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Asymptotické vlastnosti řešení rovnic matematické fyziky

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Asymptotic properties of solutions to the equations of mathematical physics

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Název práce: Asymptotické vlastnosti řešení rovnic matematické fyziky
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Abstrakt: Jedna z obecně přijímaných hypotéz mechaniky tekutin tvrdí, že vazká tekutina plně ulpívá na hranici oblasti jíž proudí, za předpokladu, že tato je nepropustná. V poslední době se objevilo množství prací, jež se snaží tuto hypotézu ospravedlnit z matematického hlediska takzvanou metodou hrubé hranice. Tato metoda předpokládá, že "reálná" hranice není nikdy dokonale hladká, ale obsahuje mikroskopické výstupky. "Ideální" oblast Ω je nahrazena třídou $\{\Omega_\varepsilon\}_{\varepsilon>0}$, kde parametr ε zastupuje amplitudu výstupků. Za dodatečných předpokladů na konvergenci oblastí $\Omega_\varepsilon \to \Omega$, stejnoměrného rozprostření výstupků a nepropustnosti hranic $\partial\Omega_\varepsilon$ lze ukázat, že pro limitní problém je nutno předepsat silnější podmínku úplné přilnavosti. V této práci chceme podrobit zkoumání optimalitu výsledků jež jsou uvedeny v Bucur et al. [3], Bucur a Feireisl [4] nebo Díaz et al. [5] a předvést několik konkrétních příkladů. Nakonec rozšíříme dosavadní výsledky pro širší třídu oblastí $\{\Omega_\varepsilon\}$, především pro oblasti Ω_ε jejichž hranice obsahuje hroty a další nelipschitzovské utvary, které znemožňují použití Kornových nerovností.

Klíčová slova: hrubá hranice, úplná přilnavost, Kornova nerovnost, hroty

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Abstract: Well-accepted hypothesis in the fluid dynamics is that if the boundary of the physical domain is impermeable then the viscous fluid adheres completely to it. Many authors recently proposed mathematical justifications for this hypothesis using the so-called method of rugous boundary. The main idea is to assume that the "real" boundary is never absolutely smooth, but it contains microscopic asperities. They use a family of rough domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$, where the parameter ε corresponds to the amplitude of asperities, as a substitute for the "ideal" domain Ω. Assuming that $\Omega_\varepsilon \to \Omega$ in some sense, asperities are uniformly distributed, and all $\partial\Omega_\varepsilon$ are impermeable, they are able to show that stronger no-slip condition has to be imposed on the solution of the limit problem. In Thesis we want to discuss optimality of results obtained in Bucur et al. [3], Bucur and Feireisl [4] or Díaz et al. [5] and we show several corresponding examples. Finally we extend results for more general case of $\{\Omega_\varepsilon\}$, mainly for Ω_ε whose boundaries contain cusps and similar non-Lipschitz shapes, which does not allow using Korn's inequalities.

Keywords: rough boundary, no-slip, Korn's inequality, cusps

1 Introduction

For a number of problems studied in fluid dynamics it is typical that suitably chosen boundary conditions play an important role. Well-accepted hypothesis is that if the boundary of the physical domain is impermeable then the viscous fluid adheres completely to it. This hypothesis is commonly used in various theoretical papers as well as in numerical experiments. Let $\mathbf{u} = \mathbf{u}(x)$ be the Eulerian velocity of the fluid at $x \in \Omega \subset \mathbb{R}^3$; the impermeability of the boundary $\partial\Omega$ means that

$$\mathbf{u}(x) \cdot \mathbf{n}(x) = 0 \text{ for any } x \in \partial\Omega, \tag{1.1}$$

where $\mathbf{n}$ will always stand for the outer normal vector. On the other hand complete adherence is formulated as the no-slip boundary condition

$$\mathbf{u}(x) = 0 \text{ for any } x \in \partial\Omega. \tag{1.2}$$

Many authors recently proposed mathematical rationalizations for this hypothesis using the so-called method of rugous boundary. The main idea is to assume that the "real" boundary is never absolutely smooth, but it contains microscopic asperities of the size significantly smaller than the characteristic length scale of the flow. Therefore we use a family of rough domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$, where the parameter ε corresponds to the amplitude of asperities, as a substitute for the "ideal" domain Ω. Assuming that $\Omega_\varepsilon \to \Omega$ in some sense, asperities are uniformly distributed, and the condition of impermeability (1.1) is satisfied for all $\partial\Omega_\varepsilon$, we are able to show that stronger no-slip condition (1.2) has to be imposed on the solution of the limit problem. Although this result may be considered as confirmation of (1.2) for viscous fluids, it seems to be in contradiction with various numerical studies based on the scale analysis of the boundary layer. This method replaces the no-slip boundary condition on a rough boundary by "milder" ones of Navier-type (see Jaeger and Mikelić [6], Mohammadi, Pironneau and Valentin [7], Basson and Varet [2], among others). This apparent contradiction disappears, however, as soon as we realize that the Navier conditions always contain a friction term proportional $\frac{1}{\varepsilon}$, which actually yields the no-slip boundary condition in the asymptotic limit $\varepsilon \to 0$. Therefore there is no problem from the purely mathematical viewpoint.

To avoid unnecessary technical details, we will always assume in the Thesis that all quantities are periodic with respect to the plane variables (x_1, x_2) with period 1. Note, however, that no periodicity of the rugous boundary restricted to the period 1 is a priori assumed. We consider a family of domains

$$\Omega_\varepsilon = \{(x_1, x_2, x_3)|\ (x_1, x_2) \in \mathcal{T}^2, -1 < x_3 < \Phi_\varepsilon(x_1, x_2)\},$$

where $\{\Phi_\varepsilon\}_{\varepsilon>0}$ is a family of functions from $W^{1,\infty}(\mathcal{T}^2)$, such that $\Phi_\varepsilon \geq 0$ and $\Phi_\varepsilon \to 0$ uniformly on $\mathcal{T}^2$, where $\mathcal{T}^2 = ([0,1]|_{\{0,1\}})^2$ is the 2-dimensional torus. We define a limit domain

$$\Omega = \{(x_1, x_2, x_3)|\ (x_1, x_2) \in \mathcal{T}^2, -1 < x_3 < 0\},$$

and the upper part of its boundary

$$\Gamma := \{(x_1, x_2, x_3)|\ (x_1, x_2) \in \mathcal{T}^2, x_3 = 0\}.$$

Next we consider a family of functions $\mathbf{u}_\varepsilon \in W^{1,2}(\Omega_\varepsilon, \mathbb{R}^3)$ that is uniformly bounded (with regard to $\varepsilon \in (0,\infty)$) with respect to some suitable norm or semi-norm on Ω_ε, and satisfying impermeability boundary condition (1.1) on the upper part of $\partial\Omega_\varepsilon$ in the sense of traces, in other words

$$\mathbf{u}_\varepsilon \cdot \mathbf{n} = 0 \text{ a.e. on } \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \mathcal{T}^2, x_3 = \Phi_\varepsilon(x_1, x_2)\}. \tag{1.3}$$

By reflexivity of $W^{1,2}(\Omega, \mathbb{R}^3)$ we can find a subsequence of $\{\mathbf{u}_\varepsilon\}$ that weakly converges to a $\mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^3)$.

What are we able to say about the trace of $\mathbf{u}$ on Γ ? First we observe that for Lipschitz domains Ω_ε, $\mathbf{u}$ satisfies boundary condition (1.1) a.e. on Γ in sense of traces.

Lemma 1.1. *Let $\{\mathbf{u}_\varepsilon\}$ satisfy condition (1.3),*

$$\|\mathbf{u}_\varepsilon\|_{W^{1,2}(\Omega_\varepsilon,\mathbb{R}^3)} \le K < \infty \text{ uniformly for } \varepsilon \in (0,\infty), \tag{1.4}$$

and $\mathbf{u}_\varepsilon \to \mathbf{u}$ weakly in $W^{1,2}(\Omega, \mathbb{R}^3)$.

Then

$$\mathbf{u} \cdot \mathbf{n} = 0 \text{ a.e. on } \Gamma,$$

in sense of traces.

Proof. Without loss of generality we may assume that the functions Φ_ϵ determining the upper part of the boundary are strictly positive. We start with Green's formula for Lipschitz domains $\Omega_\varepsilon \setminus \Omega$,

$$\int_{\partial(\Omega_\varepsilon\setminus\Omega)} \mathbf{u}_\varepsilon \cdot \mathbf{n}\varphi \,\mathrm{dS} = \int_{\Omega_\varepsilon\setminus\Omega} \operatorname{div} \mathbf{u}_\varepsilon \varphi \,\mathrm{dx} + \int_{\Omega_\varepsilon\setminus\Omega} \mathbf{u}_\varepsilon \cdot \nabla\varphi \,\mathrm{dx}, \;\; \varphi \in C^1(\mathbb{R}^3).$$

Using boundary condition (1.3) and Hölder inequality,

$$\int_\Gamma \mathbf{u}_\varepsilon \cdot \mathbf{n}\varphi \,\mathrm{dS} \le \|\varphi\|_{C^1(\mathbb{R}^3)} \|\mathbf{u}_\varepsilon\|_{W^{1,2}(\Omega_\varepsilon,\mathbb{R}^3)} |\Omega_\varepsilon \setminus \Omega|^{\frac{1}{2}}, \;\; \varphi \in C^1(\mathbb{R}^3).$$

Eventually, thanks to (1.4), uniform convergence of $\Phi_\varepsilon \to 0$, and compactness of the trace operator on the space $W^{1,2}(\Omega)$, we get

$$\int_\Gamma \mathbf{u} \cdot \mathbf{n}\varphi \,\mathrm{dS} = 0, \;\; \varphi \in C^1(\mathbb{R}^3),$$

which concludes the proof.

□

Now let us review in detail the known results that are of our interest.

In Díaz, Cara, and Simon [5], the authors considered $\Phi \in C^1(\mathcal{T}^2)$ generating $\{\Phi_\varepsilon\}_{\varepsilon>0}$ in the following way:

$$\Phi_\varepsilon(x_1, x_2) := \varepsilon\Phi(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) \text{ for } \varepsilon \in (0,\infty).$$

$$\mathbf{u}_\varepsilon \in W^{1,2}(\Omega_\varepsilon, \mathbb{R}^3), \;\; \|\nabla\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon,\mathbb{R}^{3\times3})} \le K < \infty \text{ uniformly for } \varepsilon \in (0,\infty).$$

Moreover, they assume that there is a distribution $\mathbf{u}$ on Ω such that, as $\varepsilon \to 0$, one has for all $c \in (0, \infty)$

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } L^2(\Omega_c, \mathbb{R}^{3\times 3}),$$

where $\Omega_c := \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \mathcal{T}^2,\ -1 < x_3 < -c\}$. Finally, Φ varies in any direction at least at one point, more specifically,

$$\nabla\Phi \cdot \mathbf{w} \not\equiv 0 \text{ for any } |\mathbf{w}| = 1.$$

Under these hypotheses, they showed that if the boundary condition (1.3) is satisfied for all $\mathbf{u}_\varepsilon$, then the limit $\mathbf{u}$ satisfies the no-slip boundary condition (1.2) a.e. on Γ in the sense of traces.

In Bucur et al. [3], the authors assume that

$$\Phi_\varepsilon \in W^{1,\infty}(\mathcal{T}^2), \Phi_\varepsilon > 0, \Phi_\varepsilon \to 0 \text{ uniformly on } \mathcal{T}^2,$$

$$|\Phi_\varepsilon(y_1) - \Phi_\varepsilon(y_2)| \leq L|y_1 - y_2| \text{ for any } y_1, y_2 \in \mathcal{T}^2, \tag{1.5}$$

with L independent of ε,

$$\|\mathbf{u}_\varepsilon\|_{W^{1,2}(\Omega_\varepsilon, \mathbb{R}^3)} \leq K < \infty \text{ uniformly for } \varepsilon > 0,$$

and that $\mathbf{u}_\varepsilon$ satisfy the impermeability condition (1.3).

They show that any limit function $\mathbf{u}$ satisfies the no-slip condition (1.2) providing that a certain quantity termed measure of rugosity $\{\mathcal{R}_y\}_{y\in\mathcal{T}^2}$ is non-degenerate at a.a. $y \in \mathcal{T}^2$. The measure of rugosity $\{\mathcal{R}_y\}_{y\in\mathcal{T}^2}$ is simply a Young measure associated to the family of gradients $\{\nabla\Phi_\varepsilon\}_{\varepsilon>0}$ (for more details on Young measures see *Theorem 6.2* in Pedregal [8]). A measure of rugosity is called non-degenerate at $y \in \mathcal{T}^2$ if supp$[\mathcal{R}_y]$ contains two linearly independent vectors in $\mathbb{R}^2$. Thus the measure of rugosity, associated with the directions of the normal vectors on the upper part of $\partial\Omega_\varepsilon$, vanishes on the region with none or mild asperities, while it is strictly positive in the area, where "many" microscopic asperities prevent the fluid from slipping.

Remark 1.2. *The above mentioned results (stated in Bucur et al. [3] and Díaz et al. [5]) have three common characteristics that are crucially important for their validity:*

1. *the family $\{\Phi_\varepsilon\}_{\varepsilon>0}$ is equilipschitz, which allows for using Korn's inequality, extension operators and other inequalities with constants independent of ε,*
2. *Φ_ε are regular, which is necessary for Green's and similar theorems,*
3. *amplitude and period of oscillations are both of the very same order ε.*

For closer comparison of Díaz et al. [5] with Bucur et al. [3] we refer to Corollary 4.1 in Bucur et al. [3].

Under slightly different assumptions than in Díaz et al. [5], Bucur and Feireisl [4] showed, in addition to fulfilment of (1.2) a.e. on Γ for $\mathbf{u}$, that the rate of convergence for the traces of $\mathbf{u}_\varepsilon$ on Γ can be estimated in terms of ε. For

$$\Phi \in C^\infty(\mathcal{T}^2),\ \Phi > 0 \text{ and } \nabla\Phi \cdot \mathbf{w} \not\equiv 0 \text{ for any } |\mathbf{w}| = 1,$$

generating $\{\Phi_\varepsilon\}_{\varepsilon\in(0,\infty)}$ as

$$\Phi_\varepsilon(x_1, x_2) := \varepsilon\Phi(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) \text{ for } \varepsilon \in (0,\infty),$$

they proved that there exists $\mathfrak{K} \in (0,\infty)$ independent of ε, such that

$$\int_\Gamma |\mathbf{u}_\varepsilon|^2 \,\mathrm{dS} \le \varepsilon\mathfrak{K}\|\nabla\mathbf{u}_\varepsilon\|^2_{L^2(\Omega_\varepsilon,\mathbb{R}^3)} \text{ for all } \varepsilon \in (0,\infty),$$

for any $\mathbf{u}_\varepsilon \in W^{1,2}(\Omega_\varepsilon, \mathbb{R}^3)$ satisfying the impermeability boundary condition (1.3).

