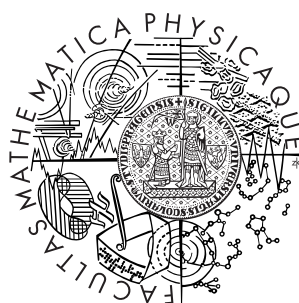


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Matematicko-fyzikální fakulta

DIPLOMOVÁ PRÁCE



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Analýza trhu úrokových měr

Katedra pravděpodobnosti a matematické statistiky

Vedoucí diplomové práce: Mgr. Karel Janeček, MBA, Ph.D.

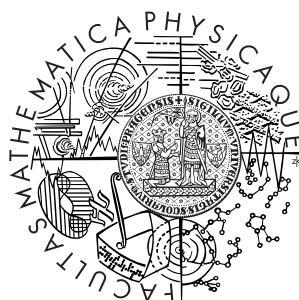
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DIPLOMA THESIS



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Interest rate markets analysis

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Ďakujem Mgr. Karlovi Janečkovi, MBA, Ph.D. za vedenie diplomovej práce, poskytnutie študijných materiálov a cenné rady a pripomienky. Zároveň ďakujem mojim rodičom a súrodencom za pomoc, podporu a zázemie po dobu celého štúdia, ako aj Mgr. Igorovi Melicherčíkovi, Ph.D. a Ing. Bronislavovi Kandříkovi za prínosné konzultácie. Najväčšia vďaka v súvislosti s mojou diplomovou prácou patrí môjmu otcovi, Prof. Ing. Vladimírovi Kvasničkovi DrSc., za sprostredkovanie odbornej literatúry a poskytnutie softvéru.

Prehlasujem, že som svoju diplomovú prácu napísala samostatne a výhradne s použitím citovaných prameňov. Súhlasím so zapožičiavaním práce.

V Prahe dňa 31. júla 2008

Eva Kvasničková

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Abstrakt

Názov práce: Analýza trhu úrokových mier

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Abstrakt: Cieľom diplomovej práce je predstaviť pravdepodobnostné stochastické modely úrokových mier v spojitom čase. Uvádzané modely sú Itôové procesy definované parametrami, pomocou ktorých sa čo najlepšie snažia popísať vývoj úrokových mier v reálnom svete. Pre vybrané modely bude diskutovaný rozdiel medzi forward a futures úrokmi, nazývaný *convexity adjustment*. V závere práce je prevedená analýza existencie arbitráže medzi úrokovými sadzbami a zmennými kurzami, aplikovaná na základný Ho-Lee model.

Kľúčové slová: Stochastický proces, Martingal, Modely úrokových mier, Forwardy, Futures

Abstract

Title: Interest Rate Markets Analysis

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Abstract: The aim of this thesis is to introduce probabilistic stochastic interest rate models in continuous time. Presented models are Itô processes defined by parameters, which are trying to describe interest rate behavior in the real world. For selected models we will discuss the difference between the forward and futures interest rates, called *convexity adjustment*. At the end of the thesis the analysis of arbitrage existence between interest rates and currency exchange rates, applied to the simplest Ho-Lee model, is presented.

Keywords: Stochastic process, Martingale, Interest Rate Models, Forwards, Futures

Chapter 1

Introduction

Financial mathematics is today going through a time of intensive development, especially in the stochastic analysis area. The methods of the general theory of random processes have turned out to be the most adequate for a suitable description of the evolution of basic and derivative securities. In this work we would like to present the methods and results of the contemporary theory of financial computations and its representation of basic techniques in stochastic analysis: martingales, Itô's formula, Girsanov's theorem and others. We will cover the main interest rate modeling techniques, most used models and the basic theory behind the formulas. We will then focus mainly on the derivative securities (forwards and futures contracts) and try to examine how much different they are from each other. Afterwards we will have a closer look on the arbitrage existence between interest rates and currency exchange rates.

In the first chapter the fundamental ideas and concepts of interest rate modeling are examined. We will discuss and define basic ideas of interest rates markets. Later we introduce interest rates, basic interest rate instruments and general modeling features. Some simple interest rate models are presented and compared. We will introduce the key ideas of martingale measure, numeraire, changes of measure and some basic modeling tools.

The second chapter brings in concrete interest rate models, presents the main model categories and frameworks. We focus specially on the Heath-Jarrow-Morton framework and the Vasicek and Cox, Ingersoll (CIR), Ross models. To illustrate some of the many issues in modeling, we give a detailed

comparison of the characteristics of the Vasicek and CIR models. We will also describe very important no-arbitrage models such as Ho-Lee and Hull-White, which are designed to reflect perfectly the today's term structure.

In the third chapter we introduce two of the most common derivative securities- forwards and futures contracts. We will look at the amount how different they are- the convexity adjustment, and investigate its value for selected models. We briefly mention the empirical research in other papers. At the end of the chapter we provide our study for real Euribor and Eurodollar rates from 2007, together with appropriate parameter estimates.

The last, fourth chapter, looks at the eventual arbitrage existence between the Euribor and Eurodollar futures quotes and appropriate currency forwards. We will use the basic Ho-Lee model to calculate the theoretical quotes in case of no-arbitrage and compare them with the real ones. We will discuss the results and possibilities of bringing the computed arbitrage into reality.

1.1 Introduction to Interest Rates

In this section we will look at some basic concepts of interest rate. We begin with an introduction of interest rate behaviour and then briefly discuss the application of interest rate models, interest rate markets and modelling.

An interest rate can represent, from one point of view, a price of substitution; given the choice between getting x units (dollars) now or 1 unit in let's say one year's time for sure, what would x have to be for the agent to be indifferent between the alternatives? Our x is the present value of one dollar received in one year's time. The today's cost of receiving one dollar at time t is the discount factor, denoted here by δ_t . At time to maturity t the continuously compounded spot interest rate r_t satisfies

$$e^{-\int_0^t r_u du} = \delta_t.$$

The discount function is the set of discount factors for all future maturities. For positive interest rates the discount factor is less than 1.

Yield Curve

The term structure of interest rates, the yield curve, is the set of interest rates for different investment maturities, or periods. A yield curve typically slopes upwards, with longer terms being higher, although there are examples of inverted term structure where long rates are smaller than short one.

Interest Rate Market

Interest rate market is an institution where the price of money (interest rate) is set. It takes into account two main factors; length of the term and instantaneous fluctuation of the interest rate market. Bonds, bond options, interest rate swaps, exotic contracts and some others are the most common derivatives of interest rate market.

Well established organizations for selling and buying all kinds of derivatives or commodities are exchanges. Many exchanges use a clearing house, which facilitates the settlement of contracts and can reduce a counterparty's exposure to credit risk. All derivative exchange contracts are *marked to market* (their value is benchmarked to current market prices). For example, a futures contract is an agreement to make a future purchase at a price agreed today. The contract is closed out at the end of each trading day and it is replaced by a new contract at the day's future settlement price. The owner's margin account is then credited/debited by the difference in the prices of two contracts. Instead of taking physical delivery of the underlying at the settlement day, the exchange settles by paying cash to the margin account, equal to the difference between the agreed price and the actual price on the delivery date.

The market is *efficient* if prices on traded assets already reflect all known information and therefore are unbiased in the sense that they reflect the collective beliefs of all investors about future prospects. This is also called as *efficient market hypothesis*. It states that it is not possible to consistently outperform the market by using any information already known on the market. Every interest model used in real market assumes that the market is efficient. (There are more precise definitions of efficiency, but the definitions here catches the intuition behind the concept so far.) When not considering an efficient market, an interest model will fail to explain prices.

Historical and Current Data

In deeper analysis we usually deal with "historical" data set, what is a set of rates/ prices covering some considered past period (for instance a set of interest rates of different maturities for the last few years). On the other hand, what means "current", depends on trader's situation. In a fast moving market, rates five minutes old might be no longer "current". However, for some daily valuation purposes, the closing rates on the day may be considered as "current" for couple of hours.

Applications

Interest rate models are needed to provide a quantitative framework for describing interest rate movements and valuing interest rate products. We would like to discover the dynamics of interest rates and the way that interest rates and derivative prices relate together by fitting a model to available interest rate data. Most of the models developed so far cannot be used to precisely predict daily ups and downs, but we can already create models describing distributional properties of interest rate movements. Interest rate models do not attempt to give accurate forecast, but describe statistical properties like distribution width and shape, or the likelihood of reaching certain levels. Then, under certain conditions, by knowing the distributional properties of interest rate movements, we can find the values of interest rate derivatives as expected discounted values. It is very difficult to compare the performance of different models, as well as to determine which model is the best one in any particular circumstances. Reasons are that most of available data are sparse and of poor quality and that through time there is a huge variability in market conditions.

Features of a Good Model

When testing and using interest rate models, the priority is pricing and hedging, where the model is fitted to available current market data. The process of calibration adjusts the model's parameters until the model prices match those seen in the market. In the market, models are often being improved and tested for their goodness of fit to current prices. However, even if it is difficult to determine which model is the best one, it is still possible to say what features a good model should have in general.

Precise valuation of market instruments: model should provide accurate prices for liquid market instruments.

Ease of calibration: In a fast moving market the speed of calibration is critical. The market instruments to which the model is calibrated should be liquid and easily observed.

Robustness: Some models are not recommended for some interest rate regimes. A good model should perform well in all markets.

Extensibility to new instruments: Possibility to value and hedge new instruments gives the institution a huge advantage in the market. Some exotic instruments cannot be valued by too simple models.

One of the main purpose, where interest rate models are needed, is obtaining prices and hedges. The traders are focused upon accurate current valuation of different financial instruments, less concerned about historical prices. Risk managers use interest rate models to simulate market behaviour, what may enable them to put limits on the range of future values that a deal may have. Another main purpose of interest rate models is explaining interest rate movements in terms of an underlying model. It is of critical importance when interest rate control is a key part of economic policy.

Modeling interest rates is a very complex problem; we will need techniques for describing interest rate movements, obtaining prices from model and estimating parameters. Interest rate behaviour does not have any definitive model, it is still not totally understood, but on the other hand, for practical uses, like valuating and hedging, there are already well-founded techniques developed.

1.2 Introduction to Interest Rate Modeling

In this section we will look in more detail at money market instruments, describing how quoted market rates convert into cashflows. We introduce the notation of stochastic process and the concept of probability space with filtration. We will present basic interest rate models and compare them using an appropriate set of criteria.

Bonds

A *bond* is a securitized form of loan. The buyer of a bond lends the issuer (national and regional governments, banks, corporations and companies) an initial price P in return for a predetermined sequence of payments. These

payments can be fixed in nominal terms (a fixed-interest bonds) or they can be linked to some index (an index-linked bond), e.g. the consumer prices index. They are often named differently, dependent on particular country; in USA- treasury bills or treasury notes, in GB- gilts, etc. Bonds that have identical characteristics but are sold by different issuer may not have the same price; for example the bond issued by company might be traded at lower price than the government bond because the market makers will take into account the possibility of default on the coupon payments.

1.2.1 Yield Curves

We will start here with idealized definitions on continuously compounded spot and forward rate and we will define corresponding instruments on the market, as Libor and forward rate agreements. As a yield it is usually referred the average interest rate offered by a bond, $r_t(T)$, which denotes the continuously compounded interest rate for a zero-coupon bond (bond with coupon rate zero and nominal value 1) sold at t with maturity T .

Discount Factors, Spot and Forward Rates

The concept of present or discounted value is used in pricing most of financial instruments. Let us denote $P(t, T)$ the value at time t of EUR 1 received for sure at time T , what means that $P(t, T)$ is the pure **discount bond** (without coupons) value with maturity at time T and maturing value 1. Then, $\{P(t, T) | t \leq T\}$ is the discount curve at time t . Suppose now a bond or bond portfolio with riskless cashflows C_i at times t_1, \dots, t_n . The present value P of the cashflow stream $\{C_i\}_{i=1, \dots, n}$, or the value of the portfolio of cashflows is then simple given by a sum of discounted cashflows

$$P = \sum_{i=1}^n C_i P(t, t_i).$$

Continuolsly-compounded spot interest rate: The term structure of interest rates is the set of yields to maturity $R(t, T)_{t < T}$,

$$R(t, T) = -\frac{1}{T-t} \ln P(t, T), \quad t < T \quad (1.1)$$

where $\{P(t, T)\}_{t < T}$ is a given set of pure discount bond prices. We often use expression *time to maturity*, $\tau = T - t$. From above, it follows that

$$e^{R(t, T)\tau} P(t, T) = 1, \quad (1.2)$$

which means that the continuously compounded spot interest rate is a constant rate that is consistent with the zero-coupon-bond prices. From the last equation we can express the bond price in terms of the continuously compounded rate r

$$P(t, T) = e^{-R(t, T)\tau}, \quad (1.3)$$

The rate of instantaneous lending or borrowing is the short rate $R(t, t) \equiv r(t)$, which we will use in order to calculate the value of money market account- a sum of 1 unit (USD) invested in the short rate at time zero and continuously rolled over. Its value, denoted by B_t at time t , is then

$$B_t = \exp \left(\int_0^t r_s ds \right).$$

Even if not exactly the same, as a surrogate for the short rate is very often a short Libor rate (e.g. three-month rate) or overnight rate used. Forward rate is rate that is possible to lock into today for borrowing or lending in the future. More concretely, **continuously compounding forward rate** $f(t, T_1, T_2)$ at time t which applies between times T_1 and T_2 ($t \leq T_1 \leq T_2$, where we borrow at time T_1 and repay at time T_2) is defined as

$$f(t, T_1, T_2) = \frac{1}{T_2 - T_1} \ln \frac{P(t, T_1)}{P(t, T_2)} \quad (1.4)$$

or

$$f(t, T_1, T_2) = \frac{1}{T_2 - T_1} [R(t, T_2)(T_2 - t) - R(t, T_1)(T_1 - t)]. \quad (1.5)$$

The forward rate is arising within the terms of forward contract, under which we agree at time t that we will invest e.g. 1 USD at time T_1 in return for $e^{(T_2 - T_1)f(t, T_1, T_2)}$ at time T_2 .

The property from the definition of the forward rate must be fulfilled, otherwise an arbitrage would be possible: For example if we suppose

$$f(t, T_1, T_2) > \frac{1}{T_2 - T_1} \ln \frac{P(t, T_1)}{P(t, T_2)},$$

we can consider that a portfolio of one forward contract (with value 0 at time t), +1 units of the T_1 -bond and $-\frac{P(t, T_1)}{P(t, T_2)}$ units of the T_2 -bond. If we will hold this portfolio (with total cost 0 at time t) to maturity of the respective contracts, it will produce a pure cashflow of zero at time T_1 and cashflow of $e^{(T_2 - T_1)f(t, T_1, T_2)} - \frac{P(t, T_1)}{P(t, T_2)}$ units at time T_2 , what is (as assumed) larger than

0. That means, we have started with zero-valued portfolio and we have sure positive profit at time T_2 . Analogously, by constructing a reverse portfolio we will deny the possibility of $f(t, T_1, T_2) < \frac{1}{T_2 - T_1} \ln \frac{P(t, T_1)}{P(t, T_2)}$ in non-arbitrage case. Summarized, if we assume a non-arbitrage case, the forward rate $f(t, T_1, T_2)$ must satisfy the equation 1.4 or 1.5 respectively.

The instantaneous forward rate at time t of maturity T is, for $t < T$,

$$f(t, T) \equiv f(t, T, T) = \lim_{T_2 \rightarrow T} f(t, T, T_2) \quad (1.6)$$

and it can be read off the discount curve (yield curve)

$$\begin{aligned} f(t, T) &= \lim_{T_2 \rightarrow T} f(t, T, T_2) = -\frac{\partial}{\partial T} (\ln P(t, T)) = -\frac{\partial P(t, T) / \partial T}{P(t, T)} \quad (1.7) \\ &= R(t, T) + (T - t) \frac{\partial R(t, T)}{\partial T} \end{aligned}$$

from what we get

$$P(t, T) = \exp \left[- \int_t^T f(t, u) du \right]. \quad (1.8)$$

Arbitrage consideration indicate that $f(t, T)$ must be positive for all $T \geq t$, so $P(t, T)$ must be a decreasing function of T .

It is important to realize that:

- We assume here that all rates are riskless, so that there is no default risk.
- Forward rates defined above correspond to FRA's, described below.
- Spot rates are in fact forward rates for immediate delivery, $R(t, T) = f(t, t, T) = \lim_{T_1 \rightarrow t} f(t, T_1, T)$, and $r_t \equiv f(t, t) = f(t, t, t)$.
- Rolling over at instantaneous forward rates is equivalent to investing at an appropriate spot rate,

$$e^{R(t, T)(T-t)} = e^{\int_t^T f(t, s) ds}.$$

Future rates are in no way the same as forward rates; to convert future prices into equivalent forward rates we will need an adjustment, called *convexity adjustment*, which we cover in detail in the third chapter.

