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Bc. Valeriia Tynianskaia

Shape of the Kerr gravitational field

Institute of Theoretical Physics

Supervisor of the master thesis: doc. RNDr. Oldřich Semerák, DSc.
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Title: Shape of the Kerr gravitational field

Author: Bc. Valeriia Tynianskaia

Department: Institute of Theoretical Physics

Supervisor: doc. RNDr. Oldřich Semerák, DSc., Institute of Theoretical Physics

Abstract:

Kerr metric is one of the most well-known and useful exact solutions of Einstein equations. We study various geometric properties of the Kerr spacetime in order to gain intuition for its spatial shape. In the review part we summarize basic features of the Kerr geometry, we write down Carter equations for geodesic motion in the Kerr spacetime, and we introduce kinematic characteristics of time-like and light-like congruences, such as expansion, shear and twist.

In the second part of the thesis we calculate scalars for acceleration, expansion, shear and twist — and plot the corresponding ”equipotential” surfaces — for several privileged congruences, namely the Carter observers, the static observers, the zero-angular-momentum observers, the principal null congruence and the recently found non-twisting null congruence(s). We also draw surfaces radially equidistant from the horizon and surfaces spatially orthogonal to the PNC and to the twist-free congruences, as well as the surfaces of constant energy and redshift for the important time-like congruences.

Keywords: general relativity, Kerr metric, observer congruences, surface shapes

Contents

Introduction

Albert Einstein published his general theory of relativity in 1915, which, together with the earlier concepts manifested in special theory of relativity, brought to light a completely new viewpoint on the theory of gravity. His equations written in a tensor form connect local geometry represented by a space-time *curvature* tensor with the energy, momentum and stress represented by *energy-momentum* tensor and, thus, give rise to a geometrical interpretation to the nature of gravity. General relativity not only brings a geometrical point of view on gravity, but also generalises Newtonian theory and after many successful probes represents modern notion of gravity.

Published in an original form, the set of differential equations was presumed by Albert Einstein himself as hardly solvable, however, shortly after its publication, the first exact solution has been provided by Karl Schwarzschild in 1916. The elegance and simplicity of the given solution was truly remarkable. The solution represented vacuum spherically symmetric space-time, where the assumed symmetry itself helped to significantly reduce the number of equations and provided a surprisingly simple solution to the remained equations. This approach introduced a great possibility for solving Einstein field equations by efficient imposing different kinds of symmetries whether on metric tensor itself or on other tensors of general relativity.

In 1963, as a result of imposing a special ansatz on metric tensor, a stationary axisymmetric solution that later has been interpreted as a generalization of Schwarzschild metric to a vacuum space-time with a rotating center, was found by Roy Patrick Kerr. The Kerr solution is found to be extremely important in astrophysics, since it involves rotation, an important property of celestial bodies.

Rotation of the central object considerably complicated the surrounding geometry and introduced some interesting phenomena such as dragging of inertial frames, the existence of static boundaries in a space-time or others. It also reinforced the meaning of coordinates used for solution of Einstein field equations, which resulted in an appearance of more sophisticated methods for finding general properties of a space-time independently of coordinates used.

At the beginning of this thesis we will provide a brief overview of a Kerr solution of Einstein equations and its main characteristics, review some special observers. In what follows we will focus on finding invariant properties of an axially-symmetric vacuum space-time using some notions of hydrodynamics applied to congruences of special observers, find some special surfaces and other quantities that can be helpful for closer understanding of the geometry of a space-time.

1. The Kerr space-time: an overview

The Kerr space-time is a vacuum stationary axially-symmetric exact solution of Einstein field equations characterized by mass M and angular momentum $J = Ma$. In this chapter we will provide a brief summary of its basic properties. We will stick to the convention $G = c = 1$.

1.1 The Kerr metric in coordinates

Generally, the metric for any stationary orthogonally transitive axially-symmetric, asymptotically flat space-time can be written in the standard form [1]:

$$\mathrm{d}s^2 = -e^{2\nu}\mathrm{d}t^2 + e^{2\psi}(\mathrm{d}\phi - \omega\mathrm{d}t)^2 + e^{2\mu_1}\mathrm{d}r^2 + e^{2\mu_2}\mathrm{d}\theta^2.$$

1.1.1 Boyer-Lindquist coordinates

In Boyer-Lindquist coordinates (t, r, θ, ϕ) the Kerr metric reads (see e.g. [2])

$$\begin{aligned}\mathrm{d}s^2 &= -\frac{\Delta\Sigma}{\mathcal{A}}\mathrm{d}t^2 + \frac{\mathcal{A}}{\Sigma}\sin^2\theta(\mathrm{d}\phi - \omega\mathrm{d}t)^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 \\ &= -\left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 - 2\frac{2Mr}{\Sigma}\,a\sin^2\theta\ \mathrm{d}t\mathrm{d}\phi + \frac{\mathcal{A}}{\Sigma}\,\sin^2\theta\ \mathrm{d}\phi^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2,\end{aligned} \tag{1.1}$$

with

$$\begin{aligned}\Sigma &\equiv r^2 + a^2\cos^2\theta, \\ \Delta &\equiv r^2 - 2Mr + a^2, \\ \mathcal{A} &\equiv (r^2 + a^2)^2 - \Delta a^2\sin^2\theta \\ &= \Sigma(r^2 + a^2) + 2Mra^2\sin^2\theta \\ &= \Sigma\Delta + 2Mr(r^2 + a^2), \\ \omega &\equiv \frac{-g_{t\phi}}{g_{\phi\phi}} = \frac{2Mar}{\mathcal{A}}.\end{aligned} \tag{1.2}$$

