FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

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Volumes of unit balls of Lorentz spaces

Department of Mathematical Analysis

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Study programme:	Mathematics
Study branch:	Mathematical Analysis

Prague 2019

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In Prague, July 18, 2019 Anna Doležalová

I would like to use this opportunity to thank my supervisor Jan Vybíral for his advice, guidance and patience, my family for their constant support and encouragement through all the years of my studies, and my friends, who made these years much better.

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Abstract: This thesis studies the volume of the unit ball of finite-dimensional Lorentz sequence spaces $\ell_n^{p,q}$. Lorentz spaces are a generalisation of Lebesgue spaces with a quasinorm described by two parameters $0 < p, q \leq \infty$. The volume of the unit ball $\mathbf{B}_n^{p,q}$ of a general finite-dimensional Lorentz space was so far an unknown quantity, even though for the Lebesgue spaces it has been well-known for many years. We present the explicit formula for $\mathrm{Vol}(\mathbf{B}_n^{p,\infty})$ and $\mathrm{Vol}(\mathbf{B}_n^{p,1})$. We also describe the asymptotic behaviour of the n-th root of $\mathrm{Vol}(\mathbf{B}_n^{p,q})$ with respect to the dimension n and show that $[\mathrm{Vol}(\mathbf{B}_n^{p,q})]^{1/n} \approx n^{-1/p}$ for all $0 < p < \infty$, $0 < q \leq \infty$. Furthermore, we study the ratio of $\mathrm{Vol}(\mathbf{B}_n^{p,\infty})$ and $\mathrm{Vol}(\mathbf{B}_n^{p})$. We conclude by examining the decay of entropy numbers of embeddings of the Lorentz spaces.

Keywords: Finite-dimensional bodies, entropy numbers, interpolation, volume estimates, Lorentz spaces

Contents

Introduction

During the last century, the scale of the Lebesgue spaces $L^p(X,\mu)$ (where (X,μ) is an arbitrary measure space) proved to be insufficiently fine to describe the properties of functions and operators. Thanks to the work of G. Lorentz [12] and R. Hunt [9], a new scale of spaces arose – the Lorentz spaces $L^{p,q}(X,\mu)$. Whereas the quasinorm in a Lebesgue space is determined by a single parameter $0 < p \leq \infty$, the quasinorm in a Lorentz space is described by two parameters $0 < p, q \leq \infty$. For $p = q$ we obtain the classical Lebesgue space $L^p(X,\mu)$. The space $L^{p,\infty}(X,\mu)$ is usually called the weak Lebesgue space. The Lorentz spaces play an important role in the interpolation theory (e.g. the Marcinkiewicz theorem, see [2, 3]) as they are interpolation spaces of the Lebesgue spaces. They also found applications in other mathematical branches such as the harmonic analysis or the analysis of PDE's (cf. [7, 13]).

When μ is the counting measure, we arrive to the Lorentz sequence spaces $\ell^{p,q}$ (for $X = \mathbb{N}$) or $\ell_n^{p,q}$ (for $X = \{1,\dots,n\}$). The space $\ell_n^{p,q}$ is then $\mathbb{R}^n$ equipped with the quasinorm

$$||a||_{p,q} = \begin{cases} \left(\sum\limits_{k=1}^{n} (a_k^*)^q k^{\frac{q}{p}-1}\right)^{\frac{1}{q}}, & q \in (0,\infty), \\ \sup\limits_{k\in\{1,\dots,n\}} \{k^{\frac{1}{p}} a_k^*\}, & q = \infty, \end{cases}$$

where $a = (a_1,\dots,a_n) \in \mathbb{R}^n$ and a^* is the nonincreasing rearrangement of $(|a_1|,\dots,|a_n|)$.

The volume of the unit ball $\mathbf{B}_n^{p,q}$ in a (general) finite-dimensional Lorentz space $\ell_n^{p,q}$ was so far an unknown quantity, even though for the Lebesgue spaces it has been well-known for many years. In that case,

$$\mathrm{Vol}(\mathbf{B}_n^p) = \frac{2^n \Gamma(1+1/p)^n}{\Gamma(1+n/p)},$$

where $\mathrm{Vol}(A)$ denotes the n-dimensional Lebesgue measure of a (measurable) set $A \subseteq \mathbb{R}^n$ and Γ denotes the gamma function. The motivation to study this problem is the fact that the properties of the unit ball describe or determine many interesting properties of the whole space and mappings between them (cf.[16]). However, despite some attempts in [14], there seems to be only very little known about the volume of the unit balls of the Lorentz spaces. The aim of this thesis is – at least to some extent – to fill this gap. Our goal is to determine the volume of the unit ball for special choices of parameter q and to get some overall results concerning the asymptotic behaviour of the volumes with respect to the dimension.

In Chapter 1 we introduce the Lorentz spaces and their basic properties. We describe the Lorentz sequence spaces and point out some of their specifics. For those who are not familiar with entropy numbers we offer a brief summary of their behaviour for the Lebesgue spaces and for the interpolation spaces in general.

The focal point of this thesis lies in Chapters 2 and 3. We present two approaches to determine the volume of the unit ball for $q = \infty$, the first through the recursive formula

$$\mathrm{Vol}(\mathbf{B}_{n,+}^{p,\infty}) = \sum_{j=1}^{n} \left[\binom{n}{j} (-1)^{j+1} \left(\frac{1}{n}\right)^{\frac{j}{p}} \mathrm{Vol}(\mathbf{B}_{n-j,+}^{p,\infty}) \right]$$

and the second through integration of a suitable function. Though we offer an explicit formula in Theorem 2.2.2, the recursive formula is more suitable for calculating the volume (see Section 2.5). For the case $q = 1$ we use another approach to obtain an elegant formula

$$\mathrm{Vol}(\mathbf{B}_n^{p,1}) = 2^n \prod_{k=1}^{n} \frac{1}{\varkappa_p(k)}, \text{ where } \varkappa_p(k) = \sum_{j=1}^{k} j^{\frac{1}{p}-1},$$

in Theorem 2.4.1 which allows us to prove that

$$\sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{p,1})} \approx n^{-1/p}$$

for all $0 < p < \infty$ (the multiplicative constants of equivalence are independent of n). Due to embeddings of the Lorentz spaces and the theory of entropy numbers we get the same asymptotic result for all choices of parameter q in Theorem 3.3.3. Figure 1 offers a summary of what is achieved (we identify $1/\infty = 0$). With the only exception of the formula for the Lebesgue spaces the results are new.

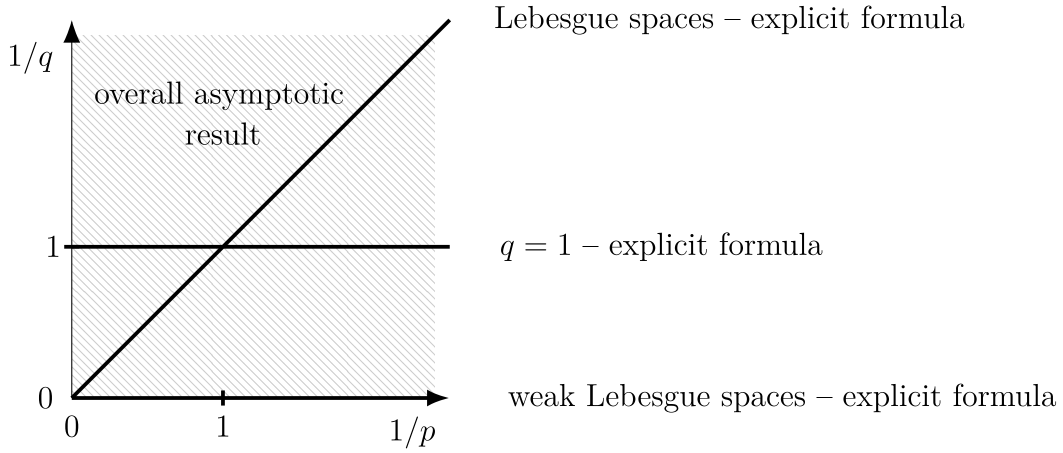


Figure 1: Comparison of the results with respect to p and q

Furthermore, we consider the volume of the unit ball in the weak Lebesgue space $\ell_n^{p,\infty}$ and in the corresponding Lebesgue space ℓ_n^p and examine the ratio

$$R_{p,n} = \frac{\mathrm{Vol}(\mathbf{B}_n^{p,\infty})}{\mathrm{Vol}(\mathbf{B}_n^p)}.$$

Even though $\mathbf{B}_n^{p,\infty}$ is generally considered as "slightly larger" than $\mathbf{B}_n^p$ and the behaviour of the n-th root of the volume was the same for both of them, we show that if the parameter p is sufficiently small, then the growth of $R_{p,n}$ is exponential in n.

Chapter 4 is dedicated to further properties of the unit ball, such as a characterisation of spaces where the ball is convex. We study the decay of entropy numbers of the embeddings of the Lorentz spaces. In most of the cases they exhibit the same behaviour as for the Lebesgue spaces, however, it is not always the case as we show that

$$e_k(\mathrm{Id} : \ell_n^{1,\infty} \to \ell_n^1) \approx \begin{cases} \log(1 + n/k), & 1 \le k \le n, \\ 2^{-\frac{k-1}{n}}, & k \ge n, \end{cases}$$

where the constants are independent of n and k (cf. Theorems 1.3.1 and 4.2.4).

1. Notation and preliminaries

In this chapter we introduce the general Lorentz spaces as well as show some of their basic properties. However, later on we will consider only special cases of these spaces. We also offer a quick glance at entropy numbers which will serve us as a useful tool later.

1.1 Lorentz spaces

Definition. *Let* $(X, \mathcal{S}, \mu)$ *be a space with a* σ*-finite measure* μ *which is defined on a* σ*-algebra* $\mathcal{S}$*. Suppose that* f *is a measurable function* $X \to \mathbb{R}$*. We define the* distribution function *of* f *as* $\mu_f : [0, \infty) \to [0, \infty]$*,*

$$\mu_f(\omega) = \mu\{x \in X : |f(x)| > \omega\}$$

and the nonincreasing rearrangement *of* f *as* $f^* : [0, \infty) \to [0, \infty]$*,*

$$f^*(t) = \inf\{\omega > 0 : \mu_f(\omega) \le t\}.$$

It is clear that both μ_f and f^* are nonincreasing (see Figure 1.1).

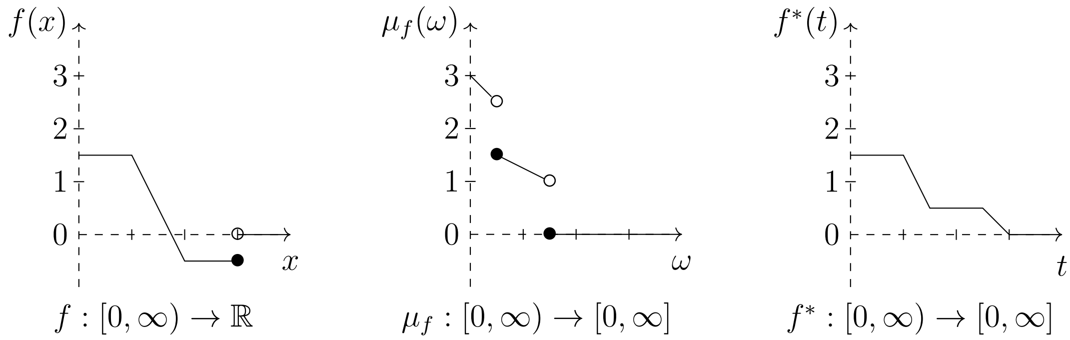


Figure 1.1: Main steps in forming f^* for $f : [0, \infty) \to \mathbb{R}$

Definition. *Let* $(X, \mathcal{S}, \mu)$ *be a space with a* σ*-finite measure* μ *which is defined on a* σ*-algebra* $\mathcal{S}$*. Suppose that* $p,\ q \in (0, \infty]$*. Then we define the* Lorentz space $L^{p,q}(X, \mu)$ *as the space of all measurable functions* $f : X \to \mathbb{R}$ *for which the following quantity is finite:*

$$||f||_{p,q} = \begin{cases} ||t^{\frac{1}{p}-\frac{1}{q}} f^*(t)||_{L^q(\mathbb{R}^+,\lambda)}, & 0 < q < \infty, \\ \sup\limits_{0<t<\infty} \{t^{\frac{1}{p}} f^*(t)\}, & q = \infty, \end{cases}$$

where f^* *is the nonincreasing rearrangement of* f*. (We identify* $1/\infty = 0$*.) The special case* $L^{p,\infty}(X, \mu)$ *is often called the weak Lebesgue space.*

Remark. To be precise, as in the Lebesgue space $L^p(X, \mu)$, two functions are considered to be equal if they differ only on a μ-negligible set. This means that formally the Lorentz space $L^{p,q}(X, \mu)$ consists not of functions, but of classes of functions with respect to this equivalence.

Recall the definition of norm, p-norm and quasinorm:

Definition. *Let X be a (real) vector space and $||\cdot|| : X \to [0,\infty)$ satisfying*

(i) *for all $x \in X$ it holds that $||x|| = 0$ if and only if $x = 0$,*

(ii) *for all $x \in X$, $t \in \mathbb{R}$ it holds that $||tx|| = |t| \cdot ||x||$,*

(iii) *for all $x, y \in X$ it holds that $||x+y|| \leq ||x|| + ||y||$.*

Then $||\cdot||$ is caled a norm. *If it satisfies*

(iii)' *there exists a constant $p \in (0,1]$ such that for all $x, y \in X$ it holds that $||x+y||^p \leq ||x||^p + ||y||^p$*

or

(iii)" *there exists a constant C such that for all $x, y \in X$ it holds that $||x+y|| \leq C(||x|| + ||y||)$*

instead of (iii)*, then it is called a p-*norm *or a* quasinorm*, respectively.*

The following proposition presents some of the basic properties of the nonincreasing rearrangement and of the Lorentz spaces. Most parts of the proof are omitted as they are well-known and can be found in [7, Chapter 1].

Proposition 1.1.1. *Let $(X, \mathcal{S}, \mu)$ be as before, p, $q \in (0,\infty]$.*

(i) *If f, $g : X \to \mathbb{R}$ are measurable, $\omega_1, \omega_2, t_1, t_2 \geq 0$, then*

$$\mu_{f+g}(\omega_1 + \omega_2) \leq \mu_f(\omega_1) + \mu_g(\omega_2) \text{ and } (f+g)^*(t_1+t_2) \leq f^*(t_1) + g^*(t_2).$$

(ii) *If $f : X \to \mathbb{R}$ is measurable, $p < \infty$, then $||f||_{L^p(X,\mu)} = ||f^*||_{L^p(\mathbb{R}^+,\lambda)}$.*

(iii) *If $f \in L^{p,q}(X,\mu)$, then $||f||_{p,q} = p^{\frac{1}{q}} ||t^{1-\frac{1}{q}} [\mu_f(t)]^{\frac{1}{p}}||_{L^q(\mathbb{R}^+,\lambda)}$.*

(iv) *The function $||\cdot||_{p,q}$ is a quasinorm on $L^{p,q}(X,\mu)$, together they form a complete space.*

(v) *If $p = q$, then $L^{p,q}(X,\mu) = L^p(X,\mu)$.*

(vi) *If $p = \infty$, $q < \infty$, then $L^{p,q}(X,\mu) = \{0\}$.*

Proof. (i) – (iv) See [7, Chapter 1].

(v) Follows from (ii), as

$$||f||_{p,p} = ||t^{\frac{1}{p}-\frac{1}{p}} f^*(t)||_{L^p(\mathbb{R}^+,\lambda)} = ||f^*||_{L^p(\mathbb{R}^+,\lambda)} = ||f||_{L^p(X,\mu)}$$

for $p < \infty$ and

$$\begin{aligned} ||f||_{\infty,\infty} &= \sup_{0<t<\infty} \{f^*(t)\} = f^*(0) = \inf\{\omega > 0 : \mu_f(\omega) \leq 0\} \\ &= \inf\{\omega > 0 : \mu\{x \in X : |f(x)| > \omega\} = 0\} = ||f||_\infty. \end{aligned}$$

(vi) Assume $f \in L^{\infty,q}(X,\mu)$. Then

$$||f||_{\infty,q}^q = \int_0^\infty t^{-1}(f^*(t))^q dt.$$

If f is nonzero, there exist $\varepsilon > 0$ and E of nonzero measure such that $f > \varepsilon$ on E. Therefore $f^* > \varepsilon$ on $(0,\mu(E))$. However, $t^{-1}\varepsilon$ is not integrable on $(0,\mu(E))$, so $||f||_{\infty,q}^q = \infty$, which is a contradiction. □

Notation. *From now on, we always assume that p is finite or that $p = q = \infty$, as for q finite the space $L^{\infty,q}$ contains only the zero function.*

Lemma 1.1.2. *Suppose that f belongs to $L^{p,q}(X,\mu)$, $0 < p < \infty$, $0 < q \leq \infty$, $E \subseteq X$ and $\varepsilon > 0$. If $f(x) \geq \varepsilon$ for every $x \in E$, then $\mu(E) < \infty$.*

Proof. We prove the statement by contradiction, let $\mu(E) = \infty$ and $f \geq \varepsilon$ on E. Then $\mu_f(\omega) = \infty$ for every $\omega \leq \varepsilon$ and it easily follows that $f^*(t) \geq \varepsilon$ for every t. The function $t^{1/p-1/q}$ does not belong to $L^q(\mathbb{R}^+,\lambda)$, therefore $||f||_{p,q} = \infty$, so f does not belong to $L^{p,q}(X,\mu)$. □

1.2 Lorentz sequence spaces

A special case of the above-mentioned spaces is when we take the positive integers or just $\{1,\dots,n\}$ (both with the canonical atomic measure) as the space X. We arrive at the definition of the Lorentz sequence spaces. We denote such spaces as $\ell^{p,q}$ and $\ell^{p,q}_n$. Although the previous definitions and propositions hold, we will reformulate some of them in order to be more clear and we introduce an equivalent quasinorm which will be used later on.

Proposition 1.2.1. *It holds that $\ell^{p,q} \subseteq \ell^\infty$. If $p < \infty$, then $\ell^{p,q} \subseteq c_0$ (where c_0 denotes the space of all sequences whose limit is zero).*

Proof. Let us firstly assume $p = \infty$, then $q = \infty$. According to Proposition 1.1.1 (iii), the space is ℓ^∞.

For p finite we proceed by contradiction. Let $a \in \ell^{p,q} \setminus c_0$, then there exists $\varepsilon > 0$, $N \subseteq \mathbb{N}$ infinite such that $|a_n| > \varepsilon$ for all $n \in N$. However, as N is infinite, we have $\mu(N) = \infty$, which contradicts Lemma 1.1.2. □

Theorem 1.2.2. *Let $0 < p < \infty$, $0 < q \leq \infty$, $a = (a_1, a_2, \dots) \in \ell^{p,q}$ (i.e., a is a function $\mathbb{N} \to \mathbb{R}$). Let us denote f_a^* the nonincreasing rearrangement of a (i.e., f_a^* is a function $[0,\infty) \to [0,\infty]$). Then there exists a sequence $b = (b_1, b_2, \dots)$ and π a permutation of $\mathbb{N}$ such that*

$$b_k \geq b_{k+1} \geq 0 \text{ and } b_k = |a_{\pi(k)}|.$$

Moreover it holds that $f_a^*(t) = b_{\lceil t \rceil}$.

Proof. We know from Proposition 1.2.1 that $a \in c_0$. Therefore absolute values of its coordinates can be reordered into nonincreasing sequence, which we denote b. For a countable set A denote $|A|$ the number of its elements.

Now for every $\varepsilon > 0$ and $k \in \mathbb{N}$ we have

$$\mu_a(b_k - \varepsilon) = |\{n \in \mathbb{N} : |a_n| > b_k - \varepsilon\}| = |\{n \in \mathbb{N} : b_n > b_k - \varepsilon\}| \geq k,$$

because b is nonincreasing and $b_k > b_k - \varepsilon$. It follows that $f_a^*(k) \geq b_k$. On the other hand,

$$\mu_a(b_k) = |\{n \in \mathbb{N} : |a_n| > b_k\}| = |\{n \in \mathbb{N} : b_n > b_k\}| < k,$$

thus we obtain the equality $f_a^*(k) = b_k$.

Since μ_a acquires its values only in $\mathbb{N}$, $\mu_a(b_k) \leq k - 1$. For $k - 1 < t < k$ we have $\mu_a(b_k) \leq k - 1 < t < k = \mu_a(b_k - \varepsilon)$, so $b_k \geq f_a^*(t) \geq b_k - \varepsilon$. This completes the proof. □

For $p = q = \infty$ the situation is slightly different, since the sequences do not have to decay to zero. For example, for the sequence $\{1 - \frac{1}{n}\}_{n=1}^{\infty}$ we have $b = (1, 1, \dots)$.

We usually denote the sequence b from the last proposition by a^* and call it the nonincreasing rearrangement of a (when we use the original meaning, we denote the nonincreasing rearrangement f_a^* as in the statement above). For illustration see Figure 1.2.