There are many different ways of quoting rates at the market. From the most conventional algorithms to convert quoted rates into actual cashflows we mention here Libor and then describe FRAs in more detail.

The Libor rate (London InterBank Offered Rate) may be introduced as the most important interbank rate usually considered as a reference for contracts. However, there are also analogous interbank rates fixing in other markets (e.g. the EURIBOR rate, fixing in Brussels), and when referring to Libor, we actually refer to any of these interbank rates.

As derived in James, Webber [13], p. 41-42, to avoid an arbitrage we must have

$$P(t, t + \tau) = \frac{1}{1 + L(t, \tau)\alpha_L(t, \tau)}, \quad (1.9)$$

for every maturity period τ^1 , where $L(\tau, t)$ denotes the Libor rate at time t with maturity τ and $\alpha_L(t, \tau)$ is the proportion of L paid out at time $t + \tau$ and is calculated as a fraction of a year.

FRAs- Forward Rate Agreements are market equivalents of the theoretical forward rates defined earlier. For the FRA rate agreed at time 0 for time t and tenor τ we will write $F_0(t, \tau) = F$, where the rate F is fixed. More generally, the simply-compounded forward interest rate prevailing at time t for the expiry T and maturity τ , where $\tau > T > t$, we denote by $F_t(T, \tau)$ and it is given by

$$F_t(T, \tau) = \frac{1}{\tau} \left(\frac{P(t, T)}{P(t, \tau)} - 1 \right). \quad (1.10)$$

At time t the holder of FRA is receiving a quantity

$$c = \frac{L(t, \tau)\alpha_L(t, \tau) - F\alpha_L(t, \tau)}{1 + L(t, \tau)\alpha_L(t, \tau)} \quad (1.11)$$

¹The maturity τ of the Libor rate is considered as the period from the point of investment to the time that interest is paid. This maturity is in literature often called as the **tenor** of an interest rate.

which is the present value (at time t) of the difference between borrowing at Libor of tenor τ at time t and borrowing at the FRA fixed rate F (premium at time 0 is 0). This quantity can be negative or positive, and one can sell or buy FRAs, analogously to future borrowing or lending at the FRA rate. To avoid an arbitrage the F rate must be related to market Libor rates, what implies

$$1 + L(0, t + \tau)\alpha_L(0, t + \tau) = (1 + L(0, t)\alpha_L(0, t))(1 + F_0(t, \tau)\alpha_L(t, \tau)), \quad (1.12)$$

what is more in detail discussed e.g. in James, Webber [13], p.41-42. Not considering the transaction costs, if the equation 1.12 is not fulfilled, an arbitrage is possible; we could lend at the more expensive rates, hedging by borrowing at the cheaper rates.

Day count conventions: Unfortunately, different markets and countries have different conventions for calculating cashflows and timings from quoted rates. The differences can amount to significant sums when large principals are involved even though they are generally small in percentage terms. Exceptional situations are handled differently in different markets, and details and possible circumstances are sometimes confusing. The usual way of calculating the size of interest cashflows to particular instrument is *cashflow = annual coupon \times the year fraction the cashflow relates to*. We denote by d_1 and d_2 the start and end dates of some calendar period. Here we will list just some conventions used in a number of major markets.

Euribor use count convention 30/360 what count the whole number of calendar months in between d_1 and d_2 and then adds on the fractions of each month at the start and end of the period. The method assumes that a year has 12 months of 30 days each. The year fraction $\alpha(d_1, d_2)$ is here broadly calculated as $\alpha(d_1, d_2) = \frac{1}{12} \left(\frac{m_2 - d_1}{30} + (n - 3) + \frac{d_2 - m_{n-1}}{30} \right)$, where m_i are month end dates², and $m_1 \leq d_1 \leq m_2 < \dots < m_{n-1} \leq d_2 \leq m_n$. As a start day Euribor and most others use same day +2. As the end day many currencies use the 'modified following business day' convention. The end day is on the following business day, unless it is in a different month, in which case it is on the previous business day.

²Markets vary in how they treat situations, for example, $d_2 = m_n$ and $m_n = 31$ or $m_n = 29$. Further description can be found e.g. in Carmona, Tehranchi [5], p.17-18, or in Reuters [24] p.161-163.

1.2.2 Interest Rate Processes

In this part we provide basic ideas for modeling financial time series, standard specification of appropriate stochastic processes and probabilistic framework that allows us to introduce important concepts such as conditional expectations, and the way they change through time. Ideas introduced here are used in the entire field of interest rate modeling. Most of them are extensively used in theoretical pricing framework, but this will not be goal of this text.

All stock prices and interest rate processes are stochastic processes. They are changing randomly over time, but the manner in which they change can be modeled. We will divide the changes in their values into two parts; the first will be a *deterministic component*, called the drift process, and the second will be a 'noise' term, which we call the *volatility component* of the process. More precisely, the process can be under certain assumptions decomposed into a finite or bounded component (drift) and a component of infinite variation (volatility). For more decomposition details we refer to Protter [18], p.88-94.

The Deterministic Component

Consider now that the deterministic drift for a stock price might be geometric growth. The stock price S_t with no noise and the exponential growth rate μ would satisfy then the differential equation

$$\frac{dS_t}{dt} = \mu S_t, \quad (1.13)$$

of which the solution is

$$S_t = S_0 e^{\mu t}. \quad (1.14)$$

Equation 1.13 can be rewritten as $dS_t = \mu S_t dt$. We might expect the interest rate processes to have a tendency to return to a mean value or some area of values, what we can model as

$$\frac{dr_t}{dt} = \alpha(\mu - r_t), \quad \alpha > 0, \quad (1.15)$$

or equivalently

$$dr_t = \alpha(\mu - r_t)dt. \quad (1.16)$$

The solution to 1.15 is

$$r_t = \mu + (r_0 - \mu)e^{-\alpha t}. \quad (1.17)$$

From 1.15 we see, that if $r_t < \mu$, then dr_t/dt is positive and r_t tends to increase. For $r_t > \mu$ the rate r_t will tend to decrease. So far we can conclude that the rate r_t trends towards the level μ , the mean reversion level, and α , the mean reversion rate, is the speed the r_t goes to μ at.

The Volatility Component

For a financial time series without jumps, the volatility component is assumed to be a function of a **Wiener process**, also called **Brownian motion**, denoted here by $W = (W_t, t \geq 0)$ that is defined as centered Gaussian process with $\text{cov}(W_s, W_t) = s \wedge t$ for all $s, t > 0$. The existence is provided by

Theorem 1.2.1 *A process W is a Brownian motion iff $W_0 = 0$, it has independent increments, and $\mathcal{L}(W_t - W_s) = N(0, |t - s|)$ holds for $t, s \in \mathbb{R}^+$. Brownian motion exists and each Brownian motion can be modified to a continuous process.*

The proof and further details are presented in Shreve [22] p.94-97, or [9], p.238-240. Here we present some most important properties of Wiener process:

- $W_0 = 0$ and $\mathcal{L}(W_t) = N(0, t)$.
- the paths of W_t are continuous, but differentiable almost nowhere
- for arbitrary time instants $0 \leq s < t$ the distribution of increments $(W_t - W_s)$ is $N(0, \sigma^2(t - s))$, $\sigma > 0$
- the increments $(W_t - W_s)$ are independent, stationary and orthogonal
- Brownian motion hits every real value, infinitely often

To introduce noise to a stock price process S_t or an interest rate r_t we add here dW_t , which is scaled by σS_t , so that returns to S_t have a constant standard deviation σ ,

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \quad (1.18)$$

This process is called geometric Brownian motion. The interest rate process will then look like

$$dr_t = \alpha(\mu - r_t)dt + \sigma dW_t, \quad (1.19)$$

called also an Ornstein-Uhlenbeck process.

The Principal Definitions of Stochastic Calculus

Prices of financial instruments are dependent upon the probabilities of events occurring in the market. Here we present some necessary concepts.

Definition 1.2.1 *Continuous-time stochastic process* X is a family of random variables, $\{X_t, t \in \mathbb{R}\}$, defined on probability space (Ω, \mathcal{F}, P) with values in some measurable space.

Definition 1.2.2 Considering a measurable space (Ω, \mathcal{F}) we call its **filtration** the set $(\mathcal{F}_t, t \geq 0)$, if any $\mathcal{F}_t \subset \mathcal{F}$ is a σ -algebra and $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$. We will denote $\mathcal{F}_\infty := \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)$. Assume E to be a metric space and X to be an E -valued process on the space (Ω, \mathcal{F}) , then we denote

$$\mathcal{F}_t^X := \sigma(X(s), s \leq t), \mathcal{F}_\infty^X := \sigma(X(s), t \geq 0). \quad (1.20)$$

Definition 1.2.3 Having a filtration (\mathcal{F}_t) of the space (Ω, \mathcal{F}) we call an E -valued process X on (Ω, \mathcal{F}) an **\mathcal{F}_t -adapted process** is $\mathcal{F}_t^X \subset \mathcal{F}_t$ for all $t \geq 0$. In other words, X is an \mathcal{F}_t -adapted process if and only if for all $t \geq 0$ the variable X_t is an \mathcal{F}_t -measurable E -valued random variable.

Definition 1.2.4 Let (Ω, \mathcal{F}, P) be a probability space and let $\mathcal{F}_t, 0 \leq t \leq T$ be a filtration of sub- σ -algebras of \mathcal{F} . For an adapted process $M_t, 0 \leq t \leq T$ we say that this process is

- **a martingale**, if $E[M_t | \mathcal{F}_s] = M_s$ for all $0 \leq s \leq t \leq T$,
- **a submartingale**, if $E[M_t | \mathcal{F}_s] \geq M_s$ for all $0 \leq s \leq t \leq T$,
- **a supermartingale**, if $E[M_t | \mathcal{F}_s] \leq M_s$ for all $0 \leq s \leq t \leq T$.

Martingales are extremely important type of stochastic processes. For example, a Wiener process W_t is a martingale by definition. A stock of share price processes $dS_t = \mu S_t dt + \sigma S_t dW_t$ is a martingale just for $\mu = 0$, otherwise it has a drift and its expected value will move farther from its value today as time goes on.

1.3 Theoretical Background

This section will provide summary of basic modeling tools, main results of no-arbitrage, the existence of an equivalent martingale measure and concept of changes of numeraire. (We will not provide all the proves here, since most of them would take large space and they are not the main point of this paper. All of them can be found with varying depth e.g. in Shreve [22], Protter [18], or Steele [23].)

1.3.1 Itô's Integral

Itô's integral is used to model the value of a portfolio that results from trading assets in continuous time. The manipulation with these integrals is based on the Itô-Doebelin formula. The difference from ordinary calculus is based on the fact that Brownian motion has a nonzero quadratic variation.

For a simple adapted process $\Delta(t)$ on $[0, T]$, i.e. for

$$\Delta(t, \omega) = \sum_{k=1}^{\infty} I_{[t_k, t_{k+1})} \cdot \Delta(t_k, \omega) \quad (1.21)$$

we will define the Itô's Integral as

$$I(t) = \int_0^t \Delta(u) dW_u := \sum_{j=0}^{k-1} \Delta(t_j) [W_{t_{j+1}} - W_{t_j}] + \Delta(t_k) [W_t - W_{t_k}], \quad t_k \leq t \leq t_{k+1} \quad (1.22)$$

where $0 = t_0 \leq t_1 \leq \dots \leq t_n = T$ is a partition of $[0, T]$. In general, it is possible to choose a sequence $\Delta_n(t)$ of simple processes, such that for $n \rightarrow \infty$ the processes converge to the continuously varying process $\Delta(t)$ in the sense of

$$\lim_{n \rightarrow \infty} \mathbb{E} \int_0^T |\Delta_n(t) - \Delta(t)|^2 dt = 0$$

Then we define the Itô's Integral of the general adapted process by

$$I(t) = \int_0^t \Delta(u) dW_u := \lim_{n \rightarrow \infty} \Delta_n(u) dW_u, \quad 0 \leq t \leq T. \quad (1.23)$$

The Itô's Integral has following properties:

- **Continuity:** the paths of $I(t)$ are continuous (they are function of the upper limit of integration t)

- **Adaptivity:** $I(t)$ is \mathcal{F}_t -measurable for each t
- **Linearity:** If $I(t) = \int_0^t \Delta(u) dW_u$ and $J(t) = \int_0^t \Gamma(u) dW_u$, then $cI(t) \pm dJ(t) = \int_0^t (c\Delta(u) \pm d\Gamma(u)) dW_u$ for every $c, d \in \mathbb{R}$
- **Martingale:** $I(t)$ is a martingale
- **Itô isometry:** $E I^2(t) = E \int_0^t \Delta^2(u) du$
- **Quadratic Variation:** $[I, I](t) = \int_0^t \Delta^2(u) du$

Definition 1.3.1 Let $W_t, t \geq 0$, be a Brownian motion and let $\mathcal{F}_t, t \geq 0$, be an associated filtration. We will define an **Itô process** as a stochastic process of the form

$$X(t) = X(0) + \int_0^t \Delta(u) dW_u + \int_0^t \Theta(u) du, \quad (1.24)$$

where $\Delta(u)$ and $\Theta(u)$ are adapted stochastic processes, $X(0)$ is nonrandom and we assume that both $\int_0^t \Delta^2(u) du$ and $\int_0^t |\Theta(u)| du$ are finite for all $t > 0$.

Definition 1.3.2 Let $X(t), t \geq 0$, be an Itô process from the definition above and let $\Gamma(t), t \geq 0$, be an adapted process. We define **the integral with respect to an Itô process** as

$$\int_0^t \Gamma(u) dX(u) = \int_0^t \Gamma(u) \Delta(u) dW_u + \int_0^t \Gamma(u) \Theta(u) du. \quad (1.25)$$

Theorem 1.3.1 (Itô-Doeblin Formula) Let $X(t), t \geq 0$, be an Itô process of the form $X(t) = X(0) + \int_0^t \Delta(u) dW_u + \int_0^t \Theta(u) du$, such that $E \int_0^t \Delta^2(u) du$ and $\int_0^t |\Theta(u)| du$ are both finite, and let $f(t, x)$ be a function with defined and continuous partial derivatives $f_t(t, x), f_x(t, x)$ and $f_{xx}(t, x)$. Then, for every $T \geq 0$,

$$\begin{aligned} f(T, X(T)) &= f(0, X(0)) + \int_0^T f_t(t, X(t)) dt + \int_0^T f_x(t, X(t)) dX(t) \\ &\quad + \frac{1}{2} \int_0^T f_{xx}(t, X(t)) d[X, X](t) \\ &= f(0, X(0)) + \int_0^T f_t(t, X(t)) dt + \int_0^T f_x(t, X(t)) \Delta(t) dW_t \\ &\quad + \int_0^T f_x(t, X(t)) \Theta(t) dt + \frac{1}{2} \int_0^T f_{xx}(t, X(t)) \Delta^2(t) dt, \end{aligned} \quad (1.26)$$

or, in differential notation,

$$\begin{aligned}
df(t, X(t)) &= f_t(t, X(t))dt + f_x(t, X(t))dX(t) + \frac{1}{2}f_{xx}(t, X(t))dX(t)dX(t) \\
&= f_t(t, X(t))dt + f_x(t, X(t))\Delta(t)dW(t) + f_x(t, X(t))\Theta(t)dt \\
&\quad + \frac{1}{2}f_{xx}(t, X(t))\Delta^2(t)dt
\end{aligned} \tag{1.27}$$

The left-hand side of 1.26, $f(T, X(T))$, is reduced to a sum of a nonrandom quantity $f(0, X(0))$, three Lebesgue integrals with respect to time and one Itô integral.

An important example of 1.27 use is e.g. modeling the value of a portfolio. We suppose here that f is the value of a portfolio denoted by S_t , so we can write $f = f(t, S_t)$ and stock is modeled by the geometric Brownian motion, $dS_t = \mu S_t dt + \sigma S_t dW_t$. Then f follows the process

$$df(t, S(t)) = \frac{\partial f}{\partial t}dt + \sigma \frac{\partial f}{\partial S}dW_t + \mu \frac{\partial f}{\partial S}dt + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial S^2}. \tag{1.28}$$

For simplicity, we will try to keep the notation as simple as possible and we will present here the Itô-Doeblin formula for two processes driven by a two-dimensional Brownian motion $W_t = (W_{1t}, W_{2t})$. Analogously, the formula generalizes to any number of processes driven by Brownian motion.

