheta_\kappa(\Lambda_n)}\mathsf{P}_n\circ\vartheta_\kappa^{-1},$$

where $\vartheta_\kappa(\Lambda_n)=\{z+\kappa,\ z\in\Lambda_n\}$. We have the following observations

1. For all $l\in\mathbb{N}$ we have that $\hat{\mathsf{P}}_n((\underline{\mathcal{M}}^l)^c)\le\bar{\mathsf{P}}_n((\underline{\mathcal{M}}^l)^c)$.

2. For any tame, local[1] measurable function $G:\mathcal{M}\to\mathbb{R}$ it holds that

$$\lim_{n\to\infty}\left|\int G(\gamma)\,\hat{\mathsf{P}}_n(\mathrm{d}\gamma)-\int G(\gamma)\,\bar{\mathsf{P}}_n(\mathrm{d}\gamma)\right|=0.\tag{2.7}$$

[1]The proof given in Rœlly and Zass [2020] was only for G being Λ-local, nevertheless we need this more general claim, therefore we modify the proof from Rœlly and Zass [2020] (using Lemma 3.5 from Dereudre [2009] as a guide).

Proof of 2): Let $G : \mathcal{M} \to \mathbb{R}$ be tame, Δ-local and measurable function and choose n_0 so that $\Lambda \cup \Delta \subset \Lambda_{n_0}$. Then we can write $\forall n \geq n_0$:

$$\begin{aligned}
\delta_0 &= \left| \int_{\mathcal{M}^{temp}} G(\gamma)\, \hat{\mathsf{P}}_n(\mathrm{d}\gamma) - \int_{\mathcal{M}^{temp}} G(\gamma)\, \bar{\mathsf{P}}_n(\mathrm{d}\gamma) \right| = \\
&= \Bigg| \frac{1}{(2n)^d} \sum_{\kappa \in \mathbb{Z}^d \cap \Lambda_n :\, \Lambda \subset \vartheta_\kappa(\Lambda_n)} \int_{\mathcal{M}^{temp}} G(\gamma)\, \mathsf{P}_n \circ \vartheta_\kappa^{-1}(\mathrm{d}\gamma) \\
&\qquad - \frac{1}{(2n)^d} \sum_{\kappa \in \mathbb{Z}^d \cap \Lambda_n} \int_{\mathcal{M}^{temp}} G(\gamma)\, \tilde{\mathsf{P}}_n \circ \vartheta_\kappa^{-1}(\mathrm{d}\gamma) \Bigg| \\
&\leq \frac{1}{(2n)^d} \sum_{\kappa \in \mathbb{Z}^d \cap \Lambda_n :\, \Lambda \cup \Delta \not\subset \vartheta_\kappa(\Lambda_n)} \left| \int_{\mathcal{M}^{temp}} G(\gamma)\, \tilde{\mathsf{P}}_n \circ \vartheta_\kappa^{-1}(\mathrm{d}\gamma) \right| \\
&\leq \frac{a}{(2n)^d} \sum_{\kappa \in \mathbb{Z}^d \cap \Lambda_n :\, \Lambda \cup \Delta \not\subset \vartheta_\kappa(\Lambda_n)} \int_{\mathcal{M}^{temp}} \left(1 + \langle \gamma_\Delta, 1 + \|m\|^{d+\delta}\right) \tilde{\mathsf{P}}_n \circ \vartheta_\kappa^{-1}(\mathrm{d}\gamma).
\end{aligned}$$

The rest of the proof follows in the same way as in Rœlly and Zass [2020] (page 989) with the modification that we use $\Delta \cup \Lambda$ instead of Λ.

3. $\hat{\mathsf{P}}_n$ is not a probability measure, but using (2.7) with $G(\gamma) = 1$ we get that $\forall \varepsilon > 0$ there exists n_0 such that $\forall n \geq n_0$ we have that $\hat{\mathsf{P}}_n(\mathcal{M}) \geq 1 - \varepsilon$.

4. It holds that $\forall n \geq i_0$ the measures $\hat{\mathsf{P}}_n$ satisfy $\mathbf{DLR}_\Lambda$:

$$\int_{\mathcal{M}} F(\gamma)\, \hat{\mathsf{P}}_n(\mathrm{d}\gamma) = \int_{\mathcal{M}} \int_{\mathcal{M}_\Lambda} F(\gamma_\Lambda \xi_{\Lambda^c})\, \Xi_\Lambda(\xi, \mathrm{d}\gamma) \hat{\mathsf{P}}_n(\mathrm{d}\xi). \tag{2.8}$$

Unfortunately the mapping $\xi \to \int_{\mathcal{M}_\Lambda} F(\gamma) \Xi_\Lambda(\xi, \mathrm{d}\gamma)$ is not local, since our range is not uniformly bounded, and therefore we cannot take a limit on both sides of (2.8). We have to use the estimating kernel $\Xi_\Lambda^{\Delta, m_0}$ and Lemma 12.

To finish the proof, we will show that $\forall \varepsilon > 0$

$$\delta_1 = \left| \int_{\mathcal{M}^{temp}} F(\gamma)\, \bar{\mathsf{P}}(\mathrm{d}\gamma) - \int_{\mathcal{M}^{temp}} \int_{\mathcal{M}_\Lambda} F(\gamma)\, \Xi_\Lambda(\xi, \mathrm{d}\gamma)\, \bar{\mathsf{P}}(\mathrm{d}\xi) \right| < \varepsilon.$$

Take $\varepsilon > 0$. By Lemmas 9 and 10 there exists $a_5 > 0$ such that $\forall t > a_5$ and $\forall n \geq 1$ it holds that

$$\bar{\mathsf{P}}(\mathcal{M}^t) \geq 1 - \varepsilon/2, \quad \bar{\mathsf{P}}_n(\underline{\mathcal{M}}^{l(t)}) \geq 1 - \varepsilon \tag{2.9}$$

Now fix $t > a_5$. For this t there exists $m_0(t) > 0$ such that

$$\bar{\mathsf{P}}(\{\gamma : \mathsf{m}(\gamma_\Lambda) \leq m_0(t)\}) \geq 1 - \varepsilon/2.$$

The last inequality comes from formula (1.6) from proof of Lemma 1. Now since $\bar{\mathsf{P}}_n \xrightarrow{\tau_{\mathcal{L}}} \bar{\mathsf{P}}$ and function $F(\gamma) = \mathbb{1}\{\mathsf{m}(\gamma_\Lambda) \leq m_0(t)\}$ is tame and local, there exist n_1 (w. l. o. g. $n_1 \geq n_0$ from 3. and $n_1 \geq i_0$) such that

$$\bar{\mathsf{P}}_n(\{\gamma : \mathsf{m}(\gamma_\Lambda) \leq m_0(t)\}) \geq 1 - \varepsilon \quad \forall n \geq n_1.$$

Using Remark 3, we can conclude that $\forall n \geq n_1$ we have

$$\hat{\mathsf{P}}_n(\underline{\mathcal{M}}^{l(t)}) \geq 1 - 2\varepsilon \text{ and } \hat{\mathsf{P}}_n(\{\gamma : \mathsf{m}(\gamma_\Lambda) \leq m_0(t)\}) \geq 1 - 2\varepsilon. \tag{2.10}$$

For our fixed t we can find $\underline{m_0}(\varepsilon, t) > 0$ and $\underline{\Delta}(\varepsilon, t)$ from Lemma 12 (recall that F and Λ are given). Take $m_0 > \max\{m_0(t), \underline{m_0}(\varepsilon, t)\}$ and take Δ such that

$$\Lambda \oplus B(0, \tau(m_0, l(t), \Lambda)) \subset \Delta \text{ and } \underline{\Delta}(\varepsilon, t) \subset \Delta.$$

Assume for simplicity that F is bounded by 1. We can write

$$\delta_1 \leq \bar{\mathsf{P}}((\mathcal{M}^t)^c) + \left| \int F(\gamma)\bar{\mathsf{P}}(\mathrm{d}\gamma) - \int_{\mathcal{M}^t} \int_{\mathcal{M}_\Lambda} F(\gamma)\Xi_\Lambda(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) \right|$$

Using Lemma 12 we get that

$$\left| \int_{\mathcal{M}^t} \int_{\mathcal{M}_\Lambda} F(\gamma)\Xi_\Lambda^{\Delta, m_0}(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) - \int_{\mathcal{M}^t} \int_{\mathcal{M}_\Lambda} F(\gamma)\Xi_\Lambda(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) \right| \leq \varepsilon,$$

and therefore we can write

$$\begin{aligned} \delta_1 &\overset{(2.9)}{\leq} 2\varepsilon + \left| \int F(\gamma)\bar{\mathsf{P}}(\mathrm{d}\gamma) - \int_{\mathcal{M}^t} \int_{\mathcal{M}_\Lambda} F(\gamma)\Xi_\Lambda^{\Delta, m_0}(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) \right| \\ &\overset{(2.9)}{\leq} 3\varepsilon + \left| \int_{\mathcal{M}^{temp}} F(\gamma)\bar{\mathsf{P}}(\mathrm{d}\gamma) - \int_{\mathcal{M}^{temp}} \int_{\mathcal{M}_\Lambda} F(\gamma)\Xi_\Lambda^{\Delta, m_0}(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) \right|. \end{aligned}$$

From Remark 2 we know that there exists n_2 (w. l. o. g. $n_2 > n_1$) such that $\forall n \geq n_2$

$$\begin{aligned} &\left| \int_{\mathcal{M}^{temp}} \int_{\mathcal{M}_\Lambda} F(\gamma)\Xi_\Lambda^{\Delta, m_0}(\xi, \mathrm{d}\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\xi) - \int_{\mathcal{M}^{temp}} \int_{\mathcal{M}_\Lambda} F(\gamma)\Xi_\Lambda^{\Delta, m_0}(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) \right| \leq \varepsilon, \\ &\left| \int_{\mathcal{M}^{temp}} F(\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\gamma) - \int_{\mathcal{M}^{temp}} F(\gamma)\bar{\mathsf{P}}(\mathrm{d}\gamma) \right| \leq \varepsilon, \end{aligned}$$

since both functions F and $\xi \to \int_{\mathcal{M}_\Lambda} F(\gamma)\Xi_\Lambda^{\Delta, m_0}(\xi, \mathrm{d}\gamma)$ are tame and local. Therefore, we can write $\forall n \geq n_2$

$$\begin{aligned} \delta_1 &\leq 5\varepsilon + \left| \int_{\mathcal{M}^{temp}} F(\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\gamma) - \int_{\mathcal{M}^{temp}} \int_{\mathcal{M}_\Lambda} F(\gamma)\Xi_\Lambda^{\Delta, m_0}(\xi, \mathrm{d}\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\xi) \right| \\ &\overset{(2.10)}{\leq} 7\varepsilon + \left| \int_{\mathcal{M}^{temp}} F(\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\gamma) - \int_{\underline{\mathcal{M}}^{l(t)}} \int_{\mathcal{M}_\Lambda} F(\gamma)\Xi_\Lambda^{\Delta, m_0}(\xi, \mathrm{d}\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\xi) \right| \\ &\leq 7\varepsilon + \left| \int_{\mathcal{M}^{temp}} F(\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\gamma) - \int_{\underline{\mathcal{M}}^{l(t)}} \int_{\mathcal{M}_\Lambda} F(\gamma)\Xi_\Lambda(\xi, \mathrm{d}\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\xi) \right| \\ &\quad + \left| \int_{\underline{\mathcal{M}}^{l(t)}} \int_{\mathcal{M}_\Lambda} F(\gamma)\Xi_\Lambda(\xi, \mathrm{d}\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\xi) - \int_{\underline{\mathcal{M}}^{l(t)}} \int_{\mathcal{M}_\Lambda} F(\gamma)\Xi_\Lambda^{\Delta, m_0}(\xi, \mathrm{d}\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\xi) \right|. \end{aligned}$$

From (2.8) and (2.10) we get that

$$\begin{aligned} &\left| \int_{\mathcal{M}^{temp}} F(\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\gamma) - \int_{\underline{\mathcal{M}}^{l(t)}} \int_{\mathcal{M}_\Lambda} F(\gamma)\Xi_\Lambda(\xi, \mathrm{d}\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\xi) \right| \\ &\qquad \leq 2\varepsilon + \left| \int F(\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\gamma) - \int \int_{\mathcal{M}_\Lambda} F(\gamma)\Xi_\Lambda(\xi, \mathrm{d}\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\xi) \right| = 2\varepsilon. \end{aligned}$$

For the other summand, we will use the conditional kernels defined in (2.6). Let

$A(\xi, m_0) = \Xi_\Lambda(\xi, \{\nu : \mathsf{m}(\nu) > m_0\})$. We can write

$$\left|\int_{\underline{\mathcal{M}}^{l(t)}} \int_{\mathcal{M}_\Lambda} F(\gamma)(\Xi_\Lambda(\xi, \mathrm{d}\gamma) - \Xi_\Lambda^{\Delta,m_0}(\xi, \mathrm{d}\gamma))\hat{\mathsf{P}}_n(\mathrm{d}\xi)\right| =$$

$$= \left| \int_{\underline{\mathcal{M}}^{l(t)}} \int_{\mathcal{M}_\Lambda} F(\gamma)\Big[\Xi_\Lambda^{\Delta,m_0}(\xi, \mathrm{d}\gamma) - \Xi_\Lambda(\xi, \mathrm{d}\gamma|\{\nu : \mathsf{m}(\nu) \le m_0\}) \cdot (1 - A(\xi, m_0)) \right.$$

$$\left. - \Xi_\Lambda(\xi, \mathrm{d}\gamma|\{\nu : \mathsf{m}(\nu) > m_0\}) \cdot A(\xi, m_0)\Big]\hat{\mathsf{P}}_n(\mathrm{d}\xi)\right|$$

$$\le \left|\int_{\underline{\mathcal{M}}^{l(t)}} \int_{\mathcal{M}_\Lambda} F(\gamma)\Big[\Xi_\Lambda^{\Delta,m_0}(\xi, \mathrm{d}\gamma) - \Xi_\Lambda(\xi, \mathrm{d}\gamma|\{\nu : \mathsf{m}(\nu) \le m_0\})\Big]\hat{\mathsf{P}}_n(\mathrm{d}\xi)\right|$$

$$+ 2\int_{\underline{\mathcal{M}}^{l(t)}} \Xi_\Lambda(\xi, \{\nu : \mathsf{m}(\nu) > m_0\})\hat{\mathsf{P}}_n(\mathrm{d}\xi).$$

Now using (2.10) and the fact that $\hat{\mathsf{P}}_n$ satisfies $\mathbf{DLR}_\Lambda$ with the local bounded function $F(\gamma) = \mathbb{1}\{\mathsf{m}(\gamma_\Lambda) > m_0\}$ we get that

$$2\int_{\underline{\mathcal{M}}^{l(t)}} \Xi_\Lambda(\xi, \{\nu : \mathsf{m}(\nu) > m_0\})\hat{\mathsf{P}}_n(\mathrm{d}\xi) \le \hat{\mathsf{P}}_n(\{\nu : \mathsf{m}(\nu_\Lambda) > m_0\}) \le 4\varepsilon.$$

Since we have chosen Δ so that $\Lambda \oplus B(0, \tau(m_0, l(t), \Lambda)) \subset \Delta$ and from $\mathcal{H}_r$ we know that $\tau(\mathsf{m}(\gamma_\Lambda), l(t), \Lambda) \le \tau(m_0, l(t), \Lambda)$ whenever $\mathsf{m}(\gamma_\Lambda) \le m_0$, we get that $\forall \xi \in \underline{\mathcal{M}}^{l(t)}$: $\Xi_\Lambda^{\Delta,m_0}(\xi, \mathrm{d}\gamma) = \Xi_\Lambda(\xi, \mathrm{d}\gamma|\{\nu : \mathsf{m}(\nu) \le m_0\})$. Altogether we can write

$$\delta_1 \le 7\varepsilon + \left|\int_{\underline{\mathcal{M}}^{l(t)}} F(\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\gamma) - \int_{\underline{\mathcal{M}}^{l(t)}} \int_{\mathcal{M}_\Lambda} F(\gamma)\Xi_\Lambda(\xi, \mathrm{d}\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\xi)\right|$$

$$+ \left|\int_{\underline{\mathcal{M}}^{l(t)}} \int_{\mathcal{M}_\Lambda} F(\gamma)\Xi_\Lambda(\xi, \mathrm{d}\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\xi) - \int_{\underline{\mathcal{M}}^{l(t)}} \int_{\mathcal{M}_\Lambda} F(\gamma)\Xi_\Lambda^{\Delta,m_0}(\xi, \mathrm{d}\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\xi)\right|$$

$$\le 13\varepsilon + \left|\int_{\underline{\mathcal{M}}^{l(t)}} F(\gamma)(\Xi_\Lambda^{\Delta,m_0}(\xi, \mathrm{d}\gamma) - \Xi_\Lambda(\xi, \mathrm{d}\gamma|\{\nu : \mathsf{m}(\nu) \le m_0\}))\hat{\mathsf{P}}_n(\mathrm{d}\xi)\right|$$

$$= 13\varepsilon,$$

which completes the proof. □

Let us note that the assumption that F is Λ-local is posed just for simplicity of notation. For F general Δ-local, we would take i_0 such that $\Lambda \cup \Delta \subset \Lambda_{i_0}$ and define $\hat{\mathsf{P}}_n$ so that $\Lambda \cup \Delta \subset \vartheta_\kappa(\Lambda_n)$. Altogether, we have the existence result.

Theorem 14 (Theorem 1 in Rœlly and Zass [2020])**.** *Under assumptions $\mathcal{H}_s$, $\mathcal{H}_l$, $\mathcal{H}_r$ and $\mathcal{H}_m$ there exists at least one infinite–volume Gibbs measure with energy function H.*

Proof. This proof is just a simple corollary of all the derivations in Section 2.2. From Section 2.2.2 we get that there exists a limit probability measure $\bar{\mathsf{P}}$. We want to show that it satisfies Definition 15. Take $\Lambda \in \mathcal{B}_b^d$ and $F : \mathcal{M} \to \mathbb{R}$ local bounded measurable function. Then by Lemma 13 we get that

$$\int_{\mathcal{M}^{temp}} F(\gamma)\bar{\mathsf{P}}(\mathrm{d}\gamma) = \int_{\mathcal{M}^{temp}} \int_{\mathcal{M}_\Lambda} F(\gamma_\Lambda\, \xi_{\Lambda^c})\Xi_\Lambda(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi)$$

holds. Since by Lemma 9 we get that $\bar{\mathsf{P}}(\mathcal{M}^{temp}) = 1$, we can conclude that $\bar{\mathsf{P}}$ satisfies the **DLR** equations and is therefore an infinite–volume Gibbs measure with energy function H. □

2.3 A Note on the Choice of the Mark Space

We require that the mark space is a space with a norm, however sometimes the model requires only some type of marks (for example non-negative numbers instead of $\mathbb{R}$ or normalized vectors). The following lemma reinforces the claim that we can then only consider configurations with marks from a chosen subset of the mark space $\mathcal{S}$.

Lemma 15. *Let* Q *be a reference mark distribution on the mark space* $(\mathcal{S}, \|\cdot\|)$ *and let* $U \in \mathcal{B}(\mathcal{S})$ *such that* $\mathsf{Q}(U) = 1$. *Then also*

$$\bar{\mathsf{P}}(\{\gamma \in \mathcal{M}(\mathbb{R}^d \times \mathcal{S}) : m \in U, \forall (x,m) \in \gamma\}) = 1,$$

where $\bar{\mathsf{P}}$ *is the limit probability measure from Section* 2.2.2.

Proof. In this proof, we will treat $\gamma \in \mathcal{M}$ as $\gamma \subset \mathbb{R}^d \times \mathcal{S}$ (see Remark 1 from Section 1.1). Recall the probability measures $\mathsf{P}_n, \tilde{\mathsf{P}}_n$ and $\bar{\mathsf{P}}_n$ defined in Section 2.2.1.

