Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

DIPLOMOVÁ PRÁCE

Petr Pošta

Bairovské a harmonické funkce

Katedra matematické analýzy

Vedoucí diplomové práce: *Prof. RNDr. Jaroslav Lukeš, DrSc.*
Studijní program: *Matematika, matematická analýza*

2008

Poděkování: Chtěl bych především poděkovat vedouímu své diplomové práce, prof. RNDr. Jaroslavu Lukešovi, DrSc., za věnovaný čas, podnětné poznámky a rady a nikoliv naposled za upozornění na řadu matematických nepřesností i stylistických chyb, které se v průběhu prací na tezi vyskytly.

Dále bych chtěl touto formou poděkovat všem, kteří jakkoliv přispěli k tomuto textu, především Mgr. Miroslavu Poštovi a Ing. Severinu Poštovi, PhD. za poskytnutou morální podporu a různé inspirativní návrhy a poznámky. Stejný dík patří také rodičům Mgr. Marii Poštové a Ing. Severínu Poštovi, jim ale patří též díky mnohonásobně větší za pěkné rodinné zázemí po pětadvacet let života.

Díky dále patří i některým spolužákům, především Bc. Pavlu Ludvíkovi, se kterým byla radost diskutovat o "diplomce", jejích matematických i nematematických aspektech (i o mnoha věcech jiných).

A v neposlední řadě patří díky také babičce Haně Kružíkové; vyjmenovat všechna její přičinění by vydalo na několikanásobně delší knihu než je celá tato teze.

A special thanks goes to professor Stephen James Gardiner for a nice, detailed and patient answers to many questions concerning harmonic approximation and potential theory.

Věnováno RNDr. Miroslavu Kružíkovi, CSc.

Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

V Praze dne 2008 Petr Pošta

Contents

Název práce: Baireovské a harmonické funkce
Autor: Petr Pošta
Katedra: Katedra matematické analýzy
Vedoucí diplomové práce: Prof. RNDr. Jaroslav Lukeš, DrSc.
E-mail vedoucího: lukes@karlin.mff.cuni.cz

Abstrakt: Je známo, že Dirichletova úloha na U omezené otevřené podmnožině Eukleidovského prostoru $\mathbb{R}^n$ nemusí mít vždy klasické řešení, ale že je vždy možné přiřadit zobecněné Perron-Wiener-Brelotovo řešení, které se shoduje s klasickým, pokud existuje. Toto zobecněné řešení je funkce první Baireovy třídy na uzávěru množiny U. Lukeš a kol. (2003) ukázali, že toto zobecněné řešení je dokonce bodovou limitou posloupnosti funkcí harmonických na množině U a spojitých do uzávěru. První část práce popisuje důkaz tohoto tvrzení pomocí bodové aproximace v simpliciálních funkčních prostorech a shrnuje k tomu potřebné poznatky z abstraktní Choquetovy teorie a funkcí první Baireovy třídy. Druhá se pak zabývá otevřeným problémem charakterizace prostoru $\mathcal{B}_1(H(U))$ bodových limit funkcí harmonických na množině U a spojitých do uzávěru. Podobný problém vyřešili Gardiner a Gustafsson (2005) pro prostor $\mathcal{B}_1(H_0(K))$ funkcí harmonických na nějakém okolí K kompaktní podmnožiny $\mathbb{R}^n$. V práci je dokázána nutná podmínka pro funkce patřící do $\mathcal{B}_1(H(U))$ a je ukázáno, že za určitých dodatečných předpokladů na množinu U je též postačující.

Klíčová slova: Dirichletův problém, harmonické funkce, funkce první Baireovy třídy, simpliciální prostory

Title: Baire and Harmonic Functions
Author: Petr Pošta
Department: Departement of mathematical analysis
Supervisor: Prof. RNDr. Jaroslav Lukeš, DrSc.
Supervisor's e-mail address: lukes@karlin.mff.cuni.cz

Abstract: It is well known that the Dirichlet problem on a bounded open subset U of $\mathbb{R}^n$ need not have a classical solution. However, it is always possible to construct a generalized (Perron-Wiener-Brelot) solution, which is identical with the classical one if it exists. This generalized solution is a Baire-one function on the closure of U. Lukeš et al. (2003) proved that PWB-solution is even a pointwise limit of a sequence of functions harmonic on U and continuous to the boundary. The first part of this thesis describes their proof which uses pointwise approximation in simplicial function spaces and compiles a necessary theoretic background, especially parts of abstract Choquet theory and a characterization of Baire-one functions. The second part treats an open problem how to characterize a space $\mathcal{B}_1(H(U))$ of pointwise limits of sequences of functions harmonic on U and continuous to the boundary. A similar problem was solved by Gardiner and Gustafsson (2005). They characterized a space $\mathcal{B}_1(H_0(K))$ of pointwise limits of sequences of functions harmonic on some neighbourhood of a compact subset K of $\mathbb{R}^n$. In the thesis, a necessary condition for functions in $\mathcal{B}_1(H(U))$ is proved and are presented several cases when, under additional assumptions on the set U, the condition is also sufficient.

Keywords: Dirichlet problem, harmonic functions, Baire-one functions, simplicial spaces

Introduction

It is almost one hundred and ten years, since René-Louis Baire (1874-1932) designed a classification of functions into, nowadays, so called Baire classes. The zero-th Baire class consists of all continuous functions and each class that follows consists of pointwise limits of functions in the previous class(es). It is often useful to know that a function is of Baire class one. It does not need to be continuous but still cannot be "too terrible". For example, if f is a Baire one function on a metric space P, then every nonempty closed subset F of P contains a point x such that f restricted to F is continuous at x. So the behavior of functions in Baire class one can be somewhat controlled.

If one says the concept of Baire classes is old, then the Potential Theory is even one or two hundred years older. Perhaps its oldest problem is the one of Dirichlet, to find a function f which solves the Laplace equation

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2} = 0$$

on a given bounded open subset U of $\mathbb{R}^n$ and can be continuously extended to the boundary where it coincides with a prescribed boundary condition. Such a function is called a *classical solution* for the Dirichlet problem and functions which satisfy the Laplace equation on U are called *harmonic* on U. For a long time, it was believed the Dirichlet problem is solvable for all open sets and all continuous boundary data; some sort of arguments which should have supported this statement can be found in Dirichlet's work in this area but the original stone was a physical interpretation of the equation: it describes a gravitational or electromagnetic field and continuous boundary conditions simply characterizes its sources, so it was natural to assume something like that every distribution of electric charge would evoke a corresponding electric field.

Karl Weierstrass published a counterexample in 1895, about one hundred year later since the Laplace equation made his first appear on the stage of science. Although his counterexample appears rather easy and natural nowadays, a shock to physical belief was not easy to overcome and lead to an extensive study of various aspects of the problem.

A set U is called *regular* if there exists a classical solution for the Dirichlet problem with any continuous boundary data. However, several methods were developed which associate a solution of the Dirichlet problem even for irregular set U, the solution is harmonic on U, continuous on the boundary except of the set which is negligible in a certain sense and this *generalized solution* coincides with the classical one if that exists.

The properties of the generalized solution have been studied quite extensively. It does not have to be continuous but it is a function of Baire class one. The original argument was based on the study of so called *fine topology* – the coarsest topology in which every superharmonic function is continuous. The fine topology has a lot of bad properties, for example, it is not normal and compact sets are exactly the finite ones. However, it has some nice properties and one of them is: every finely continuous function is of Baire class one.

In 2003, a refinement of this statement was given by Lukeš et al. They proved that every generalized solution to the Dirichlet problem on U is a pointwise limit of a sequence of harmonic functions on U continuous to the boundary. Basic ideas of the proof are given in this thesis. In the same article, the properties of pointwise limits of continuous harmonic functions are studied but a complete characterization of the space was not made.

They were, however, able to achieve a rather general theorem for pointwise approximation in simplicial function spaces which was used two years later by Gardiner and Gustafsson to give a complete characterization of a similar space of pointwise limits of functions harmonic on some neighbourhood of a given compact subset K of $\mathbb{R}^n$. Their proof is also contained in this thesis.

The plan for the thesis is the following: Chapter one provides a short look at the abstract Choquet theory of function spaces. Chapter two is devoted to deriving an approximation theorem in simplicial spaces and contains also necessary background for that, mainly elements of convex analysis, the concept of state spaces and a nontrivial characterization of Baire one functions. In Chapter three, Gardiner's characterization of pointwise limits of functions harmonic on some neighbourhood of a given compact set is presented. The Chapter four is devoted to pointwise limits of functions harmonic on a bounded open set U which are continuous to the boundary. A necessary condition for a function to be in this space, analogous to the one of Gardiner, is proved here. The rest of the chapter provides some sufficient conditions and examples, but the complete characterization of this space still remains unclear.

Before proceeding, we give here some notation used in the following. The word positive stays for greater or equal to zero, the word strictly positive for greater than zero. Similarly, negative stays for less or equal to zero. Order, partial order, ordering and partial ordering are synonyms for us, it is a binary relation which is reflexive, antisymmetric and transitive.

The n-th dimensional Euclidean space is denoted as $\mathbb{R}^n$, where mostly n is assumed to be greater or equal to two. Whenever P is a metric space and d its metric, the symbol $B(x,r)$ stands for open ball with the center $x \in P$ and radius $r > 0$,

$$B(x,r) = \{y \in P : d(y,x) < r\}.$$

By open unit ball we mean the set $B(0,1)$.

In the first two chapters, however, we work mostly in the context of compact topological space K which is always meant to be Hausdorff. By $C(K)$, we mean a space of all continuous real-valued functions on K equipped with the supremum norm $\|\cdot\|_\infty$. Where it is obvious which norm is used we will write simply $\|\cdot\|$ without an index.

If $f : X \to Y$ is a mapping and A is a subset of X, then the restriction of f to A is denoted as $f|_A$.

If X is a topological space and A is a subset of X, then χ_A denotes a *characteristic function* of A defined by one on A and zero elsewhere.

If X is a Banach space or a locally convex space, we denote X^* the corresponding dual space (of all continuous linear functionals on X). On the dual space to the Banach space, we recognized three different topologies: a natural one induced by the dual norm, a weak topology w and a weak star topology w^*.

For a subset A of a topological space (X,τ), we denote $\operatorname{int}_\tau A$ or $A^{\circ\tau}$ the interior of A in the topology τ. We write simply $\operatorname{int} A$ or A° in the case the topology is natural for the space, namely for an Euclidean topology in $\mathbb{R}^n$. The same goes for the closure $\overline{A}^\tau$ and the boundary $\partial_\tau A$, we shall write simply $\overline{A}$ and ∂A if no mistake can arise. Especially, we write $\overline{A}^{\|\cdot\|}$ for the closure of A in the topology induced by a given norm and $\overline{A}^w$, resp. $\overline{A}^{w^*}$ for the closure of A in the weak, resp. weak star topology. We use a symbol A^c for the complement of A in X, that is, $A^c = X \setminus A$.

The space of all (signed) Radon measures on a compact space K is denoted as $\mathcal{M}(K)$. We often identify this space with the dual space $(C(K))^*$ and we consider, if not said otherwise, a weak star topology on this space. By $\mathcal{M}^1(K)$, we denote a set of all positive probability Radon measures on K which is a convex and w^*-compact subset of $\mathcal{M}(K)$.

In the context of a locally convex space E, we denote $\operatorname{co} A$ a convex hull of the subset A of E, that is,

$$\operatorname{co} A = \{\sum_{i=1}^{n} \lambda_i x_i \ : \ \lambda_i > 0,\ \sum_{i=1}^{n} \lambda_i = 1 \text{ and } x_i \in A\}$$

and we denote $\overline{\operatorname{co}}\, A$ a closed convex hull, that is the smallest closed convex set which contains A. It is simple to see that

$$\overline{\operatorname{co}}\, A = \overline{\operatorname{co} A}.$$

And at last, we recall some classical theorems here for reader's convenience.

Theorem 0.1. ***(Green formula)***

Let $V \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Let $U \supset \overline{V}$ be an open set and $f, g \in C^2(U)$. Then

$$\int_V (f\Delta g - g\Delta f)\, d\lambda = \int_{\partial V} \left(f\frac{\partial g}{\partial n_e} - g\frac{\partial f}{\partial n_e} \right) d\sigma$$

where $\frac{\partial}{\partial n_e}$ denotes the exterior normal derivative at points of ∂V.

Theorem 0.2. ***(Separation theorem for LCS spaces)***

Let X be a locally convex space, A, B convex subsets of X, A compact and B closed. Then there exists continuous linear functional $F \in X^*$ and $c \in \mathbb{R}$ such that

$$F(a) < c < F(b), \qquad \text{for all } a \in A,\ b \in B.$$

Theorem 0.3. ***(Tietze's extension theorem)***

If X is a normal topological space and $f : A \to [-1, 1]$ is a continuous map from a closed subset A of X into the real numbers carrying the standard real line topology, then there exists a continuous extension of f to the whole space X.

Theorem 0.4. ***(Intersection of a system with finite intersection property)***

If X is a compact space and $\mathcal{F}$ is a collection of nonempty closed subsets of X which has a finite intersection property, that is, if $A_1, \ldots, A_n$ are elements of $\mathcal{F}$, then the intersection $\cap_{i=1}^n A_i$ is nonempty, then

$$\bigcap_{F \in \mathcal{F}} F \text{ is nonempty as well.}$$

Chapter 1

Abstract Choquet theory

1.1 Definitions and basic properties

Definition. (Function space, representing measures, Choquet boundary)

Let K be a compact space. We shall call *function space on* K any subspace of $C(K)$ which contains the constant functions and separates points of K, that is, if x, y are elements of K and $x \neq y$, then there exists a function f in the function space such that $f(x) \neq f(y)$.

Let $\mathcal{H}$ be a function space on K. Then any $\mu \in \mathcal{M}^1(K)$ is called $\mathcal{H}$*-representing measure* for $x \in K$ if

$$f(x) = \int_K f \, d\mu \qquad \text{for any } f \in \mathcal{H}.$$

A collection of all $\mathcal{H}$-representing measures for $x \in K$ will be denoted by $\mathcal{M}_x(\mathcal{H})$.

The set

$$\mathrm{Ch}_{\mathcal{H}}(K) = \{x \in K \;:\; \mathcal{M}_x(\mathcal{H}) = \{\varepsilon_x\}\}$$

is called the *Choquet Boundary* of $\mathcal{H}$.

We shall denote $\mathcal{H}$ a function space on compact space K during the rest of this section.

Definition. ($\mathcal{H}$-affine functions, barycentric formula)

A bounded Borel function $f : K \to \mathbb{R}$ is called $\mathcal{H}$*-affine function* if

$$f(x) = \int_K f \, d\mu \qquad \text{for all } x \in K \text{ and } \mu \in \mathcal{M}_x(\mathcal{H}). \tag{1.1}$$

The condition (1.1) will be called from now on *the barycentric formula*. A collection of all $\mathcal{H}$-affine function form a space of functions on K and it is denoted by $\mathcal{A}(\mathcal{H})$. Its subspace which contains *continuous* $\mathcal{H}$-affine functions is denoted by $\mathcal{A}^c(\mathcal{H})$.

In the following, we shall denote (whenever it makes sense)

$$\mu(f) = \int_K f\, d\mu.$$

Let $(\mathcal{E}(K), \|\cdot\|)$ be a linear space of all bounded Borel functions on a compact space K equipped with the supremum norm. If $F \subset \mathcal{E}(K)$, then we define

$$F^{\perp} = \{\mu \in \mathcal{M}(K) \ : \ \mu(f) = 0 \quad \text{for all } f \in F\}.$$

We recall that $\mathcal{M}(K)$ denotes the family of all (signed) Radon measures on K.

Definition. (Completely $\mathcal{H}$-affine function)

Let $f : K \to \mathbb{R}$ be a bounded Borel function. We say that f is *completely $\mathcal{H}$-affine* if

$$\mu(f) = 0 \qquad \text{for all } \mu \in \mathcal{H}^{\perp}.$$

The family of all completely $\mathcal{H}$-affine functions will be denoted by $\mathbf{A}(\mathcal{H})$. Let us remark that the set $\mathbf{A}^c(\mathcal{H})$ of continuous completely $\mathcal{H}$-affine functions coincides with the closure of $\mathcal{H}$ in the supremum norm.

$$\mathbf{A}^c(\mathcal{H}) = \overline{\mathcal{H}}.$$

Indeed, if $f \in \overline{\mathcal{H}}$, then there exists a sequence of functions $\{f_n\}$ in $\mathcal{H}$ such that f_n converges uniformly to f and by the Lebesgue dominated convergence theorem $\mu(f) = \lim \mu(f_n) = 0$. On the other hand, let us assume that there exists $f \in \mathbf{A}^c(\mathcal{H}) \setminus \overline{\mathcal{H}}$. Then one can find a Radon measure (as a member of the dual space of $C(K)$) such that $\mu(f) = 1$ and $\mu(g) = 0$ for all $g \in \overline{\mathcal{H}}$. Since $\overline{\mathcal{H}} \supset \mathcal{H}$, then $\mu \in \mathcal{H}^{\perp}$ and this argument would lead to $\mu(f) = 0$. This is an obvious contradiction.

Definition. ($\mathcal{H}$-convex functions, Choquet ordering, simplicial spaces)

Let $\mathcal{H}$ be a function space on a compact space K. A bounded Borel function f on K is called *$\mathcal{H}$-convex* if

$$f(x) \le \mu(f) \qquad \text{for all } x \in K \text{ and } \mu \in \mathcal{M}_x(\mathcal{H}).$$

The family of all $\mathcal{H}$-convex continuous functions on K forms a convex cone which will be denoted $\mathcal{K}^c(\mathcal{H})$. Now, let μ, ν be positive Radon measures on K. We define

$$\mu \prec \nu \ \overset{def}{\Longleftrightarrow} \ \mu(f) \le \nu(f) \qquad \text{for each } f \in \mathcal{K}^c(\mathcal{H}).$$

The relation "$\prec$" is a partial ordering, so called *Choquet ordering*, on the set of all positive Radon measures on K. A rather deep result of Choquet's theory is that, with respect to this ordering, there is always a maximal measure in $\mathcal{M}_x(\mathcal{H})$.

A function space $\mathcal{H}$ is called *simplicial* if for each $x \in K$ there is a unique maximal measure in $\mathcal{M}_x(\mathcal{H})$.

If $\mathcal{H}$ is a function space, then we also define an *upper envelope* for an upper bounded function f on compact space K as the function

$$f^* : x \mapsto \inf\{h(x) \ : \ h \in \mathcal{H},\ h \geq f \text{ on } K\}.$$

Theorem 1.1. ***(Mokobodzki's maximality test)***

A positive Radon measure μ on K is maximal if and only if $\mu(k) = \mu(k^*)$ for any $k \in \mathcal{K}^c(\mathcal{H})$.

Theorem 1.2. ***(Edwards separation theorem)***

Let $\mathcal{H}$ be a simplicial function space, $-f, g$ be continuous $\mathcal{H}$-convex functions and $g \leq f$. Then there exists a continuous $\mathcal{H}$-affine function h such that

$$g \leq h \leq f \quad \text{on } K.$$

We note that the Edwards theorem in fact characterizes simpliciality. These two theorems above give us the following lemma which will be essential in deriving a key result in the following chapter.