Theorem 1.3.2 (*Two-dimensional Itô-Doeblin Formula*) Let $f(t, x, y)$ be a function with defined and continuous partial derivatives $f_t, f_x, f_y, f_{xx}, f_{xy}, f_{yx}$ and f_{yy} . Let $X(t)$ and $Y(t)$ be Itô processes of the form

$$\begin{aligned}
X(t) &= X(0) + \int_0^t \Theta_1(u)du + \int_0^t \sigma_{11}(u)dW_{1u} + \int_0^t \sigma_{12}(u)dW_{2u} \\
Y(t) &= Y(0) + \int_0^t \Theta_2(u)du + \int_0^t \sigma_{21}(u)dW_{1u} + \int_0^t \sigma_{22}(u)dW_{2u}
\end{aligned}$$

The two-dimensional Itô-Doeblin formula in differential form is

$$\begin{aligned}
df(t, X(t), Y(t)) &= f_t(t, X(t), Y(t))dt + f_x(t, X(t), Y(t))dX(t) + f_y(t, X(t), Y(t))dY(t) \\
&\quad + \frac{1}{2}f_{xx}(t, X(t), Y(t))dX(t)dX(t) + f_{xy}(t, X(t), Y(t))dX(t)dY(t) \\
&\quad + \frac{1}{2}f_{yy}(t, X(t), Y(t))dY(t)dY(t).
\end{aligned} \tag{1.29}$$

We often use theorem 1.3.2 in more compact form, leaving out t , as

$$df(t, X, Y) = f_t dt + f_x dX + f_y dY + \frac{1}{2} f_{xx} dX dX + f_{xy} dX dY + \frac{1}{2} f_{yy} dY dY. \quad (1.30)$$

The right-hand side of 1.30 is based on the Taylor expansion of f out to the second order. The full expansion would have more second-order terms ($f_{tt} dt dt, \dots$), but $dt dt$, $dt dX$ and $dt dY$ are zero. This and other remarks are discussed more precisely in Shreve [22], p.164-168 or in Steele [23], p.123-128. An important feature of Itô integral is its normal distribution for deterministic integrand.

Theorem 1.3.3 *Let $W_s, s \geq 0$, be a Brownian motion, and let $\Delta(s)$ be a nonrandom function of time. Define $I(t) = \int_0^t \Delta(s) dW_s$. Then $I(t) \sim N\left(0, \int_0^t \Delta^2(s) ds\right)$ for each $t \geq 0$.*

1.3.2 Risk-Neutral Measure

Theorem 1.3.4 *Let (Ω, \mathcal{F}, P) be a probability space and let Z be an a.s. nonnegative random variable satisfying $EZ = 1$. We define for any $A \in \mathcal{F}$*

$$\tilde{P}(A) = \int_A Z(\omega) dP(\omega). \quad (1.31)$$

Then \tilde{P} is a probability measure. If we denote the expectation under \tilde{P} as \tilde{E} , then for nonnegative random variable X is

$$\tilde{E}(X) = E[XZ]. \quad (1.32)$$

If Z is strictly positive a.s., then P and \tilde{P} agree which sets have probability 0 (we say they are equivalent) and

$$EY = \tilde{E}\left[\frac{Y}{Z}\right] \quad (1.33)$$

for every nonnegative random variable Y .

Proof. Shreve [22], p.210-211.

Theorem 1.3.5 (Radon-Nikodým). *Let P and \tilde{P} be equivalent probability measures on (Ω, \mathcal{F}) . Then there exists an a.s. positive random variable Z satisfying $EZ = 1$ such that*

$$\tilde{P}(A) = \int_A Z(\omega) dP(\omega) \quad \text{for all } A \in \mathcal{F}. \quad (1.34)$$

Definition 1.3.3 Random variable Z from the previous theorem is called the **Radon-Nikodým derivative** of \tilde{P} with respect to P . We write $Z = \frac{d\tilde{P}}{dP}$. Further, we define the **Radon-Nikodým derivative process** by $Z_t = E[Z|\mathcal{F}_t]$, for $0 \leq t \leq T$.

We can remark, that the Radon-Nykodým derivative process is a martingale, and $\tilde{E}Y = E[YZ] = E[E[YZ|\mathcal{F}_t]] = E[YZ_t]$.

The original martingale measure P is not the most convenient for pricing contingent claims when interest rate are stochastic. An appropriate choice of numeraire can lead to an elegant solution of the pricing problem.

Lemma 1.3.1 Let Y be an \mathcal{F}_t -measurable random variable, $0 \leq s \leq t \leq T$. Then

$$\tilde{E}[Y|\mathcal{F}_s] = \frac{1}{Z_s} E[YZ_t|\mathcal{F}_s]. \quad (1.35)$$

We will try to show how stochastic processes change under changes in measure. The first presented, Girsanov theorem tells us, how to make drift change or disappear, how to find a probability measure that makes the present value of the stock price into a martingale.

Theorem 1.3.6 (Girsanov, one dimension). Let $W_t, 0 \leq t \leq T$, be a Brownian motion on a probability space (Ω, \mathcal{F}, P) , and let $\mathcal{F}_t, 0 \leq t \leq T$ be a filtration for this Brownian motion. Let $\Theta_t, 0 \leq t \leq T$ be an adapted process. We define

$$Z_t = \exp \left[- \int_0^t \Theta_u dW_u - \frac{1}{2} \int_0^t \Theta_u^2 du \right], \quad (1.36)$$

$$\tilde{W}_t = W_t + \int_0^t \Theta_u du \quad (1.37)$$

and assume that

$$E \int_0^T \Theta_u^2 Z_u^2 du < \infty. \quad (1.38)$$

Set $Z = Z(T)$. Then $EZ = 1$ and under probability measure \tilde{P} given by (1.34), the process $\tilde{W}_t, 0 \leq t \leq T$, is a Brownian motion.

W_t is here a Brownian motion with drift $(-\Theta_t)$ at time t . Important use of Girsanov theorem is the application to stochastic differential. Suppose X

is stochastic process with increments $dX_t = \mu_t dt + \sigma_t dW_t$ where μ_t and σ_t are adapted processes. We would like to change the drift process from μ_t into r_t and appropriate new measure. The differential can be rewritten as $dX_t = r_t dt + \sigma_t (dW_t + (\frac{\mu_t - r_t}{\sigma_t}) dt)$, where we denote $\Theta_t = \frac{\mu_t - r_t}{\sigma_t}$. We now use the Girsanov theorem with Θ_t and corresponding measure \tilde{P} with which $\tilde{W}_t = W_t + \int_0^t \Theta_s ds$ is a Brownian motion under \tilde{P} (assuming that appropriate conditions from the theorem are fulfilled). The differential of X under \tilde{P} is then $dX_t = r_t dt + \sigma_t d\tilde{W}_t$.

Similarly, we can use Girsanov theorem for the stock price process, considering the standard model

$$dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t,$$

discount process $D_t = e^{-\int_0^t r_s ds}$, where r_t is an adapted process and we would like to have the discounted price process to be a martingale. Using Girsanov theorem (Shreve, [22] p. 214-217), we get

$$d(D(t)S(t)) = \sigma_t D_t S_t d\tilde{W}_t,$$

where

$$d\tilde{W}_t = \frac{\alpha_t - r_t}{\sigma_t} dt + dW_t = \Theta_t dt + dW_t.$$

We call \tilde{P} , the measure defined in Girsanov theorem, the *risk-neutral measure* because it is equivalent to the original measure P and it renders the discounted stock price $D_t S_t$ into a martingale. The value $\Theta(t) = \frac{\alpha_t - r_t}{\sigma_t}$ is called as *market price of risk*. It determines how much the drift of S_t must be scaled in units of volatility of S_t . For more comments we refer to James, Webber [13], p.84-87.

Pricing under the Risk-Neutral Measure

Let the payoff of a derivative security at time T , V_T , be an \mathcal{F}_T -measurable random variable. Assuming the completeness of the market, we would like to know what initial capital X_0 and portfolio process $\Delta(t)$, $0 \leq t \leq T$, an agent will need in order to get $X_T = V_T$ almost surely (he wants to hedge a short position in this derivative security). Since the discounted capital process $D_t X_t$ is a martingale under \tilde{P} , we get

$$D_t X_t = \tilde{\mathbb{E}}[D_T X_T | \mathcal{F}_t] = \tilde{\mathbb{E}}[D_T V_T | \mathcal{F}_t].$$

and

$$X_0 = \tilde{\mathbb{E}}[D_T X_T]$$

in particular.

The value X_t of the portfolio is needed in order to hedge a short position at time t with the final payoff V_T . Consequently, we call X_t the price V_t of the derivative security at time t and $D_t V_t = \tilde{\mathbb{E}}[D_T V_T | \mathcal{F}_t]$. Dividing this by D_t , which is \mathcal{F}_t -measurable, we arrived at the *risk-neutral pricing formula*

$$V_t = \tilde{\mathbb{E}} \left[\frac{D_T}{D_t} V_T | \mathcal{F}_t \right] = \tilde{\mathbb{E}} \left[e^{-\int_t^T r_u du} V_T | \mathcal{F}_t \right]. \quad (1.39)$$

1.3.3 Martingale Representation

The risk-neutral pricing formula was derived under the assumption that if an agent begins with the correct initial capital, there is a portfolio process with which the security can be hedged. Under this assumption, we determined the value of the hedging portfolio at every time $t, 0 \leq t \leq T$, to be $V(t)$ given by 1.39. In this section, we will verify the assumption on which the risk-neutral pricing formula is based in the model with one stock driven by one Brownian motion.

Theorem 1.3.7 (*Martingale representation, one dimension.*) *Let $W_t, 0 \leq t \leq T$, be a Brownian motion on a probability space (Ω, \mathcal{F}, P) , and let $\mathcal{F}_t, 0 \leq t \leq T$, be the filtration generated by this Brownian motion. Let $M_t, 0 \leq t \leq T$, be a martingale with respect to this filtration. Then there is an adapted process $\Gamma_u, 0 \leq u \leq T$, such that*

$$M_t = M_0 + \int_0^t \Gamma_u dW_u, \quad 0 \leq t \leq T. \quad (1.40)$$

The theorem says that, when the filtration is generated by Brownian motion, every martingale with respect to this filtration consists from an initial condition and an Itô integral with respect to the Brownian motion. Only source of uncertainty is then the Brownian motion itself and it is the only source of uncertainty to be removed by hedging. Itô integrals are continuous, thus our assumption implies that martingales cannot have jumps. In case we need martingales with jumps, we would need different source of uncertainty than just a Brownian motion.

Another form of the martingale representation theorem is, that considering two processes $M_t, N_t, 0 \leq t \leq T$, as a martingales with respect to

the same filtration, with non-zero volatilities, there exist an \mathcal{F}_t -measurable process ϕ_t , that $P(\int_0^T \phi_t^2 \sigma_t^2 dt < \infty) = 1$ and

$$N_t = N_0 + \int_0^t \phi_s dM_s, \quad 0 \leq t \leq T,$$

where σ_t is the volatility of M_t , ϕ_t is unique, equal to the ratio of volatilities of M_t and N_t .

The theorem proves the existence of the hedge, but does not provide any particular method of finding Δ_t , what is not possible in general. Girsanov and martingale representation theorem can be stated analogously in multiple dimensions.

Chapter 2

Interest Rate Models

The first chapter described concepts underlying interest rate modeling, in this second chapter we will examine the models themselves, covering most of the main categories of models used today.

2.1 Categories of Interest Rate Model

This section introduces some elementary interest rate models. Any applicable interest model needs to have two main ingredients; it must provide a statistical description of how the state variables in the model change through time, and it should provide a procedure to price interest rate derivatives from the statistical description. With these models there are also procedures to extract prices from the model; ideally, the model will have explicit solution/formulae for the values of simple instruments such as bonds or bond options. Nevertheless, numerical methods for finding prices of any instrument other than the most simple are in most cases needed.

There are two main types of models; *equilibrium models* and *no-arbitrage models*. In an equilibrium model the initial term structure is an output from the model; in a no-arbitrage model it is an input to the model.

Equilibrium Models

Equilibrium models are built on assumptions about how the economy works. We take in account the aim to achieve a balance between the supply of bonds and other securities and the demand for these by investors. We are interested in how the economy affects the term structure of interest rates. In a one-

factor model, presented later, this means constructing stochastic model for the evolution of the risk-free rate. We invoke the fundamental theorem of asset pricing to derive a theoretical set of bond prices. Under such a model the theoretical prices evolve in an free-arbitrage way, but it may happen that the initial set of prices is different from observed market prices, giving rise to possible arbitrage opportunities. This will be the topic of the last chapter.

No-Arbitrage Models

These models are considered for pricing of short-term derivatives. They use the observed term structure at the current time as the starting point. Future price evolves in a way which is consistent with this initial price structure and which is arbitrage free. The main advantage of the no-arbitrage models is that they are designed to be exactly consistent with today's term structure. We assume that the term structure depends on only one factor and indicate how the results can be extended to several factors.

There are many other ways how to divide interest rate models into particular groups, considering different characteristics and qualities. Here we will mention some of the most known categories, described more in detail e.g. in James, Webber [13]:

1. *Affine yield models* (e.g. Vašíček, Ho-Lee or Cox-Ingersoll-Ross)
2. *Whole yield curve models* (e.g. Heath-Jarrow-Morton)
3. *Market models* (recover market pricing formulae by the direct modelling of market quoted rates, instantaneous rates are not needed and need not to be modelled)
4. *Price kernel models* (a rigorous no-arbitrage framework, specifies the market price of risk, e.g. Flesaker and Hughston)
5. *Positive models and log-r models* (guarantee the rates they generate are always positive, not very tractable though. In a log-r model the short rate is the exponential of a state variable; e.g. Black and Karasinski.)
6. *Consol models* (the Consol rate- time to maturity of a perpetual coupon bond, is taken as a surrogate for the long rate which is included into and interest rate model)

Of course many other types of model exist, that do not fit precisely into the categories mentioned above; random field models, models with jump components or particular nonlinear models. However, these models are not widely used in practice.

2.2 Simple Model

In the previous chapter we have defined the term structure of interest rate $R(t, T)$ and forward rate curve $f(t, T)$, we have shown that both of them are characterizing the situation at interest rate market and from one we can calculate another. Before we approach more difficult models, we first deal with simple model, where the forward rate process is assumed to follow

$$d_t f(t, T) = \alpha(t, T)dt + \sigma dW_t \quad (2.1)$$

where the subscript denotes differentiation with respect to t , σ is constant volatility and drift α is bounded deterministic function of t and maturity T . Integrating (2.1) we obtain

$$f(t, T) = f(0, T) + \int_0^t \alpha(s, T)ds + \sigma W_t. \quad (2.2)$$

From (2.2) is obvious that the forward interest rate is normally distributed (this fact will be used later, in the Ho-Lee model parameters estimation, see Chapter 3). Moreover, we can notice that the difference $f(t, T) - f(t, S)$ is deterministic, i.e. when we know the rate curve $f(t, t) = r_t$, we know how the entire forward curve looks like, since the only source of uncertainty is the Brownian motion W_t .

2.2.1 Simple Model under the Risk Neutral Measure

Now let us derive the explicit formula for the model drift $\alpha(t, T)$, which we use later in order to calculate the bond B_t and discounted bond $P(t, T)$ values in the risk-neutral world.