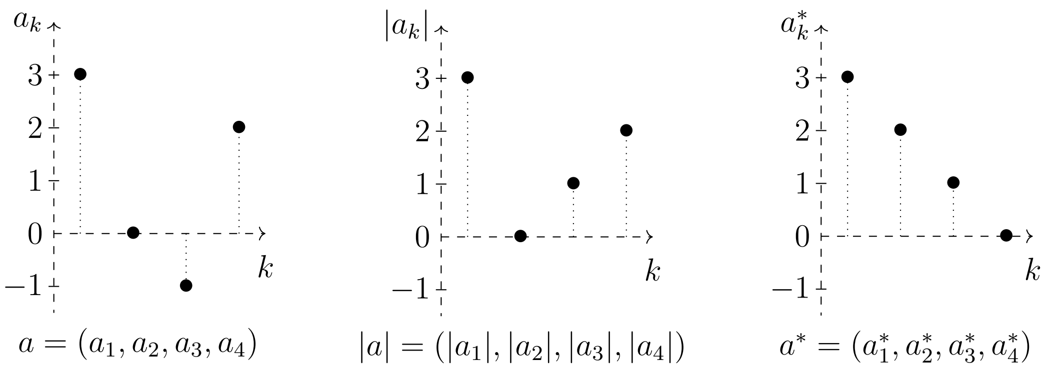


Figure 1.2: Main steps in forming a^* for $a \in \mathbb{R}^4$

Proposition 1.2.3. *Let* $a = (a_1, a_2, \dots) \in \ell^{p,q}$ *and* $a^* = (a_1^*, a_2^*, \dots)$ *be its nonincreasing rearrangement, then*

$$||a||_{p,q} = \begin{cases} \left(\dfrac{p}{q} \displaystyle\sum_{k \in \mathbb{N}} (a_k^*)^q (k^{\frac{q}{p}} - (k-1)^{\frac{q}{p}}) \right)^{\frac{1}{q}}, & q \in (0, \infty), \\ \sup\limits_{k \in \mathbb{N}} \{k^{\frac{1}{p}} a_k^*\}, & q = \infty. \end{cases}$$

For $0 < p, q < \infty$ *the original quasinorm* $|| \cdot ||_{p,q}$ *is equivalent to the function*

$$||a||'_{p,q} = ||k^{\frac{1}{p} - \frac{1}{q}} a_k^*||_q = \left(\sum_{k \in \mathbb{N}} (a_k^*)^q k^{\frac{q}{p} - 1} \right)^{\frac{1}{q}},$$

where $|| \cdot ||_q$ *is the (quasi)norm in* ℓ^q. *The function* $|| \cdot ||'_{p,q}$ *is also a quasinorm and we will use it as the canonical one from now on. By equivalence* $|| \cdot ||_{p,q} \approx || \cdot ||'_{p,q}$ *we mean that there are constants* c_1, c_2 *depending only on* p *and* q *such that*

$$c_1 || \cdot ||_{p,q} \leq || \cdot ||'_{p,q} \leq c_2 || \cdot ||_{p,q}.$$

Proof. The two equalities follow immediately from the definition of $||\cdot||_{p,q}$, as for $q<\infty$

$$\begin{aligned}||a||_{p,q}^q &= ||t^{\frac{1}{p}-\frac{1}{q}} f_a^*(t)||_{L^q(\mathbb{R}^+,\lambda)}^q = \int_0^\infty t^{\frac{q}{p}-1}(a_{\lceil t\rceil}^*)^q dt \\ &= \sum_{k=1}^\infty \int_{k-1}^k t^{\frac{q}{p}-1}(a_k^*)^q dt = \sum_{k=1}^\infty \frac{p}{q}(k^{\frac{q}{p}}-(k-1)^{\frac{q}{p}})(a_k^*)^q\end{aligned}$$

and for $q=\infty$

$$||a||_{p,q} = \sup_{0<t<\infty}\{t^{\frac{1}{p}} f_a^*(t)\} = \sup_{0<t<\infty}\{t^{\frac{1}{p}} a_{\lceil t\rceil}^*\} = \sup_{0<t<\infty}\{\lceil t\rceil^{\frac{1}{p}} a_{\lceil t\rceil}^*\} = \sup_{k\in\mathbb{N}}\{k^{\frac{1}{p}} a_k^*\}$$

(since $\sup\{\tau^{1/p}, \tau\in[t-1,t]\} = \lceil t\rceil^{\frac{1}{p}}$).

The equivalence of $||\cdot||_{p,q}$ and $||\cdot||'_{p,q}$ is obtained by using the mean value theorem. There exists $\xi\in(k-1,k)$ such that

$$k^{\frac{q}{p}}-(k-1)^{\frac{q}{p}} = \frac{q}{p}\xi^{\frac{q}{p}-1}.$$

For any $k>1$ we see that $k/2\le\xi\le k$, so $k^{\frac{q}{p}}-(k-1)^{\frac{q}{p}} \approx k^{\frac{q}{p}-1}$, where the constant is independent of k. For $k=1$ we have $k^{\frac{q}{p}} = k^{\frac{q}{p}-1}$ trivially, hence $||\cdot||_{p,q}\approx||\cdot||'_{p,q}$.

Let us denote K the constant of the quasinorm $||\cdot||_{p,q}$. Thanks to the equivalence, for every $a,b\in\ell^{p,q}$ we have

$$||a+b||'_{p,q} \le c_2||a+b||_{p,q} \le c_2K(||a||_{p,q}+||b||_{p,q}) \le c_2Kc_1(||a||'_{p,q}+||b||'_{p,q}),$$

so $||\cdot||'_{p,q}$ satisfies (iii)". Properties (i) and (ii) are satisfied trivially, hence $||\cdot||'_{p,q}$ is a quasinorm, too. □

Notation. *From now on, we use only the functional* $||\cdot||'_{p,q}$ *as the Lorentz quasinorm. Therefore we denote it again* $||\cdot||_{p,q}$*. For the case* $q=\infty$ *we still assume the quasinorm* $||a||_{p,\infty} = \sup\{k^{1/p}a_k^*\}$*.*

Remark. As we can see, the expression $||k^{1/p-1/q}a_k^*||_q$ makes sense even for q finite and $p=\infty$ (for $1/\infty=0$). However, it is not equivalent to the original Lorentz quasinorm. The corresponding spaces would be nontrivial; nevertheless, we will study them separately, as they actually behave differently, see Section 4.3.

The previous statement holds analogously for the finite-dimensional case $\ell_n^{p,q}$, since the space $\ell_n^{p,q}$ can be seen as a subspace of $\ell^{p,q}$. Therefore we can equip it with the quasinorm $||\cdot||'_{p,q}$ and we know that the constant of equivalence so not depend on the dimension n. Figures 1.3 and 1.4 show the unit balls in dimension two or three for some choice of the parameters p and q.

Notation. *By* $\mathbf{B}_n^{p,q}$ *we mean the unit ball in* $\ell_n^{p,q}$*. By* $\mathbf{B}_{n,+}^{p,q}$ *we mean those sequences from* $\mathbf{B}_n^{p,q}$ *such that all the coordinates are nonnegative. By a volume of a (measurable) subset of* $\mathbb{R}^n$ *we mean its* n*-dimensional Lebesgue measure.*

As we can choose the sign of each coordinate, which gives 2^n possibilities, it is easy to see that $\mathrm{Vol}(\mathbf{B}_n^{p,q}) = 2^n\mathrm{Vol}(\mathbf{B}_{n,+}^{p,q})$. For convenience we set $\mathrm{Vol}(\mathbf{B}_{0,+}^{p,q})=1$. Obviously, $\mathbf{B}_1^{p,q}=[-1,1]$, as the quasinorm in a one-dimensional space is always the absolute value.

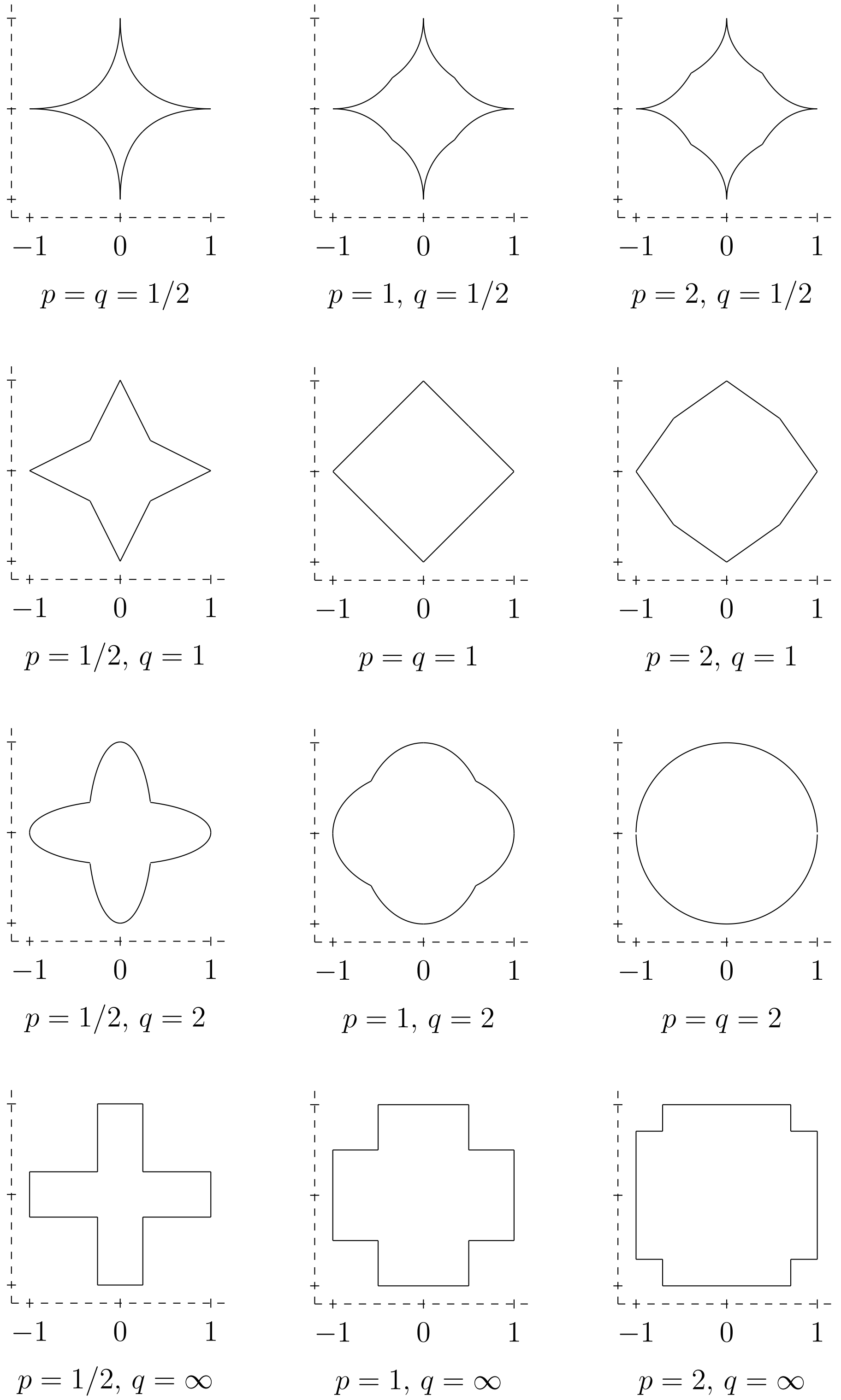


Figure 1.3: Comparison of unit balls in $\ell_2^{p,q}$ for different parameters

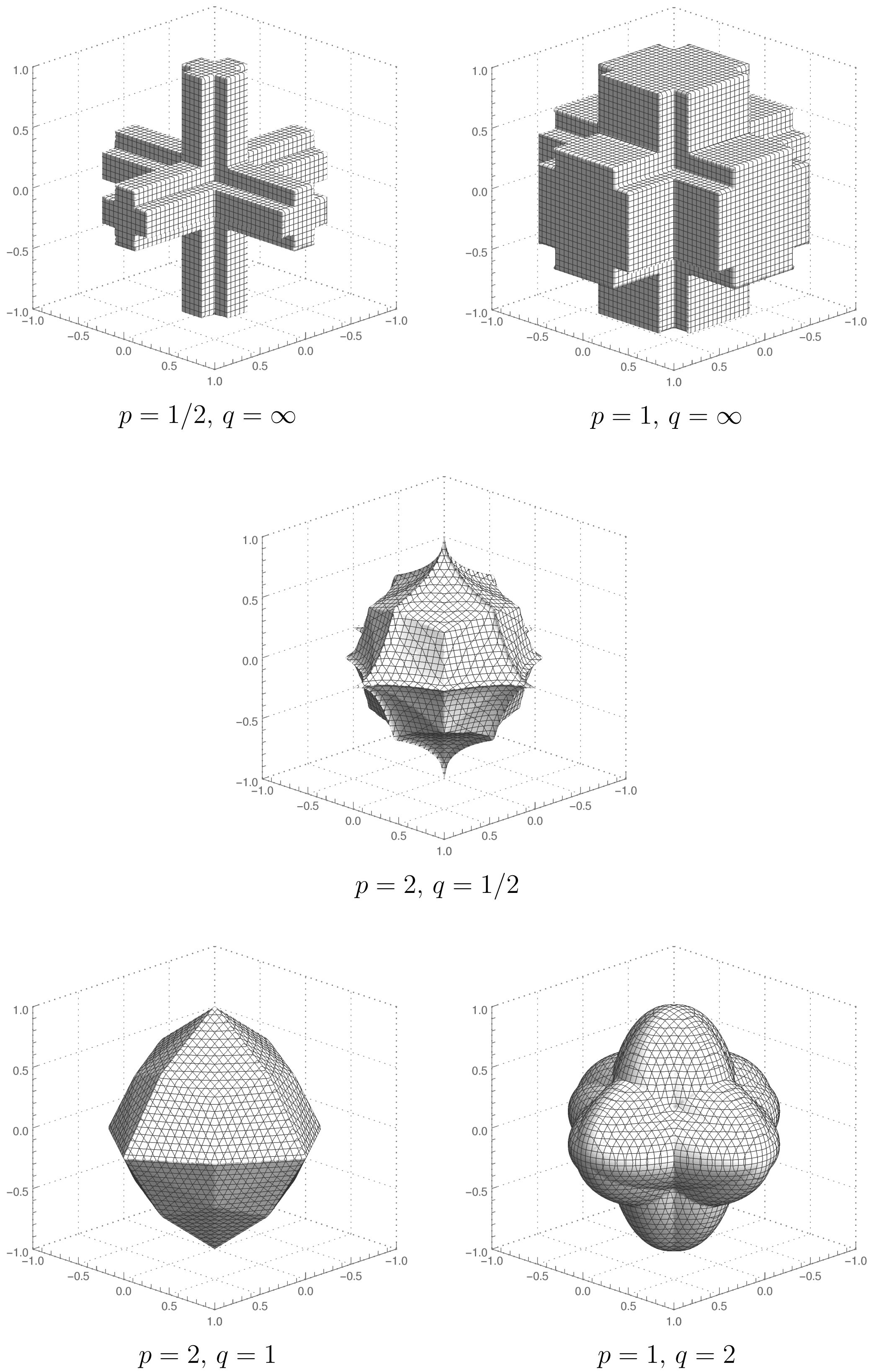


Figure 1.4: Comparison of unit balls in $\ell_3^{p,q}$ for different parameters

1.3 Entropy numbers

We introduce here the entropy numbers of an operator, since they can serve as an useful tool when dealing with asymptotic estimates of volumes of unit balls. We also study them (for the case of embeddings of the Lorentz spaces) separately in Section 4.2. For those who are interested in more details we recommend e.g. [6].

Definition. *Le X and Y be quasi-Banach spaces, $T : X \to Y$ be a bounded linear operator and $k \in \mathbb{N}$. Denote $\mathbf{B}_X$, $\mathbf{B}_Y$ the unit balls in these spaces. Then the* k-th entropy number *of T is defined as*

$$e_k(T : X \to Y) = \inf\Big\{r > 0 : \exists y_1, \dots, y_{2^{k-1}} \in Y : T(\mathbf{B}_X) \subseteq \bigcup_{i=1}^{2^{k-1}} (y_i + r\mathbf{B}_Y)\Big\}.$$

The entropy numbers are monotone in the sense that $e_k(T) \geq e_{k+1}(T)$. Consider the special case when T is the identity mapping between two finite-dimensional quasi-Banach spaces. The identity is bounded, since any linear mapping between finite-dimensional spaces is bounded, therefore the entropy numbers are well-defined. For the special case of the Lebesgue spaces, the following theorem holds. Its proof can be found in [11] and is therefore omitted.

Theorem 1.3.1. *Let p, $q \in (0, \infty]$, $n \in \mathbb{N}$.*

(i) *If $p \leq q$, then for all $k \in \mathbb{N}$ it holds that*

$$e_k(\mathrm{Id} : \ell_n^p \to \ell_n^q) \approx \begin{cases} 1, & 1 \leq k \leq \log_2 n, \\ \left(\dfrac{\log_2(1 + n/k)}{k}\right)^{\frac{1}{p} - \frac{1}{q}}, & \log_2 n \leq k \leq n, \\ 2^{-\frac{k-1}{n}} n^{\frac{1}{q} - \frac{1}{p}}, & k \geq n. \end{cases}$$

(ii) *If $q \leq p$, then for all $k \in \mathbb{N}$ it holds that*

$$e_k(\mathrm{Id} : \ell_n^p \to \ell_n^q) \approx 2^{-\frac{k-1}{n}} n^{\frac{1}{q} - \frac{1}{p}}.$$

The constants of the equivalences depend only on p and q, i.e., they are independent of k and n.

Now we present a modification of [6, Theorem 1.3.2], where the proof of the original statement can be found. By the Aoki-Rolewicz theorem, every quasinorm is equivalent to a p-norm for some $p \in (0, 1]$, on the other hand, every p-norm is a quasinorm with the constant $2^{1/p-1}$ (see e.g. [1]). Therefore the validity of the theorem does not change if we replace p-norms with quasinorms and add multiplicative constants on corresponding places. For further details on interpolation spaces, see [2] or [3].

Definition. *Let X_0, X_1 be quasi-Banach spaces, we say that (X_0, X_1) is an* interpolation couple*, if both of them are linearly continuously embedded in a common quasi-Banach space X. We endow the set $X_0 \cap X_1$ with the quasinorm* $\max\{||x||_{X_0}, ||x||_{X_1}\}$ *and the set $X_0 + X_1$ (i.e., the set $\{x : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}$) with the quasinorm* $\inf\{||x_0||_{X_0} + ||x_1||_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}$.

Theorem 1.3.2. (i) *Let X be a quasi-Banach space, (Y_0, Y_1) be an interpolation couple of quasi-Banach spaces and $\theta \in (0,1)$. Let Y_θ be a quasi-Banach space such that*

$$Y_0 \cap Y_1 \subseteq Y_\theta \subseteq Y_0 + Y_1$$

and there exists a constant C such that

$$||y||_\theta \leq C||y||_{Y_0}^{1-\theta}||y||_{Y_1}^{\theta}$$

for all $y \in Y_0 \cap Y_1$. Let $T : X \to Y_0 \cap Y_1$ be a linear bounded operator. Then there exists a constant C' such that for all $k, l \in \mathbb{N}$ it holds that

$$e_{k+l-1}(T : X \to Y_\theta) \leq C'(e_k(T : X \to Y_0))^{1-\theta}(e_l(T : X \to Y_1))^{\theta}.$$

(ii) *Let (X_0, X_1) be an interpolation couple of quasi-Banach spaces, Y a quasi-Banach space and $\theta \in (0,1)$. Let X be a quasi-Banach space such that $X \subseteq X_0 + X_1$ and there exists a constant C such that*

$$t^{-\theta}K(t,x) \leq C||x||_X$$

for all $x \in X$ and $t \in (0,\infty)$. Here $K(t,x)$ denotes the K-functional $K(t,x,X_0,X_1)$ defined as

$$K(t,x,X_0,X_1) = \inf\{||x_0||_{X_0} + t||x_1||_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}.$$

Let $T : X_0 + X_1 \to Y$ be a linear operator whose restrictions to X_0 and X_1 are bounded. Then $T : X \to Y$ is bounded, too, and there exists a constant C' such that for all $k, l \in \mathbb{N}$ it holds that

$$e_{k+l-1}(T : X \to Y) \leq C'(e_k(T : X_0 \to Y))^{1-\theta}(e_l(T : X_1 \to Y))^{\theta}.$$

Theorems 3.1.2 and 3.3.4 are two special cases which are proven separately. In Section 4.2 we use Theorem 1.3.2 in this general form.

2. Explicit formulae

We consider two special cases, $q = \infty$ and $q = 1$. We offer both recursive and explicit formulae in Sections 2.1 to 2.4 and use them to calculate the volumes for some choices of the parameter p in Section 2.5.

2.1 Recursive formula for $q = \infty$

In this section we focus on the volume of the unit ball of the space $\ell_n^{p,\infty}$. At first we deduce the recursive formula for the volume which we use later to prove the explicit formula. We apply the well-known inclusion-exclusion principle:

Theorem 2.1.1. (The inclusion–exclusion principle) *Let* $\mathbf{A}$ *be a measurable subset of* $\mathbb{R}^n$*,* $\mathbf{A}_k \subseteq \mathbf{A}$ *be measurable sets such that*

$$\mathbf{A} = \bigcup_{k=1}^{m} \mathbf{A}_k.$$

Then

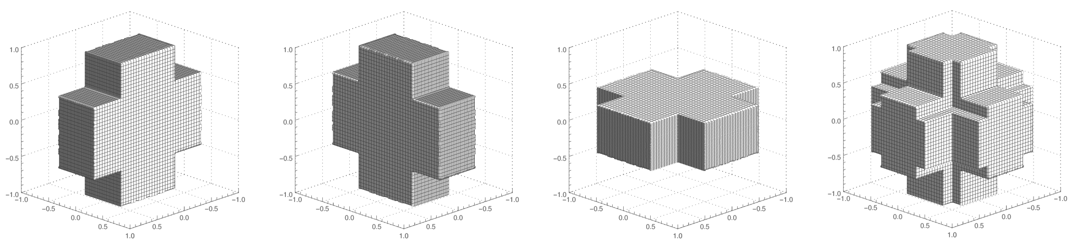
$$\mathrm{Vol}(\mathbf{A}) = \mathrm{Vol}\Big(\bigcup_{k=1}^{m} \mathbf{A}_k\Big) = \sum_{\substack{K\subseteq\{1,\dots,m\},\\ K\neq\emptyset}} (-1)^{|K|+1}\mathrm{Vol}\Big(\bigcap_{k\in K} \mathbf{A}_k\Big). \tag{2.1}$$

Let us denote

$$\mathbf{A}_k = \{a \in \mathbf{B}_n^{p,\infty} : a_k \le 1/n^{1/p}\},$$
$$\mathbf{A}_k^+ = \{a \in \mathbf{B}_{n,+}^{p,\infty} : a_k \le 1/n^{1/p}\}$$

for $k \in \{1,\dots,n\}$. For every $a \in \mathbf{B}_{n,+}^{p,\infty}$ there exists at least one coordinate, which is less or equal to $1/n^{1/p}$, otherwise the quasinorm of a would be greater than 1. Therefore, there exists $k \in \{1,\dots,n\}$ such that $a \in \mathbf{A}_k^+$ (see Figure 2.1). We obtain

$$\mathbf{B}_{n,+}^{p,\infty} = \bigcup_{k=1}^{n} \mathbf{A}_k^+.$$



$\mathbf{A}_1$ $\mathbf{A}_2$ $\mathbf{A}_3$ $\mathbf{B}_3^{1,\infty}$

Figure 2.1: Sets A_k for $n = 3$, $p = 1$ and their union $\mathbf{B}_3^{1,\infty}$

Consider $\emptyset \subsetneq K = \{k_1,\dots,k_j\} \subsetneq \{1,\dots,n\}$ and $K^C = \{1,\dots,n\} \setminus K = \{m_1,\dots,m_{n-j}\}$. We want to apply the inclusion-exclusion principle to $\mathbf{B}_{n,+}^{p,\infty}$. For that we need to know the volume of

$$\mathbf{A}_K^+ = \bigcap_{k\in K} \mathbf{A}_k^+.$$

For any sequence $a \in \mathbf{A}_K^+$ it holds that $a_{k_i} \leq 1/n^{1/p}$, $i = 1, \dots, j$. If

$$\tilde{a} = (a_{m_1}, \dots, a_{m_{n-j}})$$

is the restriction of a on K^C, then its l-th biggest coordinate can be at most $l^{-1/p}$. We get $\tilde{a} \in \mathbf{B}_{n-j,+}^{p,\infty}$. On the other hand, any sequence $\tilde{a} \in \mathbf{B}_{n-j,+}^{p,\infty}$ can be extended on K by $a_{k_i} \in [0, n^{-1/p}]$, $i = 1, \dots, j$ and the resulting sequence a is in $\mathbf{A}_K^+$. Therefore

$$\mathrm{Vol}\Big(\mathbf{A}_K^+\Big) = \mathrm{Vol}\Big([0, n^{-1/p}]^j \times \mathbf{B}_{n-j,+}^{p,\infty}\Big) = \left(\frac{1}{n}\right)^{\frac{j}{p}} \mathrm{Vol}(\mathbf{B}_{n-j,+}^{p,\infty}).$$

In the case $\{k_1, \dots, k_j\} = \{1, \dots, n\}$ there are no other coordinates left, i.e.,

$$\mathbf{A}_{\{1,\dots,n\}}^+ = \bigcap_{k=1}^{n} \mathbf{A}_k^+ = [0, n^{-1/p}]^n.$$

Since for each j there are exactly $\binom{n}{j}$ possible subsets of $\{1, \dots, n\}$ of cardinality j and the volume of $\mathbf{A}_K^+$ depends only on the number of elements of K, by (2.1) we obtain

$$\mathrm{Vol}(\mathbf{B}_{n,+}^{p,\infty}) = \sum_{j=1}^{n} \left[\binom{n}{j}(-1)^{j+1}\left(\frac{1}{n}\right)^{\frac{j}{p}} \mathrm{Vol}(\mathbf{B}_{n-j,+}^{p,\infty})\right]. \tag{2.2}$$

2.2 Explicit formula for $q = \infty$

Now we want to omit the volumes of the lower-dimensional balls in (2.2). We present two approaches to this problem in the following two sections. Although the result is the same, there are differences which make each of them more useful under certain circumstances.