Remark 1.3. *Even though the results shown in Bucur et al. [3] or Bucur and Feireisl [4] are developed in the context of particular equations, they are completely independent of them.*

Remark 1.4. *The result concerning the rate of convergence is interesting itself. The rate ε gives us a chance to get (1.2) even for non-equilipschitz domains, namely*

$$\Phi_\varepsilon(x_1, x_2) := \varepsilon^\alpha\Phi(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) \text{ for } \varepsilon \in (0,\infty) \text{ and } \alpha \in (0,\infty).$$

We see that the results stated so far concern only the case $\alpha = 1$. The counter-examples 4.3 and 5.3 show that in fact for $\alpha \in [2,\infty)$ similar estimates can not be valid. The case $\alpha \in (0,1)$ simply takes into account highly oscillating boundaries, on the contrary, the case $\alpha \in (1,2)$ treats mildly oscillating ones.

Our main goals in the Thesis are:

(i) to show a very illuminating example related to the problem of oscillating boundaries,

(ii) to show an example of $\{\Phi_\varepsilon\}_{\varepsilon>0}$ in $\mathbb{R}^2$, $\mathbb{R}^3$ that does not satisfy the assumptions of Bucur et al. [3] mentioned in *Remark 1.2* and still converges,

(iii) to show the rate of convergence for these examples and counter examples for $\alpha \in [2,\infty)$,

(iv) to discuss much more general assumptions on Φ than those used in Bucur and Feireisl [4], Díaz et al. [5] or *Corollary 4.1* in Bucur et al. [3],

(v) to show the rate of convergence for the trace of $\mathbf{u}_\varepsilon$ on Γ for more general Φ.

The Thesis is organized in the following manner. Section 2 contains notation and abbreviations used in the text. In Section 3, we show a very illuminating example of the problem. In Sections 4 and 5, we show an example of $\{\Phi_\varepsilon\}_{\varepsilon>0}$ indicating non-optimality of the assumptions in Bucur et al. [3]. We also give proofs and estimates on the rate of convergence. In Section 6, we introduce a very general Φ, for which we have convergence and rate of convergence, the relevant proofs being given in the last part of the Thesis (following the method used in Bucur and Feireisl [4]).

2 Notation

We will use the following notation or abbreviation if not stated elsewhere. We shorten $\{\mathbf{v}_n\}_{n=1}^{\infty}$ to $\{\mathbf{v}_n\}$, and also shorten $W^{1,p}(V,\mathbb{R}^n)$ and $W^{1,p}(V,\mathbb{R}^{n\times n})$ to $W^{1,p}(V)$ and the norm in $W^{1,p}(V)$ will be denoted $\|\cdot\|_{1,p}$. Norm in $C^1(V)$ is the same as in $W^{1,\infty}(V)$. Scalar product of two vectors $\mathbf{v},\mathbf{u}$ is denoted $\mathbf{v}\cdot\mathbf{u}$. We recall

$$p' = \frac{p}{p-1} \text{ and } p^\sharp = \begin{cases} \frac{p}{2-p}, & \text{for } p<2;\\ \text{arbitrarily large real}, & \text{for } p=2;\\ \infty, & \text{for } p>2. \end{cases}$$

are the Lebesgue conjugated exponent and the Sobolev trace operator exponent, respectively (for more see *Theorem 1.23* from Roubíček [10]). We denote $|\cdot|$ the Lebesgue measure or Euclidean norm for vectors. Vectors are denoted boldface.

The impermeability boundary condition can be understood in different ways. For a Lipschitz domain V, we will understand it in the sense of traces of Sobolev functions, which means

$$\int_{\partial V} \mathbf{v}\cdot\mathbf{n}\varphi \,\mathrm{dS} = 0,\ \ \varphi\in C^1(\mathbb{R}^3), \tag{2.1}$$

where $\mathbf{v}\in W^{1,p}(V)$ for some $p\in(1,\infty)$. We will abbreviate it into different form,

$$\mathbf{v}\cdot\mathbf{n} = 0 \text{ a.e. on } \partial V.$$

In this case (2.1) can be rewritten using Green's theorem as

$$\int_V \operatorname{div}\mathbf{v}\varphi\,\mathrm{dx} + \int_V \mathbf{v}\cdot\nabla\varphi\,\mathrm{dx} = 0,\ \ \varphi\in C^1(\mathbb{R}^3).$$

In Section 6 we will need boundary condition for non-Lipschitz domains, where we do not have traces in the sense of Sobolev functions. Because we will also use constraint $\operatorname{div}\mathbf{v}=0$ in Section 6, we prescribe boundary condition in form

$$\int_V \mathbf{v}\cdot\nabla\varphi\,\mathrm{dx} = 0,\ \ \varphi\in C^1(\mathbb{R}^3),$$

which does not require the existence of traces in the sense of Sobolev functions at all.

Remark 2.1. *We are aware of the fact that the boundary of Ω_ε or Ω consists of both the upper and the bottom part. However, we care only for behavior of functions $\mathbf{u}_\varepsilon$ on the upper part, or rather on some neighborhood of it, that does not contain the bottom part of the boundary. We leave the boundary conditions prescribed on the bottom part on individual concerns of the reader.*

3 Illuminating example

In this Section we will consider a special case

$$\Phi(x) = \begin{cases} x, & \text{on } [0, \frac{1}{2}]; \\ 1 - x, & \text{on } [\frac{1}{2}, 1], \end{cases}$$

with an associated family

$$\Phi_n(x) = \begin{cases} \frac{1}{n^{\alpha-1}}(x - \frac{k-1}{n}), & \text{on } [\frac{k-1}{n}, \frac{2k-1}{2n}] \text{ for } k = 1, ..., n; \\ \\ \frac{1}{n^{\alpha-1}}(\frac{k}{n} - x), & \text{on } [\frac{2k-1}{2n}, \frac{k}{n}] \text{ for } k = 1, ..., n. \end{cases}$$

We consider a family of domains

$$\Omega_n = \{(x, y) \mid x \in \mathcal{T}, -1 < y < \Phi_n(x)\},$$

and the limit domain $\Omega = \{(x, y) \mid x \in \mathcal{T},\ -1 < y < 0\}$, together with the upper boundary $\Gamma := \{(x, y) \mid x \in \mathcal{T},\ y = 0\}$.

Next we consider a function $\mathbf{v} \in C^1(\Omega_n)$ for some $n \in \mathbb{N}$, satisfying the impermeability boundary condition

$$\mathbf{v} \cdot \mathbf{n} = 0 \text{ a.e. on } \{(x, y) \mid x \in \mathcal{T}, y = \Phi_n(x)\}. \tag{3.1}$$

Proposition 3.1. *There exists $\mathfrak{K} \in (0, \infty)$ independent of n such that for any $n \in \mathbb{N}$ and for an arbitrarily chosen function $\mathbf{v} \in C^1(\Omega_n)$ satisfying (3.1), the following estimates hold true*

$$\alpha \in \begin{cases} (0, 1), & \int_\Gamma |\mathbf{v}| \,\mathrm{ds} \leq \frac{\mathfrak{K}\|\nabla \mathbf{v}\|_{L^\infty(\Omega_n)}}{n^\alpha}; \\ \\ [1, 2), & \int_\Gamma |\mathbf{v}| \,\mathrm{ds} \leq \frac{\mathfrak{K}\|\nabla \mathbf{v}\|_{L^\infty(\Omega_n)}}{n^{2-\alpha}}. \end{cases}$$

Corollary 3.2. *For an arbitrarily chosen family of functions $\{\mathbf{u}_n\}$, $\mathbf{u}_n \in C^1(\Omega_n)$ satisfying (3.1),*

$$\|\nabla \mathbf{u}_n\|_{L^\infty(\Omega_n)} \leq K < \infty \text{ uniformly for all } n \in \mathbb{N},$$

and $\mathbf{u}_n \to \mathbf{u}$ weakly-(*) in $W^{1,\infty}(\Omega)$, condition (1.2) is always satisfied, in other words, $\mathbf{u} = 0$ a.e. on Γ.

We want to point out that assumptions of *Proposition 3.1* and the method used in its proof are quite illuminating and so the results can be referred to as the "best" possible with regard to the range of convergence in α and its rate. We see that the upper bound for α is 2 (which we stated as maximum), and it looks like the lower bound can even reach 0. From the proof of *Proposition 3.1* we can easily derive another estimate, which is interesting in comparison with the results of Section 6, namely we can get,

$$\alpha \in \begin{cases} (0, 1), & \int_\Gamma |\mathbf{v}|^2 \,\mathrm{ds} \leq \frac{\mathfrak{K}\|\nabla \mathbf{v}\|^2_{L^\infty(\Omega_n)}}{n^{2\alpha}}; \\ \\ [1, 2), & \int_\Gamma |\mathbf{v}|^2 \,\mathrm{ds} \leq \frac{\mathfrak{K}\|\nabla \mathbf{v}\|^2_{L^\infty(\Omega_n)}}{n^{2(2-\alpha)}}. \end{cases}$$

Proof of Corollary 3.2. Corollary results immediately from *Proposition 3.1.* □

Proof of Proposition 3.1. The normal $\mathbf{n}$ takes its values in the two-vectors set,

$$\mathbf{n}_L := k(-n^{1-\alpha}, 1) \text{ and } \mathbf{n}_R := k(n^{1-\alpha}, 1),$$

where $k = \frac{1}{\sqrt{1+n^{2(1-\alpha)}}}$. Vectors $\mathbf{n}_L$, $\mathbf{n}_R$ are linearly independent vectors generating $\mathbb{R}^2$. We generate from them a new orthogonal basis of $\mathbb{R}^2$,

$$\begin{aligned} \mathbf{h}_1 &:= \mathbf{n}_L + \mathbf{n}_R = 2k(0,1), \\ \mathbf{h}_2 &:= \mathbf{n}_R - \mathbf{n}_L = 2k(n^{1-\alpha}, 0). \end{aligned}$$

For almost all $(x, y) \in \Omega_n$, we may write $\mathbf{u}_n(x, y)$ as a linear combination of $\mathbf{h}_1$ and $\mathbf{h}_2$,

$$\mathbf{v}(x, y) = \alpha_1(x, y)\mathbf{h}_1 + \alpha_2(x, y)\mathbf{h}_2.$$

After scalar multiplying by $\mathbf{h}_1$, $\mathbf{h}_2$ respectively, we get

$$\begin{aligned} \mathbf{v}(x, y) \cdot \mathbf{h}_1 &= \alpha_1(x, y)|\mathbf{h}_1|^2 = \alpha_1(x, y)4k^2, \\ \mathbf{v}(x, y) \cdot \mathbf{h}_2 &= \alpha_2(x, y)|\mathbf{h}_2|^2 = \alpha_2(x, y)4k^2 n^{2(1-\alpha)}. \end{aligned}$$

We can start estimating,

$$\begin{aligned} \int_\Gamma |\mathbf{v}| \,\mathrm{ds} = \int_\Gamma |\alpha_1\mathbf{h}_1 + \alpha_2\mathbf{h}_2| \,\mathrm{ds} &\le \int_\Gamma \frac{|\mathbf{v}\cdot\mathbf{h}_1|}{|\mathbf{h}_1|} + \frac{|\mathbf{v}\cdot\mathbf{h}_2|}{|\mathbf{h}_2|} \,\mathrm{ds} \\ &\le \int_\Gamma (|\mathbf{v}\cdot\mathbf{n}_L| + |\mathbf{v}\cdot\mathbf{n}_R|) \,\mathrm{ds} \left(\frac{1}{|\mathbf{h}_1|} + \frac{1}{|\mathbf{h}_2|}\right) \\ &\le \int_\Gamma (|\mathbf{v}\cdot\mathbf{n}_L| + |\mathbf{v}\cdot\mathbf{n}_R|) \,\mathrm{ds} \,\frac{\sqrt{1+n^{2(1-\alpha)}}}{2}\left(1 + \frac{1}{n^{1-\alpha}}\right). \end{aligned}$$

We know that

$$|\mathbf{v}(x_1, y_1) - \mathbf{v}(x_2, y_2)| \le \|\nabla\mathbf{v}\|_{L^\infty(\Omega_n)} |(x_1, y_1) - (x_2, y_2)|, \tag{3.2}$$

for $(x_1, y_1), (x_2, y_2) \in \Omega_n$. Let us choose an arbitrary $x \in \mathcal{T}$. Then it follows from (3.1) that there exist

$$(x_L, y_L), (x_R, y_R) \in B_{\frac{1}{n}}((x, 0)) \cap \partial\Omega_n,$$

such that $\mathbf{v}(x_L, y_L) \cdot \mathbf{n}_L = 0$ and $\mathbf{v}(x_R, y_R) \cdot \mathbf{n}_R = 0$, for better understanding see Figure 1.

Then it follows from (3.2) that

$$|\mathbf{v}(x, 0) \cdot \mathbf{n}_L| = |\mathbf{v}(x, 0) \cdot \mathbf{n}_L - \mathbf{v}(x_L, y_L) \cdot \mathbf{n}_L| \le \frac{\|\nabla\mathbf{v}\|_{L^\infty(\Omega_n)}}{n},$$

$$|\mathbf{v}(x, 0) \cdot \mathbf{n}_R| = |\mathbf{v}(x, 0) \cdot \mathbf{n}_R - \mathbf{v}(x_R, y_R) \cdot \mathbf{n}_R| \le \frac{\|\nabla\mathbf{v}\|_{L^\infty(\Omega_n)}}{n}.$$

Eventually we get estimate

$$\int_\Gamma |\mathbf{v}| \,\mathrm{ds} \le \|\nabla\mathbf{v}\|_{L^\infty(\Omega_n)} \frac{\sqrt{1+n^{2(1-\alpha)}}}{n}\left(1 + \frac{1}{n^{1-\alpha}}\right).$$

□

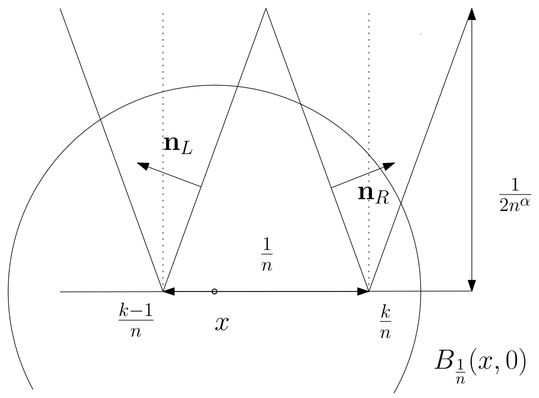

Figure 1: Graph of Φ_n

4 Example of oscillating boundaries in $\mathbb{R}^2$

In this Section we will consider a special case $\Phi(x) := \frac{1}{4}[\sin(2x - \frac{\pi}{2}) + 1],\ x \in \mathcal{T}$, where $\mathcal{T} = ([0,\pi]|_{\{0,\pi\}})$ is a 1-dimensional torus and a generated family ($\varepsilon = \frac{1}{n}$),

$$\Phi_n(x) = \frac{1}{4n^\alpha}[\sin(2nx - \frac{\pi}{2}) + 1],\ x \in \mathcal{T},\ \alpha \in (0,\infty).$$

We consider a family of domains

$$\Omega_n = \{(x,y) \mid x \in \mathcal{T}, -1 < y < \Phi_n(x)\},$$

and the limit domain $\Omega = \{(x,y) \mid x \in \mathcal{T},\ -1 < y < 0\}$, together with the upper boundary $\Gamma := \{(x,y) \mid x \in \mathcal{T},\ y = 0\}$.