Take disjoint partition $\bigcup_{k\in\mathbb{N}} B_k = \mathbb{R}^d$ and define functions

$$F_k(\gamma) = \mathbb{1}\{\gamma \in \mathcal{M}(\mathbb{R}^d \times \mathcal{S}) : \gamma_{B_k} \subset B_k \times U\},\ k \in \mathbb{N}.$$

These functions are bounded and local for all $k \in \mathbb{N}$, particularly $F_k \in \mathcal{L}, \forall k \in \mathbb{N}$. Therefore

$$\lim_{n\to\infty} \int F_k(\gamma)\bar{\mathsf{P}}_n(\mathrm{d}\gamma) = \int F_k(\gamma)\bar{\mathsf{P}}(\mathrm{d}\gamma)$$

holds for all $k \in \mathbb{N}$. From the definition of P_n and from $\mathsf{Q}(U) = 1$ we get that

$$\int F_k(\gamma)\mathsf{P}_n(\mathrm{d}\gamma) = \mathsf{P}_n(\{\gamma : \gamma_{B_k} \subset B_k \times U\}) = 1.$$

Therefore also $\int F_k(\gamma)\tilde{\mathsf{P}}_n(\mathrm{d}\gamma) = 1$ and consequently $\int F_k(\gamma)\bar{\mathsf{P}}_n(\mathrm{d}\gamma) = 1$ holds for all $k \in \mathbb{N}$ and we get that

$$\bar{\mathsf{P}}(\{\gamma : \gamma_{B_k} \subset B_k \times U\}) = \int F_k(\gamma)\bar{\mathsf{P}}(\mathrm{d}\gamma) = \lim_{n\to\infty} \int F_k(\gamma)\bar{\mathsf{P}}_n(\mathrm{d}\gamma) = 1,\ \forall k \in \mathbb{N},$$

and this finishes the proof. □

3. Gibbs Facet Process

The first model we consider will be the process of facets (presented in Večeřa and Beneš [2016]). Informally speaking, we want to model a situation where we have a random configuration of $(d-1)$-dimensional objects (called facets) in $\mathbb{R}^d$ located in random points (centres of facets) and having random tilt. We could model such situation using the particle processes (see Schneider and Weil [2008] for the general theory) or marked point processes, where the location points identify the centre of the facet and the mark space $\mathcal{S}$ is chosen so that each mark uniquely describes one facet.

Facets can interact with each other by intersecting. We would like to consider a Gibbs model, which takes into account the interactions between the facets – i.e. the energy of a configuration will be a function of the intersections. We will consider three possible assumptions on the interactions – repulsive interactions between facets (intersections lead to higher energy), attractive interactions between facets (intersections lead to lower energy) and mixed interactions – and try to verify the assumptions of Theorem 14 to show that there exists an infinite–volume Gibbs facet process.

3.1 Definition of a Facet Process

For $d \geq 2$ we denote by $\mathcal{G}_d$ the space of all $(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$. Let $\mathbb{S}^{d-1}$ denote the unit sphere in $\mathbb{R}^d$, then $A(n) \in \mathcal{G}_d$ denotes the linear subspace with unit normal vector $n \in \mathbb{S}^{d-1}$.

Definition 22. *Let $A(n) \in \mathcal{G}_d$ and $R > 0$. Then we define **facet** $V(n,R)$ as*

$$V(n,R) = A(n) \cap B(0,R).$$

*We will call R the **radius** of the facet and n the **normal vector** of the facet. Furthermore we define **the space of facets** $\mathcal{V}_d = \{A \cap B(0,R) : A \in \mathcal{G}_d, R > 0\}$.*

Remark. Facet is a $(d-1)$-dimensional object in $\mathbb{R}^d$ with centre in 0. Particularly, facet is a line segment in $\mathbb{R}^2$ (see Figure 3.1) and a "disk" in $\mathbb{R}^3$.

As we can see from the definition above, each facet is uniquely described by its radius R and its normal vector n (up to the orientation of n). It is therefore natural to choose the state space as $\mathbb{R}^d \times \mathcal{S}$, with the space of marks being $(\mathcal{S}, \|\cdot\|) = (\mathbb{R}^{d+1}, \|\cdot\|)$, the $(d+1)$-dimensional Euclidean space with standard Euclidean norm[1] $\|m\| = \sqrt{\sum_{i=1}^{d+1} m_i^2}$.

The marks will be specified using the reference mark distribution. We will restrict ourselves to distributions Q on $\mathcal{S}$ satisfying[2]

$$\mathsf{Q}(\mathbb{S}_+^{d-1} \times (0,\infty)) = 1, \tag{3.1}$$

[1] We will use the notation $\|m\|$ while talking about mark m from $\mathbb{R}^d$ and $|x|$ while talking about location point x from $\mathbb{R}^d$.

[2] Eventually, we could model a situation, where the facets have orientation and therefore $V(n,R) \neq V(-n,R)$. Then we would use $\mathsf{Q}(\mathbb{S}^{d-1} \times (0,\infty)) = 1$.

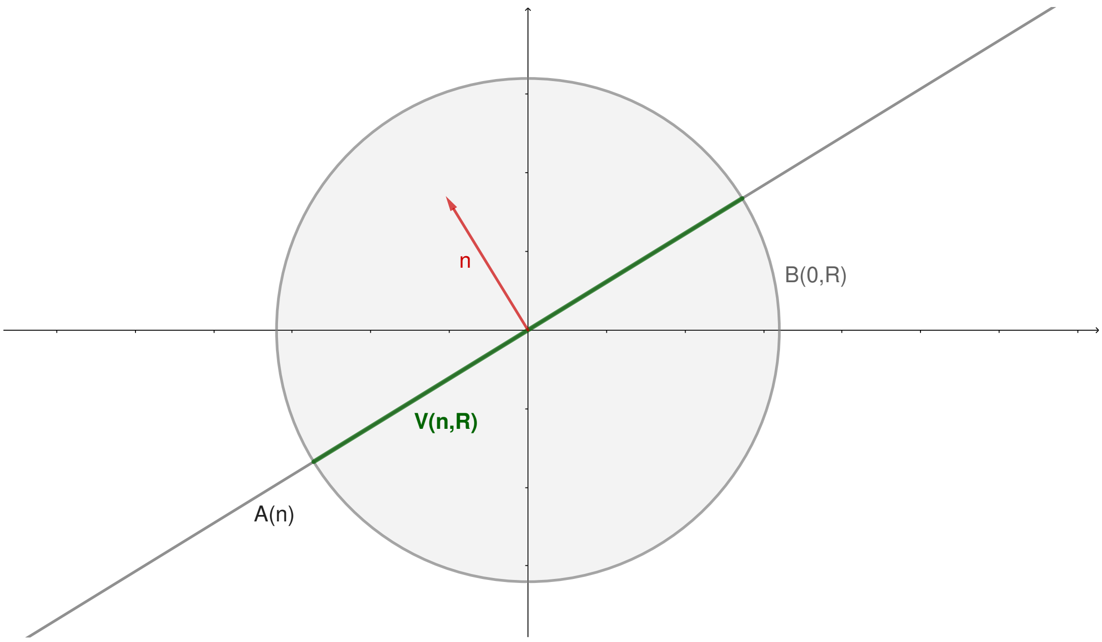


Figure 3.1: The construction of a two-dimensional facet $V(n, R)$ (green line segment), which is obtained as an intersection of the line $A(n)$ with normal vector n (in red) and the closed ball $B(0, R)$.

where $\mathbb{S}_+^{d-1}$ is the *semi–closed unit hemisphere* in $\mathbb{R}^d$,

$$\mathbb{S}_+^{d-1} = \bigcup_{i=1}^{d} \{u \in \mathbb{S}^{d-1} : u_1 = 0, \cdots, u_{i-1} = 0, u_i > 0\}.$$

Thanks to Lemma 15, we can from now on only work with configurations γ with marks $m \in \mathbb{S}_+^{d-1} \times (0, \infty)$. It is clear from the definition of $\mathbb{S}_+^{d-1}$ that there exists a bijection between $m \in \mathbb{S}_+^{d-1} \times (0, \infty)$ and the space of facets $\mathcal{V}_d$,

$$m = (n, R) \equiv V(n, R),$$

i.e. first d coordinates define the normal vector for the corresponding facet and the last coordinate defines the radius. Notice that we have $\|m\| = \sqrt{1 + R^2}$.

Definition 23. *Let $\gamma \in \mathcal{M}$, then we define*

$$\mathcal{A}(\gamma) = \{x + V(n, R) : (x, n, R) \in \gamma\}$$

*the **set of all facets** (shifted to their location) **corresponding to configuration** γ.*

An example of γ and its corresponding set of facets $\mathcal{A}(\gamma)$ in $\mathbb{R}^2$ can be seen in Figure 3.2. As we have said, the energy of a configuration will depend on the number (and volume) of intersections among facets.

Definition 24. ***The energy function of a facet process*** *is the function $H : \mathcal{M}_f \to \mathbb{R}$, where*

$$\begin{aligned} H(\gamma) &= \sum_{j=2}^{d} a_j \phi_j(\gamma), \\ \phi_j(\gamma) &= \sum_{K_1,\ldots,K_j \in \mathcal{A}(\gamma)}^{\neq} \mathbb{H}^{d-j} \left(\bigcap_{i=1}^{j} K_i \right) \cdot \mathbb{1} \left[\mathbb{H}^{d-j} (\bigcap_{i=1}^{j} K_i) < \infty \right]. \end{aligned} \tag{3.2}$$

Here $a_2, \ldots, a_d \in \mathbb{R}$ are fixed constants (and at least one of them is not 0) and $\mathbb{H}^k$ denotes the k-dimensional Hausdorff measure on $\mathbb{R}^d$.

Remark. Notice that the dimension of $\bigcap_{i=1}^{j} K_i \neq \emptyset$ is larger than or equal to $(d-j)$ and it is equal to $(d-j)$ if the facets K_i have linearly independent normal vectors.

By $\sum^{\neq}_{K_1,\ldots,K_j \in \mathcal{A}(\gamma)}$ we mean sum over all j-tuples from the set $\mathcal{A}(\gamma)$. For simplicity, we will denote the indicator in (3.2) by $\mathbb{1}_\infty$, since the j-tuple of sets it concerns is clear from the context.

We will consider the following three assumptions regarding the constants a_j:

F1: We have $a_j \geq 0$ for all $j = 2, \ldots, d$.

F2: We have $a_j \leq 0$ for all $j = 2, \ldots, d$.

F3: There exist indices $j, k \in \{2, \ldots, d\}$ such that $a_j > 0$ and $a_k < 0$.

The first assumption leads to repulsive interactions between the facets – configurations with a lot of interacting facets will have higher energy compared to those with disjoint facets. On the other hand, the second assumption leads to attractive interactions. The third assumption leads to a mixed model, as for some $j \in \{2, \ldots, d\}$ the intersection of a j-tuple will add some energy to the total energy of the configuration and for other j, it will lower the total energy. Now we can define the Gibbs facet process.

Definition 25. ***Finite (or infinite)–volume Gibbs facet process*** *with activity z is defined as the finite (infinite)–volume Gibbs process with energy function H defined in (3.2) and activity z.*

In the next chapter, we consider, whether such processes exist.

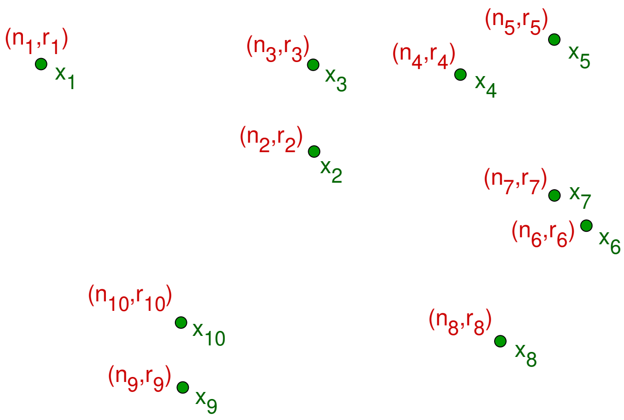
3.2 Verification of Assumptions

To verify the existence of the Gibbs facet process, we must verify the assumptions from Theorem 14: $\mathcal{H}_s, \mathcal{H}_l, \mathcal{H}_r$ and $\mathcal{H}_m$. Since the assumption $\mathcal{H}_m$ only concerns the mark distribution, we present a small note considering the relationship between the distribution of the normal vector and the radius. As we will see in the following section, the range assumption $\mathcal{H}_r$ is satisfied for all three assumptions **F1**, **F2**, **F3**. The stability and local stability assumption $\mathcal{H}_s$ and $\mathcal{H}_l$ hold under the assumption **F1**. On the other hand, for the assumptions **F2** and **F3**, we present counterexamples (for $d = 2$ and $d = 3$) showing that the stability does not hold. Furthermore we prove (for $d = 2$) that the finite–volume Gibbs measures do not exist.

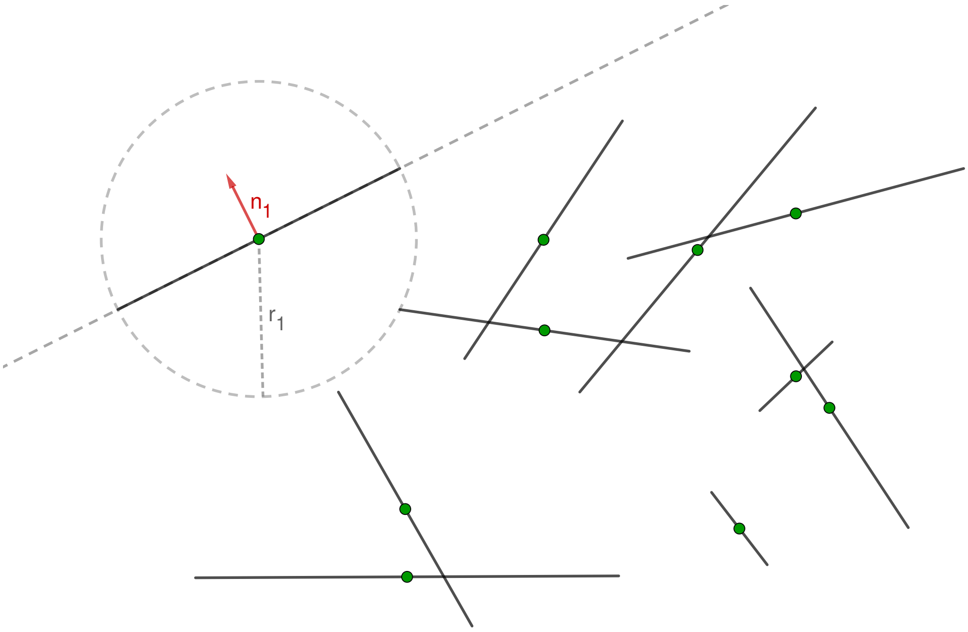
3.2.1 The moment assumption

Let us at first comment on assumption $\mathcal{H}_m$. We have to choose such mark distribution Q that

$$\infty \overset{\mathcal{H}_m}{>} \int_{\mathcal{S}} \exp(\|m\|^{d+2\delta}) \mathsf{Q}(\mathrm{d}m) = \int_{\mathbb{R}^{d+1}} \exp((\|n\|^2 + R^2)^{d/2+\delta}) \mathsf{Q}(\mathrm{d}(n, R))$$
$$\overset{(3.1)}{=} \int_{\mathbb{S}_+^{d-1} \times (0,\infty)} \exp((1 + R^2)^{d/2+\delta}) \mathsf{Q}(\mathrm{d}(n, R)).$$



(a)



(b)

Figure 3.2: (a) Marked point pattern in $\mathbb{R}^2$. Each point consist of a green location and a red mark. (b) Pattern of facets corresponding to the configuration in (a), green points denote the centres of facets. The facet corresponding to the point (x_1, n_1, r_1) has its mark shown.

Particularly, if $\mathsf{Q} = \mathsf{Q}_n \otimes \mathsf{Q}_R$, where Q_n is a probability measure on $\mathbb{S}_+^{d-1}$ and Q_R is a probability measure on $(0,\infty)$ (i.e. the tilt and the radius of the facet are independent), we can further write

$$\infty \overset{\mathcal{H}_m}{>} \int_{\mathbb{S}^{d-1}\times(0,\infty)} \exp((1+R^2)^{d/2+\delta})\mathsf{Q}(\mathrm{d}(n,R)) = \int_0^\infty \exp((1+R^2)^{d/2+\delta})\mathsf{Q}_R(\mathrm{d}R).$$

In this case, whether our model satisfies $\mathcal{H}_m$ or not depends only on the distribution of the radius of facets.

3.2.2 The range assumption

To address the assumption $\mathcal{H}_r$, recall Definition 8 of the sets $\underline{\mathcal{M}}^l$, $l \in \mathbb{N}$, from Section 1.2. In the following lemma we just rewrite this definition into the language of facets.

Lemma 16. *Take $\gamma \in \mathcal{M}^{temp}$, $\gamma \in \underline{\mathcal{M}}^{l_0}$. Then $\forall l \geq l_0$ and for all $K \in \mathcal{A}(\gamma)$ we have the following implication: $K \in \mathcal{A}(\gamma_{U(0,2l+1)^c}) \implies K \cap U(0,l) = \emptyset$.*

Proof. From the definition of $\underline{\mathcal{M}}^{l_0}$ we know that the implication

$$(x,m) \in \gamma_{U(0,2l+1)^c} \implies U(0,l) \cap B(x,\|m\|) = \emptyset$$

holds $\forall l \geq l_0$ and in our case $B(x,\|m\|) = B(x,\sqrt{1+R^2})$. Clearly we have that $K \subset B(x,\sqrt{1+R^2})$ and therefore $K \cap U(0,l) = \emptyset$. $\square$

Now we will show that the range assumption holds. Notice that the following theorem does not specify any of the situations **F1** – **F3**.

Theorem 17. *The energy function $H(\gamma)$ of a facet process defined in (3.2) satisfies the assumption $\mathcal{H}_r$.*

Proof. Fix $\Lambda \in \mathcal{B}_b^d$ and $l_0 \in \mathbb{N}$. We want to prove that for all $\gamma \in \mathcal{M}^{temp}$ such that $\gamma_{\Lambda^c} \in \underline{\mathcal{M}}^{l_0}$ there exists $\tau = \tau(\mathsf{m}(\gamma_\Lambda), l_0, \Lambda) > 0$ which is an increasing function of $\mathsf{m}(\gamma_\Lambda)$ and for which it holds that

$$H_\Lambda(\gamma) = H(\gamma_{\Lambda\oplus B(0,\tau)}) - H(\gamma_{\Lambda\oplus B(0,\tau)\setminus\Lambda}). \tag{3.3}$$

Take $i_0 \in \mathbb{N}$ large enough so that $\Lambda \subset \Lambda_{i_0} = [-i_0, i_0)^d$. From the definition of the conditional energy, we have that

$$H_\Lambda(\gamma) = \lim_{n\to\infty} \left(H(\gamma_{\Lambda_n}) - H(\gamma_{\Lambda_n\setminus\Lambda}) \right).$$

We can write for all $n \geq i_0$

$$H(\gamma_{\Lambda_n}) - H(\gamma_{\Lambda_n\setminus\Lambda}) = \sum_{j=2}^d a_j \sum_{K_1,\dots,K_j\in\mathcal{A}(\gamma_{\Lambda_n})}^{\neq} \mathbb{H}^{d-j}\left(\bigcap_{i=1}^j K_i\right) \cdot \mathbb{1}_\infty$$
$$- \sum_{j=2}^d a_j \sum_{K_1,\dots,K_j\in\mathcal{A}(\gamma_{\Lambda_n\setminus\Lambda})}^{\neq} \mathbb{H}^{d-j}\left(\bigcap_{i=1}^j K_i\right) \cdot \mathbb{1}_\infty.$$

To simplify this formula, define for general sets $A \subset B \in \mathcal{B}_b^d$ and for $j = 2, \dots, d$ the set of all j-tuples of points from γ in B (or more specifically the set of (non–ordered) j-tuples of facets represented by these points) such that at least one of these points lies in A:

$$\mathcal{C}_j(\gamma, A, B) = \{\{K_1, \dots, K_j\} : K_i \in \mathcal{A}(\gamma_B) \text{ for all } i = 1, \dots, j \\ \text{and } \exists i \text{ such that } K_i \in \mathcal{A}(\gamma_A)\}.$$

Then for any $\tau > 0$ and n large enough so that $\Lambda \oplus B(0, \tau) \subset \Lambda_n$ we can write

$$\begin{aligned} H(\gamma_{\Lambda_n}) - H(\gamma_{\Lambda_n \setminus \Lambda}) &= \sum_{j=2}^{d} a_j \sum_{\{K_1,\dots,K_j\} \in \mathcal{C}_j(\gamma,\Lambda,\Lambda_n)} \mathbb{H}^{d-j}\left(\bigcap_{i=1}^{j} K_i\right) \cdot \mathbb{1}_\infty \\ &= \sum_{j=2}^{d} a_j \sum_{\{K_1,\dots,K_j\} \in \mathcal{C}_j(\gamma,\Lambda,\Lambda\oplus B(0,\tau))} \mathbb{H}^{d-j}\left(\bigcap_{i=1}^{j} K_i\right) \cdot \mathbb{1}_\infty \\ &\quad + \sum_{j=2}^{d} a_j \sum_{\{K_1,\dots,K_j\} \in \mathcal{C}_j(\gamma,\Lambda,\Lambda_n)\setminus\mathcal{C}_j(\gamma,\Lambda,\Lambda\oplus B(0,\tau))} \mathbb{H}^{d-j}\left(\bigcap_{i=1}^{j} K_i\right) \cdot \mathbb{1}_\infty. \end{aligned} \tag{3.4}$$

Clearly the first sum does not depend on n and it is in fact equal to the desired $H(\gamma_{\Lambda\oplus B(0,r)}) - H(\gamma_{\Lambda\oplus B(0,r)\setminus\Lambda})$. So it is sufficient to show that for the right choice of τ each summand in the second sum is 0.