Hence, the stationary axially-symmetric metric becomes Kerr in Boyer-Lindquist coordinates if

$$e^{2\nu} = \frac{\Delta\Sigma}{\mathcal{A}}, \qquad e^{2\psi} = \frac{\mathcal{A}\sin^2\theta}{\Sigma},$$
$$e^{2\mu_1} = \frac{\Sigma}{\Delta}, \qquad e^{2\mu_2} = \Sigma.$$

From (1.1) it follows that for $a = 0$ the metric becomes Schwarzschild, for $r \to \infty$ the metric becomes Minkowski in spherical coordinates, therefore it is asymptotically flat. For $M = 0$ in (1.1) the metric reduces to

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{\Sigma}{r^2 + a^2}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 + (r^2 + a^2)\sin^2\theta\mathrm{d}\phi^2,$$

which for $a \neq 0$ is a flat space-time metric in *oblate spheroidal* coordinates.

Matrix form of the metric (1.1)

$$g_{\mu\nu} = \begin{pmatrix} -1+\frac{2Mr}{\Sigma} & 0 & 0 & -\frac{2Mar\sin^2\theta}{\Sigma} \\ 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ -\frac{2Mar\sin^2\theta}{\Sigma} & 0 & 0 & \frac{\mathcal{A}\sin^2\theta}{\Sigma} \end{pmatrix}, \tag{1.3}$$

its inverse [1]

$$g^{\mu\nu} = \begin{pmatrix} -1-\frac{2Mr(r^2+a^2)}{\Delta\Sigma} & 0 & 0 & -\frac{2Mar}{\Delta\Sigma} \\ 0 & \frac{\Delta}{\Sigma} & 0 & 0 \\ 0 & 0 & \frac{1}{\Sigma} & 0 \\ -\frac{2Mar}{\Delta\Sigma} & 0 & 0 & \frac{1-2Mr/\Sigma}{\Delta\sin^2\theta} \end{pmatrix}. \tag{1.4}$$

1.1.2 Kerr-Schild coordinates

The Kerr metric in Kerr-Schild coordinates (T, x, y, z) reads (see e.g.[3]):

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \frac{2Mr^3}{r^4+a^2z^2}\left[\mathrm{d}T + \frac{r(x\mathrm{d}x+y\mathrm{d}y)-a(x\mathrm{d}y-y\mathrm{d}x)}{r^2+a^2} + \frac{z\mathrm{d}z}{r}\right]^2, \tag{1.5}$$

where r satisfies the equation $r^4 - r^2(x^2+y^2+z^2-a^2) - a^2z^2 = 0$.

The relation between two sets of coordinates is the following:

$$\mathrm{d}T = \mathrm{d}t - \frac{2Mr}{\Delta}\mathrm{d}r, \qquad \mathrm{d}\psi = \mathrm{d}\phi - \frac{2Mar}{(r^2+a^2)\Delta}\mathrm{d}r,$$

$$x = \sqrt{r^2+a^2}\,\sin\theta\,\cos\psi, \qquad y = \sqrt{r^2+a^2}\,\sin\theta\,\sin\psi, \qquad z = r\,\cos\theta. \tag{1.6}$$

For $M = 0$ the metric (1.5) becomes Minkowski, thus, Kerr-Schild coordinates can be considered as a generalization of the Cartesian coordinates. From (1.6):

$$\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1, \qquad \frac{x^2+y^2}{a^2\sin^2\theta} - \frac{z^2}{a^2\cos^2\theta} = 1.$$

Thus, in Kerr-Schild coordinates (x, y, z) surfaces $r = \text{const}$ are ellipsoids and $\theta = \text{const}$ are hyperboloids.

In oblate coordinates r no longer corresponds to a circumferential radius given by the area of the surface $r = \text{const}$, $t = \text{const}$. It can be observed from a surface integral (see also [2])

$$\int_0^{2\pi}\int_0^{\pi}\sqrt{(g_{\theta\theta}g_{\phi\phi})_{r=\text{const}}}\mathrm{d}\theta\mathrm{d}\phi = \int_0^{2\pi}\int_0^{\pi}\sqrt{\mathcal{A}}\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi = 2\pi\int_0^{\pi}\sqrt{\mathcal{A}}\sin\theta\,\mathrm{d}\theta, \tag{1.7}$$

which for $a > 0$ never reduces to $4\pi r^2$, even for $M = 0$.

[1] Here $g^{rr} = \frac{1}{g_{rr}}$, $g^{\theta\theta} = \frac{1}{g_{\theta\theta}}$. Other elements $g^{\mu\nu}$ are found from an inverse of the matrix

$$\begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{t\phi} & g_{\phi\phi} \end{pmatrix}.$$