Next we consider a function $\mathbf{v} \in W^{1,p}(\Omega_n)$ for some $p \in (1,\infty)$ and some $n \in \mathbb{N}$, satisfying the impermeability boundary condition

$$\mathbf{v} \cdot \mathbf{n} = 0 \text{ a.e. on } \{(x,y) \mid x \in \mathcal{T}, y = \Phi_n(x)\}. \tag{4.1}$$

Proposition 4.1. *For any $p \in (1,\infty)$, there exists $\mathfrak{K} \in (0,\infty)$ independent of n such that for any $n \in \mathbb{N}$ and any arbitrarily chosen function $\mathbf{v} \in W^{1,p}(\Omega_n)$ satisfying (4.1), the following estimates hold true*

$$\int_\Gamma |\mathbf{v}|\,\mathrm{ds} \le \frac{\mathfrak{K}\|\mathbf{v}\|_{1,p}}{n^{\frac{(\alpha(2p-1)-p)}{p}}}, \quad \text{for } \alpha \in \left(\frac{p}{2p-1}, \frac{p(2p^\sharp-1)}{p(2p^\sharp-1)+p^\sharp(p-1)}\right);$$

$$\int_\Gamma |\mathbf{v}|\,\mathrm{ds} \le \frac{\mathfrak{K}\|\mathbf{v}\|_{1,p}}{n^{\frac{p-\alpha}{p(1+2p^{\sharp\prime})}}}, \qquad \text{for } \alpha \in \left[\frac{p(2p^\sharp-1)}{p(2p^\sharp-1)+p^\sharp(p-1)}, 1\right);$$

$$\int_\Gamma |\mathbf{v}|\,\mathrm{ds} \le \frac{\mathfrak{K}\|\mathbf{v}\|_{1,p}}{n^{\frac{2p-\alpha(p+1)}{p(1+2p^{\sharp\prime})}}}, \qquad \text{for } \alpha \in [1, \frac{2p}{p+1});$$

$$\int_\Gamma |\mathbf{v}|\,\mathrm{ds} \le \frac{\mathfrak{K}\|\mathbf{v}\|_{1,p}}{n^{\frac{1}{p'(1+2p^{\sharp\prime})}}}, \qquad \text{if } \alpha = 1.$$

Corollary 4.2. *Let* $p \in (1,\infty]$ *and* $\alpha \in (\frac{p}{2p-1}, \frac{2p}{p+1})$*. Then for an arbitrarily chosen family of functions* $\{\mathbf{u}_n\}$*,* $\mathbf{u}_n \in W^{1,p}(\Omega_n)$ *satisfying (4.1),*

$$\|\mathbf{u}_n\|_{1,p} \leq K < \infty \text{ uniformly for all } n \in \mathbb{N}, \tag{4.2}$$

and $\mathbf{u}_n \to \mathbf{u}$ *weakly in* $W^{1,p}(\Omega)$*, condition (1.2) is always satisfied, in other words* $\mathbf{u} = 0$ *a.e. on* Γ*.*

Counter-example 4.3. *Let*

$$\mathbf{u}_n(x,y) = [u_1(x,y), u_2(x,y)],\ u_1 :\equiv 1,\ u_2(x,y) := \Phi_n'(x) = \frac{1}{n^{\alpha-1}}\Phi'(nx),$$

for $(x,y) \in \Omega_n$*. Then for* $\alpha \in [2,\infty)$ *the family* $\{\mathbf{u}_n\}$ *satisfies (4.1) and (4.2), but the limit function* $\mathbf{u} = [1,0]$ *does not satisfy (1.2).*

We immediately see that $\{\Phi_n\}$ do not satisfy the condition (1.5) for $\alpha < 1$, as

$$\Phi_n'(x) = n^{1-\alpha}\Phi'(nx) \Rightarrow L(n) \to \infty,$$

and do not satisfy the non-degeneracy condition for $\alpha > 1$, because $\{\mathcal{R}_y\}_{y\in\mathcal{T}} \equiv 0$ at a.a. $y \in \mathcal{T}$ as

$$\Phi_n'(x) = \frac{1}{n^{\alpha-1}}\Phi'(nx) \to 0 \text{ uniformly on } \mathcal{T}.$$

Apart from that $\{\Phi_n\}$ satisfies all other assumptions in Bucur et al. [3]. So our example shows that condition (1.5) and non-degeneracy condition (see also *Remark 1.2*) are not optimal for this type of problem. It also shows that we can expect to evaluate the rate of convergence even for $\alpha \neq 1$, where the case $\alpha > 1$ is particularly interesting.

Proof of Corollary 4.2. Corollary results immediately from *Proposition 4.1.* □

Proof of Counter-example 4.3. Because $\alpha \in [2,\infty)$ we easily compute that

$$\mathbf{u}_n \in L^\infty(\Omega_n) \text{ and } \nabla\mathbf{u}_n = \begin{pmatrix} 0 & 0 \\ \frac{1}{n^{\alpha-2}}\Phi''(nx) & 0 \end{pmatrix} \in L^\infty(\Omega_n),$$

in other words,

$$\|\mathbf{u}_n\|_{1,\infty} \leq K < \infty \text{ uniformly for all } n \in \mathbb{N}.$$

We also have

$$\mathbf{u}_n \cdot \mathbf{n} = 0 \text{ on the upper part of } \partial\Omega_n,$$

because $\mathbf{n}(x) = \frac{(-\Phi_n'(x),1)}{\sqrt{(\Phi_n'(x))^2+1}}$.

Thus $\{\mathbf{u}_n\}$ satisfies conditions (4.2), (4.1) for any $p \in (1,\infty)$, but it is obvious that $\mathbf{u}_n \rightrightarrows [1,0]$ on $\mathcal{T}$, so (1.2) is not satisfied.

□

In the next proof we do not make a difference between subsets of $\mathbb{R}\times\{0\}$ and their canonical projection onto $\mathbb{R}$ and vice versa whenever it is useful. As well we suppose that functions defined on subsets of $\mathbb{R}$ are also defined on $\mathbb{R}\times\{0\}$ through the mapping $[x]\mapsto[x,0]$. By $\int\cdot\,\mathrm{ds}$ we mean the curvilinear integral and by $\int\cdot\,\mathrm{dx}$ the Lebesgue integral.

Proof of Proposition 4.1. We will divide the proof into three steps. In the first step we deduce estimates for $\mathbf{v}\in C^\infty(\overline{\Omega_n})$ based on subtracting the set of "bad" points of measure $\varepsilon\in(0,\frac{1}{2})$ from Γ. In the second step, we replace ε by a suitable function depending on n and prove desired estimates for $\mathbf{v}\in C^\infty(\overline{\Omega_n})$. Finally, in the third step, we extend the estimates for $\mathbf{v}\in W^{1,p}(\Omega_n)$.

1. step:

Let us choose $\varepsilon\in(0,\frac{1}{2})$ arbitrarily, fixed. We define

$$\Gamma_{n\varepsilon}:=\bigcup_{k=1}^{2n-1}\left(\frac{k\pi}{2n}-\frac{\varepsilon}{2n},\frac{k\pi}{2n}+\frac{\varepsilon}{2n}\right)\cup\left(0,\frac{\varepsilon}{2n}\right)\cup\left(\pi-\frac{\varepsilon}{2n},\pi\right),$$

and $\Gamma_n:=\Gamma\setminus\Gamma_{n\varepsilon}$, then $|\Gamma_{n\varepsilon}|=2n\frac{\varepsilon}{n}=2\varepsilon$.

Set Γ_n is a "large" subset of Γ without neighborhoods of "bad" points. Let us assume that $\mathbf{v}\in C^\infty(\overline{\Omega_n})$.

$$\int_\Gamma|\mathbf{v}|\,\mathrm{ds}=\int_{\Gamma_n}|\mathbf{v}|\,\mathrm{ds}\;+\;\int_{\Gamma_{n\varepsilon}}|\mathbf{v}|\,\mathrm{ds}\le\int_{\Gamma_n}|\mathbf{v}|\,\mathrm{ds}+\left(\int_\Gamma|\mathbf{v}|^{p^\sharp}\,\mathrm{ds}\right)^{\frac{1}{p^\sharp}}|\Gamma_{n\varepsilon}|^{\frac{1}{p^{\sharp\prime}}}$$

$$\le\;\int_{\Gamma_n}|\mathbf{v}|\,\mathrm{ds}\;+C\|\mathbf{v}\|_{W^{1,p}(\Omega)}|\Gamma_{n\varepsilon}|^{\frac{1}{p^{\sharp\prime}}}.$$

We used Hölder inequality and Sobolev trace operator for Ω (independent on n)(for more see *Theorem 1.23* from Roubíček [10]). This may be rewritten as

$$\int_\Gamma|\mathbf{v}|\,\mathrm{ds}\le\int_{\Gamma_n}|\mathbf{v}|\,\mathrm{ds}+C\|\mathbf{v}\|_{W^{1,p}(\Omega)}\varepsilon^{\frac{1}{p^{\sharp\prime}}}.$$

Outward normal to the upper boundary of Ω_n is $\mathbf{n}(x)=\frac{(-\Phi_n'(x),1)}{\sqrt{(\Phi_n'(x))^2+1}}$. Prime $'$ above a function always denotes its derivative with respective to x. We denote $k(x):=\frac{1}{\sqrt{(\Phi_n'(x))^2+1}}$. Now we introduce two vectors, which arise naturally from the form of $\mathbf{n}(x)$:

$$\begin{aligned}\mathbf{n}_1(x)&:=\mathbf{n}(x)=k(x)(-\Phi_n'(x),1),\\ \mathbf{n}_2(x)&:=k(x)(\Phi_n'(x),1).\end{aligned}$$

Vectors $\mathbf{n}_1(x),\ \mathbf{n}_2(x)$ are linearly independent for $x\in\mathcal{T}\setminus\{x\mid\Phi_n'(x)=0\}$, whence for all $x\in\Gamma_n$. So they form a normal basis of $\mathbb{R}^2$. We will generate from $\mathbf{n}_1(x),\ \mathbf{n}_2(x)$ a new orthogonal basis of $\mathbb{R}^2$. We define:

$$\begin{aligned}\mathbf{h}_1(x)&:=\mathbf{n}_1(x)+\mathbf{n}_2(x)=2k(x)(0,1),\\ \mathbf{h}_2(x)&:=\mathbf{n}_2(x)-\mathbf{n}_1(x)=2k(x)(\Phi_n'(x),0).\end{aligned}$$

So we have $|\mathbf{h}_1(x)| = 2k(x)$ a $|\mathbf{h}_2(x)| = 2k(x)|\Phi_n'(x)|$.

For all $(x,y) \in \Omega_n$, we may write $\mathbf{v}(x,y)$ as a linear combination of $\mathbf{h}_1(x)$ and $\mathbf{h}_2(x)$,

$$\mathbf{v}(x,y) = \alpha_1(x,y)\mathbf{h}_1(x) + \alpha_2(x,y)\mathbf{h}_2(x).$$

After scalar multiplying by $\mathbf{h}_1(x)$, $\mathbf{h}_2(x)$, respectively, we get

$$\begin{aligned} \mathbf{v}(x,y)\cdot\mathbf{h}_1(x) &= \alpha_1(x,y)|\mathbf{h}_1(x)|^2 = \alpha_1(x,y)4k(x)^2, \\ \mathbf{v}(x,y)\cdot\mathbf{h}_2(x) &= \alpha_2(x,y)|\mathbf{h}_2(x)|^2 = \alpha_2(x,y)4k(x)^2|\Phi_n'(x)|^2. \end{aligned}$$

Now we can continue with estimating of $\int_{\Gamma_n} |\mathbf{v}|\,\mathrm{ds}$.

$$\int_{\Gamma_n} |\mathbf{v}|\,\mathrm{ds} \le \int_{\Gamma_n} |\alpha_1\mathbf{h}_1 + \alpha_2\mathbf{h}_2|\,\mathrm{ds} \le \int_{\Gamma_n} |\alpha_1||\mathbf{h}_1| + |\alpha_2||\mathbf{h}_2|\,\mathrm{ds}$$

$$\le \int_{\Gamma_n} \frac{|\mathbf{v}\cdot\mathbf{h}_1|}{2k} + \frac{|\mathbf{v}\cdot\mathbf{h}_2|}{2k|\Phi_n'|}\,\mathrm{ds} \le \int_{\Gamma_n} (|\mathbf{v}\cdot\mathbf{n}_1| + |\mathbf{v}\cdot\mathbf{n}_2|)\frac{1}{2k}\left(1 + \frac{1}{|\Phi_n'|}\right)\mathrm{ds}$$

$$\le \int_{\Gamma_n} (|\mathbf{v}\cdot\mathbf{n}_1| + |\mathbf{v}\cdot\mathbf{n}_2|)\frac{\sqrt{(\Phi_n')^2+1}}{2}\left(1 + \frac{1}{|\Phi_n'|}\right)\mathrm{ds}$$

$$\le \int_{\Gamma_n} (|\mathbf{v}\cdot\mathbf{n}_1| + |\mathbf{v}\cdot\mathbf{n}_2|)\,\mathrm{ds}\left\|\frac{\sqrt{(\Phi_n')^2+1}}{2}\right\|_{L^\infty(\Gamma_n)}\left\|1 + \frac{1}{|\Phi_n'|}\right\|_{L^\infty(\Gamma_n)}. \tag{4.3}$$

Now we estimate the last two terms in (4.3).

$$\Phi_n'(x) = \frac{1}{2n^{\alpha-1}}\cos(2nx - \frac{\pi}{2}).$$

Set Γ_n was constructed according to the form of the first of them, so we immediately read

$$\left\| 1 + \frac{1}{|\Phi_n'|}\right\|_{L^\infty(\Gamma_n)} = \left\| 1 + \frac{2n^{\alpha-1}}{|\cos(2n\cdot - \frac{\pi}{2})|}\right\|_{L^\infty(\Gamma_n)} = 1 + \frac{2n^{\alpha-1}}{\cos(-\frac{\pi}{2}+\varepsilon)} = 1 + \frac{2n^{\alpha-1}}{\sin(\varepsilon)},$$

$$\left\|\frac{\sqrt{(\Phi_n')^2+1}}{2}\right\|_{L^\infty(\Gamma_n)} = \left\|\frac{\sqrt{n^{2(1-\alpha)}\cos^2(2n\cdot - \frac{\pi}{2}) + 4}}{4}\right\|_{L^\infty(\Gamma_n)} \le \frac{\sqrt{n^{2(1-\alpha)}+4}}{4}.$$

We still need to estimate the term $\int_{\Gamma_n}(|\mathbf{v}\cdot\mathbf{n}_1| + |\mathbf{v}\cdot\mathbf{n}_2|)\,\mathrm{ds}$. First we will get estimates of terms $\int_\Gamma |\mathbf{v}\cdot\mathbf{n}_1|\,\mathrm{ds}$ and $\int_{\Gamma_n} |\mathbf{v}\cdot\mathbf{n}_2|\,\mathrm{ds}$ and then desired estimate will follow.