Let us discuss the choice of τ so that it satisfies $\mathcal{H}_r$.

1. Since $\mathsf{m}(\gamma_\Lambda)$ is finite (there is only finitely many points in γ_Λ) there exists $l_1(\mathsf{m}(\gamma_\Lambda), \Lambda) = \min\{l \in \mathbb{N} : \Lambda \oplus B(0, \mathsf{m}(\gamma_\Lambda)) \subset U(0, l)\} < \infty$.
2. Let $l_2(\mathsf{m}(\gamma_\Lambda), l_0, \Lambda) = \max\{l_0, l_1(\mathsf{m}(\gamma_\Lambda), \Lambda)\}$.
3. Take

$$\tau(\mathsf{m}(\gamma_\Lambda), l_0, \Lambda) = \min\{k \in \mathbb{N} : U(0, 2l_2(\mathsf{m}(\gamma_\Lambda), l_0, \Lambda) + 1) \subset \Lambda \oplus B(0, k)\}.$$

Then clearly for $a < b$ we have that $\tau(a, l_0, \Lambda) \le \tau(b, l_0, \Lambda)$. Now let n_0 be the smallest n such that $\Lambda \oplus B(0, \tau(\mathsf{m}(\gamma_\Lambda), l_0, \Lambda)) \subset \Lambda_n$. Let $n \ge n_0$ and fix $j \in \{2, \dots, d\}$. For simplicity denote $\tau = \tau(\mathsf{m}(\gamma_\Lambda), l_0, \Lambda)$ and $l_2 = l_2(\mathsf{m}(\gamma_\Lambda), l_0, \Lambda)$ from the second step in the definition of τ.

Take $\{K_1, \dots, K_j\} \in \mathcal{C}_j(\gamma, \Lambda, \Lambda_n) \setminus \mathcal{C}_j(\gamma, \Lambda, \Lambda \oplus B(0, \tau))$. From the definition of $\mathcal{C}_j$ there exist indices $i, k \in \{1, \dots, j\}, i \ne k$ such that $K_i = x + V(n, R) \in \mathcal{A}(\gamma_\Lambda)$ and $K_k \in \mathcal{A}(\gamma_{\Lambda_n \setminus \Lambda \oplus B(0,\tau)})$.

In particular, considering the choice of τ above, it holds that

i) $K_i \subset U(0, l_2)$ from the first and second step,

ii) $K_k \in \mathcal{A}(\gamma_{U(0,2l_2+1)^c})$ from the third step,

iii) $l_2 \ge l_0$ from the second step.

We get from Lemma 16 that $K_i \cap K_k = \emptyset$ and so $\mathbb{H}^{d-j}\left(\bigcap_{i=1}^{j} K_i\right) = 0$. This holds for all $\{K_1, \ldots, K_j\} \in \mathcal{C}_j(\gamma, \Lambda, \Lambda_n) \setminus \mathcal{C}_j(\gamma, \Lambda, \Lambda \oplus B(0, \tau))$ and therefore we have that $\forall n \geq n_0$, using (3.4) for $\tau = \tau(\mathsf{m}(\gamma_\Lambda), l_0, \Lambda)$:

$$H(\gamma_{\Lambda_n}) - H(\gamma_{\Lambda_n \setminus \Lambda}) = \sum_{j=2}^{d} a_j \sum_{\{K_1,\ldots,K_j\} \in \mathcal{C}_j(\gamma,\Lambda,\Lambda\oplus B(0,\tau))} \mathbb{H}^{d-j}\left(\bigcap_{i=1}^{j} K_i\right) \cdot \mathbb{1}_\infty.$$

Therefore we get that

$$\begin{aligned}
H_\Lambda(\gamma) &= \lim_{n\to\infty} H(\gamma_{\Lambda_n}) - H(\gamma_{\Lambda_n \setminus \Lambda}) \\
&= \sum_{j=2}^{d} a_j \sum_{\{K_1,\ldots,K_j\} \in \mathcal{C}_j(\gamma,\Lambda,\Lambda\oplus B(0,\tau))} \mathbb{H}^{d-j}\left(\bigcap_{i=1}^{j} K_i\right) \cdot \mathbb{1}_\infty. \\
&= \sum_{j=2}^{d} a_j \sum_{K_1,\ldots,K_j \in \mathcal{A}(\gamma_{\Lambda\oplus B(0,\tau)})}^{\neq} \mathbb{H}^{d-j}\left(\bigcap_{i=1}^{j} K_i\right) \cdot \mathbb{1}_\infty \\
&\qquad - \sum_{j=2}^{d} a_j \sum_{K_1,\ldots,K_j \in \mathcal{A}(\gamma_{\Lambda\oplus B(0,\tau)\setminus\Lambda})}^{\neq} \mathbb{H}^{d-j}\left(\bigcap_{i=1}^{j} K_i\right) \cdot \mathbb{1}_\infty \\
&= H(\gamma_{\Lambda\oplus B(0,\tau)}) - H(\gamma_{\Lambda\oplus B(0,\tau)\setminus\Lambda}).
\end{aligned}$$

□

Remark. Notice that analogous choice of τ would also work for the energy functions from (1.8).

3.2.3 Stability and local stability assumptions

Recall the assumptions $\mathcal{H}_l$ and $\mathcal{H}_s$ from Section 2.1.2 and the assumptions **F1**, **F2** and **F3** for the energy function of facet process. Then we present the following two results. The first one is that under **F1** (i.e. the repulsive model), the infinite volume Gibbs facet process exists.

Theorem 18. *Let the energy function of a facet process satisfy **F1** and assume that the reference mark distribution* Q *satisfies* $\mathcal{H}_m$*. Then the infinite–volume Gibbs facet process exists.*

Proof. If we consider the situation **F1**, i.e. $a_j \geq 0, \forall j \in \{2, \ldots, d\}$, then the energy function of a facet process H is non–negative and therefore the stability assumption $\mathcal{H}_s$ holds. Since H clearly satisfies Claim 4, also the local stability assumption $\mathcal{H}_l$ holds. Theorem 17 shows that also the range assumption $\mathcal{H}_r$ is satisfied and therefore the assumption of Theorem 14 hold and the existence is proven. □

Considering situations **F2** and **F3** (i.e. the clustering and mixed models), we bring following results.

3.2.4 A counterexample for negative a_j in $\mathbb{R}^2$

In this subsection we consider facet process in $\mathbb{R}^2$, i.e.

$$H(\gamma) = a_2 \sum_{K_1, K_2 \in \mathcal{A}(\gamma)}^{\neq} \mathbb{H}^0 (K_1 \cap K_2) \cdot \mathbb{1}_\infty. \tag{3.5}$$

Suppose that $a_2 < 0$, we can assume for simplicity that $a_2 = -1$. We will show that not only are we able to find a counterexample proving that the stability assumption does not hold, but we are in fact able to prove that the finite volume Gibbs measures do not exist at all.

The first step will be to find a sequence $\{\gamma_N\}_{N \in \mathbb{N}} \subset \mathcal{M}_f$ contradicting the stability assumption $\mathcal{H}_s$. In the second step we show that we can for some $\Lambda \in \mathcal{B}_b^2$ (and under some mild assumptions on the mark distribution Q) modify these configurations to form a sequence of subsets $A_{\Lambda,N} \subset \mathcal{M}_f$ such that

i) $\pi_\Lambda^z(A_{\Lambda,N}) > 0$, $\forall N \in \mathbb{N}$,

ii) $\mathcal{H}_s$ does not hold on $\bigcup_{N \in \mathbb{N}} A_{\Lambda,N}$.

In the final step we use sets $A_{\Lambda,N}$ to show that the partition function Z_Λ is infinite.

Step 1) Consider the following lemma.

Lemma 19. *The energy function of a facet process in $\mathbb{R}^2$ (i.e. (3.5)) does not satisfy the stability assumption $\mathcal{H}_s$ for $a_2 < 0$.*

Proof. Take $N \in \mathbb{N}$ even, $n_1, n_2 \in \mathbb{S}_+^1$ and $R > 0$ and take $\gamma_N \in \mathcal{M}_f$ satisfying

i) **supp** $\gamma_N = \{(x_1, n_1, R), \dots, (x_{\frac{N}{2}}, n_1, R), (x_{\frac{N}{2}+1}, n_2, R), \dots, (x_N, n_2, R)\}$,

ii) normal vectors $n_1, n_2 \in \mathbb{S}_+^1$ satisfy $n_1 \neq n_2$,

iii) location points satisfy $x_i = (x_i^1, 0)^T$, where $1 = x_1^1 > x_2^1 > \dots > x_{\frac{N}{2}}^1 > 0$ and $-1 = x_{\frac{N}{2}+1}^1 < x_{\frac{N}{2}+2}^1 < \dots < x_N^1 < 0$,

iv) R is a large enough constant (depending on n_1, n_2) such that the facets $x_1 + V(n_1, R)$ and $x_{\frac{N}{2}+1} + V(n_2, R)$ intersect.

For such configuration it holds that each facet given by points (x_i, n_1, R), $i \in \{1, \dots, \frac{N}{2}\}$ intersects all facets given by the second half of the points and there are no intersections within the first half and within the second half. So we have that

$$H(\gamma_N) = - \sum_{K_1, K_2 \in \mathcal{A}(\gamma_N)}^{\neq} \mathbb{H}^0 (K_1 \cap K_2) \cdot \mathbb{1}_\infty = -\frac{N}{2} \cdot \frac{N}{2}.$$

At the same time

$$\left\langle \gamma_N, 1 + \|m\|^{2+\delta} \right\rangle = \sum_{i=1}^{N} (1 + (1 + R^2)^{1+\frac{\delta}{2}}) = N \cdot (1 + (1 + R^2)^{1+\frac{\delta}{2}}).$$

Denote by $b := (1 + (1 + R^2)^{1+\frac{\delta}{2}}) < \infty$ the constant, which does not depend on N. Assume for contradiction that $\mathcal{H}_s$ holds, i.e. there $\exists c > 0$ such that $\forall \gamma \in \mathcal{M}_f : H(\gamma) \geq -c \left\langle \gamma, 1 + \|m\|^{2+\delta} \right\rangle$. Then we get that $\forall N \in \mathbb{N}$ even

$$-\frac{N}{2} \cdot \frac{N}{2} \geq -c \cdot N \cdot b,$$

which is clearly a contradiction. Therefore, the assumption $\mathcal{H}_s$ cannot hold for $a_2 < 0$, since we would need the energy to grow somehow linearly, but we sum over all pairs of points, i.e. the growth can be, in the worst case, quadratic. □

Remark. Consequently, the energy function of a facet process in $\mathbb{R}^2$ also does not satisfy $\mathcal{H}_l$.

Step 2) From now on we assume that there exist vectors $u, v \in \mathbb{S}^1_+$, constants $0 < a \leq b < \infty$ and $\varepsilon > 0$ such that

$$\begin{aligned} &\mathsf{Q}(U(u \pm \varepsilon) \times (a, b)) > 0,\ \mathsf{Q}(U(v \pm \varepsilon) \times (a, b)) > 0, \\ &U(u \pm \varepsilon) \cap U(v \pm \varepsilon) = \emptyset. \end{aligned} \tag{3.6}$$

where $U(u \pm \varepsilon) = \{w \in \mathbb{S}^1_+ : |\sphericalangle(u, w)| \leq \varepsilon\}$ (here by $\sphericalangle(u, w)$ we denote the angle between the vectors u and w).

We are able to find a set Λ such that if two facets have centres inside Λ, their normal vectors do not differ too much from u and v, respectively, and their length is at least a, then they must intersect in one point.

Lemma 20. *For given constants $a, \varepsilon > 0$ and two different vectors $u, v \in \mathbb{S}^1_+$ there exists set $\Lambda \in \mathcal{B}^2_b$ such that*

$$|(x + V(n, R)) \cap (y + V(m, T))| = 1$$

holds for all $x, y \in \Lambda, x \neq y$, $n \in U(u \pm \varepsilon)$, $m \in U(v \pm \varepsilon)$ and $R > a, T > a$.

Proof. Set $\Lambda_0 = [-1, 1]^2$ and take $x, y \in \Lambda_0$, $n \in U(u \pm \varepsilon)$, $n = (n_1, n_2)^T$, and $m \in U(v \pm \varepsilon)$, $m = (m_1, m_2)^T$. We will denote by $\langle x, y \rangle$ the standard dot product on $\mathbb{R}^2$. Denote by

$$p(x, n) = \{z \in \mathbb{R}^2 : \langle z, n \rangle = \langle n, x \rangle\}$$

the line given by a point x and a normal vector n and analogously line $p(y, m)$ given by a point y and a normal vector m. Then because of assumption (3.6) $n \neq \pm m$ and these two lines intersect in one point:

$$P(x, y, n, m) = A^{-1} b,$$

where $b = (\langle n, x \rangle, \langle m, y \rangle)^T$ and

$$A = \begin{pmatrix} n_1 & n_2 \\ m_1 & m_2 \end{pmatrix}.$$

Then we can define a function f_1 as the distance from point x to the intersection $P(x, y, n, m)$:

$$f_1(x, y, n, m) = \|x - P(x, y, n, m)\|.$$

This is a continuous function on $\Lambda_0 \times \Lambda_0 \times U(u \pm \varepsilon) \times U(v \pm \varepsilon)$, which is a compact subset of $\mathbb{R}^8$. Therefore, function f_1 has a maximum M_1 on this set. Analogously, we can define f_2 as the distance from point y to the intersection $P(x, y, n, m)$ and there exists its maximum M_2 on $\Lambda_0 \times \Lambda_0 \times U(u \pm \varepsilon) \times U(v \pm \varepsilon)$. Now we only need the following observation. Take any $s > 0$, then

$$\begin{aligned} f_1(sx, sy, n, m) &= \|sx - P(sx, sy, n, m)\| = \\ &= \|sx - A^{-1}(\langle n, sx\rangle, \langle m, sy\rangle)^T\| = s\|x - A^{-1}b\| = \\ &= sf_1(x, y, n, m). \end{aligned}$$

Therefore the maximum of f_1 on $s\Lambda_0 \times s\Lambda_0 \times U(u \pm \varepsilon) \times U(v \pm \varepsilon)$ is sM_1 and analogously the maximum of f_2 on $s\Lambda_0 \times s\Lambda_0 \times U(u \pm \varepsilon) \times U(v \pm \varepsilon)$ is sM_2. Now it is enough to find $s > 0$ small enough such that $\max\{sM_1, sM_2\} < a$ and take $\Lambda = s\Lambda_0 = [-s, s]^2$. □

Now we take Λ from Lemma 20 and denote

$$\begin{gathered} G_u = \Lambda \times U(u \pm \varepsilon) \times (a, b) \text{ and } \Gamma_u = (z\lambda_\Lambda \otimes \mathsf{Q})(G_u), \\ G_v = \Lambda \times U(v \pm \varepsilon) \times (a, b) \text{ and } \Gamma_v = (z\lambda_\Lambda \otimes \mathsf{Q})(G_v), \\ D = \Lambda \times \mathcal{S} \setminus (G_u \cup G_v) \text{ and } \Delta = (z\lambda_\Lambda \otimes \mathsf{Q})(D). \end{gathered}$$

Then we define, $\forall k \in \mathbb{N}$, the following set of configurations

$$A_{\Lambda,2k} = \{\gamma \in \mathcal{M}_f : |\gamma| = 2k,\ \gamma(G_u) = k,\ \gamma(G_v) = k\} \subset \mathcal{M}_\Lambda. \tag{3.7}$$

Thanks to the assumption (3.6), it holds that

$$\pi^z_\Lambda(A_{\Lambda,2k}) = e^{-\Delta} \cdot e^{-\Gamma_u} \cdot \frac{\Gamma_u^k}{k!} \cdot e^{-\Gamma_v} \cdot \frac{\Gamma_v^k}{k!} > 0 \tag{3.8}$$

and thanks to Lemma 20 we have that $\forall k \in \mathbb{N}$ and $\forall \gamma \in A_{\Lambda,2k}$

$$\begin{aligned} H(\gamma) &= -\sum_{K_1, K_2 \in \mathcal{A}(\gamma)} \mathbb{H}^0(K_1 \cap K_2) \cdot \mathbb{1}_\infty = -k \cdot k, \\ \left\langle \gamma, 1 + \|m\|^{2+\delta} \right\rangle &= \sum_{i=1}^{2k} (1 + (1 + R_i^2)^{1+\frac{\delta}{2}}) \leq 2k \cdot (1 + (1 + b^2)^{1+\frac{\delta}{2}}). \end{aligned} \tag{3.9}$$

Therefore we have the following claim.

Claim 21. *1. Assumption $\mathcal{H}_s$ does not hold on the set $\bigcup_{k\in\mathbb{N}} A_{\Lambda,2k}$.*

2. It holds that $\pi^z_\Lambda(A_{\Lambda,2k}) > 0$, $\forall k \in \mathbb{N}$ and consequently $\pi^z_\Lambda(\bigcup_{k\in\mathbb{N}} A_{\Lambda,2k}) > 0$.

Proof. Analogously like in step 1), if for contradiction there existed $c > 0$ such that

$$\forall \gamma \in \bigcup_{k\in\mathbb{N}} A_{\Lambda,2k} : H(\gamma) \geq -c \left\langle \gamma, 1 + \|m\|^{2+\delta} \right\rangle,$$

then we would get, using (3.9), that $\forall k \in \mathbb{N}$

$$-k \cdot k \geq -c \cdot 2k \cdot (1 + (1 + b^2)^{1+\frac{\delta}{2}}).$$

Part 2) is proven in (3.8). □

Step 3) So far, we have only shown that the assumptions $\mathcal{H}_s$ (and consequently $\mathcal{H}_l$) are not satisfied for negative a_2 and therefore we cannot use Theorem 14 to show that the infinite volume Gibbs measure for the facet process exists. However, using the sets $A_{\Lambda,2k}$ defined above, we are in fact capable to prove that the finite–volume Gibbs measures do not exist.

Recall the Stirling's formula:

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}(\frac{n}{e})^n} = 1. \tag{3.10}$$

Theorem 22. *If $a_2 < 0$ and mark distribution Q satisfies (3.6) then it holds that $Z_{\tilde{\Lambda}} = +\infty$, $\forall \tilde{\Lambda} \in \mathcal{B}_b^2$, and therefore the finite volume Gibbs measures do not exist.*

Proof. Take Λ from Lemma 20 and $A_{\Lambda,2k}$, $k \in \mathbb{N}$, defined in (3.7). Then we can write $\forall k \in \mathbb{N}$ that

$$\begin{aligned} Z_\Lambda &= \int_{\mathcal{M}_\Lambda} e^{-H(\gamma)} \pi^z_\Lambda(\mathrm{d}\gamma) \geq \int_{A_{\Lambda,2k}} e^{-H(\gamma)} \pi^z_\Lambda(\mathrm{d}\gamma) = \\ &= e^{k^2} \cdot \pi^z_\Lambda(A_{\Lambda,2k}) = e^{k^2} \cdot e^{-\Delta} \cdot e^{-\Gamma_u} \cdot \frac{(\Gamma_u)^k}{k!} \cdot e^{-\Gamma_v} \cdot \frac{(\Gamma_v)^k}{k!}. \end{aligned}$$

We have used (3.8) and (3.9). Thanks to the Stirling's formula (3.10) the right side converges to ∞ with $k \to \infty$ and therefore $Z_\Lambda = \infty$. Now take any $\tilde{\Lambda} \in \mathcal{B}_b^d$. Since H is assumed to be translation invariant we can, without loss of generality, assume that there exists a constant $1 \geq t > 0$ such that $t\Lambda \subset \tilde{\Lambda}$. Going back to the proof of Lemma 20, we could have used the approach from Step 2) for $t\Lambda$ and everything would have worked in the same way, so we can assume, without loss of generality, that $\Lambda \subset \tilde{\Lambda}$.

Now denote $\tilde{D} = \tilde{\Lambda} \times \mathcal{S} \setminus (G_u \cup G_v)$ and $\tilde{\Delta} = (z\lambda_{\tilde{\Lambda}} \otimes \mathsf{Q})(\tilde{D})$. Then we can write

$$\begin{aligned} Z_{\tilde{\Lambda}} &= \int_{\mathcal{M}_{\tilde{\Lambda}}} e^{-H(\gamma)} \pi^z_{\tilde{\Lambda}}(\mathrm{d}\gamma) \geq \int_{A_{\Lambda,2k}} e^{-H(\gamma)} \pi^z_{\tilde{\Lambda}}(\mathrm{d}\gamma) = \\ &= e^{k^2} \cdot \pi^z_{\tilde{\Lambda}}(A_{\Lambda,2k}) = e^{k^2} \cdot e^{-\tilde{\Delta}} \cdot e^{-\Gamma_u} \cdot \frac{(\Gamma_u)^k}{k!} \cdot e^{-\Gamma_v} \cdot \frac{(\Gamma_v)^k}{k!}, \end{aligned}$$

and we can again use the Stirling's formula to get that $Z_{\tilde{\Lambda}} = \infty$. □

3.2.5 A counterexample for negative a_j in $\mathbb{R}^3$

In this subsection we consider the facet process in $\mathbb{R}^3$. We have

$$H(\gamma) = a_3 \sum^{\neq}_{K_1,K_2,K_3 \in \mathcal{A}(\gamma)} \mathbb{H}^0 \left(\bigcap_{i=1}^{3} K_i \right) \cdot \mathbb{1}_\infty + a_2 \sum^{\neq}_{K_1,K_2 \in \mathcal{A}(\gamma)} \mathbb{H}^1 \left(\bigcap_{i=1}^{2} K_i \right) \cdot \mathbb{1}_\infty. \tag{3.11}$$

The number of triplets is of order N^3 while the number of pairs is only of order N^2, where $N = |\gamma|$. Therefore for large N the second sum will be negligible with respect to the first sum. For $a_3 < 0$ and $a_2 \in \mathbb{R}$ we will run into similar problems as in the previous example. It remains to consider whether we could take $a_3 > 0$ and $a_2 < 0$.