Lemma 1.3. *Let $\mathcal{H}$ be a simplicial function space and δ_x be the (unique) maximal measure in $\mathcal{M}_x(\mathcal{H})$. Then*

$$\delta_x(g) = \delta_x(g^*) = g^*(x) \qquad \text{for any } g \in \mathcal{K}^c(\mathcal{H}).$$

Proof. (cf Lukeš et al. [17], Lemma 2.1–2.3.)

At first, we will show that whenever f is a continuous function on K and x a fixed point of K, then there exists $\mu \in \mathcal{M}_x(\mathcal{H})$ such that $f^*(x) = \mu(f)$.

The mapping $p : g \mapsto g^*(x)$ is a sublinear functional on $C(K)$ and by the Hahn-Banach theorem, we get a linear functional μ_f on $C(K)$ such that $\mu_f(f) = f^*(x)$ and $\mu_f(g) \leq g^*(x)$ for any $g \in C(K)$. Let now g be negative, then identical zero function on K is an element of $\mathcal{H}$ and majorizes g, hence

$$g \leq 0 \implies \mu_f(g) \leq g^*(x) = \inf\{h(x) \ : \ h \in \mathcal{H},\ h \geq g \text{ on } K\} \leq 0.$$

This means that μ_f is a positive linear functional, thus positive Radon measure on K. If now $h \in \mathcal{H}$, then

$$\mu_f(h) \leq h^*(x) = h(x), \qquad -\mu_f(h) = \mu_f(-h) \leq (-h)^*(x) = -h(x).$$

So $\mu_f(h) = h(x)$ for any $h \in \mathcal{H}$. Therefore, μ is a $\mathcal{H}$-representing measure of x.

This claim is still valid if we take f only upper semicontinuous on K. Let us consider a lower directed family $\mathcal{G}$ of all continuous functions which majorizes f on K. For each $g \in \mathcal{G}$ we can find a $\mathcal{H}$-representing measure μ_g of x such that $\mu_g(g) = g^*(x)$. Given $\varphi \in \mathcal{G}$, we denote

$$M_\varphi = \{\mu_g \ : \ g \in \mathcal{G},\ g \leq \varphi\}.$$

Since $\mathcal{M}_x(\mathcal{H})$ is w^*-compact and $\overline{M_\varphi}^{w^*}$ is a w^*-closed subset of $\mathcal{M}_x(\mathcal{H})$, for every $\varphi \in \mathcal{G}$ is the set $\overline{M_\varphi}^{w^*}$ a w^*-compact set. Therefore, the intersection taken over the entire family $\mathcal{G}$ is nonempty and there is

$$\mu \in \bigcap_{\varphi \in \mathcal{G}} \overline{M_\varphi}^{w^*}.$$

It is obvious that μ is an element of $\mathcal{M}_x(\mathcal{H})$. We observe

$$\inf\{\nu(\varphi) : \nu \in M_\varphi\} = \inf\{\nu(\varphi) : \nu \in \overline{M_\varphi}^{w^*}\} \leq \mu(\varphi) \qquad \text{for each } \varphi \in \mathcal{G}.$$

Hence

$$\begin{aligned} f^*(x) &\leq \inf\{g^*(x) \ : \ g \in \mathcal{G}\} = \inf\{\mu_g(g) \ : \ g \in \mathcal{G}\} \\ &\leq \inf\{\inf\{\mu_g(\varphi) \ : \ g \in \mathcal{G},\ g \leq \varphi\} : \varphi \in \mathcal{G}\} \leq \inf\{\mu(\varphi) \ : \ \varphi \in \mathcal{G}\} \\ &= \mu(f) \leq \inf\{\mu(h) \ : \ h \geq f,\ h \in \mathcal{H}\} = \inf\{h(x) \ : \ h \geq f,\ h \in \mathcal{H}\} = f^*(x) \end{aligned}$$

We shall need one other claim before proving the statement. In the definition of the upper envelope, we take an infimum over functions in the function space. We will show that no difference is caused by taking an infimum over all continuous $\mathcal{H}$-affine functions.

Given a bounded function f on K, fixed $x \in K$ and $g \in \mathcal{A}^c(\mathcal{H})$ such that $g \geq f$ on K, by the previous part we can find a measure $\mu \in \mathcal{M}_x(\mathcal{H})$ which satisfies $\mu(g) = g^*(x)$. Since $\mathcal{H}$ is a subset of $\mathcal{A}^c(\mathcal{H})$, it follows

$$\begin{aligned} g(x) = \mu(g) = g^*(x) \geq f^*(x) &= \inf\{h(x) \ : \ h \in \mathcal{H},\ h \geq f\} \\ &\geq \inf\{\tilde{g}(x) \ : \ \tilde{g} \in \mathcal{A}^c(\mathcal{H}),\ \tilde{g} \geq f\} \end{aligned}$$

and by taking an infimum over all functions in $\mathcal{A}^c(\mathcal{H})$ on the left side, we arrive to the equality

$$f^*(x) = \inf\{g(x) \ : \ g \in \mathcal{A}^c(\mathcal{H}),\ g \geq f\}.$$

We are now prepared for the proof of our lemma. Fix now $g \in \mathcal{K}^c(\mathcal{H})$, we want to show that

$$\delta_x(g) = \delta_x(g^*) = g^*(x).$$

The first equality is a direct consequence of the Mokobodzki test. For each $h \in \mathcal{H}$, since the measure δ_x is an element of $\mathcal{M}_x(\mathcal{H})$, we have $\delta_x(h) = h(x)$. Furthermore, the family $\{h \in \mathcal{A}^c(\mathcal{H}) \ : \ h \geq g\}$ is lower directed due to the Edwards theorem, since a simple reasoning shows that the function $-\min\{f_1, \ldots, f_n\}$ is a $\mathcal{H}$-convex function if $f_1, \ldots, f_n$ are elements of $\mathcal{H}$. Hence,

$$\begin{aligned} g^*(x) &= \inf\{h(x) \ : \ h \in \mathcal{A}^c(\mathcal{H}),\ h \geq g\} \\ &= \inf\{\delta_x(h) \ : \ h \in \mathcal{A}^c(\mathcal{H}),\ h \geq g\} \\ &= \delta_x(\inf\{h \ : \ h \in \mathcal{A}^c(\mathcal{H}),\ h \geq g\}) = \delta_x(g^*). \end{aligned}$$

The proof is complete. □

1.2 Subclasses of Baire-one functions

Definition. (Subclasses of Baire-one functions)

We recall that f is a *Baire-one function* on K if there exists a sequence $\{f_n\}$ of continuous functions on K such that $f_n \to f$ pointwise on K.

Now, we define the set

$$\begin{aligned} \mathcal{B}_1(\mathcal{H}) &= \{f : K \to \mathbb{R}, \text{ there exists a sequence } \{f_n\} \text{ in } \mathcal{H} \text{ such that} \\ &\quad f_n \to f \text{ pointwise on } K\}. \end{aligned}$$

Furthermore, we shall denote by $\mathcal{B}_1^b(\mathcal{H})$ a family of bounded elements of $\mathcal{B}_1(\mathcal{H})$ and $\mathcal{B}_1^{bb}(\mathcal{H})$ a subset of $\mathcal{B}_1(\mathcal{H})$ defined as functions which are pointwise limit of bounded sequences of elements of $\mathcal{H}$

$$\begin{aligned} \mathcal{B}_1^{bb}(\mathcal{H}) &= \{f : K \to \mathbb{R}, \text{ there exists a bounded sequence } \{f_n\} \text{ in } \mathcal{H} \text{ such that} \\ &\quad f_n \to f \text{ pointwise on } K\}. \end{aligned}$$

In the following, we shall denote $\mathcal{B}_1(K) = \mathcal{B}_1(C(K))$ where $\mathcal{B}_1(C(K))$ is in fact the space of Baire-one functions on K. Analogously, we shall denote $\mathcal{B}_1^b(K) = \mathcal{B}_1^b(C(K))$ and $\mathcal{B}_1^{bb}(K) = \mathcal{B}_1^{bb}(C(K))$.

Chapter 2

Approximation theorem in simplicial spaces

The purpose of this section is to establish the following theorem about pointwise approximation of bounded, Baire-one and $\mathcal{H}$-affine functions by continuous and $\mathcal{H}$-affine functions.

Theorem 2.1. ***(Pointwise approximation of bounded $\mathcal{H}$-affine functions)***

Let $K \subset \mathbb{R}^n$ be compact and $\mathcal{H}$ be a simplicial function space on K. Let $f : K \to \mathbb{R}$ be bounded, Baire-one and $\mathcal{H}$-affine. Then there exists a bounded sequence $\{h_n\}$ of continuous $\mathcal{H}$-affine functions which converges pointwise to f on K.

2.1 Affinity on compact convex sets

In this section, we shall recall a small part of functional analysis on compact convex sets and derive results that will be of use in the theory of state spaces. The concept of state space is an idea to inject a general function space into a suitable compact convex space in which we can use a lot of means presented here.

Definition. (Affine function, extreme points, barycenter)

Let X be a compact convex subset of a locally convex space. Then we call $f : X \to \mathbb{R}$ an *affine function* on X if

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda) f(y) \qquad \text{for all } x, y \in X \text{ and } \lambda \in [0,1].$$

We denote the space of all *continuous* affine functions on X by $A(X)$ and the space of all affine functions on X by $\mathfrak{A}(X)$. Obviously, $A(X)$ is a function space on X.

We call $z \in X$ an *extreme point* if there do not exist different points x, y of X and $\lambda \in (0,1)$, such that $z = \lambda x + (1-\lambda)y$. Alternatively, z is an extreme point if and only if $z = \frac{x+y}{2}$ for some x, y of X implies that $x = y$. We denote

$$\operatorname{ext} X = \{x \in X : x \text{ is an extreme point of } X\}.$$

We recall that the set $\mathcal{M}^1(X)$ denotes the set of all (positive) probabilistic Radon measures on X. Let μ be in $\mathcal{M}^1(X)$. We say that a point x of X is a *barycenter* of the measure μ and we also say that the measure μ *represents* x if

$$\mu(h) = h(x) \qquad \text{for each } h \in A(X).$$

Then we denote the barycenter of μ as $r(\mu)$.

Theorem 2.2. ***(Existence and uniqueness of the barycenter)***

Let E be a locally convex space, $X \subset E$ be a compact convex set and $\mu \in \mathcal{M}^1(X)$. Then there is a unique point $x \in X$ such that $x = r(\mu)$.

Furthermore, the mapping $r : \mathcal{M}^1(X) \to X$ defined as $r : \mu \mapsto r(\mu)$ is surjective, affine and continuous (if we take the space $\mathcal{M}^1(X)$ with w^*-topology and X with the original topology given by E).

Proof. *Uniqueness:* Let us consider $x, y \in X$ which satisfy the condition on barycenter. Then $h(x) = \mu(h) = h(y)$ for all $h \in A(X)$ and it implies that $x = y$, since the space of all continuous affine functions contains all continuous linear functionals and these functionals separate points of E.

Existence: For $h \in A(X)$, consider the set

$$X_h = \{x \in X \ : \ \mu(h) = h(x)\}.$$

We would like to show that

$$\bigcap_{h \in A(X)} X_h \neq \emptyset.$$

If $x_\alpha \to x$, then $h(x_\alpha) \to h(x)$ by continuity. But $h(x_\alpha) = \mu(h)$, hence $h(x) = \mu(h)$ and this implies that X_h is closed, and therefore compact. Therefore, it is sufficient to prove that the system $\{X_h,\ h \in \mathcal{H}\}$ has a finite intersection property.

Choose $h_1, \dots, h_n \in A(X)$ and define the mapping

$$\varphi : X \to \mathbb{R}^n, \qquad \varphi(x) = (h_1(x), \dots, h_n(x)).$$

Then φ is continuous and $\varphi(X)$ is a compact convex subset of $\mathbb{R}^n$. We denote $c = (\mu(h_1), \dots, \mu(h_n))$. If $c \in \varphi(X)$, there is nothing else to prove.

Let us assume that $c \notin \varphi(X)$. Then there exist $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ and $d \in \mathbb{R}$ such that

$$c \cdot \alpha < d < \min_{x \in X} \alpha \cdot \varphi(x).$$

(It is a consequence of the earlier mentioned separation theorem and the fact that in Hilbert spaces the continuous linear functionals can be represented by the Frèchet-Riesz theorem as elements of the space through scalar product.) Thus we have

$$\sum_{i=1}^{n} \mu(h_i)\alpha_i < d < \min_{x \in X} \sum_{i=1}^{n} h_i(x)\alpha_i$$

and hence

$$\mu\left(\sum_{i=1}^{n}\alpha_i h_i\right) < d < \min_{x\in X}\left(\sum_{i=1}^{n}\alpha_i h_i\right)(x).$$

But $f = \sum_{i=1}^{n}\alpha_i h_i \in A(X)$ and $\mu \in \mathcal{M}^1(X)$, so that

$$\mu(f) = \int_X f(x)\, d\mu(x) \geq \min_{x\in X} f(x) \cdot \int_X 1\, d\mu = \min_{x\in X} f(x)\cdot \mu(X) = \min_{x\in X} f(x).$$

This is a contradiction.

The mapping r is obviously surjective. For any $x \in X$ it follows that $r(\varepsilon_x) = x$.

The mapping r is affine on $\mathcal{M}^1(X)$. Given $0 < \alpha < 1$ and $\mu_1, \mu_2 \in \mathcal{M}^1(X)$, we get for every $h \in A(X)$

$$\begin{aligned} h(r(\alpha\mu_1 + (1-\alpha)\mu_2) &= (\alpha\mu_1 + (1-\alpha)\mu_2)(h) \\ &= \alpha\mu_1(h) + (1-\alpha)\mu_2(h) \\ &= \alpha h(r(\mu_1)) + (1-\alpha)\, h(r(\mu_2)) \end{aligned}$$

The mapping r is continuous if we consider $\mathcal{M}^1(X)$ with the w^*-topology and X with the w-topology. If an arbitrary net $\{\mu_\alpha\}$ converges in the w^*-topology to μ, then

$$\mu_\alpha(h) \to \mu(h) \qquad \text{and then} \qquad h(r(\mu_\alpha)) \to h(r(\mu)) \quad \text{for all } h \in A(X)$$

and the family of all continuous affine functions obviously contains the family of all continuous linear functionals on X. But it is well known fact that on convex compact subsets of locally convex spaces the weak topology and the initial locally convex topology coincide. □

Theorem 2.3. ***(Krein-Milman theorem)***

Let X be a compact convex set in a locally convex space. Then X is equal to the closed convex hull of the set of extreme points of X, that is,

$$X = \overline{\text{co}}\,\text{ext}\, X.$$

We will omit the proof here since it can be found in almost every textbook of functional analysis, for example [15], [18] and others. As its consequence, we shall derive a theorem on integral representation and another theorem of Milman which will be needed later.

Theorem 2.4. ***(Integral representation theorem)***

Let X be a compact convex set in a locally convex space. Then for every point x of X, there exists a representing measure μ for x which is supported by the closure of the set of the extreme points of X.

Proof. If $x \in X$, then, by the Krein-Milman theorem, $x \in \overline{\text{co}}\,\text{ext}\,X$. So there exists a net of finite sums $\{\sum_{i=1}^{n_\alpha} c_i^\alpha x_i^\alpha\}_\alpha$ where $x_i^\alpha \in \text{ext}\,X$ and $\sum_{i=1}^{n_\alpha} c_i^\alpha = 1$ with $c_i^\alpha > 0$. Then the family of measures

$$\{\sum_{i=1}^{n_\alpha} c_i^\alpha \varepsilon_{x_i^\alpha}\}_\alpha \text{ is a net in } \mathcal{M}^1(\overline{\text{ext}\,X})$$

and, since $\mathcal{M}^1(\overline{\text{ext}\,X})$ is a w^*-compact, there exists a subnet which converges in the w^*-topology to a measure μ. It is not difficult to prove that $\mu \in \mathcal{M}^1(X)$ and μ is a representing measure of x because for every $h \in A(X)$ we get

$$\mu(h) = \lim_\alpha \left(\sum c_i^\alpha \varepsilon_{x_i^\alpha}\right)(h) = \lim_\alpha \sum c_i^\alpha h(x_i^\alpha) = \lim_\alpha h\left(\sum c_i^\alpha x_i^\alpha\right) = h(x).$$

Obviously, μ (as a measure on X) is carried by $\overline{\text{ext}\,X}$. □

Theorem 2.5. ***(Milman)***

Let X be a compact convex set in a locally convex space. Then

$$F \subset X,\ \overline{\text{co}}\,F = X \implies \text{ext}\,X \subset \overline{F}.$$

Proof. Let us suppose that there exists $x \in \text{ext}\,X \setminus \overline{F}$. We claim there exists a continuous affine function f on X such that $f(x) > 0 > \max f(\overline{F})$. If that is so, then

$$f(x) > 0 > \max f(\overline{\text{co}}\,F)$$

and then $\overline{\text{co}}\,F \neq X$ which is a contradiction.

What remains is to prove the claim. Let x be an extreme point of X and U be its open neighbourhood, then $W = X \setminus U$ is closed set. If now $x \in \overline{\text{co}}\,W$ then due to the theorem of integral representation there exists a representing measure μ such that $\mu \in \mathcal{M}^1(W)$. In a while, we will show that the only representing measure for an extreme point is a Dirac measure and that would be a contradiction. Let μ be a representing measure for x and $\mu \neq \varepsilon_x$. Then there is a compact set K such that $\mu|_K$ and $\mu|_{X\setminus K}$ are nontrivial measures. Then $\mu_1 = \frac{\mu|_K}{\mu(K)}$ is in $\mathcal{M}^1(X)$ and so it is $\mu_2 = \frac{\mu|_{X\setminus K}}{\mu(X\setminus K)}$. The measure μ is then a convex combination

$$\mu = \mu(K)\mu_1 + \mu(X \setminus K)\mu_2,$$

which implies that x is not an extreme point of X since

$$x = r(\mu) = \mu(K)r(\mu_1) + \mu(X \setminus K)r(\mu_2)$$

and at least one of these measures does not have x as its barycenter. □

We close this section with a lemma on density of continuous linear functionals in affine functions which we shall need later in the theory of state spaces and with a theorem of Mokobodzki which is an essential tool in deriving the approximation theorem mentioned at the beginning of this chapter.