Theorem 2.2.1. *The volume of the unit ball in $\ell_n^{1,\infty}$ is given by the formula*

$$\begin{aligned}
\mathrm{Vol}(\mathbf{B}_{n,+}^{1,\infty}) &= \sum_{k \in \mathbf{K}_n} \prod_{l=1}^{j} \binom{n - \sum_{i=1}^{l-1} k_i}{k_l} \frac{(-1)^{k_l+1}}{(n - \sum_{i=1}^{l-1} k_i)^{k_l}} \\
&= n! \sum_{k \in \mathbf{K}_n} (-1)^{n+j} \prod_{l=1}^{j} \frac{1}{(n - \sum_{i=1}^{l-1} k_i)^{k_l} (k_l)!} \\
&= \sum_{m \in \mathbf{M}_n} \prod_{l=0}^{j-1} \binom{n - m_l}{m_{l+1} - m_l} \frac{(-1)^{m_{l+1} - m_l + 1}}{(n - m_l)^{m_{l+1} - m_l}},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{K}_n &= \{k = (k_1, \dots, k_j) : k_i \in \mathbb{N}, \sum_{i=1}^{j} k_i = n\}, \\
\mathbf{M}_n &= \{m = (m_0, \dots, m_j) \subseteq \{0, \dots, n\} : 0 = m_0 \leq m_1 \leq \dots \leq m_j = n\}
\end{aligned}$$

and the sum over an empty set is zero.

Proof. Firstly we prove that the first and the third formulae are equal. We identify each element of $\mathbf{K}_n$ with an element of $\mathbf{M}_n$ this way:

$$(k_1, \ldots, k_j) \mapsto (0, k_1, k_1 + k_2, \ldots, \sum_{i=1}^{j} k_i), \text{ i.e., } m_l = \sum_{i=1}^{l} k_i.$$

Clearly this mapping is both injective and surjective. By substituting for m_l in the third formula we obtain the first one. The second equality follows easily from the definition of the binomial coefficient and from $\prod_{l=1}^{j}(-1)^{k_l+1} = (-1)^{n+j}$, since $\sum_{l=1}^{j} k_l = n$.

Now let us prove that the volume is given by the first formula. We proceed by induction. For the case $n = 1$ the expressions are equal, as the only element of $\mathbf{K}_1$ is the sequence (1) and $\mathbf{B}_{1,+}^{1,\infty} = [0, 1]$.

Assume now that the formula holds for all $n < n_0$. Furthermore we put $\mathbf{K}_0 = \emptyset$. By (2.2) we know that

$$\mathrm{Vol}(\mathbf{B}_{n_0,+}^{1,\infty}) = \sum_{l=1}^{n_0} \left[\binom{n_0}{l} (-1)^{l+1} \left(\frac{1}{n_0}\right)^l \mathrm{Vol}\left(\mathbf{B}_{n_0-l,+}^{1,\infty}\right) \right].$$

By using the induction hypothesis we obtain

$$\begin{aligned}\mathrm{Vol}(\mathbf{B}_{n_0,+}^{1,\infty}) &= \sum_{l=1}^{n_0} \left[\binom{n_0}{l} \frac{(-1)^{l+1}}{n_0^l} \sum_{k\in\mathbf{K}_{n_0-l}} \prod_{l=1}^{j} \binom{n_0 - l - \sum_{i=1}^{l-1} k_i}{k_l} \frac{(-1)^{k_l+1}}{(n_0 - l - \sum_{i=1}^{l-1} k_i)^{k_l}} \right] \\ &= \sum_{l=1}^{n_0} \left[\sum_{k\in\mathbf{K}_{n_0-l}} \binom{n_0}{l} \frac{(-1)^{l+1}}{n_0^l} \prod_{l=1}^{j} \binom{n_0 - l - \sum_{i=1}^{l-1} k_i}{k_l} \frac{(-1)^{k_l+1}}{(n_0 - l - \sum_{i=1}^{l-1} k_i)^{k_l}} \right].\end{aligned}$$

Now take $k \in \mathbf{K}_{n_0-l}$ and define $h = (l, k_1, \ldots, k_j)$ (in the case $l = n_0$ define $h = (n_0)$). This is an element of $\mathbf{K}_{n_0}$ and each element of this set can be uniquely produced this way for some $l \leq n_0$. Therefore

$$\mathrm{Vol}(\mathbf{B}_{n_0,+}^{1,\infty}) = \sum_{h\in\mathbf{K}_{n_0}} \left[\prod_{l=1}^{j} \binom{n_0 - \sum_{i=1}^{l-1} h_i}{h_l} \frac{(-1)^{h_l+1}}{(n_0 - \sum_{i=1}^{l-1} h_i)^{h_l}} \right],$$

which completes the proof. □

This approach can be generalised for arbitrary $p \in (0, \infty)$.

Theorem 2.2.2. *Let $p \in (0, \infty)$, then the volume of the unit ball $\mathbf{B}_{n,+}^{p,\infty}$ is given by the formula*

$$\begin{aligned}\mathrm{Vol}(\mathbf{B}_{n_0,+}^{p,\infty}) &= \sum_{k\in\mathbf{K}_n} \prod_{l=1}^{j} \binom{n - \sum_{i=1}^{l-1} k_i}{k_l} \frac{(-1)^{k_l+1}}{(n - \sum_{i=1}^{l-1} k_i)^{\frac{k_l}{p}}} \\ &= \sum_{m\in\mathbf{M}_n} \prod_{l=1}^{j} \binom{n - m_{l-1}}{m_l - m_{l-1}} \frac{(-1)^{m_l - m_{l-1}+1}}{(n - m_{l-1})^{\frac{m_l - m_{l-1}}{p}}},\end{aligned}$$

where $\mathbf{K}_n$ and $\mathbf{M}_n$ are as before.

Proof. The proof goes exactly the same way with the only difference being the power $1/p$. Therefore we show here just the most important part, the induction step for $n_0 > 1$.

$$\begin{aligned}
\mathrm{Vol}(\mathbf{B}^{p,\infty}_{n_0,+}) &= \sum_{l=1}^{n_0} \left[\binom{n_0}{l} (-1)^{l+1} \left(\frac{1}{n_0}\right)^{l/p} \mathrm{Vol}(\mathbf{B}^{p,\infty}_{n_0-l,+}) \right] \\
&= \sum_{l=1}^{n_0} \left[\sum_{k \in \mathbf{K}_{n_0-l}} \binom{n_0}{l} \frac{(-1)^{l+1}}{n_0^{l/p}} \prod_{l=1}^{j} \binom{n_0 - l - \sum_{i=1}^{l-1} k_i}{k_l} \frac{(-1)^{k_l+1}}{(n_0 - l - \sum_{i=1}^{l-1} k_i)^{k_l/p}} \right] \\
&= \sum_{h \in \mathbf{K}_{n_0}} \left[\prod_{l=1}^{j} \binom{n_0 - \sum_{i=1}^{l-1} h_i}{h_l} \frac{(-1)^{h_l+1}}{(n_0 - \sum_{i=1}^{l-1} h_i)^{h_l/p}} \right].
\end{aligned}$$

□

Remark. As the inclusion-exclusion principle works also for $p = \infty$ (in which case $A_k = \mathbf{B}^{\infty}_n$), we can use the fact that $\mathrm{Vol}(\mathbf{B}^{p,\infty}_{n,+}) = 1$ to obtain a combinatorial identity

$$\sum_{k \in \mathbf{K}_n} (-1)^{n+j} \binom{n}{k_1, \ldots, k_j} = n! \sum_{k \in \mathbf{K}_n} (-1)^{n+j} \prod_{l=1}^{j} \frac{1}{(k_l)!} = 1.$$

In contrast with the multinomial theorem, we allow only $k_l > 0$.

2.3 Integral approach

In this section we determine the volume of $\mathbf{B}^{p,\infty}_n$ by using integration.

Definition. *Let $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $a \in \mathbb{R}^n$ such that $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. We denote*

$$\mathbf{V}^{(m)}(n, a) = \int_0^{a_n} \int_{x_n}^{a_{n-1}} \cdots \int_{x_2}^{a_1} x_1^m dx_1 \ldots dx_{n-1} dx_n.$$

Furthermore, we set $\mathbf{V}^{(m)}(0, ()) = 1$.

When we set $\tilde{a}_k = k^{-\frac{1}{p}}$, the domain of the integration is a subset of $\mathbf{B}^{p,\infty}_{n,+}$. Moreover, when we consider all possible permutations of the n coordinates, these sets cover $\mathbf{B}^{p,q}_{n,+}$ and their intersections are sets of zero (n-dimensional Lebesgue) measure. Therefore an integral over the ball can be written as a sum of integrals over these sections. Since there are $n!$ such permutations (and values of all corresponding integrals are equal), we obtain that

$$\mathrm{Vol}(\mathbf{B}^{p,\infty}_{n,+}) = n! \int_0^{n^{-\frac{1}{p}}} \int_{x_n}^{(n-1)^{-\frac{1}{p}}} \cdots \int_{x_2}^{1} 1 dx_1 \ldots dx_{n-1} dx_n = n! \mathbf{V}^{(0)}(n, \tilde{a}). \tag{2.3}$$

Theorem 2.3.1. *Let $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $a \in \mathbb{R}^n$ such that $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. Then*

$$\mathbf{V}^{(m)}(n, a) = \sum_{i=1}^{n} (-1)^{i+1} \frac{a_i^{m+i} m!}{(m+i)!} \mathbf{V}^{(0)}(n - i, (a_{i+1}, \ldots, a_n)).$$

Proof. We prove the assertion by induction for n. If $n = 1$, then we have $\mathbf{V}^{(m)}(1,(a_1)) = \frac{a_1^{m+1}}{m+1}$. Assume that m and a are now arbitrary admissible and $n > 1$. We use the integration by parts:

$$\begin{aligned}
\mathbf{V}^{(m)}(n,a) &= \int_0^{a_n}\int_{x_n}^{a_{n-1}}\cdots\int_{x_2}^{a_1} x_1^m dx_1\dots dx_{n-1}dx_n \\
&= \int_0^{a_n}\int_{x_n}^{a_{n-1}}\cdots\int_{x_3}^{a_2}\left(\frac{a_1^{m+1}}{m+1}-\frac{x_2^{m+1}}{m+1}\right)dx_2\dots dx_{n-1}dx_n \\
&= \frac{a_1^{m+1}}{m+1}\mathbf{V}^{(0)}(n-1,(a_2,\dots,a_n)) - \frac{1}{m+1}\mathbf{V}^{(m+1)}(n-1,(a_2,\dots,a_n)).
\end{aligned}$$

Let the theorem hold for all $k < n$ and denote

$$\begin{aligned}
&a[l,i] = (a_l,\dots,a_i) \text{ for all } l,i : 1 \le l \le i \le n, \\
&a[l,i] = () \text{ otherwise.}
\end{aligned}$$

By the induction hypothesis we have

$$\begin{aligned}
\mathbf{V}^{(m)}(n,a) &= \frac{a_1^{m+1}}{m+1}\mathbf{V}^{(0)}(n-1,a[2,n]) - \frac{1}{m+1}\mathbf{V}^{(m+1)}(n-1,a[2,n]) \\
&= \frac{a_1^{m+1}}{m+1}\mathbf{V}^{(0)}(n-1,a[2,n]) \\
&\quad - \frac{1}{m+1}\left(\sum_{i=1}^{n-1}(-1)^{i+1}\frac{a_{i+1}^{m+i+1}(m+1)!}{(m+i+1)!}\mathbf{V}^{(0)}(n-i-1,a[i+2,n])\right).
\end{aligned}$$

By putting $i + 1 = j$ we obtain

$$\begin{aligned}
\mathbf{V}^{(m)}(n,a) &= \frac{a_1^{m+1}}{m+1}\mathbf{V}^{(0)}(n-1,a[2,n]) \\
&\quad - \frac{1}{m+1}\sum_{j=2}^{n}(-1)^{j}\frac{a_j^{m+j}(m+1)!}{(m+j)!}\mathbf{V}^{(0)}(n-j,a[j+1,n]) \\
&= \sum_{j=1}^{n}(-1)^{j+1}\frac{a_j^{m+j}m!}{(m+j)!}\mathbf{V}^{(0)}(n-j,a[j+1,n]),
\end{aligned}$$

which is the sought formula. □

As we want to proceed from (2.3), we set $m = 0$. Then

$$\mathbf{V}^{(0)}(n,a) = \sum_{i=1}^{n}(-1)^{i+1}\frac{a_i^i}{i!}\mathbf{V}^{(0)}(n-i,a[i+1,n]).$$

It is easy to show by induction that

$$\mathbf{V}^{(0)}(n,a) = \sum_{m\in\mathbf{M}_n}(-1)^{n+j}\prod_{l=0}^{j-1}\frac{a_{n-m_l}^{m_{l+1}-m_l}}{(m_{l+1}-m_l)!}.$$

For $n = 1$ there is nothing to prove ($\mathbf{V}^{(0)}(1, a) = a_1$) and for $n > 1$ we proceed with the recursive formula, assuming the assertion holds for $k < n$:

$$\begin{aligned}\mathbf{V}^{(0)}(n, a) &= \sum_{i=1}^{n} (-1)^{i+1} \frac{a_i^i}{i!} \mathbf{V}^{(0)}(n - i, a[i + 1, n]) \\ &= \sum_{i=1}^{n} (-1)^{i+1} \frac{a_i^i}{i!} \sum_{m \in \mathbf{M}_{n-i}} (-1)^{n-i+j} \prod_{l=0}^{j-1} \frac{a_{n-i-m_l+i}^{m_{l+1}-m_l}}{(m_{l+1} - m_l)!} \\ &= \sum_{i=1}^{n} \sum_{m \in \mathbf{M}_{n-i}} (-1)^{n+j+1} \frac{a_i^i}{i!} \prod_{l=0}^{j-1} \frac{a_{n-m_l}^{m_{l+1}-m_l}}{(m_{l+1} - m_l)!}.\end{aligned}$$

Now denote $h = (m_0, \ldots, m_j, n)$ for each $(m_0, \ldots, m_j) \in \mathbf{M}_{n-i}$ (in the case $i = n$ define $h = (0, n)$), we obtain each element of $\mathbf{M}_n$ exactly once. Therefore,

$$\begin{aligned}\mathbf{V}^{(0)}(n, a) &= \sum_{i=1}^{n} \sum_{m \in \mathbf{M}_{n-i}} (-1)^{n+j+1} \frac{a_i^i}{i!} \prod_{l=0}^{j-1} \frac{a_{n-m_l}^{m_{l+1}-m_l}}{(m_{l+1} - m_l)!} \\ &= \sum_{h \in \mathbf{M}_n} (-1)^{n+j+1} \prod_{l=0}^{j} \frac{a_{n-h_l}^{h_{l+1}-h_l}}{(h_{l+1} - h_l)!} \\ &= \sum_{m \in \mathbf{M}_n} (-1)^{n+j} \prod_{l=0}^{j-1} \frac{a_{n-m_l}^{m_{l+1}-m_l}}{(m_{l+1} - m_l)!}.\end{aligned}$$

For $a = \tilde{a}$ and after multiplication by $n!$ we obtain the same expression as in Theorem 2.2.2.

Remark. This approach gives us a formula for the weighted weak Lebesgue space (i.e., the weights are not $k^{1/p}$, but some arbitrary positive numbers). The same formula can be also obtained by a slight modification of the first approach. Denote

$$\mathbf{B}_{n,+}^{\infty}(a) = \{x \in \mathbb{R}^n : x_k \geq 0, a_k x_k^* \leq 1\}$$

for $a \in \mathbb{R}^n$ such that $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. For $a_k = k^{1/p}$ we obtain $\mathbf{B}_{n,+}^{p,\infty}$. The recursive formula works the same way as before, since

$$\mathrm{Vol}(\mathbf{A}_K^+) = \mathrm{Vol}(\{x \in \mathbb{R}_+^n : x_k \leq a_n, k \in K\}) = (a_n)^{|K|} \mathrm{Vol}(\mathbf{B}_{n-|K|,+}^{\infty}(a)).$$

We get

$$\mathrm{Vol}(\mathbf{B}_{n,+}^{\infty}(a)) = \sum_{j=1}^{n} \left[\binom{n}{j} (-1)^{j+1} (a_n)^j \mathrm{Vol}(\mathbf{B}_{n-j,+}^{\infty}(a[1, n-j])) \right].$$

The proof of the explicit formula works analogously (considering $p = 1$) and we get

$$\begin{aligned}\mathrm{Vol}(\mathbf{B}_{n,+}^{\infty}(a)) &= \sum_{m \in \mathbf{M_n}} \prod_{l=0}^{j-1} \binom{n - m_l}{m_{l+1} - m_l} (-1)^{m_{l+1}-m_l+1} a_{n-m_l}^{m_{l+1}-m_l} \\ &= \sum_{m \in \mathbf{M_n}} (-1)^{n+j} \prod_{l=0}^{j-1} \binom{n - m_l}{m_{l+1} - m_l} a_{n-m_l}^{m_{l+1}-m_l} \\ &= \sum_{m \in \mathbf{M_n}} (-1)^{n+j} \prod_{l=0}^{j-1} \frac{a_{n-m_l}^{m_{l+1}-m_l}}{(m_{l+1} - m_l)!}.\end{aligned}$$

2.4 Explicit formula for $q = 1$

In this section we consider another special case, $q = 1$. As the approaches from the previous sections fail in this case, we use a technique similar to the one used when calculating the volume of the unit ball in the classical ℓ_n^p space.

Let us consider a smooth function $f : [0,\infty) \to [0,\infty)$ with a rapid decay (to zero) at infinity. For such function it holds by the fundamental theorem of calculus and the Fubini theorem that

$$\begin{aligned}\int_{\mathbb{R}^n} f(||x||_{p,1})dx &= \int_{\mathbb{R}^n}\int_{||x||_{p,1}}^{\infty} -f'(t)dtdx = -\int_0^{\infty} \int\limits_{\{x\in\mathbb{R}^n:||x||_{p,1}<t\}} f'(t)dxdt \\ &= -\int_0^{\infty} f'(t) \int\limits_{\{x\in\mathbb{R}^n:||x||_{p,1}<t\}} 1dxdt \\ &= -\int_0^{\infty} f'(t)\mathrm{Vol}(\{x\in\mathbb{R}^n : ||x||_{p,1} < t\})dt \\ &= -\mathrm{Vol}(\mathbf{B}_n^{p,1})\int_0^{\infty} t^n f'(t)dt\end{aligned} \tag{2.4}$$

since all assumptions of used theorems easily hold by the smoothness and decay of f. By choosing $f(t) = e^{-t}$ we obtain

$$\int_{\mathbb{R}^n} e^{-||x||_{p,1}}dx = -\mathrm{Vol}(\mathbf{B}_n^{p,1})\int_0^{\infty} -t^n e^{-t}dt = \mathrm{Vol}(\mathbf{B}_n^{p,1})\Gamma(n+1) = n!\mathrm{Vol}(\mathbf{B}_n^{p,1}).$$

That leads us to the following statement:

Theorem 2.4.1. *Let $p \in (0,\infty)$, $n \in \mathbb{N}$, then the volume of the unit ball $\mathbf{B}_n^{p,1}$ is given by the formula*

$$\mathrm{Vol}(\mathbf{B}_n^{p,1}) = 2^n \prod_{k=1}^{n} \frac{1}{\varkappa_p(k)}, \tag{2.5}$$

where

$$\varkappa_p(k) = \sum_{j=1}^{k} j^{\frac{1}{p}-1}.$$

Proof. All that remains is to compute the left-hand side of (2.4), i.e., $\int_{\mathbb{R}^n} e^{-||x||_{p,1}}dx$. Let us denote

$$\mathcal{C}_n(t) = \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \cdots \geq x_n \geq t\}$$

for t positive and

$$A(n,p,t) = \int_{\mathcal{C}_n(t)} \exp(-\sum_{k=1}^{n} k^{1/p-1}x_k)dx.$$

Since $||x||_{p,1} = \sum_{k=1}^n k^{1/p-1}x_k^*$, we obtain

$$\int_{\mathbb{R}^n} e^{-||x||_{p,1}}dx = \int_{\mathbb{R}^n} \exp(-\sum_{k=1}^{n} k^{1/p-1}x_k^*)dx.$$

The set $\mathbb{R}_+^n$ is covered by sections $\mathcal{C}_n(0)$ for all possible permutations of coordinates and their intersections are sets of measure zero. Therefore

$$\int_{\mathbb{R}^n} \exp(-\sum_{k=1}^{n} k^{1/p-1} x_k^*) dx = 2^n n! \int_{\mathcal{C}_n(0)} \exp(-\sum_{k=1}^{n} k^{1/p-1} x_k) dx = 2^n n! A(n,p,0).$$

We observe that $A(1,p,t) = \int_t^\infty e^{-y} dy = e^{-t}$ and that

$$\begin{aligned} A(n,p,t) &= \int_t^\infty \int_{x_n}^\infty \cdots \int_{x_2}^\infty \exp(-\sum_{k=1}^{n} k^{1/p-1} x_k) dx_1 \dots dx_{n-1} dx_n \\ &= \int_t^\infty e^{-n^{1/p-1}x_n} \int_{x_n}^\infty e^{-(n-1)^{1/p-1}x_{n-1}} \cdots \int_{x_2}^\infty e^{-x_1} dx_1 \dots dx_{n-1} dx_n \\ &= \int_t^\infty e^{-n^{1/p-1}x_n} A(n-1,p,x_n) dx_n. \end{aligned}$$

It can be now easily proved by induction that

$$A(n,p,t) = e^{-t\varkappa_p(n)} \prod_{k=1}^{n} \frac{1}{\varkappa_p(k)}, \text{ where } \varkappa_p(k) = \sum_{j=1}^{k} j^{1/p-1},$$

as $\varkappa_p(1) = 1$ and

$$\int_t^\infty e^{-(n^{1/p-1}+\varkappa_p(n-1))x_n} dx_n = \frac{1}{\varkappa_p(n)} e^{-t\varkappa_p(n)}.$$

We finish by

$$\mathrm{Vol}(\mathbf{B}_n^{p,1}) = \frac{1}{n!} \int_{\mathbb{R}^n} e^{-||x||_{p,1}} dx = 2^n \prod_{k=1}^{n} \frac{1}{\varkappa_p(k)}.$$

□

Remark. For $p = 1$ we obtain $\prod_1^n \varkappa_1(k) = n!$, therefore we can look at this expression as a generalisation of the factorial.