Drift

Since the short rate is $r_t = f(t, t)$, we can thanks to (2.2) write

$$r_t = f(0, t) + \int_0^t \alpha(s, t)ds + \sigma W_t. \quad (2.3)$$

Plugging (2.3) into the relationship for bond price $B_t = \int_0^t r_u du$ we obtain

$$\begin{aligned} B_t &= \exp \left(\int_0^t f(0, u) du + \int_0^t \int_0^u \alpha(s, u) ds du + \sigma \int_0^t W_u du \right) \\ &= \exp \left(\int_0^t f(0, u) du + \int_0^t \int_s^t \alpha(s, u) du ds + \sigma \int_0^t W_u du \right). \end{aligned} \quad (2.4)$$

Using $P(t, T) = e^{-\int_t^T f(t, u) du}$, we easily get

$$\begin{aligned} P(t, T) &= \exp - \left(\int_t^T f(0, u) du + \int_t^T \int_0^t \alpha(s, u) ds du + \sigma(T - t)W_t \right) \\ &= \exp - \left(\int_t^T f(0, u) du + \int_0^t \int_t^T \alpha(s, u) du ds + \sigma(T - t)W_t \right) \end{aligned} \quad (2.5)$$

Consider now the process Z_t as a value of discount bond discounted by bond price. We will try to find a risk-neutral measure under which Z_t is a martingale. This will imply that all relative derivative values with respect to bond price are martingales.

$$\begin{aligned} Z_t &= B_t^{-1} P(t, T) \\ &= \exp - \left(\int_0^T f(0, u) du + \int_0^t \int_s^T \alpha(s, u) du ds + \sigma \int_0^t W_u du + \sigma(T - t)W_t \right) \end{aligned} \quad (2.6)$$

Increment of a martingale must have a zero drift and we will need to calculate the increment dZ_t . In order to obtain the differential equation for Z_t we denote $e^{A_t} = Z_t$;

$$\begin{aligned} dA_t &= d \left(- \int_0^T f(0, u) du - \int_0^t \int_s^T \alpha(s, u) du ds - \sigma \int_0^t W_u du - \sigma(T - t)W_t \right) \\ &= d(-\sigma(T - t))W_t - \sigma(T - t)dW_t - \sigma d \left(\int_0^t W_u du \right) \\ &\quad - d \left(\int_0^T f(0, u) du \right) - d \left(\int_0^t \int_s^T \alpha(s, u) du ds \right) \\ &= \sigma W_t dt - \sigma(T - t)dW_t - \sigma W_t dt - \left(\int_t^T \alpha(t, u) du \right) dt \\ &= -\sigma(T - t)dW_t - \left(\int_t^T \alpha(t, u) du \right) dt. \end{aligned}$$

Using the Itô-Deoblin formula we obtain

$$dZ_t = d(e^{A_t})$$

$$\begin{aligned}
&= e^{A_T} dA_T + \frac{1}{2} e^{A_t} \sigma^2 (T-t)^2 dt \\
&= Z_t dA_t + \frac{1}{2} Z_t \sigma^2 (T-t)^2 dt \\
&= Z_t \left(dA_t + \frac{1}{2} \sigma^2 (T-t)^2 dt \right) \\
&= Z_t \left(\left[\frac{1}{2} \sigma^2 (T-t)^2 - \int_t^T \alpha(t, u) du \right] dt - \sigma(T-t) dW_t \right). \tag{2.7}
\end{aligned}$$

Changing the measure to the risk-neutral one, using the Girsanov theorem with $\tilde{W}_t = W_t + \int_0^t \Theta_s ds$ as Brownian motion, we can continue calculating dZ_t by using relationship $dW_t = d\tilde{W}_t - \Theta_t dt$;

$$dZ_t = Z_t \left(-\sigma(T-t) d\tilde{W}_t + \sigma(T-t) \Theta_t dt + \left(\frac{1}{2} \sigma^2 (T-t)^2 - \int_t^T \alpha(t, u) du \right) dt \right)$$

In order to have from Z_t a martingale under risk-neutral measure we set the part of time-increment dt equal to zero;

$$\Theta_t = \frac{1}{\sigma(T-t)} \left(\int_t^T \alpha(t, u) du - \frac{1}{2} \sigma^2 (T-t)^2 \right) \tag{2.8}$$

or

$$\sigma(T-t) \Theta_t = \frac{1}{2} \sigma^2 (T-t)^2 + \int_t^T \alpha(t, u) du \tag{2.9}$$

respectively.

Differentiating the last relationship with respect to time T we will get explicit formula for the drift $\alpha(t, T)$ and the process Θ_t ;

$$\sigma \Theta_t = -\sigma^2 (T-t) + \alpha(t, T). \tag{2.10}$$

Bond and Discounted Bond Values

We plug the drift relationship $\alpha(t, T) = \sigma^2 (T-t) + \sigma \Theta_t$ into (2.1) and rewrite the forward-rate curve as

$$\begin{aligned}
d_t f(t, T) &= (\sigma^2 (T-t) + \sigma \Theta_t) dt + \sigma dW_t \\
&= \sigma^2 (T-t) dt + \sigma d\tilde{W}_t
\end{aligned} \tag{2.11}$$

The forward interest rate under the risk-neutral measure will be, integrating (2.11),

$$f(t, T) = f(0, T) + \sigma^2 \left(T - \frac{1}{2} t \right) t + \sigma \tilde{W}_t, \tag{2.12}$$

and the short rate $r_t = f(t, t)$,

$$r_t = f(0, t) + \frac{1}{2}\sigma^2 t^2 + \sigma \widetilde{W}_t. \quad (2.13)$$

Now we can finally calculate both the value of bond B_t and value of discounted bond $P(t, T)$ at time t in the risk-neutral world,

$$B_t = \exp\left(\int_0^t r_u du\right) = \exp\left(\int_0^t f(0, u) du + \frac{1}{6}\sigma^2 t^3 + \sigma \int_0^t \widetilde{W}_u du\right), \quad (2.14)$$

$$\begin{aligned} P(t, T) &= \exp\left(-\int_t^T f(t, u) du\right) \\ &= \exp\left(-\int_t^T f(0, u) du + \int_t^T \sigma^2 \left(u - \frac{1}{2}t\right) t du + \sigma(T-t)\widetilde{W}_t\right) \\ &= \exp\left(-\int_t^T f(0, u) du + \frac{\sigma^2 t T(T-t)}{2} + \sigma(T-t)\widetilde{W}_t\right). \end{aligned} \quad (2.15)$$

Risk Neutral Measure in Interest Rate Models

The most common way of introducing the risk-adjusted measure is to define it straight away into the model and not bother with the objective probabilities at all. We have shown this for the above simple model and we will do so once again for Vasicek model. Same as in previous model, Vasicek also specifies the form of \widetilde{E} and then calculates prices consistent with it. Given the process under objective measure

$$dr_t = a(b - r_t)dt + \sigma dW_t,$$

for constant a, b, σ , we can rewrite it, similarly as in previous model using Girsanov theorem, as a risk-adjusted process

$$dr_t = (a(b - r_t) - \lambda\sigma)dt + \sigma d\widetilde{W}_t,$$

for some constant λ (which is equal to Θ_t from Girsanov theorem). We denote $\widetilde{b} = b - \frac{\lambda\sigma}{a}$, so that the risk-adjusted process is

$$dr_t = a(\widetilde{b} - r_t)dt + \sigma d\widetilde{W}_t,$$

where \widetilde{W}_t is a Brownian motion under \widetilde{P} with which $\widetilde{W}_t = W_t + \lambda t$. (The value λ , market price of risk, determines the return in excess of the risk-free rate that the market implies as a compensation for taking the risk.)

Analogous change of measure can be done also in other models; in the next pages we will use notation without tildes, introducing the risk-adjusted measure straight forward.

2.3 Heath-Jarrow-Morton Framework (HJM)

There are several ways to represent the yield curve. The one chosen in the HJM model is in terms of the forward rates that can be locked in at one time for borrowing at a later time. In the following two subchapters we introduce the one-factor and multifactor Heath-Jarrow-Morton frameworks.

One-factor HJM

We have already presented the simple model with the forward rate curve given as $d_t f(t, T) = \alpha(t, T)dt + \sigma dW_t$, with constant volatility σ . In the Heath-Jarrow-Morton framework we will generalize the above presented stochastic differential equation to

$$d_t f(t, T) = \alpha(t, T)dt + \sigma(t, T)dW_t, \quad (2.16)$$

where both $\alpha(t, T)$ and $\sigma(t, T)$ are \mathcal{F}_t^W -adapted processes, i.e. the forward rate curve is modeled by an Itô process without any concrete description of volatility $\sigma(t, T)$. The drift component $\alpha(t, T)$ is from the pricing viewpoint not the most important one, since after change of measure it elapses into the standard form. For constant volatility, $\sigma(t, T) = \sigma$, we get the above discussed basic model. Integrating the (2.16) and with given initial value $f(0, T)$ we obtain for $0 \leq t \leq T$,

$$f(t, T) = f(0, T) + \int_0^t \alpha(u, T)du + \int_0^t \sigma(u, T)dW_u, \quad (2.17)$$

The difference $f(t, T) - f(t, S)$ is not deterministic anymore, since it contains the term $\int_0^t (\sigma(u, T) - \sigma(u, S))dW_u$, which is non-zero here, unlike in the basic model. We denote

$$\Sigma(t, T) = - \int_t^T \sigma(t, u)du, \quad (2.18)$$

and similarly as in the basic model, it can be shown (see [15], p.203-206), that under no-arbitrage assumptions we get

$$\alpha(t, T) = \sigma(t, T)(\Theta_t - \Sigma(t, T)), \quad (2.19)$$

with \mathcal{F}_t^W -adapted market price of risk Θ_t . In the risk-neutral world, i.e. if $\Theta_t = 0$ the drift is then

$$\alpha(t, T) = -\sigma(t, T)\Sigma(t, T), \quad (2.20)$$

and the forward rate is given by

$$\begin{aligned} d_t f(t, T) &= -\sigma(t, T)\Sigma(t, T) + \sigma(t, T)d\widetilde{W}_t, \\ d\widetilde{W}_t &= dW_t + \Theta_t dt. \end{aligned} \quad (2.21)$$

Integrating (2.21) we can calculate relationships for the forward rate curve $f(t, T)$, short-term interest rate r_t , and for the bond value $P(t, T)$, where all this values are functions of initial forward-rate curve, volatility $\sigma(t, T)$ and $\Sigma(t, T)$.

$$\begin{aligned} f(t, T) &= f(0, T) - \int_0^t \sigma(u, T)\Sigma(u, T)du + \int_0^t \sigma(u, T)d\widetilde{W}_u \\ r_t &= f(t, t) = f(0, t) - \int_0^t \sigma(u, t)\Sigma(u, t)du + \int_0^t \sigma(u, t)d\widetilde{W}_u \\ P(t, T) &= \exp\left(-\int_t^T f(t, u)du\right) \\ &= \exp\left(-\int_t^T f(0, u)du - \int_0^t \int_t^T \sigma(s, u)\Sigma(s, u)ds + \int_0^t \int_t^T \sigma(s, u)d\widetilde{W}_s\right) \end{aligned} \quad (2.22)$$

Multifactor HJM

In the multifactor HJM model is the stochastic evolution of the forward-rate curve modeled with n -dimensional Brownian motion $W = (W_t^1, \dots, W_t^n)$,

$$d_t f(t, T) = \alpha(t, T)dt + \sum_{i=1}^n \sigma_i(t, T)dW_t^i, \quad (2.23)$$

where $\alpha(t, T)$ and all $\sigma_i(t, T)$ are \mathcal{F}_t^W -adapted, i.e. they depend on to history of Brownian motion up to time t . Very similarly as in one-factor HJM model it can be derived how the forward rate curve, short-term interest rate and the bond value depend on volatilities $\sigma_i(t, T)$ and $\Sigma_i(t, T) = -\int_t^T \sigma_i(t, u)du$ in the risk-neutral world.

Short Rate Models and HJM

In the following text we will show that the short-rate model is nothing but one-factor HJM model, because we can find particular transformation leading from one to another.

Assume that the short-rate process r_t in the risk-neutral world follows $dr_t = \mu_t dt + \omega_t dW_t$, with \mathcal{F}_t^W -adapted processes μ_t and ω_t . We know that

every one-factor HJM model can be written as a model for short-rate,

$$r_t = f(0, t) - \int_0^t \sigma(u, t) \Sigma(u, t) du + \int_0^t \sigma(u, t) d\widetilde{W}_u,$$

and differential equation for the forward rate is

$$d_t f(t, T) = -\sigma(t, T) \Sigma(t, T) + \sigma(t, T) d\widetilde{W}_t.$$

On the other hand, every short-rate model can be written as one-factor HJM model as well, which is not that obvious. We need to find such a volatility $\sigma(t, T)$, that the short-rate process from HJM model is equal to the original one, r_t . This is possible for every process r_t , but it is easier in special case, where r_t is Markov.

Let us assume that r_t is Markov process with both deterministic drift $\mu(r_t, t)$ and volatility $\omega(r_t, t)$, i.e.

$$dr_t = \mu_{r_t, t} dt + \omega_{r_t, t} dW_t$$

The bond price $P(t, T)$ is then just a function of r_t and T , $P(t, T) = \exp\left(-\int_t^T f(t, u) du\right)$. Let us denote, as in [15], p.213-214,

$$g(r_t, t, T) := -\ln P(t, T) = \int_t^T f(t, u) du, \quad (2.24)$$

where $g(x, t, T)$ is a deterministic function:

$$g(x, t, T) = -\ln \tilde{E} \left[\exp \left(-\int_t^T r_s ds \right) | r_t = x \right], \quad (2.25)$$

which does not depend on history up to time t , but just value r_t . Applying the Itô-Doeblin formula on the relationship $f(t, T) = \frac{\partial g}{\partial T}(r_t, t, T)$ we get

$$d_t f(t, T) = \frac{\partial^2 g}{\partial x \partial T} [\mu(r_t, t) dt + \omega^2(r_t, t) dW_t] + \frac{\partial^2 g}{\partial t \partial T} dt + \frac{1}{2} \frac{\partial^3 g}{\partial x^2 \partial T} \omega^2(r_t, t) dt.$$

Volatility of this process has to match to $\sigma(t, T)$,

$$\sigma(t, T) = \omega(r_t, t) \frac{\partial^2 g}{\partial x \partial T}(r_t, t, T),$$

from what we get

$$\Sigma(t, T) = -\int_t^T \sigma(t, u) du = -\omega(r_t, t) \frac{\partial g}{\partial x}(r_t, t, T).$$

Moreover, the initial forward rate curve $f(0, T)$ follows from $f(t, T) = \frac{\partial g}{\partial T}(r_t, t, T)$ as

$$f(0, T) = \frac{\partial g}{\partial T}(r_0, 0, T). \quad (2.26)$$

2.4 Equilibrium Models

Equilibrium models assume particular properties of economic variables and derive a process for the short-term risk-free rate r (sometimes referred as instantaneous short rate), and explore what the process implies for the prices of the particular derivatives. As already derived in (1.39), the value of an interest-rate derivative with payoff V_T at time T is

$$V_t = \tilde{E}[e^{-\bar{r}(T-t)} V_T | \mathcal{F}_t], \quad (2.27)$$

where \tilde{E} denotes expected value in a risk-neutral world and \bar{r} is the "average" value of r between time t and T , or more precisely, $\bar{r}(T-t) = \int_t^T r_u du$. For simplicity we will not write conditioning on \mathcal{F}_t anymore, if not necessarily needed.

We define $P(t, T)$ as the price at time t of discount bond that pays off 1 unit (USD, EUR, ...) at time T . From (2.27) with $V_T = 1$, we get

$$P(t, T) = \tilde{E}[e^{-\bar{r}(T-t)}]. \quad (2.28)$$

If $R(t, T)$ is the continuously compounded interest rate at time t , $P(t, T) = e^{-R(t, T)(T-t)}$, or $R(t, T) = -\frac{1}{T-t} \ln P(t, T)$ analogously, from (2.28) we obtain

$$R(t, T) = -\frac{1}{T-t} \ln \tilde{E}[e^{-\bar{r}(T-t)}]. \quad (2.29)$$

This equation shows that once we have fully defined the process for r , we have also fully defined the initial term structure and how it behaves at future times. In other words, we can obtain the term structure of interest rates at any given time from the value r at that time and from the risk-neutral process for r .

2.4.1 One-Factor Models

The process for the short-term risk-free rate r involves only one source of uncertainty. It is usually described by an Itô process of the form

$$dr = m(r)dt + s(r)dW_t, \quad (2.30)$$

where $W_t, 0 \leq t \leq T$ is the standard Brownian motion under the real world measure P , processes m (*instantaneous drift*) and s (*instantaneous standard deviation*) are assumed to be adapted functions of r , independent of time itself (in sense that $m(r) = m(r_t), s(r) = s(r_t)$). It implies that all rates move in the same direction over any short time interval, but not all move by the same amount.