Lemma 2.6. ***(Density of continuous linear functionals in continuous affine functions)***

Let E be a locally convex space and $X \subset E$ be a compact convex set. Then

$$\overline{E^*|_X + \mathbb{R}}^{\|\cdot\|_\infty} = A(X).$$

That is, for every $f \in A(X)$ and $\varepsilon > 0$ there exist $g \in E^*|_X$ and $c \in \mathbb{R}$ such that

$$\|f - (g + c)\|_\infty < \varepsilon.$$

Proof. Let $f \in A(X)$ and $\varepsilon > 0$ be given. Without any loss of generality, let us assume that $0 \leq f \leq 1$ (since continuous functions on compact set are bounded and attain their minimum and maximum, we can rescale f by shifting and multplying with a suitable real constant). We denote

$$K_1 = \{(x,t) \in X \times \mathbb{R} \ : \ 0 \leq t \leq f(x)\},$$

$$K_2 = \{(x,t) \in X \times \mathbb{R} \ : \ f(x) + \varepsilon \leq t \leq 2\}.$$

Then K_1, K_2 are compact, convex and disjoint. Hence, by the separation theorem (0.2) there exist $\varphi \in (X \times \mathbb{R})^*$ and $c \in \mathbb{R}$ such that

$$\max \varphi(K_1) < c < \min \varphi(K_2).$$

But the equality

$$(E \times \mathbb{R})^* = E^* \times \mathbb{R}$$

implies that

$$\varphi(x,t) = g(x) + \alpha\, t, \qquad \text{where } g \in E^* \text{ and } \alpha \in \mathbb{R}.$$

Thus we get

$$\varphi(x, f(x)) < c < \varphi(x, f(x) + \varepsilon)$$

and by the equality above

$$g(x) + \alpha f(x) < c < g(x) + \alpha((f(x) + \varepsilon).$$

It is obvious that $\alpha > 0$ (because $\alpha\varepsilon > 0$) and therefore

$$f(x) < \frac{c - g(x)}{\alpha} < f(x) + \varepsilon.$$

Let us define

$$g(x) = \frac{c - g(x)}{\alpha} = \frac{c}{\alpha} - \frac{1}{\alpha} g(x).$$

Then $g \in E^*|_X + \mathbb{R}$ and $\|g - f\| < \varepsilon$. □

The proof of the following theorem is omitted here. One can find it, for example, in Lukeš et al. [17]. It is similar to the theorem we want to derive; in fact, with a suitable chosen function space $\mathcal{H}$ one can see it as a special case. In the following, we will be doing nothing less than finding a suitable connection between general function spaces and compact convex spaces presented in this section. Such a relation provides the concept of state spaces which we develop in a while later. And then we will be able to carry the following theorem in a more general settings of function space.

Theorem 2.7. ***(Mokobodzki approximation theorem)***

Let X be a compact convex set in a locally convex space and f a Baire-one affine function on X. Then there exists a bounded sequence of continuous affine functions on X which converges pointwise to f on X.

2.2 State space

In this section, let $\mathcal{H}$ be a function space on a compact space K. We denote by $\mathcal{H}^*$ its dual space, that is, the space of all continuous linear functionals on $\mathcal{H}$.

Definition. (State space)

We define the *state space* of $\mathcal{H}$ as a topological subspace of the dual space $\mathcal{H}^*$ equipped with w^*-topology

$$\mathcal{S}(\mathcal{H}) = \{\varphi \in \mathcal{H}^* \ : \ \varphi \geq 0, \ \varphi(1) = 1.\}.$$

Proposition 2.8. *Let $\mathcal{H}$ be a function space on compact space K. Then there exists a mapping $(C(K))^*/\mathcal{H}^\perp \to \mathcal{H}^*$ which is an isomorphism and homeomorphism if we endow $(C(K))^*$ and $\mathcal{H}^*$ with w^*-topology and the quotient space $(C(K))^*/\mathcal{H}^\perp$ with the corresponding quotient topology.*

Proof. At first, let us describe a dual space $\mathcal{H}^*$. By the definition of the dual space

$$\mathcal{H}^* = \{f : \mathcal{H} \to \mathbb{R}, \ f \text{ is linear and continuous}\},$$

$$(C(K))^* = \{f : C(K) \to \mathbb{R}, \ f \text{ is linear and continuous}\}.$$

Since $\mathcal{H}$ is a subspace of $C(K)$, we may construct the dual space $\mathcal{H}^*$ by simple restriction of elements of $(C(K))^*$ on $\mathcal{H}$ and identifying those functionals which give the same value on the elements of $\mathcal{H}$. As for the representing functional, the natural option is to take ones that are zero on the complement of $\mathcal{H}$.

Now, we consider a mapping

$$\Psi : (C(K))^*/\mathcal{H}^\perp \to \mathcal{H}^*, \qquad \Psi([f]) = f|_{\mathcal{H}}, \text{ where } f \in (C(K))^*.$$

We shall prove that this mapping is an isomorphism and w^*-homeomorphism.

1. The mapping Ψ is defined correctly. Indeed, if $f|_{\mathcal{H}} \neq g|_{\mathcal{H}}$, then $(f-g)|_{\mathcal{H}} \neq 0$ on $\mathcal{H}$. It follows that $\Psi[f-g] \neq [0]$ which implies $\Psi[f] \neq \Psi[g]$. (If that was true, then f would be equal to $g+h$, where h is in $\mathcal{H}^{\perp}$. Then $f-g$ belongs to $\mathcal{H}^{\perp}$ which means $f-g=0$ on $\mathcal{H}$.)

2. The mapping Ψ is injective. Whenever $f|_{\mathcal{H}} = g|_{\mathcal{H}}$, then $(f-g)|_{\mathcal{H}} = 0$ on $\mathcal{H}$ which means that $(f-g) \in \mathcal{H}^{\perp}$ and $[f-g]=[0]$ (the equivalence class is fully and uniquely determined by any of its elements). Thus $[f]=[g]$.

3. The mapping Ψ is obviously surjective.

4. The mapping Ψ is w^*-homeomorphism. Recall, that Ψ is continuous if and only if $\Psi \circ \pi$ is continuous where π is the quotient mapping, namely $\pi : f \mapsto [f]$. Hence, it is enough to prove that for every net $f_\alpha \xrightarrow{w^*} f$ in $(C(K))^*$ the net $\Psi(\pi(f_\alpha))$ converges in w^*-topology to $\Psi(\pi(f))$. However, that is trivial because

$$f_\alpha \xrightarrow{w^*} f \iff f_\alpha(x) \to f(x)\ \forall x \in C(K) \implies f_\alpha|_{\mathcal{H}}(x) \to f|_{\mathcal{H}}(x)\ \forall x \in \mathcal{H} \iff$$

$$\iff \Psi(\pi(f_\alpha))(x) \to \Psi(\pi(f))(x)\ \forall x \in \mathcal{H} \iff \Psi(\pi(f_\alpha)) \xrightarrow{w^*} \Psi(\pi(f)).$$

The inverse mapping Ψ^{-1} is continuous either because $\Psi^{-1} = \pi \circ I$, where I is an injection of $\mathcal{H}^*$ into $(C(K))^*$ given by $I(f|_H) = \tilde{f}$, where $\tilde{f} = f|_H$ on H and zero elsewhere. Then the proof of continuity follows the scheme above in the reverse direction and the only non-equivalent step is easily overcome. □

It is well known that the Riesz representation theorem allows us to identify spaces $(C(K))^*$ and $\mathcal{M}(K)$. Hence, we can identify the space $\mathcal{H}^*$ with the quotient space

$$(\mathcal{M}(K), w^*)/\mathcal{H}^{\perp}.$$

The quotient mapping will be denoted by π and

$$\mathcal{S}(\mathcal{H}) = \pi(\mathcal{M}^1(K)).$$

Indeed, if $\mu \in \mathcal{M}^1(K)$, then $\mu \in (C(K))^*$, $\mu(1)=1$ and $\mu \geq 0$. On the other hand, if $\varphi \in \mathcal{S}(\mathcal{H})$, then by the Hahn-Banach theorem there exists $\mu \in (C(K))^*$ such that $\mu = \varphi$ on $\mathcal{H}$ and $\|\mu\| = \|\varphi\|$. But $1 \in \mathcal{H}$, hence $\mu(1) = \varphi(1) = 1$ and since μ is positive, $\|\mu\| = \mu(1) = 1$. It implies that $\mu \in \mathcal{M}^1(K)$.

Now, we define two mappings. The first is a mapping which "identifies" points in K and in the state space $\mathcal{S}(\mathcal{H})$. The second mapping "identifies" the function space $\mathcal{H}$ with affine functions in the state space.

Let us note that the state space $\mathcal{S}(\mathcal{H})$ is a convex and w^*-closed subset of the unit ball in $\mathcal{H}^*$ (hence w^*-compact set). Hence, to talk about affine functions in the following definition make sense.

Definition. Let K be a compact space and $\mathcal{H}$ be a function space on K. We define

$$\phi : K \to \mathcal{S}(\mathcal{H}), \qquad \phi(x) = s_x, \qquad \text{where } s_x(h) = h(x) \text{ for } h \in \mathcal{H}.$$

It easily follows that

$$\phi(x)(h) = s_x(h) = h(x) = \varepsilon_x|_H(h) = \pi(\varepsilon_x)(h) \implies \phi(x) = \pi(\varepsilon_x).$$

Since $\mathcal{S}(\mathcal{H})$ is a convex and w^*-compact set, it makes sense to denote by $A(\mathcal{S}(\mathcal{H}))$ the collection of all affine functions on $\mathcal{S}(\mathcal{H})$. Then we define a mapping

$$\Phi : \mathcal{H} \to A(\mathcal{S}(\mathcal{H})), \qquad \Phi(h)(s) = s(h), \qquad s \in \mathcal{S}(\mathcal{H}).$$

Let $\mu \in \mathcal{M}^1(K)$. Then we naturally define $\phi\mu$ as a functional on $\mathcal{S}(\mathcal{H})$ by

$$(\phi\mu)(f) = \mu(f \circ \phi), \qquad f \in \mathcal{S}(\mathcal{H}).$$

Then $\phi\mu \in \mathcal{M}^1(\mathcal{S}(\mathcal{H}))$ and we denote by $r(\phi\mu)$ the barycenter of the measure $\phi\mu$.

The basic properties of mappings ϕ and Φ are presented in the following proposition.

Proposition 2.9. ***(Basic properties of ϕ and Φ mappings)***

Let K be a compact space and $\mathcal{H}$ be a function space on K. Then

(i) $\Phi(\mathcal{H})$ is a dense set (in the norm topology) in $A(\mathcal{S}(\mathcal{H}))$,

(ii) the barycenter mapping r satisfy the equality

$$r(\phi\mu) = \pi(\mu).$$

Especially,

$$r(\phi\mu) = \phi(x), \qquad \text{for all } \mu \in \mathcal{M}_x(\mathcal{H}),$$

(iii) the mapping $\phi : K \to \mathcal{S}(\mathcal{H})$ is a homeomorphism into $\mathcal{S}(\mathcal{H})$ and $\phi(\mathrm{Ch}_{\mathcal{H}}(K)) = \mathrm{ext}\,\mathcal{S}(\mathcal{H})$,

(iv) the mapping Φ is an isometric isomorphism between $\mathcal{H}$ and $\mathcal{A}(\mathcal{S}(\mathcal{H}))$,

(v) moreover, Φ is surjective if and only if $\mathcal{H} = \overline{\mathcal{H}}$. Then there exists an inverse mapping Φ^{-1} and it satisfy

$$\Phi^{-1}(F) = F \circ \phi, \qquad F \in \mathcal{A}(\mathcal{S}(\mathcal{H})).$$

Proof. We shall omit several techniqualities in the proof of several parts of the lemma.

(i) We know from the density lemma (2.6) that

$$((H^*, w^*)^* + \mathbb{R})\,|_{\mathcal{S}(\mathcal{H})} \text{ is norm-dense in } A(\mathcal{S}(\mathcal{H})).$$

Hence

$$(\Phi(\mathcal{H}) + \mathbb{R})\,|_{\mathcal{S}(\mathcal{H})} \text{ is norm-dense in } A(\mathcal{S}(\mathcal{H}))$$

and because

$$[\Phi(h) + c](h) = s(h) + c = s(h + c) = [\Phi(h + c)](h)$$

we have that $\Phi(\mathcal{H})$ is norm-dense in $A(\mathcal{S}(\mathcal{H}))$.

(ii) We want to prove the equality $r(\phi\mu) = \pi(\mu)$ for every $\mu \in \mathcal{M}^1(K)$. Because of the part (i), it is enough to prove this on elements of $\Phi(\mathcal{H})$. Let $h \in \mathcal{H}$, then by the definition of the barycenter, the definition of the measure $\phi\mu$, the definition of the mappings Φ, ϕ, π and again Φ we get

$$\Phi(h)(r(\phi\mu)) = (\phi\mu)(\Phi(h)) = \mu(\Phi(h) \circ \phi) = \int_K (\Phi(h))(\phi(x))\, d\mu(x) =$$

$$= \int_K \phi(x)(h)\, d\mu(x) = \int_K h(x)\, d\mu(x) = \mu(h) = (\pi(\mu))(h) = (\Phi(h))(\pi(\mu)).$$

Thus the equality $r(\phi\mu) = \pi(\mu)$ is established. Now, if $\mu \in \mathcal{M}_x(\mathcal{H})$, then

$$r(\phi\mu) = \pi(\mu) = \pi(\varepsilon_x) = \varphi(x)$$

because $\mu(h) = h(x) = \varepsilon_x(h)$ for all $h \in \mathcal{H}$ and therefore μ and ε_x represent functionals in the same equivalence class.

(iii) The mapping ϕ is correctly defined. Indeed,

$$\phi(x) = \phi(y) \implies \phi(x)(h) = \phi(y)(h)\ \forall h \in \mathcal{H} \implies$$

$$\implies h(x) = h(y)\ \forall h \in \mathcal{H} \implies x = y$$

because $\mathcal{H}$ separates the points of K.

The mapping ϕ is injective. Since $\mathcal{H}$ separates the points of K, if $x \neq y$ then there exists $h \in \mathcal{H}$ such that $h(x) \neq h(y)$. Therefore, $\phi(x) \neq \phi(y)$ because

$$\phi(x)(h) = h(x) \neq h(y) = \phi(y)(h).$$

The mapping ϕ is continuous. It is sufficient to prove that if the net $x_\alpha \to x$ in K, then $\phi(x_\alpha) \xrightarrow{w^*} \phi(x)$. By continuity of functionals in $\mathcal{H}$, we get

$$x_\alpha \to x \implies h(x_\alpha) \to h(x)\ \forall h \in \mathcal{H} \implies$$

$$\implies \phi(x_\alpha)(h) \to \phi(x)(h)\ \forall h \in \mathcal{H} \implies \phi(x_\alpha) \xrightarrow{w^*} \phi(x).$$

Now, it remains to prove that $\phi(\mathrm{Ch}_{\mathcal{H}}(K)) = \operatorname{ext} \mathcal{S}(\mathcal{H})$. At first, we will show that $\mathcal{S}(\mathcal{H}) = \overline{\mathrm{co}}\, \phi(K)$. Since $\phi(K) \subset \mathcal{S}(\mathcal{H})$, $\mathcal{S}(\mathcal{H})$ convex and w^*-closed, it follows that $\overline{\mathrm{co}}\, \phi(K) \subset \mathcal{S}(\mathcal{H})$.

On the other hand, consider $s \in \mathcal{S}(\mathcal{H}) \setminus \overline{\mathrm{co}}\, \phi(K)$. Then by separating theorem in a locally convex space (0.2) there exists $F \in (\mathcal{H}^*, w^*)^*$ and $c \in \mathbb{R}$ such that

$$F(s) > c > \sup F(\overline{\mathrm{co}}\, \phi(K)).$$

Then there exists $h \in \mathcal{H}$ such that $F(s) = s(h)$ if $s \in \mathcal{S}(\mathcal{H})$. Hence

$$s(h) > c > \sup_{x \in K} (\phi(x))\,(h) = \sup_{x \in K} h(x).$$

However, there is $\mu \in \mathcal{M}^1(K)$ such that $\mu|_H = s$ a this implies

$$\mu(h) > \sup_{x \in K} h(x).$$

Since μ is nonnegative and $\|\mu\| = 1$, this is an evident contradiction.

Now, consider $s \in \operatorname{ext} \mathcal{S}(\mathcal{H})$. Then by Milman theorem (2.5) $s \in \phi(K)$ and there is $x \in K$ such that $s = \phi(x)$. We want to show that the only $\mathcal{H}$-representing measure for $x \in K$ is the Dirac measure. So, let us choose $\mu \in \mathcal{M}_x(\mathcal{H})$. Then by (ii) $r(\phi\mu) = \phi(x)$ and therefore $\phi\mu \in \mathcal{M}_{\phi(x)}(\mathcal{S}(\mathcal{H}))$. But $s = \phi(x)$ is an extreme point so $\phi\mu = \varepsilon_{\phi(x)}$. Since ϕ is injective, μ has to be the Dirac measure ε_x.

For the converse inclusion, consider $x \in \operatorname{Ch}_{\mathcal{H}}(K)$ and assume that $\phi(x) = \frac{1}{2}(s_1 + s_2)$ and $s_1 \neq s_2$. Then there are $\mu_1, \mu_2 \in \mathcal{M}^1(K)$ such that $\mu_1|_{\mathcal{H}} = s_1$ and $\mu_2|_{\mathcal{H}} = s_2$. Therefore,

$$\frac{1}{2}(\mu_1 + \mu_2)(h) = \frac{1}{2}(s_1 + s_2)(h) = \frac{1}{2}(h(x) + h(x)) = h(x) \qquad \forall h \in \mathcal{H}$$

so it follows because the Dirac measures are the extreme points in $\mathcal{M}^1(K)$ that

$$\frac{1}{2}(\mu_1 + \mu_2) \in \mathcal{M}_x(\mathcal{H}) \implies \frac{1}{2}(\mu_1 + \mu_2) = \varepsilon_x \implies$$
$$\implies \mu_1 = \mu_2 = \varepsilon_x \implies s_1 = s_2 = \phi(x).$$

This is a contradiction.

(iv) We should prove at first a little debt that the mapping Φ is defined correctly. So let $h_1, h_2 \in \mathcal{H}$ and it follows

$$\Phi(h_1) = \Phi(h_2) \implies \Phi(h_1)(s) = \Phi(h_2)(s)\ \forall s \in \mathcal{S}(\mathcal{H}) \implies$$
$$\implies s(h_1) = s(h_2)\ \forall s \in \mathcal{S}(\mathcal{H})$$

and since $\varepsilon_x|_H \in \mathcal{S}(\mathcal{H})$ for each $x \in K$, we have

$$\varepsilon_x(h_1) = \varepsilon_x(h_2)\ \forall x \in K \implies h_1(x) = h_2(x)\ \forall x \in K \implies h_1 = h_2.$$

Since every isometric mapping is injective, it is sufficient to prove

$$\|\Phi(h)\|_\infty = \|h\|_\infty \quad \text{for an arbitrary } h \in \mathcal{H}.$$

That follows from the following inequalities. The first one is justified by formerly proved inclusion $\phi(K) \subset \mathcal{S}(\mathcal{H})$, the second one by the elementary fact that $|\mu(h)| \leq \|\mu\| \cdot \sup_{x \in K} |h(x)|$.

$$\|\Phi(h)\|_\infty = \sup_{s \in \mathcal{S}(\mathcal{H})} |\Phi(h)(s)| = \sup_{s \in \mathcal{S}(\mathcal{H})} |s(h)| \geq \sup_{x \in K} |(\phi(x))(h)| = \sup_{x \in K} |h(x)| =$$
$$= \|h\|_\infty \geq \sup_{\mu \in \mathcal{M}^1(K)} |\mu(h)| \geq \sup_{s \in \mathcal{S}(\mathcal{H})} |s(h)| = \|\Phi(h)\|_\infty$$

(v) Let us assume that $\overline{\mathcal{H}} = \mathcal{H}$, especially $\mathcal{H}$ is a complete metric space. Since $\Phi(\mathcal{H})$ is dense in $A(\mathcal{S}(\mathcal{H}))$ and Φ is an isometric mapping (hence uniformly continuous) and therefore $\Phi(\mathcal{H})$ is complete – hence closed in $A(\mathcal{S}(\mathcal{H}))$ – it has to be $\Phi(\mathcal{H}) = A(\mathcal{S}(\mathcal{H}))$.