The formula (2.5) can be easily rewritten in a recursive manner. This is used in the next section.

2.5 Numerics

In this section we use the formulae to actually calculate the volumes for some specific choices of p, namely 1/2, 1, 2 and 100. These values were chosen to illustrate the behaviour both when p and q are close and far from each other. First we present table of the values and compare the behaviour of the volumes of $\mathrm{Vol}(\mathbf{B}_{n,+}^{p,\infty})$ and $\mathrm{Vol}(\mathbf{B}_n^{p,\infty})$, then we do the same for the case $q = 1$. Note that the precision of the results is limited by the means of storage of numbers in computer memory. In the following two tables we use a precision satisfying our purpose of illustrating the behaviour of the volumes. We use the recursive formulae

$$\begin{aligned} \mathrm{Vol}(\mathbf{B}_{n,+}^{p,\infty}) &= \sum_{j=1}^{n} \left[\binom{n}{j} (-1)^{j+1} \left(\frac{1}{n}\right)^{\frac{j}{p}} \mathrm{Vol}(\mathbf{B}_{n-j,+}^{p,\infty}) \right], \\ \mathrm{Vol}(\mathbf{B}_{n,+}^{p,1}) &= \frac{1}{\varkappa_p(n)} \mathrm{Vol}(\mathbf{B}_{n-1,+}^{p,1}), \end{aligned}$$

with initial conditions $\mathrm{Vol}(\mathbf{B}_{0,+}^{p,\infty}) = 1$, $\mathrm{Vol}(\mathbf{B}_{1,+}^{p,1}) = 1$ and $\varkappa_p(n)$ defined as before (again, which can be written in a recursive manner to speed up the calculations). The time complexity is quadratic or linear, respectively, in n, which allows us to compute the volumes for high dimensions with sufficient precision in reasonable time.

n	$p = 1/2$	$p = 1$	$p = 2$	$p = 100$
1	2	2	2	2
2	1.75	3	3.657	4
3	0.881	3.63	6.207	7.999
4	0.292	3.697	9.888	15.995
5	$6.894 \cdot 10^{-2}$	3.26	14.901	31.985
6	$1.224 \cdot 10^{-2}$	2.541	21.376	63.955
7	$1.699 \cdot 10^{-3}$	1.776	29.333	127.873
8	$1.898 \cdot 10^{-4}$	1.126	38.659	255.662
9	$1.746 \cdot 10^{-5}$	0.654	49.1	511.132
10	$1.347 \cdot 10^{-6}$	0.351	60.262	1,021.834
11	$8.846 \cdot 10^{-8}$	0.175	71.648	2,042.716
12	$5.011 \cdot 10^{-9}$	$8.138 \cdot 10^{-2}$	82.691	4,083.343
13	$2.475 \cdot 10^{-10}$	$3.555 \cdot 10^{-2}$	92.81	8,162.143
14	$1.076 \cdot 10^{-11}$	$1.464 \cdot 10^{-2}$	101.467	16,314.474
15	$4.148 \cdot 10^{-13}$	$5.698 \cdot 10^{-3}$	108.207	32,607.879

Table 2.1: $\mathrm{Vol}(\mathbf{B}_n^{p,\infty})$ for $p = 1/2; 1; 2; 100$ for dimension up to 15

The quantity $\mathrm{Vol}(\mathbf{B}_{n,+}^{100,\infty})$ seems to be almost constant in Figure 2.2. This is due to the fact that p is rather large and therefore this volume for small dimensions is behaving more like $\mathrm{Vol}(\mathbf{B}_{n,+}^{\infty})$, which is constantly 1. However, there is a decay, $\mathrm{Vol}(\mathbf{B}_{n,+}^{100,\infty})$ tends to zero, but the decay is much slower (the quantity gets below 0.9 firstly for dimension 71 and below 0.5 for dimension 195). Therefore we omitted the case $p = 100$ in the second plot, as the growth for the first twenty dimensions is almost exponential (and thus much larger than the other data), see Table 2.1.

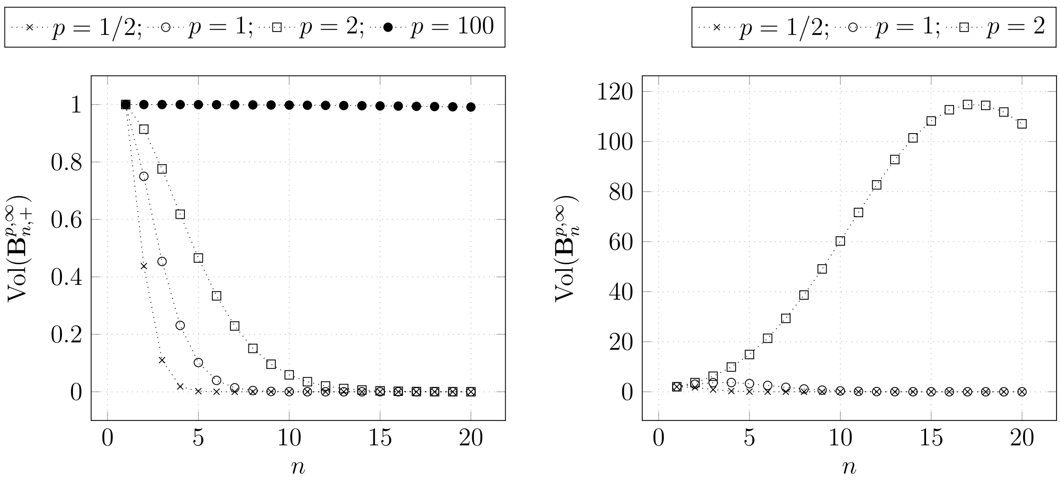


Figure 2.2: Volumes for $p = 1/2; 1; 2; 100$ and $q = \infty$ for dimension up to 20

As can be seen, the multiplicative factor 2^n changes the decreasing sequence of volumes into one which can have a peak in some higher dimension. (It is well known that e.g. for $\mathbf{B}_n^2$ the maximal volume is obtained at $n = 5$. For $\mathbf{B}_n^{2,\infty}$ the maximum is obtained at $n = 17$ and for $\mathbf{B}_n^{2,1}$ at $n = 2$.)

n	$p = 1/2$	$p = 1$	$p = 2$	$p = 100$
1	2	2	2	2
2	1.333	2	2.343	2.66
3	0.444	1.333	2.051	2.891
4	$8.889 \cdot 10^{-2}$	0.667	1.473	2.761
5	$1.185 \cdot 10^{-2}$	0.267	0.912	2.404
6	$1.129 \cdot 10^{-3}$	$8.889 \cdot 10^{-2}$	0.501	1.949
7	$8.062 \cdot 10^{-5}$	$2.54 \cdot 10^{-2}$	0.249	1.492
8	$4.479 \cdot 10^{-6}$	$6.349 \cdot 10^{-3}$	0.114	1.089
9	$1.991 \cdot 10^{-7}$	$1.411 \cdot 10^{-3}$	$4.851 \cdot 10^{-2}$	0.763
10	$7.239 \cdot 10^{-9}$	$2.822 \cdot 10^{-4}$	$1.932 \cdot 10^{-2}$	0.516
11	$2.194 \cdot 10^{-10}$	$5.131 \cdot 10^{-5}$	$7.26 \cdot 10^{-3}$	0.339
12	$5.625 \cdot 10^{-12}$	$8.551 \cdot 10^{-6}$	$2.588 \cdot 10^{-3}$	0.216
13	$1.236 \cdot 10^{-13}$	$1.316 \cdot 10^{-6}$	$8.789 \cdot 10^{-4}$	0.134
14	$2.355 \cdot 10^{-15}$	$1.879 \cdot 10^{-7}$	$2.856 \cdot 10^{-4}$	$8.183 \cdot 10^{-2}$
15	$3.924 \cdot 10^{-17}$	$2.506 \cdot 10^{-8}$	$8.904 \cdot 10^{-5}$	$4.877 \cdot 10^{-2}$

Table 2.2: $\mathrm{Vol}(\mathbf{B}_n^{p,1})$ for $p = 1/2; 1; 2; 100$ for dimension up to 15

In the case $q = 1$ the decay of $\mathrm{Vol}(\mathbf{B}_{n,+}^{p,1})$ is rapid even for p large (cf. Table 2.2 and Figure 2.3). For example, for $p = 100$ (even for $p = 1000$) the quantity is below 0.1 for dimension 5. This is due to the fact that the limiting case for $p \to \infty$ is not $\mathbf{B}_n^{\infty}$ as it was before, but $\mathbf{B}_n^{\infty,1}$ (see Section 4.3).

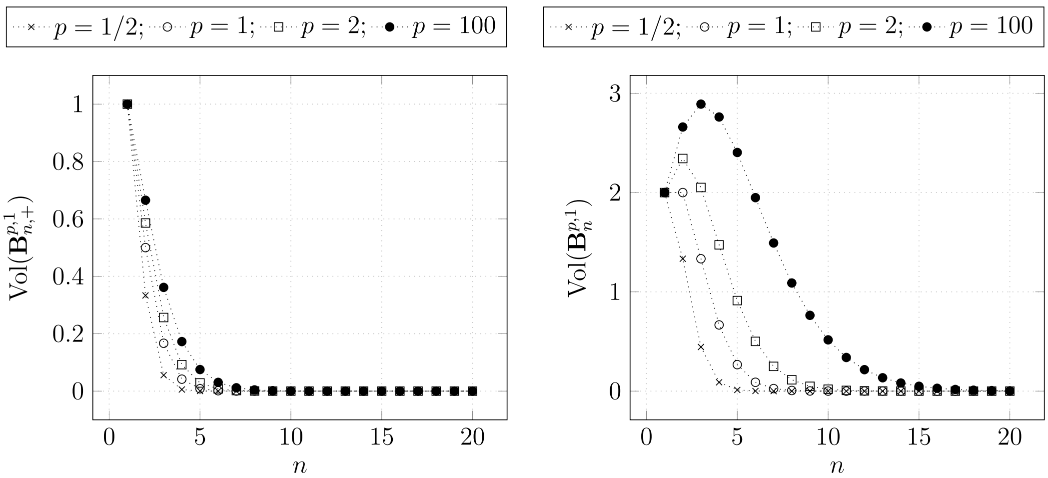


Figure 2.3: Volumes for $p = 1/2; 1; 2; 100$ and $q = 1$ for dimension up to 20

3. Asymptotic results

In this chapter we study two quantities – we investigate the asymptotic behaviour of the term $\sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{p,q})}$ when n approaches ∞, and the ratio of the volume of the unit ball in the n-dimensional weak Lebesgue space to the volume of the unit ball in the corresponding n-dimensional Lebesgue space. We use embeddings as well as the interpolation theory.

3.1 Asymptotic estimate for $q = \infty$

Firstly consider the case $q = \infty$. It is well-known that for the space ℓ_n^p the volume of the unit ball is given by the formula

$$\mathrm{Vol}(\mathbf{B}_n^p) = \frac{2^n \Gamma(1+1/p)^n}{\Gamma(1+n/p)},$$

where Γ is the gamma function defined as $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ for $x > 0$ (see [16]). The following estimate is one of the many forms of the Stirling formula and follows from [18, Section 12.33]. (More approachable proof can be found in [10].) If $x > 0$, then

$$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \leq \Gamma(1+x) \leq \sqrt{2\pi x}\left(\frac{x}{e}\right)^x e^{1/(12x)}. \tag{3.1}$$

From the Stirling formula we get that $\sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^p)} \approx n^{-1/p}$, i.e., there exist constants $c_1, c_2 > 0$ independent of n such that for all positive integers n it holds that

$$c_1 n^{-1/p} \leq \sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^p)} \leq c_2 n^{-1/p}$$

(for further details see [11]).

Proposition 3.1.1. *For every $p \in (0, \infty]$ it holds that $\mathbf{B}_n^p \subseteq \mathbf{B}_n^{p,\infty}$.*

Proof. For $p = \infty$ it is true since the spaces are the same (Proposition 1.1.1 (iii)). We can therefore assume $p < \infty$. Let a be an element of $\mathbf{B}_n^p$. Then we have

$$k\,(a_k^*)^p \leq \sum_{i=1}^k (a_i^*)^p\,,$$

since a^* is nonincreasing. It follows that

$$k^{\frac{1}{p}}\,(a_k^*) = (k\,(a_k^*)^p)^{\frac{1}{p}} \leq \left(\sum_{i=1}^k (a_i^*)^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n (a_i^*)^p\right)^{\frac{1}{p}} = ||a||_p \leq 1.$$

The statement is obtained by taking the supremum. ☐

This provides us with the one-sided estimate

$$\sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{p,\infty})} \gtrsim n^{-1/p}.$$

We want to show that it is, in fact, an equivalence.

The case $p = \infty$ is already done, since $\ell_n^{\infty,\infty} = \ell_n^\infty$. We now deal with the case $p \in (0, \infty)$. To obtain the required result we use the entropy numbers and their interpolation property. In general, our argumentation goes as follows: Let e_k be the k-th entropy number of $\mathrm{Id} : \ell_n^{p,\infty} \to \ell_n^\infty$, then for any $\varepsilon > 0$ we have $x_1, \dots, x_{2^{k-1}}$ such that

$$\mathbf{B}_n^{p,\infty} \subseteq \bigcup_{i=1}^{2^{k-1}} \left(x_i + (e_k + \varepsilon)\mathbf{B}_n^\infty\right).$$

Therefore

$$\mathrm{Vol}(\mathbf{B}_n^{p,\infty}) \leq 2^{k-1}(e_k + \varepsilon)^n \mathrm{Vol}(\mathbf{B}_n^\infty) = 2^{k+n-1}(e_k + \varepsilon)^n$$

and, by letting ε go to zero,

$$\sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{p,\infty})} \leq 2^{1+\frac{k-1}{n}} e_k. \tag{3.2}$$

We use the next theorem to estimate e_k.

Theorem 3.1.2. *Let $0 < p < \infty$ and $k, l \in \mathbb{N}$. Then*

$$e_{k+l-1}(\mathrm{Id} : \ell_n^{p,\infty} \to \ell_n^\infty) \leq 2^{\frac{2}{p}+2} \left(e_k(\mathrm{Id} : \ell_n^{p/2} \to \ell_n^\infty) e_l(\mathrm{Id} : \ell_n^\infty \to \ell_n^\infty)\right)^{\frac{1}{2}}.$$

Proof. Take arbitrary $\varepsilon > 0$, fix k and l and set

$$r_0 = (1+\varepsilon)e_k(\mathrm{Id} : \ell_n^{p/2} \to \ell_n^\infty), r_1 = (1+\varepsilon)e_l(\mathrm{Id} : \ell_n^\infty \to \ell_n^\infty).$$

Then we have $y_1, \dots, y_{2^{k-1}}, z_1, \dots, z_{2^{l-1}}$ such that

$$\mathbf{B}_n^{p/2} \subseteq \bigcup_{i=1}^{2^{k-1}} (y_i + r_0 \mathbf{B}_n^\infty) \text{ and } \mathbf{B}_n^\infty \subseteq \bigcup_{j=1}^{2^{l-1}} (z_j + r_1 \mathbf{B}_n^\infty).$$

The idea of the proof is the following: For every $a \in \mathbf{B}_n^{p,\infty}$ we want to find suitable a^0, $\tilde{y}_i \in \{\tilde{y}_1, \dots, \tilde{y}_{2^{k-1}}\}$, a^1, $\tilde{z}_j \in \{\tilde{z}_1, \dots, \tilde{z}_{2^{l-1}}\}$ (all elements of $\mathbb{R}^n$) such that

$$a^0 + a^1 = a \text{ and } ||a - \tilde{y}_i - \tilde{z}_j||_\infty \leq ||a^0 - \tilde{y}_i||_\infty + ||a^1 - \tilde{z}_j||_\infty \leq 2^{\frac{2}{p}+2}(r_0 r_1)^{1/2}.$$

We prove that for arbitrary $H > 0$ and $a \in \mathbf{B}_n^{p,\infty}$ there exist a^0, a^1 satisfying

$$a^0 + a^1 = a \text{ and } H^{-1}||a^0||_{p/2} + H||a^1||_\infty \leq 2^{\frac{2}{p}+1}||a||_{p,\infty}. \tag{3.3}$$

First observe that it is enough to prove (3.3) only for $||a||_{p,\infty} = 1$ and $a^* = a$, i.e., $0 \leq a_i \leq 1/i^{1/p}$. We distinguish three cases according to the value of H:

(i) $H \leq 1$

Set $a^0 = (0, \dots, 0), a^1 = a$, then

$$H||a^1||_\infty \leq H||a||_{p,\infty} = H \leq 1 \leq 2^{\frac{2}{p}+1}.$$

(ii) $H > n^{1/p}$

Set $a^0 = a$, $a^1 = (0, \dots, 0)$, then

$$H^{-1}||a^0||_{p/2} = H^{-1}\left(\sum_{i=1}^{n} |a_i^0|^{p/2}\right)^{2/p} \leq H^{-1}\left(\sum_{i=1}^{n} |1/i^{1/p}|^{p/2}\right)^{2/p}$$
$$= H^{-1}\left(\sum_{i=1}^{n} 1/i^{1/2}\right)^{2/p} \leq n^{-1/p}(2\sqrt{n}-1)^{2/p}$$
$$= \left(\frac{2\sqrt{n}-1}{\sqrt{n}}\right)^{2/p} \leq 2^{2/p},$$

where we used that

$$\sum_{i=1}^{n}(1/\sqrt{i}) = 1 + \sum_{i=2}^{n}(1/\sqrt{i}) \leq 1 + \int_1^n (1/\sqrt{x})dx = 2\sqrt{n} - 1.$$

(iii) $H \in (1, n^{1/p}]$

Let $m \in \{1, \dots, n-1\}$ and r such that $a_{m+1} \leq r < a_m$. Set

$$a^0 = (a_1 - r, \dots, a_m - r, 0, \dots, 0)$$

and

$$a^1 = (r, \dots, r, a_{m+1}, \dots, a_n).$$

This is a decomposition of a. We have

$$||a^0||_{p/2} = \left(\sum_{i=1}^{m}(a_i - r)^{p/2}\right)^{2/p} \leq \left(\sum_{i=1}^{m}(a_i)^{p/2}\right)^{2/p}$$
$$\leq \left(\sum_{i=1}^{m} i^{-1/2}\right)^{2/p} \leq (2\sqrt{m}-1)^{2/p}$$

and

$$||a^1||_\infty = r.$$

Set now $r = a_{m+1} \leq (m+1)^{-1/p}$, we obtain

$$H^{-1}||a^0||_{p/2} + H||a^1||_\infty \leq H^{-1}(2\sqrt{m}-1)^{2/p} + H(m+1)^{-1/p}.$$

Now it is enough to choose m such that $H \in [m^{1/p}, (m+1)^{1/p}]$ and we acquire

$$H^{-1}(2\sqrt{m}-1)^{2/p} + H(m+1)^{-1/p} \leq \left(\frac{2\sqrt{m}-1}{\sqrt{m}}\right)^{2/p} + (m+1)^{1/p-1/p}$$
$$\leq 2^{2/p} + 1 \leq 2^{2/p+1}.$$

As a by-product we obtain that $a^0 \in 2^{2/p+1}H\mathbf{B}_n^{p/2}$, $a^1 \in 2^{2/p+1}H^{-1}\mathbf{B}_n^\infty$. Therefore there exist i, j such that

$$||a - 2^{2/p+1}Hy_i - 2^{2/p+1}H^{-1}z_j||_\infty \leq ||a^0 - 2^{2/p+1}Hy_i||_\infty + ||a^1 - 2^{2/p+1}H^{-1}z_j||_\infty$$
$$\leq 2^{2/p+1}Hr_0 + 2^{2/p+1}H^{-1}r_1.$$

To complete the proof we just need to set $H = (r_1/r_0)^{1/2}$ and let ε go to zero. $\square$

This theorem can be viewed as a special case of Theorem 1.3.2.

Theorem 3.1.3. *Let* $0 < p \le \infty$, $n \in \mathbb{N}$, *then it holds that* $\sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{p,\infty})} \approx n^{-1/p}$, *where the constants do not depend on* n.

Proof. We already have the lower estimate from the inclusion $\mathbf{B}_n^p \subseteq \mathbf{B}_n^{p,\infty}$. The upper estimate is obtained by combining (3.2) with Theorems 1.3.1 and 3.1.2. Let again be $e_k = e_k(\mathrm{Id} : \ell_n^{p,\infty} \to \ell_n^\infty)$. For $l = 1$ and $k = n$ this gives

$$\sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{p,\infty})} \le 2^{1+\frac{n-1}{n}} e_n \le C\left(e_n(\mathrm{Id} : \ell_n^{p/2} \to \ell_n^\infty) e_1(\mathrm{Id} : \ell_n^\infty \to \ell_n^\infty)\right)^{\frac{1}{2}} \lesssim n^{-\frac{1}{p}},$$

where C is a constant which depends only on p. □

3.2 Asymptotic estimate for $q = 1$

We proceed from the explicit formula (2.5). Let us estimate the value of $\varkappa_p(k)$.

Proposition 3.2.1. *Let* $0 < p \le \infty$, $k \in \mathbb{N}$, *then* $\varkappa_p(k) \approx k^{\frac{1}{p}}$, *where the constants depend only on* p.

Proof. Let us firstly assume $p \le 1$. Therefore $x^{1/p-1}$ is a nondecreasing function and we have

$$pk^{\frac{1}{p}} = \int_0^k x^{\frac{1}{p}-1} dx \le \sum_{j=1}^k j^{\frac{1}{p}-1} = \varkappa_p(k) \le \int_1^{k+1} x^{\frac{1}{p}-1} dx \le p(k+1)^{\frac{1}{p}} \le p2^{\frac{1}{p}} k^{\frac{1}{p}}.$$

Now consider the case $p > 1$, then $x^{1/p-1}$ is a decreasing function (but still integrable on $[0, k]$) and

$$p(k^{\frac{1}{p}} - 1) \le p((k+1)^{\frac{1}{p}} - 1) = \int_1^{k+1} x^{\frac{1}{p}-1} dx \le \sum_{j=1}^k j^{\frac{1}{p}-1} \le \int_0^k x^{\frac{1}{p}-1} dx = pk^{\frac{1}{p}}.$$

For k big enough we have $k^{1/p} > 2$, so $\varkappa_p(k) \ge pk^{1/p}/2$. The number of k's for which this does not hold is finite and depends only on p. For them we know that $\varkappa_p(k) \ge p(2^{\frac{1}{p}} - 1) = c_p > 0$, therefore we obtain the wanted result at the cost of the multiplicative constant. □

Theorem 3.2.2. *Let* $0 < p \le \infty$, $n \in \mathbb{N}$, *then it holds that* $\sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{p,1})} \approx n^{-1/p}$, *where the constants do not depend on* n.