To get estimates on $\int_\Gamma |\mathbf{v}\cdot\mathbf{n}_1|\,\mathrm{ds}$ and $\int_{\Gamma_n} |\mathbf{v}\cdot\mathbf{n}_2|\,\mathrm{ds}$ we use Fubini's theorem. We integrate with respect to x and y axes stepwise over the set

$$\Upsilon_n := \{(x,y) \mid x \in \mathcal{T}, 0 < y < \Phi_n(x)\}.$$

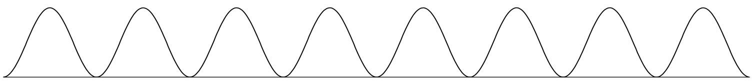

Figure 2: Set Υ_n for $n = 8$

(i) integrating with respect to x

Set Υ_n is the union of n geometrically identical domains (see Figure 2), we provide estimate on the first part and the rest holds in the same way. So we work only on the set

$$\Delta := \{(x, y) \mid x \in (0, \frac{\pi}{n}), 0 < y < \Phi_n(x)\}.$$

For all $y \in (0, \frac{1}{2n^\alpha})$ we arrive

$$\mathbf{v}(L_y, y) - \mathbf{v}(R_y, y) = \int_{R_y}^{L_y} \frac{\partial}{\partial x}\mathbf{v}(x, y)\,\mathrm{dx}, \tag{4.4}$$

where $L_y := \Phi_n^{-1}(y) < \frac{\pi}{2n}$ and $R_y := \Phi_n^{-1}(y) > \frac{\pi}{2n}$ (for better understanding see Figure 3).

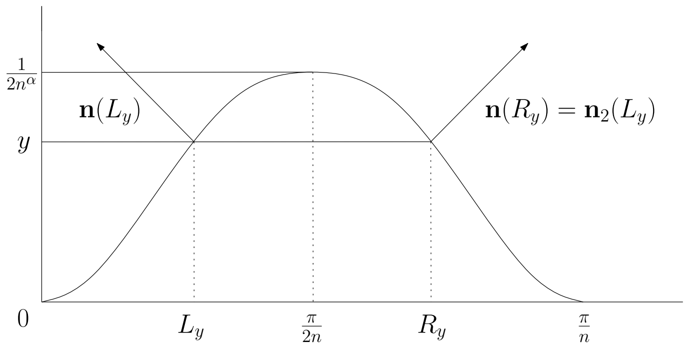

Figure 3: Graph of Φ_n on $(0, \frac{\pi}{n})$ alias Δ

Scalar multiplication of (4.4) by normal at exact point $\mathbf{n}(R_y)$ gives,

$$\mathbf{v}(L_y, y) \cdot \mathbf{n}(R_y) - \mathbf{v}(R_y, y) \cdot \mathbf{n}(R_y) = \int_{R_y}^{L_y} \frac{\partial}{\partial x}\mathbf{v}(x, y)\,\mathrm{dx} \cdot \mathbf{n}(R_y).$$

Take absolute values of both sides and use Schwartz inequality with $|\mathbf{n}| = 1$,

$$|\mathbf{v}(L_y, y) \cdot \mathbf{n}(R_y) - \mathbf{v}(R_y, y) \cdot \mathbf{n}(R_y)| \le \int_{L_y}^{R_y} |\frac{\partial}{\partial x}\mathbf{v}(x, y)|\,\mathrm{dx}\,.$$

Integrating over $y \in (0, \frac{1}{2n^\alpha})$ gives rise to

$$\int_0^{\frac{1}{2n^\alpha}} |\mathbf{v}(L_y, y) \cdot \mathbf{n}(R_y) - \mathbf{v}(R_y, y) \cdot \mathbf{n}(R_y)|\,\mathrm{dy} \le \int_0^{\frac{1}{2n^\alpha}}\int_{L_y}^{R_y} |\frac{\partial}{\partial x}\mathbf{v}(x, y)|\,\mathrm{dxdy}\,.$$

From the boundary condition (4.1) imposed on $\mathbf{v}$ we immediately see that for all $y \in (0, \frac{1}{2n^\alpha})$ we get $\mathbf{v}(R_y, y) \cdot \mathbf{n}(R_y) = 0$; whence

$$\int_0^{\frac{1}{2n^\alpha}} |\mathbf{v}(L_y, y) \cdot \mathbf{n}(R_y)| \,\mathrm{dy} \le \int_0^{\frac{1}{2n^\alpha}} \int_{L_y}^{R_y} |\frac{\partial}{\partial x} \mathbf{v}(x, y)| \,\mathrm{dxdy}\,.$$

From the definition of $\mathbf{n}_2$, it is obvious that $\mathbf{n}(R_y) = \mathbf{n}_2(L_y)$ (for better understanding see Figure 3). Consequently, using Fubini's theorem gives us,

$$\int_0^{\frac{1}{2n^\alpha}} |\mathbf{v}(L_y, y) \cdot \mathbf{n}_2(L_y)| \,\mathrm{dy} \le \int_\Delta |\frac{\partial}{\partial x} \mathbf{v}| \,\mathrm{dxdy}\,.$$

Using the change of notation $L_y = x$ and substituting $y = \Phi_n(x)$ for $x \in (0, \frac{\pi}{2n})$, we get

$$\int_0^{\frac{1}{2n^\alpha}} |\mathbf{v}(L_y, y) \cdot \mathbf{n}_2(L_y)| \,\mathrm{dy} = \int_0^{\frac{\pi}{2n}} |\mathbf{v}(x, \Phi_n(x)) \cdot \mathbf{n}_2(x)||\Phi_n'(x)| \,\mathrm{dx}\,.$$

If we change scalar multiplication of (4.4) by $\mathbf{n}(R_y)$ for multiplication of (4.4) by $\mathbf{n}(L_y)$, we get, after similar calculation, estimate

$$\int_{\frac{\pi}{2n}}^{\frac{\pi}{n}} |\mathbf{v}(x, \Phi_n(x)) \cdot \mathbf{n}_2(x)||\Phi_n'(x)| \,\mathrm{dx} \le \int_\Delta |\frac{\partial}{\partial x} \mathbf{v}| \,\mathrm{dxdy}\,.$$

Eventually, we arrive at the following estimate,

$$\int_{\mathcal{T}} |\mathbf{v}(x, \Phi_n(x)) \cdot \mathbf{n}_2(x)||\Phi_n'(x)| \,\mathrm{dx} \le 2 \int_{\Upsilon_n} |\frac{\partial}{\partial x} \mathbf{v}| \,\mathrm{dxdy}\,.$$

(ii) integrating with respect to y

For all $x \in \mathcal{T}$ we have

$$\mathbf{v}(x, 0) - \mathbf{v}(x, \Phi_n(x)) = \int_{\Phi_n(x)}^0 \frac{\partial}{\partial y} \mathbf{v}(x, y) \,\mathrm{dy},$$

which we rewrite into the form

$$\mathbf{v}(x, 0) = \mathbf{v}(x, \Phi_n(x)) + \int_{\Phi_n(x)}^0 \frac{\partial}{\partial y} \mathbf{v}(x, y) \,\mathrm{dy}\,. \tag{4.5}$$

Scalar multiplication of (4.5) by $\mathbf{n}_1(x)$ gives

$$\mathbf{v}(x, 0) \cdot \mathbf{n}_1(x) = \mathbf{v}(x, \Phi_n(x)) \cdot \mathbf{n}_1(x) + \int_{\Phi_n(x)}^0 \frac{\partial}{\partial y} \mathbf{v}(x, y) \,\mathrm{dy} \cdot \mathbf{n}_1(x).$$

Take absolute values of both sides and use Schwartz inequality with $|\mathbf{n}_1| = 1$,

$$|\mathbf{v}(x, 0) \cdot \mathbf{n}_1(x)| \le |\mathbf{v}(x, \Phi_n(x)) \cdot \mathbf{n}_1(x)| + \int_0^{\Phi_n(x)} |\frac{\partial}{\partial y} \mathbf{v}(x, y)| \,\mathrm{dy}\,.$$

Integrating over $x \in \mathcal{T}$, using boundary condition (4.1) and Fubini's theorem, we obtain

$$\underbrace{\int_{\mathcal{T}} |\mathbf{v}(x,0)\cdot \mathbf{n}_1(x)|\,\mathrm{dx}}_{=\int_\Gamma |\mathbf{v}\cdot\mathbf{n}_1|\,\mathrm{ds}} \le \underbrace{\int_{\mathcal{T}} |\mathbf{v}(x,\Phi_n(x))\cdot \mathbf{n}_1(x)|\,\mathrm{dx}}_{=0} + \int_{\Upsilon_n} |\frac{\partial}{\partial y}\mathbf{v}|\,\mathrm{dxdy}\,.$$

If we change scalar multiplication of (4.5) by $\mathbf{n}_1(x)$ for multiplication of (4.5) by $\mathbf{n}_2(x)$, we get after similar calculation and integrating over $x \in \Gamma_n$ estimate

$$\underbrace{\int_{\Gamma_n} |\mathbf{v}(x,0)\cdot \mathbf{n}_2(x)|\,\mathrm{dx}}_{=\int_{\Gamma_n} |\mathbf{v}\cdot\mathbf{n}_2|\,\mathrm{ds}} \le \int_{\Gamma_n} |\mathbf{v}(x,\Phi_n(x))\cdot \mathbf{n}_2(x)|\,\mathrm{dx} + \int_{\Upsilon_n} |\frac{\partial}{\partial y}\mathbf{v}|\,\mathrm{dxdy}\,.$$

According to construction of Γ_n we have $\Phi_n' \neq 0$ on Γ_n so we can use the following estimate,

$$\int_{\Gamma_n} |\mathbf{v}(x,\Phi_n(x))\cdot \mathbf{n}_2(x)|\,\mathrm{dx} \le \int_{\mathcal{T}} |\mathbf{v}(x,\Phi_n(x))\cdot \mathbf{n}_2(x)||\Phi_n'(x)|\,\mathrm{dx}\left\|\frac{1}{\Phi_n'}\right\|_{L^\infty(\Gamma_n)}.$$

Eventually, we get

$$\begin{aligned}
\int_{\Gamma_n} (|\mathbf{v}\cdot\mathbf{n}_1| + |\mathbf{v}\cdot\mathbf{n}_2|)\,\mathrm{ds} &\le 2\int_{\Upsilon_n} |\frac{\partial}{\partial x}\mathbf{v}|\,\mathrm{dxdy}\left\|\frac{1}{\Phi_n'}\right\|_{L^\infty(\Gamma_n)} + \\
2\int_{\Upsilon_n} |\frac{\partial}{\partial y}\mathbf{v}|\,\mathrm{dxdy} &\le 2\|\mathbf{v}\|_{1,p}|\Upsilon_n|^{\frac{1}{p'}}\left(1+\left\|\frac{1}{\Phi_n'}\right\|_{L^\infty(\Gamma_n)}\right)\\
&= 2\|\mathbf{v}\|_{1,p}\left(\int_{\mathcal{T}}\Phi_n(x)\,\mathrm{dx}\right)^{\frac{1}{p'}}\left(1+\left\|\frac{1}{\Phi_n'}\right\|_{L^\infty(\Gamma_n)}\right)\\
&= 2\|\mathbf{v}\|_{1,p}\left(\frac{\pi}{4n^\alpha}\right)^{\frac{1}{p'}}\left\|1+\frac{1}{|\Phi_n'|}\right\|_{L^\infty(\Gamma_n)}.
\end{aligned}$$

Now if we put all so-far obtained estimates together we read out

$$\int_\Gamma |\mathbf{v}|\,\mathrm{ds} \le \int_{\Gamma_n} |\mathbf{v}|\,\mathrm{ds} + C\|\mathbf{v}\|_{W^{1,p}(\Omega)}\varepsilon^{\frac{1}{p^{\sharp\prime}}}$$

$$\le \int_{\Gamma_n} (|\mathbf{v}\cdot\mathbf{n}_1|+|\mathbf{v}\cdot\mathbf{n}_2|)\,\mathrm{ds}\left\|\frac{\sqrt{(\Phi_n')^2+1}}{2}\right\|_{L^\infty(\Gamma_n)}\left\|1+\frac{1}{|\Phi_n'|}\right\|_{L^\infty(\Gamma_n)} + C\|\mathbf{v}\|_{W^{1,p}(\Omega)}\varepsilon^{\frac{1}{p^{\sharp\prime}}}$$

$$\le \|\mathbf{v}\|_{1,p}\left(\frac{\pi}{4n^\alpha}\right)^{\frac{1}{p'}}\left\|\sqrt{(\Phi_n')^2+1}\right\|_{L^\infty(\Gamma_n)}\left\|1+\frac{1}{|\Phi_n'|}\right\|^2_{L^\infty(\Gamma_n)} + C\|\mathbf{v}\|_{1,p}\,\varepsilon^{\frac{1}{p^{\sharp\prime}}}$$

$$\le \mathfrak{C}\|\mathbf{v}\|_{1,p}\left(\frac{1}{n^{\frac{\alpha}{p'}}}\left(1+\frac{2n^{\alpha-1}}{\sin(\varepsilon)}\right)^2\sqrt{n^{2(1-\alpha)}+4}+\varepsilon^{\frac{1}{p^{\sharp\prime}}}\right). \qquad (4.6)$$

We see that $\mathfrak{C}$ is independent of n.

2. step:

Because ε was chosen arbitrary in the previous step, we can fit ε to n. In this step we replace ε by suitable function (denoted f) of n. The idea is the following. For each $n \in \mathbb{N}$ we can take ε as small as we wish nevertheless we don't reach good estimates because of the term $\frac{1}{\sin(\varepsilon)}$. We need to balance the dependence of ε on n so that we get optimal decay at both terms of addition in (4.6). We do so through f such that both terms will have the same decay. Using Taylor expansion at 0 and $\varepsilon \in (0, \frac{1}{2})$ we have estimates

$$\sin(\varepsilon) \geq \frac{\varepsilon(6-\varepsilon^2)}{6} \geq \frac{\varepsilon}{2},$$

$$\frac{1}{n^{\frac{\alpha}{p'}}}\left(1+\frac{2n^{\alpha-1}}{\sin(\varepsilon)}\right)^2\sqrt{n^{2(1-\alpha)}+4}+\varepsilon^{\frac{1}{p^{\sharp\prime}}} \leq \frac{1}{n^{\frac{\alpha}{p'}}}\left(1+\frac{2n^{\alpha-1}}{\frac{\varepsilon}{2}}\right)^2\sqrt{n^{2(1-\alpha)}+4}+\varepsilon^{\frac{1}{p^{\sharp\prime}}}.$$

For each $n \in \mathbb{N}$ we choose $\varepsilon = f(n)$, where f is a nonnegative function such that $\lim_{n\to\infty} f(n) = 0$. We calculate $f(n)$ by comparing the decay of

$$\frac{1}{n^{\frac{\alpha}{p'}}}\left(1+\frac{4n^{\alpha-1}}{f(n)}\right)^2\sqrt{n^{2(1-\alpha)}+4} \text{ and } f(n)^{\frac{1}{p^{\sharp\prime}}}.$$

We will distinguish two cases with regard to α:

(i) case $\alpha \in [1, \infty)$:

In this case, we have $2 \leq \sqrt{n^{2(1-\alpha)}+4} \leq C(\alpha) < \infty$, which means that this term has no influence on decay, so we can omit it. We can also omit any constant that has no influence on decay. We have

$$\frac{n^{\alpha-1}}{f(n)} \geq 1 \text{ for } n \in \mathbb{N},$$

too. Altogether it means that we need to evaluate $f(n)$ from

$$\frac{n^{2(\alpha-1)}}{f(n)^2 n^{\frac{\alpha}{p'}}} = f(n)^{\frac{1}{p^{\sharp\prime}}} \Rightarrow f(n) = \frac{1}{n^{\frac{p^{\sharp\prime}(2p-\alpha(p+1))}{p(1+2p^{\sharp\prime})}}},$$

where $f(n)$ decays to 0 if and only if $\alpha < \frac{2p}{p+1}$.