On the contrary, if Φ is onto, then $A(\mathcal{S}(\mathcal{H})) = \Phi(\mathcal{H})$ is complete. Every isometric isomorphism is obviously an uniform homeomorphism. Therefore, $\mathcal{H}$ has to be complete and therefore closed.

Lastly, let us take an arbitrary $F \in A(\mathcal{S}(\mathcal{H}))$. Then there $h \in \mathcal{H}$ such that $F = \Phi(h)$ and

$$(F \circ \varphi)(x) = F(\varphi(x)) = \Phi(h)(\varphi(x)) = \\ = \varphi(x)(h) = h(x) = \Phi^{-1}(F)(x), \qquad \forall x \in K.$$

So we have proved that

$$\Phi^{-1} = F \circ \varphi$$

and the proof is complete. □

2.3 Characterization of Baire-one functions

Theorem 2.10. ***(Characterization of Baire-one functions)***

Let P be a metric space and $f : P \to \mathbb{R}$. Then the following are equivalent:

(i) f is of Baire class one,

(ii) for each $a \in \mathbb{R}$, the sets $\{f \geq a\}$ and $\{f \leq a\}$ are G_δ-sets.

Proof. $(i) \implies (ii)$: Let $f : P \to \mathbb{R}$ be a Baire-one function. Then there are continuous functions $f_n : P \to \mathbb{R}$ such that $f(x) = \lim f_n(x)$, $x \in P$.

Since $f(x) = \lim f_n(x) = \sup_{n\geq 1}(\inf_{k\geq n} f_k(x))$, we see that f is an increasing limit of upper semicontinuous functions. If $f(x) < \alpha$, then $\inf_{k\geq n} f_k(x) < \alpha$ as well, so

$$\{f \leq a\} = \bigcap_{n=1}^{\infty} \{f < a + \frac{1}{n}\} = \bigcap_{n=1}^{\infty} \{\inf_{k\geq n} f_k < a + \frac{1}{n}\}.$$

But the set $\{g < \alpha\}$ is open whenever $\alpha \in \mathbb{R}$ and g is an upper semicontinuous function.

$(ii) \implies (i)$: We shall give the proof a little while later. □

Let P be a metric space and $\mathcal{F}$ be a family of real valued functions on P which satisfies the following conditions:

L1. $\mathcal{F}$ is a lattice cone, that is, for every $f, g \in \mathcal{F}$ and $\alpha > 0$ we have

$$f + g \in \mathcal{F}, \quad \alpha f \in \mathcal{F}, \quad \max\{f, g\} \in \mathcal{F}, \quad \min\{f, g\} \in \mathcal{F}.$$

L2. $\mathcal{F}$ contains constant functions.

L3. $\mathcal{F}$ is closed on uniform convergence.

Then we denote

$$\mathcal{F}^{\uparrow} = \{f \ : \text{ there exists an increasing sequence } \{f_n\} \text{ of } \mathcal{F} \text{ such that } f_n \to f \text{ pointwise}\}$$

$$\mathcal{F}^{\downarrow} = \{f \ : \text{ there exists a decreasing sequence } \{f_n\} \text{ of } \mathcal{F} \text{ such that } f_n \to f \text{ pointwise}\}$$

We say that a set A is $\mathcal{F}$-separated from B if there exists $f \in \mathcal{F}$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Obviously, A is $\mathcal{F}$-separated from B if and only if for every $a < b$ there exists $g \in \mathcal{F}$ such that $g(A) = \{a\}$, $g(B) = \{b\}$ and $a \leq g \leq b$.

Lemma 2.11. *Let $\mathcal{F}$ be a system of functions which satisfies conditions (L1) and (L2). Then both of systems $\mathcal{F}^{\uparrow}$ and $\mathcal{F}^{\downarrow}$ satisfy all conditions (L1)–(L3).*

Proof. (L1) and (L2) are almost obvious. For (L3), let us assume that a sequence $\{f_n\}$ in $\mathcal{F}^{\uparrow}$ converges uniformly to f. Without any loss of generality, we may assume that

$$|f_k - f| < \frac{1}{k} \qquad \text{for each } k \in \mathbb{N},$$

otherwise we may choose a suitable subsequence. Now, we define

$$h_k = \max\left\{f_1 - 1,\ f_2 - \frac{1}{2},\ \ldots,\ f_k - \frac{1}{k}\right\}.$$

Then $h_k \in \mathcal{F}^{\uparrow}$ and $h_k \nearrow f$. Hence $f \in \mathcal{F}^{\uparrow}$. □

Lemma 2.12. ***(Abstract in-between theorem)***

Let $\mathcal{F}$ be a system of functions which satisfies (L1)–(L3) and let $t \leq s$ be bounded functions on P. Then the following are equivalent:

(i) there exists $f \in \mathcal{F}$ such that $t \leq f \leq s$,

(ii) if $a < b$, then $\{s \leq a\}$ is $\mathcal{F}$-separated from $\{t \geq b\}$.

Proof. $(i) \implies (ii)$: The function

$$h(x) = \max\{a, \min\{f(x), b\}$$

is obviously a member of the lattice cone $\mathcal{F}$, $h(x) = a$ on $\{s \leq a\}$, $h(x) = b$ on $\{t \geq b\}$ and $a \leq h(x) \leq b$ for each $x \in P$. In view of the definition, the sets $\{t \geq b\}$ and $\{s \leq a\}$ are $\mathcal{F}$-separated by the function h.

$(ii) \implies (i)$: Without any loss of generality, we may assume that $0 \le t \le s \le 1$, otherwise we can rescale both of these functions (by adding and multiplying with suitable constants).

Let now $\varepsilon > 0$. Then there exists $p \in \mathbb{N}$ such that $\frac{1}{p} < \varepsilon$ and for each $k = 1, 2, \dots, p$, due to the assumption on $\mathcal{F}$-separation, we can find $f_k \in \mathcal{F}$ such that

$$0 \le f_k \le \frac{k}{p} \qquad \text{on } P,$$

$$f_k = 0 \qquad \text{on } \left\{ s \le \frac{k-1}{p} \right\},$$

$$f_k = \frac{k}{p} \qquad \text{on } \left\{ t \ge \frac{k}{p} \right\}.$$

Let

$$f(x) = \max\{f_1(x), \dots, f_p(x)\}.$$

Then $f \in \mathcal{F}$ and f satisfies the inequality for each $x \in P$

$$t(x) - \varepsilon \le t(x) - \frac{1}{p} \le f(x) \le s(x) + \frac{1}{p} \le s(x) + \varepsilon.$$

(If $\frac{i}{p} \le t(x) \le \frac{i+1}{p}$ for $i \in \{0, 1, \dots, p-1\}$, then $f(x) \ge f_i(x) = \frac{i}{p} = \frac{i+1}{p} - \frac{1}{p} \ge t(x) - \frac{1}{p}$. The other part of inequality is valid due to similar reason.)

So, for every $\varepsilon > 0$, there exists $f_\varepsilon \in \mathcal{F}$ such that

$$t - \varepsilon \le f_\varepsilon \le s + \varepsilon \qquad \text{on } P.$$

Hence, there exists a sequence $\{f_n\}$ in $\mathcal{F}$ such that

$$t - \frac{1}{2^n} \le f_n \le s + \frac{1}{2^n}.$$

We put

$$h_1 = f_1, \quad h_n = \max\{h_{n-1} - \frac{1}{2^n}, \min\{h_{n-1} + \frac{1}{2^n}, f_n\}\}, \quad f = \lim h_n.$$

Obviously, each $h_n \in \mathcal{F}$,

$$t - \frac{1}{2^n} \le h_n \le s + \frac{1}{2^n} \qquad \text{and if } n \to \infty \text{ we get} \qquad t \le f \le s.$$

Hence, f has to be in $\mathcal{F}$ because h_n converges uniformly to f. □

Lemma 2.13. ***(Baire-one functions are uniformly closed)***

Let P be a metric space. Then the space $B_1(P)$ is closed under uniform convergence.

Proof. Let f be a Baire-one function. Then

$$\begin{aligned} f(x) = \lim f_n(x) &= \sup_{n\in\mathbb{N}}(\inf_{k\geq n} f_k(x)) \\ &= \inf_{n\in\mathbb{N}}(\sup_{k\geq n} f_k(x)) \end{aligned}$$

It implies that

$$B_1(P) \subset C^{\uparrow\downarrow}(P) \cap C^{\downarrow\uparrow}(P).$$

We will prove the converse inclusion. Let now be $h \in C^{\uparrow\downarrow}(P) \cap C^{\downarrow\uparrow}(P)$. Then there exist sequences $\{g_n\}$ in $C^{\downarrow}(P)$ and $\{f_n\}$ in $C^{\uparrow}(P)$ such that $g_n \nearrow h$ and $f_n \searrow h$. Obviously, $g_n \leq h \leq f_n$ for each $n \in \mathbb{N}$. We want to show that there exists $h_n \in C(P)$ such that $g_n \leq h_n \leq f_n$ for each $n \in \mathbb{N}$ because then $h_n \to h$ pointwise and therefore $h \in B_1(P)$.

For this purpose, we will use the previous Lemma (2.12) (with $\mathcal{F} = C(P)$), so we have to show that for each $a < b$ there exists a continuous function which separates $\{g_n \leq a\}$ and $\{f_n \geq b\}$. In view of Tietze's extension theorem, it is enough to show that both of these sets are closed, since they are obviously disjoint. But $g_n \in C^{\downarrow}(P)$, that is, there exists an increasing sequence $(g_{n,k}) \subset C(P)$ such that $g_{n,k} \to g_n$ as $k \to \infty$. So

$$\{g_n \leq a\} = \bigcap_{k=1}^{\infty}\{g_{n,k} \leq a\}.$$

The inverse image of closed set is a closed set and every intersection of closed sets is a closed set. The set $\{f_n \geq b\}$ is closed due to similar reason.

So we have proved that

$$B_1(P) = C^{\uparrow\downarrow}(P) \cap C^{\downarrow\uparrow}(P).$$

Due to Lemma (2.11), the space $B_1(P)$ is closed under uniform convergence. □

Lemma 2.14. *Let G be a G_δ subset of a metric space P. Then there exists $f \in C^{\downarrow}(P)$ such that $0 \leq f \leq 1$ and $G = \{f = 0\}$.*

Proof. Let $G = \cap_{n=1}^{\infty} G_n$ where G_n are open sets. We consider the function

$$f = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n}\chi_{G_n}.$$

In view of Lemma (2.11) it is enough to prove that χ_{G_n} belongs to $C^{\uparrow}(P)$. However, if U is an open set, there exists $u \in C(P)$ such that $u \geq 0$ on P and $U = \{u > 0\}$ (for example, $u(x) = \text{dist}(x, P \setminus G)$) and obviously

$$\min\{1, ng\} \nearrow \chi_U.$$

So $\chi_U \in C^{\uparrow}(P)$ for every U open. □

Proof of the implication (ii) $\implies$ (i) of the main theorem. It follows from the condition (*ii*) that for each $a < b$ in $\mathbb{R}$ the sets $\{f \leq a\}$ and $\{f \geq b\}$ are disjoint G_δ sets. Hence, there are functions $h_1, h_2 \in C^{\downarrow}(P)$ such that

$$0 \leq h_i \leq 1 \quad i = 1, 2; \qquad \{f \leq a\} = \{h_1 = 0\}, \qquad \{f \geq b\} = \{h_2 = 0\}.$$

Now, we consider a function

$$h = \frac{h_1}{h_1 + h_2}.$$

Obviously $h \in B_1(P)$, $h = 0$ on $\{f \leq a\}$ and $h = 1$ on $\{f \geq b\}$; so these sets are $B_1(P)$-separated and by Lemma (2.12), there exists $u \in B_1(P)$ such that $f \leq u \leq f$. Inevitably, f is a function of Baire class one. □

2.4 Preparatory results

Lemma 2.15. *Let K be a compact space, $\mu \in \mathcal{M}^1(K)$ and f be a pointwise limit of a bounded sequence of continuous functions on K. Then the function*

$$g : \mu \mapsto \mu(f)$$

is a Baire-one function on $\mathcal{M}^1(K)$.

Proof. There is a bounded sequence $\{f_n\}$ in $C(K)$ such that $f_n \to f$ pointwise on K. Then the function $g_n : \mu \mapsto \mu(f_n)$ is w^*-continuous by the definition of w^*-convergence

$$\mu_\alpha \xrightarrow{w^*} \mu \implies \mu_\alpha(f_n) \to \mu(f_n).$$

Then

$$g_n(\mu) = \mu(f_n) \to \mu(f) = g(f)$$

so we have a sequence of w^*-continuous functions g_n which converges pointwise to g. Thus, by the definition, g is a Baire-one function on $(\mathcal{M}^1(K), w^*)$. □

Proposition 2.16. *Let $\mathcal{H}$ be a function space on a compact space K and f be a bounded Baire-one function on K. Then the following conditions are equivalent:*

(i) $f \in \mathcal{B}_1^{bb}(\mathcal{H})$,

(ii) f is completely $\mathcal{H}$-affine,

Proof. (cf. Lukeš et al. 2003 [17])

(i) $\implies$ (ii): Since $f \in \mathcal{B}_1^{bb}(\mathcal{H})$, there exists a bounded sequence $\{f_n\}$ in $\mathcal{H}$ such that $f_n \to f$ pointwise and if $\mu \in \mathcal{H}^\perp$, then we have by the Lebesgue dominated convergence theorem

$$\mu(f) = \lim \mu(f_n) = 0.$$

(ii) $\implies$ (i): Let us define

$$F : s \in \mathcal{S}(\mathcal{H}) \mapsto \mu_s(f) \qquad \text{where } \mu_s \in \mathcal{M}^1(K) \text{ and } \pi(\mu_s) = s.$$

The function F is correctly defined because such a measure exists (π as a quotient mapping is automatically surjective) and the value does not depend on the choice of the measure. If $\pi(\mu_s) = \pi(\lambda_s)$, then both measures are in the same equivalence class and therefore $\mu_s - \lambda_s \in \mathcal{H}^\perp$. Hence,

$$(\mu_s - \lambda_s)f = 0 \implies \mu_s(f) = \lambda_s(f).$$

The function F is an affine function on $\mathcal{S}(\mathcal{H})$. If $s_1, s_2 \in \mathcal{S}(\mathcal{H})$ and $\lambda \in [0,1]$, then we have (by the Lebesgue theorem)

$$\begin{aligned} F(\lambda s_1 + (1-\lambda)s_2) &= \mu_{\lambda s_1+(1-\lambda)s_2}(f) \\ &= \lim \mu_{\lambda s_1+(1-\lambda)s_2}(f_n) \\ &= \lim(\lambda s_1 + (1-\lambda)s_2)(f_n) \\ &= \lambda \lim s_1(f_n) + (1-\lambda) \lim s_2(f_n) \\ &= \lambda \lim \mu_{s_1}(f_n) + (1-\lambda) \lim \mu_{s_2}(f_n) \\ &= \lambda \mu_{s_1}(f) + (1-\lambda)\mu_{s_2}(f) \\ &= \lambda F(s_1) + (1-\lambda)F(s_2). \end{aligned}$$

We have to verify that $f = F \circ \phi$. Each measure $\mu \in \mathcal{M}^1(K)$ belongs to one equivalence class determined by the quotient mapping π and uniquely determines an element s in the state space $\mathcal{S}(\mathcal{H})$. Hence,

$$F(\pi(\mu)) = F(s) = \mu_s(f) = \mu(f)$$

because we have proved in the beginning that the value $F(s)$ does not depend on the choice of a measure in the equivalence class.

But we have proved in the Proposition (2.9) that $\pi(\mu) = \phi(x)$ for all $\mu \in \mathcal{M}_x(\mathcal{H})$ and thus

$$f(x) = \varepsilon_x(f) = F(\pi(\varepsilon_x)) = F(\phi(x)), \qquad x \in X,$$

so the equality $f = F \circ \phi$ is successfully verified.

It remains to prove that F is a bounded and Baire-one function on $\mathcal{S}(\mathcal{H})$. It is quite obvious that F is bounded because $F(s) = \mu_s(f)$, $\mu_s \in \mathcal{M}^1(K)$ and f is bounded on K

$$|F(s)| = |\mu_s(f)| \le \sup_{x\in K} |f(x)| \cdot \|\mu\| = \sup_{x \in K} |f(x)|, \qquad s \in \mathcal{S}(\mathcal{H}).$$

To show that F is a Baire-one function, it is sufficient to prove that $F^{-1}(U)$ is an F_σ-set whenever U is an open subset of $\mathbb{R}$ (due to Theorem (2.10)). We have $F(\pi(\mu)) = \mu(f)$, so for an open set $U \subset \mathbb{R}$,

$$F^{-1}(U) = \{\pi(\mu) \ : \ \mu \in \mathcal{M}^1(K),\ F(\pi(\mu)) \in U\} = \pi(\{\mu \ : \ \mu \in \mathcal{M}^1(K),\ \mu(f) \in U\}).$$

But the quotient mapping π is continuous and closed and the mapping $g : \mu \mapsto \mu(f)$ is a Baire-one mapping by the previous Lemma (2.15). So

$$F^{-1}(U) = \pi(g^{-1}(U))$$

and since mappings preserve unions, closed mappings map closed sets to closed sets and due to (2.10) Baire-one mappings map open sets to F_σ-sets.

Now, by Mokobodzki's approximation theorem (2.7), there exists a bounded sequence $\{F_n\}$ in $A(\mathcal{S}(\mathcal{H}))$ such that $F_n \to F$ on $\mathcal{S}(\mathcal{H})$. Then $f_n = F_n \circ \phi$ is an element of $\overline{H}$, this is a consequence of the Proposition (2.9), part (v) in which we assume that the function space is uniformly closed.

Thus we have $f_n = F_n \circ \phi \to F \circ \phi = f$, so $f \in \mathcal{B}_1^{bb}(\overline{\mathcal{H}})$.

It remains to prove that $\mathcal{B}_1^{bb}(\mathcal{H}) = \mathcal{B}_1^{bb}(\overline{\mathcal{H}})$. One inclusion is trivial, for the other: if there are $f_n \in \overline{\mathcal{H}}$, $f_n \to f$ pointwise, then for given f_n we have $g_n \in \mathcal{H}$ such that $\|g_n - f_n\| < \frac{1}{n}$. So for each $x \in K$ and $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon/2$ and for every $n > k$ it is $|f_n(x) - f(x)| < \varepsilon/2$ so

$$|g_n(x) - f(x)| \leq |g_n(x) - f_n(x)| + |f_n(x) - f(x)| < \varepsilon.$$

Hence $g_n \to f$ pointwise. The proof is complete. □

2.5 Proof of the main theorem

This section follows closely the work of Lukeš et al.

Proof of the main theorem. We need to proof that every bounded Baire-one $\mathcal{H}$-affine function on K is a pointwise limit of bounded sequence of continuous $\mathcal{H}$-affine functions, that is, [1]

$$\mathcal{B}_1^b(K) \cap \mathcal{A}(H) \subset \mathcal{B}_1^{bb}(\mathcal{A}^c(H)).$$

For that purpose, it is enough to show that every bounded Baire-one $\mathcal{H}$-affine function on K is completely $\mathcal{A}^c(\mathcal{H})$-affine. (Here, it is essential that the function space $\mathcal{H}$ is simplicial.)