As it turns out, this approach also does not work as we are capable to construct such configuration γ_N for which any triplet of facets $K_1, K_2, K_3 \in \mathcal{A}(\gamma_N)$ does not intersect, but "enough" (i.e. of quadratic order) pairs intersect.

Fix $N \in \mathbb{N}$ even and consider the following construction. Set $\gamma_N \in \mathcal{M}_f$, **supp** $\gamma_N = \{(x_1, n_1, R), \ldots, (x_{\frac{N}{2}}, n_1, R), (x_{\frac{N}{2}+1}, n_2, R), \ldots, (x_N, n_2, R)\}$, where

i) $n_1 = \frac{1}{\sqrt{2}}(1,0,1)^T$ and $n_2 = \frac{1}{\sqrt{2}}(-1,0,1)^T$,

ii) location points satisfy $x_i = (x_i^1, 0, 0)^T$, where $1 = x_1^1 > x_2^1 > \cdots > x_{\frac{N}{2}}^1 > 0$ and $-1 = x_{\frac{N}{2}+1}^1 < x_{\frac{N}{2}+2}^1 < \cdots < x_N^1 < 0$,

iii) R is a large enough constant (e.g. $R = 2$).

We can see these configurations in Figure 3.3.

For such γ_N, no triplet of facets intersects and each facet belonging to the first half of the points intersects every facet belonging to the second half of the points. Therefore we have $\frac{N}{2} \cdot \frac{N}{2}$ pairs of facets $K_1, K_2 \in \mathcal{A}(\gamma_N)$ such that $K_1 \cap K_2 \neq \emptyset$.

Moreover, if we denote by K and L the facets belonging to the furtherest apart points (x_1, n_1, R) and $(x_{\frac{N}{2}+1}, n_2, R)$ (from the construction of γ_N these do not depend on N), we have that $\mathbb{H}^1(K \cap L) \leq \mathbb{H}^1(K_i \cap K_j)$ for any other intersecting pair $K_i, K_j \in \mathcal{A}(\gamma_N)$, $\forall N \in \mathbb{N}$ even.

Suppose again for contradiction, that there $\exists c > 0$ such that

$$\forall \gamma \in \mathcal{M}_f : H(\gamma) \geq -c \left\langle \gamma, 1 + \|m\|^{3+\delta} \right\rangle$$

for H defined in (3.11) for $a_3 > 0$ and $a_2 < 0$ (again, we can assume that $a_2 = -1$). Then we can write $\forall N \in \mathbb{N}$ even

$$\begin{aligned} \frac{N^2}{4} \cdot \mathbb{H}^1(K \cap L) &\leq \sum_{K_1, K_2 \in \mathcal{A}(\gamma_N)} \mathbb{H}^1\left(\bigcap_{i=1}^{2} K_i\right) \cdot \mathbb{1}_\infty = -H(\gamma_N) \\ &\overset{\mathcal{H}_s}{\leq} c \left\langle \gamma, 1 + \|m\|^{3+\delta} \right\rangle = N(1 + (1+R)^{\frac{3+\delta}{2}}). \end{aligned}$$

This implies $N \leq \frac{4}{\mathbb{H}^1(K \cap L)} \cdot (1 + (1+R)^{\frac{3+\delta}{2}})$ for all $N \in \mathbb{N}$ even, which is clearly a contradiction.

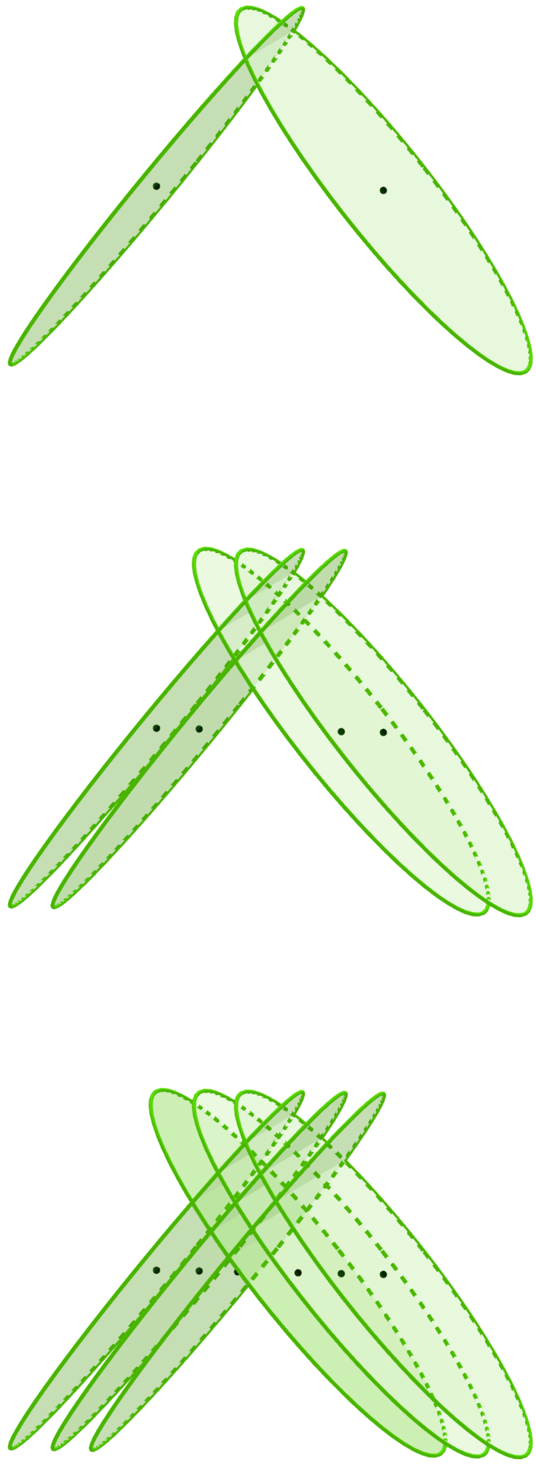


Figure 3.3: Examples of configurations γ_N defined in Section 3.2.5 with the number of points being in order $N = 2$, $N = 4$ and $N = 6$ and with $R = 2$.

4. Gibbs-Laguerre Process

In this section we will consider the class of Gibbs-Laguerre processes, that present a model for random tessellations. We only consider tessellations of $\mathbb{R}^2$. We were not able to use the existence theorem from article Rœlly and Zass [2020] to prove that an infinite-volume Gibbs-Laguerre processes exist in general, but we were able to derive new existence theorem for a particular energy function, under the assumption that we almost surely see a point.

The theory for tessellations and Laguerre diagrams in $\mathbb{R}^2$, presented in Section 1.5 and Chapter 2 in Lautensack [2007] for general $\mathbb{R}^d$, is summarized in Section 4.1 and enlarged by our own auxiliary lemmas in Section 4.1.2. Then in Section 4.2 we consider Gibbs process with energy function (4.10).

In this chapter, it will be useful to regard $\gamma \in \mathcal{M}$ as a locally finite subsets of the state space rather then a locally finite measure (see Remark 1. in Section 1.1).

4.1 Tessellations and Laguerre Geometry

Recall that a *convex polytope* in $\mathbb{R}^2$ is defined as a convex hull of finitely many points and it holds that a bounded intersection of finitely many closed half–planes is a convex polytope. For a convex polytope P we define an **edge of the polytope** P (more generally called 1–face) as a 1-dimensional intersection of P with its supporting hyperplanes and we define a **vertex of the polytope** P (more generally called 0–face) as a 0-dimensional intersection of P with its supporting hyperplanes[1]. We denote the set of all edges of P by $\Delta_1(P)$ and the set of all vertices of P by $\Delta_0(P)$

Remark. For $A \subset \mathbb{R}^2$ we denote by $\text{int}(A)$ the *interior* of the set A, by $\text{clo}(A)$ the *closure* of A and by $\text{bd}(A) = \text{clo}(A) \setminus \text{int}(A)$ the *boundary* of A.

Definition 26. *We say that a set $T = \{C_i : i \in \mathbb{N}\}$, where $C_i \subset \mathbb{R}^2$, is a* ***tessellation of*** $\mathbb{R}^2$*, if*

i) $\text{int}(C_i) \cap \text{int}(C_j) = \emptyset$ *for* $i \neq j$,

ii) $\bigcup_i C_i = \mathbb{R}^2$ *(it is space filling)* ,

iii) $|\{C_i \in T : C_i \cap B \neq \emptyset\}| < \infty$ *for all* $B \subset \mathbb{R}^2$ *bounded (T is locally finite),*

iv) the sets C_i (called ***cells****) are convex compact sets with interior points.*

It holds that the cells of a tessellation are convex polytopes (see Lemma 10.1.1 in Schneider and Weil [2008]).

Particularly we have the sets of all vertices and edges of a cell C denoted by $\Delta_0(C)$ and $\Delta_1(C)$, respectively. Then we can define *the set of edges of cells of a tessellation* T as $\Delta_1(T) = \bigcup_{C \in T} \Delta_1(C)$. We can also define the set of edges of a tessellation.

[1] See Schneider [1993], Section 2.4., for the theoretical background.

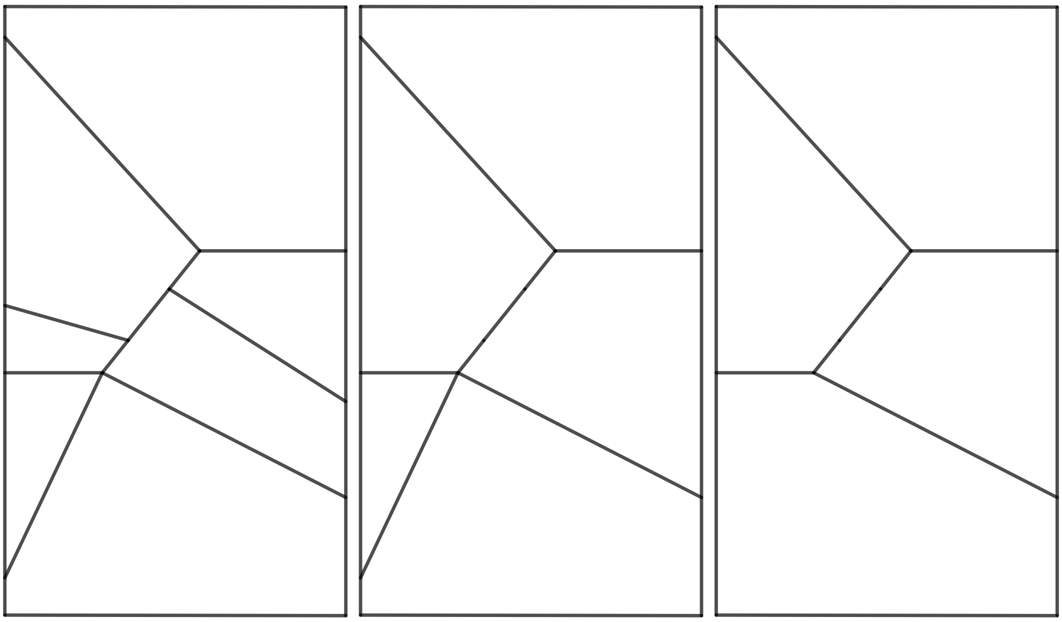


Figure 4.1: Three examples of a tessellation of $\mathbb{R}^2$. Left: a general tessellation, that is not face-to-face, middle: a face-to-face tessellation, that is not normal and right: a normal tessellation.

Definition 27. *We define **the set of edges of a tessellation T** as*

$$S_1(T) = \left\{ F(y) : \dim(F(y)) = 1,\, y \in \mathbb{R}^2 \right\},$$

where the set $F(y)$ is the intersection of all cells of T containing the point y,

$$F(y) = \bigcap_{C \in T:\, y \in C} C.$$

Analogously, we could define $S_0(T)$, the **set of vertices of a tessellation T**. It always holds that $\Delta_0(T) = S_0(T)$, but it can happen that $\Delta_1(T) \neq S_1(T)$. We will not consider such tessellations in our work.

Definition 28. *A tessellation T of $\mathbb{R}^2$ is called **face-to-face**, if the edges of the cells and the edges of the tessellation coincide, i.e. $\Delta_1(T) = S_1(T)$.*

Not every tessellation is face-to-face, as we can see in Figure 4.1. We will also pose assumptions on the vertices of the tessellation.

Definition 29. *A tessellation T is called **normal**, if it is face-to-face, every edge is contained in the boundary of exactly two cells and every vertex is contained in the boundary of exactly three cells.*

An example of a normal and non-normal tessellation can be seen in Figure 4.1. We will now focus only on a special kind of tessellations, so-called Laguerre diagrams, which are based on the power distance from some fixed set of weighted points.

Definition 30. *For $x, z \in \mathbb{R}^2$ and $u \geq 0$ define **the power distance** of z and weighted point (x, u) as $\rho(z, (x, u)) = |x - z|^2 - u^2$.*

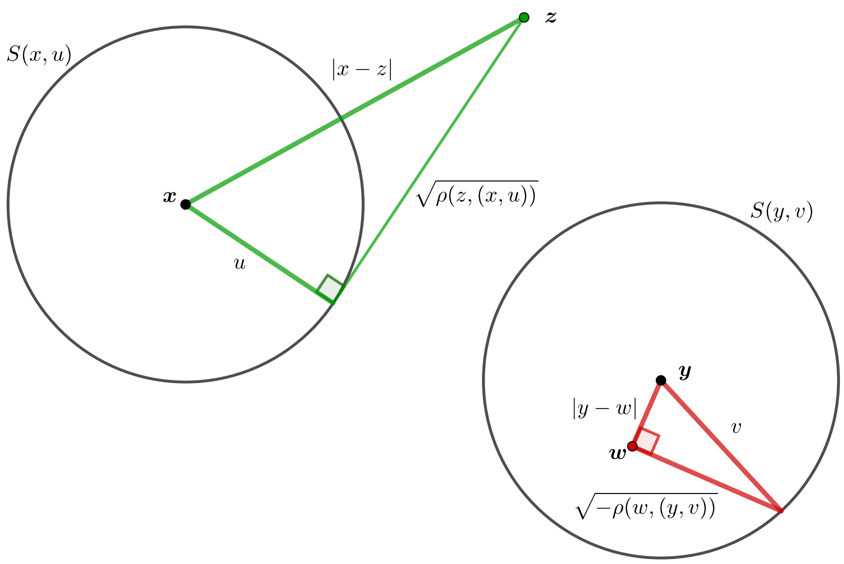


Figure 4.2: The geometric interpretation of the power distance.

The geometric interpretation of the power distance can be seen in Figure 4.2. The power distance $\rho(z,(x,u))$ is a power of the point z w. r. t. the circle $S(x,u)$ with centre x and radius u. Particularly we have that $\rho(z,(x,u)) > 0$ for $z \in B(x,u)^c$, $\rho(z,(x,u)) = 0$ for $z \in \mathrm{bd}(B(x,u))$ and $\rho(z,(x,u)) < 0$ for $z \in U(x,u)$. Denote for points $x, y \in \mathbb{R}^2$ and weights $u, v \geq 0$

$$\begin{aligned} HP\left((x,u),(y,v)\right) &= \left\{z \in \mathbb{R}^2 : \rho(z,(x,u)) = \rho(z,(y,v))\right\} \\ &= \left\{z \in \mathbb{R}^2 : 2\left\langle y-x, z\right\rangle = |y|^2 - |x|^2 + u^2 - v^2\right\} \end{aligned}$$

the line separating $\mathbb{R}^2$ into two half-planes based on the power distances to (x,u) and (y,v) and

$$\begin{aligned} P\left((x,u),(y,v)\right) &= \left\{z \in \mathbb{R}^2 : \rho(z,(x,u)) \leq \rho(z,(y,v))\right\} \\ &= \left\{z \in \mathbb{R}^2 : 2\left\langle y-x, z\right\rangle \leq |y|^2 - |x|^2 + u^2 - v^2\right\} \end{aligned} \tag{4.1}$$

the closed half-plane, whose points are closer to (x,u) then to (y,v) w. r. t. to the power distance. Particularly the line $HP((x,u),(y,v))$ for two weighted points $(x,u),(y,v)$ is the *radical axis* of the circles $S(x,u)$ and $S(y,v)$ and is perpendicular to the line going through x and y.

Now take at most countable subset $\gamma \subset \mathbb{R}^2 \times (0,\infty)$ of weighted points. We will use the notation $x = (x', x'')$ for $x \in \gamma$, where x' denotes the location and x'' the weight of the point. We consider the following assumption

$$\text{(R0)} \qquad \forall z \in \mathbb{R}^2 \qquad \exists \min_{x \in \gamma} \rho(z,x).$$

Definition 31. *Take at most countable subset $\gamma \subset \mathbb{R}^2 \times (0,\infty)$ satisfying the assumption (R0). Then we can define* ***the Laguerre diagram*** *of γ as*

$$L(\gamma) = \{L(x,\gamma) : x \in \gamma, L(x,\gamma) \neq \emptyset\},$$

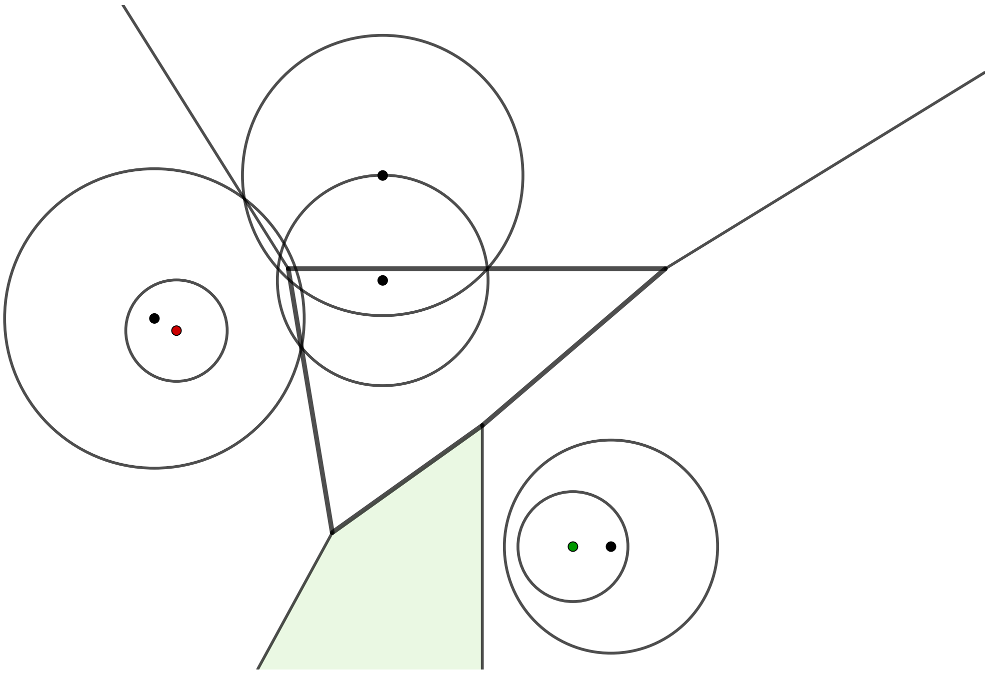


Figure 4.3: An example of a Laguerre diagram with a point with empty cell (red nucleus) and a point whose nucleus does not lie in its cell (green nucleus and green cell). The reference for this figure is Lautensack [2007].

where $L(x, \gamma)$ *is* ***the Laguerre cell of*** x ***in*** γ *defined as*

$$L(x, \gamma) = \{z \in \mathbb{R}^2 : \rho(z, x) \leq \rho(z, y) \ \forall y \in \gamma\}.$$

We call x' *the* ***nucleus*** *of the cell* $L(x, \gamma)$ *and* γ *the* ***set of generators*** *of* $L(\gamma)$.

Special case of Laguerre diagram is the **Voronoi diagram**, which is generated by a configuration of points with constant weights. Voronoi diagram generated by γ is a partition of $\mathbb{R}^2$ into sets of points closest to each nucleus in the Euclidean norm. Particularly, each nucleus produces a cell and lies in it.