Proof. From the explicit formula and the previous proposition we have

$$\mathrm{Vol}(\mathbf{B}_n^{p,1}) \approx 2^n c^n (n!)^{-1/p},$$

where c arises from the product of $\varkappa_p(k)$'s ($\varkappa_p(k)$ is equivalent to $k^{\frac{1}{p}}$ up to a multiplicative constant). Therefore

$$\sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{p,1})} \approx (n!)^{-1/np}$$

and from the Stirling formula we get the desired result. □

3.3 Asymptotic estimate for the general case

Proposition 3.3.1. *Let $n \in \mathbb{N}$. Then we have:*

(i) *For $0 < p < \infty$, $0 < q_0 \le q_1 \le \infty$ there exists $c_{p,q_0,q_1} > 0$ (independent of n) such that $\mathbf{B}_n^{p,q_0} \subseteq c_{p,q_0,q_1}\mathbf{B}_n^{p,q_1}$. Moreover, if $q_0 \le p$, then $\mathbf{B}_n^{p,q_0} \subseteq \mathbf{B}_n^{p,q_1}$.*

(ii) *For $0 < p_0 \le p_1 < \infty, q \in (0,\infty]$ it holds that $\mathbf{B}_n^{p_0,q} \subseteq \mathbf{B}_n^{p_1,q}$.*

Proof. Firstly we note that (i) is a generalisation of Proposition 3.1.1.

(i) We begin with proving the assertion for $q_1 = \infty$. For every $l \in \{1,\dots,n\}$ we have that

$$||x||_{p,q_0}^{q_0} = \sum_{k=1}^{n} k^{\frac{q_0}{p}-1}(x_k^*)^{q_0} \ge \sum_{k=1}^{l} k^{\frac{q_0}{p}-1}(x_k^*)^{q_0} \ge (x_l^*)^{q_0}\sum_{k=1}^{l} k^{\frac{q_0}{p}-1}.$$

Therefore

$$\begin{aligned}||x||_{p,\infty} = \sup_{l\in\mathbb{N}}\{l^{\frac{1}{p}}x_l^*\} &\le \sup_{l\in\mathbb{N}}\Big\{l^{\frac{1}{p}}\Big(\sum_{k=1}^{l} k^{\frac{q_0}{p}-1}\Big)^{-\frac{1}{q_0}}||x||_{p,q_0}\Big\}\\ &= ||x||_{p,q_0}\sup_{l\in\mathbb{N}}\left\{\left(l^{-\frac{q_0}{p}}\sum_{k=1}^{l} k^{\frac{q_0}{p}-1}\right)^{-1/q_0}\right\}.\end{aligned}$$

Firstly assume that $q_0 \le p$ and denote $f(x) = x^{\frac{q_0}{p}-1}$. This function is convex and nonincreasing on $(0,\infty)$, so we have

$$\begin{aligned}l^{-\frac{q_0}{p}}\sum_{k=1}^{l} k^{\frac{q_0}{p}-1} &= \frac{1}{l}\sum_{k=1}^{l}(k/l)^{\frac{q_0}{p}-1} = \frac{1}{l}\sum_{k=1}^{l} f(k/l) \ge f\left(\frac{1}{l}\sum_{k=1}^{l} k/l\right)\\ &= f\left(\frac{1}{l^2}\frac{l(l+1)}{2}\right) = f\left(\frac{1}{2}+\frac{1}{2l}\right) \ge f(1) = 1,\end{aligned}$$

as $1/(2l) \le 1/2$.

We obtain

$$\inf_{l\in\mathbb{N}}\Big\{l^{-\frac{q_0}{p}}\sum_{k=1}^{l} k^{\frac{q_0}{p}-1}\Big\} \ge 1$$

and since $-1/q_0$ is negative, we finally get

$$||x||_{p,\infty} \le ||x||_{p,q_0}\sup_{l\in\mathbb{N}}\left\{\left(l^{-\frac{q_0}{p}}\sum_{k=1}^{l} k^{\frac{q_0}{p}-1}\right)^{-1/q_0}\right\} \le ||x||_{p,q_0}.$$

Now we assume $p < q_0$. The function f is now increasing, so $f(k/l) \ge f(x)$ for all $x \in [(k-1)/l, k/l]$. Therefore, from the Riemannian definition of integral,

$$l^{-\frac{q_0}{p}}\sum_{k=1}^{l} k^{\frac{q_0}{p}-1} = \frac{1}{l}\sum_{k=1}^{l} f(k/l) \ge \int_0^1 f(x)dx$$

and the sum on the left-hand side goes to the integral on the right-hand side as l goes to infinity.

We have an estimate

$$\left(l^{-\frac{q_0}{p}}\sum_{k=1}^{l} k^{\frac{q_0}{p}-1}\right)^{-1/q_0} \leq \left(\int_0^1 f(x)dx\right)^{-1/q_0} = \left(\frac{p}{q_0}\right)^{-1/q_0}$$

and

$$\sup_{l\in\mathbb{N}}\left\{\left(l^{-\frac{q_0}{p}}\sum_{k=1}^{l} k^{\frac{q_0}{p}-1}\right)^{-1/q_0}\right\} = \left(\frac{p}{q_0}\right)^{-1/q_0}.$$

This completes the proof for the case $q_1 = \infty$.

We now return to the general case where $q_0 \leq q_1 < \infty$. We have

$$\begin{aligned}||x||_{p,q_1} &= \left(\sum_{k=1}^{n} k^{\frac{q_1}{p}-1}(x_k^*)^{q_1}\right)^{1/q_1} = \left(\sum_{k=1}^{n} k^{\frac{q_1-q_0}{p}}(x_k^*)^{q_1-q_0}k^{\frac{q_0}{p}-1}(x_k^*)^{q_0}\right)^{1/q_1}\\ &\leq \left(\sum_{k=1}^{n} ||x||_{p,\infty}^{q_1-q_0}k^{\frac{q_0}{p}-1}(x_k^*)^{q_0}\right)^{1/q_1} = (||x||_{p,\infty})^{\frac{q_1-q_0}{q_1}}(||x||_{p,q_0})^{\frac{q_0}{q_1}}\\ &\leq (c_{p,q_0,\infty})^{\frac{q_1-q_0}{q_1}}(||x||_{p,q_0})^{\frac{q_1-q_0}{q_1}+\frac{q_0}{q_1}}.\end{aligned}$$

Therefore $c_{p,q_0,q_1} = (c_{p,q_0,\infty})^{\frac{q_1-q_0}{q_1}}$ and the proof is complete.

(ii) This part of the proposition follows from the fact that, as $\frac{1}{p_0} \geq \frac{1}{p_1}$,

$$||x||_{p_0,q}^q = \sum_{k=1}^{n}(k^{\frac{1}{p_0}-\frac{1}{q}}x_k^*)^q \geq \sum_{k=1}^{n}(k^{\frac{1}{p_1}-\frac{1}{q}}x_k^*)^q = ||x||_{p_1,q}^q.$$

□

Thanks to this embedding and the results for $q = 1$ and $q = \infty$ we easily obtain the asymptotics for q in-between.

Theorem 3.3.2. *For arbitrary $0 < p < \infty$, $1 < q \leq \infty$, $n \in \mathbb{N}$ it holds that $\sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{p,q})} \approx n^{-1/p}$, where the constants do not depend on n.*

Proof. The proof is a consequence of Proposition 3.3.1, as

$$\mathbf{B}_n^{p,1} \subseteq c_{p,1,q}\mathbf{B}_n^{p,q} \subseteq c_{p,1,q}c_{p,q,\infty}\mathbf{B}_n^{p,\infty},$$

where the constants are independent of the dimension. From Theorem 3.1.3 and Theorem 3.2.2 we have immediately

$$n^{-1/p} \approx \sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{p,1})} \lesssim \sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{p,q})} \lesssim \sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{p,\infty})} \approx n^{-1/p}.$$

□

Next we present a more general result, which has, however, more complicated proof than the previous theorem. We firstly state the theorem and prove the interpolation theorem for entropy numbers afterwards.

Theorem 3.3.3. *For arbitrary $0 < p < \infty$, $0 < q \leq \infty$, $n \in \mathbb{N}$ it holds that $\sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{p,q})} \approx n^{-1/p}$, where the constants do not depend on n.*

Proof. The upper estimate is obtained the same way as before, i.e., by

$$\mathbf{B}_n^{p,q} \subseteq c_{p,q,\infty} \mathbf{B}_n^{p,\infty}.$$

We prove the lower estimate with a similar argument as we did for the upper bound when $q = \infty$.

Let e_k be the k-th entropy number for $\mathrm{Id} : \ell_n^{p/2} \to \ell_n^{p,q}$, then for any $\varepsilon > 0$ we have that

$$\mathbf{B}_n^{p/2} \subseteq \bigcup_{i=1}^{2^{k-1}} \left(x_i + (e_k + \varepsilon)\mathbf{B}_n^{p,q}\right).$$

Therefore

$$\mathrm{Vol}(\mathbf{B}_n^{p/2}) \leq 2^{k-1} e_k^n \mathrm{Vol}(\mathbf{B}_n^{p,q})$$

and

$$n^{-2/p} \lesssim 2^{\frac{k-1}{n}} e_k \sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{p,q})}.$$

From Theorem 3.3.4 for the choice $l = 1$, $k = n$ we have

$$e_n \leq c_{p,q} \left(e_n(\mathrm{Id} : \ell_n^{p/2} \to \ell_n^{\infty}) e_1(\mathrm{Id} : \ell_n^{p/2} \to \ell_n^{p/2})\right)^{\frac{1}{2}} \approx n^{-1/p},$$

so

$$\sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{p,q})} \gtrsim n^{1/p-2/p} = n^{-1/p}.$$

□

As the reader may notice, we could have used just this version of the theorem and omit Theorem 3.3.2. However, it gave us the result for $q \geq 1$ much easier.

Theorem 3.3.4. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $k, l \in \mathbb{N}$. Then*

$$e_{k+l-1}(\mathrm{Id} : \ell_n^{p/2} \to \ell_n^{p,q}) \leq c_{p,q} \left(e_k(\mathrm{Id} : \ell_n^{p/2} \to \ell_n^{\infty}) e_l(\mathrm{Id} : \ell_n^{p/2} \to \ell_n^{p/2})\right)^{\frac{1}{2}}.$$

Proof. Take arbitrary $\varepsilon > 0$, fix k and l and set

$$r_0 = (1+\varepsilon) e_k(\mathrm{Id} : \ell_n^{p/2} \to \ell_n^{\infty}),$$
$$r_1 = (1+\varepsilon) e_l(\mathrm{Id} : \ell_n^{p/2} \to \ell_n^{p/2}).$$

Therefore we have $y_1, \ldots, y_{2^{k-1}}$ such that

$$\mathbf{B}_n^{p/2} \subseteq \bigcup_{i=1}^{2^{k-1}} \left(y_i + r_0 \mathbf{B}_n^{\infty}\right).$$

Let us denote

$$B_i = \mathbf{B}_n^{p/2} \cap \left(y_i + r_0 \mathbf{B}_n^{\infty}\right),$$

then this set can be covered by 2^{l-1} balls in $\ell_n^{p/2}$ with radius r_1 (as it is a subset of $\mathbf{B}_n^{p/2}$), i.e., we have $z_{i,1}, \ldots, z_{i,2^{l-1}}$ such that

$$B_i \subseteq \bigcup_{j=1}^{2^{l-1}} \left(z_{i,j} + r_1 \mathbf{B}_n^{p/2}\right).$$

We add the condition that centres of these balls must lie in B_i. Let us denote $s_{i,j}$ an arbitrary fixed point from $B_i \cap \left(z_{i,j} + r_1 \mathbf{B}_n^{p/2}\right)$ and take another point x from this set. Then

$$||s_{i,j} - x||_{p/2} \leq \max\{1, 2^{\frac{1}{p}-1}\} \left(||s_{i,j} - z_{i,j}||_{p/2} + ||z_{i,j} - x||_{p/2}\right) \leq \max\{2, 2^{\frac{1}{p}}\} r_1.$$

In other words, the set B_i is covered by collection of balls in $\ell_n^{p/2}$ with centres $s_{i,j} \in B_i$ and radius $\max\{2, 2^{\frac{1}{p}}\} r_1$. Now it is easy to see that for any $x \in \mathbf{B}_n^{p/2}$ there exist i, j such that

$$||x - s_{i,j}||_\infty \leq ||x - y_i||_\infty + ||y_i - s_{i,j}||_\infty \leq 2r_0$$

and

$$||x - s_{i,j}||_{p/2} \leq \max\{2, 2^{\frac{1}{p}}\} r_1.$$

The remaining step is to use an inequality which is a special case of [5, Lemma 4]:

$$||2^{sk/2} a_k||_q \leq C ||a_k||_\infty^{1/2} ||2^{sk} a_k||_\infty^{1/2} \tag{3.4}$$

for arbitrary $s \in \mathbb{R}$, $q \in (0, \infty)$ and some $C > 0$ (which is independent of the sequence a). We can set for convenience $x_k = 0$ for $k > n$. Assume now that $q \leq p$. Then

$$\begin{aligned} ||x||_{p,q}^q &= \sum_{k=1}^\infty k^{\frac{q}{p}-1} (x_k^*)^q = \sum_{m=0}^\infty \sum_{k=2^m}^{2^{m+1}-1} k^{\frac{q}{p}-1} (x_k^*)^q \leq \sum_{m=0}^\infty (2^m)^{\frac{q}{p}-1} \sum_{k=2^m}^{2^{m+1}-1} (x_k^*)^q \\ &\leq \sum_{m=0}^\infty (2^m)^{\frac{q}{p}-1} 2^m (x_{2^m}^*)^q = \sum_{m=0}^\infty (2^m)^{\frac{q}{p}} (x_{2^m}^*)^q = ||2^{(m-1)/p} x_{2^{m-1}}^*||_q^q. \end{aligned}$$

In the case $q > p$ there will occur $(2^{m+1})^{\frac{q}{p}-1}$ instead of $(2^m)^{\frac{q}{p}-1}$ in the first estimate. However, they differ only by a multiplicative constant depending on p and q, so the rest of the proof will proceed the same way. Now use (3.4):

$$\begin{aligned} ||2^{(m-1)/p} x_{2^{m-1}}^*||_q^q &\leq \left(c ||x_{2^{m-1}}^*||_\infty ||2^{2(m-1)/p} x_{2^{m-1}}^*||_\infty\right)^{q/2} \\ &= C ||x^*||_\infty^{q/2} \left(\sup_{m \in \mathbb{N}} \left\{ \left(2^{m-1}\right)^{2/p} x_{2^{m-1}}^* \right\}\right)^{q/2} \\ &\leq C ||x^*||_\infty^{q/2} \left(\sup_{k \in \mathbb{N}} \{k^{2/p} x_k^*\}\right)^{q/2} = C ||x||_\infty^{q/2} ||x||_{p/2,\infty}^{q/2} \\ &\leq C ||x||_\infty^{q/2} ||x||_{p/2}^{q/2}. \end{aligned}$$

This yields

$$||x||_{p,q} \leq C ||x||_\infty^{1/2} ||x||_{p/2}^{1/2},$$

which leads to the final estimate

$$||x - s_{i,j}||_{p,q} \leq C (2r_0)^{1/2} (\max\{2, 2^{\frac{1}{p}}\} r_1)^{1/2} = c_{p,q} (r_0 r_1)^{1/2}.$$

We finish by letting ε go to zero.

□

3.4 Ratios of volumes

The unit balls $\mathbf{B}_n^{p,\infty}$ of the weak Lebesgue spaces are often considered to be "slightly larger" than $\mathbf{B}_n^p$. Indeed, they have much in common, as we saw in the previous section that $\sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{p,q})} \approx \sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{p,q})}$. In this section, we study the ratio

$$R_{p,n} = \frac{\mathrm{Vol}(\mathbf{B}_n^{p,\infty})}{\mathrm{Vol}(\mathbf{B}_n^p)}.$$

We will show that for p small enough the ratio is growing exponentially.

Theorem 3.4.1. *For every $p \in (0,2]$ there exists $c > 1$ depending only on p such that*

$$R_{p,n} \gtrsim c^n,$$

where the constant depends only on p.

Proof. We show the proof for even n. For odd n which is large enough (at least 7) the proof is based on the same idea but is slightly more technical. The remaining cases ($n = 1,3,5$) are negligible as the ratio is some positive number and the statement allows a multiplicative constant.

Let us assume that

$$\begin{aligned} &x_1 \in [1/2^{1/p}, 1],\\ &x_2 \in [1/3^{1/p}, 1/2^{1/p}],\\ &\dots\\ &x_{n/2} \in [1/(n/2+1)^{1/p}, 1/(n/2)^{1/p}],\\ &x_{n/2+1}, \dots, x_n \in [0, 1/n^{1/p}]. \end{aligned}$$

Such x belongs to $\mathbf{B}_{n,+}^{p,\infty}$. There are $\binom{n}{n/2}$ ways how to choose the $n/2$ largest coordinates and $(n/2)!$ of ways how to order them. Moreover, there are no overlaps in the sense that by two different choices we never get the same element x. Therefore

$$\begin{aligned} R_{p,n} &= \frac{\mathrm{Vol}(\mathbf{B}_n^{p,\infty})}{\mathrm{Vol}(\mathbf{B}_n^p)} = \frac{\mathrm{Vol}(\mathbf{B}_{n,+}^{p,\infty})}{\mathrm{Vol}(\mathbf{B}_{n,+}^p)}\\ &= \frac{\Gamma(1+n/p)}{\Gamma(1+1/p)^n} \cdot \binom{n}{n/2} \cdot (n/2)! \cdot \prod_{i=1}^{n/2} \left(\frac{1}{i^{1/p}} - \frac{1}{(i+1)^{1/p}} \right) \cdot \left(\frac{1}{n^{1/p}} \right)^{n/2}\\ &= \frac{\Gamma(1+n/p)}{\Gamma(1+1/p)^n} \cdot \binom{n}{n/2} \cdot (n/2)! \cdot \prod_{i=1}^{n/2} \frac{(i+1)^{1/p} - i^{1/p}}{i^{1/p}(i+1)^{1/p}} \cdot \left(\frac{1}{n^{1/p}} \right)^{n/2}. \end{aligned}$$

We divide the proof into three steps now. In the first one, we estimate part of this figure. In the other two we distinguish cases $p \in (0,1)$ and $p \in [1,2]$.

(i) Let us denote

$$\begin{aligned} P &= \Gamma(1+n/p) \cdot \binom{n}{n/2} \cdot (n/2)! \cdot \prod_{i=1}^{n/2} \frac{1}{i^{1/p}(i+1)^{1/p}} \cdot \left(\frac{1}{n^{1/p}} \right)^{n/2}\\ &= \frac{\Gamma(1+n/p) n! (n/2)!}{[(n/2)!]^{2+1/p} \left[(n/2+1)! \right]^{1/p} n^{n/(2p)}}. \end{aligned}$$

We remind that the equivalence is up to multiplicative constants which are independent of n. By the Stirling formula (3.1) we have

$$\begin{aligned}P &\approx \frac{\sqrt{2\pi n/p}\left(\frac{n}{pe}\right)^{n/p}\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}}{(n/2+1)^{1/p}\left(\sqrt{\pi n}\right)^{1+2/p}\left(\frac{n}{2e}\right)^{n/2+n/p}n^{n/(2p)}}\\
&\approx \frac{2^{n/2+n/p}n^{1+n/p+n}}{p^{n/p}n^{1/2+2/p+n/2+n/p+n/(2p)}e^{n/2}}\\
&\approx \left[\frac{2^{1/2+1/p}}{p^{1/p}e^{1/2}}\right]^n\cdot n^{n/2-n/(2p)+1/2-2/p}.\end{aligned}$$

We obtain

$$R_{p,n}\approx \frac{P}{\Gamma(1+1/p)^n}\cdot\prod_{i=1}^{n/2}\left((i+1)^{1/p}-i^{1/p}\right).$$

(ii) Assume now that $p\in(0,1)$. Therefore $1/p>1$ and we can estimate

$$(i+1)^{1/p}-i^{1/p}=\int_i^{i+1}(1/p)x^{1/p-1}dx\geq(1/p)i^{1/p-1}.$$

Together we have

$$\begin{aligned}R_{p,n}&\gtrsim\left[\frac{2^{1/2+1/p}}{\Gamma(1+1/p)p^{1/p}e^{1/2}}\right]^n\cdot n^{n/2-n/(2p)+1/2-2/p}\cdot p^{-n/2}\cdot[(n/2)!]^{1/p-1}\\
&\approx\left[\frac{2^{1+1/(2p)}}{\Gamma(1+1/p)p^{1/p+1/2}e^{1/(2p)}}\right]^n\cdot n^{-1/(2p)-1/p}.\end{aligned}$$

While estimating $\Gamma(1+1/p)$ we need to be more careful as the n-th power of the corresponding multiplicative constant (which depends on p) must be taken into consideration. Recall from (3.1) that

$$\Gamma(1+x)\leq\sqrt{2\pi x}\left(\frac{x}{e}\right)^x e^{1/(12x)}.$$

We obtain

$$\begin{aligned}R_{p,n}&\gtrsim\left[\frac{2^{1+1/(2p)}}{\sqrt{2\pi/p}\left(\frac{1}{pe}\right)^{1/p}e^{p/12}p^{1/p+1/2}e^{1/(2p)}}\right]^n\cdot n^{-1/(2p)-1/p}\\
&=\left[\frac{2^{1+1/(2p)}}{\sqrt{2\pi}e^{1/(2p)-1/p+p/12}}\right]^n\cdot n^{-3/(2p)}\\
&=\left[\frac{2}{e^{p/12}}\right]^n\left[\sqrt{\frac{(2e)^{1/p}}{2\pi}}\right]^n\cdot n^{-3/(2p)}.\end{aligned}$$

By using $p<1$ we conclude

$$R_{p,n}\gtrsim\left[\frac{2}{e^{1/12}}\sqrt{\frac{e}{\pi}}\right]^n\cdot n^{-3/(2p)}\geq\left[\frac{2}{e^{1/12}}\sqrt{\frac{e}{4}}\right]^n\cdot n^{-3/(2p)}\geq\left[e^{1/3}\right]^n\cdot n^{-3/(2p)},$$

which is what we need since $e^{1/3}>1$.