(ii) case $\alpha \in (0, 1)$:

In this case we have $n^{1-\alpha} < \sqrt{n^{2(1-\alpha)}+4} < C(\alpha)n^{1-\alpha}$. We also have

$$\frac{n^{\alpha-1}}{f(n)} = 1 \Leftrightarrow f(n) = n^{\alpha-1} \Rightarrow \frac{n^{(1-\alpha)}}{n^{\frac{\alpha}{p'}}} = n^{\frac{\alpha-1}{p^{\sharp\prime}}} \Rightarrow \alpha = \frac{p(2p^{\sharp}-1)}{p(2p^{\sharp}-1)+p^{\sharp}(p-1)}.$$

Above $f(n)$ decays to 0 because $\alpha < 1$.

We easily deduce that for $\alpha \geq \frac{p(2p^{\sharp}-1)}{p(2p^{\sharp}-1)+p^{\sharp}(p-1)}$ one has $\frac{n^{\alpha-1}}{f(n)} \geq 1$, and, consequently, we need to calculate $f(n)$ from

$$\frac{n^{(\alpha-1)}}{f(n)^2 n^{\frac{\alpha}{p'}}} = f(n)^{\frac{1}{p^{\sharp\prime}}} \Rightarrow f(n) = \frac{1}{n^{\frac{p^{\sharp\prime}(p-\alpha)}{p(1+2p^{\sharp\prime})}}},$$

where $f(n)$ always decays to 0 because $\alpha < 1 < p$.

Furthermore for $\alpha < \frac{p(2p^\sharp-1)}{p(2p^\sharp-1)+p^\sharp(p-1)}$ we have $\frac{n^{\alpha-1}}{f(n)} \leq 1$; whence we need to calculate $f(n)$ from

$$\frac{n^{(1-\alpha)}}{n^{\frac{\alpha}{p'}}} = f(n)^{\frac{1}{p^{\sharp\prime}}} \Rightarrow f(n) = \frac{1}{n^{\frac{p^{\sharp\prime}(\alpha(2p-1)-p)}{p}}},$$

where $f(n)$ decays to 0 if and only if $\alpha > \frac{p}{2p-1}$.

Now we calculate the desired $\mathfrak{K}$ from $\mathfrak{C}$ and the constants omitted during the previous calculations. We see that $\mathfrak{K}$ is independent of n and α.

3. step:

To get above estimates for $\mathbf{v} \in W^{1,p}(\Omega_n)$ we will use the approximation *Theorem 3.22* from Adams and Fournier [1].

Theorem 3.22 from [1]: *Let $V \in C^{0,1}$ and $\mathbf{v} \in W^{1,p}(V)$ for $p \in [1,\infty)$. Then there exists $\{\mathbf{v}_k\} \subset C^\infty(\overline{V})$ such that $\mathbf{v}_k \to \mathbf{u}$ in $W^{1,p}(V)$ strongly.*

Direct using of *Theorem 3.22* from [1] for $V = \Omega_n$, which satisfies assumption $\Omega_n \in C^{0,1}$, gives us desired estimates for $\mathbf{v} \in W^{1,p}(\Omega_n)$.

□

5 Example of oscillating boundaries in $\mathbb{R}^3$

In this Section we will consider a special case

$$\Phi(x_1,x_2) := \frac{1}{8}[\sin(2x_1 - \frac{\pi}{2}) + 1][\sin(2x_2 - \frac{\pi}{2}) + 1],\ (x_1,x_2) \in \mathcal{T}^2,$$

where $\mathcal{T}^2 = ([0,\pi]|_{\{0,\pi\}})^2$ and a generated family ($\varepsilon = \frac{1}{n}$),

$$\Phi_n(x_1,x_2) = \frac{1}{8n^\alpha}[\sin(2nx_1 - \frac{\pi}{2}) + 1][\sin(2nx_2 - \frac{\pi}{2}) + 1],\ (x_1,x_2) \in \mathcal{T}^2,\ \alpha \in (0,\infty).$$

We consider a family of domains

$$\Omega_n = \{(x_1,x_2,x_3) \mid (x_1,x_2) \in \mathcal{T}^2, -1 < x_3 < \Phi_n(x_1,x_2)\},$$

and the limit domain $\Omega = \{(x_1,x_2,x_3) \mid (x_1,x_2) \in \mathcal{T}^2,\ -1 < x_3 < 0\}$, together with the upper boundary $\Gamma := \{(x_1,x_2,x_3) \mid (x_1,x_2) \in \mathcal{T}^2,\ x_3 = 0\}$.

Next we consider a function $\mathbf{v} \in W^{1,p}(\Omega_n)$ for some $p \in (1,\infty)$ and some $n \in \mathbb{N}$, satisfying the impermeability boundary condition

$$\mathbf{v} \cdot \mathbf{n} = 0 \text{ a.e. on } \{(x_1,x_2,x_3) \mid (x_1,x_2) \in \mathcal{T}^2, x_3 = \Phi_n(x_1,x_2)\}. \tag{5.1}$$

Proposition 5.1. *For any $p \in (1,\infty)$, there exists $\mathfrak{K} \in (0,\infty)$ independent of n such that for any $n \in \mathbb{N}$ and any arbitrarily chosen function $\mathbf{v} \in W^{1,p}(\Omega_n)$ satisfying (5.1), the following estimates hold true*

$\int_\Gamma |\mathbf{v}|\,\mathrm{ds} \leq \frac{\mathfrak{K}\|\mathbf{v}\|_{1,p}}{n^{\frac{(\alpha(2p-1)-p)}{p}}}$, *for* $\alpha \in (\frac{p}{2p-1}, \frac{p(4p^\sharp-1)}{p(4p^\sharp-1)+3p^\sharp(p-1)})$;

$\int_\Gamma |\mathbf{v}|\,\mathrm{ds} \leq \frac{\mathfrak{K}\|\mathbf{v}\|_{1,p}}{n^{\frac{p-\alpha}{p(1+6p^{\sharp\prime})}}}$, *for* $\alpha \in [\frac{p(4p^\sharp-1)}{p(4p^\sharp-1)+3p^\sharp(p-1)}, 1)$;

$\int_\Gamma |\mathbf{v}|\,\mathrm{ds} \leq \frac{\mathfrak{K}\|\mathbf{v}\|_{1,p}}{n^{\frac{2p-\alpha(p+1)}{p(1+6p^{\sharp\prime})}}}$, *for* $\alpha \in [1, \frac{2p}{p+1})$;

$\int_\Gamma |\mathbf{v}|\,\mathrm{ds} \leq \frac{\mathfrak{K}\|\mathbf{v}\|_{1,p}}{n^{\frac{1}{p'(1+6p^{\sharp\prime})}}}$, *if* $\alpha = 1$.

Corollary 5.2. *Let* $p \in (1, \infty]$ *and* $\alpha \in (\frac{p}{2p-1}, \frac{2p}{p+1})$. *Then for an arbitrarily chosen family of functions* $\{\mathbf{u}_n\}$, $\mathbf{u}_n \in W^{1,p}(\Omega_n)$ *satisfying (5.1),*

$$\|\mathbf{u}_n\|_{1,p} \leq K < \infty \text{ uniformly for all } n \in \mathbb{N}, \tag{5.2}$$

and $\mathbf{u}_n \to \mathbf{u}$ *weakly in* $W^{1,p}(\Omega)$, *condition (1.2) is always satisfied, in other words* $\mathbf{u} = 0$ *a.e. on* Γ.

Counter-example 5.3. *Let*

$$\mathbf{u}_n(x_1, x_2, x_3) = [u_1(x_1, x_2, x_3), u_2(x_1, x_2, x_3), u_3(x_1, x_2, x_3)],$$

$$u_1 :\equiv 1,\ u_2 :\equiv 1,\ u_3(x_1, x_2) := \frac{\partial}{\partial x_1}\Phi_n(x_1, x_2) + \frac{\partial}{\partial x_2}\Phi_n(x_1, x_2),$$

for $(x_1, x_2, x_3) \in \Omega_n$. *Then for* $\alpha \in [2, \infty)$ *the family* $\{\mathbf{u}_n\}$ *satisfies (5.1) and (5.2), but the limit function* $\mathbf{u} = [1, 1, 0]$ *does not satisfy (1.2).*

We again immediately see that $\{\Phi_n\}$ do not satisfy the condition (1.5) for $\alpha < 1$, as

$$\nabla\Phi_n(x_1, x_2) = n^{1-\alpha}\nabla\Phi(nx_1, nx_2) \Rightarrow L(n) \to \infty,$$

and do not satisfy non-degeneracy condition for $\alpha > 1$, because $\{\mathcal{R}_y\}_{y\in\mathcal{T}^2} \equiv 0$ at a.a. $y \in \mathcal{T}^2$ as

$$\nabla\Phi_n(x) = \frac{1}{n^{\alpha-1}}\nabla\Phi(nx) \to 0 \text{ uniformly on } \mathcal{T}^2.$$

Apart from that $\{\Phi_n\}$ satisfies all other assumptions in Bucur et al. [3]. So our example shows that condition (1.5) and non-degeneracy condition (see also *Observation 1.2*) are not optimal for this type of problem. It also shows that we can expect to evaluate the rate of convergence even for $\alpha \neq 1$, where the case $\alpha > 1$ is particularly interesting. Comparing results obtained in 2D and 3D case shows that the dimension should not be significant factor in this type of problem.

Proof of Corollary 5.2. Corollary results immediately from *Proposition 5.1.* □

Proof of Counter-example 5.3. The proof is done in the same way as in $\mathbb{R}^2$ case. □

In the next proof we do not make a difference between subsets of $\mathbb{R}^2 \times \{0\}$ and their canonical projection onto $\mathbb{R}^2$ and vice versa whenever it is useful. As well we suppose that functions defined on subsets of $\mathbb{R}^2$ are also defined on $\mathbb{R}^2 \times \{0\}$ through mapping $[x_1, x_2] \mapsto [x_1, x_2, 0]$. By $\int \cdot \, \mathrm{dS}$ we mean the surface integral and by $\int \cdot \, \mathrm{dx}$ the Lebesgue integral.

Proof of Proposition 5.1. The proof will be done in a similar way to the proof of *Theorem 4.1.* So we will omit some calculations and comments that are identical to those already made.

1. step:

Let us choose $\varepsilon \in (0, \frac{1}{2})$ arbitrarily, fixed. We define $\Gamma_{n\varepsilon}$ the set of neighborhoods of "bad" points. For simplicity we define it through Figure 4.

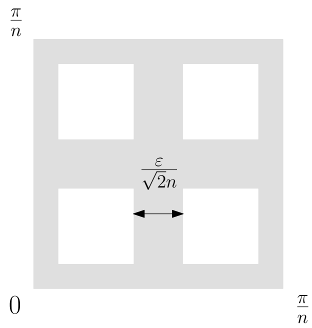

Figure 4: One of n^2 parts of $\Gamma_{n\varepsilon}$

$$\Gamma_n := \Gamma \setminus \Gamma_{n\varepsilon} \text{ and } |\Gamma_{n\varepsilon}| = 2\varepsilon(\sqrt{2}\pi - \varepsilon) \leq 2\sqrt{2}\pi\varepsilon.$$

Let us assume that $\mathbf{v} \in C^\infty(\overline{\Omega_n})$.

$$\int_\Gamma |\mathbf{v}| \, \mathrm{dS} \leq \int_{\Gamma_n} |\mathbf{v}| \, \mathrm{dS} + C\|\mathbf{v}\|_{W^{1,p}(\Omega)} \varepsilon^{\frac{1}{p^{\sharp'}}}.$$

We denote

$$k(x_1, x_2) := \frac{1}{\sqrt{(\frac{\partial}{\partial x_1}\Phi_n(x_1, x_2))^2 + (\frac{\partial}{\partial x_2}\Phi_n(x_1, x_2))^2 + 1}}.$$

Outward normal to the upper boundary of Ω_n is

$$\mathbf{n}(x_1, x_2) = k(x_1, x_2)(-\frac{\partial}{\partial x_1}\Phi_n(x_1, x_2), -\frac{\partial}{\partial x_2}\Phi_n(x_1, x_2), 1).$$

Now we introduce three vectors, which arise naturally from form of $\mathbf{n}(x_1, x_2)$.

$$\begin{aligned}
\mathbf{n}_1(x_1, x_2) &:= \mathbf{n}(x_1, x_2), \\
\mathbf{n}_2(x_1, x_2) &:= k(x_1, x_2)(\frac{\partial}{\partial x_1}\Phi_n(x_1, x_2), -\frac{\partial}{\partial x_2}\Phi_n(x_1, x_2), 1), \\
\mathbf{n}_3(x_1, x_2) &:= k(x_1, x_2)(-\frac{\partial}{\partial x_1}\Phi_n(x_1, x_2), \frac{\partial}{\partial x_2}\Phi_n(x_1, x_2), 1).
\end{aligned}$$

Vectors $\mathbf{n}_1(x_1,x_2),\ \mathbf{n}_2(x_1,x_2),\ \mathbf{n}_3(x_1,x_2)$ are linearly independent for

$$(x_1,x_2)\in\mathcal{T}^2\setminus\{(x_1,x_2)\mid \frac{\partial}{\partial x_1}\Phi_n(x_1,x_2)=0\vee\frac{\partial}{\partial x_2}\Phi_n(x_1,x_2)=0\},$$

whence for all $(x_1,x_2)\in\Gamma_n$. So they form a normal basis of $\mathbb{R}^3$. We will generate from $\mathbf{n}_1,\ \mathbf{n}_2,\ \mathbf{n}_3$ a new orthogonal basis of $\mathbb{R}^3$. We define:

$$\begin{aligned}
\mathbf{h}_1(x_1,x_2) &:= \mathbf{n}_2(x_1,x_2)-\mathbf{n}_1(x_1,x_2)=2k(x_1,x_2)(\frac{\partial}{\partial x_1}\Phi_n(x_1,x_2),0,0),\\
\mathbf{h}_2(x_1,x_2) &:= \mathbf{n}_3(x_1,x_2)-\mathbf{n}_1(x_1,x_2)=2k(x_1,x_2)(0,\frac{\partial}{\partial x_2}\Phi_n(x_1,x_2),0),\\
\mathbf{h}_3(x_1,x_2) &:= \mathbf{n}_2(x_1,x_2)+\mathbf{n}_3(x_1,x_2)=2k(x_1,x_2)(0,0,1).
\end{aligned}$$