Several one-factor equilibrium models:

Model	$m(r)$	$s(r)$
Merton	μ	σ
Rendleman and Bartter (Dothan)	μr	σr
Vasicek	$a(b - r)$	σ
Cox-Ingersoll-Ross	$a(b - r)$	$\sigma\sqrt{r}$

2.4.2 Vasicek Model

Vasicek assumed that the instantaneous spot rate evolves as an Ornstein-Uhlenbeck process with constant coefficients,

$$dr_t = a(b - r_t)dt + \sigma dW_t, \quad (2.31)$$

where a, b, σ are positive constants. In this model, b represents the risk-neutral long-term mean risk-free rate; a represents the rate at which r_t reverts back to this long-term mean; and σ represents the local volatility of short-term interests rate. The key feature is that the interest rates appear to be pulled back to some long-run average level over time. This, so called *mean reversion structure*, implies that for high r the model tends to have a negative drift, for low r it tends to have a positive drift. (When rates are high, the economy is slowing down and there are less borrowers, consequently, rates decline.) More exactly, for $r_t = b$, the drift term (the dt term) is zero; for $r_t > b$, the drift term is negative what pushes r_t back downward b . Analogously, for $r_t < b$ the drift term will be positive what pushes r_t back upward b . Considering this as expectation, if $r_0 \neq b$, then $\lim_{t \rightarrow \infty} Er_t = b$; and if $r_0 = b$, then $Er_t = b$ for all $t \geq 0$.

Now we will try to determine the short-term interest rate process, using the Itô-Doeblin formula. We will denote

$$h(t, x) = e^{-at}r_0 + b(1 - e^{-at}) + \sigma e^{-at}x, \quad X(t) = \int_0^t e^{as}dW_s$$

and we calculate all partial derivatives needed for applying Itô lemma on $h(t, X_t)$;

$$\begin{aligned} h_t(t, x) &= -ae^{-at}r_0 + abe^{-at} - \sigma ae^{-at}x = ab - ah(t, x), \\ h_x(t, x) &= \sigma e^{-at}, \quad h_{xx}(t, x) = 0, \end{aligned}$$

and $dX(t) = e^{at}dW_t$. Since $h_{xx}(t, x) = 0$, we will not need $dX_t dX_t = e^{2at}$. Hence

$$\begin{aligned} dh(t, X_t) &= h_t(t, X_t)dt + h_x(t, X_t)dX_t + \frac{1}{2}h_{xx}(t, X_t)dX_t dX_t \\ &= a(b - h(t, X_t))dt + \sigma dW_t \end{aligned}$$

This shows that here defined $h(t, X_t)$ satisfies (2.31), that defines r_t and moreover, has the same initial condition, $h(0, X_0) = r_0$, what implies that $h(t, X_t) = r_t$ for all $t \geq 0$. Thus, the short-term interest rate in Vasicek model is of the form

$$r_t = e^{-at}r_0 + b(1 - e^{-at}) + \sigma e^{-at} \int_0^t e^{as} dW_s, \quad (2.32)$$

which we can rewrite as

$$r_t = e^{-at}r_0 + b(1 - e^{-at}) + \sigma e^{-at}Z, \quad (2.33)$$

where

$$Z := \int_0^t e^{as} dW_s \sim N\left(0, \frac{e^{2at} - 1}{2a}\right) \quad (2.34)$$

We have used here the normal-distribution property of an Itô integral with deterministic integrand (theorem 1.3.3). From (2.34) it follows that

$$r_t \sim N\left(e^{-at}r_0 + b(1 - e^{-at}), \frac{\sigma^2}{2a}(1 - e^{-2at})\right). \quad (2.35)$$

In Vasicek model both bond value B_t and discount bond with price $P(t, T)$ are lognormal distributed ([15]. p.220-221).

For large value of time t the distribution of r_t converges to $N(b, \sigma^2/2a)$. In particular, there is positive probability that r_t is negative; an undesirable property of Vasicek interest rate model.

Cairns shows in [4], p.249-253, that using the HJM Framework we can express the price at time t of a zero-coupon bond that pays 1 unit at T as

$$P(t, T) = e^{A(t, T) - B(t, T)r_t}, \quad (2.36)$$

where (for $a \neq 0$)

$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a},$$

$$A(t, T) = \frac{(B(t, T) - (T-t))(a^2b - \sigma^2/2)}{a^2} - \frac{\sigma^2 B(t, T)^2}{4a}.$$

For $a = 0$ we have $B(t, T) = (T-t)$ and $A(t, T) = \exp[\sigma^2(T-t)^3/6]$. Using the relationship $R(t, T) = -\frac{1}{T-t} \ln P(t, T)$, we obtain the Vasicek continuously compounded interest rate at time t as

$$R(t, T) = \frac{1}{T-t} [B(t, T)r_t - A(t, T)], \quad (2.37)$$

which shows us that once the parameters a, b, σ have been chosen, the entire term structure is determined as a function of r_t . Equation (2.37) also shows, that $R(t, T)$ is linearly dependent on r_t , i.e. that the value r_t exactly determines the level of the term structure at time t . The general shape of the term structure at time t does depend on time t itself, but is independent of r_t .

Generalized Vasicek Model

The Vasicek model is sometimes generalized as

$$dr_t = (\theta_t - \alpha_t r_t)dt + \sigma_t dW_t, \quad (2.38)$$

where all θ_t, α_t and σ_t are deterministic functions of time. The bond B_t and discount bond with price $P(t, T)$ are still lognormal distributed (see [15], p.221).

2.4.3 Cox-Ingersoll, and Ross Model (CIR)

As we noticed before, the short-term interest rate r_t in Vasicek model can become negative, which implies that all spot rates and forward rates for finite maturity can become negative. An additional minus of the Vasicek model is that many empirical evidences suggest that the volatility of r_t is not constant, but increasing function of r_t . Cox, Ingersoll and Ross have proposed an alternative one-factor model for the risk-free rate of interest where rates are always nonnegative. The risk-neutral process in their model is

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t, \quad (2.39)$$

where a, b, σ are positive constants. In the Vasicek model r_t could reach negative values, in case of CIR model this is not possible. If r_t reaches zero, the term multiplying dW_t vanishes and the positive drift term αdt pushes the interest rate back into positive territory. Like the Vasicek model, CIR has the same mean-reverting drift, but on the other hand, the stochastic term has a standard deviation proportional to $\sqrt{r_t}$, i.e. with increasing short-term interest rate its standard deviation increases as well.

To find an explicit solution for r_t or its distribution is not easy, and it would take us too far afield. Instead we can derive the expected value and variance of r_t . For this, we will apply the Itô-Doebelin formula on the function $h(t, x) = e^{at}x$;

$$\begin{aligned}
d(e^{at}r_t) &= dh(t, r_t) \\
&= h_t(t, r_t)dt + h_x(t, r_t)dr_t + \frac{1}{2}h_{xx}(t, r_t)dr_tdr_t \\
&= ae^{at}r_tdt + e^{at}(ab - ar_t)dt + e^{at}\sigma\sqrt{r_t}dW_t \\
&= abe^{at}dt + \sigma e^{at}\sqrt{r_t}dW_t.
\end{aligned} \tag{2.40}$$

Integrating (2.40) we get

$$\begin{aligned}
e^{at}r_t &= r_0 + ab \int_0^t e^{au}du + \sigma \int_0^t e^{au}\sqrt{r_u}dW_u \\
&= r_0 + b(e^{at} - 1) + \sigma \int_0^t e^{au}\sqrt{r_u}dW_u,
\end{aligned}$$

from what we can calculate the expectation,

$$e^{at}Er_t = r_0 + b(e^{at} - 1)$$

or

$$Er_t = e^{-at}r_0 + b(1 - e^{-at}), \tag{2.41}$$

where we used the fact, that the expectation of an Itô integral is zero. We can notice, that this is the same expectation as in Vasicek model. For calculating the variance of r_t , we set $X_t = e^{at}r_t$, of which expectation and dX_t we have already calculated. According to the Itô-Doebelin formula,

$$\begin{aligned}
d(X_t^2) &= 2X_t dX_t + dX_t dX_t \\
&= 2abe^{at}X_t dt + 2\sigma e^{at/2}X_t^{3/2}dW_t + \sigma^2 e^{at}X_t dt.
\end{aligned} \tag{2.42}$$

Integrating (2.42), taking its expectation and using the zero expectation of an Itô integral we obtain

$$\begin{aligned} EX_t^2 &= X_0^2 + (2ab + \sigma^2) \int_0^t e^{au} EX_u du \\ &= r_0^2 + \frac{2ab + \sigma^2}{a} (r_0 - b) (e^{at} - 1) + \frac{2ab + \sigma^2}{2a} b (e^{2at} - 1). \end{aligned}$$

The variance of r_t is then

$$\begin{aligned} \text{Var}(r_t) &= Er_t^2 - (Er_t)^2 = e^{-2at} EX_t^2 - (Er_t)^2 \\ &= \frac{\sigma^2}{a} r_0 (e^{-at} - e^{-2at}) + \frac{b\sigma^2}{2a} (1 - 2e^{-at} + e^{-2at}). \end{aligned} \quad (2.43)$$

For large values of t ,

$$\lim_{t \rightarrow \infty} \text{Var}(r_t) = \frac{b\sigma^2}{2a}.$$

Similarly as in Vasicek model, the bond prices have in the CIR model the general form

$$P(t, T) = e^{A(t, T) - B(t, T)r_t},$$

(derived e.g. in Cairns [4], p.253-263), but with different functions $A(t, T)$ and $B(t, T)$, where

$$\begin{aligned} B(t, T) &= \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma}, \\ A(t, T) &= (2ab/\sigma^2) \ln \left[\frac{2\gamma e^{(a+\gamma)(T-t)/2}}{(a + \gamma)(e^{\gamma(T-t)} - 1) + 2\gamma} \right], \end{aligned}$$

with $\gamma = \sqrt{a^2 + 2\sigma^2}$. The long rate $R(t, T)$ is again, similarly as in Vasicek model, linearly dependent on r_t , with analogous consequences.

Generalized CIR Model

The CIR model is sometimes considered with time-dependent coefficients, where the short rate dynamics are given by

$$dr_t = (\theta_t - a_t r_t) dt + \sigma_t \sqrt{r_t} dW_t, \quad (2.44)$$

where a_t, θ_t and σ_t are deterministic functions of time. Such extension however is not analytically tractable. So far, no analytical expression for θ_t in terms of the observed yield curve is available in the literature. and there is no guarantee that a numerical approximation of θ_t would keep the rate r_t positive. Consequently, this extension has been less successful than original no-extended form.

2.5 No-Arbitrage Models

The main disadvantage of the equilibrium models presented in previous section is that they do not automatically fit today's term structure. In this section we present some general theoretical background on no-arbitrage models, which are designed to be exact-reflecting today's term structure. One way of introducing such models is specifying Markov model for the short rate where the drift is a function of time. As we will see in next few pages, many models developed in this way are natural extensions of the equilibrium models described earlier.

2.5.1 Ho and Lee Model

Ho and Lee considered the following model for the risk-free rate

$$dr_t = \theta_t dt + \sigma dW_t, \quad (2.45)$$

where the instantaneous standard deviation of the short rate, σ , is constant and θ_t is time dependent function chosen to ensure that the model fits the initial term structure. The simple model from the section 2.1 was its HJM formulation and it is a more general version of the random-walk model under which is θ_t constant. Using the HJM framework, with parameters $\sigma(s, t) = \sigma$ and $\Sigma(s, t) = -(t - s)\sigma$, it can be shown (e.g. in Cairns[4], p.96-97) that

$$d_t f(t, T) = \sigma^2(T - t)dt + \sigma dW_t, \quad (2.46)$$

where

$$f(0, T) = r_0 - \frac{1}{2}\sigma^2 T^2 + \int_0^T \theta_u du. \quad (2.47)$$

Deriving (2.47), we can find θ_T ,

$$\frac{\partial}{\partial T} f(0, T) = \theta_T - \sigma^2 T, \quad (2.48)$$

and solution for r_t ,

$$\begin{aligned} r_t &= r_0 + \int_0^t \theta_s ds + \sigma W_t \\ &= r_0 + f(0, t) - r_0 + \frac{1}{2}\sigma^2 t^2 + \sigma W_t \\ &= f(0, t) + \frac{1}{2}\sigma^2 t^2 + \sigma W_t. \end{aligned} \quad (2.49)$$

The value of r_t can be also easily found using the HJM formula (2.22).

The Ho and Lee model has the advantage that it is a Markov tractable model, it is easy to apply and it provides an exact fit to the current term structure of interest rates. The main disadvantage is the little flexibility in choosing the volatility structure, since all spot and forward rates have the same standard deviation σ .

The model can be generalized quite easily to have σ_t deterministic but time dependent. The equation (2.45) will then look like

$$dr_t = \theta_t dt + \sigma_t dW_t, \quad (2.50)$$

and the formulation of HJM model will be

$$d_t f(t, T) = \sigma_t^2 (T - t) dt + \sigma_t dW_t,$$

with the initial forward rate curve given as

$$f(0, T) = r_0 + \int_0^T \theta_s ds - \int_0^T \sigma_s^2 (T - s) ds.$$

2.5.2 Hull and White Model

Hull and White proposed a simple generalization of the Vasicek model, with time-dependent reversion level, in which

$$dr_t = \alpha(\mu_t - r_t)dt + \sigma dW_t, \quad (2.51)$$

where μ_t is a deterministic function of time. It is often expressed in the form

$$dr_t = (\theta_t - \alpha r_t)dt + \sigma dW_t,$$

but the first notation (2.51) has straightforward interpretation of a local mean-reversion level. The $\theta_t \equiv \alpha\mu_t$ function can be calculated as in Hull [11], p.433-434, from

$$\mu_t = \frac{1}{\alpha} \frac{\partial}{\partial t} f(0, t) + f(0, t) + \frac{\sigma^2}{2\alpha^2} (1 - e^{-2\alpha t}). \quad (2.52)$$

The drift of the process is then

$$\begin{aligned} \alpha\mu_t - \alpha r_t &= \frac{\partial}{\partial t} f(0, t) + \alpha f(0, t) + \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha t}) - \alpha r_t \\ &\approx \frac{\partial}{\partial t} f(0, t) + \alpha(f(0, t) - r_t), \end{aligned} \quad (2.53)$$

since the term $\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})$ is usually fairly small.

The HJM parameters are $\sigma(s, t) = \sigma e^{-\alpha(t-s)}$ and $\Sigma(s, t) = -\frac{\sigma}{\alpha}(1 - e^{-\alpha(t-s)})$ from what, using (2.22), we can express r_t as

$$r_t = f(0, t) + \frac{\sigma^2}{2\alpha^2}(1 - e^{-\alpha t})^2 + \sigma \int_0^t e^{-\alpha(t-s)} dW_s. \quad (2.54)$$

The model can be generalized to have α_t and σ_t deterministic but time dependent.

2.5.3 Black and Karasinski Model

To ensure to have positive values of short-term interest rates r_t , we will use exponential function. We will denote here $r_t = \exp X_t$ and assume that the process X_t is generalized Ornstein-Uhlenbeck process from Vasicek model,

$$dX_t = \alpha_t(\mu_t - X_t)dt + \sigma_t dW_t, \quad (2.55)$$

where α_t, μ_t and σ_t are deterministic functions of time. Applying Itô-Doeblin formula to (2.55) we get the stochastic differential equation for r_t ,

$$dr_t = \alpha_t r_t \left[\mu_t + \frac{\sigma_t^2}{2\alpha_t} - \log r_t \right] dt + \sigma_t r_t dW_t. \quad (2.56)$$

Functions α_t and σ_t are often considered to be constant. A drawback of the Black-Karanski model is that the expected accumulation of cash over any positive time interval (t, T) , i.e. $\tilde{E}[B(T)/B(t)|\mathcal{F}_t]$, is infinite (Sandmann and Sondermann [20]) and it cannot be used to price Eurodollar- and many other futures contracts.

Chapter 3

Convexity Adjustment

Money market instruments are often constructed from relatively few instruments. This means that it may indeed be possible to find a curve with as many parameters as there are prices that can exactly reconstruct market prices. Instruments include Libor, futures, FRAs and swaps. The idea is to build up the yield curve from shorter maturities to longer maturities. There are many methods developed so far, the main ideas and concrete examples are presented e.g. in [13], [4] or in [3].

Constructing the yield curves, we should calculate discount factors. In many practical applications, an approximation is used when we treat futures as if they were FRAs. In reality futures rates are greater than corresponding FRA rates and an adjustment is required to convert futures prices to equivalent FRAs. The arbitrage is to short the future; if rates rise, then margin payments on the future contract are received immediately whereas the loss on the FRA is not crystallized until later. If rates fall, the converse will happen. The amount by which the futures rate needs to be decreased is called the *convexity adjustment* (CA). It is determined by the market's expectations of future changes in rates, so that different interest rate model imply different convexity adjustment.

Assumptions

For simplicity, all bonds considered here have a nominal of one unit of currency. Moreover in all text below, we assume that:

- There are no market frictions (zero-coupon bonds of all maturities can be traded).

- There is no credit risk.
- Markets are competitive and market participants act as price takers.
- The market is arbitrage-free and complete.