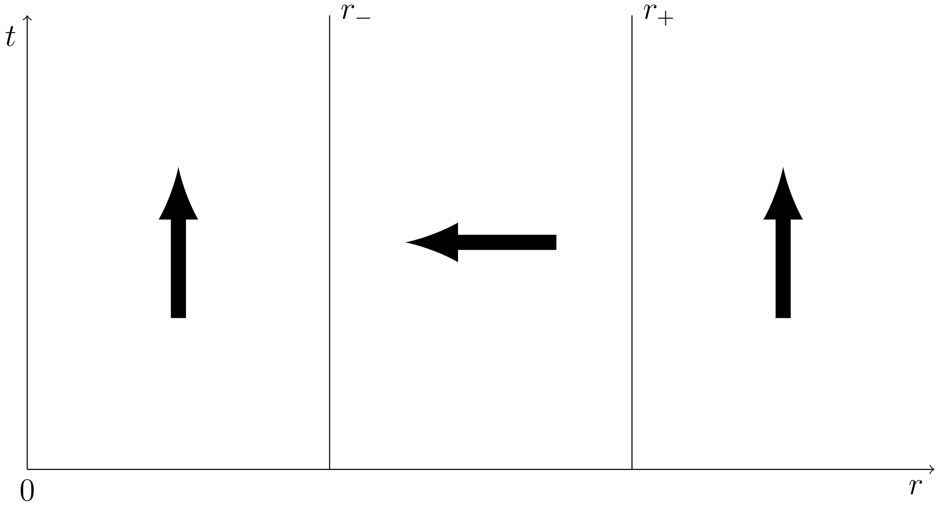

Figure 1.1: Causal future of light-cones in regions $r < r_-$, $r_- < r < r_+$ and $r > r_+$

1.2 Main features of the metric

1.2.1 Space-time symmetries

Space-time symmetry means constancy of the metric in the direction of some vector field, say ξ^μ. When the symmetries of the space-time itself are in question, it is instructive to turn to the vanishing of the Lie derivative of the metric tensor. Lie derivative of the metric tensor is given by

$$\pounds_\xi g_{\mu\nu} = g_{\mu\nu;\kappa}\, \xi^\kappa + \xi^\kappa{}_{;\mu} g_{\kappa\nu} + \xi^\kappa{}_{;\nu} g_{\mu\kappa} = \xi_{\mu;\nu} + \xi_{\nu;\mu},$$

where the first term vanishes because of the vanishing covariant derivative of the metric. The metric does not change along ξ^μ, if

$$\pounds_\xi g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu} = 0. \tag{1.8}$$

Equation (1.8) is called the **Killing equation**, and its solution ξ^μ is called the **Killing vector field**.

Given a vector field $\xi^\mu = \frac{\partial x^\mu}{\partial x^\alpha} = \delta^\mu_\alpha$, where x^α is some specific coordinate from a coordinate system adapted to this vector field, for a Lie derivative of any tensor T along ξ^μ, it holds (see e.g. [4], Appendix C):

$$\pounds_\xi T = T_{,\nu}\, \xi^\nu = T_{,\nu}\, \delta^\nu_\alpha = T_{,\alpha}\,.$$

So, for the metric tensor:

$$\pounds_\xi g_{\mu\nu} = g_{\mu\nu,\alpha}.$$

In order to satisfy Killing equation, and for ξ^μ to be the corresponding Killing vector field, $g_{\mu\nu,\alpha} = 0$ must be valid.

The metric (1.1) does not depend on t and ϕ coordinates, thus, vector fields $\xi^\mu_{(t)} = \frac{\partial x^\mu}{\partial t}$, $\xi^\mu_{(\phi)} = \frac{\partial x^\mu}{\partial \phi}$ are Killing vectors as well as any linear combination of the two with constant coefficients. In addition, there are invariants given by products of Killing vectors [2]

$$g_{tt} = g_{\alpha\beta}\xi^\alpha_{(t)}\xi^\beta_{(t)}, \qquad g_{t\phi} = g_{\alpha\beta}\xi^\alpha_{(t)}\xi^\beta_{(\phi)}, \qquad g_{\phi\phi} = g_{\alpha\beta}\xi^\alpha_{(\phi)}\xi^\beta_{(\phi)}$$

The metric in Boyer-Lindquist coordinates shows reflection symmetry with respect to the equatorial plane $\theta = \frac{\pi}{2}$. The metric is also invariant under transformations $(a \to -a,\ t \to -t)$, $(a \to -a,\ \phi \to -\phi)$, from which it can be seen that a is a parameter responsible for the rotation of a space-time.

1.2.2 Singularities, static limits and light-cones

There are three singularities in the Kerr space-time. As it can be seen from (1.1), singularities are given by $\Sigma = 0$ and $\Delta = 0$. By calculating Kretschmann invariant [2]

$$K := R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} = \frac{48M^2}{\Sigma^6}(r^2 - a^2\cos^2\theta)(\Sigma^2 - 16r^2a^2\cos^2\theta),$$

it can be deduced which singularity is physical, and which stems from the coordinates used. Hence, $\Sigma = 0$ is a true physical singularity, which requires from (1.2) $r = 0$ as well as $\theta = \frac{\pi}{2}$. Therefore, the singularity at $r = 0$ can only be reached from the equatorial plane. For θ other than $\frac{\pi}{2}$, the central point in Boyer-Lindquist coordinates appears non-singular.

Physical singularity in the Kerr space-time has an interesting feature, it is no longer a point-like singularity, but a *ring singularity*. The ring character of the singularity follows from calculations of its proper radius and a proper area of the disc $r = 0$, both calculations give the radius of a singular ring being a.

The coordinate singularities are given by $\Delta = 0$ and correspond to two horizons

$$r_\pm = M \pm \sqrt{M^2 - a^2}, \tag{1.9}$$

where for $a = M$ there exists only one horizon (*an extreme black hole*), for $a > M$ horizons do not exist (*a naked singularity*). For $0 < a < M$ two horizons are present.

Static limits are defined by $g_{tt} = 0$, i.e. $\Sigma = 2Mr$, and by solving quadratic equation we have

$$r = r_{0,1} = M \pm \sqrt{M^2 - a^2\cos^2\theta}. \tag{1.10}$$

Static limits in the Kerr space-time do not coincide with horizons and are special surfaces by themselves. Horizons and static limits are arranged as $r_0 \geq r_+ > r_- \geq r_1$, static limits touch the horizons only at the axis $(\theta = 0,\ \pi)$. In figure 1.2 the Kerr space-time is pictured in Boyer-Lindquist and in Kerr-Schild coordinates.