So we have

$$\begin{aligned}
|\mathbf{h}_1(x_1,x_2)| &= 2k(x_1,x_2)|\frac{\partial}{\partial x_1}\Phi_n(x_1,x_2)|,\\
|\mathbf{h}_2(x_1,x_2)| &= 2k(x_1,x_2)|\frac{\partial}{\partial x_2}\Phi_n(x_1,x_2)|,\\
|\mathbf{h}_3(x_1,x_2)| &= 2k(x_1,x_2).
\end{aligned}$$

For all $x\in\Omega_n$, we may write $\mathbf{v}(x)$ as linear combination of $\mathbf{h}_1(x_1,x_2)$, $\mathbf{h}_2(x_1,x_2)$ and $\mathbf{h}_3(x_1,x_2)$,

$$\mathbf{v}(x)=\alpha_1(x)\mathbf{h}_1(x_1,x_2)+\alpha_2(x)\mathbf{h}_2(x_1,x_2)+\alpha_3(x)\mathbf{h}_3(x_1,x_2).$$

After scalar multiplying by $\mathbf{h}_1(x_1,x_2),\ \mathbf{h}_2(x_1,x_2),\ \mathbf{h}_3(x_1,x_2)$, respectively, we get

$$\begin{aligned}
\mathbf{v}(x)\cdot\mathbf{h}_1(x_1,x_2) &= \alpha_1(x)|\mathbf{h}_1(x_1,x_2)|^2=\alpha_1(x)4k(x_1,x_2)^2(\frac{\partial}{\partial x_1}\Phi_n(x_1,x_2))^2,\\
\mathbf{v}(x)\cdot\mathbf{h}_2(x_1,x_2) &= \alpha_2(x)|\mathbf{h}_2(x_1,x_2)|^2=\alpha_2(x)4k(x_1,x_2)^2(\frac{\partial}{\partial x_2}\Phi_n(x_1,x_2))^2,\\
\mathbf{v}(x)\cdot\mathbf{h}_3(x_1,x_2) &= \alpha_3(x)|\mathbf{h}_3(x_1,x_2)|^2=\alpha_3(x)4k(x_1,x_2)^2.
\end{aligned}$$

Now we can continue with estimating of $\int_{\Gamma_n}|\mathbf{v}|\,\mathrm{dS}$.

$$\int_{\Gamma_n}|\mathbf{v}|\,\mathrm{dS}\le\int_{\Gamma_n}|\alpha_1\mathbf{h}_1+\alpha_2\mathbf{h}_2+\alpha_3\mathbf{h}_3|\,\mathrm{dS}\le\int_{\Gamma_n}\frac{|\mathbf{v}\cdot\mathbf{h}_1|}{2k|\frac{\partial}{\partial x_1}\Phi_n|}+\frac{|\mathbf{v}\cdot\mathbf{h}_2|}{2k|\frac{\partial}{\partial x_2}\Phi_n|}+\frac{|\mathbf{v}\cdot\mathbf{h}_3|}{2k}\,\mathrm{dS}$$

$$\le\int_{\Gamma_n}(|\mathbf{v}\cdot\mathbf{n}_1|+|\mathbf{v}\cdot\mathbf{n}_2|+|\mathbf{v}\cdot\mathbf{n}_3|)\,\mathrm{dS}\left\|\frac{1}{2k}\right\|_{L^\infty(\Gamma_n)}\left\|1+\frac{1}{|\frac{\partial}{\partial x_1}\Phi_n|}+\frac{1}{|\frac{\partial}{\partial x_2}\Phi_n|}\right\|_{L^\infty(\Gamma_n)}.$$

Now we estimate the last two terms in the previous line.

$$\frac{\partial}{\partial x_1}\Phi_n(x_1,x_2) = \frac{1}{4n^{\alpha-1}}\cos(2nx_1-\frac{\pi}{2})[\sin(2nx_2-\frac{\pi}{2})+1],$$
$$\frac{\partial}{\partial x_2}\Phi_n(x_1,x_2) = \frac{1}{4n^{\alpha-1}}[\sin(2nx_1-\frac{\pi}{2})+1]\cos(2nx_2-\frac{\pi}{2}).$$

Set Γ_n was constructed according to the form of the first of them, so we immediately read

$$\begin{aligned}\left\|1+\frac{1}{|\frac{\partial}{\partial x_1}\Phi_n|}+\frac{1}{|\frac{\partial}{\partial x_2}\Phi_n|}\right\|_{L^\infty(\Gamma_n)} &\leq \left\|1+\frac{2}{|\frac{\partial}{\partial x_1}\Phi_n|}\right\|_{L^\infty(\Gamma_n)}\\ &= 1+\frac{8n^{\alpha-1}}{\sin(\varepsilon)(1-\cos(\varepsilon))},\end{aligned}$$

$$\left\|\frac{1}{2k}\right\|_{L^\infty(\Gamma_n)} = \left\|\frac{\sqrt{(\frac{\partial}{\partial x_1}\Phi_n(\cdot,\cdot))^2+(\frac{\partial}{\partial x_2}\Phi_n(\cdot,\cdot))^2+1}}{2}\right\|_{L^\infty(\Gamma_n)} \leq \frac{\sqrt{2n^{2(1-\alpha)}+4}}{4}.$$

We still need to estimate the term $\int_{\Gamma_n}(|\mathbf{v}\cdot\mathbf{n}_1|+|\mathbf{v}\cdot\mathbf{n}_2|+|\mathbf{v}\cdot\mathbf{n}_3|)\,\mathrm{dS}$. First we will get estimates of terms $\int_\Gamma|\mathbf{v}\cdot\mathbf{n}_1|\,\mathrm{dS}$, $\int_{\Gamma_n}|\mathbf{v}\cdot\mathbf{n}_2|\,\mathrm{dS}$ and $\int_{\Gamma_n}|\mathbf{v}\cdot\mathbf{n}_3|\,\mathrm{dS}$. To get estimates on these terms, we will integrate with respect to x_1, x_2 and x_3 axes stepwise over set

$$\Upsilon_n := \{(x_1,x_2,x_3)\mid(x_1,x_2)\in\mathcal{T}^2, 0<x_3<\Phi_n(x_1,x_2)\}.$$

(i) integrating with respect to x_1

We work only on the set

$$\Delta := \{(x_1,x_2,x_3)\mid(x_1,x_2)\in(0,\frac{\pi}{n})\times(0,\frac{\pi}{n}), 0<x_3<\Phi_n(x_1,x_2)\},$$

while the rest is done in the same way.

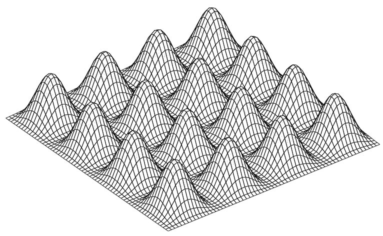

Figure 5: Graph of Φ_n for $n=4$

We denote

$$V(x_2) := \max_{\{x_1\in(0,\frac{\pi}{n})\}}\Phi_n(x_1,x_2).$$

For $(x_2,x_3)\in(0,\frac{\pi}{n})\times(0,V(x_2))$ we obtain

$$\mathbf{v}(L_{x_2x_3},x_2,x_3)-\mathbf{v}(R_{x_2x_3},x_2,x_3) = \int_{R_{x_2x_3}}^{L_{x_2x_3}}\frac{\partial}{\partial x_1}\mathbf{v}(x_1,x_2,x_3)\,\mathrm{dx}_1, \qquad (5.3)$$

where

$$L_{x_2x_3} := \Phi_n^{-1}(x_3)_1 < \frac{\pi}{2n} \wedge \Phi_n^{-1}(x_3)_2 = x_2,$$

and

$$R_{x_2x_3} := \Phi_n^{-1}(x_3)_1 > \frac{\pi}{2n} \wedge \Phi_n^{-1}(x_3)_2 = x_2,$$

for better understanding see Figure 6.

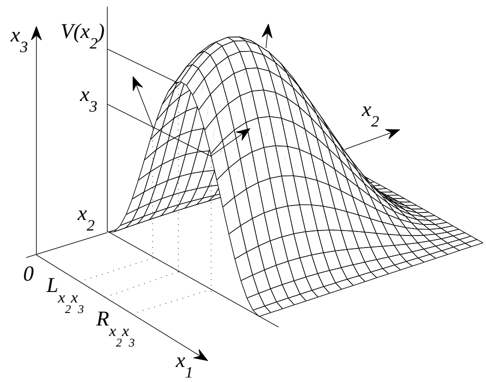

Figure 6: Graph of Φ_n on $(0, \frac{\pi}{n}) \times (0, \frac{\pi}{n})$ alias Δ

Scalar multiplication of (5.3) by the normal at the exact point $\mathbf{n}(R_{x_2x_3}, x_2)$ gives,

$$\mathbf{v}(L_{x_2x_3}, x_2, x_3) \cdot \mathbf{n}(R_{x_2x_3}, x_2) - \mathbf{v}(R_{x_2x_3}, x_2, x_3) \cdot \mathbf{n}(R_{x_2x_3}, x_2) = \int_{R_{x_2x_3}}^{L_{x_2x_3}} \frac{\partial}{\partial x_1} \mathbf{v}(x_1, x_2)\, \mathrm{dx}_1 \cdot \mathbf{n}(R_{x_2x_3}, x_2).$$

Take absolute values of both sides, use Schwartz inequality with $|\mathbf{n}| = 1$, and integrate first over $x_3 \in (0, V(x_2))$ and secondly over $x_2 \in (0, \frac{\pi}{n})$. Using the boundary conditions and definition of $\mathbf{n}_2$ (for better understanding see Figure 6) we can use Fubini's theorem to get

$$\int_0^{\frac{\pi}{n}} \int_0^{V(x_2)} |\mathbf{v}(L_{x_2x_3}, x_2, x_3) \cdot \mathbf{n}_2(L_{x_2x_3}, x_2)|\, \mathrm{dx}_3 \mathrm{dx}_2 \le \int_\Delta |\frac{\partial}{\partial x_1} \mathbf{v}|\, \mathrm{dx}\,.$$

For $x_2 \in (0, \frac{\pi}{n})$ arbitrary, fixed, using change of notation $L_{x_2x_3} = x_1$ and substitution $x_3 = \Phi_n(x_1, x_2)$ for $x_1 \in (0, \frac{\pi}{2n})$ yield

$$\int_0^{V(x_2)} |\mathbf{v}(L_{x_2x_3}, x_2, x_3) \cdot \mathbf{n}_2(L_{x_2x_3}, x_2)|\, \mathrm{dx}_3 = \int_0^{\frac{\pi}{2n}} |\mathbf{v}(x_1, x_2, \Phi_n(x_1, x_2)) \cdot \mathbf{n}_2(x_1, x_2)||\frac{\partial}{\partial x_1} \Phi_n(x_1, x_2)|\, \mathrm{dx}_1\,.$$

If we change scalar multiplication of (5.3) by $\mathbf{n}(R_{x_2x_3}, x_2)$ for multiplication of (5.3) by $\mathbf{n}(L_{x_2x_3}, x_2)$, we get after similar calculation estimate

$$\int_0^{\frac{\pi}{n}} \int_{\frac{\pi}{2n}}^{\frac{\pi}{n}} |\mathbf{v}(x_1, x_2, \Phi_n(x_1, x_2)) \cdot \mathbf{n}_2(x_1, x_2)||\frac{\partial}{\partial x_1} \Phi_n(x_1, x_2)|\, \mathrm{dx}_1 \mathrm{dx}_2 \le \int_\Delta |\frac{\partial}{\partial x_1} \mathbf{v}|\, \mathrm{dx}\,.$$

Eventually we arrive at the following estimate,

$$\iint_{\mathcal{T}^2} |\mathbf{v}(x_1,x_2,\Phi_n(x_1,x_2))\cdot\mathbf{n}_2(x_1,x_2)||\frac{\partial}{\partial x_1}\Phi_n(x_1,x_2)|\,\mathrm{dx_1dx_2} \le 2\int_{\Upsilon_n}|\frac{\partial}{\partial x_1}\mathbf{v}|\,\mathrm{dx}\,.$$

(ii) integrating with respect to x_2

Repeating the whole procedure done above for x_1 we arrive at

$$\iint_{\mathcal{T}^2} |\mathbf{v}(x_1,x_2,\Phi_n(x_1,x_2))\cdot\mathbf{n}_3(x_1,x_2)||\frac{\partial}{\partial x_2}\Phi_n(x_1,x_2)|\,\mathrm{dx_1dx_2} \le 2\int_{\Upsilon_n}|\frac{\partial}{\partial x_2}\mathbf{v}|\,\mathrm{dx}\,.$$

(iii) integrating with respect to x_3

Analogously to the 2-dimensional case we get the desirable results.

Eventually we get

$$\begin{aligned}\int_{\Gamma_n}(|\mathbf{v}\cdot\mathbf{n}_1|+|\mathbf{v}\cdot\mathbf{n}_2|+|\mathbf{v}\cdot\mathbf{n}_3|)\,\mathrm{dS} &\le 2\int_{\Upsilon_n}|\frac{\partial}{\partial x_1}\mathbf{v}|\,\mathrm{dx}\left\|\frac{1}{\frac{\partial}{\partial x_1}\Phi_n}\right\|_{L^\infty(\Gamma_n)}\\ &+ 2\int_{\Upsilon_n}|\frac{\partial}{\partial x_2}\mathbf{v}|\,\mathrm{dx}\left\|\frac{1}{\frac{\partial}{\partial x_2}\Phi_n}\right\|_{L^\infty(\Gamma_n)}\\ &+ 3\int_{\Upsilon_n}|\frac{\partial}{\partial x_3}\mathbf{v}|\,\mathrm{dx}\end{aligned}$$

$$\le 3\|\mathbf{v}\|_{1,p}\left(\iint_{\mathcal{T}^2}\Phi_n(x_1,x_2)\,\mathrm{dx_1dx_2}\right)^{\frac{1}{p'}}\left(\left\|\frac{1}{\frac{\partial}{\partial x_1}\Phi_n}\right\|_{L^\infty(\Gamma_n)}+\left\|\frac{1}{\frac{\partial}{\partial x_2}\Phi_n}\right\|_{L^\infty(\Gamma_n)}+1\right)$$

$$\le 3\|\mathbf{v}\|_{1,p}\left(\frac{\pi^2}{8n^\alpha}\right)^{\frac{1}{p'}}\left\|1+\frac{2}{|\frac{\partial}{\partial x_1}\Phi_n|}\right\|_{L^\infty(\Gamma_n)}.$$

Now if we put all so-far obtained estimates together we read out

$$\begin{aligned}\int_\Gamma|\mathbf{v}|\,\mathrm{dS} &\le \int_{\Gamma_n}(|\mathbf{v}\cdot\mathbf{n}_1|+|\mathbf{v}\cdot\mathbf{n}_2|+|\mathbf{v}\cdot\mathbf{n}_3|)\,\mathrm{dS}\left\|\frac{1}{2k}\right\|_{L^\infty(\Gamma_n)}\left\|1+\frac{2}{|\frac{\partial}{\partial x_1}\Phi_n|}\right\|_{L^\infty(\Gamma_n)}\\ &\quad + C\|\mathbf{v}\|_{W^{1,p}(\Omega)}\varepsilon^{\frac{1}{p^{\sharp\prime}}}\\ &\le 3\|\mathbf{v}\|_{1,p}\left(\frac{\pi^2}{8n^\alpha}\right)^{\frac{1}{p'}}\left\|\frac{1}{2k}\right\|_{L^\infty(\Gamma_n)}\left\|1+\frac{2}{|\frac{\partial}{\partial x_1}\Phi_n|}\right\|^2_{L^\infty(\Gamma_n)}+C\|\mathbf{v}\|_{1,p}\,\varepsilon^{\frac{1}{p^{\sharp\prime}}}\\ &\le \mathfrak{C}\|\mathbf{v}\|_{1,p}\left(\frac{1}{n^{\frac{\alpha}{p'}}}\left(1+\frac{8n^{\alpha-1}}{\sin(\varepsilon)(1-\cos(\varepsilon))}\right)^2\sqrt{2n^{2(1-\alpha)}+4}+\varepsilon^{\frac{1}{p^{\sharp\prime}}}\right).\end{aligned}$$

We see that $\mathfrak{C}$ is independent of n.