(iii) The remaining case is $p \in [1, 2]$. Therefore $1/p \leq 1$ and

$$(i+1)^{1/p} - i^{1/p} = \int_i^{i+1} (1/p) x^{1/p-1} dx \geq (1/p)(i+1)^{1/p-1}.$$

Now continue as in (ii):

$$\begin{aligned} R_{p,n} &\gtrsim \left[\frac{2^{1/2+1/p}}{\Gamma(1+1/p) p^{1/p} e^{1/2}}\right]^n \cdot n^{n/2-n/(2p)+1/2-2/p} \cdot p^{-n/2} \cdot [(n/2+1)!]^{1/p-1} \\ &\gtrsim \left[\frac{2^{1+1/(2p)}}{\sqrt{2\pi} e^{-1/(2p)+p/12}}\right]^n \cdot n^{-1/(2p)-1}. \end{aligned}$$

By using $p \in [1, 2]$ we obtain

$$R_{p,n} \gtrsim \left[\frac{2^{1+1/(2p)} e^{1/(2p)}}{\sqrt{2\pi} e^{p/12}}\right]^n \cdot n^{-1/(2p)-1} \geq \left[\frac{2^{1+1/4} e^{1/4}}{\sqrt{2\pi} e^{1/6}}\right]^n \cdot n^{-1/(2p)-1},$$

and since $2^{3/4} e^{1/12} \pi^{-1/2} \doteq 1.03$ we may conclude as before. $\square$

Remark. As it follows from the proof, the assumption holds even for some p slightly bigger than 2. In fact, we assume that it holds for all $p \in (0, \infty)$, however, our proof does not work in that general case – probably more precise estimate of $\mathrm{Vol}(\mathbf{B}_{n,+}^{p,\infty})$ is needed – and so far it is an open problem.

4. Further properties

This chapter is devoted to study some further properties of the unit balls of the Lorentz spaces and the relation between the unit balls in the Lebesgue and Lorentz spaces. We also deal with the case $p = \infty$ for $q = 1$ in Section 4.3. We want to remind that in the first two sections we still consider p to be finite or $p = q = \infty$.

4.1 Properties of the Lorentz quasinorm

In this section we study the Lorentz quasinorm $||\cdot||_{p,q}$. As we are in the finite dimension, there always exists a norm which is equivalent to the Lorentz quasinorm. However, the constants of equivalence may depend on the dimension. We offer a characterisation of the spaces where the quasinorm is actually a norm.

Proposition 4.1.1. *Let $n \in \mathbb{N}$ and denote $S(n)$ the set of all permutations of $\{1, \dots, n\}$. If $p, q \in (0, \infty]$, $x \in \mathbb{R}^n$, then*

$$||x||_{p,q} = \begin{cases} \min\{||k^{\frac{1}{p}-\frac{1}{q}} x_{\pi(k)}||_q, \pi \in S(n)\}, & p \le q \\ \max\{||k^{\frac{1}{p}-\frac{1}{q}} x_{\pi(k)}||_q, \pi \in S(n)\}, & p \ge q. \end{cases}$$

Proof. For $n = 1$ there is nothing to prove as well as for x being a constant sequence. If $p = q$, then $k^{1/p-1/q} = 1$ for all k and the permutation of coordinates of x does not play a role. We can assume for simplicity that $x_k \ge 0$ for all $1 \le k \le n$. Denote by R the set of permutations which corresponds to the nonincreasing rearrangement of x, i.e., for $\sigma \in R$ we have $x_{\sigma(k)} = x^*_k$. Set $c_k = k^{1/p-1/q}$. It is obvious that for $\sigma, \sigma' \in R$ it holds that $||c_k x_{\sigma(k)}||_q = ||c_k x_{\sigma'(k)}||_q$.

To prove the first part by contradiction, let π be the permutation for which $||c_k x_{\pi(k)}||_q$ is minimal and assume that $\pi \notin R$. Therefore there has to be an index k_0 such that $x_{\pi(k_0+1)} > x_{\pi(k_0)}$. Since $1/p - 1/q > 0$ we obtain

$$(x_{\pi(k_0+1)} - x_{\pi(k_0)})(c_{k_0+1} - c_{k_0}) > 0,$$

therefore

$$c_{k_0+1} x_{\pi(k_0+1)} + c_{k_0} x_{\pi(k_0)} > c_{k_0+1} x_{\pi(k_0)} + c_{k_0} x_{\pi(k_0+1)},$$

so we can define $\tilde{\pi}(k_0) = \pi(k_0+1)$, $\tilde{\pi}(k_0+1) = \pi(k_0)$ and $\tilde{\pi}(k) = \pi(k)$ otherwise. Thus we obtained a permutation for which $||c_k x_{\pi(k)}||_q > ||c_k x_{\tilde{\pi}(k)}||_q$, which is a contradiction with the assumption on π.

The case $p > q$ can be done analogously. □

Proposition 4.1.2. *Let $p, q \in (0, \infty]$, $k, n \in \mathbb{N}$, $k \le n$, $\pi \in S(n)$ and let us denote*

$$D_\pi = \{x \in \mathbb{R}^n : ||k^{1/p-1/q} x_{\pi(k)}||_q \le 1\},$$

$$\mathbf{A}^{p,q}_{k,n} = \bigcup_{\substack{\pi \in S(n): \\ \pi(k)=n}} D_\pi \quad \text{for } p \le q,$$

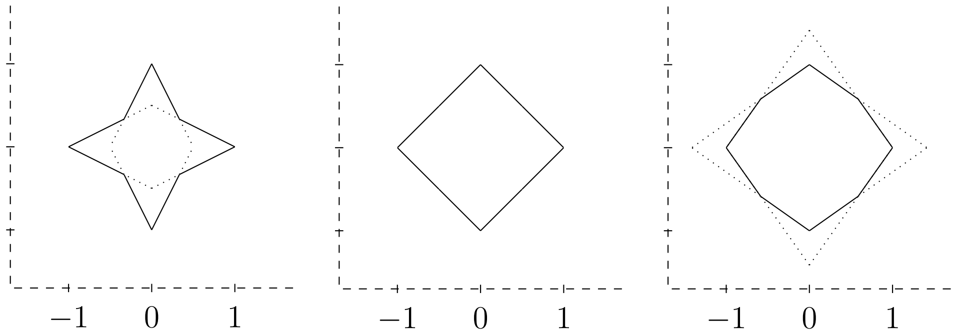
$$\mathbf{A}^{p,q}_{k,n} = \bigcup_{\substack{\pi \in S(n): \\ \pi(k)=n}} D_\pi \quad \text{for } p \ge q,$$

then it holds that

$$\mathbf{B}_n^{p,q} = \bigcup_{k=1}^{n} \mathbf{A}_{k,n}^{p,q} = \bigcup_{\pi \in S(n)} D_\pi \quad \text{for } p \le q,$$

$$\mathbf{B}_n^{p,q} = \bigcap_{k=1}^{n} \mathbf{A}_{k,n}^{p,q} = \bigcap_{\pi \in S(n)} D_\pi \quad \text{for } p \ge q.$$

Proof. The proof follows immediately from Proposition 4.1.1, as taking minimum corresponds to a union and taking maximum to an intersection, see Figure 4.1. □



$p = 1/2, q = 1$ $\quad$ $p = q = 1$ $\quad$ $p = 2, q = 1$

Figure 4.1: Unit ball as a union or an intersection

Remark. We show how to use Proposition 4.1.2 to get one-sided estimates of $\text{Vol}(\mathbf{B}_{n,+}^{p,q})$ by different approach (these estimates are the same as in Theorem 3.3.3, i.e., optimal). Let us denote

$$A_k^+ = \{x \in A_{k,n}^{p,q} : \forall i \in \{1, \dots, n\} : x_i \ge 0\}$$

(we omit the indices p, q and n for brevity). We can see that for $q = \infty$ the sets A_k^+ are exactly the sets we used in the calculation of volume of $\mathbf{B}_n^{p,\infty}$.

However, though the principle of computation of the volume for $q < \infty$ remains the same, the computation itself seems hard to be done. We can at least get a one-sided estimate – the same one as from the embedding of $\mathbf{B}_n^{p,q}$ and $\mathbf{B}_n^p$. We show it only for the case $p \le q$, as the procedure works similarly for the second one. We have

$$\text{Vol}(A_k^+) \le \text{Vol}(\mathbf{B}_{n,+}^{p,q}) = \text{Vol}(\bigcup_{k=1}^{n} A_k^+) \le n\text{Vol}(A_k^+).$$

For any sequence a from $\mathbf{A}_k^+$ it holds that $a_k \le 1/n^{1/p-1/q}$. Set

$$\tilde{a} = (a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n),$$

then

$$\sum_{i=1}^{n-1} \left(i^{1/p-1/q} \tilde{a}_i^*\right)^q \le 1 - \left(n^{1/p-1/q} a_k\right)^q.$$

We obtain

$$\tilde{a} \in \left(1 - n^{q/p-1} a_k^q\right)^{1/q} \mathbf{B}_{n-1,+}^{p,q},$$

where $a_k \in [0, n^{-1/p+1/q}]$, and vice versa, for each b element of this set there exists $c \in A_k^+$ such that $\tilde{c} = b$ and $c_k = a_k$. We may proceed with the calculation:

$$\mathrm{Vol}(A_k^+) = \mathrm{Vol}(\mathbf{B}_{n-1,+}^{p,q}) \int_0^{n^{-1/p+1/q}} \left(1 - n^{q/p-1} x^q\right)^{(n-1)/q} dx.$$

First use substitution $x = n^{-1/p+1/q} y$ and then $y = s^{1/q}$:

$$\begin{aligned}
\mathrm{Vol}(A_k^+) &= \mathrm{Vol}(\mathbf{B}_{n-1,+}^{p,q}) \cdot n^{-1/p+1/q} \int_0^1 (1 - y^q)^{(n-1)/q} dy \\
&= \mathrm{Vol}(\mathbf{B}_{n-1,+}^{p,q}) \cdot \frac{n^{-1/p+1/q}}{q} \int_0^1 s^{1/q-1} (1-s)^{(n-1)/q} ds \\
&= \mathrm{Vol}(\mathbf{B}_{n-1,+}^{p,q}) \cdot \frac{n^{-1/p+1/q}}{q} \cdot \mathrm{B}\left(\frac{1}{q}, \frac{n-1}{q} + 1\right),
\end{aligned}$$

where B denotes the beta function. To sum it up, we have

$$\begin{aligned}
\mathrm{Vol}(\mathbf{B}_{n,+}^{p,q}) \geq \mathrm{Vol}(A_k^+) &= \mathrm{Vol}(\mathbf{B}_{n-1,+}^{p,q}) \cdot \frac{n^{-1/p+1/q}}{q} \cdot \mathrm{B}\left(\frac{1}{q}, \frac{n-1}{q} + 1\right) \\
&\geq \mathrm{Vol}(\mathbf{B}_{1,+}^{p,q}) q^{-n+1} (n!)^{-1/p+1/q} \cdot \prod_{i=1}^{n-1} \mathrm{B}\left(\frac{1}{q}, \frac{i}{q} + 1\right)
\end{aligned}$$

and as we can rewrite the beta function using gamma function,

$$\begin{aligned}
&= q^{-n+1} (n!)^{-1/p+1/q} \cdot \prod_{i=1}^{n-1} \frac{\Gamma\left(\frac{1}{q}\right) \Gamma\left(\frac{i}{q} + 1\right)}{\Gamma\left(\frac{i+1}{q} + 1\right)} \\
&= q^{-n+1} (n!)^{-1/p+1/q} \cdot \frac{\Gamma\left(\frac{1}{q}\right)^{n-1} \Gamma\left(\frac{1}{q} + 1\right)}{\Gamma\left(\frac{n}{q} + 1\right)} \\
&= q^{-n} \Gamma\left(1/q\right)^n \cdot \frac{(n!)^{-1/p+1/q}}{\Gamma\left(\frac{n}{q} + 1\right)}.
\end{aligned}$$

We finish by using (3.1) to get

$$\mathrm{Vol}(\mathbf{B}_{n,+}^{p,q}) \geq q^{-n} \Gamma\left(1/q\right)^n \cdot \frac{(\sqrt{2\pi n}(n/e)^n)^{-1/p+1/q}}{2\sqrt{2\pi n/q}\,(n/(eq))^{n/q}},$$

thus for an appropriate constant c, which is independent of n,

$$\sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{p,q})} \geq c \cdot n^{-1/p+1/q} \cdot n^{-1/q} \approx n^{-1/p}.$$

The case $p \geq q$ uses the same formula for the volume of A_k^+, however, this time it holds that

$$\mathrm{Vol}(\mathbf{B}_{n,+}^{p,q}) = \mathrm{Vol}(\bigcap_{k=1}^{n} A_k^+) \leq \mathrm{Vol}(A_k^+),$$

so the estimate would be from above.

Theorem 4.1.3. *For arbitrary* $1 \le q \le p \le \infty$ *the unit ball in* $\ell_n^{p,q}$ *is a convex set. Furthermore,* $||\cdot||_{p,q}$ *is a norm.*

Proof. We use the fact that a quasinorm is a norm if and only if the unit ball is a convex set (see e.g. [17, Chapter 1]). Let us consider π a permutation of $\{1, \dots, n\}$. The set D_π is a unit ball in weighted ℓ_n^q, therefore it is convex whenever $q \ge 1$. According to the Proposition 4.1.2,

$$\mathbf{B}_n^{p,q} = \bigcap_\pi D_\pi.$$

Since an intersection of convex sets is convex, the proof is complete. □

Theorem 4.1.4. *Let* $n \in \mathbb{N}$*, then the functional* $||\cdot||_{p,q}$ *is a norm on* $\ell_n^{p,q}$ *if and only if* $1 \le q \le p \le \infty$ *or* $n = 1$.

Proof. For $n = 1$ we know that $||\cdot||_{p,q} = |\cdot|$, so there is nothing to prove. From now on, consider only $n > 1$.

We already have one of the implications from the Theorem 4.1.3. To prove the other one we distinguish three cases.

(i) $p < q < 1$

We know that $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0) \in \mathbf{B}_n^{p,q}$. If the set is convex, then $||(1/2, 1/2, 0, \dots, 0)||_{p,q} \le 1$. However, by simple computation,

$$||(1/2, 1/2, 0, \dots, 0)||_{p,q}^q = 2^{-q} + 2^{q/p-1}2^{-q} = 2^{-q}(1 + 2^{q/p-1}) > 1,$$

since $q/p - 1 > 0$ and $1/2 < 2^{-q} < 1$. This yields that the unit ball is not convex.

(ii) $q < 1$, $q < p$

Let us denote

$$v = \left(\frac{1}{(1 + 2^{q/p-1})^{1/q}}, \frac{1}{(1 + 2^{q/p-1})^{1/q}}, 0, \dots, 0 \right).$$

Then

$$||v||_{p,q} = (1 + 2^{q/p-1}) \cdot \frac{1}{1 + 2^{q/p-1}} = 1,$$

so $v \in \mathbf{B}_n^{p,q}$. The same holds for e_1. Let us denote s the segment connecting these two elements, i.e.,

$$S = \{te_1 + (1-t)v, t \in [0, 1]\}.$$

For every $x \in S$ it holds that $x = x^*$, so

$$||x||_{p,q} = (x_1^q + 2^{q/p-1}x_2^q)^{1/q}.$$

Now let us define $f(t) = ||te_1 + (1-t)v||_{p,q}^q$ for $t \in [0, 1]$. This is a continuous function of one variable and $f(0) = f(1) = 1$.

We have

$$f'(t) = q\left(t + \frac{1-t}{(1+2^{q/p-1})^{1/q}}\right)^{q-1}\left(1 - \frac{1}{(1+2^{q/p-1})^{1/q}}\right)$$
$$+2^{q/p-1}q\left(\frac{1-t}{(1+2^{q/p-1})^{1/q}}\right)^{q-1}\frac{(-1)}{(1+2^{q/p-1})^{1/q}}.$$

In particular,

$$f'_{+}(0) = q\left(\frac{1}{(1+2^{q/p-1})^{1/q}}\right)^{q-1}\left[\frac{(1+2^{q/p-1})^{1/q}-1}{1+2^{q/p-1}} - \frac{2^{q/p-1}}{1+2^{q/p-1}}\right] > 0,$$

since $1/q > 1$. Therefore there exists $t_0 \in (0,1)$ such that $f(t_0) > 1$, i.e., there is a convex combination of e_1 and v which has the norm strictly greater than 1. We obtain that the ball $\mathbf{B}_n^{p,q}$ is not convex.

(iii) $q \geq 1$, $q > p$

First let us consider $q < \infty$. We proceed similarly to the previous case, set

$$v_1 = (2^{-1/p}, 2^{-1/q}, 0, \ldots, 0), v_2 = (2^{-1/q}, 2^{-1/p}, 0, \ldots, 0),$$

and

$$S = \{tv_1 + (1-t)v_2, t \in [0, 1/2]\}.$$

It again holds that $x = x^*$ whenever $x \in S$. By simple calculation both v_1 and v_2 are elements of $\mathbf{B}_n^{p,q}$. Define $f(t) = ||tv_1 + (1-t)v_2||_{p,q}^q$ and differentiate

$$f'(t) = q(2^{-1/p}t + 2^{-1/q}(1-t))^{q-1}(2^{-1/p} - 2^{-1/q})$$
$$+ 2^{q/p-1}q(2^{-1/q}t + 2^{-1/p}(1-t))^{q-1}(2^{-1/q} - 2^{-1/p}).$$

We obtain

$$f'_{+}(0) = q(2^{-1/q} - 2^{-1/p})\left(-2^{(1-q)/q} + 2^{q/p-1+(1-q)/p}\right) > 0,$$

since $2^{1/p-1} > 2^{1/q-1}$, as $1/p > 1/q$. This is what we required.

The case $q = \infty$ works the same way with

$$v_1 = (2^{-1/p}, 1, 0, \ldots, 0), v_2 = (1, 2^{-1/p}, 0, \ldots, 0)$$

and S defined as before. Then

$$f(t) = ||tv_1 + (1-t)v_2||_{p,\infty} = 1 + t(2^{1/p} - 1)$$

on S and $f'_{+}(0) = 2^{1/p} - 1 > 0$. □

In the case that $||\cdot||_{p,q}$ is a quasinorm, we are interested in the corresponding constant. The next theorem offers a summary of this matter, cf. Figure 4.2.

Theorem 4.1.5. *In the following cases the function $||\cdot||_{p,q}$ is a quasinorm on the space $\ell^{p,q}$ such that*

(i) *for $p < q$, $q \geq 1$ the constant is at worst $2^{\frac{1}{p}}$, i.e., $||a+b||_{p,q} \leq 2^{\frac{1}{p}}(||a||_{p,q} + ||b||_{p,q})$,*

(ii) *for $p < q < 1$ the constant is at worst $2^{\frac{1}{p}+\frac{1}{q}-1}$,*

(iii) *for $p \geq q$, $q < 1$ the constant is at worst $2^{\frac{1}{p}+\frac{2}{q}-1}$.*

In general, we can say that the constant is globally at worst $2^{\frac{1}{p}+\frac{2}{q}}$, where we consider $1/\infty = 0$.

Proof. We use Proposition 1.1.1 (i), which tells us that $f^*_{a+b}(k) \leq f^*_a(k/2) + f^*_b(k/2)$. In other words,

$$(a+b)^*_k \leq a^*_{\lceil k/2 \rceil} + b^*_{\lceil k/2 \rceil}. \tag{4.1}$$

(i) First assume that $q = \infty$, $p < q$. Thus we have

$$\begin{aligned} ||a+b||_{p,\infty} &= \sup_{k\in\mathbb{N}}\{k^{\frac{1}{p}}(a+b)^*_k\} \leq \sup_{k\in\mathbb{N}}\{k^{\frac{1}{p}}(a^*_{\lceil k/2\rceil} + b^*_{\lceil k/2\rceil})\} \\ &= \sup_{k\in\mathbb{N}}\{(2k/2)^{\frac{1}{p}}(a^*_{\lceil k/2\rceil} + b^*_{\lceil k/2\rceil})\} \\ &\leq 2^{\frac{1}{p}} \sup_{k\in\mathbb{N}}\{\lceil k/2\rceil^{\frac{1}{p}}(a^*_{\lceil k/2\rceil} + b^*_{\lceil k/2\rceil})\} \\ &\leq 2^{\frac{1}{p}}\left(\sup_{k\in\mathbb{N}}\{\lceil k/2\rceil^{\frac{1}{p}}a^*_{\lceil k/2\rceil}\} + \sup_{k\in\mathbb{N}}\{\lceil k/2\rceil^{\frac{1}{p}}b^*_{\lceil k/2\rceil}\}\right) \\ &= 2^{\frac{1}{p}}\left(||a||_{p,\infty} + ||b||_{p,\infty}\right). \end{aligned}$$

Now assume $1 \leq q < \infty$, $p < q$. We use that

$$||\lceil k/2\rceil^{\frac{1}{p}-\frac{1}{q}}a^*_{\lceil k/2\rceil}||^q_q = 2||k^{\frac{1}{p}-\frac{1}{q}}a^*_k||^q_q = 2||a||^q_{p,q}. \tag{4.2}$$

Thanks to the triangle inequality for $||\cdot||_q$ and (4.1) we obtain

$$\begin{aligned} ||a+b||_{p,q} &= ||k^{\frac{1}{p}-\frac{1}{q}}(a+b)^*_k||_q \leq ||k^{\frac{1}{p}-\frac{1}{q}}a^*_{\lceil k/2\rceil}||_q + ||k^{\frac{1}{p}-\frac{1}{q}}b^*_{\lceil k/2\rceil}||_q \\ &= ||2^{\frac{1}{p}-\frac{1}{q}}(k/2)^{\frac{1}{p}-\frac{1}{q}}a^*_{\lceil k/2\rceil}||_q + ||2^{\frac{1}{p}-\frac{1}{q}}(k/2)^{\frac{1}{p}-\frac{1}{q}}b^*_{\lceil k/2\rceil}||_q. \end{aligned}$$

Now we use that $1/p - 1/q > 0$ and (4.2) to conclude

$$\begin{aligned} ||a+b||_{p,q} &\leq ||2^{\frac{1}{p}-\frac{1}{q}}\lceil k/2\rceil^{\frac{1}{p}-\frac{1}{q}}a^*_{\lceil k/2\rceil}||_q + ||2^{\frac{1}{p}-\frac{1}{q}}\lceil k/2\rceil^{\frac{1}{p}-\frac{1}{q}}b^*_{\lceil k/2\rceil}||_q \\ &= 2^{\frac{1}{p}-\frac{1}{q}}\left(2^{\frac{1}{q}}||a||_{p,q} + 2^{\frac{1}{q}}||b||_{p,q}\right) = 2^{\frac{1}{p}}\left(||a||_{p,q} + ||b||_{p,q}\right). \end{aligned}$$