Unfortunately, Laguerre diagram does not keep these properties in general, as can be seen in Figure 4.3. Nucleus does not necessarily lie in its cell and some nuclei may not generate a cell at all. The necessary condition for a point x to produce an empty cell is that

$$B(x', x'') \subset \bigcup_{y \in \gamma,\, y \neq x} B(y', y''),$$

but unfortunately, it is not a sufficient condition. We will denote the set of points from γ, whose Laguerre cells are empty, as $E(\gamma) = \{x \in \gamma : L(x, \gamma) = \emptyset\}$.

Clearly from the definition, the (possibly empty) Laguerre cell can be written as

$$L(x, \gamma) = \bigcap_{y \in \gamma} P(x, y). \tag{4.2}$$

We would like to know whether $L(\gamma)$ is, under some conditions, a normal tessellation. The definition of a tessellation assumes that the cells are bounded subsets of $\mathbb{R}^2$, however this will not be true for some Laguerre cells if the set of generators is finite. Therefore we treat the situation with a finite set of generators separately in the next section. For the countable set of generators we have two sets of assumptions.

Definition 32. *We say that $\gamma \subset \mathbb{R}^2 \times (0, \infty)$ fulfils* ***regularity conditions****, if it satisfies*

(R1) for all $(z, t) \in \mathbb{R}^2 \times \mathbb{R}$ only finitely many $x \in \gamma$ satisfy $|z - x'|^2 - (x'')^2 \leq t$,

(R2) conv$\{x' : (x', x'') \in \gamma\} = \mathbb{R}^2$.

Notice that (R1) $\implies$ (R0).

Definition 33. *We say that $\gamma \subset \mathbb{R}^2 \times (0, \infty)$ is* ***in general position****, if the following conditions hold*

(GP1) no 3 nuclei are contained in a 1-dimensional affine subspace of $\mathbb{R}^2$,

(GP2) no 4 points have equal power distance to some point in $\mathbb{R}^2$.

Then the following theorem holds.

Theorem 23. *Let γ satisfy (R1) and (R2). Then every cell $L(x, \gamma)$, where $x \in \gamma$, is compact, $L(\gamma)$ is locally finite and space filling and*

$$\tilde{L}(\gamma) = \{L(x, \gamma) \in L(\gamma) : \text{int}(L(x, \gamma)) \neq \emptyset\}$$

is a face-to-face tessellation. If γ satisfies (R1),(R2), (GP1) and (GP2), then all cells of $L(\gamma)$ have dimension 2 and the Laguerre diagram $L(\gamma)$ is a normal tessellation.

Proof. Lautensack [2007], from Proposition 2.2.2 to Theorem 2.2.8. □

Finite set of generators will not satisfy the condition (R2), hence Theorem 23 cannot be used in this case.

4.1.1 A finite set of generators

Assume that $\gamma \subset \mathbb{R}^2 \times (0, \infty)$ is finite, $\gamma = \{x_1, \ldots, x_N\}$ for some $N \in \mathbb{N}$. Then the assumption (R0) surely holds and therefore the Laguerre cell $L(x, \gamma)$ is well defined $\forall x \in \gamma$. We have from (4.2) that each cell is an intersection of finitely many closed hyperplanes. Bounded $L(x, \gamma)$ are therefore convex polytopes. Clearly

$$\mathbb{R}^2 = \bigcup_{i=1}^{N} L(x_i, \gamma)$$

and for two points $x_i, x_j \in \gamma$ such that their cells have non-empty interiors, we get that $\text{int}\,(L(x_i, \gamma)) \cap \text{int}\,(L(x_j, \gamma)) = \emptyset$.

Analogously as in the previous part we define the sets

- $S_1(\gamma) = \{F(y) : \dim(F(y)) = 1,\, y \in \mathbb{R}^2\}$
- $S_0(\gamma) = \{F(y) : \dim(F(y)) = 0,\, y \in \mathbb{R}^2\}$

of edges and vertices of diagram $L(\gamma)$. We can also define the sets of vertices and edges of the Laguerre cell $L(x, \gamma)$

- $\Delta_i(\gamma) = \bigcup_{x\in\gamma} \Delta_i(x,\gamma)$ for $i = 0, 1$,

where $\Delta_i(x,\gamma)$ denotes the set of i-dimensional intersections of the cell $L(x,\gamma)$ with the hyperplanes $HP(x,y)$, $y \in \gamma$.

Claim 24. *The diagram $L(\gamma)$ is well defined for a finite set of generators γ. Assume that γ satisfies (GP1) and (GP2). Then it holds that the cell $L(x,\gamma)$ is either empty or it has dimension 2, $S_1(\gamma) = \Delta_1(\gamma)$, each vertex $v \in \Delta_0(\gamma)$ lies in the boundary of exactly three cells and each edge $e \in \Delta_1(\gamma)$ lies in the boundary of exactly two cells.*

Proof. Assume that $L(x,\gamma) \neq \emptyset$ and $\dim(L(x,\gamma)) \leq 1$. If $\dim(L(x,\gamma)) = 1$, then it must hold that $L(x,\gamma) \subset HP(x_i,x_j)$ for some x_i, $x_j \in \gamma$, $x_i \neq x_j \neq x$. But this would mean that the three points x_i, x_j, x lie on a line, since then $HP(x_i,x_j) = HP(x,x_j) = HP(x_i,x)$ and these hyperplanes are perpendicular to the lines going through the corresponding pairs of points. Hence we get a contradiction with (GP1). Analogously if an edge u lies in the boundary of three or more cells, then their nuclei lie on a line, a contradiction with (GP1).

If $\dim(L(x,\gamma)) = 0$ i.e. $L(x,\gamma) = \{z\}$ for some $z \in \mathbb{R}^2$, then there exist x_i, x_j, $x_k \in \gamma$ such that x, x_i, x_j, x_k have the same power distance to z, which is a contradiction with (GP2). Analogously if a vertex lies in the boundary of four or more cells, then the power distance of the vertex and the nuclei of these cells would be the same, a contradiction with (GP2). □

For finite γ in general position we say that $L(\gamma)$ is a **generalized normal tessellation**.

4.1.2 Auxiliary lemmas

In this section we present several technical lemmas about Laguerre diagram, its cells and the preservation of regularity conditions and the general position. The main goal of this section is to rigorously derive properties, which will be used in the following sections rather intuitively. We will also connect the tempered configurations and Laguerre theory, as we know from the previous sections that Gibbs measures are concentrated on the set $\mathcal{M}^{temp}$.

Lemma 25. *Let $\gamma \subset \mathbb{R}^2 \times (0,\infty)$ be at most countable set of points such that it satisfies (GP1) and (GP2) and $E(\gamma) = \emptyset$. Then for all $x \in \gamma$ also $E(\gamma \setminus x) = \emptyset$ and $\gamma \setminus x$ satisfies (GP1) and (GP2).*

Proof. Conditions (GP1) and (GP2) cannot be broken by removing a point. The rest holds since $\forall x, y \in \gamma, y \neq x$ we can write $L(y,\gamma) \subset L(y, \gamma \setminus \{x\})$. □

Remark. The fact that $L(x,\gamma) \subset L(x,\nu)$ for $x \in \nu \subset \gamma$ is a simple but useful property, which is important to keep in mind for the future derivations.

Denote for $\gamma \subset \mathbb{R}^2 \times (0,\infty)$ the set of its nuclei $\gamma' = \{x' : (x', x'') \in \gamma\}$.

Lemma 26. *Let $\gamma \subset \mathbb{R}^2 \times (0,\infty)$ satisfy (R1) and (R2). Then it holds that also $\gamma \setminus \{x\}$ satisfies (R1) and (R2), $\forall x \in \gamma$.*

Proof. Let γ satisfy (R1) and (R2) and take $x \in \gamma$. Clearly the condition (R1) is satisfied for $\gamma \setminus \{x\}$. Concerning condition (R2), we will at first show that for $A \subset \mathbb{R}^2$ we have that $\text{clo}(\text{conv}\{A\}) = \mathbb{R}^2 \implies \text{conv}\{A\} = \mathbb{R}^2$.

If for contradiction $\exists\, z \in \text{clo}(\text{conv}\{A\}) \setminus \text{conv}\{A\}$, then by a separating theorem (see Schneider [1993], Theorem 1.3.4.) there exists a closed half-plane L such that $\text{conv}\{A\} \subset L$, which implies that $\mathbb{R}^2 = \text{clo}(\text{conv}\{A\}) \subset L$, a contradiction.

Now assume for contradiction that $\text{conv}\{\gamma'\} = \mathbb{R}^2$ but $\text{conv}\{\gamma' \setminus \{x'\}\} \neq \mathbb{R}^2$. Then $x' \notin \text{clo}(\text{conv}\{\gamma' \setminus \{x'\}\})$ (since if $x' \in \text{clo}(\text{conv}\{\gamma' \setminus \{x'\}\})$, then we would have $\text{clo}(\text{conv}\{\gamma' \setminus \{x'\}\}) = \mathbb{R}^2 \implies \text{conv}\{\gamma' \setminus \{x'\}\} = \mathbb{R}^2$). Therefore we can strongly separate point x' and closed convex set $\text{clo}(\text{conv}\{\gamma' \setminus \{x'\}\})$ by a closed half-plane H (again see Schneider [1993], Theorem 1.3.4.), i.e.

$$\text{clo}(\text{conv}\{\gamma' \setminus \{x'\}\}) \subset H \text{ and } x' \notin H.$$

We can choose $z \in \mathbb{R}^2$ such that $x' \in H + z$ and therefore $\gamma' \subset H + z$ which in turn implies $\mathbb{R}^2 = \text{conv}\{\gamma'\} \subset H + z$, which is a contradiction.

□

Next we will show that Laguerre cells can be represented as a finite intersections of the closed half-planes $P(x, y)$ (see (4.1)). This proof is just a slight modification of the proof of Lemma 10.1.1. in Schneider and Weil [2008], however, since the formula (4.3) is a key property, we include it in here.

Lemma 27. *Let $\gamma \subset \mathbb{R}^2 \times (0, \infty)$ be such that $L(\gamma)$ is a tessellation. Then $\forall L(x, \gamma) \in L(\gamma)$ there exist $k_x \in \mathbb{N}$ and $y_i^x \in \gamma \setminus E(\gamma)$, $i = 1, \ldots, k_x$, such that*

$$L(x, \gamma) = \bigcap_{i=1}^{k_x} P(x, y_i^x). \tag{4.3}$$

Proof. Clearly $\forall x \in \gamma$ we can write

$$L(x, \gamma) = \bigcap_{y \in \gamma} P(x, y). \tag{4.4}$$

Take $L(x, \gamma) \in L(\gamma)$. Since $L(\gamma)$ is a tessellation, then from Definition 26 iii) and iv) we get that

$$k_x + 1 = |\{L(y, \gamma) \in L(\gamma) : L(y, \gamma) \cap L(x, \gamma) \neq \emptyset\}| < \infty. \tag{4.5}$$

Denote $y_1^x, \ldots, y_{k_x}^x$ such points from $\gamma \setminus \{x\}$ that $L(x, \gamma) \cap L(y_i^x, \gamma) \neq \emptyset$. We want to show that $L(x, \gamma) = \bigcap_{i=1}^{k_x} P(x, y_i^x)$.

Clearly from (4.4) we get that $L(x, \gamma) \subset \bigcap_{i=1}^{k_x} P(x, y_i^x)$. Assume for contradiction that there exists $z \in \bigcap_{i=1}^{k_x} P(x, y_i^x)$ such that $z \notin L(x, \gamma)$. From Definition 26 iv) we get that $\exists w \in \text{int}(L(x, \gamma)) \subset \text{int}\left(\bigcap_{i=1}^{k_x} P(x, y_i^x)\right)$. Denote by $U = \{\lambda z + (1 - \lambda) w : \lambda \in [0, 1]\}$ the line segment with end points z and w. Then $\exists z' \in U \cap \text{bd}\,(L(x, \gamma))$, $z \neq z'$, and we have the following two properties (see the end of this proof for the arguments why they hold)

1) $\text{bd}\,(L(x, \gamma)) = \bigcup_{i=1}^{k_x} \left(L(x, \gamma) \cap L(y_i^{k_x}, \gamma)\right)$,

2) $z' \in \text{int}\left(\bigcap_{i=1}^{k_x} P(x, y_i^x)\right)$.

But since $z' \in \mathrm{bd}\,(L(x,\gamma))$, 1) implies that there exists $j \in \{1,\dots,k_x\}$ such that $z' \in L(y_j^x,\gamma)$. On the other hand 2) implies that $\forall i \in \{1,\dots,k_x\}$ we have $z' \in \mathrm{int}(P(x,y_i^x))$. Therefore 1) and 2) imply that $\rho(z',y_j^x) \overset{1)}{\leq} \rho(z',x) \overset{2)}{<} \rho(z',y_j^x)$, which is a contradiction. Therefore $\bigcap_{i=1}^{k_x} P(x,y_i^x) \subset L(x,\gamma)$, which completes the proof.

Proof of 1): We have that

$$\mathrm{bd}\,(L(x,\gamma)) \overset{\text{D.26}}{=} \mathrm{bd}\,(L(x,\gamma)) \cap \bigcup_{y\in\gamma} L(y,\gamma) = \mathrm{bd}\,(L(x,\gamma)) \cup \bigcup_{i=1}^{k_x} (L(x,\gamma) \cap L(y_i^x,\gamma))$$

from the choice of y_i^x. We denote

$$A = \bigcup_{i=1}^{k_x} L(x,\gamma) \cap L(y_i^x,\gamma),$$

then clearly $A \subset \mathrm{bd}\,(L(x,\gamma))$. Let $z \in \mathrm{bd}\,(L(x,\gamma))$, then for all $n \in \mathbb{N}$ there exists $z_n \in B(z,\frac{1}{n}) \cap \bigcup_{y\in\gamma, y\neq x} L(y,\gamma)$. Therefore $z_n \to z$ and

$$\{z_n, n\in\mathbb{N}\} \subset B(z,1) \cap \bigcup_{y\in\gamma, y\neq x} L(y,\gamma) \overset{\text{D.26,iii)}}{=} B(z,1) \cap \bigcup_{j=1}^{N} L(y_j,\gamma),$$

hence $\{z_n, n\in\mathbb{N}\} \subset \bigcup_{j=1}^{N} L(y_j,\gamma)$ which is closed, so also $z \in \bigcup_{j=1}^{N} L(y_j,\gamma)$ and $\exists j$ such that $z \in L(y_j,\gamma)$ (i.e. $L(x,\gamma) \cap L(y_j,\gamma) \neq \emptyset$). But from the choice of y_i^x it must hold that $\exists i$ such that $y_j = y_i^x$ and therefore $z \in A$.

Proof of 2): Since $z' \neq z$ we have that $z' \in \{\lambda z + (1-\lambda)w : \lambda \in [0,1)\}$. Since $w \in \mathrm{int} \bigcap_{i=1}^{k_x} P(x,y_i^x)$ and $z \in \bigcap_{i=1}^{k_x} P(x,y_i^x)$, we get that

$$\{\lambda z + (1-\lambda)w : \lambda \in [0,1)\} \subset \mathrm{int}\left(\bigcap_{i=1}^{k_x} P(x,y_i^x)\right).$$

□

Remark. For $x \in \gamma$, we will call $\{y_1^x,\dots,y_{k_x}^x\}$, defined by (4.5) as those points whose cells intersect the cell $L(x,\gamma)$, the set of **neighbours** of the point x.

For γ finite and in general position, each non-empty Laguerre cell can also be written (using (4.2) and analogous proof as in Lemma 27) as

$$L(x,\gamma) = \bigcap_{i=1}^{k_x} P(x,y_i^x),$$

where y_i^x are as in Lemma 27.

If we furthermore take into consideration the definition of tempered configurations, we can get a similar result for empty Laguerre cells.

Lemma 28. *Let $\gamma \subset \mathbb{R}^2 \times (0,\infty)$ be such that $\gamma \in \mathcal{M}^{temp}$, it satisfies regularity conditions and it is in general position. Then $\forall x \in E(\gamma) = \{x \in \gamma : L(x,\gamma) = \emptyset\}$ there exist $k_x \in \mathbb{N}$ and $y_1^x,\dots,y_{k_x}^x \in \gamma$ such that $L(x,\gamma) = \bigcap_{i=1}^{k_x} P(x,y_i^x)$.*

Proof. Let $x \in E(\gamma)$, i.e. $L(x,\gamma) = \emptyset$. Then it must hold that

$$B(x', x'') \subset \bigcup_{y \in \gamma, y \neq x} B(y', y'').$$

The set $B(x', x'')$ is bounded and γ is tempered, therefore there exists l such that $l \geq l(t)$, where $l(t)$ is from Lemma 1, and $B(x', x'') \subset U(0, l)$. Therefore we know that $\forall y \in \gamma_{U(0,2l+1)^c}$ we have that $B(x', x'') \cap B(y', y'') = \emptyset$. So we can write

$$B(x', x'') \subset \bigcup_{y \in \gamma_{U(0,2l+1)},\, y \neq x} B(y', y'').$$

Let $\varphi = \gamma_{U(0,2l+1)^c} \cup \{x\}$, then according to Lemmas 25 and 26 it holds that φ satisfies regularity condition and is in general position. Particularly $L(\varphi)$ is a (normal) tessellation. Furthermore it holds that $B(x', x'') \not\subset \bigcup_{y \in \varphi, y \neq x} B(y', y'')$ and therefore $\emptyset \neq L(x, \varphi) \in L(\varphi)$. This allows us to use Lemma 27 and we get that there exist $y_1^x, \dots, y_{n_x}^x \in \varphi$ such that $L(x, \varphi) = \bigcap_{i=1}^{n_x} P(x, y_i^x)$.

Altogether we get that

$$\begin{aligned} L(x,\gamma) = \bigcap_{y \in \gamma} P(x,y) &= \bigcap_{y \in \gamma_{U(0,2l+1)}} P(x,y) \cap \bigcap_{y \in \gamma_{U(0,2l+1)^c}} P(x,y) = \\ &= \bigcap_{y \in \gamma_{U(0,2l+1)}} P(x,y) \cap L(x,\varphi) = \bigcap_{y \in \gamma_{U(0,2l+1)}} P(x,y) \cap \bigcap_{i=1}^{n_x} P(x, y_i^x). \end{aligned}$$

Thanks to the local finiteness of γ, there is only finitely many points in $\gamma_{U(0,2l+1)}$, which completes the proof. ☐

Notice that, in contrast with non-empty cells, we do not have a specific formula for the "neighbours" of an empty Laguerre cell, we can only say that it is empty thanks to finitely many points. What follows now is an auxiliary lemma for the proof that tempered configurations satisfy (R0) and (R1).

Lemma 29. *Let $l \in \mathbb{N}$. Then $\forall z \in U\left(0, \frac{1}{2}l\right)$ and $\forall y' \in U(0, 2l+1)^c$ the following inequalities hold*

$$\rho(z, (y', |y'| - l)) > l^2 \geq \sup_{w \in U(0, \frac{1}{2}l)} |w - z|^2.$$

Proof. Clearly the second inequality holds. For the first one, we can simply write

$$\begin{aligned} \rho(z, (y', |y'| - l)) &= |z - y'|^2 - (|y'| - l)^2 \\ &= |z|^2 + |y'|^2 - 2\langle z, y' \rangle - |y'|^2 + 2l\,|y'| - l^2 \\ &\geq |z|^2 - 2\,|z|\,|y'| + 2\,|y'|\,l - l^2 \\ &= |z|^2 + l(|y'| - l) + |y'|\,(l - 2\,|z|) \geq l^2 + l > l^2. \end{aligned}$$

We have used the Cauchy-Schwartz inequality and the fact that

$$|y'| - l \geq 2l + 1 - l \geq l + 1 \text{ and } l - 2\,|z| \geq 0.$$

☐

Lemma 30. *It holds that all $\gamma \in \mathcal{M}^{temp}$ satisfy (R0) and therefore the Laguerre cells $L(x,\gamma)$ are well defined. Furthermore it holds that all $\gamma \in \mathcal{M}^{temp}$ satisfy the first regularity condition (R1).*

Proof. Take $z \in \mathbb{R}^2$ and $\gamma \in \mathcal{M}^t$, $t \in \mathbb{N}$. We want to show that there exists $\min_{x\in\gamma}\rho(z,x)$. Clearly, if $\gamma \in \mathcal{M}_f$, the assumption is satisfied. Consider infinite configuration γ.

We will use the property of tempered configurations given by Lemma 1, which states that there exist $l(t)$ such that $\forall l \geq l(t)$ the following implication holds:

$$(x', x'') \in \gamma_{(U(0,2l+1))^c} \implies B(x', x'') \cap U(0,l) = \emptyset. \tag{4.6}$$

Choose l large enough so that

i) $l \geq l(t)$,

ii) $z \in U\left(0, \frac{1}{2}l\right)$ and there exists $x \in \gamma_{U\left(0,\frac{1}{2}l\right)}$.