2. step:

We will again proceed analogously to 2-dimensional case. Using Taylor expansion at 0 and $\varepsilon \in (0, \frac{1}{2})$ we have estimates

$$1 - \cos(\varepsilon) \geq \frac{\varepsilon^2(6-\varepsilon^2)}{12} \geq \frac{\varepsilon^2}{3} \text{ and } \sin(\varepsilon) \geq \frac{\varepsilon}{2}.$$

For each $n \in \mathbb{N}$ we choose $\varepsilon = f(n)$, where f is a nonnegative function such that $\lim_{n\to\infty} f(n) = 0$. We calculate $f(n)$ by comparing the decay of

$$\frac{1}{n^{\frac{\alpha}{p'}}} \left(1 + \frac{48n^{\alpha-1}}{f(n)^3}\right)^2 \sqrt{2n^{2(1-\alpha)}+4} \text{ and } f(n)^{\frac{1}{p^{\sharp\prime}}}.$$

We will distinguish two cases with regard to α:

(i) case $\alpha \in [1, \infty)$:

In this case, we have $2 \leq \sqrt{2n^{2(1-\alpha)}+4} \leq C(\alpha) < \infty$, which means that this term has no influence on the decay, so we can omit it. We can also omit any constant that has no influence on the decay. We have

$$\frac{n^{\alpha-1}}{f(n)^3} \geq 1 \text{ for } n \in \mathbb{N},$$

too. Altogether it means that we need to calculate $f(n)$ from

$$\frac{n^{2(\alpha-1)}}{f(n)^6 n^{\frac{\alpha}{p'}}} = f(n)^{\frac{1}{p^{\sharp\prime}}} \Rightarrow f(n) = \frac{1}{n^{\frac{p^{\sharp\prime}(2p-\alpha(p+1))}{p(1+6p^{\sharp\prime})}}},$$

where $f(n)$ decays to 0 if and only if $\alpha < \frac{2p}{p+1}$.

(ii) case $\alpha \in (0, 1)$:

In this case we have $n^{1-\alpha} < \sqrt{2n^{2(1-\alpha)}+4} < C(\alpha)n^{1-\alpha}$. We also have

$$\frac{n^{\alpha-1}}{f(n)^3} = 1 \Leftrightarrow f(n) = n^{\frac{\alpha-1}{3}} \Rightarrow \frac{n^{(1-\alpha)}}{n^{\frac{\alpha}{p'}}} = n^{\frac{\alpha-1}{3p^{\sharp\prime}}} \Rightarrow \alpha = \frac{p(4p^\sharp - 1)}{p(4p^\sharp-1) + 3p^\sharp(p-1)}.$$

Above $f(n)$ decays to 0 because $\alpha < 1$.

We easily deduce that for $\alpha \geq \frac{p(4p^\sharp-1)}{p(4p^\sharp-1)+3p^\sharp(p-1)}$ one has $\frac{n^{\alpha-1}}{f(n)^3} \geq 1$ and we need to calculate $f(n)$ from

$$\frac{n^{(\alpha-1)}}{f(n)^6 n^{\frac{\alpha}{p'}}} = f(n)^{\frac{1}{p^{\sharp\prime}}} \Rightarrow f(n) = \frac{1}{n^{\frac{p^{\sharp\prime}(p-\alpha)}{p(1+6p^{\sharp\prime})}}},$$

where $f(n)$ always decays to 0 because $\alpha < 1 < p$.

Furthermore for $\alpha < \frac{p(4p^\sharp-1)}{p(4p^\sharp-1)+3p^\sharp(p-1)}$ is $\frac{n^{\alpha-1}}{f(n)^3} \leq 1$ and we need to calculate $f(n)$ from

$$\frac{n^{(1-\alpha)}}{n^{\frac{\alpha}{p'}}} = f(n)^{\frac{1}{p^{\sharp\prime}}} \Rightarrow f(n) = \frac{1}{n^{\frac{p^{\sharp\prime}(\alpha(2p-1)-p)}{p}}},$$

where $f(n)$ decays to 0 if and only if $\alpha > \frac{p}{2p-1}$.

Now we calculate the desired $\mathfrak{K}$ from $\mathfrak{C}$ and the constants omitted during the previous calculations. We see that $\mathfrak{K}$ is independent of n and α.

3. step:

To get above estimates for $\mathbf{v} \in W^{1,p}(\Omega_n)$ we will use the approximation *Theorem 3.22* from Adams and Fournier [1].

Theorem 3.22 from [1]: *Let $V \in C^{0,1}$ and $\mathbf{v} \in W^{1,p}(V)$ for $p \in [1,\infty)$. Then there exists $\{\mathbf{v}_k\} \subset C^\infty(\overline{V})$ such that $\mathbf{v}_k \to \mathbf{u}$ in $W^{1,p}(V)$ strongly.*

Direct using of *Theorem 3.22* from [1] for $V = \Omega_n$, which satisfies assumption $\Omega_n \in C^{0,1}$, gives us desired estimates for $\mathbf{v} \in W^{1,p}(\Omega_n)$. $\square$

6 General functions Φ

In this Section we will consider the case $\alpha = 1$ for as much as general function $\Phi > 0$. We consider a family of domains

$$\Omega_\varepsilon = \{(x_1,x_2,x_3) \mid (x_1,x_2) \in \mathcal{T}^2, -1 < x_3 < \Phi_\varepsilon(x_1,x_2)\},$$

and the limit domain $\Omega = \{(x_1,x_2,x_3) \mid (x_1,x_2) \in \mathcal{T}^2,\ -1 < x_3 < 0\}$, together with the upper boundary $\Gamma := \{(x_1,x_2,x_3) \mid (x_1,x_2) \in \mathcal{T}^2,\ x_3 = 0\}$. Moreover we set

$$\Phi_\varepsilon(x_1,x_2) := \varepsilon\Phi(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}),\ (x_1,x_2) \in \mathcal{T}^2 \text{ for } \ \varepsilon \in (0,\infty),$$

and $\mathcal{T}^2 = ([0,1]|_{\{0,1\}})^2$.

Next we consider a function $\mathbf{v} \in W^{1,2}(\Omega_\varepsilon)$ for some $\varepsilon \in (0,\infty)$ satisfying the impermeability boundary condition in form

$$\int_{\Omega_\varepsilon} \mathbf{v} \cdot \nabla\varphi \,\mathrm{dx} = 0,\ \varphi \in C^1(\mathbb{R}^3). \tag{6.1}$$

Proposition 6.1. *There exists $\mathfrak{K} \in (0,\infty)$ independent of ε such that for any $\varepsilon \in (0,\infty)$ and any arbitrarily chosen function $\mathbf{v} \in W^{1,2}(\Omega_\varepsilon)$ satisfying (6.1), it holds true*

$$\int_\Gamma |\mathbf{v}|^2 \,\mathrm{dS} \le \varepsilon\mathfrak{K}\|\nabla\mathbf{v}\|^2_{L^2(\Omega_\varepsilon)}, \tag{6.2}$$

if the following conditions on Φ are satisfied

(i) $\Phi \in C(\mathcal{T}^2)$,

(ii) $\exists\{\Phi_k\}$ *of Lipschitz functions and* $\exists\{M_k\}$ *of open subsets of* $\mathcal{T}^2$ *such that*

$$0 < \Phi_k \le \Phi_{k+1} \le \Phi,\ \Phi_k = \Phi \text{ on } M_k,$$

$$|\mathcal{T}^2 \setminus M_k| \to 0 \text{ as } k \to \infty,$$

(iii) $\nabla\Phi \cdot \mathbf{w} \not\equiv 0$ *a.e. on* $\mathcal{T}^2$ *for* $|\mathbf{w}| = 1$.

Corollary 6.2. *Let conditions (i)-(iii) on* Φ *from Proposition 6.1 are satisfied. Then for an arbitrarily chosen family of functions* $\{\mathbf{u}_\varepsilon\}$, $\mathbf{u}_\varepsilon \in W^{1,2}(\Omega_\varepsilon)$ *satisfying (6.1),*

$$\|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le K < \infty \text{ uniformly for } \varepsilon \in (0,\infty),$$

and $\mathbf{u}_\varepsilon \to \mathbf{u}$ *weakly in* $W^{1,2}(\Omega)$, *condition (1.2) is always satisfied in the limit, in other words* $\mathbf{u} = 0$ *a.e. on* Γ.

Remark 6.3. *We see that Proposition 6.1 holds under very mild hypotheses on* Φ. *Conditions (i)-(iii) on* Φ *seem maybe strange at first sight, but they are quite natural. Our intention is to have (6.2) for domains where Korn's inequality does not hold, for example domains whose boundary contains cusps etc. (for better understanding see Figure 7). Conditions (i) and (ii) only state that the normal to the boundary has sense at a.a. points and we can always "cut off" the non-Lipschitz part of the boundary. (ii) also guarantees existence of* $\nabla\Phi$ *at a.a. points of* $\mathcal{T}^2$. *(i) and (ii) are always easily verified for* Φ *Lipschitz with constant sequence and empty sets. Condition (iii) guarantees that* Φ *is not constant at any direction, which is a very reasonable condition also used in Bucur et al. [3], Bucur and Feireisl [4] or Díaz et al. [5].*

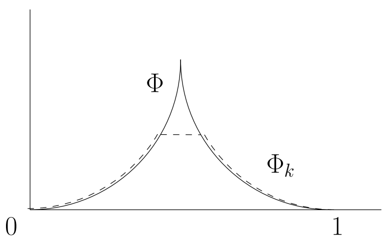

Figure 7: Graph of Φ and $\{\Phi_k\}$ for cusp

Remark 6.4. *Using boundary condition in the form (6.1) is not accidental. Although results obtained in this work are independent of particular equations, the motivation was to get justification of the no-slip boundary condition for the incompressible flows governed by the Navier-Stokes equations in impermeable containers. Incompressibility of the flow gives us a divergenceless motion and together with impermeability of the boundary we obtain equivalent condition (6.1) in the case of Lipschitz domains. Because we are now interested in the non-Lipschitz case, somehow locally Lipschitz, prescribing condition (6.1) seems very natural. This condition also contains the constraint of free divergence in the container. With regard to the Remark 2.1 we should use in (6.1)* φ, *such that their support does not contain the bottom part of the boundary. We omit this to make notation easier.*

Proof of Corollary 6.2. Corollary results immediately from *Proposition 6.1.* □

Proof of Proposition 6.1. Without loss of generality we can suppose that $\frac{1}{\varepsilon}$ is a positive integer. For $n_1, n_2 \in [0, \frac{1}{\varepsilon}) \cap \mathbb{N}$ we introduce sets

$$C_{n_1,n_2} := \{(x_1,x_2,x_3) : (x_1,x_2) \in (n_1\varepsilon, n_1\varepsilon+\varepsilon) \times (n_2\varepsilon, n_2\varepsilon+\varepsilon), -\varepsilon < x_3 < \Phi_\varepsilon(x_1,x_2)\}.$$

We observe that

$$\bigcup_{n_1,n_2} C_{n_1,n_2} \subseteq \Omega_\varepsilon,$$

and moreover we have

$$\int_\Gamma |\mathbf{v}|^2 \,\mathrm{dS} = \sum_{n_1,n_2} \int_{\underline{C_{n_1,n_2}}} |\mathbf{v}|^2 \,\mathrm{dS} \;\text{ and }\; \sum_{n_1,n_2} \varepsilon\mathfrak{K}\|\nabla\mathbf{v}\|^2_{L^2(C_{n_1,n_2})} \le \varepsilon\mathfrak{K}\|\nabla\mathbf{v}\|^2_{L^2(\Omega_\varepsilon)},$$

where the underlined set always denotes its intersection with plane $\{x_3 = 0\}$, otherwise intersection with Γ. This means that it is enough to prove

$$\int_{\underline{C_{n_1,n_2}}} |\mathbf{v}|^2 \,\mathrm{dS} \le \varepsilon\mathfrak{K}\|\nabla\mathbf{v}\|^2_{L^2(C_{n_1,n_2})},$$

for all $n_1, n_2 \in [0, \frac{1}{\varepsilon}) \cap \mathbb{N}$ with the same $\mathfrak{K}$. Fortunately, since every set C_{n_1,n_2} can be expressed through shift as an image of the set $C_{0,0}$, it is enough to prove:

Let

$$V := \{(x_1, x_2, x_3) : (x_1, x_2) \in (0, \varepsilon)^2, -\varepsilon < x_3 < \Phi_\varepsilon(x_1, x_2)\}.$$

If assumptions from *Theorem 6.1* are satisfied, then there exists $\mathfrak{K} \in (0, \infty)$ such that for any $\varepsilon \in (0, \infty)$,

$$\int_{\underline{V}} |\mathbf{v}|^2 \,\mathrm{dS} \le \varepsilon\mathfrak{K}\|\nabla\mathbf{v}\|^2_{L^2(V)}.$$

Now we introduce the scaling $x \approx \frac{x}{\varepsilon}$. We define a mapping

$$\mathbf{H}_\varepsilon : \mathbb{R}^3 \to \mathbb{R}^3,\; \mathbf{H}_\varepsilon(x_1, x_2, x_3) = (\frac{1}{\varepsilon}x_1, \frac{1}{\varepsilon}x_2, \frac{1}{\varepsilon}x_3).$$

We get sets

$$\mathbf{H}_\varepsilon(V) = \{(x_1, x_2, x_3) : (x_1, x_2) \in \mathcal{T}^2, -1 < x_3 < \Phi(x_1, x_2)\},$$

$$\mathbf{H}_\varepsilon(\underline{V}) = \underline{\mathbf{H}_\varepsilon(V)}.$$