If f is $\mathcal{H}$-affine function, then

$$f(x) = \mu(f), \qquad x \in K,\ \mu \in \mathcal{M}_x(\mathcal{H}).$$

Since $\mathcal{H}$ is simplicial, there exists a unique maximal measure $\delta_x \in \mathcal{M}_x(\mathcal{H})$ and it is obvious that

$$\delta_x(f) = f(x).$$

So it is sufficient to prove that the function $x \mapsto \delta_x(f)$ is completely $\mathcal{A}^c(H)$-affine, that is, if $\mu \in (\mathcal{A}^c(\mathcal{H}))^\perp$, then $\mu(H^f) = 0$.

[1]) the other inclusion is obviously valid due to the Lebesgue dominated convergence theorem

1. step: let now g be a continuous $\mathcal{H}$-convex continuous function and $H^g(x) := \delta_x(g)$. Then (see Lemma (1.3))

$$H^g(x) = \delta_x(g) = g^*(x)$$

and it follows that H^g is an upper semicontinuous on K, hence obviously Baire one on K. Let $\mu \in (A^c(\mathcal{H}))^\perp$ be given and let $\mu = \mu_1 - \mu_2$ where μ_1, μ_2 are positive Radon measures on K. Then

$$\begin{aligned}
\mu_1(H^g) &= \mu_1(g^*) = \mu_1(\inf\{h \ : \ h \in \mathcal{A}^c(\mathcal{H}), \ h \geq g\}) \\
&= \inf\{\mu_1(h) \ : \ h \in \mathcal{A}^c(\mathcal{H}), \ h \geq g\} \\
&= \inf\{\mu_2(h) \ : \ h \in \mathcal{A}^c(\mathcal{H}), \ h \geq g\} \\
&= \mu_2(\inf\{h \ : \ h \in \mathcal{A}^c(\mathcal{H}), \ h \geq g\}) = \mu_2(g^*) = \mu_2(H^g)
\end{aligned}$$

since $\mu(h) = 0$ for any $h \in \mathcal{A}^c(\mathcal{H})$ (hence $\mu_1(h) = \mu_2(h)$) and since the family of functions $\{h \in \mathcal{A}^c(\mathcal{H}) \ : \ h \geq g\}$ is lower directed due to the Edwards separation theorem.

2. step: let now g be a continuous function on K. Due to the Stone-Weierstrass theorem, the space $\mathcal{K}^c(\mathcal{H}) - \mathcal{K}^c(\mathcal{H})$ is uniformly dense in $C(K)$ and therefore, the function H^g is again completely $\mathcal{H}$-affine.

3. step: let now $\mathcal{F}$ be a family of bounded Borel functions on K such that for $f \in \mathcal{F}$ the function H^f is a Borel function and completely $\mathcal{A}^c(\mathcal{H})$-affine. Then $\mathcal{F}$ contains all continuous functions on K and, obviously, $\mathcal{F}$ is closed with respect to limits of bounded sequences. Thus $\mathcal{F}$ contains all bounded Borel functions. □

Chapter 3

Pointwise limits of functions in $H_0(K)$

From now on, we divert our look from abstract theory in simplicial spaces and close our attention on the classical harmonic case on $\mathbb{R}^n$. At first, some definitions and notations. Let U be an open subset of $\mathbb{R}^n$. We say that $h : U \to \mathbb{R}$ is harmonic on U if

$$\Delta h = 0 \quad \text{on } U.$$

Let now U be a bounded open subset of $\mathbb{R}^n$ and K be a compact subset of $\mathbb{R}^n$. We will consider two spaces of harmonic functions in the following.

$$H(U) = \{f : \overline{U} \to \mathbb{R},\ f \text{ is harmonic on } U \text{ and continuous on } \overline{U}\},$$

$$H_0(K) = \{f : K \to \mathbb{R},\ f \text{ is harmonic on some neighbourhood of } K\}.$$

In this chapter, we will be concerned mostly about the pointwise limits of functions in $H_0(K)$ and give here a complete characterizations of these functions which was proved in 2005 by Gardiner and Gustafsson [14].

Theorem 3.1. ***(Gardiner, Gustafsson (2005))*** *Let $K \subset \mathbb{R}^n$ be a compact set and $f : K \to \mathbb{R}$. Then f is a pointwise limit of functions in $H_0(K)$ if and only if the there exists a sequence of compact sets $K_k \nearrow K$ and the following conditions are satisfied:*

(i) $f|K_k$ is bounded, $H_0(K_k)$-affine and Baire-one,

(ii) every bounded component of $\mathbb{R}^n \setminus K$ intersects $\mathbb{R}^n \setminus K$.

In the following, we will represent their proof of this characterization.

3.1 Preparatory results

At first, we present a simple lemma. See, for example, Dautray and Lions [8], Lemma II.4.2.1.

Lemma 3.2. *Let U be an open set in $\mathbb{R}^n$. There exists an increasing sequence of regular bounded open sets with boundaries of class C^∞ whose union is U.*

Sketch of the proof. Let us choose for every $x \in U$ an open ball $B(x, r_x)$ such that $\overline{B(x, r_x)} \subset U$. Then the collection $\mathcal{U} = \{B(x, r_x),\ x \in U\}$ is a covering of U and there exists a sequence of balls $\{B_n\}$ from $\mathcal{U}$ such that $\overline{B}_k \subset U$ and the balls of the sequence covers U.

Now, given $\overline{B}_1, \ldots, \overline{B}_n$, there exists a regular bounded open set U_n of class C^∞ containing $\overline{B}_1 \cup \ldots \cup \overline{B}_n$ such that $\overline{U}_n \subset U$. (There exists an open set G such that $U \supset \overline{G} \supset G \supset \cup_{i=1}^n \overline{B}_i$ and C^∞ function f which is 1 on the union and zero on G^c. Now, consider the set $\{f \geq \frac{1}{2}\}$. It is known that a set with C^2-boundary is regular due to the external ball touching criterion.) □

For the proof of the main result in this chapter, we shall need several theorems on harmonic approximation outside compact sets. The first one is a simple application of Green's identity and states that a function harmonic on some neighbourhood of a compact set can be approximated uniformly on this compact by a finite sum of potentials (with singularities outside the compact set).

The second important result relies on the technique of pole-pushing. Roughly, it says that a function harmonic outside connected open set with singularity inside can be uniformly approximated outside this set by functions which have a singularity in the set either but anywhere else.

For each $y \in \mathbb{R}^n$ we define

$$U_y(x) = \begin{cases} \|x-y\|^{2-n} & \text{if } n > 2, \\ -\log\|x-y\| & \text{if } n = 2. \end{cases}$$

Lemma 3.3. *Let $K \subset \mathbb{R}^n$ be compact and h be a function which is harmonic on some neighbourhood of K. For $\varepsilon > 0$ chosen arbitrarily there exists real numbers $\alpha_1, \ldots, \alpha_m$ and $y_1, \ldots, y_m \in \mathbb{R}^n \setminus K$ such that $0 < \operatorname{dist}\{y_i, K\} < \varepsilon$ for $i = 1, \ldots, m$ and*

$$\left| h - \sum_{i=1}^{m} \alpha_i U_{y_i} \right| < \varepsilon \qquad \text{on } K.$$

Proof. Since h is harmonic on some bounded open set $U \supset K$, we can choose a bounded open set V such that $U \supset \overline{V} \supset V \supset K$ and V has a smooth boundary.

Let us choose $x \in K$. Then the Green formula applied for $f = U_x$ and $g = h$ on $L_\delta := \overline{V} \setminus B(x, \delta)$ (with δ sufficiently small to be $\overline{B}(x, \delta) \subset V$) gives

$$\int_{L_\delta} (U_x(y)\Delta h(y) - h(y)\Delta U_x(y))\, d\lambda(y) = \int_{\partial L_\delta} \left(U_x(y)\frac{\partial h}{\partial n_e}(y) - h(y)\frac{\partial U_x}{\partial n_e}(y) \right) d\sigma(y).$$

Since h and U_x are harmonic on L_δ

$$0 = \int_{\partial L_\delta} \left(U_x(y)\frac{\partial h}{\partial n_e}(y) - h(y)\frac{\partial U_x}{\partial n_e}(y) \right) d\sigma(y)$$

and therefore

$$\begin{aligned} - \int_{\partial B(x,\delta)} \left(U_x(y)\frac{\partial h}{\partial n_e}(y) - h(y)\frac{\partial U_x}{\partial n_e}(y) \right) d\sigma(y) = \\ = \int_{\partial V} \left(U_x(y)\frac{\partial h}{\partial n_e}(y) - h(y)\frac{\partial U_x}{\partial n_e}(y) \right) d\sigma(y). \quad (3.1) \end{aligned}$$

Since U_x is constant on $\partial B(x, \delta)$, Green's formula applied on the set $B(x, \delta)$ and functions $f = 1$ and $g = h$ gives

$$\int_{\partial B(x,\delta)} U_x(y)\frac{\partial h}{\partial n_e}(y)\, d\sigma(y) = C_\delta \cdot \int_{\partial B(x,\delta)} \frac{\partial h}{\partial n_e}(y)\, d\sigma(y) = 0.$$

Using mean value theorem for harmonic functions with the direct calculation of normal derivative for the potential $U_x(y)$, we get

$$\int_{\partial B(x,\delta)} h(y)\frac{\partial U_x(y)}{\partial n_e}(y)\, d\sigma(y) = -a_n h(x),$$

where $a_n = \sigma_n \max\{1, n-2\}$ and σ_n is a surface area of unit sphere in $\mathbb{R}^n$. Thus, from the equation (3.1), we get

$$h(x) = \frac{1}{a_n} \int_{\partial V} \left(U_x(y)\frac{\partial h}{\partial n_e}(y) - h(y)\frac{\partial U_x}{\partial n_e}(y) \right) d\sigma(y), \qquad x \in K.$$

The integrand is uniformly continuous as a function of (x, y) on $K \times \partial L$. Therefore, there are $y_j \in \partial L$ such that the Riemannian sum

$$\sum_{j=1}^{n} b_j \left(U_x(y_j)\frac{\partial h}{\partial n_e}(y_j) - h(y_j)\frac{\partial U_x}{\partial n_e}(y_j) \right)$$

uniformly approximates the integrand as close as we need. Furthermore, the derivation $\frac{\partial U_x}{\partial n_e}(y_j)$ can be suitably approximated by linear combination of $U_x(y_j), U_x(y_j)$ for some $y_j' \in \mathbb{R}^n$ (using the definition of the derivation). So the lemma is established by rellabeling the points y_j, y_j'. □

The proof of the following lemma requires a lot of work which we will omit here and give only a short overview of the facts in background. It is known (cf. Armitage and Gardiner [4], sections 2.4–2.6 or Gardiner [13], section 1.6) that every function h harmonic on the annulus $\{x \in \mathbb{R}^n \ : \ r < |x-y| < R\}$, where $y \in \mathbb{R}^n$ is fixed, can be written as

$$h(x) = a + bU_x(y) + \sum_{k=1}^{\infty} H_k(x-y) + \sum_{k=1}^{\infty} \frac{I_k(x-y)}{\|x-y\|^{2k+n-2}}$$

where H_k, I_k are harmonic polynomials of degree k on $\mathbb{R}^n$. The series is convergent on the annulus and convergent absolutely and uniformly on

$$\{x \in \mathbb{R}^n \ : \ r+\varepsilon < |x-y| < R-\varepsilon\} \qquad \text{for every } \varepsilon > 0.$$

By truncating such a series, we can derive this statement:

If h is harmonic on $\mathbb{R}^n \setminus \overline{B(y,r)}$ and $R > r$, then, for each $\varepsilon > 0$, there exists a function H harmonic everywhere except y and $|H-h| < \varepsilon$ on $\mathbb{R}^n \setminus \overline{B(y,R)}$.

It is now easy to derive the following statement which is sometimes refered as "pole-pushing lemma".

Definition. (path, tract)

If z_1, z_2 are points in $\mathbb{R}^n$, then by a *path* from z_1 to z_2 we mean a continuous function $g : [0,1] \to \mathbb{R}^n$ such that $g(0) = z_1$ and $g(1) = z_2$. By a *tract of this path* from z_1 to z_2 we mean a connected open set containing this path. Generally, by a *tract* from z_1 to z_2 we mean any connected open set containing some path from z_1 to z_2.

Lemma 3.4. ***(Pole-pushing lemma)***

Let y_0, y_1 be points in $\mathbb{R}^n$ and T be a tract from y_0 to y_1. If $\varepsilon > 0$ and u is harmonic on $\mathbb{R}^n \setminus \{y_0\}$, then there exists a harmonic function w on $\mathbb{R}^n \setminus \{y_1\}$ such that $|w-u| < \varepsilon$ on $\mathbb{R}^n \setminus T$.

Proof. There exist a finite number of balls $B(x_j, r_j) \subset T$ where $x_1 = y_0$, $x_m = y_1$ and every x_{j-1} is an element of $B(x_j, r_j)$. We put $g_1 = u$. In view of the statement before, we can now recursively find functions g_j harmonic everywhere except the point x_j such that

$$|g_j - g_{j-1}| < 2^{-j}\varepsilon \qquad \text{outside the ball } B(x_j, r_j),\ j = 2, \dots, m.$$

Hence

$$|u - g_m| \le \sum_{j=1}^{m} 2^{-j}\varepsilon < \varepsilon \qquad \text{outside the tract } T.$$

So we put simply $g_m = w$ and the proof is complete. ☐

Let us remark that the lemma is still valid if the point y_1 is the point in infinity: by a path from y_0 to ∞, we mean then a continuous function $g : [0, 1) \to \mathbb{R}^n$ such that $g(0) = y_0$ and $\lim_{t\to 1-} g(t) = \infty$ and by a tract of this path we mean a given connected open subset of $\mathbb{R}^n$ containing this path. Then the lemma can be read this way:

Lemma 3.5. ***(Pole-pushing to the infinity)***

Let y_0 be a point in $\mathbb{R}^n$ and T be a tract from y_0 to ∞. If $\varepsilon > 0$ and u is harmonic on $\mathbb{R}^n \setminus \{y_0\}$, then there exists a harmonic function w on $\mathbb{R}^n$ such that $|w - u| < \varepsilon$ on $\mathbb{R}^n \setminus T$.

3.2 Proof of the main theorem

1. part: the condition is necessary

Let us assume that $f : K \to \mathbb{R}$ is a pointwise limit of functions $h_n \in H_0(K)$. We shall consider sets

$$K_k = \{x \in K,\ |h_n(x)| \le k \text{ for each } n \in \mathbb{N}\}, \qquad k \in \mathbb{N}.$$

Obviously, $K_{k+1} \supset K_k$ and every K_k is closed (and therefore compact) because

$$K_k = \bigcap_{n\in\mathbb{N}} h_n^{-1}([-k, k]).$$

For fixed $x \in K$, the sequence $h_n(x)$ is convergent and hence bounded, so there is $k_x \in \mathbb{N}$ such that $|h_n(x)| \le k_x$. So for every $x \in K$ there is $k \in \mathbb{N}$ such that $x \in K_k$. Hence,

$$\bigcup_{k\in\mathbb{N}} K_k = K.$$

Now, we have to verify that $f|_{K_k}$ is bounded, Baire-one and $H_0(K_k)$-affine. But $f|_{K_k}$ is a limit of bounded sequence $\{h_n|_{K_k}\}$ of continuous functions. Therefore, f is obviously bounded and Baire-one and the Lebesgue dominated convergence theorem assures its affinity.

So the last thing remains: whether every bounded component of $\mathbb{R}^n \setminus K_k$ intersects $\mathbb{R}^n \setminus K$. If $K_k = K$, then there is nothing to discuss. If $K_k \ne K$ and U is bounded open component in $\mathbb{R}^n \setminus K_k$, then $\partial U \subset K_k$. If $(\mathbb{R}^n \setminus K) \cap U = \emptyset$, then $K \supset U$ and hence $K \cap U = U$. But $|h_n| \le k$ on $\partial U \subset K_k$ and hence $|h_n| \le k$ on $\overline{U}$ due to maximum principle. Therefore $\overline{U} \subset K_k$ which is an obvious contradiction.

2. part: the condition is sufficient

We require the following result (cf. Debiard and Gaveau [9], theorem 1).

Theorem 3.6. ***(Debiard, Gaveau; 1973)***

Let K be a compact subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$. The following statements are equivalent:

(a) there exists a sequence $\{h_m\}$ in $H_0(K)$ such that $h_m \to f$ uniformly on K,

(b) the function f is continuous on K and $H_0(K)$-affine, that is,

$$f(x) = \int f\, d\mu \qquad \text{for all } x \in K \text{ and } \mu \in \mathcal{M}_x(H_0(K)).$$

Now we can continue in the proof with the sufficiency part. For reader's convenience, we shall divide the proof in several steps.

"Simplicial" approximation. We assume that there are compact sets $K_k \nearrow K$ such that $f|_{K_k}$ is bounded, Baire-one and $H_0(K_k)$-affine. Since the function space $H_0(K_k)$ is simplicial there are continuous $H_0(K_k)$-affine functions $g_{n,k}$ such that $g_{n_k} \xrightarrow{n\to\infty} f|_{K_k}$ pointwise. Due to the Debiard-Gaveau theorem (3.6), we may assume without any loss of generality that the functions $g_{n,k}$ are harmonic on some neighbourhood of $H_0(K_k)$.

Reconstruction of compact sets into more suitable form. For $n \in \mathbb{N}$, we now define

$$L_n = K_1 \cup \{x \in K_2,\ \operatorname{dist}(x, K_1) \geq \frac{1}{n}\} \cup \ldots \cup \{x \in K_n,\ \operatorname{dist}(x, K_{n-1}) \geq \frac{1}{n}\}.$$

Then (since the distance function is continuous), L_n is a finite union of disjoint compact sets and hence L_n is compact. We put

$$\begin{aligned} v_n &= g_{n,1} && \text{on } L_n \cap K_1, \\ v_n &= g_{n,k} && \text{on } L_n \cap (K_k \setminus K_{k-1}),\ k = 2, \ldots, n. \end{aligned}$$

Hence, we may assume that v_n is harmonic on some neigbourhood of L_n. By Lemma (3.3), there are points $y_{n,1}, \ldots, y_{n,i_n} \in \mathbb{R}^n \setminus L_n$ and real numbers $\alpha_{n,1}, \ldots, \alpha_{n,i_n}$ such that

$$\left| v_n - \sum_{i=1}^{i_n} \alpha_{n,i} U_{y_{n,i}} \right| < \frac{1}{n} \qquad \text{on } L_n.$$

By detail examination of the proof of the Lemma (3.3), we can arrange that the singularity points $y_{n,1}, \ldots, y_{n,i_n}$ are distinct and as close to L_n as we wish.

Construction and approximation on tracts. We have a lot of singularities outside compact sets L_n. So we put

$$A = \{y_{n,i}\ :\ n \geq 1,\ 1 \leq i \leq i_n\}, \qquad A_k = A \cap (K_{k+1} \setminus K_k),\ A_\infty = A \setminus K.$$

Obviously $A \cap K_1 = \emptyset$, so

$$A = (\cup_k A_k) \cup A_\infty.$$

We can arrange that any limit points in A_k belongs to K_k. We now may choose inductively a countable collection of tracts $\{T_x,\ x \in A_k\}$ such that

(i) T_x is a tract from x to some point x' in $\mathbb{R}^n \setminus K$ such that $\mathbb{R}^n \setminus \overline{T}_x$ is connected and $\overline{T}_x \subset \mathbb{R}^n \setminus K_k$ (This is due to the assumption that every bounded component of $\mathbb{R}^n \setminus K_k$ intersects $\mathbb{R}^n \setminus K$.)