Clearly such l can be chosen. Lemma 29 states that the following inequality holds $\forall y' \in U(0, 2l+1)^c$:

$$\rho(z, (y', |y'| - l)) \geq \sup_{w\in U(0,\frac{1}{2}l)} |w - z|^2 . \tag{4.7}$$

We know, because of the property (4.6), that

$$\forall y = (y', y'') \in \gamma_{(U(0,2l+1))^c} : y'' \leq |y'| - l \tag{4.8}$$

and therefore

$$\rho(z, y) = |y' - z|^2 - (y'')^2 \geq |y' - z|^2 - (|y'| - l)^2 = \rho(z, (y', |y'| - l)). \tag{4.9}$$

Then, using (4.7) together with point ii) above, we get that $\forall y \in \gamma_{(U(0,2l+1))^c}$

$$\rho(z, y) \geq \sup_{w\in U(0,\frac{1}{2}l)} |w - z|^2 \geq |x' - z|^2 \geq \rho(z, x)$$

and this completes the proof as then

$$\min_{x\in\gamma}\rho(z, x) = \min_{x\in\gamma_{U(0,2l+1)}}\rho(z, x),$$

which exists thanks to the local finiteness of γ.

Now consider (R1). We want to show that for every $z \in \mathbb{R}^2$ and $t \in \mathbb{R}$ only finitely many elements $y \in \gamma$ satisfy $|z - y'|^2 - (y'')^2 \leq t$. But this is a clear consequence of the derivations above. Take $z \in \mathbb{R}^2$ and $t \in \mathbb{R}$. Then there exists l large enough such that $l^2 > t$ and such that it satisfies i) and ii). Then we have that $\forall y \in \gamma_{(U(0,2l+1))^c}$

$$|z - y'|^2 - (y'')^2 \geq |z - y'|^2 - (|y'| - l)^2 \geq l^2 > t,$$

and therefore only the points $y \in \gamma_{U(0,2l+1)}$ (and there is finitely many of them) can satisfy $|z - y'|^2 - (y'')^2 \leq t$. □

Before we move to the next section, where we add randomness to Laguerre diagrams, let us emphasize the points (4.8) and (4.9) as these two properties of tempered configurations will be useful in the next part as well.

4.2 Gibbs-Laguerre Measures

To model a random Laguerre diagram, we consider Laguerre diagram with random generator $L(\Psi)$, where Ψ is a marked point process in the space $\mathbb{R}^2 \times (0, \infty)$. Our aim was to consider Ψ to be an infinite-volume Gibbs measure with energy function depending on the geometric properties of $L(\Psi)$ and use article Rœlly and Zass [2020] to show that there exists an infinite-volume Gibbs-Laguerre measure with unbounded weights. Unfortunately, the range assumption $\mathcal{H}_r$ turned out to be an insurmountable obstacle.

Let us briefly discuss the reason behind this. The main problem lies in the fact that we would need a uniform range for all boundary conditions $\xi \in \underline{\mathcal{M}}^l$. However, the behaviour of $L(\gamma_\Lambda \xi_{\Lambda^c})$ depends heavily on the actual locations of points from ξ. Imagine that for fixed $\xi \in \underline{\mathcal{M}}^l$ we have a range $r > 0$, i.e. $\forall x \in \gamma_\Lambda$ we have that $L(x, \gamma_\Lambda \xi_{\Lambda^c}) = L(x, \gamma_\Lambda \xi_{\Lambda \oplus B(0,r) \setminus \Lambda})$. Then also $\varphi = \xi_{(\Lambda \oplus B(0,r))^c}$ belongs to $\underline{\mathcal{M}}^l$ but at the same time there also exists $x \in \gamma_\Lambda$ such that

$$L(x, \gamma_\Lambda \varphi) \neq L(x, \gamma_\Lambda) = L(x, \gamma_\Lambda \varphi_{\Lambda \oplus B(0,r) \setminus \Lambda}).$$

However, for a non-negative energy function and reference mark distribution satisfying $\mathcal{H}_m$ the first three parts of the existence proof from Rœlly and Zass [2020] still work. For the energy function defined in (4.10), we were able to prove that the limit measure $\bar{\mathsf{P}}$ is an infinite-volume Gibbs measure, under the condition that $\bar{\mathsf{P}}(\bar{o}) = 0$.

4.2.1 Energy function and finite–volume Gibbs measures

Let the state space be $\mathcal{E} = \mathbb{R}^2 \times \mathbb{R}$ with mark space $(\mathbb{R}, \|\cdot\|)$ and take mark distribution Q such that $\mathsf{Q}((0, \infty)) = 1$ and such that $\mathcal{H}_m$ holds.

Recall the notation:

1. for $\gamma \in \mathcal{M}$ and $x \in \gamma$ we use the notation $x = (x', x'')$ with x' being the nuclei and x'' being the weight.
2. $\Delta_0(x, \gamma)$ is the set of all vertices of the cell $L(x, \gamma)$, $\Delta_0(\gamma) = \bigcup_{x \in \gamma} \Delta_0(x, \gamma)$.
3. $E(\gamma)$ denotes the set of points from γ with empty cells.

Consider the following energy function $H : \mathcal{M}_f \to \mathbb{R} \cup \{+\infty\}$

$$H(\gamma) = \begin{cases} \sum_{x \in \gamma} |\Delta_0(x, \gamma)| & \text{if } E(\gamma) = \emptyset, \\ +\infty & \text{if } E(\gamma) \neq \emptyset, \end{cases} \qquad \gamma \in \mathcal{M}_f. \tag{4.10}$$

We sum the number of vertices for each Laguerre cell and we forbid the configurations for which there exists an empty cell.

Remark. If γ is in general position, then $H(\gamma) = 3 \cdot |\Delta_0(\gamma)|$.

Clearly H is non-negative and therefore the stability assumption $\mathcal{H}_s$ is satisfied. According to Lemma 3, the partition function Z_Λ is finite for all $\Lambda \in \mathcal{B}_b^2$ and the finite-volume Gibbs measure in Λ with energy function H and activity z is well defined for all $z > 0$,

$$\mathsf{P}_\Lambda(\mathrm{d}\gamma) = \frac{1}{Z_\Lambda} e^{-H(\gamma)} \pi_\Lambda^z(\mathrm{d}\gamma).$$

It holds $\forall \Lambda \in \mathcal{B}_b^2$, $\forall z > 0$ that

$$\pi_\Lambda^z \left(\{\gamma \in \mathcal{M} : \gamma \text{ is in general position}\}\right) = 1 \tag{4.11}$$

(see Lautensack [2007], Proposition 3.1.5, or Zessin [2008]). Therefore also

$$\mathsf{P}_\Lambda(\{\gamma \in \mathcal{M} : \gamma \text{ is in general position}\}) = 1,$$

particularly for P_Λ-a.a. γ the Laguerre diagram $L(\gamma)$ is a generalized normal tessellation. Since configurations with empty cells are forbidden, we also get that

$$\mathsf{P}_\Lambda(\{\gamma : E(\gamma) = \emptyset\}) = 1. \tag{4.12}$$

In the following proposition, we present the key observation for the energy function H from (4.10). This observation will later allow us to show that the conditional energy H_Λ is attained as soon as all of the cells belonging to the points in Λ are bounded.

Proposition 31. *Let H be the energy function defined in (4.10) and take $\gamma \in \mathcal{M}_f$ such that it satisfies (GP1), (GP2) and $E(\gamma) = \emptyset$. Assume that the Laguerre cell $L(x, \gamma)$ of a point $x \in \gamma$ is bounded. Then we have that*

$$H(\gamma) - H(\gamma \setminus \{x\}) = 6.$$

Proof. Let γ and x be as assumed. Then $L(\gamma)$ (and also $L(\gamma \setminus x)$ thanks to Lemma 25) is a generalized normal tessellation and we know that

$$L(x, \gamma) = \bigcap_{i=1}^{k} P(x, y_i^x)$$

for $y_i \in \gamma$ such that $L(x, \gamma) \cap L(y_i, \gamma) \neq \emptyset$, $k \in \mathbb{N}$. Particularly, since $L(x, \gamma)$ is bounded, we have $|\Delta_0(x, \gamma)| = k$. The Laguerre cells of points $y \in \gamma \setminus \{y_1^x, \dots, y_k^x\}$ do not change by removing the point x and therefore we can write

$$\begin{aligned} H(\gamma) - H(\gamma \setminus \{x\}) &= |\Delta_0(x, \gamma)| + \sum_{i=1}^{k} |\Delta_0(y_i^x, \gamma)| - |\Delta_0(y_i^x, \gamma \setminus \{x\})| \\ &= k + \sum_{i=1}^{k} |\Delta_0(y_i^x, \gamma)| - |\Delta_0(y_i^x, \gamma \setminus \{x\})| \,. \end{aligned}$$

By removing the point x, the neighbours of x partition the cell $L(x, \gamma)$ into k non-empty bounded convex polytopes $K_1, \dots, K_k$ such that

$$L(y_i^x, \gamma \setminus \{x\}) = K_i \cup L(y_i^x, \gamma),$$

(see Figure 4.4). Denote by v_i the number of new vertices attained by the nucleus y_i^x, $v_i = |\Delta_0(K_i)| - |\Delta_0(K_i) \cap \Delta_0(L(y_i^x, \gamma))| = |\Delta_0(K_i)| - 2$, (each neighbour y_i^x shares 2 vertices with the nucleus x). Altogether, we can write

$$\begin{aligned} H(\gamma) - H(\gamma \setminus \{x\}) &= k + \sum_{i=1}^{k} |\Delta_0(y_i^x, \gamma)| - |\Delta_0(y_i^x, \gamma \setminus \{x\})| \\ &= k + \sum_{i=1}^{k} 2 - v_i = 3k - \sum_{i=1}^{k} v_i. \end{aligned} \tag{4.13}$$

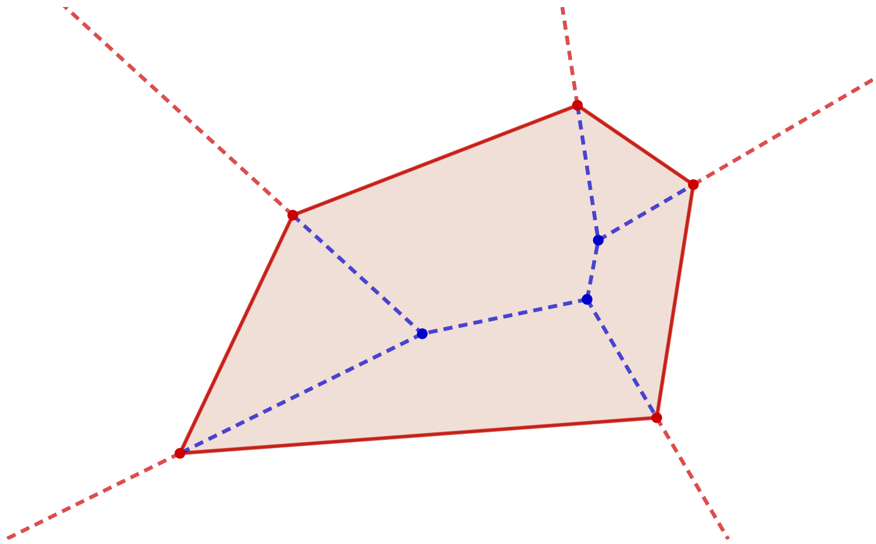


Figure 4.4: Comparison of a Laguerre diagram with and without the point x. The Laguerre cell $L(x,\gamma)$ is the red pentagon, full red lines are its edges, the edges of the neighbouring cells in $L(\gamma)$ are the red dashed lines, red points are the vertices in $L(\gamma)$. Blue points are the additional vertices in $L(\gamma \setminus \{x\})$ and blue dashed lines are the additional edges of the cells $L(y_i^x, \gamma \setminus \{x\})$ arising from the removal of the point x.

The partition of the cell $L(x,\gamma)$ by its neighbours defines a graph structure (see Figure 4.4) with vertices $V = \Delta_0(x,\gamma) \cup V_2$, where V_2 is the set of new vertices, which appear after the removal of the point x, $V_2 = \Delta_0(\gamma \setminus \{x\}) \setminus \Delta_0(\gamma)$. The set of edges is defined as $E = \Delta_1(x,\gamma) \cup E_2$, where E_2 is the set of new edges (intersected with $L(x,\gamma)$), which appear after the removal of the point x. Since both $L(\gamma)$ and $L(\gamma \setminus \{x\})$ are normal, all of the vertices have degree 3. Thus we have that

$$3 \cdot |V| = 2 \cdot |E| \qquad \Longrightarrow \qquad 3(k + |V_2|) = 2(k + |E_2|). \tag{4.14}$$

Since we assume that there are no empty cells, the graph (V, E_2) is a connected graph without cycles (i.e. a tree) and we know that

$$|V| = |E_2| + 1 \qquad \Longrightarrow \qquad k + |V_2| = |E_2| + 1. \tag{4.15}$$

Putting together (4.14) and (4.15), we get that $|V_2| = k - 2$. From the normality we also get that $\sum_{i=1}^{k} v_i = 3 \cdot |V_2|$ and that together with (4.13) completes the proof. □

4.2.2 The existence of $\bar{\mathsf{P}}$ and its support

Consider finite volume Gibbs measure $\mathsf{P}_n = \mathsf{P}_{\Lambda_n}$ with energy function H from (4.10) and activity $z > 0$ in the window $\Lambda_n = [-n,n)^2$, $n \in \mathbb{N}$ and recall Sections 2.2.1, 2.2.2 and 2.2.3. We have the periodic extension to the whole $\mathbb{R}^2$ of the finite-volume Gibbs measure denoted by $\tilde{\mathsf{P}}_n$ (see (2.2)) and stationarised empirical field $\bar{\mathsf{P}}_n$ (see Definition 20). Since the energy function H satisfies the stability assumption, the results of Lemmas 7, 8, 9 and 10 are valid. Particularly, we have the following claim.

Claim 32. *There exists a probability measure* $\bar{\mathsf{P}}$ *such that*

i) $\bar{\mathsf{P}}$ *is invariant under translations by* $\kappa \in \mathbb{Z}^2$*,*

ii) (w. l. o. g.) $\bar{\mathsf{P}}_n \xrightarrow{\tau_{\mathcal{L}}} \bar{\mathsf{P}}$*,*

iii) $\bar{\mathsf{P}}(\mathcal{M}^{temp}) = \bar{\mathsf{P}}_n(\mathcal{M}^{temp}) = 1$*, for all* $n \in \mathbb{N}$*.*

We also get that $\forall \varepsilon > 0$ *there exists* $l \in \mathbb{N}$ *such that*

iv) $\bar{\mathsf{P}}_n(\underline{\mathcal{M}}^l) \geq 1 - \varepsilon$ *for all* $n \in \mathbb{N}$*.*

We would like to show that $\bar{\mathsf{P}}$ satisfies Definition 15 for our energy function H, however since the range assumption $\mathcal{H}_r$ is not satisfied, we cannot use Theorem 14. In the last section of this chapter, we will prove that $\bar{\mathsf{P}}$ is an infinite-volume Gibbs measure under the condition that $\bar{\mathsf{P}}(\{\bar{o}\}) = 0$.

At first we need to prepare some preliminary results. We will show that $\bar{\mathsf{P}}$-a.a. configurations satisfy that $L(\gamma)$ is a normal tessellation with no empty cells. We already know, thanks to Lemma 30 and Claim 32 iii), that $\bar{\mathsf{P}}$-a.a. γ satisfy (R0) and (R1).

For the condition (R2), we need the following lemma (as was remarked in Lautensack [2007]). In its proof we work with the notion of *random closed set.* We refer to Chapter 2 in Schneider and Weil [2008] for the definition and general theory.

Lemma 33. *It holds that if* Ψ *is a simple marked point process whose distribution is invariant under translation by* $\kappa \in \mathbb{Z}^2$ *then it almost surely satisfies the assumption (R2) or it is empty, i.e.* $\mathbb{P}(conv\{x' : (x', x'') \in \Psi\} \in \{\mathbb{R}^2, \emptyset\}) = 1$*.*

Proof. We will proceed in three steps.

Step 1) The following claim holds:

Let Z *be a convex random closed set such that* $\forall \kappa \in \mathbb{Z}^2$ *it holds that* $Z \overset{D}{=} Z + \kappa$*. Then* $\mathbb{P}(Z \in \{\emptyset, \mathbb{R}^2\}) = 1$*.*

Since the proof of this claim is just a slight modification of the proof of Theorem 2.4.4. in Schneider and Weil [2008] (which assumes invariance under all translations, not just by integer-valued vectors), we only show the part where they differ. The proof is the same up to a definition of the set A_k. We have $x, y \in \mathbb{Q}^2, y \neq 0$ such that

$$\mathbb{P}(\emptyset \neq Z \cap K(x, y) \subset x + |y|\, B(0, 1)) =: p > 0.$$

Choose $m \in \mathbb{N}$ such that $my \in \mathbb{Z}^2$ and define

$$A_k = \{\emptyset \neq Z \cap K(x + 2mky, y) \subset x + 2kmy + |y|\, B(0, 1)\}.$$

The rest of the proof follows as in Schneider and Weil [2008].

Step 2) If Ψ is a simple marked point process whose distribution is invariant under translations by all $\kappa \in \mathbb{Z}^2$, then $\mathbf{supp}\Psi'$ is a random closed set (see Chapter 3 in Schneider and Weil [2008]) $\implies$ clo(conv$\{\mathbf{supp}\Psi'\}$) is a convex random closed set (see Theorem 2.4.3. in Schneider and Weil [2008]) which satisfies

$$\text{clo}(\text{conv}\{\mathbf{supp}\Psi'\}) \overset{D}{=} \text{clo}(\text{conv}\{\mathbf{supp}\Psi' + \kappa\}) = \text{clo}(\text{conv}\{\mathbf{supp}\Psi'\}) + \kappa$$

$\forall \kappa \in \mathbb{Z}^2$ and therefore step 1. gives us that $\mathbb{P}(\text{clo}(\text{conv}\{\mathbf{supp}\Psi'\}) \in \{\emptyset, \mathbb{R}^2\}) = 1$.

Step 3) We have already shown in the proof of Lemma 26 that for $A \subset \mathbb{R}^2$ we have that $\text{clo}(\text{conv}\{A\}) = \mathbb{R}^2 \implies \text{conv}\{A\} = \mathbb{R}^2$, which finishes the proof. □

For the assumptions (GP1) and (GP2) and the non-emptiness of the cells, we use the convergence in the $\tau_{\mathcal{L}}$ topology.