Immediately we see that $\nabla\mathbf{H}_\varepsilon^{-1} = \varepsilon I$ implying $|\det \nabla\mathbf{H}_\varepsilon^{-1}| = \varepsilon^3$, moreover

$$\nabla(\mathbf{v} \circ \mathbf{H}_\varepsilon^{-1}) = \nabla\mathbf{v}(\mathbf{H}_\varepsilon^{-1}) \cdot \nabla\mathbf{H}_\varepsilon^{-1},$$

therefore $|\nabla(\mathbf{v} \circ \mathbf{H}_\varepsilon^{-1})|^2 = \varepsilon^2|\nabla\mathbf{v}(\mathbf{H}_\varepsilon^{-1})|^2$. Using substitution in the right part of our inequality we get

$$\varepsilon\mathfrak{K}\|\nabla\mathbf{v}\|^2_{L^2(V)} = \varepsilon\mathfrak{K}\int_V |\nabla\mathbf{v}|^2 \,\mathrm{dx} = \varepsilon\mathfrak{K}\int_{\mathbf{H}_\varepsilon(V)} |\nabla\mathbf{v}(\mathbf{H}_\varepsilon^{-1})|^2 |\det \nabla\mathbf{H}_\varepsilon^{-1}| \,\mathrm{dx}$$

$$= \varepsilon^4\mathfrak{K}\int_{\mathbf{H}_\varepsilon(V)} \frac{|\nabla(\mathbf{v} \circ \mathbf{H}_\varepsilon^{-1})|^2}{\varepsilon^2} \,\mathrm{dx} = \varepsilon^2\mathfrak{K}\int_{\mathbf{H}_\varepsilon(V)} |\nabla(\mathbf{v} \circ \mathbf{H}_\varepsilon^{-1})|^2 \,\mathrm{dx}\,.$$

Using the definition of the surface integral of the first type in the left part of our inequality with parametrization $\varphi_\varepsilon : \{(x_1, x_2) \in (0, \varepsilon)^2\} \to \underline{V},\ [x_1, x_2] \mapsto [x_1, x_2, 0]$ we obtain,

$$\int_{\underline{V}} |\mathbf{v}|^2 \,\mathrm{dS} = \iint_{\{(x_1,x_2)\in(0,\varepsilon)^2\}} |\mathbf{v} \circ \varphi_\varepsilon(x_1, x_2)|^2 \| \frac{\partial}{\partial x_1}\varphi_\varepsilon \times \frac{\partial}{\partial x_2}\varphi_\varepsilon \| \,\mathrm{dx_1 dx_2}$$

$$= \iint_{\{(x_1,x_2)\in(0,\varepsilon)^2\}} |\mathbf{v} \circ \mathbf{H}_\varepsilon^{-1} \circ \mathbf{H}_\varepsilon \circ \varphi_\varepsilon(x_1, x_2)|^2 \,\mathrm{dx_1 dx_2}$$

$$= \iint_{\{(x_1,x_2)\in(0,\varepsilon)^2\}} |(\mathbf{v}\circ\mathbf{H}_\varepsilon^{-1})\circ(\mathbf{H}_\varepsilon\circ\varphi_\varepsilon)(x_1, x_2)|^2 \| \frac{\partial}{\partial x_1}(\mathbf{H}_\varepsilon\circ\varphi_\varepsilon)\times\frac{\partial}{\partial x_2}(\mathbf{H}_\varepsilon\circ\varphi_\varepsilon)\|\varepsilon^2 \,\mathrm{dx_1 dx_2}$$

$$= \varepsilon^2 \int_{\underline{\mathbf{H}_\varepsilon(V)}} |\mathbf{v} \circ \mathbf{H}_\varepsilon^{-1}|^2 \,\mathrm{dS}\,.$$

Eventually we get the inequality

$$\int_{\underline{\mathbf{H}_\varepsilon(V)}} |\mathbf{v} \circ \mathbf{H}_\varepsilon^{-1}|^2 \,\mathrm{dS} \le \mathfrak{K} \int_{\mathbf{H}_\varepsilon(V)} |\nabla(\mathbf{v} \circ \mathbf{H}_\varepsilon^{-1})|^2 \,\mathrm{dx}\,.$$

Thanks to the fact that $\mathbf{H}_\varepsilon \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ and $\mathbf{H}_\varepsilon(V) = \Omega_1,\ \mathbf{H}_\varepsilon(\underline{V}) = \Gamma$, it is enough to prove the *Proposition 6.1* only in the case $\varepsilon = 1$. We prove the following:

There exists $\mathfrak{K} \in (0, \infty)$ such that for any arbitrarily chosen function $\mathbf{v} \in W^{1,2}(\Omega_1)$, satisfying boundary condition in form

$$\int_{\Omega_1} \mathbf{v} \cdot \nabla\varphi \,\mathrm{dx} = 0,\ \varphi \in C^1(\mathbb{R}^3), \tag{6.3}$$

it holds true

$$\int_\Gamma |\mathbf{v}|^2 \,\mathrm{dS} \le \mathfrak{K}\|\nabla\mathbf{v}\|^2_{L^2(\Omega_1)},$$

if the following conditions on Φ are satisfied

(i) $\Phi \in C(\mathcal{T}^2)$,

(ii) $\exists\{\Phi_k\}$ of Lipschitz functions and $\exists\{M_k\}$ of open subsets of $\mathcal{T}^2$ such that

$$0 < \Phi_k \le \Phi_{k+1} \le \Phi,\ \Phi_k = \Phi \text{ on } M_k,$$

$$|\mathcal{T}^2 \setminus M_k| \to 0 \text{ as } k \to \infty,$$

(iii) $\nabla\Phi \cdot \mathbf{w} \not\equiv 0$ a.e. on $\mathcal{T}^2$ for $|\mathbf{w}| = 1$.

We prove the above statement by contradiction. Let there exists a family of functions $\{\mathbf{v}_n\} \subset W^{1,2}(\Omega_1)$ such that

$$\int_\Gamma |\mathbf{v}_n|^2 \,\mathrm{dS} > n\|\nabla \mathbf{v}_n\|^2_{L^2(\Omega_1)},$$

satisfying (6.3).

In other words there exists a family of functions $\{\mathbf{v}_n\} \subset W^{1,2}(\Omega_1)$ such that

$$\int_\Gamma |\mathbf{v}_n|^2 \,\mathrm{dS} = 1 \text{ and } \frac{1}{n} > \|\nabla \mathbf{v}_n\|^2_{L^2(\Omega_1)},$$

satisfying (6.3). Let us use condition (ii). Choose subsequence of $\{\Phi_k\}$(do not change labeling) such that

$$|\mathcal{T}^2 \setminus M_k| \leq \frac{1}{2k}.$$

We define sets

$$\mathfrak{U}_k := \{(x_1, x_2, x_3) : (x_1, x_2) \in \mathcal{T}^2, -1 < x_3 < \Phi_k(x_1, x_2)\}.$$

Then for all $k \in \mathbb{N}$, it holds $\mathfrak{U}_k \subset \Omega_1$ and $\underline{\mathfrak{U}_k} = \Gamma$ and $\mathfrak{U}_k$ are Lipschitz domains. Obviously $\mathfrak{U}_k \nearrow \Omega_1$ as $k \to \infty$.

Let us choose $k \in \mathbb{N}$ arbitrary, fixed. Then for all $n \in \mathbb{N}$ it holds true

$$\int_{\underline{\mathfrak{U}_k}} |\mathbf{v}_n|^2 \,\mathrm{dS} = 1 \text{ and } \frac{1}{n} > \|\nabla \mathbf{v}_n\|^2_{L^2(\mathfrak{U}_k)}.$$

Thanks to the fact that $\mathfrak{U}_k$ is Lipschitz we can use following Poincaré-type inequality. There exists $C_1 \in (0, \infty)$ depending only on $\mathfrak{U}_k$ such that

$$\|\mathbf{u}\|_{L^2(\mathfrak{U}_k)} \leq C_1(\|\nabla \mathbf{u}\|_{L^2(\mathfrak{U}_k)} + \|\mathbf{u}\|_{L^2(\underline{\mathfrak{U}_k})}),$$

for more details see Poincaré [9].

Applying inequality on the family $\{\mathbf{v}_n\}$ we get for all $n \in \mathbb{N}$,

$$\|\mathbf{v}_n\|_{W^{1,2}(\mathfrak{U}_k)} \leq C_2 < \infty.$$

By reflexivity of $W^{1,2}(\mathfrak{U}_k)$ we can find a subsequence of $\{\mathbf{v}_n\}$ (we do not change labeling) that weakly converges to some $\mathbf{v} \in W^{1,2}(\mathfrak{U}_k)$. From lower semi-continuity of the norm we have $\|\nabla \mathbf{v}\|_{L^2(\mathfrak{U}_k)} = 0$ in other words $\nabla \mathbf{v} \equiv 0$ a.e. in $\mathfrak{U}_k \Rightarrow \mathbf{v} \equiv C \in \mathbb{R}^3$ constant. From compactness of the trace operator (see *Theorem 1.23* from Roubíček [10]) we have

$$\int_{\underline{\mathfrak{U}_k}} |\mathbf{v}|^2 \,\mathrm{dS} = 1 \ \Rightarrow \mathbf{v} \equiv C_k \neq 0. \tag{6.4}$$

We reach the desired contradiction from the boundary condition (6.3) and condition (iii) od Φ.

$$\int_{\Omega_1} \mathbf{v}_n \cdot \nabla\varphi \,\mathrm{dx} = 0, \ \varphi \in C^1(\mathbb{R}^3) \text{ and } \mathfrak{U}_k \subset \Omega_1 \Rightarrow$$
$$\int_{\mathfrak{U}_k} \mathbf{v}_n \cdot \nabla\varphi \,\mathrm{dx} = 0, \ \varphi \in C^1(\mathbb{R}^3), \ \operatorname{supp}\varphi \subset (\overline{\Omega_1 \setminus \mathfrak{U}_k})^c.$$

From weak convergence

$$\int_{\mathfrak{U}_k} \mathbf{v} \cdot \nabla\varphi \,\mathrm{dx} = 0,\ \ \varphi \in C^1(\mathbb{R}^3),\ \ \operatorname{supp}\varphi \subset (\overline{\Omega_1 \setminus \mathfrak{U}_k})^c.$$

Using Green's formula for $\mathfrak{U}_k$ gives us

$$\int_{\partial\mathfrak{U}_k} (\mathbf{v} \cdot \mathbf{n})\varphi \,\mathrm{dS} = 0,\ \ \varphi \in C^1(\mathbb{R}^3),\ \ \operatorname{supp}\varphi \subset (\overline{\Omega_1 \setminus \mathfrak{U}_k})^c.$$

This gives us

$$\mathbf{v} \cdot \mathbf{n} = 0 \text{ a.e. on } \Gamma_k,$$

where

$$\Gamma_k := \{(x_1, x_2, x_3) \mid (x_1, x_2) \in M_k,\ x_3 = \Phi_k(x_1, x_2) = \Phi(x_1, x_2)\}.$$

From condition (ii) imposed on Φ it is obvious that $\nabla\Phi = \nabla\Phi_k$ at a.a. points of M_k. Since normal vector $\mathbf{n}$ to the upper part of the boundary of Ω_1 is given as

$$\mathbf{n}(x_1, x_2) = \left(-\frac{\partial}{\partial x_1}\Phi(x_1, x_2), -\frac{\partial}{\partial x_2}\Phi(x_1, x_2), 1\right) \text{ at a.a. } (x_1, x_2) \in \mathcal{T}^2,$$

we know that

$$C_k \cdot \left(-\frac{\partial}{\partial x_1}\Phi(x_1, x_2), -\frac{\partial}{\partial x_2}\Phi(x_1, x_2), 1\right) = 0 \text{ at a.a. } (x_1, x_2) \in M_k.$$

We will use abbreviate form

$$C_k \cdot (-\nabla\phi, 1) = 0 \text{ a.e. on } M_k.$$

We know that $C_k = (c_1, c_2, c_3) \neq 0$. From $C_k \cdot (-\nabla\phi, 1) = 0$ a.e. on M_k we get

$$c_3 = c_1\frac{\partial}{\partial x_1}\Phi + c_2\frac{\partial}{\partial x_2}\Phi \text{ a.e. on } M_k. \tag{6.5}$$

Let us define

$$\mathbf{w}_k := \frac{1}{\sqrt{c_1^2 + c_2^2}}(c_1, c_2).$$

We will distinguish two cases with regard to c_3.

(i) If $c_3 = 0$ then from $C_k \neq 0$ we have $c_1 \neq 0$ or $c_2 \neq 0$. So we have $\nabla\Phi \cdot \mathbf{w}_k \equiv 0$ a.e. on M_k, where $|\mathbf{w}_k| = 1$ and definition of $\mathbf{w}_k$ is correct.

(ii) If $c_3 \neq 0$ then from (6.5) is again $c_1 \neq 0$ or $c_2 \neq 0$. So we have

$$\nabla\Phi \cdot \mathbf{w}_k \equiv \frac{c_3}{\sqrt{c_1^2 + c_2^2}} \text{ a.e. on } M_k,$$

where $|\mathbf{w}_k| = 1$ and definition of $\mathbf{w_k}$ is again correct. Since $\frac{c_3}{\sqrt{c_1^2+c_2^2}} \neq 0$ we see that derivative of Φ in direction $\mathbf{w}_k$ is non-zero constant, in other words Φ has linear growth in direction $\mathbf{w}_k$.

Both cases leads to contradiction for sufficiently large k. From now on k is not fixed.

(i) From condition (ii) and (iii) on Φ there exists $k_0 \in \mathbb{N}$, such that for all $k \geq k_0$ it holds $\nabla \Phi_k \cdot \mathbf{w} \not\equiv 0$ a.e. on M_k for any $|\mathbf{w}| = 1$. We immediately see that for $k \geq k_0$ we have contradiction with case (i).

(ii) Since Φ is defined on $\mathcal{T}^2$, otherwise is periodic, it can not have a linear growth in any direction $\mathbf{w}$, $|\mathbf{w}| = 1$. From condition (ii) there exists $k_1 \in \mathbb{N}$, such that for all $k \geq k_1$ it holds that Φ_k do not have a linear growth in any direction $\mathbf{w}$, $|\mathbf{w}| = 1$. Again we immediately see contradiction for $k \geq k_1$ with case (ii).

Eventually for $k \geq \max\{k_0, k_1\}$ there is always contradiction for Φ_k, which guarantees contradiction for Φ.

□

References

[1] Adams R.A. and Fournier J.J.F.:*Sobolev spaces*, Elsevier, Second edition, 2007.

[2] Basson A. and Gérard-Varet D.:*Wall laws for fluid flows at a boundary with random roughness*, 2006. Preprint.

[3] Bucur D., Feireisl E., Nečasová Š. and Wolf J.:*On the asymptotic limit of the Navier-Stokes system on domains with rough boundaries*, J.Differential Equations, 2006. To appear.

[4] Bucur D., Feireisl E.:*The incompressible limit of the full Navier-Stokes-Fourier system on domains with rough boundaries*, Nonlinear Anal. Real World Appl. Submitted.

[5] Casado-Díaz J., Fernández-Cara E., and Simon J.:*Why viscous fluids adhere to rugose walls: A mathematical explanation.* J. Differential Equations, **189**:526-537, 2003.

[6] Jaeger W. and Mikelić A:*On the roughness-induced effective boundary conditions for an incompressible viscous flow*, J. Differential Equations, **170**:96-122, 2001.

[7] Mohammadi B., Pironneau O., and Valentin F.:*Rough boundaries and wall laws*, Int. J. Numer. Meth. Fluids, **27**:169-177, 1998.

[8] Pedregal P.:*Parametrized measures and variational principles*, Birkhäuser, Basel, 1997.

[9] Poincaré H.:*Sur les équations aux dérivées partialles de la physique mathématique*, Amer. J. Math. **12** (1890), 211-294.

[10] Roubíček T.:*Nonlinear partial differential equations with applications*, Birkhäuser, Basel, 2005.