3.1 Forward and Future Contracts

3.1.1 Forwards

Assume D_t to be a discount process, given at time t by $D_t = e^{-\int_0^t r_u du}$. The price of a zero-coupon bond paying 1 at time T is then given (as in 1.39) by

$$B_t^T = \frac{1}{D_t} \tilde{E}[D_T | \mathcal{F}_t], \quad 0 \leq t \leq T. \quad (3.1)$$

A **forward contract** is an agreement to pay a predetermined delivery price K at a predetermined delivery date T for the asset whose price at time t is S_t . The **forward price** $For_S(t, T)$ of this asset at time t is the value of K that makes the forward contract have no-arbitrage price zero at time t .

Theorem 3.1.1 *The forward price $For_S(t, T)$ defined above, denoted for simplicity as For_t^T , satisfies*

$$For_t^T = \frac{S_t}{B_t^T}, \quad 0 \leq t \leq T. \quad (3.2)$$

Proof. Suppose that the forward price K is higher than S_t/B_t^T . We could borrow then at time t money in the amount of S_t (by selling short $S_t/B_t^T > S_t$) and buy one asset. At time T we obtain the payoff $K - S_T$, sell the one asset for S_T and pay off our debt, S_t/B_t^T . Consequently, the remaining cash is $K - S_T + S_t - S_t/B_t^T = K - S_t/B_t^T > 0$, an arbitrage. We could do the opposite analogy for K lower than S_t/B_t^T . The forward price can be derived also from the standard pricing formula with risk-neutral measure,

$$V_t = \frac{1}{D_t} \tilde{E}[D_T(S_T - K) | \mathcal{F}_t] = S_t - KB_t^T = 0.$$

□

3.1.2 Futures

Consider now partition of our interval $(0, T)$ given by $0 = t_0 \leq t_1 \leq \dots \leq t_n = T$ and suppose that the discount process is predictable, i.e. that $D_{t_{k+1}}$ is \mathcal{F}_{t_k} -measurable for every $k = 0, \dots, n-1$. We could consider the daily rolling of the forward contract described above; and at day $(k+1)$ generate the cashflow

$$V_{k,k+1} = \frac{1}{D_{t_{k+1}}} \tilde{E} \left[D_T S_T - \frac{S_{t_k}}{B_{t_k}^T} | \mathcal{F}_{t_{k+1}} \right] = S_{t_{k+1}} - S_{t_k} \cdot \frac{B_{t_{k+1}}^T}{B_{t_k}^T},$$

what is in reality not very practical. This serves as motivation for introducing the future contracts (with future prices Fut_t^T), which we want to satisfy three natural requirements:

- Fut_t^T is \mathcal{F}_t -measurable
- $Fut_T^T = S_T$
- We can change position in our futures at no cost, or more general, the value of holding the future contract is zero.

The last condition implies that receiving a payment $Fut_{t_{k+1}}^T - Fut_{t_k}^T$ as a holder of a long futures between t_k and t_{k+1} , it must be satisfied

$$0 = \frac{1}{D_{t_k}} E^Q [D_{t_{k+1}} (Fut_{t_{k+1}}^T - Fut_{t_k}^T) | \mathcal{F}_{t_k}] = \frac{D_{t_{k+1}}}{D_{t_k}} (E^Q [Fut_{t_{k+1}}^T | \mathcal{F}_{t_k}] - Fut_{t_k}^T)$$

or

$$Fut_{t_k}^T = E^Q [Fut_{t_{k+1}}^T | \mathcal{F}_{t_k}]$$

from which follows that $Fut_{t_k}^T$ is a discrete-time martingale. By adding the second condition ($Fut_T^T = S_T$) we get

$$Fut_t^T = E^Q [S_T | \mathcal{F}_{t_k}], \quad k = 0, 1, \dots, n.$$

Using predictability of D_{t_j} for every $j \geq k+1$ and iterated conditioning we can see that the value of the payment at time t_k to be received at time t_j is zero. This leads to the following definition of futures price.

Definition 3.1.1 *The futures price is given by formula*

$$Fut_t^T = E^Q[S_T|\mathcal{F}_t]$$

A long position in the futures contract on the interval $[s, t]$ is an agreement to receive the changes in the future price, i.e. $Fut_t^T - Fut_s^T$, as a cash flow.

Theorem 3.1.2 *The futures price is a martingale under Q , satisfying $Fut_T^T = S_T$, and the value of any strategy (futures position) is zero.*

Proof. The agent is assumed to hold Δ_t futures contracts. His profit is then given by

$$dX_t = \Delta_t dFut_t^T + r_t X_t dt$$

or by

$$X_t = X_0 + \int_0^t \Delta_u dFut_u^T + \int_0^t r_u X_u du$$

respectively. The discounted value of portfolio is then

$$dD_t X_t = D_t \Delta_t dFut_t^T,$$

so if we set $X_0 = 0$, at any time $t \geq s$ is the agent's profit given by

$$D_t X_t = \int_s^t D_u \Delta_u dFut_u^T.$$

Using the properties of stochastic integrals with respect to a general martingale process we obtain

$$\begin{aligned} E^Q[D_t X_t | \mathcal{F}_s] &= E^Q \left[\int_0^t D_u \Delta_u dFut_u^T - \int_0^s D_u \Delta_u dFut_u^T | \mathcal{F}_s \right] = \\ &= E^Q \left[\int_0^t D_u \Delta_u dFut_u^T | \mathcal{F}_s \right] - \int_0^s D_u \Delta_u dFut_u^T = \\ &= 0, \end{aligned} \tag{3.3}$$

since the term $\int_0^t D_u \Delta_u dFut_u^T$ is martingale. From above we can see that $Fut_t^T = E^Q[S_T | \mathcal{F}_t]$ is the correct futures price. \square

3.1.3 Forward-Futures Spread

First, we will assume non-stochastic interest rate. Then $B_t^T = \exp\left(-\int_t^T r_u du\right)$ so the forward price is

$$For_t^T = \frac{S_t}{B_t^T} = \exp\left(\int_t^T r_u du\right) S_t$$

The futures price in a nonrandom interest rate case is

$$\begin{aligned} Fut_t^T &= E^Q[S_T|\mathcal{F}_t] \\ &= \exp\left(\int_0^T r_u du\right) \cdot E^Q\left[\exp\left(-\int_0^T r_u du\right) S_T|\mathcal{F}_t\right] \\ &= \exp\left(\int_0^T r_u du\right) \cdot D_t S_t \\ &= \exp\left(\int_t^T r_u du\right) \cdot S_t, \end{aligned} \tag{3.4}$$

so the forward and futures prices agree.

In general, stochastic interest rate case, the forward and futures prices differ from each other. For simplicity, we begin at time zero, the spread is then given by

$$\begin{aligned} For_0^T - Fut_0^T &= \frac{S_0}{B_0^T} - E^Q S_T \\ &= \frac{1}{B_0^T} [S_0 - B_0^T \cdot E^Q S_T] \\ &= \frac{1}{B_0^T} [E^Q[D_T S_T] - E^Q D_T \cdot E^Q S_T] \\ &= \frac{1}{B_0^T} \cdot \text{cov}(D_T, S_T) \end{aligned} \tag{3.5}$$

This spread is often called **convexity adjustment**.

With stochastic interest rates, we demonstrated that the difference between forward and futures price is given by the "local" covariance between the rate of return on the futures contract and the rate of return on a risk-free pure discount bond. We can interpret this result by considering the case when the price of asset S is strongly positive correlated with interest rates

(as a consequence is then $For_0^T > Fut_0^T$). With increasing S , an investor in long futures position makes an immediate gain because of the daily settlement procedure. This gain will tend to be invested in higher rate of interest, since increases in the asset price S occur at the same time as increases in interest rate. The converse happens when S decreases; the investor will make an immediate loss, which tend to be financed at a lower interest rate. To be not affected in this way by interest rate movements requires to hold a forward contract rather than a futures contract. It follows that a long futures contract will be more attractive in this sense than a long forward contract. Consequently, for S strongly positively correlated with interest rates, futures prices tend to be higher than forward prices. Analogous arguments show that futures prices tend to be lower than forward prices when S is strongly negatively correlated with interest rates.

The no-arbitrage model is most often used in empirically testing the pricing of share price index futures contracts. In fact, it is actually a forward, not a futures, pricing model. To apply the model to share price index futures, we assume the equality of forward and futures prices, which is not obviously appropriate assumption. (In particular, if deeper analysis will provide a support for non-zero local covariance (implying a non-zero forward-futures price differential), the use of the no-arbitrage model may be questioned.) This paper will try to analyze the in/appropriateness of assuming the equality of forward and futures prices.

3.1.4 Empirical Research Done So Far

We will mention some empirical research that has been carried out studying the forward-futures spread.

Spread in Metal- and Treasury Bill Markets

French [10] studied copper and silver during 1968-1980. Significant difference between the futures price and forward price (at 5% confidence level) was confirmed for silver; the results for copper were less clear. Park and Chen [16] studied gold, silver, platinum, copper and plywood between 1977 and 1981; the spread was here confirmed with the futures price above the forward price. Rendeleman [19] studied the Treasury bill market between 1976 and 1978, they also found statistically significant spread here.

Spread in Currency Market

Cornell and Reinganum [8] studied the difference between futures and forward prices on the British pound, German mark, Canadian dollar, Japanese yen, and Swiss franc between 1974 and 1979. Here, in contrast to "metal-studies", they found very few statistically significant spread. These results were also confirmed by Park and Chen [16], who also looked at the British pound, German mark, Japanese yen and Swiss franc between 1977 and 1981.

Hull [11] summarized their results observing that the theoretical spread for contracts that last only a few months are in most circumstances sufficiently small to be ignored. In reality, there are many factors (such as transaction costs, taxes and the treatment of margins), not included in theoretical models, that may cause the spread. As the life of a futures contract increases, the spread is becoming more significant and it is then not appropriate to assume that forward and futures price are perfect substitutes for each other. This point is particularly relevant to Eurodollar futures contracts since they have maturities up to 10 years. Eurodollar futures are regularly used to calculate zero-coupon LIBOR rates. For contracts lasting one or two years it is reasonable to assume the zero-spread, or equivalently, that the rate calculated from the futures price is a forward interest rate. We will look at this important point later in the paper.

3.2 General Framework

The difference between futures and forward rates is determined by the market's expectations of future changes in rates, so that different interest rate model will lead to different convexity adjustment. Theoretical forward rates are computed from bond prices whereas futures are expected future spot rates computed under risk-neutral measure Q .

In the following parts we will have closer look at a convexity adjustment (CA) in particular models.

3.3 CA in Hull-White Model

In this part we will use the results of Hull [11], derived for Hull-White model defined here in (2.51). The notation in Hull is slightly different than notation presented in our paper, with $a \equiv \alpha$ and $\mu_t \equiv \theta_t$, the model itself has then

form

$$dr_t = a \left[\frac{\theta_t}{a} - r_t \right] dt + \sigma dW_t. \quad (3.6)$$

The θ_t function can be calculated as

$$\theta_t = \frac{\partial}{\partial t} f(0, t) + a f(0, t) + \frac{\sigma^2}{2a} (1 - e^{-2at}) \quad (3.7)$$

and the bond prices are given by

$$P(t, T) = A(t, T) e^{-B(t, T) r_t} \quad (3.8)$$

where

$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a} \quad (3.9)$$

and

$$\begin{aligned} \ln A(t, T) &= \ln \frac{P(0, T)}{P(0, t)} - B(t, T) \frac{\partial \ln P(0, t)}{\partial t} - \frac{1}{4a^3} \sigma^2 (e^{-aT} - e^{-at})^2 (e^{2at} - 1) \\ &= -(T - t) f(0, t, T) + B(t, T) f(0, t) - \frac{1}{4a} \sigma^2 B(t, T)^2 (1 - e^{2at}) \end{aligned} \quad (3.10)$$

From (2.54) we can derive the exact formula for the risk-neutral expectation of r_t ;

$$E^Q[r_t] = f(0, t) + \frac{\sigma^2}{2a^2} (1 - e^{-at})^2 \quad (3.11)$$

or, using (3.9),

$$E^Q[r_t] = f(0, t) + \frac{\sigma^2 B(0, t)^2}{2}. \quad (3.12)$$

From the relationship $R(t, T) = -\frac{1}{T-t} \ln P(t, T)$ and (3.8),

$$R(t_1, t_2) = -\frac{1}{t_2 - t_1} \ln[A(t_1, t_2)] + \frac{1}{t_2 - t_1} B(t_1, t_2) r_t \quad (3.13)$$

which yields, using (3.12) and (3.10), to

$$\begin{aligned} E^Q[R(t_1, t_2)] &= -\frac{1}{t_2 - t_1} \ln[A(t_1, t_2)] + \frac{1}{t_2 - t_1} B(t_1, t_2) \left[f(0, t_1) + \frac{\sigma^2 B(0, t_1)^2}{2} \right] \\ &= f(0, t_1, t_2) + \frac{B(t_1, t_2)}{(t_2 - t_1)} [B(t_1, t_2)(1 - e^{-2at_1}) + 2aB(0, t_1)^2] \frac{\sigma^2}{4a}. \end{aligned} \quad (3.14)$$

In the risk-neutral world, the expected future price equals to the futures price, which means that the $E^Q [R(t_1, t_2)]$ value is the futures rate between t_1 and t_2 . As a consequence, we can observe that the relationship (3.14) shows how much the future rate should be reduces in order to obtain the forward rate ;

$$\frac{B(t_1, t_2)}{(t_2 - t_1)} [B(t_1, t_2)(1 - e^{-2at_1}) + 2aB(0, t_1)^2] \frac{\sigma^2}{4a}, \quad (3.15)$$

which is the Hull-White convexity adjustment.

If we consider $a = 0$, or very close to zero respectively, it will become simple Ho-Lee convexity adjustment $\sigma^2 t_1 t_2 / 2$, which we will derive more in detail in the following section. The Hull-White model is a version of Ho-Lee model with mean reversion. Therefore, it describes better the volatility environment. Lognormal one-factor models' main advantage is that they avoid possibilities of negative interest rates, but unfortunately (unlike the Ho-Lee model) they have no analytic tractability.

In the following example we will see can be quite significant for long-maturity futures contracts.

Example 3.3.1 Consider the Hull-White model with estimated parameters $\sigma = 0.015$, $a = 0.05$. Let us calculate the forward rate when 8-year Eurodollar futures prices 95. In this case the futures rate per annum with quarterly compounding is 0.05 or 5%. Futures rate with continuous compounding is therefore $\frac{\ln(0.25 \times 0.05 + 1)}{0.25} = 0.04969$ or 4.969%.

We have $t_1 = 8$, $t_2 = 8.25$,
 $B(8, 8.25) = \frac{1 - e^{-0.05 \times 0.25}}{0.05} = 0.2484$,
 $B(0, 8) = \frac{1 - e^{-0.05 \times 8}}{0.05} = 6.5936$
and the convexity adjustment:

$$\frac{0.2484}{8.25 - 8} [0.2484(1 - e^{-2 \times 0.05 \times 8}) + 2 \times 0.05 \times 6.5936^2] \frac{0.015^2}{4 \times 0.05} = 0.005013,$$

or 0.5013%. Since the futures rate with continuous compounding is 4.969%, we will calculate the forward rate with continuous compounding simply as $4.969 - 0.5013 = 4.4677\%$.

Figure Fig. (3.1) plots the Hull-White convexity adjustment for futures contracts on the 3-month Libor for different maturity dates, with $\sigma = 0.02$ and $a = 0.2$. James and Webber in [13] have shown that the convexity adjustment is approximately five basis point at two years while it is staggering 40 basis points at 10 years. However, at longer maturities the yield curve

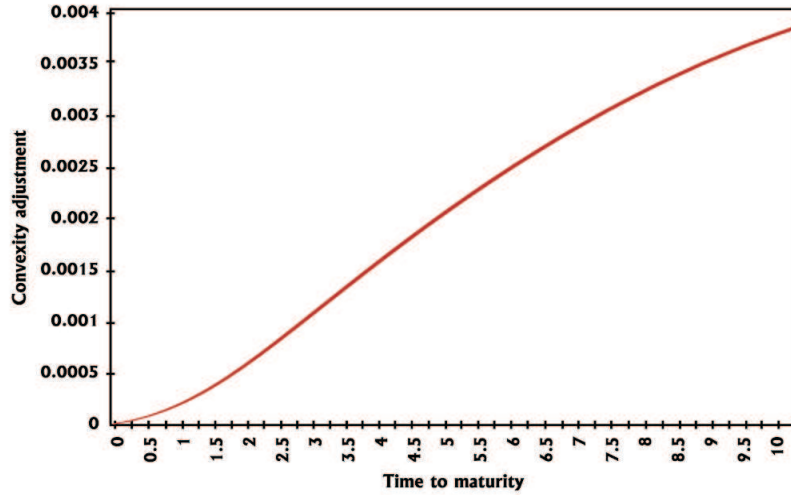


Figure 3.1: The convexity adjustment calculated by James and Webber in [13]

may be constructed only as a spread over Treasury yields. In reality, the yield curve may only use Eurodollar futures of short maturities. Therefore, the usefulness of convexity adjustment is moot- small at short maturities and not used at long maturities. However, it can be used in calculations to obtain the forward rates or discussing the arbitrage possibilities, as we will do later.