Lemma 34. *For $\bar{\mathsf{P}}$-a.a. γ we get that it is in general position and $E(\gamma) = \emptyset$.*

Proof. Denote

$$\mathcal{M}_{gp} = \{\gamma \in \mathcal{M} : \gamma \text{ is in general position}\},$$
$$\mathcal{M}_{gp}^k = \{\gamma \in \mathcal{M} : \gamma_{\Lambda_k} \text{ is in general position}\}, \quad k \in \mathbb{N}, \quad \Lambda_k = [-k,k)^2 .$$

Then we have that $\mathcal{M}_{gp} = \bigcap_{k\in\mathbb{N}} \mathcal{M}_{gp}^k$ and $\mathcal{M}_{gp}^k \subset \mathcal{M}_{gp}^{k-1}$. Therefore for any probability measure P on $\mathcal{M}$ we have that

$$\lim_{k\to\infty} \mathsf{P}(\mathcal{M}_{gp}^k) = \mathsf{P}(\mathcal{M}_{gp}).$$

Now fix $k \in \mathbb{N}$. Then according to (4.11) we have for all $\Lambda \in \mathcal{B}_b^2$ and for all $z > 0$ that $\pi_\Lambda^z(\mathcal{M}_{gp}) = 1$, so also $\pi_\Lambda^z(\mathcal{M}_{gp}^k) = 1$. Therefore for $n \geq k$ we have that

$$\mathsf{P}_n(\mathcal{M}_{gp}^k) = \int_{\mathcal{M}_{gp}^k} \frac{1}{Z_{\Lambda_n}} e^{-H(\gamma_{\Lambda_n})} \pi_{\Lambda_n}^z(\mathrm{d}\gamma) = 1$$

and since $\Lambda_k \subset \Lambda_n$ we also have $\tilde{\mathsf{P}}_n(\mathcal{M}_{gp}^k) = 1$. Now for $\bar{\mathsf{P}}_n$:

$$\bar{\mathsf{P}}_n = \frac{1}{(2n)^2} \sum_{\kappa\in\Lambda_n\cap\mathbb{Z}^2} \tilde{\mathsf{P}}_n \circ \vartheta_\kappa^{-1}.$$

It holds that if $\Lambda_k + \kappa \subset \Lambda_n$ then $\tilde{\mathsf{P}}_n \circ \vartheta_\kappa^{-1}(\mathcal{M}_{gp}^k) = 1$. It also holds that for all $\kappa \in \Lambda_{n-k-1} \cap \mathbb{Z}^2$ we have $\Lambda_k + \kappa \subset \Lambda_n$, so we can write $\forall n \geq k+1$

$$\begin{aligned}
\bar{\mathsf{P}}_n(\mathcal{M}_{gp}^k) &= \frac{1}{(2n)^2} \sum_{\kappa\in\Lambda_n\cap\mathbb{Z}^2} \tilde{\mathsf{P}}_n \circ \vartheta_\kappa^{-1}(\mathcal{M}_{gp}^k) \\
&= \frac{(2(n-k-1))^2}{(2n)^2} + \frac{1}{(2n)^2} \sum_{\kappa\in\Lambda_n\setminus\Lambda_{n-k-1}\cap\mathbb{Z}^2} \tilde{\mathsf{P}}_n \circ \vartheta_\kappa^{-1}(\mathcal{M}_{gp}^k).
\end{aligned}$$

Therefore $\lim_{n\to\infty} \bar{\mathsf{P}}_n(\mathcal{M}_{gp}^k) = 1$. Since $\bar{\mathsf{P}}$ is a limit of $\{\bar{\mathsf{P}}_n\}_{n\in\mathbb{N}}$ in the $\tau_{\mathcal{L}}$ topology and $\mathbb{1}\left[\gamma \in \mathcal{M}_{gp}^k\right]$ is a tame and local function, we get that

$$1 = \lim_{n\to\infty} \bar{\mathsf{P}}_n(\mathcal{M}_{gp}^k) = \bar{\mathsf{P}}(\mathcal{M}_{gp}^k).$$

This holds $\forall k \in \mathbb{N}$ and therefore $\bar{\mathsf{P}}(\mathcal{M}_{gp}) = 1$. For the second part, we define sets

$$\mathcal{M}_z = \{\gamma \in \mathcal{M} : E(\gamma) = \emptyset\},$$
$$\mathcal{M}_z^k = \{\gamma \in \mathcal{M} : E(\gamma_{\Lambda_k}) = \emptyset\}, \quad k \in \mathbb{N}, \quad \Lambda_k = [-k,k)^2 .$$

Because of Lemma 28 and the fact that $\bar{\mathsf{P}}$-a.a. $\gamma \in \mathcal{M}$ satisfy regularity conditions and are in general position, we can write

$$\lim_{k\to\infty} \bar{\mathsf{P}}(\mathcal{M}_z^k) = \bar{\mathsf{P}}(\mathcal{M}_z).$$

For fixed $k \in \mathbb{N}$ and $n \geq k$ we have $\mathsf{P}_n(\mathcal{M}_z^k) = 1$ thanks to (4.12) and the fact that

$$E(\gamma_{\Lambda_n}) = \emptyset \implies E(\gamma_{\Lambda_k}) = \emptyset.$$

The rest of the proof follows analogously as in the previous case. $\square$

Altogether, we have the following proposition.

Proposition 35. *Define the set of* ***admissible configurations***

$$\overline{\mathcal{M}} = \{\gamma \in \mathcal{M}^{temp} : \gamma \text{ satisfies (R1),(R2),(GP1),(GP2) and } E(\gamma) = \emptyset\} \cup \{\bar{o}\}.$$

It holds that $\bar{\mathsf{P}}(\overline{\mathcal{M}}) = 1$. *Particularly for* $\bar{\mathsf{P}}$*-a.a.* $\gamma \neq \bar{o}$ *we have that* $L(\gamma)$ *is a normal tessellation.*

Proof. Lemma 30 gives (R1), (R2) is implied by Lemma 33, since $\bar{\mathsf{P}}$ is invariant under translations by $\kappa \in \mathbb{Z}^2$ and conditions (GP1), (GP2) and non-emptiness of the cells are implied by Lemma 34. □

4.2.3 An infinite-volume Gibbs-Laguerre measure

Recall Definition 12 of the conditional energy of configuration γ in Λ,

$$H_\Lambda(\gamma) = \lim_{n\to\infty} H(\gamma_{\Lambda_n}) - H(\gamma_{\Lambda_n \setminus \Lambda}).$$

Thanks to Proposition 31, we know how this function looks for admissible configurations.

Lemma 36. *Take* $\gamma \in \overline{\mathcal{M}}$*. Then for all* $\Lambda \in \mathcal{B}_b^2$ *we obtain* $H_\Lambda(\gamma) = 6 \cdot |\gamma_\Lambda|$.

Proof. If $\gamma = \bar{o}$, then it clearly holds. For $\gamma \neq \bar{o}$ we have that $\gamma_\Lambda = \{x_1, \dots, x_M\}$ for some $M \in \mathbb{N}$. Denote $\gamma_\Lambda^i = \{x_1, \dots, x_i\}$. From the definition of conditional energy

$$H_\Lambda(\gamma) = \lim_{n\to\infty} H(\gamma_{\Lambda_n}) - H(\gamma_{\Lambda_n\setminus\Lambda}) = \sum_{i=1}^{M} \lim_{n\to\infty} H(\gamma_{\Lambda_n\setminus\Lambda}\gamma_\Lambda^i) - H(\gamma_{\Lambda_n\setminus\Lambda}\gamma_\Lambda^{i-1}).$$

Thanks to the assumptions on γ we get that $L(\gamma)$ is a normal tessellation with no empty cells and therefore for all $i = 1, \dots, M$ there exists n large enough so that $L(x_i, \gamma_{\Lambda_n\setminus\Lambda}\gamma_\Lambda^i)$ is bounded. With the help of Proposition 31 we get that

$$\lim_{n\to\infty} H(\gamma_{\Lambda_n\setminus\Lambda}\gamma_\Lambda^i) - H(\gamma_{\Lambda_n\setminus\Lambda}\gamma_\Lambda^{i-1}) = 6,$$

which finishes the proof. □

Recall that $\mathcal{M}_a = \{\gamma \in \mathcal{M} : \mathsf{m}(\gamma) \le a\}$, $a \in \mathbb{N}$, is the set of configurations whose marks are at most a. We define an increasing sequence of *local sets* (i.e. subsets of $\mathcal{M}$ whose indicator is a local function). Take $\Lambda \in \mathcal{B}_b^2$ and $l, n, a \in \mathbb{N}$ and define

$$\begin{aligned} C(\Lambda, a, l, n) = \{\xi \in \mathcal{M} : \xi \text{ satisfies assumptions (C1) and (C2)}\}\,,\ \text{where} \\ \text{(C1)}: \text{ there exists } u \in \xi_{\Lambda_n\setminus\Lambda} : u' \in U\left(0, \frac{1}{2}l\right), \\ \text{(C2)}: \forall\gamma \in \mathcal{M}_a, \forall x \in \gamma_\Lambda \text{ we have } L(x, \xi_{\Lambda_n\setminus\Lambda} \cup \{x\}) \subset U\left(0, \frac{1}{2}l\right). \end{aligned} \tag{4.16}$$

Put

$$B(\Lambda, a, l) = \bigcup_{n\in\mathbb{N}} C(\Lambda, a, l, n),$$
$$A(\Lambda, a) = \bigcup_{l\in\mathbb{N}} B(\Lambda, a, l),$$

then clearly $\forall \Lambda \in \mathcal{B}_b^2$, $\forall a, l, n \in \mathbb{N}$:

$$\begin{aligned}
&C(\Lambda, a, l, n) \subset C(\Lambda, a, l+1, n),\ C(\Lambda, a, l, n) \subset C(\Lambda, a, l, n+1),\\
&B(\Lambda, a, l) \subset B(\Lambda, a, l+1),\\
&A(\Lambda, a) \supset A(\Lambda, a+1).
\end{aligned}$$

We also have the following equality.

Lemma 37. *For all $\Lambda \in \mathcal{B}_b^2$ we have that*

$$\overline{\mathcal{M}} = \bigcap_{a\in\mathbb{N}} \bigcup_{l\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \overline{\mathcal{M}} \cap C(\Lambda, a, l, n) \cup \{\bar{o}\}.$$

Proof. The relation $\supset$ clearly holds. Take $\xi \in \overline{\mathcal{M}}$, $\xi \neq \bar{o}$. We would like to show that $\forall a \in \mathbb{N}$ there exist $l, n \in \mathbb{N}$ such that $\xi \in C(\Lambda, a, l, n)$. Take $a \in \mathbb{N}$ and consider

- $n_0 = \min\{n \in \mathbb{N} : \exists u \in \xi_{\Lambda_n \setminus \Lambda}\}$,
- $l_0 = \min\{l \in \mathbb{N} : \Lambda_{n_0} \subset U(0, \frac{1}{2}l)\}$.

We will consider $n \geq n_0$ and $l \geq l_0$. This will assure that the assumption (C1) is satisfied. Now w.l.o.g. assume that Λ is closed (otherwise work with $\mathrm{clo}(\Lambda)$) and recall that we have chosen fixed $a \in \mathbb{N}$. We will use the observation that $L((x', x''), \gamma) \subset L((x', a), \gamma)$, whenever $x'' \leq a$. Therefore to prove (C2), it is enough to prove that for some $n, l \in \mathbb{N}$ and $\forall x' \in \Lambda$ we have that

$$L\left((x', a), \xi_{\Lambda_n \setminus \Lambda} \cup \{(x', a)\}\right) \subset U\left(0, \frac{1}{2}l\right).$$

It holds (since $\xi \in \overline{\mathcal{M}}$) that $\forall x' \in \Lambda$ there exist n_x, l_x such that

$$L\left((x', a), \xi_{\Lambda_{n_x} \setminus \Lambda} \cup \{(x', a)\}\right) \subset U\left(0, \frac{1}{2}l_x\right).$$

Then, because of the representation (4.2) and the openness of $U\left(0, \frac{1}{2}l_x\right)$, there exists $\varepsilon_x > 0$ such that also $\forall y' \in U(x', \varepsilon_x)$ we have that

$$L\left((y', a), \xi_{\Lambda_{n_x} \setminus \Lambda} \cup \{(y', a)\}\right) \subset U\left(0, \frac{1}{2}l_x\right).$$

Therefore we have an open cover of Λ, $\Lambda \subset \bigcup_{x' \in \Lambda} U(x', \varepsilon_x)$ and since Λ is a compact set, there exists a finite cover $\Lambda \subset \bigcup_{i=1}^N U(x_i', \varepsilon_{x_i})$. To finish the proof, it is enough to take $n = \max\{n_0, n_{x_1}, \dots, n_{x_N}\}$ and $l = \max\{l_0, l_{x_1}, \dots, l_{x_N}\}$. $\square$

Recall Definition 14 of Gibbs kernel Ξ_Λ:

$$\Xi_\Lambda(\xi, \mathrm{d}\gamma) = \frac{e^{-H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c})}}{Z_\Lambda(\xi)} \pi_\Lambda^z(\mathrm{d}\gamma).$$

Since we do not in general have the local stability assumption $\mathcal{H}_l$, we need to make sure that this quantity is well defined, at least for almost all configurations. To do that we need the following observation.

Lemma 38. *Take $\xi \in \mathcal{M}$ such that it is in general position. Then for all $\Lambda \in \mathcal{B}_b^2$ and for all $z > 0$ we get that for π_Λ^z-a.a. $\gamma \in \mathcal{M}$ also $\xi_{\Lambda^c}\gamma_\Lambda$ is in general position.*

Proof. Using (4.11), it is enough to show that for fixed points $A, B, C \in \Lambda^c$ and fixed weights $a, b, c > 0$ we have that for π_Λ^z-a.a. $\gamma \in \mathcal{M}$

1. no 2 points from γ' lie on a line with A and no point from γ' lies on a line going through A and B,
2. no 3 points from γ and (A, a), no 2 points from γ and $(A, a), (B, b)$ and no point from γ and $(A, a), (B, b), (C, c)$ have equal power distance to some $x \in \mathbb{R}^2$.

All of these can be proven similarly as (4.11) in Lautensack [2007], Proposition 3.1.5. (see also Møller [1994], Proposition 4.1.2.). □

Now we can show that the Gibbs kernel is well defined for all $\xi \in \overline{\mathcal{M}} \cup \mathcal{M}_f$.

Lemma 39. *Let $\xi \in \overline{\mathcal{M}}$ or $\xi \in \mathcal{M}_f$ such that it is in general position and $E(\xi) = \emptyset$, then $\forall \Lambda \in \mathcal{B}_b^2$, $\forall z > 0$ we have that $0 < Z_\Lambda(\xi) < \infty$.*

Proof. At first take $\xi \in \overline{\mathcal{M}}$. Then we know that ξ is in general position, satisfies regularity conditions and also $E(\xi) = \emptyset$. Then thanks to Lemma 26 we have that $\forall \gamma \in \mathcal{M}$ also $\xi_{\Lambda^c}\gamma_\Lambda$ satisfies regularity conditions and according to Lemma 38 we have that for π_Λ^z-a.a. $\gamma \in \mathcal{M}$ it holds that $\xi_{\Lambda^c}\gamma_\Lambda$ is in general position. If $E(\xi_{\Lambda^c}\gamma_\Lambda) \neq \emptyset$, then $H_\Lambda(\xi_{\Lambda^c}\gamma_\Lambda) = +\infty$. Otherwise thanks to Lemma 36 we get that $H_\Lambda(\xi_{\Lambda^c}\gamma_\Lambda) = 6\,|\gamma_\Lambda|$. Altogether $H_\Lambda(\xi_{\Lambda^c}\gamma_\Lambda) \geq 0$ for π_Λ^z-a.a. γ, and therefore

$$Z_\Lambda(\xi) = \int e^{-H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c})} \pi_\Lambda^z(\mathrm{d}\gamma) \leq 1 < \infty.$$

Now take $\xi \in \mathcal{M}_f$, which is in general position and has no empty cells and denote $M = |\xi_{\Lambda^c}|$. Then thanks to Lemma 38 we can only work with such γ_Λ so that $L(\xi_{\Lambda^c}\gamma_\Lambda)$ is a generalized normal tessellation and we can write

$$H_\Lambda(\xi_{\Lambda^c}\gamma_\Lambda) = H(\xi_{\Lambda^c}\gamma_\Lambda) - H(\xi_{\Lambda^c}) \geq -H(\xi_{\Lambda^c}) \geq -3 \cdot \binom{M}{3},$$

since $L(\xi_{\Lambda^c})$ can have at most $\binom{M}{3}$ vertices. Therefore

$$Z_\Lambda(\xi) = \int e^{-H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c})} \pi_\Lambda^z(\mathrm{d}\gamma) \leq e^{3 \cdot \binom{M}{3}} < \infty.$$

The part $0 < Z_\Lambda(\xi)$ can be shown in the same way as in Lemma 3. □

Particularly, Ξ_Λ is well defined for all $\xi \in \overline{\mathcal{M}}$ and $\xi \in \mathcal{M}_f$ which are in general position and satisfy $E(\xi) = \emptyset$. Recall Definition 21 of the cut-off $\Xi_\Lambda^{\Delta, m_0}$. We will denote $\Xi_\Lambda^{n,a} := \Xi_\Lambda^{\Lambda_n, a}$, i.e.

$$\Xi_\Lambda^{n,a}(\xi, \mathrm{d}\gamma) = \frac{\mathbb{1}\{\gamma_\Lambda \in \mathcal{M}_a\} \cdot e^{-H_\Lambda(\gamma_\Lambda \xi_{\Lambda_n \setminus \Lambda})}}{Z_\Lambda^{n,a}(\xi_{\Lambda_n \setminus \Lambda})} \pi_\Lambda^z(\mathrm{d}\gamma).$$

Using the second part of the proof of Lemma 39, we can see that $\Xi_\Lambda^{n,a}$ is well defined for all ξ in general position with $E(\xi) = \emptyset$.

Recall Definition 8 of the sets $\underline{\mathcal{M}}^l$. This final auxiliary lemma will justify the definition in (4.16) of the sets $C(\Lambda, a, l, n)$. These sets are in fact chosen so that the conditional energy depends only on the boundary condition inside Λ_n.

Lemma 40. *Let $\Lambda \in \mathcal{B}_b^2$, and take $a, n, l \in \mathbb{N}$ such that $U(0, 2l+1) \subset \Lambda_n$ and $\Lambda \oplus B(0, a) \subset U(0, \frac{1}{2}l)$. Then for all $\xi \in C(\Lambda, a, l, n) \cap \underline{\mathcal{M}}^l$ and for all $\gamma \in \mathcal{M}_a$ such that $\xi_{\Lambda^c}\gamma_\Lambda$ are in general position and $E(\xi_{\Lambda^c}) = \emptyset$ we have that*

i) $E(\xi_{\Lambda^c}\gamma_\Lambda) \neq \emptyset \iff E(\xi_{\Lambda_n \setminus \Lambda}\gamma_\Lambda) \neq \emptyset$,

ii) $H_\Lambda(\xi_{\Lambda^c}\gamma_\Lambda) = H_\Lambda(\xi_{\Lambda_n \setminus \Lambda}\gamma_\Lambda)$.

Proof. At first we assume i) and prove ii). Take ξ, γ satisfying the assumptions and assume that i) holds. We have $\gamma_\Lambda = \{x_1, \dots, x_M\}$ for some $M \in \mathbb{N}$. Denote $\gamma_\Lambda^i = \{x_1, \dots, x_i\}$, $i = 1, \dots, M$.

If $E(\xi_{\Lambda^c}\gamma_\Lambda) \neq \emptyset$ then according to i) also $E(\xi_{\Lambda_n \setminus \Lambda}\gamma_\Lambda) \neq \emptyset$ and we have

$$H_\Lambda(\xi_{\Lambda^c}\gamma_\Lambda) = +\infty = H_\Lambda(\xi_{\Lambda_n \setminus \Lambda}\gamma_\Lambda).$$

If $E(\xi_{\Lambda^c}\gamma_\Lambda) = E(\xi_{\Lambda_n \setminus \Lambda}\gamma_\Lambda) = \emptyset$, then thanks to the definition of the set $C(\Lambda, a, l, n)$ we have that the cells $L(x_i, \xi_{\Lambda_n \setminus \Lambda}\gamma_\Lambda^i)$ are bounded $\forall i \in \{1, \dots, M\}$. Recalling the key Proposition 31 for our energy function H, we can write

$$\begin{aligned} H_\Lambda(\xi_{\Lambda_n \setminus \Lambda}\gamma_\Lambda) &= H(\xi_{\Lambda_n \setminus \Lambda}\gamma_\Lambda) - H(\xi_{\Lambda_n \setminus \Lambda}) \\ &= \sum_{i=1}^{M} H(\xi_{\Lambda_n \setminus \Lambda}\gamma_\Lambda^i) - H(\xi_{\Lambda_n \setminus \Lambda}\gamma_\Lambda^{i-1}) \overset{\text{P.31}}{=} \sum_{i=1}^{M} 6 = 6 \cdot |\gamma_\Lambda| \,. \end{aligned}$$

Using Lemma 36 we also have that $H_\Lambda(\xi_{\Lambda^c}\gamma_\Lambda) = 6 \cdot |\gamma_\Lambda|$.

Now it remains to prove i). Take ξ, γ satisfying the assumptions. The implication $\Longleftarrow$ always holds, so we only have to prove that if there exists an empty cell for $\xi_{\Lambda^c}\gamma_\Lambda$, then it is already empty in $\xi_{\Lambda_n \setminus \Lambda}\gamma_\Lambda$ (remember that $E(\xi_{\Lambda^c}) = \emptyset$).

Let there exist $x \in \xi_{\Lambda^c}\gamma_\Lambda$ such that $L(x, \xi_{\Lambda^c}\gamma_\Lambda) = \emptyset$ and assume for contradiction that $E(\xi_{\Lambda_n \setminus \Lambda}\gamma_\Lambda) = \emptyset$. This means that either $x \in \xi_{\Lambda_n^c}$ or $L(x, \xi_{\Lambda_n \setminus \Lambda}\gamma_\Lambda) \neq \emptyset$. Recall Lemma 29 and consider the three possible locations of the point x:

1) $x \in \gamma_\Lambda$: Then for all $z \in L(x, \xi_{\Lambda_n \setminus \Lambda}\gamma_\Lambda)$ there exists $y \in \xi_{\Lambda_n^c}$ such that $\rho(z, y) \leq \rho(z, x)$. However, from the choice of n and l and from the definition of the set $C(\Lambda, a, l, n)$ we know that $y' \in U(0, 2l+1)^c$, $x' \in U(0, \frac{1}{2}l)$ and $z \in U(0, \frac{1}{2}l)$. Using Lemma 29 we get that

$$\rho(z, y) \leq \rho(z, x) \leq l^2 \overset{\text{L.29}}{<} \rho(z, (y', |y'| - l)) \leq \rho(z, y),$$

which is clearly a contradiction.