(ii) the sets $\overline{T}_x$ are pairwise disjoint (due to the distinction of singularities and absence of limit points)

For each choice of n, i such that $y_{n,i} \in K$ we apply Lemma (3.4) to the function $\alpha_{n,i}U_{y_{n,i}}$. Hence, there exists a function $w_{n,i}$ harmonic on $\mathbb{R}^n$ apart from a singularity outside K such that

$$|w_{n,i} - \alpha_{n,i}U_{y_{n,i}}| < \frac{1}{ni_n} \qquad \text{on } \mathbb{R}^n \setminus T_{y_{n,i}}.$$

If now $y_{n,i}$ is outside the compact K, we simply put $w_{n,i} = \alpha_{n,i}U_{y_{n,i}}$ and $T_{y_{n,i}} = \emptyset$. Now we consider a function

$$w_n = \sum_{i=1}^{i_n} w_{n,i}.$$

Final estimate. It is obvious that w is harmonic on some neigbourhood of K. It remains to show that $w_m \to f$ pointwise.

We have

$$|v_n - w_n| \leq \left| v_n - \sum_{i=1}^{i_n} \alpha_{n,i}U_{y_{n,i}} \right| + \sum_{i=1}^{i_n} \left| \alpha_{n,i}U_{y_{n,i}} - w_{n,i} \right| <$$

$$< \frac{1}{n} + \sum_{i=1}^{i_n} \frac{1}{ni_n} < \frac{2}{n} \qquad \text{on } L_n \setminus \cup_{i=1}^{i_n} T_{y_{n,i}}.$$

Let now $x_0 \in K$. Then there is $k_0 \in \mathbb{N}$ such that $x_0 \in K_{k_0} \setminus K_{k_0 - 1}$ and $n_0 \geq k_0$ such that $\mathrm{dist}(x, K_{k_0} - 1) > \frac{1}{n_0}$. We conclude that $x_0 \in L_n$ whenever $n \geq n_0$.

Since the tracts $\{T_x,\ x \in A_k\}$ are pairwise disjoint, the point x_0 can belong only in one of these for each $k \in \mathbb{N}$. Furthermore, x_0 cannot belong in the tract T_x if $x \in A_k$ whenever $k \geq k_0$ since $\overline{T}_x \subset \mathbb{R}^n \setminus K_k$. We conclude that x_0 belongs to at most $k_0 - 1$ tracts. So there is n_1 large enough such that

$$x_0 \notin \bigcup_{n \geq n_1} \bigcup_{i=1}^{i_n} T_{y_{n,i}}.$$

Hence

$$|v_n(x_0) - w_n(x_0)| < \frac{2}{n} \qquad \text{for } n \geq n_1 \implies$$

$$\implies |g_{k_0,n}(x_0) - w_n(x_0)| < \frac{2}{n} \qquad \text{for } n \geq n_1.$$

But $g_{k_0,n}(x_0) \to f(x_0)$, so the proof is complete. □

Let us remark, that the theorem remains valid for K being an open set if $H_0(K)$ is interpreted as all functions which are harmonic on K.

3.3 Properties and examples

It is well known that if a sequence $\{h_n\}$ of harmonic functions on an open set U, which is locally uniformly bounded from below, converges pointwise to a function f, then the sequence $\{h_n\}$ converges locally uniformly and f is harmonic on U (see, for example, Armitage and Gardiner [4], Theorem 1.5.8).

In the following, we shall give several examples that, in general, pointwise limits of functions harmonic on a neighbourhood of a compact set K need not be harmonic on the interior of K. They, however, have to be harmonic on a dense subset of the interior of K. That follows easily from the mentioned theorem. If $\{h_n\}$ is a sequence of functions harmonic on a neighbourhood of K and converges pointwise to f on K, then for compact sets

$$K_k = \{x \in K : |h_n(x)| \leq k \text{ for each } n \in \mathbb{N}\}$$

we have $K_k \nearrow K$ and f has to be harmonic on the set $\cup_{k=1}^{\infty} K_k^{\circ}$ which has to be dense in the interior of K.

The same fact is true with respect to fine harmonicity of the limit function, it has to be finely harmonic on a finely open finely dense subset of the fine interior of K. The reasoning is almost the same, we only refer to the pointwise convergence theorem of Fuglede ([11], Theorem 11.9) and the fact that $\mathbb{R}^n$ endowed with the fine topology is a Baire space. For the exact definition of fine harmonicity, see Section 4.1.

In the following two examples, we present a function which is harmonic on a dense subset of closed unit ball but it is not a pointwise limit of functions harmonic on a neighbourhood of this closed unit ball. The examples are from Lukeš et al. [17] and Gardiner, Gustafsson [14].

Example 3.7. Let $K = \overline{B(0,1)}$ be a closed unit ball and let $B_j = B(x_j, r_j)$ be a sequence of pairwise disjoint open balls of which union V is a dense subset of K. Let us consider the characteristic function χ_V. Then χ_V is lower semicontinuous and harmonic on V but it is not a pointwise limit of any sequence of functions in $H_0(K)$.

If it was so, then there would have to be a sequence of compact sets $K_k \nearrow K$ on which f would be bounded, Baire-one and $H_0(K_k)$-affine and every bounded component of $\mathbb{R}^n \setminus K_k$ would have to intersect the set $\mathbb{R}^n \setminus K$. Let $x_0 \in K \setminus V$ and U be an open neighbourhood of x, then U would contain $\overline{B_{j_0}}$ for some j_0.

But $\overline{B_{j_0}}$ is not a subset of K_k for any k; otherwise, the function f would be $H_0(\overline{B_{j_0}})$-affine and normalized surface measure $\sigma_{x_{j_0}, r_{j_0}}$ on ∂B_{j_0} is a $H_0(\overline{B_{j_0}})$-representing measure for the center x_{j_0} of B_{j_0}. And we know that $0 = \sigma_{x_{j_0}, r_{j_0}}(f) \neq f(x_j) = 1$. Then

$$(\mathbb{R}^n \setminus K_k) \cap (K \setminus V) \cap U \supset (\mathbb{R}^n \setminus K_k) \cap \partial B_{j_0} \neq \emptyset,$$

so that $\mathbb{R}^n \setminus K_k$ would be dense in $K \setminus V$ for each k and hence $\mathbb{R}^n \setminus K = \cap_k(\mathbb{R}^n \setminus K_k)$ would be dense in $K \setminus V$. This is a contradiction.

Example 3.8. Let K, B_j and V be as in the previous example. Let us further remark that the function χ_V is not even a pointwise limit of functions h_n which are harmonic on V and continuous on ∂V. The reasoning is similar. Let $x \in \overline{V} \setminus V$ and U be an open neighbourhood of x. Then U contains B_k for some natural k. Let σ_k be a normalized surface measure on ∂B_k. Since $h_n \in H(B_k)$ for every $n \in \mathbb{N}$, we get $\sigma_k(h_n) = h_n(x) \to f(x) \neq 0$ and since $h_n \to 0$ on ∂B_k, the sequence $\{h_n\}$ cannot be bounded on ∂B_k in view of the Lebesgue dominated convergence theorem. Then

$$(\overline{V} \setminus V) \cap U \cap \bigcup_{i \geq n} \{|h_i| \geq 1\} \supset \partial B_k \cap \bigcup_{i \geq n} \{|h_i| \geq 1\} \neq \emptyset \qquad \text{for each } n \in \mathbb{N}.$$

So the set $(\overline{V} \setminus V) \cap \bigcup_{i \geq n} \{|h_i| \geq 1\}$ intersects every neighbourhood of an arbitrary point x in $(\overline{V} \setminus V)$, namely, the set $\bigcup_{i \geq n} \{|h_i| \geq 1\}$ is dense in $\overline{V} \setminus V$ for every n and hence

$$\bigcap_{n \in \mathbb{N}} \bigcup_{i \geq n} \{(\overline{V} \setminus V) \cap |h_i| \geq 1\} \neq \emptyset.$$

This is a contradiction since $h_n \to 0$ on $\overline{V} \setminus V$.

Chapter 4

On the space $\mathcal{B}_1(H(U))$

In the Chapter 3, the complete characterization of the space $\mathcal{B}_1(H_0(K))$ was given. Namely, if $K \subset \mathbb{R}^n$ is compact, then the function $f : K \to \mathbb{R}$ is a pointwise limit of functions harmonic on some neighbourhood of K if and only if there exists an increasing sequence $K_k \nearrow K$ such that each bounded component of $\mathbb{R}^n \setminus K_k$ intersects $\mathbb{R}^n \setminus K$ and $f|K_k$ is bounded, Baire-one and $H_0(K_k)$-affine.

We shall prove that the functions belonging to $\mathcal{B}_1(H(U))$ have to satisfy an analogous condition. We recall that by U we mean throughout this thesis a bounded open set in $\mathbb{R}^n$ where $n \geq 2$.

4.1 Fine topology and fine harmonicity

At first, we review some basic facts of the fine topology. Let V be an open subset of $\mathbb{R}^n$. The function $s : V \to [-\infty, +\infty]$ is called *hyperharmonic* on V if it is lower semicontinuous on V and

$$s(x) \geq \sigma_{x,r}(s)$$

whenever $\overline{B(x,r)} \subset V$. We recall that $\sigma_{x,r}$ denotes a normalized surface measure on $\partial B(x,r)$. The function s is called *superharmonic* if it is hyperharmonic and finite on a dense subset of V.

The *fine topology* is the coarsest topology $\mathbb{R}^n$ in which every function superharmonic on $\mathbb{R}^n$ is continuous. Since the fine topology is generated by a family of functions, one can show that the fine topology is completely regular. However, it is not normal.

If we talk about, say, an interior of a set $A \subset \mathbb{R}^n$ with respect the fine topology, we will simply say the fine interior of A and denote it by $\operatorname{int}_f A$. The same goes for the closure, the boundary and so on.

We say that a subset A of $\mathbb{R}^n$ is *thin* at a point $x \in \mathbb{R}^n$ if $A \setminus \{x\}$ contains a fine neighbourhood of x, that is, a finely open set containing x. We remark that the set of boundary points of U where $\mathbb{R}^n \setminus U$ is not thin is precisely the set of all regular points where the Perron-Wiener-Brelot generalized solution of the Dirichlet problem is continuous to this points for any continuous boundary condition.

Now, and let u be a hyperharmonic function on Ω where $\Omega = \mathbb{R}^n$ if $n \geq 3$ or a (sufficiently large but bounded) open ball if $n = 2$. Let A be an arbitrary subset of Ω. We define R_u^A *the reduite* of u on A by

$$R_u^A = \inf\{v \text{ hyperharmonic on } \Omega \ : v \geq u \text{ on } A\}$$

and $\hat{R}_u^A$ *the balayage* of u on A as the greatest lower semicontinuous minorant of R_u^A. One can show that there is a unique Radon measure ε_x^A, called *the balayaged measure* of ε_x on A, such that

$$\hat{R}_u^A(x) = \varepsilon_x^A(u) \qquad \text{for every positive hyperharmonic function } u \text{ on } \Omega.$$

We say, that a function f is *finely harmonic* on a finely open subset V of Ω if it is finely continuous and the fine topology on V has a basis consisting of finely open sets W with $\overline{W}^f \subset V$ such that f is integrable with respect to $\varepsilon_x^{W^c}$ for every $x \in W$ and

$$f(x) = \int f \, d\varepsilon_x^{W^c} \qquad \text{for every } x \in W.$$

4.2 Necessary condition

If $K \subset \overline{U}$ is compact, we define

$$H_K(U) = \{f \in C(K) \ : \ f \text{ is finely harmonic on the fine interior of } K \cap U\}.$$

Theorem 4.1. ***(Necessary condition)***

Let U be a bounded open set in $\mathbb{R}^n$ and $\{h_n\}$ be a sequence in $H(U)$ converging pointwise on $\overline{U}$ to a real-valued function f. Then there exists an increasing sequence of compact sets $K_k \nearrow \overline{U}$ such that

(i) f restricted to K_k is bounded, Baire-one and $H_{K_k}(U)$-affine,

(ii) either $K_k = \overline{U}$ for some $k \in \mathbb{N}$ or every (bounded) component of $\mathbb{R}^n \setminus K_k$ intersects ∂U.

We would like to remark before proving this theorem that if $K_k = \overline{U}$ for some $k \in \mathbb{N}$, then f is $H(U)$-affine. In view of Theorem (2.1) and the fact that $\mathcal{A}^c(H(U)) = H(U)$, there exists even a bounded sequence in $H(U)$ which converges pointwise to f.

Proof. We define

$$K_k = \{x \in \overline{U} \ : \ |h_n(x)| \leq k \text{ for each } n \in \mathbb{N}\}.$$

Then obviously f is bounded and Baire-one on K_k. Since every h_n is $H(U)$-affine (by definition) and hence $h_n|_{K_k}$ is $H_{K_k}(U)$-affine, the Lebesgue dominated convergence theorem gives

$$\mu(f) = \lim \mu(h_n) = \lim h_n(x) = f(x), \qquad x \in K_k,\ \mu \in \mathcal{M}_x(H_{K_k}(U))$$

so the function f restricted to K_k is also $H_{K_k}(U)$-affine.

For the condition (ii): let V be a bounded component of $\mathbb{R}^n \setminus K_k$. If $V \cap \partial U = \emptyset$, then $V \subset U$ and h_m is defined on V. Since $\partial V \subset K_k$, it follows that $|h_m| \leq k$ on ∂V and in view of the maximum principle for harmonic functions, $|h_m| \leq k$ on V. Hence, $V \subset K_k$ by the definition of K_k, but this is an obvious contradiction with the assumption V being the component of $\mathbb{R}^n \setminus K_k$. □

We do not know, whether the conditions in Theorem (4.1) are also sufficient. If one repeats the scheme of the proof of the Gardiner and Gustafsson result he may get the following: there exists a countable set of points $\{y_i,\ i = 1, 2, \ldots\}$ which is a subset of $\mathbb{R}^n \setminus U$ (but a lot of points can be on the boundary) and a sequence of functions which are finite sums of U_{y_i}. This sequence converges pointwise to f but the functions within are far from being continuous on the boundary. The proof will be given a little while later.

4.3 Concept of stability

The first question is how far are functions in $H(U)$ and $H_0(\overline{U})$, that is, whether functions harmonic inside U and continuous on the boundary cannot be extended or uniformly approximated by functions harmonic on some neighbourhood of the compact set $\overline{U}$.

The following theorem is due to Deny [10], the formulation follows [19].

Theorem 4.2. ***(Deny)***

Let U be a bounded open subset of $\mathbb{R}^n$. The following statements are equivalent:

(i) $\mathbb{R}^n \setminus U$ and $\mathbb{R}^n \setminus \overline{U}$ are thin at the same points

(ii) for every $f \in H(U)$ and for each $\varepsilon > 0$ there exists a function $h \in H_0(\overline{U})$ and $|h - f| < \varepsilon$ on $\overline{U}$.

Example 4.3. Let U be an open unit ball in $\mathbb{R}^n$ without its center. Then U is not regular for the Dirichlet problem but since $\mathbb{R}^n \setminus U$ and $\mathbb{R}^n \setminus \overline{U}$ are thin at the same points (open unit ball exactly), the function spaces $H(U)$ and $H_0(\overline{U})$ coincides (together with the function space $H(B(0, 1))$.

Example 4.4. Let now U be an open unit ball in $\mathbb{R}^2$ without a line segment $I = [-\frac{1}{2}, \frac{1}{2}] \times \{0\}$. Since the line segment is not thin at any of its points in $\mathbb{R}^2$, it follows that $\mathbb{R}^n \setminus U$ and $\mathbb{R}^n \setminus \overline{U}$ are not thin at the same points. So there has to be a harmonic function on U which is continuous on $\overline{U}$ but cannot be uniformly approximated by functions harmonic on some neigbourhoods of U.

To give an example of such a function is quite easy. The set U is regular for the Dirichlet problem and therefore, there exists a harmonic function f which is equal to zero on $\partial B(0,1)$ and to one on the line segment I inside the ball. If now g is harmonic on some neighbourhood of $\overline{U} = \overline{B(0,1)}$ and $|g - f| < \varepsilon$ on $\partial B(0,1)$, then by the maximum principle $|g - f| < \varepsilon$ on the entire ball.

This example relies on non-thin segment of boundary which is part of the interior of $\overline{U}$. Every such a set obviously does not satisfy conditions of the previous theorem. However, there are more sophisticated examples of compact sets K in $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus K$ and $\mathbb{R}^n \setminus K^\circ$ are not thin at the same points. See, for example, a so called "Swiss cheese" in Gardiner's book [13], Example 1.2.

Remark 4.5. This remark serves as a motivation for the definitions which follow.

The points where the set $\mathbb{R}^n \setminus U$ is not thin are precisely the regular points for Dirichlet problem and the points where the set $\mathbb{R}^n \setminus \overline{U}$ is not thin are the *stable* points for the Dirichlet problem.

The definition of a regular point is well known: these are points where the generalized PWB-solution of the Dirichlet problem coincides with boundary condition for every continuous function on the boundary.

The meaning of a stable point is roughly the following: every $f \in C(\partial U)$ can be extended continuously on $\mathbb{R}^n$ (by Tietze's theorem). Then we can choose a decreasing sequence of (regular) open sets ω_n containing $\overline{U}$ such as $\omega_n \searrow \overline{U}$ and define h_n on $\overline{\omega}_n$ as the solution to the Dirichlet problem with the boundary data $f|_{\partial\omega_n}$. The sets can be chosen regular because every open set can be exhausted by open sets of C^∞-class. One can prove that the sequence $\{h_n\}$ converges on $\overline{U}$, the limit is harmonic function on U (but can be different from PWB-solution) and the limit function does not depend on the choice of the extension f or of the sequence $\{\omega_n\}$. We call a boundary point *stable* if the limit function is equal to f in this point (independently of the chosen boundary condition, of course).

A trivial observation is that every irregular point is also an unstable point. Thus irregular points are not something which should concern the stability problem. In the example (4.3) we presented an irregular but "stable" set. Another example is so called Lebesgue's cup (cf. Arendt and Daners, [3]).

The theorem of Deny can now be read this way: for a bounded open subset U of $\mathbb{R}^n$, the space $H_0(\overline{U})$ is uniformly dense in $H(U)$ if and only if every unstable point of U is an irregular point as well. Equivalently, if and only if there are not regular unstable points.

But it is not necessary true that every regular point is also a stable point as we have already shown in the previous example.

Definition. Let U be a bounded open subset of $\mathbb{R}^n$. We recall that every boundary point of U in which the set $\mathbb{R}^n \setminus U$ is not thin is called a *regular* boundary point (for the Dirichlet problem) and every boundary point of U in which the set $\mathbb{R}^n \setminus U$ is thin is called an *irregular* boundary point (for the Dirichlet problem).

We say that a boundary point of U is a *stable* point of U (for the Dirichlet problem) if the set $\mathbb{R}^n \setminus \overline{U}$ is not thin.

We say that the set U is *stable* (for the Dirichlet problem) if every regular point is also a stable point, that is, the sets $\mathbb{R}^n \setminus U$ and $\mathbb{R}^n \setminus \overline{U}$ are thin at the same points.

In view of the previous definition, we can reformulate the theorem of Deny:

Let U be a bounded open subset of $\mathbb{R}^n$. The set U is stable if and only if for each f harmonic on U and continuous on the boundary there exists a sequence h_n of functions harmonic on some neighbourhood of $\overline{U}$ and $h_n \to f$ uniformly on $\overline{U}$.