Since several horizons are present in the metric, we can expect non-trivial light-cone behaviour in space-time regions $r < r_-$, $r_- < r < r_+$ and $r > r_+$. Investigation of the metric in the radial direction $\mathrm{d}\theta = 0$, $\mathrm{d}\phi = 0$, using e.g. analytical extension with Eddington-Finkelstein coordinates as it is described in [3], implies that r and t reverse their roles every time radial photons pass through each of the horizons. The character of causal future of light-cones as photons radially approach the center is described in figure 1.1.

An interesting situation occurs below the inner horizon, here causal future points back in the t-direction, so future-oriented causal world lines need not necessarily end up in a singularity. For the case of an *extreme* black hole, when the region $r_- < r < r_+$ is reduced to a single horizon, it is possible to stay at a constant r both below and above such a horizon, see figure 1.1.

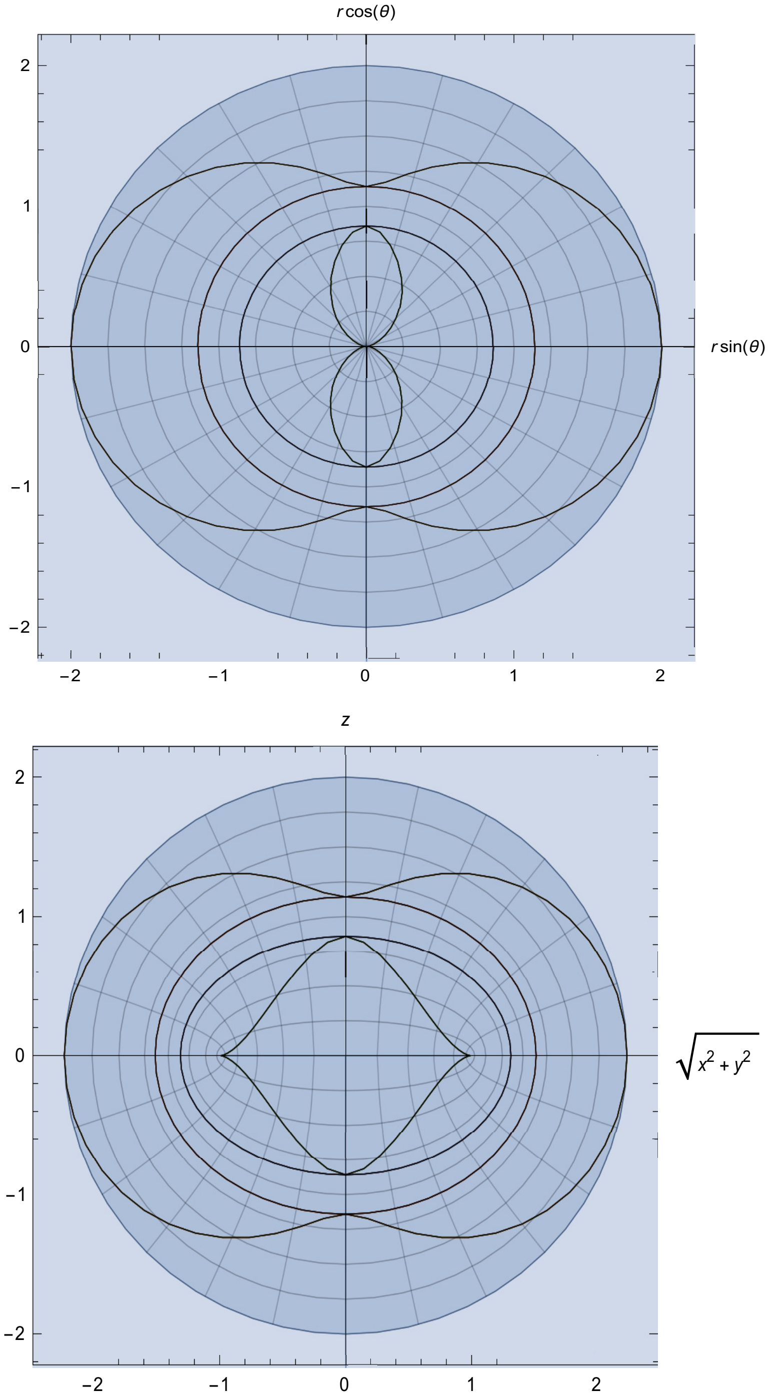

Figure 1.2: Kerr space-time with a spin parameter $a = 0.99M$, $M = 1$ in Boyer-Lindquist coordinates (upper) and in Kerr-Schild coordinates (lower). Bold black lines correspond to surfaces $r_0 \geq r_+ > r_- \geq r_1$. Ellipsoids of $r/M = 0.25, 0.5, 0.75, ..$ and hyperboloids of $\theta = 15°, 30°, 45°, ..$ are represented by gray faint lines.