2) $x \in \xi_{\Lambda_n \setminus \Lambda}$: We know that $L(x, \xi_{\Lambda^c}\gamma_\Lambda) = \emptyset$ but $L(x, \xi_{\Lambda_n \setminus \Lambda}\gamma_\Lambda) \neq \emptyset$ and $L(x, \xi_{\Lambda^c}) \neq \emptyset$. Therefore

$$\begin{aligned} &\forall z \in L(x, \xi_{\Lambda^c})\, \exists u \in \gamma_\Lambda \text{ such that } z \in L(u, \xi_{\Lambda^c}\gamma_\Lambda), \\ &\forall z \in L(x, \xi_{\Lambda_n \setminus \Lambda}\gamma_\Lambda)\, \exists y \in \xi_{\Lambda_n^c} \text{ such that } z \in L(y, \xi_{\Lambda^c}\gamma_\Lambda). \end{aligned} \tag{4.17}$$

If $\exists z \in L(x, \xi_{\Lambda^c}) \cap L(x, \xi_{\Lambda_n \setminus \Lambda}\gamma_\Lambda)$, then there exist $u \in \gamma_\Lambda$ and $y \in \xi_{\Lambda_n^c}$ such that $\rho(z, y) = \rho(z, u)$ and we again get a contradiction with Lemma 29. Therefore $L(x, \xi_{\Lambda^c}) \cap L(x, \xi_{\Lambda_n \setminus \Lambda}\gamma_\Lambda) = \emptyset$. Then there exists $z \in L(x, \xi_{\Lambda_n \setminus \Lambda}\gamma_\Lambda)$ such that $\exists u \in \gamma_\Lambda$ such that $\rho(z, x) = \rho(z, u)$, i.e. $z \in L(u, \xi_{\Lambda_n \setminus \Lambda}\gamma_\Lambda)$. Since by (4.17) there also exists $y \in \xi_{\Lambda_n^c}$ such that $z \in L(y, \xi_{\Lambda^c}\gamma_\Lambda)$, we again get the contradiction $\rho(z,y) \le \rho(z,u)$.

3) $x \in \xi_{\Lambda_n^c}$: We know that $L(x, \xi_{\Lambda^c}\gamma_\Lambda) = \emptyset$ and $L(x, \xi_{\Lambda^c}) \neq \emptyset$. Therefore $\forall z \in L(x, \xi_{\Lambda^c})\, \exists u \in \gamma_\Lambda$ such that $\rho(z,u) < \rho(z,x)$. Particularly we can assume that $\rho(z,u) \le \rho(z,v)$ for all $v \in \gamma_\Lambda$ and therefore $z \in L(u, \xi_{\Lambda^c}\gamma_\Lambda) \subset U(0, \frac{1}{2}l)$. Therefore $L(x, \xi_{\Lambda^c}) \subset U(0, \frac{1}{2}l)$. Notice that $x' \in U(0, 2l+1)^c$. From the definition of the set C there exists $y \in \xi_{\Lambda_n \setminus \Lambda}$ such that $y' \in U(0, \frac{1}{2}l) \implies \forall z \in L(x, \xi_{\Lambda^c})$ we have that

$$\rho(z,x) \le \rho(z,y) \le l^2 \overset{\text{L.29}}{<} \rho(z, (x', |x'| - l)) \le \rho(z,x),$$

which is the final contradiction and the proof is finished. □

Now we are ready to prove our main result.

Theorem 41. *Consider the probability measure $\bar{\mathsf{P}}$ from Claim 32 and assume that it satisfies $\bar{\mathsf{P}}(\{\bar{o}\}) = 0$. Then for all $\Lambda \in \mathcal{B}_b^2$ and for all measurable bounded local functions F the* $\boldsymbol{DLR}_\Lambda$ *equations hold. Particularly, $\bar{\mathsf{P}}$ is an infinite-volume Gibbs measure with energy function H defined in (4.10) and activity $z > 0$.*

Proof. Take $\Lambda \in \mathcal{B}_b^2$ and measurable bounded Λ-local function F. We will show that $\forall \varepsilon > 0$

$$\delta_0 = \left| \int F(\gamma)\bar{\mathsf{P}}(\mathrm{d}\gamma) - \int \int F(\gamma_\Lambda)\Xi_\Lambda(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) \right| < \varepsilon.$$

Fix $\varepsilon > 0$. Find i_0 smallest such that $\Lambda \subset \Lambda_{i_0}$. We will w.l.o.g. assume that $i_0 = 1$ (otherwise work with $n \ge i_0$ in the whole proof). Then there exists $a \in \mathbb{N}$ such that

1. $\pi_\Lambda^z(\mathcal{M}_a) \ge 1 - \varepsilon$.

For this a find $l \in \mathbb{N}$ such that

2. $\Lambda \oplus B(0,a) \subset U(0, \frac{1}{2}l)$,
3. $\bar{\mathsf{P}}(\underline{\mathcal{M}}^l) \ge 1 - \varepsilon$, $\bar{\mathsf{P}}_n(\underline{\mathcal{M}}^l) \ge 1 - \varepsilon$ for all $n \in \mathbb{N}$ (from Claim 32),
4. $\bar{\mathsf{P}}(B(\Lambda, a, l)) \ge 1 - \varepsilon$ (from Proposition 35, Lemma 37 and $\bar{\mathsf{P}}(\{\bar{o}\}) = 0$).

For these a and l we can find $k \in \mathbb{N}$ such that

5. $U(0, 2l+1) \subset \Lambda_k$
6. $\bar{\mathsf{P}}(C(\Lambda, a, l, k)) \ge 1 - 2\varepsilon$.

Fix a, l, k and recall the definition of the measures $\hat{\mathsf{P}}_n$ from the proof of Lemma 13:

$$\hat{\mathsf{P}}_n = \frac{1}{|\Lambda_n|} \sum_{\kappa \in \mathbb{Z}^2 \cap \Lambda_n :\, \Lambda \subset \vartheta_\kappa(\Lambda_n)} \mathsf{P}_n \circ \vartheta_\kappa^{-1}.$$

We know that $\hat{\mathsf{P}}_n$ satisfy $(\mathbf{DLR})_\Lambda$ and they are asymptotically equivalent to $\bar{\mathsf{P}}_n$ in the sense that for any $G \in \boldsymbol{\mathcal{L}}$ we get that

$$\lim_{n \to \infty} \left| \int G(\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\gamma) - \int G(\gamma)\bar{\mathsf{P}}_n(\mathrm{d}\gamma) \right| = 0.$$

Particularly there exists n_0 such that $\forall n \ge n_0$ we get that $\hat{\mathsf{P}}_n(\mathcal{M}) \ge 1 - \varepsilon$. It also holds that $\bar{\mathsf{P}}_n((\underline{\mathcal{M}}^l)^c) \ge \hat{\mathsf{P}}_n((\underline{\mathcal{M}}^l)^c)$. Therefore there exists $n_1 \ge n_0$ such that

7. $\hat{\mathsf{P}}_n(\underline{\mathcal{M}}^l) \geq 1 - 2\varepsilon$ for all $n \geq n_1$,

8. $\hat{\mathsf{P}}_n(C(\Lambda, a, l, k)) \geq 1 - 3\varepsilon$ for all $n \geq n_1$.

The second part is true thanks to 6. and the fact that $G(\gamma) = \mathbb{1}\{\gamma \in C(\Lambda, a, l, k)\}$ is a bounded and Λ_k-local function.

Now we have everything we need to estimate δ_0. Assume w. l. o. g. that $|F| \leq 1$ and recall that $\bar{\mathsf{P}}(\overline{\mathcal{M}}) = 1$.

$$\begin{aligned}
\delta_0 &= \left| \int F(\gamma)\bar{\mathsf{P}}(\mathrm{d}\gamma) - \int\int F(\gamma_\Lambda)\Xi_\Lambda(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) \right| \leq \bar{\mathsf{P}}((C(\Lambda, a, l, k) \cap \underline{\mathcal{M}}^l)^c) \\
&\qquad + \left| \int F(\gamma)\bar{\mathsf{P}}(\mathrm{d}\gamma) - \int_{C(\Lambda,a,l,k)\cap\underline{\mathcal{M}}^l} \int F(\gamma_\Lambda)\Xi_\Lambda(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) \right| \\
&\overset{3.,6.}{\leq} 3\varepsilon + \left| \int F(\gamma)\bar{\mathsf{P}}(\mathrm{d}\gamma) - \int_{C(\Lambda,a,l,k)\cap\underline{\mathcal{M}}^l} \int F(\gamma_\Lambda)\Xi^{k,a}_\Lambda(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) \right| \\
&\qquad + \left| \int_{C(\Lambda,a,l,k)\cap\underline{\mathcal{M}}^l} \left[\int_{\mathcal{M}_a} F(\gamma_\Lambda)\Xi^{k,a}_\Lambda(\xi, \mathrm{d}\gamma) - \int_{\mathcal{M}_a} F(\gamma_\Lambda)\Xi_\Lambda(\xi, \mathrm{d}\gamma) \right] \bar{\mathsf{P}}(\mathrm{d}\xi) \right| \\
&\qquad + \left| \int_{C(\Lambda,a,l,k)\cap\underline{\mathcal{M}}^l} \int_{(\mathcal{M}_a)^c} F(\gamma_\Lambda)\Xi_\Lambda(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) \right|.
\end{aligned}$$

Now we have for some $b < \infty$:

$$\begin{aligned}
&\left| \int_{C(\Lambda,a,l,k)\cap\underline{\mathcal{M}}^l} \int_{(\mathcal{M}_a)^c} F(\gamma_\Lambda)\Xi_\Lambda(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) \right| \\
&\leq \int\int_{(\mathcal{M}_a)^c} \frac{1}{Z_\Lambda(\xi)} \pi^z_\Lambda(\mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) \leq \pi^z_\Lambda((\mathcal{M}_a)^c) \cdot \frac{1}{\pi^z_\Lambda(\{\bar{o}\})} \overset{1.}{\leq} b \cdot \varepsilon.
\end{aligned} \tag{4.18}$$

Now for $\bar{\mathsf{P}}$-a.a. $\xi \in C(\Lambda, a, l, k) \cap \underline{\mathcal{M}}^l$ we can use Lemma 40 to show that

$$\begin{aligned}
&\left| \int_{\mathcal{M}_a} F(\gamma_\Lambda)\Xi^{k,a}_\Lambda(\xi, \mathrm{d}\gamma) - \int_{\mathcal{M}_a} F(\gamma_\Lambda)\Xi_\Lambda(\xi, \mathrm{d}\gamma) \right| \\
&\qquad\qquad = \left| \int_{\mathcal{M}_a} F(\gamma_\Lambda) e^{-H_\Lambda(\xi_{\Lambda^c}\gamma_\Lambda)} \left(\frac{Z_\Lambda(\xi) - Z^{k,a}_\Lambda(\xi)}{Z_\Lambda(\xi) \cdot Z^{k,a}_\Lambda(\xi)} \right) \pi^z_\Lambda(\mathrm{d}\gamma) \right| \\
&\leq \int_{\mathcal{M}_a} \frac{\left| Z_\Lambda(\xi) - Z^{k,a}_\Lambda(\xi) \right|}{\pi^z_\Lambda(\{\bar{o}\})^2} \pi^z_\Lambda(\mathrm{d}\gamma) \\
&\leq b^2 \cdot \left| Z_\Lambda(\xi) - Z^{k,a}_\Lambda(\xi) \right| = b^2 \cdot \left| \int_{(\mathcal{M}_a)^c} e^{-H_\Lambda(\xi_{\Lambda^c}\gamma_\Lambda)} \pi^z_\Lambda(\mathrm{d}\gamma) \right| \overset{1.}{\leq} b^2 \cdot \varepsilon.
\end{aligned} \tag{4.19}$$

Therefore we can estimate

$$\delta_0 \leq c \cdot \varepsilon + \left| \int F(\gamma)\bar{\mathsf{P}}(\mathrm{d}\gamma) - \int_{C(\Lambda,a,l,k)\cap\underline{\mathcal{M}}^l} \int F(\gamma_\Lambda)\Xi^{k,a}_\Lambda(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) \right| =: c \cdot \varepsilon + \delta_1,$$

where $c = 3 + b + b^2$. We continue with δ_1:

$$\begin{aligned}
\delta_1 &= \left| \int F(\gamma)\bar{\mathsf{P}}(\mathrm{d}\gamma) - \int_{C(\Lambda,a,l,k)\cap\underline{\mathcal{M}}^l} \int F(\gamma_\Lambda)\Xi^{k,a}_\Lambda(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) \right| \\
&\leq \bar{\mathsf{P}}((C(\Lambda, a, l, k) \cap \underline{\mathcal{M}}^l)^c) + \left| \int F(\gamma)\bar{\mathsf{P}}(\mathrm{d}\gamma) - \int\int F(\gamma_\Lambda)\Xi^{k,a}_\Lambda(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) \right| \\
&\overset{3.,6.}{\leq} 3\varepsilon + \left| \int F(\gamma)\bar{\mathsf{P}}(\mathrm{d}\gamma) - \int\int F(\gamma_\Lambda)\Xi^{k,a}_\Lambda(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) \right| =: 3\varepsilon + \delta_2.
\end{aligned}$$

Now we use the asymptotic equivalence for $\hat{\mathsf{P}}_n$ and $\bar{\mathsf{P}}_n$ and the fact that $F(\gamma)$ and $G(\gamma) = \int F(\nu)\Xi_\Lambda^{k,a}(\gamma, \mathrm{d}\nu)$ are bounded (and therefore tame) local functions. Let $n \geq n_1$, then we have the following estimate for δ_2:

$$\begin{aligned}
\delta_2 &= \left| \int F(\gamma)\bar{\mathsf{P}}(\mathrm{d}\gamma) - \int\int F(\gamma_\Lambda)\Xi_\Lambda^{k,a}(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) \right| \\
&\leq \left| \int F\mathrm{d}\bar{\mathsf{P}} - \int F\mathrm{d}\hat{\mathsf{P}}_n \right| + \left| \int F(\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\gamma) - \int\int F(\gamma_\Lambda)\Xi_\Lambda^{k,a}(\xi, \mathrm{d}\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\xi) \right| \\
&+ \left| \int\int F(\gamma_\Lambda)\Xi_\Lambda^{k,a}(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) - \int\int F(\gamma_\Lambda)\Xi_\Lambda^{k,a}(\xi, \mathrm{d}\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\xi) \right|.
\end{aligned}$$

We can choose $n_2 \geq n_1$ so that $\forall n \geq n_2$ we have

$$\begin{aligned}
&\left| \int F(\gamma)\bar{\mathsf{P}}(\mathrm{d}\gamma) - \int F(\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\gamma) \right| \leq \varepsilon, \\
&\left| \int\int F(\gamma_\Lambda)\Xi_\Lambda^{k,a}(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) - \int\int F(\gamma_\Lambda)\Xi_\Lambda^{k,a}(\xi, \mathrm{d}\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\xi) \right| \leq \varepsilon.
\end{aligned}$$

Therefore for $n \geq n_2$ we can write

$$\begin{aligned}
\delta_2 &= \left| \int F(\gamma)\bar{\mathsf{P}}(\mathrm{d}\gamma) - \int\int F(\gamma_\Lambda)\Xi_\Lambda^{k,a}(\xi, \mathrm{d}\gamma)\bar{\mathsf{P}}(\mathrm{d}\xi) \right| \\
&\leq 2\varepsilon + \left| \int F(\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\gamma) - \int\int F(\gamma_\Lambda)\Xi_\Lambda^{k,a}(\xi, \mathrm{d}\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\xi) \right| =: 2\varepsilon + \delta_3.
\end{aligned}$$

Now for our last estimate. Since $\hat{\mathsf{P}}_n$ satisfies $\mathbf{DLR}_\Lambda$ we can write

$$\begin{aligned}
\delta_3 &= \left| \int F(\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\gamma) - \int\int F(\gamma_\Lambda)\Xi_\Lambda^{k,a}(\xi, \mathrm{d}\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\xi) \right| \\
&\leq 0 + \left| \int\int F(\gamma_\Lambda)\Xi_\Lambda(\xi, \mathrm{d}\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\xi) - \int\int F(\gamma_\Lambda)\Xi_\Lambda^{k,a}(\xi, \mathrm{d}\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\xi) \right| \\
&\leq \left| \int_{C(\Lambda,a,l,k)\cap\underline{\mathcal{M}}^l} \left[\int F(\gamma_\Lambda)\Xi_\Lambda(\xi, \mathrm{d}\gamma) - \int F(\gamma_\Lambda)\Xi_\Lambda^{k,a}(\xi, \mathrm{d}\gamma) \right] \hat{\mathsf{P}}_n(\mathrm{d}\xi) \right| \\
&\qquad\qquad + 2 \cdot \hat{\mathsf{P}}_n((C(\Lambda, a, l, k) \cap \underline{\mathcal{M}}^l)^c) \\
&\overset{7.,8.}{\leq} \left| \int_{C(\Lambda,a,l,k)\cap\underline{\mathcal{M}}^l} \left[\int_{\mathcal{M}_a} F(\gamma_\Lambda)\Xi_\Lambda(\xi, \mathrm{d}\gamma) - \int_{\mathcal{M}_a} F(\gamma_\Lambda)\Xi_\Lambda^{k,a}(\xi, \mathrm{d}\gamma) \right] \hat{\mathsf{P}}_n(\mathrm{d}\xi) \right| \\
&\qquad\qquad + \left| \int\int_{(\mathcal{M}_a)^c} F(\gamma_\Lambda)\Xi_\Lambda(\xi, \mathrm{d}\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\xi) \right| + 10 \cdot \varepsilon.
\end{aligned}$$

Now analogously as in (4.18) and (4.19) we can estimate

$$\begin{aligned}
&\left| \int\int_{(\mathcal{M}_a)^c} F(\gamma_\Lambda)\Xi_\Lambda(\xi, \mathrm{d}\gamma)\hat{\mathsf{P}}_n(\mathrm{d}\xi) \right| \leq b \cdot \varepsilon \\
&\left| \int_{C(\Lambda,a,l,k)\cap\underline{\mathcal{M}}^l} \int_{\mathcal{M}_a} F(\gamma_\Lambda) \left[\Xi_\Lambda(\xi, \mathrm{d}\gamma) - \Xi_\Lambda^{k,a}(\xi, \mathrm{d}\gamma) \right] \hat{\mathsf{P}}_n(\mathrm{d}\xi) \right| \leq b^2 \cdot \varepsilon.
\end{aligned}$$

Putting everything together we get that (recall that $c = 3 + b + b^2$):

$$\delta_0 \leq c \cdot \varepsilon + \delta_1 \leq (c + 3)\varepsilon + \delta_2 \leq (c + 5)\varepsilon + \delta_3 \leq (2c + 12)\varepsilon.$$

This finishes the proof. □

Conclusion

To conclude this work, let us summarize our main results and comment on possible future extensions for the considered processes. We believe that particularly the results from Chapter 4 deserve an additional examination as they could potentially be extended to other energy functions.

This work was concentrated on marked Gibbs point processes. In the second chapter, we considered the recent existence theorem from Rœlly and Zass [2020] and expressed our objections (justified by a counterexample) to the formulation of the range assumption. We presented a reformulation of the range assumption and checked that the proof of the existence theorem still holds.

Therefore, we could use this theorem in the third chapter, where we studied the Gibbs facet process. As was expected, we proved that for the repulsive model **F1** (i.e. the model with non–negative energy function) the infinite volume Gibbs facet process in $\mathbb{R}^d$ exists. We also considered the case of non–positive and real energy function, i.e. the clustering and mixed models **F2** and **F3**. In $\mathbb{R}^2$ we proved that the finite–volume Gibbs facet processes with negative energy function do not exist. In $\mathbb{R}^3$ we found a counterexample showing that the stability assumption is not satisfied for positive interactions between triplets and negative interactions between pairs of facets. We believe that after a careful analysis, we should be able to find counterexamples for any situation from **F2** and **F3** in any dimension d and modify them similarly as in the two-dimensional case to show that the finite–volume Gibbs measures do not exist.

In the last chapter, we considered the Gibbs–Laguerre tessellations of $\mathbb{R}^2$ and proved that under the assumption that we almost surely see a point, the infinite–volume Gibbs–Laguerre process exists for the energy function given in (4.10).

The important tool for this proof was the definition of the sets $C(\Lambda, a, l, n)$ in (4.16) and Proposition 31, which allowed us to prove the equality of the conditional energies in Lemma 40. However, the proof of the first part of Lemma 40 raises a question, whether the definition of the sets $C(\Lambda, a, l, n)$ is not in itself enough to show that the conditional energies are already equal. We hope to further examine this situation and potentially consider other energy functions in a future work.

Another problem which we wish to address in the future is the assumption $\bar{\mathsf{P}}(\{\bar{o}\}) = 0$. We have not yet been able to prove that this assumption holds, using the tools of the local convergence. We hope to find a proof or an estimating rule that would enable us to get rid of this assumption altogether. Alternatively, we could try to show that the family of sets $\{C(\Lambda, a, l, n)\}_{n,l\in\mathbb{N}}$ forms an uniform estimate of the support of the measures $\bar{\mathsf{P}}_n$ (as in Proposition 3 in Rœlly and Zass [2020]).

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