We refer to Vicent-Smith [19] for a proof which carries on in more general harmonic spaces.

It is kind of obvious that if U is a stable set, then $\mathcal{B}_1(H(U))$ and $\mathcal{B}_1(H_0(\overline{U}))$ coincides. In a while, we shall give examples of unstable sets in which the equality is still valid. But for the equality of subclasses $\mathcal{B}_1^{bb}(H(U))$ and $\mathcal{B}_1^{bb}(H_0(\overline{U}))$, the stability of U is essential.

Theorem 4.6. *Let U be a bounded open subset of $\mathbb{R}^n$ and $\mathcal{F}$ be a family of functions $f : \overline{U} \to \mathbb{R}$ such as there exists a bounded sequence $\{h_n\}$ in $H(U)$ and $f_n \to f$ pointwise on $\overline{U}$. Then the following statements are equivalent:*

(i) *Each member of $\mathcal{F}$ is a pointwise limit of a bounded sequence of functions which are harmonic on some neighbourhood of $\overline{U}$,*

(ii) *the set U is stable,*

(iii) *$H_0(\overline{U})$ is uniformly dense in $H(U)$.*

Proof. The equivalence between (*ii*) and (*iii*) is the theorem of Deny. It is obvious that (*iii*) implies (*i*), so what remains is to prove that (*i*) implies (*iii*).

However, if (*iii*) is not valid, then there exists $f \in H(U) \setminus \overline{H_0(\overline{U})}$. If such a function f was a pointwise limit of a bounded sequence of functions which are harmonic on some neighbourhood of $\overline{U}$, that is, $f \in \mathcal{B}_1^{bb}(H_0(\overline{U}))$, then it follows from the Proposition (2.16) that f has to be $H_0(\overline{U})$-affine. Since f is continuous on $\overline{U}$, f has to be in $\overline{H_0(\overline{U})}$ due to the theorem of Debiard and Gaveau (3.6). This is a contradiction. □

4.4 Examples and problems

In the last section, we treated the case when the bounded open subset set U of $\mathbb{R}^n$ was stable and thus the function spaces $H(U)$ and $H_0(\overline{U})$ coincided. We shall now present a simple example ofan unstable set for which the spaces $\mathcal{B}_1(H(U))$ and $\mathcal{B}_1(H_0(\overline{U}))$ will coincide.

For that purpose, the following corollary of the theorem of Deny (4.2) and the pole-pushing technique (3.5) would be useful.

Theorem 4.7. *Let U be a bounded open subset of $\mathbb{R}^n$ and suppose that*

(i) U is stable,

(ii) $\mathbb{R}^n \setminus \overline{U}$ is connected.

Then for every $\varepsilon > 0$ and every function u harmonic on U and continuous to the boundary, there exists a function w harmonic on $\mathbb{R}^n$ such that $|w - u| < \varepsilon$ on the closure of U.

Proof. By the theorem of Deny, there exists a function v_1 harmonic on some neighbourhood of $\overline{U}$ such that $|u - v_1| < \varepsilon/3$ on $\overline{U}$. The function v_1 can be uniformly approximated by a function v_2 which would be a finite sum of potentials with singularities outside $\overline{U}$ (Lemma 3.3), so we can assume that $|v_1 - v_2| < \varepsilon/3$ on $\overline{U}$. And each of these potentials can be uniformly approximated outside a tract from its pole to the infinity which does not intersect $\overline{U}$, so we can construct a function w such that w is harmonic on $\mathbb{R}^n$ and $|w - v_2| < \varepsilon/3$ on $\overline{U}$. □

Example 4.8. Let U be a subset of $\mathbb{R}^n$ defined by

$$U = B(0,1) \setminus S(0, \tfrac{1}{2}).$$

Since the sphere $S(0, \frac{1}{2})$ is not thin at any of its points, the set U is not stable and so $H(U) \neq H_0(\overline{U})$ and also $\mathcal{B}_1^{bb}(H(U)) \neq \mathcal{B}_1^{bb}(H_0(\overline{U}))$.

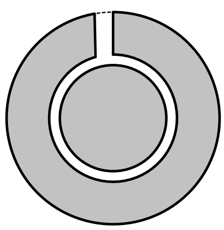
We will now prove that every function in $\mathcal{B}_1(H(U))$ also lies in $\mathcal{B}_1(\mathcal{H}(\mathbb{R}^n))$.

Consider $f \in \mathcal{B}_1(H(U))$. There exists a sequence of functions f_n which are harmonic on U, continuous to the boundary and converging pointwise to f on the closure of U. Let us consider sets U_n which are like in the figure below. Explicitly,

$$\overline{U_n} = \overline{U} \setminus \left(\{x : \tfrac{1}{2} < |x| < \tfrac{1}{2} + \tfrac{1}{n}\} \cup \{x = (x_1, \dots, x_n) : |x| > \tfrac{1}{2} \text{ and } 0 > x_1 > -\tfrac{1}{n}\}\right)$$

and

$$U_n = \operatorname{int} \overline{U_n}.$$



It is obvious, that the sets U_n are stable and $\mathbb{R}^n \setminus \overline{U_n}$ is connected. So for a given sequence of positive numbers $\varepsilon_n \searrow 0$, we can find functions w_n harmonic on $\mathbb{R}^n$ such that $|w_n - f_n| < \varepsilon_n$ on $\overline{U_n}$. Since for every $x \in \overline{U}$ there exists n such that $x \in \overline{U_n}$, we observe that functions w_n are converging to the function f pointwise on $\overline{U}$.

The main idea of the previous example was to exhaust the closure of the given bounded open set U with stable sets such that their complements are connected. Using this idea, we can get a sufficient condition for the spaces $\mathcal{B}_1(H(U))$ and $\mathcal{B}_1(H_0(\overline{U}))$ to coincide.

Proposition 4.9. *Let U be a bounded open subset of $\mathbb{R}^n$ and suppose that there exists a sequence $\{U_n\}$ such that*

(i) $\overline{U}_n \nearrow \overline{U}$,

(ii) the set U_n is stable for each $n \in \mathbb{N}$,

(iii) every component of $\mathbb{R}^n \setminus \overline{U}_n$ meets $\mathbb{R}^n \setminus \overline{U}$.

Then the spaces $\mathcal{B}_1(H(U))$ and $\mathcal{B}_1(H_0(\overline{U}))$ coincides.

Proof. The proof is simple. If f is a pointwise limit of a sequence $\{f_n\}$ of functions harmonic on $\overline{U}$ and continuous on $\overline{U}$, then each f_n can be uniformly approximated on $\overline{U}_n$ by finite sums of potentials with singularites outside U (due to the theorem of Deny and pole-pushing lemma). □

Obviously, every stable set satisfies trivially these assumptions and in the previous example, we presented an unstable set which can be exhausted by "stable sets" sequence. One can go a little further along this way and discuss whenever such an exhaustion is possible. This brings us back to a question, how to decide whether a given set is stable or not. One useful criterion is that if the set is topologically regular (that means that $\operatorname{int} \overline{U} = U$) and has a continuous boundary, then the set is stable (cf. Arendt and Daners, [3], Proposition 1.2). Therefore, it follows from Lemma (3.2) that every open set can be exhausted by stable sets; but it should be noted that the lemma says nothing about exhausting the set up to the boundary.

Whether such an exhaustion is possible for every bounded open set, it seems to be an open problem.

We shall now present the promised proof that if the function meets necessary conditions, then it is approximable by functions harmonic on $\mathbb{R}^n$ except some singularities on the boundary. For that, we shall need stronger approximation theorem than in Chapter 3.

Theorem 4.10. ***(Gardiner, 1997)***

Let Ω be an open of $\mathbb{R}^n$, where $n \geq 2$ and let E be a relatively closed subset of Ω. Let u be a function continuous on E with the continuous extension to $\overline{E}^{\infty}$, the closure of E in compactified space $\mathbb{R}^n \cup \{\infty\}$. Furthermore, assume that u is finely harmonic on the fine interior of E.

Then for each $\varepsilon > 0$, there exists an open subset V of $\mathbb{R}^n \cup \{\infty\}$ such that $\overline{E}^\infty \subset V$ and a function w continuous on the space $\mathbb{R}^n \cup \{\infty\}$ and harmonic on $V \cap \Omega$ such that $|w - u| < \varepsilon$ on E.

The proof can be found in Gardiner (1997) [12], Theorem 2, part (a). Using this theorem and another general result of Armitage and Goldstein (1990) [5], we are ready for the promised proof.

Proposition 4.11. *Let U be a bounded open subset of $\mathbb{R}^n$ and $\{h_n\}$ be a sequence in $H(U)$ which converges pointwise to a real valued function f on $\overline{U}$.*

Then there exists a sequence of functions harmonic on $\mathbb{R}^n$, except from at most countable set of singularities on the boundary ∂U, which converges pointwise to f on $\overline{U}$.

Proof. We know that there exists a sequence of compact sets $K_k \nearrow \overline{U}$ such that f restricted to K_k is bounded, Baire-one and $H_{K_k}(U)$-affine. The space $H_K(U)$ is simplicial for any compact set $K \subset \overline{U}$. This follows from the work of Bliedtner and Hansen (1975), see Example III.3.1.2. and Corollary III.3.8. So as a consequence of the approximation theorem in simplicial spaces (2.1), there is a bounded sequence $\{g_{n,k}\}$ of continuous and $H_{K_k}(U)$-affine functions which converges pointwise to f on K_k. Due to the theorem of Debiard and Gaveau (3.6), we may assume that $g_{n,k}$ are continuous on K_k and finely harmonic on the fine interior of $K_k \cap U$.

We construct a new sequence of compact sets L_k as we have once done before. Let

$$\begin{aligned}
L_1 &= K_1, \\
L_2 &= K_1 \cup \{x \in K_2 : \mathrm{dist}(x, K_1) \geq \frac{1}{2}\}, \\
L_3 &= K_1 \cup \{x \in K_2 : \mathrm{dist}(x, K_1) \geq \frac{1}{3}\} \cup \{x \in K_3 : \mathrm{dist}(x, K_2) \geq \frac{1}{3}\}, \\
&\vdots \\
L_n &= K_1 \cup \{x \in K_2 : \mathrm{dist}(x, K_1) \geq \frac{1}{n}\} \cup \ldots \cup \{x \in K_n : \mathrm{dist}(x, K_{n-1}) \geq \frac{1}{n}\}.
\end{aligned}$$

and define a new function v_n on L_n by

$$\begin{aligned}
v_n(x) &= g_{n,1}(x) \qquad \text{on } L_n \cap K_1, \\
v_n(x) &= g_{n,k}(x) \qquad \text{on } L_n \cap (K_k \setminus K_{k-1}),\ k = 2, \ldots, n.
\end{aligned}$$

Hence, we may assume that the function v_n is continuous on L_n and finely harmonic on the fine interior of $L_n \cap U$. Then there exists an open set V containing L_n and a function w_n continuous on $\mathbb{R}^n$ and harmonic on $V \cap U$ such that $|w_n - v_n| < \frac{1}{n}$ on L_n. This is a direct consequence of the theorem (4.10) mentioned earlier (since both of the functions w_n and v_n are uniformly continuous on L_n and therefore, the approximation can be extended up to the boundary).

Then we can use a general result of Armitage and Goldstein (1990) (cf. the main result of [5]) to approximate the function w_n on L_n by functions harmonic on some neighbourhood of U apart from certain isolated singularities (outside L_n) and pole-pushing technique to push all of these singularities, which are in U, on the boundary of U. This is possible since every component of set $\mathbb{R}^n \setminus K_n$ intersects ∂U. The rest of the proof copies the appropriate parts of Section 3.2. □

The fact that singularities can be wiped from U to the boundary immediately yields to the following corollaries which are simple consequences of the pole-pushing technique and, with some additional assumptions on the set U, gives a sufficiency of presented necessary conditions. Whether the necessary conditions are sufficient even in the general case, seems to be an open problem.

Corollary 4.12. *Let us assume that in every neighbourhood of any boundary point of U, there is a point which does not belong to $\overline{U}$. Then $\mathcal{B}_1(H(U)) = \mathcal{B}_1(H_0(\overline{U}))$ and the necessary conditions are also sufficient.*

Especially, every topologically regular set U (that is, sets for which $\operatorname{int}(\overline{U}) = U$) has this property.

The proof is simple since any tract used in pole-pushing technique would contain a point outside $\overline{U}$.[1]

We presented before an example 4.8 (an open unit ball with an inner cut on the sphere with half radius) which does not meet the assumptions of this corollary yet still the conclusion is valid. The example can be derived from the following weaker version.

Corollary 4.13. *Let us assume that there are open sets ω_n such that for every $x \in \overline{U}$, there exists $n_x \in \mathbb{N}$ such that $x \notin \omega_n$ for every $n > n_x$, and*

(i) every boundary point of U either has in its every neighbourhood a point which does not belong to $\overline{U}$, or

(ii) in its every neighbourhood, there exists a point belonging to ω_n and in the same component of ω_n there is a point which does not belong to $\overline{U}$.

Then every function in $\mathcal{B}_1(H(U))$ is also in $\mathcal{B}_1(H_0(\overline{U})$. Hence, the necessary conditions are also sufficient.

[1]) The topologically regular sets are important in similar problems in numerical analysis (we refer, for example, for papers of Babuška and Chleboun on this topic which were devoted to numerical estimates in unstable domains for Dirichlet problems).

Appendix: Simpliciality of $H^f(K)$

In this short addition to the thesis, we will follow closely the work of Bliedtner and Hansen [7]. Without proofs, we shall present several ideas of their view of simpliciality in potential theory and use it to justify the fact that the space $H^f(K)$ (of all continuous functions on compact set K finely harmonic on the fine interior of K) is simplicial in the sense we defined at the beginning of the thesis.

A.1 Simplicial cones

Let Y be a locally compact space with a countable base and $C(Y)$ the space of all real continuous functions on Y. A convex cone $S \subset C(Y)$ is called *admissible* if

(a) $S^+ \neq \{0\}$ and S^+ is *linearly separating*. This means: for every pair of points $x \neq y$ of Y and every $\lambda \geq 0$, there exists $f \in S^+$ such that

$$f(x) \neq \lambda f(y).$$

(b) S is dominated by S^+. That means: for every $f \in S$, there exists $g \in S^+$ such that for any $\varepsilon > 0$ there exists a compact set $K \subset Y$ and

$$|f(x)| \leq \varepsilon g(x) \qquad \text{for any } x \in Y \setminus K.$$

We denote $W(S)$ a min-stable convex cone consisting of all functions $\min\{s_1, \ldots, s_n\}$ where $s_1, \ldots, s_n \in S$. $W(S)$ is an admissible cone.

Let $\mathcal{M}(S)$ be the convex cone of positive Radon measures on Y for which the functions of S are integrable. The admissible cone S determines an ordering on $\mathcal{M}(S)$:

$$\mu \prec \nu \text{ if } \mu(s) \leq \nu(s) \qquad \text{for all } s \in W(S).$$

We denote

$$\mathcal{M}_x(S) = \{\mu \in \mathcal{M}(S) \ : \ \mu(s) \leq s(x) \text{ for all } s \in S\}$$

and we call $\mathcal{M}_x(S)$ a set of *S-representing measures* for x. A measure $\mu \in \mathcal{M}(S)$ is *minimal* if it is minimal with respect to the ordering $\prec$. Note that for every $\mu \in \mathcal{M}(S)$ there is a minimal measure ν such that $\nu \prec \mu$.

We call an admissible cone *simplicial* if for every $x \in Y$ there exists a *unique* minimal measure $\mu_x \in \mathcal{M}_x(S)$.

We define $\mathcal{F}_S(Y)$ as the set of all lower semicontinuous numerical functions f on Y which are lower S-bounded. A function f is lower S-bounded if there exists $s \in S^+$ such that $f \geq -s$. For any $f \in \mathcal{F}_S(Y)$ and $g \in -\mathcal{F}_S(Y)$ we define *lower S-envelope* and *upper S-envelope* as

$$\check{f} = \sup\{t \in -S : t \leq f\}$$

$$\hat{g} = \inf\{t \in S : t \geq g\}$$

Finally, we denote $\hat{S}$ the set of all lower semi-continuous *S-concave* functions on Y,

$$\hat{S} = \{v \in \mathcal{F}_S(Y) \ : \ \mu(v) \leq s(x) \text{ for all } x \in Y \text{ and } \mu \in \mathcal{M}_x(S)\}.$$

By a space of all continuous *S-affine* functions we mean

$$H(S) = \hat{S} \cap (-\hat{S}) = \{h \in C_S(Y) \ : \ \mu(h) = h(x) \text{ for all } x \in Y \text{ and } \mu \in \mathcal{M}_x(S)\},$$

where $C_S(Y)$ denotes the space of all continuous S-bounded functions. A function f is S-bounded if it is lower and upper S-bounded. Namely, $f \in C_S(Y)$ if and only if there exists $s \in S^+$ such that $|f| \leq s$ on Y.

The set

$$\mathrm{Ch}_S\, Y = \{x \in Y \ : \ \mathcal{M}_x(S) = \{\varepsilon_x\}\}$$

is called *the Choquet boundary* of Y with respect to S.

Proposition A.1. *Let $S \subset C(Y)$ be a simplicial cone and let $S_0 \subset C_S(Y)$ be an admissible cone such that*

$$H(S) \subset S_0 \subset \hat{S}.$$

Then the following statements hold:

(i) $H(S_0) = H(S)$.

(ii) S_0 is a simplicial cone.

(iii) $\mathrm{Ch}_{S_0}\, Y = \mathrm{Ch}_S(Y)$.

(iv) For any $x \in Y$, the minimal measures in $\mathcal{M}_x(S_0)$ and $\mathcal{M}_x(S)$ coincide.

Proof. See [7] Bliedtner, Hansen (1975), Proposition I.2.6. □

Let now X be a closed subset of a locally compact space Y with a countable base and let $\mathcal{P} \subset C^+(Y)$ be a convex cone such that $\mathcal{P}|_X$ is an admissible cone on X. Let

$$\mathcal{H}_+^* = \{\sup p_n \ : \ p_n \in \mathcal{P} \text{ and } (p_n) \text{ is an increasing sequence}\}.$$

For any function $f : Y \to [0, +\infty]$ and an arbitrary subset A of Y, we define *reduced function* by

$$R_f^A = \inf\{v \in \mathcal{H}_+^* \ : \ v \geq f \text{ on } A\}.$$

If $R_f^Y \in \mathcal{P}$ for every $f \in C_{\mathcal{P}}^+(Y)$, then we call $\mathcal{P}$ a *cone of potentials.*

If $\mathcal{P}$ is a cone of potentials, then by $\mathcal{P}$*-dilation* on X we mean a kernel[2] T on Y such that

(a) for every $p \in \mathcal{P}$, $Tp \in \mathcal{H}_+^*$ and $Tp \le p$,

(b) for every $x \in X$, the measure $T(x,\cdot)$ is supported by X and for every $y \in X^c$ we have $T(y,\cdot) = \varepsilon_y$.

The definition allows to apply $\mathcal{P}$-dilations on X to functions defined on X only.

For any $\mathcal{P}$-dilation T on X, let

$$A(T) = \{y \in Y \ : \ T(y,\cdot) = \varepsilon_y\}.$$

Obviously, $X^c \subset A(T)$.