3.4 CA in Ho-Lee Model

In the simple Ho-Lee model the risk-neutral process for the short rate r_t is given by

$$dr_t = \theta_t dt + \sigma dW_t,$$

the bond price $P(t, T)$ has the form $P(t, T) = A(t, T)e^{-r(T-t)}$, for some deterministic function $A(t, T)$, further described e.g. in Hull [11]. From the Itô's lemma the process followed by the bond price in a risk-neutral world is

$$dP(t, T) = r_t P(t, T) dt - (T - t) \sigma P(t, T) dW_t.$$

Recalling now $f(t, t_1, t_2) = \frac{1}{t_2 - t_1} \ln \frac{P(t, t_1)}{P(t, t_2)}$, we can obtain, using again the Itô's lemma, the process for $f(t, t_1, t_2)$,

$$df(t, t_1, t_2) = \frac{\sigma^2(t_2 - t)^2 - \sigma^2(t_1 - t)^2}{2(t_2 - t_1)} dt + \sigma dW_t. \quad (3.16)$$

The forward rate equals the spot rate at time t_1 . Therefore, the expected value of the forward rate at t_1 is the expected value of the spot rate at t_1 . Since we consider our model in the traditional risk-neutral world, the expected value of the spot rate is the same as the futures rate. As a consequence, the futures rate is greater than the forward rate by the expected change in the forward rate between times 0 and t_1 . This change can be computed easily from (3.16), it is determined by integrating the coefficient of dt between 0 and t_1 . It is:

$$\begin{aligned} \int_0^{t_1} \frac{\sigma^2(t_2 - t)^2 - \sigma^2(t_1 - t)^2}{2(t_2 - t_1)} dt &= \frac{\sigma^2}{2} \int_0^{t_1} (t_2 - 2t + t_1) dt \\ &= \frac{\sigma^2 t_1 t_2}{2}. \end{aligned} \quad (3.17)$$

As Hull in [11] explains, this convexity adjustment is composed actually from two components:

- The difference between a futures contract that is settled daily and a similar contract that is settled entirely at time t_1 .
- The difference between the contract that is settled at time t_1 and a similar contract that is settled at time t_2 .

The Ho-Lee model is the simplest interest rate model. This has the advantage that it is analytically tractable, on the other hand, its main disadvantage is that it implies that all rates are equally variable at all times. Other, more complicated models introduced in this work, have various descriptive advantages, such as precious description and avoiding the possibility of negative interest rates, but, unfortunately, they have no analytic tractability. For this reason, we will further focus on the simple Ho-Lee model and we will use it in our calculations.

Example 3.4.1 *Again, it is possible to show that the difference between forward and futures rates is not small in case of long-maturity contracts. If we go back to our example*

from previous section, we will not need the estimate of a anymore, and calculate the convexity adjustment as

$$\frac{1}{2} \times 0.015^2 \times 8 \times 8.25 = 0.007425$$

or 0.7425%. The forward rate with continuous compounding will be $4.696 - 0.7425 = 3.9535\%$.

3.5 CA in Other Models

There are several papers studying the convexity adjustment in detail for more complicated models or trying to describe the calculations in model-independent way.

Hunt and Kennedy in [12] used the approach derived here in (3.5), that the convexity adjustment is given by $For_0^T - Fut_0^T = \frac{1}{B_0^T} \cdot \text{cov}(D_T, S_T)$, where the covariance is taken under the risk neutral measure. No model-specific calculations were given.

Vaillant in [25] defined the convexity adjustment as a quotient between the forward rate and futures rate. He derived it is given by:

$$CA_t = \frac{For_t^T}{Fut_t^T} = \exp \left(-r_\infty \int_t^T (T-s) \sigma_r(s) \sigma_{Fut}(s) \sigma(s) dt \right),$$

where σ_r^2 and σ_{Fut}^2 are the variances of the spot zero rate and futures rate, σ is from the model relationship $d\langle W_1(t), W_2(t) \rangle = \sigma(t)dt$, where W_1 and W_2 are two Brownian motions used in model for spot zero rate and futures rate modeling. Parameter r_∞ is some asymptotic value that zero rate reaches. (For deeper model description please see [25].) From the assumptions it is not clear whether the model is arbitrage free, furthermore, no provision for the critical input $\sigma(t)$ is given.

Piterbarg and Renedo in [17] have applied an expansion technique to derive a model-independent relationship for calculating convexity adjustment. They divided the variance terms into parameters that are easily observed and change often (volatility parameters), and those that require calibration to be estimated but do not fluctuate often (correlations parameters). However, even this deep analysis still leaves us the problem of solving the derived equations to obtain forward rates from market-observed futures.

3.6 Our Data Set

In our further analysis we will focus on our data set from year 2007 (provided by Bloomberg L.P.), measured between time 08/01/07 and 09/03/07. We dispose of intraday 3-month futures prices quotes on EUR currency (Euribor) as well as on USD currency (Eurodollar- U.S. dollars deposited in commercial banks outside the United States), in 7 different maturities: 19/03/07, 18/06/07, 17/09/07, 17/12/07, 17/03/08, 16/06/08 and 15/09/08. The future prices quotes are stated in terms of a maturity value of 100, so a typical price would be e.g 94.98. Rates are measured during the trading hours every minute and in case that an observation in particular minute is missing, we use the rate from previous minute. (The problem is that bid and ask quotes are not both available throughout the entire sample period in the forward market. The problem is not that data for specific moments are missing, but rather that the market did not report the quotes during entire time.)

As a fair price for the forward quote we set the observed bid price plus one-half the bid-ask spread. (Although this calculation is very rough, for more precise fair price calculations we would need complete traded-volume data set for entire time period and all maturities, which is not available.) This is often referred as a *MID Price* in financial markets. (The main disadvantage of quoting the MID price is that the bid or offer price may be unrealistic and distort the MID price.)

Let us denote the quotes as $FutQuote_{e_i}$ and $FutQuote_{u_j}$ for $i, j = 1, \dots, 7$, where e, u are the currency indexes (EUR and USD), and i, j the maturity indexes.

For simplicity we will now skip the currency index e, u and the maturity index i, j , since the calculations will be the same for all of them. (These coefficients will be used below only if specially needed and they will be noted in the same form as above.) Analogously as in the example 3.3.1 we will compute the futures discrete rates $FutD$ as

$$FutD = 1 - \frac{FutQuote}{100}$$

and the futures rates with continuous compounding $FutC$ as

$$FutC = \frac{\ln(0.25FutD + 1)}{0.25}$$

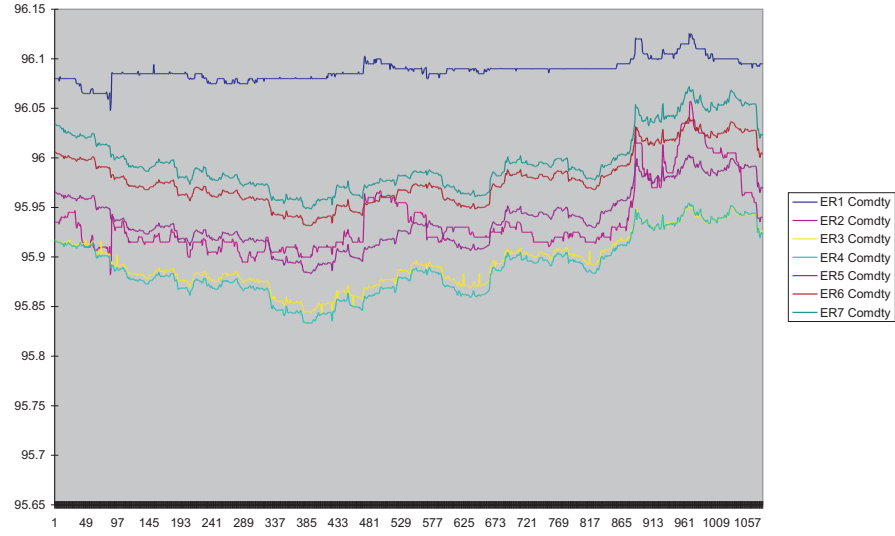


Figure 3.2: Euribor quotes

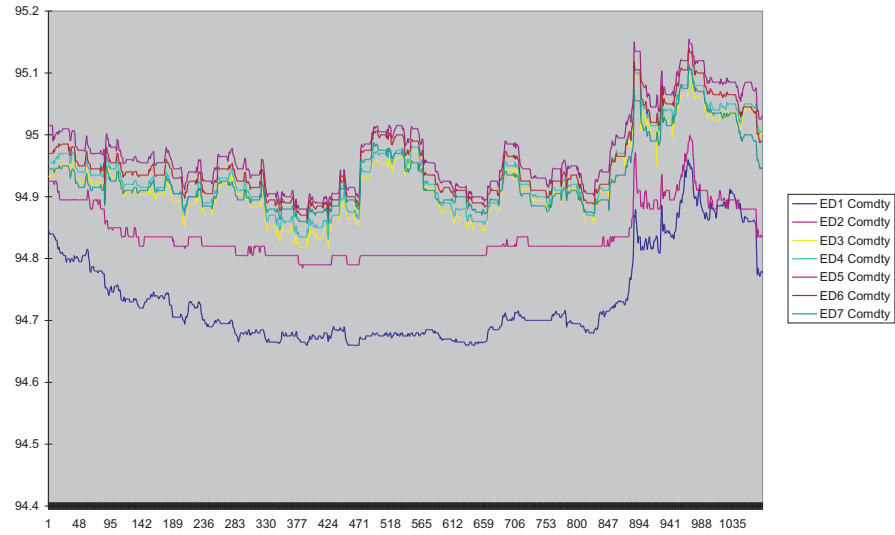


Figure 3.3: Eurodollar quotes

for both currencies in all seven maturities. In figures Fig.(3.2) and Fig.(3.3) we plot the Euribor and Eurodollar quotes, as reposted from the market. (We plot here the MID price and, because of the extremely large data set,

only the hour averages. In calculations, of course, we use the full intraday data set and not the hour averages anymore.)

3.6.1 Ho-Lee Model Parameters Estimation

For computing the forward rates we need the convexity adjustment applied on the simplest Ho-Lee model. We denote $FutC_m^d$ as a futures rate in the time moment determined by the day d and its minute m . In our case we have observations for $D = 45$ whole days ($d = 1, \dots, D$) and within each day observations for all its M minutes ($m = 1, \dots, M$). We remove the nontrading days and minutes from our consideration so the resulting time series can be considered as regular (with minute time intervals).

We want to estimate the standard deviation of daily changes of the futures rate and possibly recompute it to annual basis. A model behind our formulas is that the minute sequence of the futures rates is assumed here to form a random walk. (I.e. we assume the changes to have zero expected value, are uncorrelated and homoscedastic. These assumptions are partly based on the results in section 2.2, where we noticed that the forward interest rate in Ho-Lee model is normally distributed.)

Specially $FutC_m^d - FutC_m^{d-1}$, $d = 2, \dots, D$, $m = 1, \dots, M$ is a collection of identically distributed random variables with zero mean and finite variance σ_{day}^2 . Although these variables are correlated (and so do not form a random sample), the expression

$$\hat{\sigma}_{day}^2 = \frac{\sum_{d=2}^D \sum_{m=1}^M (FutC_m^d - FutC_m^{d-1})^2}{(D-1) \cdot M} \quad (3.18)$$

(sample variance) is an unbiased estimator of σ_{day}^2 . The estimate of a standard deviation $\sigma_{day} = \sqrt{\sigma_{day}^2}$ will be obviously $\hat{\sigma}_{day} = \sqrt{\hat{\sigma}_{day}^2}$.

Since the annual change of $FutC$ is a sum of individual daily changes through the year, its variance is simply

$$\sigma_{year}^2 = D_{year} \cdot \sigma_{day}^2, \quad (3.19)$$

where D_{year} is the number of (trading) days in one year (in our case is $D_{year} = 260$). Our annual estimators then will obviously be

$$\hat{\sigma}_{year}^2 = D_{year} \cdot \hat{\sigma}_{day}^2 \quad \text{and} \quad \hat{\sigma}_{year} = \sqrt{\hat{\sigma}_{year}^2}. \quad (3.20)$$

Estimated standard deviation of daily changes of the futures rates

Maturity	EUR	USD
19/03/07	0.97 E-02	1.47 E-02
18/06/07	1.03 E-02	1.31 E-02
17/09/07	1.15 E-02	1.60 E-02
17/12/07	0.85 E-02	1.53 E-02
17/03/08	1.17 E-02	1.61 E-02
16/06/08	1.21 E-02	1.59 E-02
15/09/08	1.13 E-02	1.51 E-02

As we can notice from the table, the estimated parameters for the Euribor are slightly below the Eurodollar rates. This can also lead us to conclusion that the forward interest rate market is more stable -in the daily changes point of view- for the EUR currency. (Analyzing this in more detail is not the aim of our paper though.)

The standard deviation of the short rate changes is in financial markets also often briefly estimated as a current value of ATR (Average True Value). ATR is defined as a moving exponential average of the TR (True Ranges), where TR indicator is the maximum of yesterday's range to today's range. It is the greatest of the following:

- current high less the current low
- the absolute value of the current high less the previous close
- the absolute value of the current low less the previous close.

It tells us the maximum distance that this market traveled over a 24-hour period. However, for our purpose we will use the more precise estimate computed from the sample variance.

3.6.2 CA for Our Data Set

After the estimating the futures rates' standard deviation of daily changes - the only parameter in the Ho-Lee model, we can compute straightforward the exact amount of Ho-Lee Convexity adjustment, as in (3.17):

$$CA = \frac{1}{2}\sigma^2 t_1 t_2.$$

As σ we use now the just estimated daily changes standard deviation, the first time variable t_1 is remaining time to appropriate maturity of the contract and the second time variable t_2 will be here simply set as $t_1 + 0.25$, since we work with 3-month futures prices.

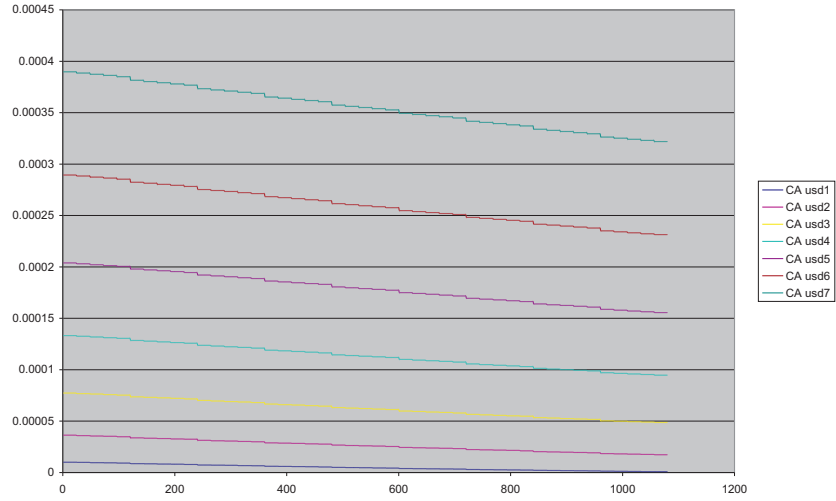


Figure 3.4: Convexity Adjustment for the Eurodollar Futures

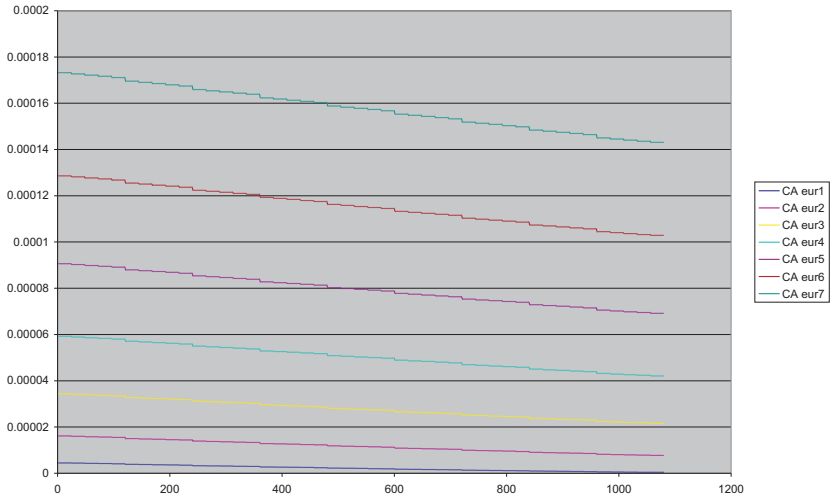


Figure 3.5: Convexity Adjustment for the Euribor Futures

In figures Fig.(3.4) and Fig.(3.5) we plot the calculated convexity adjustment for all 7 maturities. As already expected from the model, the exact amount of the adjustment increases with longer maturity. For a fixed maturity, convexity adjustment in Ho-Lee model is decreasing. (Follows directly from the formula (3.17), since closer we are to the moment of expiration, time to maturity t_1 approaches zero.)