1.3 Stationary circular motion

Since the space-time is axially-symmetric, the only possibility for observers to remain stationary is to move along with the symmetries of a space-time, i.e. to perform circular motion with a uniform angular velocity $\Omega = \frac{\mathrm{d}\phi}{\mathrm{d}t}$ on $r = \text{const}$, $\theta = \text{const}$. This kind of motion is characterized by the 4-velocity with Boyer-Lindquist components (see also [2])

$$u^\mu = (u^t, 0, 0, u^\phi) = u^t(1, 0, 0, \Omega). \tag{1.11}$$

From normalization condition $g_{\mu\nu}u^\mu u^\nu = -1$:

$$(u^t)^2(g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2) = -1.$$

Then,

$$u^t = \frac{1}{\sqrt{-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}}. \tag{1.12}$$

Normalized 4-velocity of a circular motion written as a linear combination of Killing vector fields, i.e. symmetries of a space-time, is [2]

$$u^\mu = \frac{\xi^\mu_{(t)} + \Omega\xi^\mu_{(\phi)}}{|\xi^\mu_{(t)} + \Omega\xi^\mu_{(\phi)}|} = u^t(\xi^\mu_{(t)} + \Omega\xi^\mu_{(\phi)}). \tag{1.13}$$

The motion is limited by the light-like motion, which is given by the condition $g_{\mu\nu}u^\mu u^\nu = 0$. This leads to the quadratic equation

$$(u^t)^2(g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2) = 0, \tag{1.14}$$

the solution of which provides us with limits on angular velocity Ω:

$$\Omega_{\text{max,min}} = \omega \pm \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}} = \omega \pm \frac{\Sigma\sqrt{\Delta}}{\mathcal{A}\sin\theta}, \tag{1.15}$$

where central value ω is interpreted as dragging of the surrounding geometry by the rotating center. From (1.15) it can be seen that at the horizon r_+ ($\Delta = 0$), there exists only one value of angular velocity Ω_H, which is interpreted as an angular velocity of the horizon with respect to an asymptotic inertial frame.

Interval $\Omega_{\text{min,max}}$ of the allowed angular velocities becomes strictly positive towards the horizon r_+ after reaching r_0, where $\Omega_{\text{min}} = 0$:

$$\Omega_{\text{min}} = 0 \Leftrightarrow g_{tt} = 0 \Leftrightarrow r = r_0.$$

After passing r_0, Ω_{min} increases to positive values for decreasing r making the whole interval $\Omega_{\text{min,max}}$ strictly positive, and thus excluding the possibility to stay in the ϕ-direction. This is the strongest evidence of dragging, which forces any observer performing circular motion to move in the ϕ-direction after static limit r_0 was reached (see e.g. [2]).

For velocity (1.11) we can calculate its 4-acceleration:

$$\begin{aligned} a_\mu = u_{\mu,\nu}u^\nu - \Gamma^\rho{}_{\mu\sigma}u_\rho u^\sigma = -\Gamma_{\rho\mu\sigma}u^\rho u^\sigma = -\frac{1}{2}(g_{\rho\mu,\sigma} + g_{\sigma\rho,\mu} - g_{\mu\sigma,\rho})u^\rho u^\sigma = \\ = -\frac{1}{2}g_{\sigma\rho,\mu}u^\rho u^\sigma = -\frac{1}{2}(u^t)^2(g_{tt,\mu} + 2g_{t\phi,\mu}\Omega + g_{\phi\phi,\mu}\Omega^2) = \\ = \frac{1}{2}\frac{g_{tt,\mu} + 2g_{t\phi,\mu}\Omega + g_{\phi\phi,\mu}\Omega^2}{g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2}. \end{aligned} \tag{1.16}$$

From (1.16) it is evident that a_t, a_ϕ are zero.

The specific energy and azimuthal angular momentum with respect to infinity for the circular motion with $\Omega = \frac{\mathrm{d}\phi}{\mathrm{d}t}$ and 4-velocity $u^\mu = u^t(1,0,0,\Omega)$, are (see [5])

$$\mathcal{E} := \frac{E}{m} = -u_t = -u^t(g_{tt} + g_{t\phi}\Omega), \tag{1.17}$$

$$\ell := \frac{L}{m} = u_\phi = u^t(g_{t\phi} + g_{\phi\phi}\Omega) = u^t g_{\phi\phi}(\Omega - \omega). \tag{1.18}$$

1.3.1 Stationary observers in the Kerr space-time

Stationary circular motion follows symmetries of a space-time and is privileged in that sense. Dragging of inertial frames in the Kerr space-time arises a question of what it means to stay in this type of geometry. There are several possibilities for an observer to be considered standing, depending on the point of reference. Considering different reference points relatively to which observers are viewed as standing leads to different types of stationary observers.

ZAMO

One possibility to stay is to stay relatively to geometry, which means to orbit with $\Omega = \omega$ with respect to infinity. This type of stationary observer is called **ZAMO** (*Zero Angular Momentum Observer*), see e.g.[2]. Indeed, for observers with $\Omega = \omega$, from (1.18) it follows $\ell = u_\phi = u^t g_{\phi\phi}(\Omega - \omega) = 0$.

Since $u_\phi = 0$, $u_\mu = (u_t, 0, 0, 0)$ is orthogonal to hypersurfaces $t = \text{const}$ (tangent vectors to the latter have non-zero only their spatial components). According to Frobenius theorem (see e.g. [4]), this implies that ZAMO congruence has zero vorticity, which we will also verify in the next chapter.

Due to the orthogonality to $t = \text{const}$ hypersurfaces, ZAMO congruence is important in $3+1$ formalism for stationary axially-symmetric space-times. Lapse there is then given by

$$-N^2 = g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2 = g_{tt} + g_{t\phi}\omega. \tag{1.19}$$

Thus, specially for ZAMO, 4-acceleration (1.16) can be written using lapse [2]

$$a_\mu^{\mathrm{ZAMO}} = \frac{N_{,\mu}}{N}. \tag{1.20}$$

In Boyer-Lindquist coordinates lapse is given by

$$N = \sqrt{\frac{\Delta\Sigma}{\mathcal{A}}}. \tag{1.21}$$

The motion is bounded by light-like motion, so from (1.14), (1.19), (1.15) it follows that ZAMO are physical only above the horizon, where lapse is positive.