For any family $\mathcal{T}$ of $\mathcal{P}$-dilations on X we shall consider the convex cone $\mathcal{S}(\mathcal{T})$ and the linear space $\mathcal{H}(\mathcal{T})$ defined as

$$\mathcal{S}(\mathcal{T}) = \{s \in C_{\mathcal{P}}(X) \ : \ Ts \le s \text{ for every } T \in \mathcal{T}\},$$

$$\mathcal{H}(\mathcal{T}) = \{h \in C_{\mathcal{P}}(X) \ : \ Th = h \text{ for every } T \in \mathcal{T}\}.$$

One can see that $\mathcal{S}(\mathcal{T}) \supset \mathcal{P}|_X$ so $\mathcal{S}(\mathcal{T})$ is an admissible cone and

$$H(\mathcal{S}(\mathcal{T})) = \mathcal{S}(\mathcal{T}) \cap (-\mathcal{S}(\mathcal{T})) = \mathcal{H}(\mathcal{T}).$$

Proposition A.2. *$\mathcal{S}(\mathcal{T})$ is a simplicial cone and if $\mathcal{H}(\mathcal{T})$ is linearly separating, then $\mathcal{H}(\mathcal{T})$ is simplicial as well. Furthermore, the minimal representing measures of $\mathcal{H}(\mathcal{T})$ and $\mathcal{S}(\mathcal{T})$ coincide.*

Proof. See [7] Bliedtner, Hansen (1975), Theorem II.3.3. and Corollary II.3.8. □

We end this section by connecting this abstract theory to the case we need. Let X be a closed subset of a locally compact space Y with a countable base. Let U be an open subset of Y which is contained in X and consider a family of $\mathcal{P}$-dilations

$$\mathcal{T} = \{(x \mapsto \varepsilon_x^{V^c} \ : \ V \text{ is open and relatively compact, } \overline{V} \subset U\}.$$

Then

$$\mathcal{S}(\mathcal{T}) = S(X,U) = \{s \in C_{\mathcal{P}}(X) : s \text{ is superharmonic on } U\}$$

and

$$\mathcal{H}(\mathcal{T}) = H(X,U) = \{h \in C_{\mathcal{P}}(X) : h \text{ is harmonic on } U\}.$$

[2]) By a kernel T, we mean simultaneously an integral operator T and a function $T(\cdot,\cdot)$ on $Y \times Y$ such that $Tf(x) = \int_Y T(x,y)f(y)\,dy$.

We denote

$$\mathcal{I} = \{\mathcal{T}(U) \ : \ U \text{ open}, \ X \subset U\}$$

where

$$\mathcal{T}(U) = \{(y \mapsto \varepsilon_y^{V^c} \ : \ V \text{ open and relatively compact}, \ \overline{V} \subset U\}.$$

Then $\mathcal{I}$ is a decreasingly filtered family of sets of $\mathcal{P}$-dilations on X. We define $\mathcal{S}(\mathcal{I})$ as the set

$$\mathcal{S}(\mathcal{I}) = \bigcup_{\mathcal{T} \in \mathcal{I}} \mathcal{S}(\mathcal{T}) = \bigcup_{\substack{U \text{ open} \\ X \subset U}} S(Y, U),$$

$$\mathcal{H}(\mathcal{I}) = \bigcup_{\mathcal{T} \in \mathcal{I}} \mathcal{H}(\mathcal{T}) = \bigcup_{\substack{U \text{ open} \\ X \subset U}} H(Y, U).$$

One can show that $\mathcal{S}(\mathcal{I})$ is a simplicial cone. If X is compact, then $\overline{\mathcal{H}(\mathcal{I})} = \mathcal{H}(S(\mathcal{I}))$. And if $\mathcal{H}(\mathcal{I})$ is linearly separating and nontrivial, then $\mathcal{H}(\mathcal{I})$ is simplicial and the minimal representing measures of $\mathcal{H}(\mathcal{I})$ and $\mathcal{S}(\mathcal{I})$ coincide again. However, the space $\overline{\mathcal{H}(\mathcal{I})|_X}$ is exactly the space of all functions $f : X \to \mathbb{R}$ which are continuous on X and finely harmonic on the fine interior of X as follows from Theorem III.3.15. in Bliedtner, Hansen [7].

A.2 Simpliciality of $H^f(K)$

What remains is to show that the previous simpliciality result is the same "simpliciality" we need. We recall that by $H_0(K)$ we mean a space of functions harmonic on some neighbourhood of a compact subset K of $\mathbb{R}^n$. We denote

$$H^f(K) \text{ the closure of } H_0(K) \text{ in the supremum norm in } C(K).$$

The Debiard-Gaveau theorem states that

$$H^f(K) = \{f : K \to \mathbb{R}, f \text{ is continuous on } K \text{ and finely harmonic on the fine interior of } K.\}$$

Thus $H^f(K)$ is a function space. Further, let

$$\mathcal{W} = \{\min(h_1, \dots, h_m) \ : \ h_1, \dots, h_m \in H^f(K), \, m \in \mathbb{N}\}.$$

For a while, we shall denote $\mathcal{H} = H^f(K)$ and $\mathcal{K} = -\mathcal{K}^c(H^f(K))$, that is,

$$\mathcal{K} = \{f \in C(K) \ : \ \mu(f) \le f(x) \text{ whenever } x \in K \text{ and } \mu \in \mathcal{M}_x(\mathcal{H})\}$$

and let $\mathcal{M}_x^{\mathcal{K}}$ denote the collection of all probability measures μ on K satisfying $\mu(g) \le g(x)$ whenever $g \in \mathcal{K}$.

Lemma A.3. $\mathcal{M}_x^{\mathcal{K}}$ *coincides with* $\mathcal{M}_x(\mathcal{H})$.

Proof. Let $\mu \in \mathcal{M}_x(\mathcal{H})$. Then $\mu(g) \leq g(x)$ whenever $g \in \mathcal{K}$ by definition of $\mathcal{K}$. On the contrary, let $\mu \in \mathcal{M}_x^{\mathcal{K}}$. We have to show that for every $f \in H^f(K)$ we have $\mu(f) = f(x)$. But that is obvious because $f \in \mathcal{K}$ and $-f \in \mathcal{K}$, hence

$$\mu(f) \leq f(x) \qquad \text{and} \qquad -\mu(f) = \mu(-f) \leq -f(x) \implies \mu(f) \geq f(x)$$

and that completes the proof. □

We recall that we have defined a *simplicial* function space as a subspace of $C(K)$ which separates points and contains constant functions and with this additional property: for each $x \in K$, there exists a unique maximal measure in $\mathcal{M}_x(\mathcal{H})$ with respect to the Choquet ordering. This ordering is defined defined as

$$\mu \prec \nu \qquad \text{if} \qquad \mu(f) \leq \nu(f) \;\text{ whenever } f \in \mathcal{K}^c(H^f(K)).$$

It is trivial that it is the same as the existence of a unique minimal measure in the ordering given by

$$\mu \prec \nu \qquad \text{if} \qquad \mu(f) \leq \nu(f) \;\text{ whenever } f \in \mathcal{K},$$

since the first one is an ordering which uses $\mathcal{H}$-convex functions and the second definition uses $\mathcal{H}$-concave functions.

However, the version of simpliciality presented in this chapter, minimizes the measures in $\mathcal{M}_x(\mathcal{H})$ with respect to the ordering

$$\mu \prec \nu \qquad \text{if} \qquad \mu(f) \leq \nu(f) \;\text{ whenever } f \in \mathcal{W},$$

so we have to check whether minimal measures with respect to one ordering are different from the other ones. But in view of Proposition (A.1) used on $\mathcal{S} = H^f(K)$ and $\mathcal{S}_0 = \mathcal{K}$, we see that the minimal measures in both ordering are the same, since $\mathcal{K}$ is obviously min-stable and hence $W(\mathcal{K}) = \mathcal{K}$. So the results of Bliedtner and Hansen transfer as we need and the assumption of simpliciality of $H^f(K)$ is justified by Theorem III.3.15. in [7].

A.3 Proof of the Debiard-Gaveau theorem

The theorem of Debiard and Gaveau (3.6) was an essential tool for the proof of pointwise approximation theorems presented in this thesis. So we decided to present here a proof. In fact, we will prove a generalized version which is due to Gardiner. It has been used in Section (4.4).

In Section (4.1), we defined a balayaged measure of Dirac measure ε_x. We extend this definition on all admissible measures. Let Ω be a bounded open ball if $n = 2$ and $\Omega = \mathbb{R}^n$ if $n > 2$. We call a positive Radon measure μ on Ω *admissible* if for each finite and continuous potential p on Ω harmonic off some compact set

$$\mu(p) < +\infty.$$

A *potential* on Ω is a positive superharmonic function whose greatest harmonic minorant on Ω is zero.

For every admissible measure μ and any subset A of Ω, there exists precisely one measure μ^A on Ω such that

$$\mu^A(u) = \mu(\hat{R}_u^A)$$

for every positive hyperharmonic function on Ω.

Theorem 4.10. (**Gardiner, 1997**)

Let Ω be an open of $\mathbb{R}^n$, where $n \geq 2$ and let E be a relatively closed subset of Ω. Let u be a function continuous of E with the continuous extension to $\overline{E}^\infty$, the closure of E in compactified space $\mathbb{R}^n \cup \{\infty\}$. Furthermore, let us assume that u is finely harmonic on the fine interior of E.

Then for each $\varepsilon > 0$, there exists an open subset V of $\mathbb{R}^n \cup \{\infty\}$ such that $\overline{E}^\infty \subset V$ and a function w continuous on the space $\mathbb{R}^n \cup \{\infty\}$ and harmonic on $V \cap \Omega$ such that $|w - u| < \varepsilon$ on E.

Note, that if E is compact, then this is exactly the nontrivial implication of the theorem of Debiard and Gaveau. The original proof used a probabilistic potential theory, another proof can be found in Bliedtner and Hansen [7]. The proof presented here relies on deep results of Fuglede and Ancona [1], [2], [11].

Proof. (cf. Gardiner, [12])

When $E = \Omega$, there is little to prove. When $n = 2$, we may assume without any loss of generality that $\overline{\Omega}$ is contained in some open ball B (if not, then we use Kelvin's transformation). We define $\Omega_0 = B$ when $n = 2$ and $\Omega_0 = \mathbb{R}^n$ when $n \geq 3$.

The function u is continuous on compact set $\overline{E}^\infty$. If $n \geq 3$, then we may add a suitable constant so that $u(\infty) = 0$. Suppose that μ is a signed Radon measure on $\overline{E}^\infty$ such that $\mu(f) = 0$ whenever $f \in C(\mathbb{R}^n \cup \{\infty\}) \cap \mathcal{H}(V \cap \Omega)$ for some open subset V of $\mathbb{R}^n \cup \{\infty\}$ that contains $\overline{E}^\infty$. Since such Radon measures represent continuous linear functionals on continuous functions on compact set $\overline{E}^{\infty}$, then, due to the Hahn-Banach theorem, it will be enough to show that $\mu(u) = 0$.

We define a so called *chordal metric* on $\mathbb{R}^n \cup \{\infty\}$ by

$$d_\infty(x, y) = \frac{|x - y|}{\sqrt{(1 + |x|^2)(1 + |y|^2)}} \qquad \text{where } x, y \in \mathbb{R}^n$$

and

$$d_\infty(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}} \qquad \text{where } x \in \mathbb{R}^n,$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. Furthermore, if $A \subset \mathbb{R}^n \cup \{\infty\}$ we define

$$\operatorname{dist}_\infty(x, A) = \inf\{d_\infty(x, y) \ : \ y \in A\} \qquad \text{where } x \in \mathbb{R}^n.$$

Next, let $\{K_m\}$ be an increasing sequence of closed subsets of $\overline{\Omega}_0 \setminus \Omega$ such that each set K_m is non-thin at each of its points and such that the set Z defined by

$$Z = \Omega_0 \setminus \left(\Omega \cup \left[\bigcup_{m=1}^{\infty} K_m \right] \right)$$

is polar (that is, there exists a superharmonic function on some open neighbourhood of Z which is equal to $+\infty$ on Z). This can be done as a straightforward consequence of Ancona's result [1], [2]. Now, let $\{U_m\}$ be a decreasing sequence of open sets which are regular for the Dirichlet problem and which satisfy

$$\{x \in \Omega_0 : \operatorname{dist}_\infty(x, \overline{E}^{\infty} \leq \frac{1}{m+1}\} \subset U_m \subset \{x \in \Omega_0 : \operatorname{dist}_\infty(x, \overline{E}^{\infty} < \frac{1}{m}\}$$

and let $V_m = U_m \setminus K_m$ for each m. It follows that the open sets V_m are also regular for the Dirichlet problem.

Let w be a continuous potential on Ω_0 whose associated measure has compact support in Ω_0. Then we have

$$\hat{R}_w^{\Omega_0 \setminus V_m} \in C(\mathbb{R}^n \cup \{\infty\} \cup \mathcal{H}(U_m \cap \Omega)$$

if we extend $\hat{R}_w^{\Omega_0 \setminus V_m}$ by zero outside Ω_0. Hence

$$\mu(\hat{R}_w^{\Omega_0 \setminus V_m}) = 0$$

since that is true for all functions in $C(\mathbb{R}^n \cup \{\infty\} \cup \mathcal{H}(U_m \cap \Omega)$. Since Z is polar,

$$\hat{R}_w^{\Omega_0 \setminus V_m}(x) \nearrow \hat{R}_w^{\Omega_0 \setminus [E \cup (\partial E \cap Z)]}(x) = \hat{R}_w^{\Omega_0 \setminus E}(x) \qquad x \in \overline{E}^{\infty}, \ m \to \infty.$$

Let us denote $\mu_1 = \mu|_{\overline{E}}$, then

$$\mu_1^{\Omega_0 \setminus E}(w) = \mu_1(\hat{R}_w^{\Omega_0 \setminus E}) = 0.$$

But it is well known that if $f \in C(\mathbb{R}^n \cup \{\infty\})$ with value 0 outside Ω_0, then f can be uniformly approximated by differences of continuous potentials with the same properties as above. Hence

$$\mu_1^{\Omega_0 \setminus E} = 0 \qquad \text{in the sense that} \qquad (\mu_1^+)^{\Omega_0 \setminus E} = (\mu_1^-)^{\Omega_0 \setminus E}.$$

We claim that

$$u(x) = \varepsilon_x^{\Omega_0 \setminus E}(u) \qquad x \in \overline{E}.$$

If $x \in \operatorname{int}_f E$ that this is true from the definition of the fine harmonicity. (If E is unbounded, then

$$u(x) = \varepsilon_x^{\Omega_0 \setminus \{y \in E \,:\, |y| \leq m\}}(u) \xrightarrow{m \to \infty} \varepsilon_x^{\Omega_0 \setminus E}(u) \qquad x \in \operatorname{int}_f E$$

in view of the continuity of u at ∞).

If $x \in \overline{E} \setminus \operatorname{int}_f E$ and $\Omega_0 \setminus E$ is non-thin at x, then $\varepsilon_x^{\Omega_0 \setminus E} = \varepsilon_x$ and the equality is trivial. The remaining points form a polar set and we can redefine u there as we need without touching any of its assumed properties (see Fuglede [11], Theorem 9.14).

Hence,

$$\mu(u) = \mu_1(u) = \mu_1(\varepsilon_x^{\Omega_0 \setminus E}(u)) = \mu_1^{\Omega_0 \setminus E}(u) = 0$$

and the proof is complete. □

The theorem of Debiard and Gaveau follows this way.

Theorem 3.6. (Debiard, Gaveau; 1973)

Let K be a compact subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$. The following statements are equivalent:

(a) there exists a sequence $\{h_m\}$ in $H_0(K)$ such that $h_m \to f$ uniformly on K,

(b) the function f is continuous on K and $H_0(K)$-affine, that is,

$$f(x) = \int f \, d\mu \qquad \text{for all } x \in K \text{ and } \mu \in \mathcal{M}_x(H_0(K)).$$

(c) f is continuous on K nad finely harmonic on the fine interior of K.

Proof. $(a) \implies (b)$ is obvious.

$(b) \implies (c)$ follows from the definition of fine harmonicity with respect to the fact that measures $\varepsilon_x^{W^c}$ in the definition are in $\mathcal{M}_x(H_0(K))$. This is obvious because functions harmonic on an open set are also finely harmonic there.

$(c) \implies (a)$ is a special case of the previous theorem. □

Bibliography

[1] Ancona A.: *Démonstration d'une conjecture sur la capacité et l'effilement*, C.R. Acad. Sci. Paris Série 1, 297 (1983), 393-395.

[2] Ancona A.: *Sur une conjecture concernant la capacité et l'effilement.* In: Mokobodzki G., Pinchon D. (ed.) Théorie du Potentiel (Orsay, 1983), Springer, Berlin, 1984, pp. 34-68.

[3] Arendt W., Daners D.: *Varying Domains: Stability of the Dirichlet and the Poisson Problem*, Discrete and Continuous Dynamical Systems 21, 1 (2008), 21-39.

[4] Armitage D. H., Gardiner S. J.: *Classical Potential Theory*, Springer-Verlag, London, 2001.

[5] Armitage D. H., Goldstein M.: *Better than uniform approximation on closed sets by harmonic functions with singularities*, Proc. London Math. Soc. (3), 60 (1990), 319-343.

[6] Boboc N., Bucur G., Cornea A.: *Order and Convexity in Potential Theory: H-Cones*, Springer-Verlag, Berlin Heidelberg, 1981.

[7] Bliedtner J., Hansen W.: *Simplicial Cones in Potential Theory*, Inventiones math. 29 (1975), 83-110.

[8] Dautray R., Lions J. L.: *Mathematical Analysis and Numerical Methods for Science and Technology*, vol. 1, Springer-Verlag, Berlin Heidelberg, 1990.

[9] Debiard A., Gaveau B.: *Potentiel Fin et Algèbres de Fonctions Analytiques I*, Journal of Functional Analysis, 16, 3 (1974).

[10] Deny J.: *Systèmes totaux de fonctions harmoniques*, Ann. Inst. Fourier, Grenoble, 1 (1949), 103-113.

[11] Fuglede B.: *Finely harmonic functions*, Lecture Notes in Mathematics 289, Springer-Verlag, Berlin, 1972.

[12] Gardiner S. J.:*Decomposition of Approximable Harmonic Functions*, Math. Ann. 308 (1997), 175-185.

[13] Gardiner S. J.: *Harmonic approximation*, Cambridge University Press, 1995.

[14] Gardiner S. J., Gustafsson A.: *Pointwise convergence and radial limits of harmonic functions*, Israel Journal of Mathematics 145 (2005), 243-256.

[15] Lukeš J.: *Zápisky z funkcionální analýzy*, Univerzita Karlova v Praze – Karolinum, 2003.

[16] Lukeš J., Malý J., Zajíček L.: *Fine Topology Methods in Real Analysis and Potential Theory*, Springer-Verlag, Berlin Heidelberg, 1986.

[17] Lukeš J., Malý J., Netuka I., Smrčka M., Spurný J.: *On approximation of affine baire-one functions*, Israel Journal of Mathematics 134 (2003), 255-287.

[18] Royden H. L.: *Real Analysis*, Prentice-Hall, Englewood Cliffs, New Jersey, 1988.

[19] Vincent-Smith G. F.: *Uniform approximation of harmonic functions*, Ann. Inst. Fourier, Grenoble, 19, 2 (1969), 339-353.