Smallest convexity adjustment varies around 0.00001 (0.001%) for the Euribor and 0.00004 (0.004%) for the Eurodollar. Largest adjustment occurs in case of the last maturity and it is 0.000175 (0.0175%) for the Euribor and 0.0004 (0.004%) for the Eurodollar. Even in this largest case, it is not of a big impact for the forward rate consideration. (More significant difference would appear for longer maturities, but usefulness of these calculations for long maturities is questionable.)

However, most of calculations on the real market simply assumes that the forward and futures prices are equal, or use the same parameters in calculations for all maturities. For example, Hull in [11] recommends to use $\sigma = 0.015$, which is very close to our deviation estimates for USD futures prices model.

3.6.3 Forward rates

Eurodollar futures reflect market expectations of forward 3-month rates. An implied forward rate indicates approximately where short-term rates may be expected to be sometime in the future. The forward rates for both currencies and all seven maturities can be now easily obtained from futures rates reduced by the convexity adjustment calculated above:

$$For_{e_i} = FutC_{e_i} - CA_{e_i}, i = 1, \dots, 7, \quad (3.21)$$

$$For_{u_j} = FutC_{u_j} - CA_{u_j}, j = 1, \dots, 7. \quad (3.22)$$

We will use these in the following chapter in order to arrive to possible arbitrage. (However, in reality it is often assumed that both forward and futures are the same and appropriate convexity adjustment is not included in the calculations.)

Chapter 4

Arbitrage Analysis

In previous chapter we have used the currency futures prices to calculate the appropriate currency forward prices. All calculations were done for seven different maturities, for both USD and EUR currencies, during entire trading time between 08/01/07 and 09/03/07. In this chapter, we will look at the eventual arbitrage existence between interest rates (the Euribor and Eurodollar futures quotes) and currency exchange rates (corresponding currency forwards).

4.1 Currency Forwards

Considering now the most common definition of arbitrage -as a process with positive probability of gain and zero probability of lose- we will try construct the arbitrage possibilities using our Euribor and Eurodollar rates and appropriate currency forward rates.

A currency forward contract is defined on the market as forward contract in the forex market that locks in the price at which an entity can buy or sell a currency on a future date. Also often referred as “outright forward currency transaction”, “forward outright” or “FX forward”. In our further calculations we will denote it as *ForFX*, which will mean the Euro FX futures. This rate assesses the relative value of the U.S. dollar compared to the euro, provides a way to manage risks associated with currency rate fluctuations in the FX markets and to take advantage of profit opportunities stemming from changes in those rates.

Since we are analyzing the futures interest rates measured between the days 08/01/07 and 09/03/07 with maturities from 19/03/07 until 15/09/08, for the arbitrage construction we will use the corresponding forward exchanges rates: the spot rate, 1 week, 1-, 2-, 3-, 4-, 5-, 6-, 9-, 12-, 15-, 18- month and 2- year rate. These are often labeled on market as *EUR Curncy* (spot rate), *EUR1W Curncy* (one week rate), ..., *EUR2Y Curncy* or *EUR24M Curncy* (two year rate).

Currency forward rate for 21 months is not quoted and we will have to roughly approximate it using the *15M*, *18M*, *21M* and *24M Curncy* rates. Let x be the forward change between 18 and 21, and y the change between 21 and 24. Then

$$x + y = EUR24MCurncy - EUR18MCurncy.$$

Furthermore, we assume that the trend of the currency forward will keep the same trend across the time and so that

$$x/y = (EUR18MCurncy - EUR15MCurncy) / x.$$

Putting both equations together we get the quadratic equation (here with short notation), of which solution gives us approximated *EUR21M Curncy* rate:

$$x^2 + x(18M - 15M) - (18M - 15M)(24M - 18M) = 0$$

Now, after obtaining the *EUR21M Curncy*, we can do the full linear interpolation between these currency forwards (according to the relevant maturity) in order to obtain the approximate forward rate for every trading day considered in our analysis. Nevertheless, the currency forward computation presented here might be in some cases very vague and not explaining the real market behaviour. In general though, it should be sufficient for our further calculations, as it takes into consideration the main estimative currency forward trend.

4.2 Construction of the Arbitrage

The main idea of our arbitrage consideration is comparing the two possibilities: having one EUR unit we can first exchange it to the USD using

forward exchanges rates and then deposit it with the forward dollar interest rate. Or, as a second case, we can start with deposition using the forward euro interest rate and then exchange it to USD currency using the matching forward exchanges rates. At the end of both of these we should arrive to the same amount of USD.

Since we already have done all calculations needed to obtain the appropriate forward dollar/euro interest rates and the forward exchanges rates are derived directly from the data, the 2-step-arbitrage considered above can be now easily computed.

Let us assume now we start with one EUR unit at time 0. We move to time t within the first step (exchange or deposit as first) and arrive to the second step (deposition after exchange or exchange after deposition) at time T . We denote here the corresponding exchange rates as $ForFX(t)$, $ForFX(T)$. In case of no arbitrage appearance we have:

$$ForFX(t) \cdot e^{For_u(t) \cdot (T-t)} = e^{For_e(t) \cdot (T-t)} \cdot ForFX(T), \quad (4.1)$$

or

$$For_e(t) = For_u(t) + \frac{1}{0.25} \ln \left(\frac{ForFX(t)}{ForFX(T)} \right), \quad (4.2)$$

since we work with 3-month futures prices and so $T - t = 0.25$.

4.3 Futures Quotes vs. Implied Quotes

Using the results just derived in previous section, we will present the arbitrage as a difference between the real market Euribor quotes and Euribor quotes calculated using the exchange rates from above. More precisely, after computing the implied forward rate for EUR as in (4.2) we can next obtain the implied EUR futures interest rate (by addition of a convexity adjustment, already calculated in previous chapter). Finally, we will compare the implied Euribor quotes with the real ones, observed from the market.

In Fig. (4.1) we plot the difference between calculated EUR currency futures quotes and those observed from the market. It is plotted here in the common price fluctuation units - *basis points (bps)*, which means we multiplied the computed difference by 100.

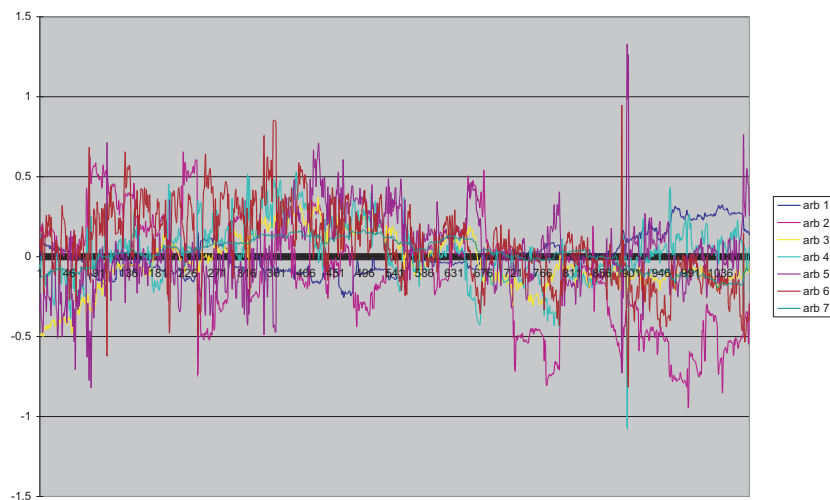


Figure 4.1: The difference between the real market Euribor quotes and implied Euribor quotes (in bps)

Price Fluctuation Measurements

One basis points in the Eurodollar contracts reflects the dollar value of a $1/100$ of one percent change in a \$1 million, 90-day deposit. It is determined by: $\$1,000,000 \times 0.0001 \times \frac{90}{360} = \25 . The smaller fluctuation measurements on a Eurodollar futures contract is often 0.005 or half of a basis point and even smaller units. (The minimum price fluctuation for a contract is called minimum tick, which is the smallest increment a given futures market can move). However, many different definitions for these fluctuations are recently provided and we will use only the basis points plots in our paper.

Statistical Analysis

The exact results are presented in the table below. The most striking fact is the small sample mean of the arbitrage. For two maturities the sample mean is negative, which indicate that the market quotes are below the implied, calculated ones. In the rest of the cases, sample mean is positive, but still very close to zero. However, on the second contract we can notice, how misleading this might be. Sample mean of arbitrage is in this case negative (and larger than for other contract), but the median value is positive.

Futures Quotes vs. Implied Quotes			
Maturity	Sample Mean	Mean Standard Error	Median
19/03/07	0.00654	0.00368	-0.01669
18/06/07	-0.10107	0.01053	0.05179
17/09/07	-0.03733	0.00528	-0.06815
17/12/07	0.03386	0.00545	0.03529
17/03/08	0.03067	0.00690	0.00897
16/06/08	0.06887	0.00719	0.05593
15/09/08	0.01909	0.00279	0.02755
	Sample Variance	Max	Min
	0.01466	0.32335	-0.25452
	0.11989	0.64867	-0.94135
	0.03013	0.36850	-0.49857
	0.03216	0.52581	-1.04880
	0.05145	1.30983	-0.82011
	0.05589	0.94792	-0.80177
	0.00839	0.16775	-0.19173

Trying to test the data for the zero-hypothesis makes no moderate statistical sense, since most of considerable hypotheses would be strongly rejected according to the high number of observation. Therefore, simple look at the pictures plotting the arbitrage will make more sense this time.

Figure (4.2) plots the calculated arbitrage possibilities for the contract in different maturities. (Denoted here on this figure as $arb1, \dots, arb7$.) In some moments, the amount of the arbitrage exceeds 1 basis point in both negative and positive sense. Most of the time they oscillate in a narrow range around the zero value.

However, for some certain moments of time, the trend for some maturities seems to be strongly biased in negative or positive direction. This may be caused by imperfect estimation of the appropriate currency forwards in our previous calculations.

Furthermore, not all rates which we used here, were recorded at a same time instant. We had to achieve the appropriate data by assuming the rates not to change dramatically at the specific moment. This might have also

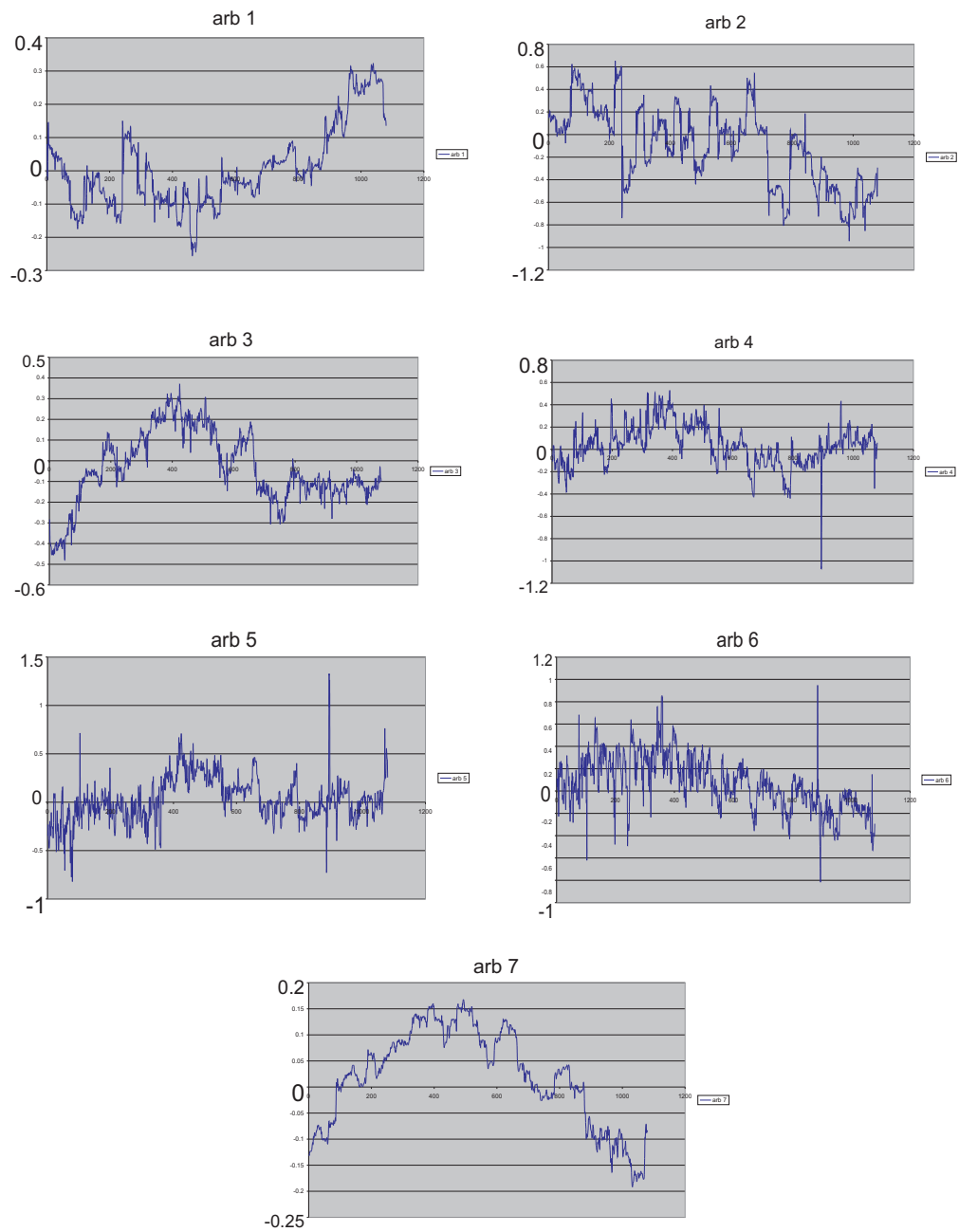


Figure 4.2: The difference between the real market Euribor quotes and implied Euribor quotes (in bps)

caused some bias though. (Since all the rates are not recorded at the same time instants, some random variation between them can be observed. This random error, though, will not bias the results.)

Moreover, the question whether this kind of arbitrage is tradable in reality remains as major. First reason is that we have used the MID prices for our calculations. This might occur as a problem in case that the bid-ask spread is too large. (If e.g. the ask quote is too high, the arbitrageur would find it impossible to make a profit, even if here appears existence of an arbitrage in our analysis.)

Secondly, and maybe even more important, is the transaction costs appearance. For the futures rates the costs are relatively small, but in case of the currency forwards they are sometimes significantly higher. Specially, in cases of large bid-ask spread, the transaction costs are increasing and make the arbitrage opportunity not tradable anymore. (There are analytical studies and papers about the transaction costs- and making the profit taking them into consideration- done so far, but this was not the aim of our work.)

4.4 Overview

In this chapter we had a closer look to the possible arbitrage existence between interest rates and currency exchange rates. More concretely, we have analyzed the Euribor and Eurodollar futures quotes and corresponding currency forwards.

We first had to compute the convexity adjustment- difference between the futures and forward rates. For all considered seven maturities it has appeared in a very small amount, anyway, we used it in order to calculate the corresponding futures interest rate. We have compared the computed (implied) Euribor futures quotes with the data reported from the market.

The difference has shown up in all cases fairly small, oscillating around zero. In couple of moments we have observed stronger deviation from zero or even biased trend as well. However, this does not indicate the significant arbitrage appearance. The two main reasons for the resulting bias are usage of MID prices and transactions costs for the currency forwards and futures interest rates.

All the calculations were done using the software programs Excel 2003 and Matlab 7.1.

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