(ii) For $0 < p < q < 1$ we proceed the same way as in (i), but since $||\cdot||_q$ is a quasinorm with constant $2^{1/q-1}$, this factor has to appear when using the quasitriangle inequality. Therefore

$$\begin{aligned} ||a+b||_{p,q} &= ||k^{\frac{1}{p}-\frac{1}{q}}(a+b)^*_k||_q \leq 2^{\frac{1}{q}-1}\left(||k^{\frac{1}{p}-\frac{1}{q}}a^*_{\lceil k/2\rceil}||_q + ||k^{\frac{1}{p}-\frac{1}{q}}b^*_{\lceil k/2\rceil}||_q\right) \\ &\leq 2^{\frac{1}{q}-1}\left(||2^{\frac{1}{p}-\frac{1}{q}}\lceil k/2\rceil^{\frac{1}{p}-\frac{1}{q}}a^*_{\lceil k/2\rceil}||_q + ||2^{\frac{1}{p}-\frac{1}{q}}\lceil k/2\rceil^{\frac{1}{p}-\frac{1}{q}}b^*_{\lceil k/2\rceil}||_q\right) \\ &= 2^{\frac{1}{p}-1}\left(||2^{\frac{1}{q}}a||_{p,q} + ||2^{\frac{1}{q}}b||_{p,q}\right) = 2^{\frac{1}{p}+\frac{1}{q}-1}\left(||a||_{p,q} + ||b||_{p,q}\right). \end{aligned}$$

(iii) The last case is $q < 1$, $p \geq q$. Then $(k/2)^{\frac{1}{p}-\frac{1}{q}} \geq \lceil k/2 \rceil^{\frac{1}{p}-\frac{1}{q}}$, so we need to proceed differently. Let us split the following sum into three parts:

$$\begin{aligned} ||k^{\frac{1}{p}-\frac{1}{q}} a^*_{\lceil k/2 \rceil}||_q^q &= \sum_{k=1}^{\infty} \left(2^{\frac{1}{p}-\frac{1}{q}} \left(\frac{k}{2}\right)^{\frac{1}{p}-\frac{1}{q}} a^*_{\lceil k/2 \rceil} \right)^q = \sum_{k \text{ even}} \left(2^{\frac{1}{p}-\frac{1}{q}} \left(\frac{k}{2}\right)^{\frac{1}{p}-\frac{1}{q}} a^*_{\lceil k/2 \rceil} \right)^q \\ &+ \sum_{\substack{k \text{ odd}, \\ k \neq 1}} \left(2^{\frac{1}{p}-\frac{1}{q}} \left(\frac{k}{2}\right)^{\frac{1}{p}-\frac{1}{q}} a^*_{\lceil k/2 \rceil} \right)^q + \left(2^{\frac{1}{p}-\frac{1}{q}} \left(\frac{1}{2}\right)^{\frac{1}{p}-\frac{1}{q}} a^*_{\lceil k/2 \rceil} \right)^q \\ &= \sum_{n=1}^{\infty} \left(2^{\frac{1}{p}-\frac{1}{q}} n^{\frac{1}{p}-\frac{1}{q}} a^*_n \right)^q + \sum_{\substack{k \text{ odd}, \\ k \neq 1}} \left(2^{\frac{1}{p}-\frac{1}{q}} \left(\frac{k}{2}\right)^{\frac{1}{p}-\frac{1}{q}} a^*_{\lceil k/2 \rceil} \right)^q + (a_1^*)^q. \end{aligned}$$

The middle term can be estimated because $(k/2)^{\frac{1}{p}-\frac{1}{q}} \leq ((k-1)/2)^{\frac{1}{p}-\frac{1}{q}}$ and $a^*_{\lceil k/2 \rceil} \leq a^*_{\lceil (k-1)/2 \rceil}$. Therefore

$$||k^{\frac{1}{p}-\frac{1}{q}} a^*_{\lceil k/2 \rceil}||_q^q \leq ||2^{\frac{1}{p}-\frac{1}{q}} a||_{p,q}^q + \sum_{\substack{k \text{ odd}, \\ k \neq 1}} \left(2^{\frac{1}{p}-\frac{1}{q}} \left(\frac{k-1}{2}\right)^{\frac{1}{p}-\frac{1}{q}} a^*_{\lceil (k-1)/2 \rceil} \right)^q + ||a||_{p,q}^q$$

We can rewrite

$$\begin{aligned} \sum_{\substack{k \text{ odd}, \\ k \neq 1}} \left(2^{\frac{1}{p}-\frac{1}{q}} \left(\frac{k-1}{2}\right)^{\frac{1}{p}-\frac{1}{q}} a^*_{\lceil (k-1)/2 \rceil} \right)^q &= \sum_{n \text{ even}} \left(2^{\frac{1}{p}-\frac{1}{q}} \left(\frac{n}{2}\right)^{\frac{1}{p}-\frac{1}{q}} a^*_{\lceil n/2 \rceil} \right)^q \\ &= ||2^{\frac{1}{p}-\frac{1}{q}} a||_{p,q}^q. \end{aligned}$$

Together we get

$$||2^{\frac{1}{p}-\frac{1}{q}} (k/2)^{\frac{1}{p}-\frac{1}{q}} a^*_{\lceil k/2 \rceil}||_q^q \leq ||2^{\frac{1}{p}-\frac{1}{q}} a||_{p,q}^q + ||2^{\frac{1}{p}-\frac{1}{q}} a||_{p,q}^q + ||a||_{p,q}^q = (2^{\frac{q}{p}} + 1)||a||_{p,q}^q.$$

Now use (4.1) and property of $||\cdot||_q$ to conclude

$$\begin{aligned} ||a+b||_{p,q} = ||k^{\frac{1}{p}-\frac{1}{q}} (a+b)^*_k||_q &\leq 2^{\frac{1}{q}-1} \left(||k^{\frac{1}{p}-\frac{1}{q}} a^*_{\lceil k/2 \rceil}||_q + ||k^{\frac{1}{p}-\frac{1}{q}} b^*_{\lceil k/2 \rceil}||_q \right) \\ &\leq 2^{\frac{1}{q}-1} (2^{\frac{q}{p}} + 1)^{\frac{1}{q}} (||a||_{p,q} + ||b||_{p,q}) \\ &\leq 2^{\frac{1}{q}-1} (2^{\frac{q}{p}+1})^{\frac{1}{q}} (||a||_{p,q} + ||b||_{p,q}) \\ &= 2^{\frac{1}{p}+\frac{2}{q}-1} (||a||_{p,q} + ||b||_{p,q}). \end{aligned}$$

□

As we are in the finite-dimensional case, the Lorentz quasinorm is always equivalent to some norm. In the case $1 < p \leq \infty$, $1 \leq q \leq \infty$ it is well-known even in the infinite-dimensional case. (The norm is defined similarly to the Lorentz quasinorm, where f^* is replaced by its maximal operator f^{**}, for further details see [15].) Therefore for that choice of parameters the constant of equivalence are independent of the dimension, which may not be true in the rest of cases.

As the last proposition of this section we offer a summary of relations between the Lorentz quasinorms. This proposition is (again) a generalisation of Propositions 3.1.1 and 3.3.1.

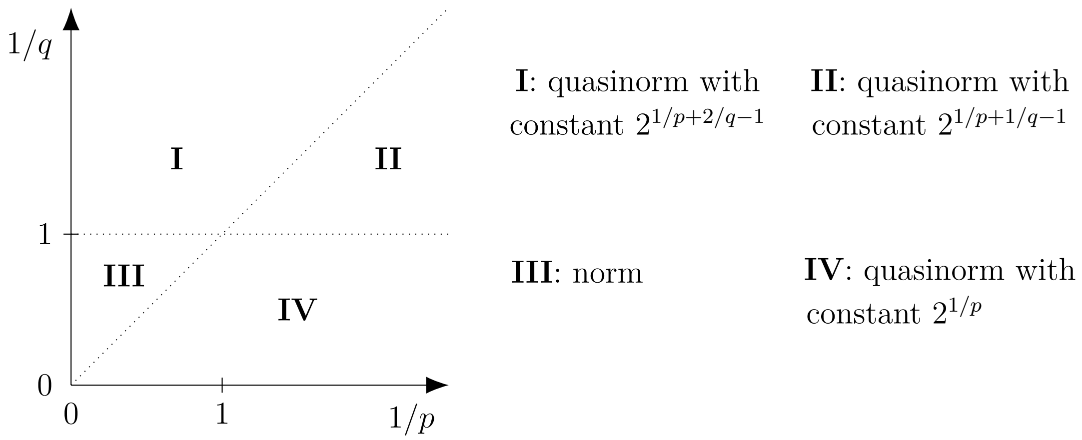


Figure 4.2: Comparison of the $\ell^{p,q}$-quasinorm with respect to p and q

Proposition 4.1.6. *Let $0 < p_0, p_1, q_0, q_1 \leq \infty$, $n \in \mathbb{N}$, then $\mathbf{B}_n^{p_0,q_0} \subseteq K(n)\mathbf{B}_n^{p_1,q_1}$, where*

$$K(n) \approx \begin{cases} 1, & p_0 < p_1, \text{ or } p_0 = p_1 \text{ and } q_0 \leq q_1 \\ (1+\log(n))^{1/q_1 - 1/q_0}, & p_0 = p_1 \text{ and } q_0 > q_1, \\ n^{1/p_1 - 1/p_0}, & p_0 > p_1, \end{cases}$$

(the constants of equivalence do not depend on n).

Proof. We can assume that $n \geq 2$, since the case $n = 1$ is trivial.

The first case can be easily proven by Proposition 3.3.1. If $p_0 = p_1$ and $q_0 \leq q_1$, we have

$$\mathbf{B}_n^{p_0,q_0} \subseteq c\mathbf{B}_n^{p_0,q_1} = c\mathbf{B}_n^{p_1,q_1}.$$

If $p_0 < p_1$, we use the fact that $\mathbf{B}_n^{p_0,q_0} \subseteq c\mathbf{B}^{p_0,\infty}$ and that $x_k^* \leq ||x||_{p_0,\infty} k^{-1/p_0}$. It remains to show that $\mathbf{B}^{p_0,\infty} \subseteq \tilde{c}\mathbf{B}^{p_1,q_1}$ for some $\tilde{c} > 0$. Indeed,

$$\begin{aligned} ||x||_{p_1,q_1}^{q_1} &= \sum_{k=1}^{n} (x_k^*)^{q_1} k^{q_1/p_1 - 1} \leq \sum_{k=1}^{n} ||x||_{p_0,\infty}^{q_1} k^{q_1/p_1 - q_1/p_0 - 1} \\ &\leq ||x||_{p_0,\infty}^{q_1} \sum_{k=1}^{\infty} k^{q_1(1/p_1 - 1/p_0) - 1}. \end{aligned}$$

Since $1/p_1 - 1/p_0 < 0$, the sum converges (to $\tilde{c}^{q_1}$). The case $q_1 = \infty$ is trivial (from Proposition 3.3.1).

The second part is also rather straightforward. If $q_0 < \infty$, then using the Hölder inequality and $\sum_{k=2}^{n} k^{-1} \leq \int_1^n x^{-1} dx$ yields

$$\begin{aligned} ||x||_{p_1,q_1}^{q_1} &= \sum_{k=1}^{n} (x_k^*)^{q_1} k^{q_1/p_1 - 1} \\ &\leq \left[\sum_{k=1}^{n} \left((x_k^*)^{q_1} k^{q_1/p_1 - q_1/q_0} \right)^{q_0/q_1} \right]^{q_1/q_0} \left[\sum_{k=1}^{n} \left(k^{q_1/q_0 - 1} \right)^{q_0/(q_0 - q_1)} \right]^{1 - q_1/q_0} \\ &= ||x||_{p_0,q_0}^{q_1} \left[1 + \sum_{k=2}^{n} k^{-1} \right]^{1 - q_1/q_0} \leq ||x||_{p_0,q_0}^{q_1} (1 + \log(n))^{1 - q_1/q_0}. \end{aligned}$$

If $q_0 = \infty$, we use again that $x_k^* \leq ||x||_{p_0,\infty} k^{-1/p_0}$.

The third case can be shown in a similar manner as the first one, it is enough to show $||x||_{p_1,q_1} \leq K(n)||x||_{p,0,\infty}$. If $q_1 < \infty$, then

$$\begin{aligned} ||x||_{p_1,q_1}^{q_1} &\leq ||x||_{p_0,\infty}^{q_1} \sum_{k=1}^{n} k^{q_1(1/p_1-1/p_0)-1} \leq ||x||_{p_0,\infty}^{q_1} \int_0^{n+1} x^{q_1(1/p_1-1/p_0)-1} dx \\ &= c||x||_{p_0,\infty}^{q_1}(n+1)^{q_1(1/p_1-1/p_0)} \leq \tilde{c}||x||_{p_0,\infty}^{q_1} n^{q_1(1/p_1-1/p_0)}, \end{aligned}$$

as $q_1(1/p_1 - 1/p_0) > 0$ ($c, \tilde{c}$ are constants independent of n). If $q_1 = \infty$, then

$$||x||_{p_1,q_1} = \max_{k\in\{1,\dots,n\}} \{x_k^* k^{1/p_1}\} \leq \max_{k\in\{1,\dots,n\}} \{||x||_{p_0,\infty} k^{1/p_1-1/p_0}\} = ||x||_{p_0,\infty} n^{1/p_1-1/p_0}.$$

□

4.2 Entropy numbers for Lorentz spaces

In this section we focus on the quantity $e_k(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{p_1,q_1})$. As the entropy numbers were for us more a tool than a focal point of this work, we offer sketches of proofs where some technical aspects are presented only briefly. For the details we refer to [11], as the procedure follows a structure very similar to the proof of their Theorem 2.

We present a (known) result for the case $0 < p_0 \neq p_1 < \infty$, $1 \leq q_0, q_1 \leq \infty$ and a new result for the special case $p_0 = p_1 = 1$, $q_0 = \infty$, $q_1 = 1$. However, firstly we state a technical lemma:

Lemma 4.2.1. *For any real $x > 2$ it holds that*

$$\log_2(x) \approx \log_2(x+1) \approx \log_2(2x),$$

where the constants do not depend on x, and

$$\log_2(\log_2(x)) \leq \frac{1}{2}\log_2(x).$$

Proof. The first part is a consequence of the fact that ratios of these functions are monotone with positive upper and lower bounds. The second part is obvious as $0 < \log_2(x) \leq x \leq \sqrt{2}x$. The statement holds for $b > 1$ an arbitrary base of the logarithm (for $x > b$). □

Theorem 4.2.2. *Let $0 < p_0 \neq p_1 < \infty$, $1 \leq q_0, q_1 \leq \infty$, then*

$$e_k(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{p_1,q_1}) \approx e_k(\mathrm{Id} : \ell_n^{p_0} \to \ell_n^{p_1}),$$

where the constants of equivalence do not depend on k and n.

Proof. (sketch) We use the following two facts: due to the definition of the entropy numbers and an estimate for the entropy numbers of identity from any quasi-Banach space into the space itself (see [8, Lemma 2.1]),

$$\begin{aligned} e_k(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{p_1,q_1}) &\leq e_k(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{p_0,q_0}) e_1(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{p_1,q_1}) \\ &\leq c2^{-\frac{k-1}{n}} e_1(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{p_1,q_1}), \end{aligned} \tag{4.3}$$

where c may depend on p_0 and q_0, but not on n.. The second fact is based on [3, Theorem 5.3.1] combined with [2, Chapter 5, Propositions 1.8, 1.10 and 2.10], which implies that the assumptions of Theorem 1.3.2 are met. (We can apply it on ℓ^{p_0,q_0} and ℓ^{p_1,q_1}, so the constants do not depend on n.) We get an estimate for $e_k(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{p_1,q_1})$:

To get an upper bound we start with choosing $p_0' < p_0 < p_0''$ such that $p_1 \notin [p_0', p_0'']$ to obtain

$$e_{2k-1}(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{p_1}) \le c \left(e_k(\mathrm{Id} : \ell_n^{p_0'} \to \ell_n^{p_1})\right)^{1-\theta} \left(e_k(\mathrm{Id} : \ell_n^{p_0''} \to \ell_n^{p_1})\right)^{\theta},$$

where $0 < \theta < 1$ is a parameter such that $\ell_n^{p_0,q_0} = (\ell_n^{p_0'}, \ell_n^{p_0''})_{\theta,q_0}$. (The interpolation space $(\ell_n^{p_0'}, \ell_n^{p_0''})_{\theta,q_0}$ is defined through the K-functional, for further details on real interpolation see e.g. [2, Chapter 5] or [3, Chapter 3].) We proceed by choosing $p_1' < p_1 < p_1''$ such that $p_0 \notin [p_1', p_1'']$ and obtain

$$e_{2k-1}(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{p_1,q_1}) \le c \left(e_k(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{p_1'})\right)^{1-\theta'} \left(e_k(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{p_1''})\right)^{\theta'}, \tag{4.4}$$

where $\ell_n^{p_1,q_1} = (\ell_n^{p_1'}, \ell_n^{p_1''})_{\theta',q_1}$.

The lower estimate is done in a similar manner, choose $p_0 < \tilde{p}_0' < \tilde{p}_0''$ such that $p_1 \notin [p_0, \tilde{p}_0'']$ or $p_1 < \tilde{p}_1' < \tilde{p}_1''$ such that $p_0 \notin [p_1, \tilde{p}_1'']$, respectively. Then

$$e_{2k-1}(\mathrm{Id} : \ell_n^{\tilde{p_0}'} \to \ell_n^{p_1}) \le c \left(e_k(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{p_1})\right)^{1-\omega} \left(e_k(\mathrm{Id} : \ell_n^{\tilde{p_0}''} \to \ell_n^{p_1})\right)^{\omega},$$

so

$$e_{2k-1}(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{\tilde{p_1}'}) \le c \left(e_k(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{p_1,q_1})\right)^{1-\omega'} \left(e_k(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{\tilde{p}_1''})\right)^{\omega'}, \tag{4.5}$$

where $\ell_n^{\tilde{p}_0'} = (\ell_n^{p_0,q_0}, \ell_n^{\tilde{p}_0''})_{\omega,q_0}$ and $\ell_n^{p_1,q_1} = (\ell_n^{p_1'}, \ell_n^{p_1''})_{\omega',q_1}$.

Now let us divide the proof into four cases.

(i) $p_0 > p_1$

Let k be a positive integer and let $\mathbf{B}_n^{p_0,q_0}$ be covered by 2^{k-1} balls in $\ell_n^{p_1,q_1}$ with the radius τ. Then (by Theorem 3.3.3)

$$\tau \ge 2^{-\frac{k-1}{n}} \left(\frac{\mathrm{Vol}(\mathbf{B}_n^{p_0,q_0})}{\mathrm{Vol}(\mathbf{B}_n^{p_1,q_1})}\right)^{1/n} \ge c 2^{-\frac{k-1}{n}} n^{1/p_1 - 1/p_0}$$

for some $c > 0$ independent of n.

On the other hand, thanks to (4.3) and Proposition 4.1.6 we get

$$e_k(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{p_1,q_1}) \le c 2^{-\frac{k-1}{n}} ||\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{p_1,q_1}|| \le \tilde{c} 2^{-\frac{k-1}{n}} n^{1/p_1 - 1/p_0},$$

which finishes the proof for this case.

(ii) $p_0 < p_1$, $1 \le k \le \frac{1}{2}\log_2(n)$

From (4.4) and (4.5) (combined with Theorem 1.3.1) we have

$$\begin{aligned} e_{2k-1}(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{p_1,q_1}) &\le c, \\ e_k(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{p_1,q_1}) &\ge \tilde{c}, \end{aligned}$$

therefore from the monotonicity of the entropy numbers we obtain $e_k(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{p_1,q_1}) \approx 1$.

(iii) $p_0 < p_1$, $\log_2(n) \le k \le n/4$

For $n \ge 16$, by the same approach as in (ii) combined with Lemma 4.2.1 we obtain $e_k(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{p_1,q_1}) \approx \left(\frac{\log_2(1+n/k)}{k}\right)^{\frac{1}{p_0}-\frac{1}{p_1}}$. The case $n < 16$, $\log_2(n) \le k \le n/4$ contains only finitely many situations, so they are covered at the cost of the constant of equivalence.

(iv) $p_0 < p_1$, $k \ge \gamma n$

The lower bound is obtained as in (i). For the upper bound, let $\tau > 0$ and let $\{x_1, \dots, x_N\} \subseteq \mathbf{B}_n^{p_0,q_0}$ be a maximal τ-distant set in the $\ell_n^{p_1,q_1}$-quasinorm, i.e., for $1 \le i \ne j \le N$ it holds that $||x_i - x_j||_{p_1,q_1} > \tau$ and for $z \in \mathbf{B}_n^{p_0,q_0}$ there exists $1 \le i \le N$ such that $||x_i - z||_{p_0,q_0} \le \tau$. According to Theorem 4.1.5 and Proposition 4.1.6 we have

$$(x_i + \tau \mathbf{B}_n^{p_1,q_1}) \subseteq 2^{1/p_0+2/q_0}(1 + \tau n^{1/p_0-1/p_1})\mathbf{B}_n^{p_0,q_0}$$

and (for $i \ne j$)

$$\left(x_i + \frac{\tau}{2^{1/p_1+2/q_1}}\mathbf{B}_n^{p_1,q_1}\right) \cap \left(x_j + \frac{\tau}{2^{1/p_1+2/q_1}}\mathbf{B}_n^{p_1,q_1}\right) = \emptyset.$$

Hence

$$N\left(\frac{\tau}{2^{1/p_1+2/q_1}}\right)^n \mathrm{Vol}(\mathbf{B}_n^{p_1,q_1}) \le 2^{n/p_0+2n/q_0}\left(1 + \tau n^{1/p_0-1/p_1}\right)^n \mathrm{Vol}(\mathbf{B}_n^{p_0,q_0}),$$

i.e.,

$$N \le 2^{n\alpha}\left(\frac{1 + \tau n^{1/p_0-1/p_1}}{\tau}\right)^n \frac{\mathrm{Vol}(\mathbf{B}_n^{p_0,q_0})}{\mathrm{Vol}(\mathbf{B}_n^{p_1,q_1})},$$

where $\alpha = 1/p_0 + 1/p_1 + 2/q_0 + 2/q_1$.

In order to have 2^{k-1} on the right-hand side we put

$$\tau = \left[2^{\frac{k-1}{n}-\alpha}\left(\frac{\mathrm{Vol}(\mathbf{B}_n^{p_0,q_0})}{\mathrm{Vol}(\mathbf{B}_n^{p_1,q_1})}\right)^{-1/n} - n^{1/p_0-1/p_1}\right]^{-1}.$$

Thanks to Theorem 3.3.3, there exists a suitable integer γ such that for $k \ge \gamma n$ it holds that

$$2^{\frac{k-1}{n}-\alpha} \ge 2\left(\frac{\mathrm{Vol}(\mathbf{B}_n^{p_0,q_0})}{\mathrm{Vol}(\mathbf{B}_n^{p_1,q_1})}\right)^{1/n} n^{1/p_0-1/p_1},$$

so

$$\begin{aligned}2^{\frac{k-1}{n}-\alpha}\left(\frac{\mathrm{Vol}(\mathbf{B}_n^{p_0,q_0})}{\mathrm{Vol}(\mathbf{B}_n^{p_1,q_1})}\right)^{-1/n} - n^{1/p_0-1/p_1} &\ge \frac{2^{\frac{k-1}{n}-\alpha}}{2}\left(\frac{\mathrm{Vol}(\mathbf{B}_n^{p_0,q_0})}{\mathrm{Vol}(\mathbf{B}_n^{p_1,q_1})}\right)^{-1/n}\\ &\ge c2^{\frac{k-1}{n}} n^{1/p_0-1/p_1},\end{aligned}$$

where c does not depend on n. We may conclude

$$e_k(\mathrm{Id} : \ell_n^{p_0,q_0} \to \ell_n^{p_1,q_1}) \le \tau \le c^{-1}2^{-\frac{k-1}{n}} n^{1/p_1-1/p_0}.$$

This finishes the proof for $k \ge \gamma n$.