Static observers

Another possibility to stay is relatively to the asymptotic inertial frame, i.e. to orbit with $\Omega = 0$. This kind of observers is called *static observers* (see e.g [2]), they exist only above static limit r_0. In equatorial plane $\theta = \frac{\pi}{2}$ static observers exist only for $r > 2M$, which can be observed from (1.14), (1.10).

Static observers have zero shear as it will be calculated later.

Carter observers

Special place belongs to observers linked to the curvature structure. Kerr space-time is of *type D* according to Petrov classification of space-times (see e.g.[3]), which means it possesses two principal null directions (PND) k^μ, l^μ. Congruence of observers with angular velocity $\Omega = \frac{a}{r^2+a^2}$, which perceive PND as purely radial is called *Carter congruence* (Carter observers, [2]). For Carter observers it holds

$$-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2 = \frac{\Sigma\Delta}{(r^2+a^2)^2}, \tag{1.22}$$

which implies that they are only physical above the horizon.

1.4 Geodesic motion in the Kerr space-time

As in Schwarzschild, geodesic motion in the Kerr space-time is completely integrable. Since the Kerr metric does not depend on t and ϕ coordinates, it admits integrals of motion $E = -mu_t$, $L = mu_\phi$. However, unlike Schwarzschild, the motion in Kerr is not planar, thus $u^\theta \neq 0$ in general[2]. It means, that constants of motion E, L together with normalization condition $g_{\mu\nu}u^\mu u^\nu = -1$ are not sufficient for the complete determination of u^μ.

It turns out that the Kerr space-time admits a deeper symmetry that allows additional *fourth* constant of motion to exist. Additional fourth constant of motion $\mathcal{K}$ was discovered by B. Carter (see e.g. [6], [3]), who found it while showing the separability of the Hamilton-Jacobi equation. The full set of Carter equations for the Kerr space-time for geodesics of massive particles reads (see also [2])

$$m\Delta\Sigma u^t = (r^2+a^2)R - \Delta\Theta\, a\, \sin^2\theta = \mathcal{A}E - 2MraL, \tag{1.23a}$$

$$m\Delta\Sigma u^\phi = aR - \Delta\Theta = 2MraE + (\Delta - a^2\sin^2\theta)\frac{L}{\sin^2\theta}, \tag{1.23b}$$

$$(m\Sigma u^r)^2 = R^2 - \Delta(m^2r^2 + \mathcal{K}), \tag{1.23c}$$

$$(m\Sigma u^\theta)^2 = \mathcal{K} - (ma\,\cos\theta)^2 - \Theta^2\sin^2\theta, \tag{1.23d}$$

where

$$R = R(r) := (r^2+a^2)E - aL,$$
$$\Theta = \Theta(\theta) := aE - \frac{L}{\sin^2\theta}.$$

For the case of massless particles we write p^μ instead of mu^μ and assume $m = 0$ on the right-hand side of equations (1.23). Hence, Carter equations for photons

[2] Equatorial plane $\theta = \frac{\pi}{2}$ is a special case of motion, where $u^\theta = 0$ and the motion is given by u^t, u^ϕ components together with normalization condition $g_{\mu\nu}u^\mu u^\nu = -1$, the latter would imply u^r component. In this case, constants E, L together with normalization condition would be sufficient for the description of motion.

become

$$\Delta\Sigma p^t = (r^2+a^2)R - \Delta\Theta a\sin^2\theta = \mathcal{A}E - 2MraL, \tag{1.24a}$$

$$\Delta\Sigma p^\phi = aR - \Delta\Theta = 2MraE + (\Delta - a^2\sin^2\theta)\frac{L}{\sin^2\theta}, \tag{1.24b}$$

$$(\Sigma p^r)^2 = R^2 - \Delta\mathcal{K} = \Big[(r^2+a^2)E - aL\Big]^2 - \Delta\mathcal{K}, \tag{1.24c}$$

$$(\Sigma p^\theta)^2 = \mathcal{K} - \Theta^2\sin^2\theta = \mathcal{K} - \left(aE\sin\theta - \frac{L}{\sin\theta}\right)^2. \tag{1.24d}$$

Special choice $\mathcal{K} = 0$, $L = aE\sin^2\theta$ completely annuls right-hand side of equation (1.24d) and leads to the set

$$\Delta\Sigma p^t = E[\mathcal{A} - 2Mra^2\sin^2\theta] = E\Sigma(r^2+a^2), \tag{1.25a}$$

$$\Delta\Sigma p^\phi = E[2Mar + a(\Delta - a^2\sin^2\theta)] = E\Sigma a, \tag{1.25b}$$

$$(\Sigma p^r)^2 = E^2(r^2+a^2-a^2\sin^2\theta)^2 = E^2\Sigma^2. \tag{1.25c}$$

For the components of p^μ from (1.25) it follows

$$p^t = E\frac{r^2+a^2}{\Delta}, \tag{1.26a}$$

$$p^\phi = E\frac{a}{\Delta}, \tag{1.26b}$$

$$p^r = \pm E, \tag{1.26c}$$

where p^r has two solutions of different sign. Solutions with positive sign are usually denoted by k^μ (outgoing) and solutions with negative sign are denoted by l^μ (ingoing). Factor E plays the role of the normalization and can e.g. be fixed by condition $g_{\mu\nu}k^\mu l^\nu = -1$. Hence, both congruences outgoing and ingoing read (see also [2])

$$k^\mu = \frac{1}{\Delta}(r^2+a^2, \Delta, 0, a), \quad k_\mu = \left(-1, \frac{\Sigma}{\Delta}, 0, a\ \sin^2\theta\right), \tag{1.27}$$

$$l^\mu = \frac{1}{2\Sigma}(r^2+a^2, -\Delta, 0, a), \quad l_\mu = \frac{\Delta}{2\Sigma}\left(-1, -\frac{\Sigma}{\Delta}, 0, a\ \sin^2\theta\right). \tag{1.28}$$

Congruences of photons (1.27), (1.28) are called **principal null congruences** (PNC) and represent principal null directions (PND), these terms are interchangeable.

Angular velocity of PNC photons is given by $\Omega = \frac{d\phi}{d\tau}\frac{d\tau}{dt} = \frac{k^\phi}{k^t} = \frac{l^\phi}{l^t} = \frac{a}{r^2+a^2}$. Note that Carter observers, whose angular velocity Ω coincides with the angular velocity of PNC photons, perceive PND as purely radial.

According to the corollary to the Goldberg-Sachs theorem [3], in the type II space-time the repeated null direction forms a congruence that is geodesic and shear free. Type II space-time admits one double and two simple principal null directions, type D admits two double PNDs, so we can expect the corollary to the Goldberg-Sachs theorem to hold for at least one of the two degenerated null directions in the Kerr space-time. From normalization chosen in (1.27), (1.28), it follows that only vector field k^μ forms a congruence that is geodesic and shear free.

Normalization of energy chosen in (1.27), (1.28) represents one of many possible choices (see normalization in e.g. [7]). For (1.26) normalization condition reads $g_{\mu\nu}k^\mu l^\nu = -\frac{2\Sigma}{\Delta}E$, where we have chosen the energy to be $E = \frac{\Delta}{2\Sigma}$ and multiplied vector l^μ by this factor in order to have normalization condition in the form $g_{\mu\nu}k^\mu l^\nu = -1$. However, factor $\frac{\Delta}{\Sigma}$ is not a constant factor, so field l^μ is no longer a geodesic (or at least affinely parametrized one) and/or shear free. Normalization condition $g_{\mu\nu}k^\mu l^\nu = -1$ is often assumed to hold while working with tetrad, thus one should keep in mind that depending on energy E chosen in the solution to Carter equations, one of the vectors of principal null congruence may not be geodesic (or at least affinely parametrized one) and/or may not have zero shear.

1.5 Killing tensor

As it was mentioned in the previous section, the existence of an additional constant of motion stems from deeper symmetries of a space-time. This kind of deeper symmetry for the Kerr space-time is represented by the existence of the second rank Killing tensor $\xi_{\mu\nu}$. The existence of the second rank Killing tensor itself follows from the existence of the second rank Killing-Yano tensor $Y_{\mu\nu}$ [3].

Killing and Killing-Yano tensors of the second rank and higher usually belong to *hidden* symmetries of a space-time not only in four dimensions, but also in higher dimensions as it is discussed in e.g. [8]. In four dimensions and in stationary axisymmetric space-time, the second rank Killing tensor $\xi_{\mu\nu}$ generates the fourth constant of motion $\mathcal{K}$.

Killing tensors $\xi_{\alpha..\beta}$ are totally symmetric tensors of any rank, which obey the generalized Killing equation [4]

$$\xi_{(\alpha..\beta;\mu)} = 0.$$

Hence, Killing vectors are the simplest Killing tensors, i.e. Killing tensors of the first rank. A quantity $\xi_{\mu..\nu}u^\mu...u^\nu$ is conserved along geodesics with tangent vector u^μ (for proof see e.g. [3]). Metric tensor is an example of Killing tensor and it generates the constant of geodesic motion $g_{\mu\nu}u^\mu u^\nu = -1$.

Killing-Yano tensors are totally antisymmetric tensors of any rank, which satisfy the Killing equation in a form

$$Y_{\alpha..(\beta;\mu)} = 0.$$

Killing tensor is given by the "square" of Killing-Yano tensors $\xi_{\mu\nu} = Y_{\mu\alpha}Y_\nu{}^\alpha$. Thus, the existence of Killing-Yano tensor implies the existence of Killing tensor.

In the Kerr space-time there exists second rank Killing-Yano tensor $Y_{\mu\nu}$ with non-zero components in Boyer-Lindquist coordinates [2]

$$Y_{\alpha r} = a\,\cos\theta\;(1,0,0,-a\,\sin^2\theta),\;\; Y_{\alpha\theta} = r\,\sin\theta\;(-a,0,0,r^2+a^2),$$

where $\alpha = t, \phi$. Hence, from the relation $\xi_{\mu\nu} = Y_{\mu\alpha}Y_\nu{}^\alpha$ it follows for Killing tensor components [2]

$$\xi_{tt} = \frac{a^2}{\Sigma}\ (\Delta\cos^2\theta + r^2\sin^2\theta),\quad \xi_{t\phi} = -\frac{a\sin^2\theta}{\Sigma}\ (\Sigma\Delta + 2Mr^3),$$
$$\xi_{rr} = -\frac{\Sigma}{\Delta}\ a^2\cos^2\theta,\quad \xi_{\theta\theta} = r^2\Sigma,\quad \xi_{\phi\phi} = \frac{\sin^2\theta}{\Sigma}(r^2\mathcal{A} + \Sigma\Delta a^2\sin^2\theta).$$

Killing tensor in the Kerr space-time with normalization used in (1.27), (1.28), can be written as

$$\xi_{\mu\nu} = 2\Sigma\ k_{(\mu}l_{\nu)} + r^2 g_{\mu\nu}. \tag{1.29}$$

For the second rank Killing tensor the conserved quantity along geodesics is $\xi_{\mu\nu}u^\mu u^\nu$. For test particles the conserved quantity along their world lines is given by $\mathcal{K} = \xi_{\mu\nu}p^\mu p^\nu$, which is exactly Carter constant that is present in Carter equations and is required for the complete description of geodesic motion in the Kerr space-time [4].