To complete the whole proof we need to "fill the gaps" for k's between $\frac{1}{2}\log_2(n)$ and $\log_2(n)$ or $n/4$ and γn, respectively. This follows from the monotonicity of the entropy numbers and the fact that the entropy numbers in the endpoints of these intervals differ only by a multiplicative constant, therefore these k's can be incorporated at the cost of larger constants of equivalence. $\square$

As a complementary statement, we show that the entropy numbers in the limiting case where $p_0 = p_1$ (specifically $e_k(\mathrm{Id} : \ell_n^{1,\infty} \to \ell_n^1)$) behave differently. First we present a combinatorial lemma from the coding theory which will be used in the main proof. The proof can be found in [4] and therefore we omit it.

Lemma 4.2.3. *Let $k \le n \in \mathbb{N}$, then there exist M subsets $T_1, \dots, T_M$ of $\{1, \dots, n\}$ such that*

(i) $M \ge \left(\frac{n}{4k}\right)^{k/2}$,

(ii) $|T_i| = k$ *for all* $i \in \{1, \dots, M\}$,

(iii) $|T_i \cap T_j| \le k/2$ *for all* $i \neq j$, $i, j \in \{1, \dots, M\}$.

Theorem 4.2.4. *Let $k, n \in \mathbb{N}$, then*

$$e_k(\mathrm{Id} : \ell_n^{1,\infty} \to \ell_n^1) \approx \begin{cases} \log(1 + n/k), & 1 \le k \le n, \\ 2^{-\frac{k-1}{n}}, & k \ge n, \end{cases}$$

where the constants do not depend on k and n.

Proof. (sketch) We divide the proof into four steps, in which we obtain upper and lower estimates for the both cases.

(i) the lower bound for $k \ge n$

Let $\mathbf{B}_n^{1,\infty}$ be covered by 2^{k-1} balls in ℓ_n^1 with radius $r > 0$, then using the volume argument gives (3.2)

$$\sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{1,\infty})} \le 2^{\frac{k-1}{n}} r \sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^1)}.$$

Combined with Theorem 3.3.3 we obtain a lower bound for all $k \in \mathbb{N}$. (However, for $1 \le k \le n$ it is not optimal.)

(ii) the upper bound for $k \ge \gamma n$

We use the volume argument again, mimicking the part (iv) in the proof of Theorem 4.2.2. Let us have $\tau > 0$, which will be specified later, and let $\{x_1, \dots, x_N\} \subseteq \mathbf{B}_n^{1,\infty}$ be a maximal ε-distant set in the ℓ_n^1-norm. Due to Theorem 4.1.5 and Proposition 4.1.6 we have

$$(x_i + \tau \mathbf{B}_n^1) \subseteq 2(1+\tau)\mathbf{B}_n^{1,\infty}$$

and (for $i \neq j$)

$$\left(x_i + \frac{\tau}{2}\mathbf{B}_n^1\right) \cap \left(x_j + \frac{\tau}{2}\mathbf{B}_n^1\right) = \emptyset.$$

Therefore

$$N \leq 4^n \left(\frac{1+\tau}{\tau}\right)^n \frac{\mathrm{Vol}(\mathbf{B}_n^{1,\infty})}{\mathrm{Vol}(\mathbf{B}_n^1)}.$$

By setting the right-hand side to 2^{k-1} we get

$$\tau = \left[\frac{2^{\frac{k-1}{n}}}{4}\left(\frac{\mathrm{Vol}(\mathbf{B}_n^{1,\infty})}{\mathrm{Vol}(\mathbf{B}_n^1)}\right)^{-1/n} - 1\right]^{-1}.$$

As the ratio of the volumes is equivalent to 1, there exists γ a suitable integer such that for $k \geq \gamma n$ it holds that

$$\frac{2^{\frac{k-1}{n}}}{8}\left(\frac{\mathrm{Vol}(\mathbf{B}_n^{1,\infty})}{\mathrm{Vol}(\mathbf{B}_n^1)}\right)^{-1/n} \geq 1,$$

i.e.,

$$\frac{2^{\frac{k-1}{n}}}{4}\left(\frac{\mathrm{Vol}(\mathbf{B}_n^{1,\infty})}{\mathrm{Vol}(\mathbf{B}_n^1)}\right)^{-1/n} - 1 \geq \frac{2^{\frac{k-1}{n}}}{8}\left(\frac{\mathrm{Vol}(\mathbf{B}_n^{1,\infty})}{\mathrm{Vol}(\mathbf{B}_n^1)}\right)^{-1/n} \geq c 2^{\frac{k-1}{n}},$$

where c does not depend on n. We may conclude

$$e_k(\mathrm{Id} : \ell_n^{1,\infty} \to \ell_n^1) \leq \tau \leq c^{-1} 2^{-\frac{k-1}{n}}.$$

(iii) the lower bound for $1 \leq k \leq n/200$

Let $n \geq 200$ and $1 \leq k \leq n/200$. Denote ν the largest integer such that $12 \cdot 4^\nu \leq n$ and η an integer between 1 and ν. We apply Lemma 4.2.3 with k replaced by 4^l for every $\eta \leq l \leq \nu$. We obtain a system $T_1^l, \dots, T_{M_l}^l$ of subsets of $\{1, \dots, n\}$ such that for every $1 \leq i \neq j \leq M_l$ it holds that $|T_i^l| = 4^l$ and $|T_i^l \cap T_j^l| < 4^l/2$. Moreover, we know that

$$M_l \geq \left(\frac{n}{4^{l+1}}\right)^{4^l/2} \geq \left(\frac{n}{4^{\eta+1}}\right)^{4^\eta/2}$$

as the function $f(x) = \left(\frac{n}{4^{x+1}}\right)^{4^x/2}$ is increasing on $[1, \nu]$. We set

$$M = \left(\frac{n}{4^{\eta+1}}\right)^{4^\eta/2}.$$

For $j \in \{1, \dots, M\}$ set

$$\begin{aligned}
\tilde{T}_j^\eta &= T_j^\eta, \\
\tilde{T}_j^{\eta+1} &= T_j^{\eta+1} \setminus T_j^\eta, \\
&\vdots \\
\tilde{T}_j^\nu &= T_j^\nu \setminus (T_j^{\nu-1} \cup \dots \cup T_j^\eta).
\end{aligned}$$

The sets $\tilde{T}^l_j$ for fixed j are mutually disjoint with at most 4^l elements. Furthermore, $|\tilde{T}^\eta_j| = 4^\eta$ and for $\eta < l \le \nu$ we have

$$|\tilde{T}^l_j| \ge |T^l_j| - \sum_{s=\eta}^{l-1} |T^s_j| = 4^l - \sum_{s=\eta}^{l-1} 4^s \ge \frac{2}{3}4^l.$$

We associate these sets for l fixed with a vector $x^l \in \mathbb{R}^n$ such that

$$x^l = \sum_{l=\eta}^{\nu} \frac{1}{4^l}\chi_{\tilde{T}^l_j},$$

where $\chi_{\tilde{T}^l_j}$ is the indicator function of the set $\tilde{T}^l_j$. We obtain vectors $x^1, \dots, x^M$.

Observe now that $u \in \{1, \dots, n\}$ belongs to $\tilde{T}^l_j$ for at most one l between η and ν, and therefore

$$\begin{aligned}||x^j||_{1,\infty} &\le \max\Big\{4^\eta \frac{1}{4^\eta}, \left(4^\eta + 4^{\eta+1}\right)\frac{1}{4^{\eta+1}}, \dots, \left(4^\eta + 4^{\eta+1} + \dots + 4^\nu\right)\frac{1}{4^\nu}\Big\}\\ &\le 1 + \frac{1}{4} + \frac{1}{4^2} + \dots = \frac{4}{3}.\end{aligned}$$

Let now be $i \ne j$ and $u \in \tilde{T}^l_i \setminus \tilde{T}^l_j$. Then

$$|(x^i)_u - (x^j)_u| \ge \frac{1}{4^l} - \frac{1}{4^{l+1}} = \frac{3}{4}\cdot\frac{1}{4^l}.$$

Together combined, we get

$$\begin{aligned}||x^i - x^j||_1 &\ge \sum_{l=\eta}^{\nu}\sum u \in \tilde{T}^l_i \setminus \tilde{T}^l_j |(x^i)_u - (x^j)_u| \ge \sum_{l=\eta}^{\nu} \frac{3}{4}\cdot\frac{1}{4^l}|\tilde{T}^l_i \setminus \tilde{T}^l_j|\\ &= \frac{3}{4}\left[\sum_{l=\eta}^{\nu}\frac{1}{4^l}|\tilde{T}^l_i| - \sum_{l=\eta}^{\nu}\frac{1}{4^l}|\tilde{T}^l_i \cap \tilde{T}^l_j|\right]\\ &\ge \frac{3}{4}\left[1 + \sum_{l=\eta+1}^{\nu}\frac{1}{4^l}\cdot\frac{2}{3}4^l - \sum_{l=\eta}^{\nu}\frac{1}{4^l}|\tilde{T}^l_i \cap \tilde{T}^l_j|\right]\\ &\ge \frac{3}{4}\left[1 + \frac{2}{3}(\nu - \eta) - \sum_{l=\eta}^{\nu}\frac{1}{4^l}\cdot\frac{4^l}{2}\right]\\ &= \frac{3}{4}\left[1 + \frac{2}{3}(\nu - \eta) - \frac{1}{2}(\nu - \eta + 1)\right] \ge \frac{1}{8}(\nu - \eta + 1).\end{aligned}$$

This yields that $\{x^1, \dots, x^M\}$ is a $\frac{1}{8}(\nu - \eta + 1)$-distant set and $||x^i||_{1,\infty} \le 4/3$. Now if k satisfies the condition $2^{k-1} \le M$, we may conclude that $e_k(\mathrm{Id} : \ell_n^{1,\infty} \to \ell_n^1) \ge c(\nu - \eta + 1)$, where $c > 0$ (it can be taken $3/64$).

Due to our assumptions on k and n we can now take η as the smallest integer such that $k \le 4^\eta/2$. Therefore $n/4^{\eta+1} \ge 2$ and so $2^{k-1} \le M$. We finish by

$$4^{\nu+1-\eta+1} \ge \frac{n}{12}\cdot\frac{1}{2k} = \frac{n}{24},$$

which with Lemma 4.2.1 implies that

$$\nu - \eta + 1 \geq \log_4\left(\frac{n}{96k}\right) \gtrsim \log(1+n/k).$$

There are only finitely many $1 \leq k \leq n < 200$, so they are covered at the cost of the multiplicative constants.

(iv) the lower bound for $1 \leq k \leq n$

Let us have an integer $1 \leq l \leq n/2$ and $x \in \mathbf{B}_n^{1,\infty}$. We set $S \subseteq \{1,\dots,n\}$ to be the indices of its l largest coordinates (in absolute value). Denote $x_S \in \mathbb{R}^n$ its restriction to S, i.e., $x_S = x\chi_S$, where χ_S is the indicator function of S. Then

$$||x - x_S||_1 \leq \sum_{k=l+1}^{n} \frac{1}{k} \leq \int_l^n \frac{1}{x} dx = \log(n/l).$$

Thanks to (ii) there exist constants γ and $c > 0$ (both independent of n and l) such that $e_{\gamma l}(\mathrm{Id} : \ell_l^{1,\infty} \to \ell_l^1) < c$. Therefore there is a set $N \subseteq \mathbb{R}^l$ such that $|N| = 2^{\gamma l - 1}$ which is a c-net of $\mathbf{B}_n^{1,\infty}$ in the ℓ_n^1-norm. We can embed N into $\mathbb{R}^n$ by extending it by zero outside of S. We get $N_S \subseteq \mathbb{R}^n$ which is a c-net of

$$\{x \in \mathbf{B}_n^{1,\infty} : x_i = 0 \text{ for } i \notin S\}.$$

By taking union of all those sets N_S such that $|S| = l$ we obtain a set of $2^{\gamma l-1}\binom{n}{l}$ points which is a $(c + \log(n/l))$-net of $\mathbf{B}_n^{1,\infty}$ in the ℓ_n^1-norm. Therefore whenever $2^{k-1} \geq 2^{\gamma l - 1}\binom{n}{l}$, we may conclude that

$$e_k(\mathrm{Id} : \ell_n^{1,\infty} \to \ell_n^1) \leq c + \log(n/l).$$

We use the estimate $\binom{n}{l} \leq (en/l)^l$ and we may assume that $\gamma \geq 2$ to obtain $\gamma l \log(en/l) \geq l \log_2(en/l)$ and

$$k \geq \gamma l\,(1 + \log(en/l)) \implies e_k(\mathrm{Id} : \ell_n^{1,\infty} \to \ell_n^1) \lesssim 1 + \log(n/l).$$

Let $2\alpha\gamma\log(n) \leq k \leq n$, where $\alpha \geq 1$ will be chosen later. Let

$$\frac{k}{2\alpha\gamma\log(en/k)} \leq l \leq \frac{k}{\alpha\gamma\log(en/k)}.$$

As $\alpha\gamma\log(en/k) \geq 2\log(e)$, the right-hand side is at most $n/2$ and

$$\begin{aligned}
\gamma l(1 + \log(en/l)) &\leq \gamma\frac{k}{\alpha\gamma\log(en/k)}\left[1 + \log\left(en\alpha\gamma\log(en/k)/k\right)\right] \\
&= \frac{k}{\alpha\log(en/k)}\left[1 + \log(en/k) + \log(\alpha\gamma) + \log(\log(en/k))\right] \\
&\leq \frac{k}{\alpha}\left[1 + 1 + \log(\alpha\gamma) + 1/2\right] \leq \frac{3 + \log(\alpha\gamma)}{\alpha}k,
\end{aligned}$$

because $f(x) = x(1 + \log(en/x))$ is increasing on $[1, n]$.

If α is chosen in such a way that $e^3\alpha\gamma \leq e^\alpha$ (which is easily met), we may conclude that

$$k \geq \frac{3 + \log(\alpha\gamma)}{\alpha} k \geq \gamma l(1 + \log(en/l)).$$

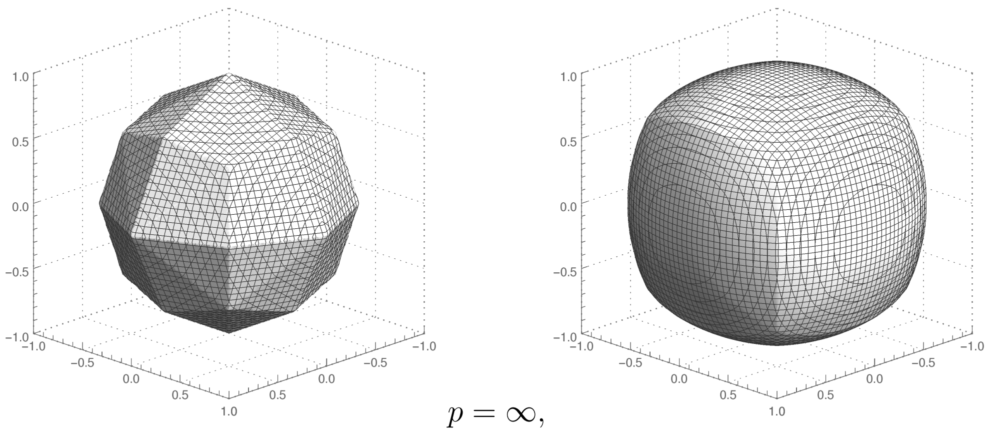
This finishes the proof for n large enough and $2\alpha\gamma \log(n) \leq k \leq n$. For $1 \leq k \leq 2\alpha\gamma \log(n)$ we use Proposition 4.3, which implies that

$$e_1(\mathrm{Id} : \ell_n^{1,\infty} \to \ell_n^1) \approx 1 + \log(n),$$

and the monotonicity of the entropy numbers. The rest of n's (the small ones) can be incorporated again at the cost of the constants.

The gaps between n and γn or $n/200$ and n, respectively, can be covered as in the proof of Theorem 4.2.2. □

4.3 Concerning $p = \infty$



$p = \infty$, $q = 1$ $p = \infty$, $q = 2$

Figure 4.3: Sets $\mathbf{B}_3^{\infty,1}$ and $\mathbf{B}_3^{\infty,2}$

In this section we consider the up-to-now omitted case when $p = \infty$ and q is finite. We have

$$||a||_{\infty,q} = ||k^{-\frac{1}{q}} a_k^*||_q = \Big(\sum_{k\in\mathbb{N}} (a_k^*)^q k^{-1} \Big)^{\frac{1}{q}}.$$

Though there is no big change at the first sight (cf. Figure 4.3), we show that the geometry of this space might differ from the geometry of the Lorentz spaces. This is due to the fact that in this case the quasinorm $||\cdot||_{\infty,q}$ is not equivalent to the original Lorentz quasinorm. As we now demand q to be finite, we study the case $q = 1$, since we have an explicit formula for the volume of the unit ball in $\ell_n^{p,1}$ for all p finite. When we inspect the course of the proof, we realize that it works also for the case $p = \infty$ (identify $1/\infty = 0$). Therefore we have

$$\mathrm{Vol}(\mathbf{B}_n^{\infty,1}) = 2^n \prod_{k=1}^{n} \frac{1}{\varkappa_\infty(k)}, \text{ where } \varkappa_\infty(k) = \sum_{j=1}^{k} \frac{1}{j}.$$

We can easily deduce from

$$\log(k+1) = \int_1^{k+1} x^{-1} dx \leq \sum_{j=1}^{k} j^{-1} \leq \int_0^k \min\{1, x^{-1}\} dx = 1 + \log(k).$$

that

$$\varkappa_\infty(k) \approx \log(k+1)$$

(the constants are independent of k). This yields

$$\sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{\infty,1})} \approx \left(\prod_{k=1}^{n} \frac{1}{\log(k+1)}\right)^{1/n} \geq \left(\prod_{k=1}^{n} \frac{1}{\log(n+1)}\right)^{1/n} = (\log(n+1))^{-1}.$$

We want to show that it is, in fact, an equivalence. The second inequality can be obtained by using the inequality between the geometric and arithmetic mean:

$$\left(\prod_{k=1}^{n} \frac{1}{\log(k+1)}\right)^{1/n} \leq \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\log(k+1)} \leq \frac{1}{n\log(2)} + \frac{1}{n} \int_2^{n+1} \frac{1}{\log(t)} dt.$$

By using L'Hospital's rule we can deduce that $\int_2^{x+1} \frac{1}{\log(t)} dt \approx \frac{x}{\log(x+1)}$ for all $x > 1$. Therefore

$$\left(\prod_{k=1}^{n} \frac{1}{\log(k+1)}\right)^{1/n} \lesssim \frac{1}{\log(n+1)}.$$

Together we have

$$\sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{\infty,1})} \approx (\log(n+1))^{-1},$$

which is different behaviour than for the Lorentz spaces. As we know that

$$\sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{\infty})} = 2,$$

it is reasonable to presume that probably

$$\sqrt[n]{\mathrm{Vol}(\mathbf{B}_n^{\infty,q})} \approx (\log(n+1))^{-1/q}.$$

However, we do not pursue this hypothesis further.

Bibliography

[1] Aoki, Tosio: *Locally bounded linear topological spaces.* Proceedings of the Imperial Academy 18(10):588-594. Do Gakushuin, 1942.

[2] Bennett, Colin, Sharpley, Robert C.: *Interpolation of Operators.* Pure and Applied Mathematics vol. 129. Academic Press, 1988.

[3] Bergh, Jöran, Löfström, Jörgen: *Interpolation Spaces: An Introduction.* Grundlehren der mathematischen Wissenschaften vol. 223. Springer, 1976.

[4] Boche, Holger, Calderbank, Robert, Kutyniok, Gitta, Vybíral, Jan:: A Survey of Compressed Sensing. In: *Compressed Sensing and its Applications*:1-39. Birkhäuser, 2015.

[5] Brezis, Haïm, Mironescu, Petru: Gagliardo-Nirenberg, composition and products in fractional Sobolev spaces. *Journal of Evolution Equations* 1(4):387-404. Springer, 2001.

[6] Edmunds, David E., Triebel, Hans: *Function Spaces, Entropy Numbers, Differential Operators.* Cambridge Tracts in Mathematics vol. 120. Cambridge University Press, 2008.

[7] Grafakos, Loukas: *Classical Fourier Analysis.* Graduate Texts in Mathematics vol. 249. Springer, 2008.

[8] Hinrichs, Aicke, Kolleck, Anton, Vybíral, Jan: Carl's inequality for quasi-Banach spaces. *Journal of Functional Analysis* 271(8):2293-2307. Elsevier, 2016.

[9] Hunt, Richard A.: An extension of the Marcinkiewicz interpolation theorem to Lorentz spaces. *Bulletin of the American Mathematical Society* 70(6):803-807. American Mathematical Society, 1964.

[10] Jameson, Graham J. O.: A simple proof of Stirling's formula for the gamma function. *The Mathematical Gazette* 99(544):68-74. Cambridge University Press, 2015.

[11] Kossaczká, Marta, Vybíral, Jan: Entropy numbers of finite-dimensional embeddings. To appear in *Expositiones Mathematicae.* Accessible at `https://doi.org/10.1016/j.exmath.2019.04.001`.

[12] Lorentz, George G.: Some new functional spaces. *Annals of Mathematics* 51:37-55. Princeton University, 1950.

[13] Meyer, Yves: Wavelets, paraproducts, and Navier-Stokes equations. *Current developments in mathematics* 1996(1):105-212. International Press, 1996.

[14] Nasdala, Robert: *Volumen von Einheitskugeln in Folgenräumen.* Bachelor thesis. Accessible at `https://tu-chemnitz.de/~mnas/paper/BA.pdf`. Technical University Chemnitz, 2016.

[15] Opic, Bohumír, Pick, Luboš: On generalized Lorentz-Zygmund spaces. *Mathematical Inequalities and Applications* 2(3):391-467. Element, 1999.

[16] Pisier, Gilles: *The Volume of Convex Bodies and Banach Space Geometry*. Cambridge Tracts in Mathematics vol. 94. Cambridge University Press, 1999.

[17] Rudin, Walter: *Functional Analysis*. 2nd Edition. McGraw-Hill, 1991.

[18] Whittaker, Edmund T., Watson, George N.: *A Course of Modern Analysis*. 3rd Edition. Cambridge University Press, 1920.

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