1.6 Hydrodynamic properties of congruences

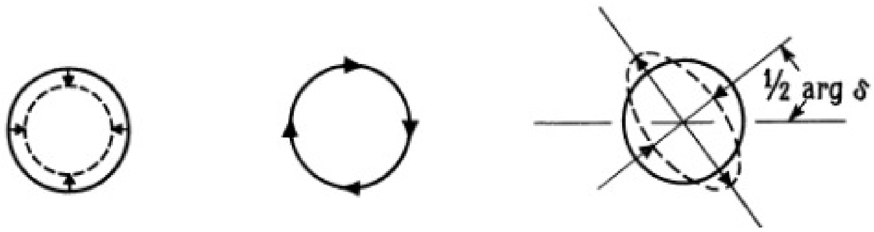

Figure 1.3: Schematic picture of deformations carried by the extension scalar (left), the twist scalar (middle) and the shear scalar (right). Adopted from [3].

In the present section we will focus on how geometry of a space-time affects the collective properties of a world lines. The collection of curves, which are integral curves of some smooth vector field u^μ, is called a congruence.

1.6.1 Time-like congruences

For the time-like congruences we define a tangent vector field

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}, \tag{1.30}$$

where τ is a proper time along the world line $x^\mu(\tau)$. We will also assume normalization condition $g_{\mu\nu}u^\mu u^\nu = -1$.

Since we are interested in the collective properties of a word lines, we can employ some concepts familiar from classical hydrodynamics. According to classical hydrodynamics for the velocity field it holds [2]

$$u_{\mu;\nu} = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}\Theta h_{\mu\nu} - a_\mu u_\nu, \tag{1.31}$$

where $a_\mu = u_{\mu;\nu}u^\nu$ is the acceleration of a tangent vector field u^μ, $\omega_{\mu\nu}$ is the vorticity (twist) tensor, $\sigma_{\mu\nu}$ is the shear tensor, Θ is the expansion scalar and $h_{\mu\nu}$

is the projection tensor, that at any point projects on the 3-space orthogonal to the local u^μ, given by

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu. \tag{1.32}$$

The vorticity tensor (antisymmetric)

$$\omega_{\mu\nu} = u_{[\mu;\nu]} + a_{[\mu} u_{\nu]},$$

the expansion tensor (symmetric)

$$\Theta_{\mu\nu} = u_{(\mu;\nu)} + a_{(\mu} u_{\nu)},$$

the expansion scalar

$$\Theta = h^{\mu\nu}\Theta_{\mu\nu} = u^\mu{}_{;\mu},$$

the shear tensor

$$\sigma_{\mu\nu} = \Theta_{\mu\nu} - \frac{1}{3}\Theta h_{\mu\nu}.$$

From the above tensors we can build the following scalars [2]

$$\kappa_1^2 = a_\mu a^\mu, \tag{1.33}$$

$$\omega^2 = \frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu} = \frac{1}{2}u_{[\mu;\nu]}u^{\mu;\nu} + \frac{1}{4}\kappa_1^2, \tag{1.34}$$

$$\Theta = u^\mu{}_{;\mu}, \tag{1.35}$$

$$\sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} = \frac{1}{2}u_{(\mu;\nu)}u^{\mu;\nu} + \frac{1}{4}\kappa_1^2 - \frac{1}{6}\Theta^2. \tag{1.36}$$

Each of the above scalars carries a certain type of information about a congruence. If we follow a volume element, that confines curves of a congruence, along the congruence, the volume element can be isotropically expanded (or contracted), warped (twisted) and/or sheared in the course. These deformations are carried by the corresponding scalars as schematically pictured in figure 1.3.

1.6.2 Light-like congruences

The case of the light-like congruences should be considered separately, since in this case it becomes not so clear, which direction is tangent and which is normal. Projection tensor (1.32) in this case reads

$$h_{\mu\nu} = g_{\mu\nu} + k_\mu l_\nu + l_\mu k_\nu, \tag{1.37}$$

where we introduced two null tangent vector fields l^μ, k^μ, which together locally form a plane, and can be normalized, so that $g_{\mu\nu}k^\mu l^\nu = -1$. Projection tensor (1.37) projects onto a 2-plane normal to both l^μ and k^μ and represents a 2D metric in that plane.

The scalars formed by expansion, twist and shear should be modified in the light-like case as follows [3]

$$\Theta = k^\mu{}_{;\mu}, \tag{1.38}$$

$$\omega^2 = \frac{1}{2}k_{[\mu;\nu]}k^{\mu;\nu}, \tag{1.39}$$

$$\sigma^2 = \frac{1}{2}k_{(\mu;\nu)}k^{\mu;\nu} - \frac{1}{4}\Theta^2. \tag{1.40}$$

In the light-like case these scalars are called *optical scalars*.

2. Gravitational field of the Kerr space-time

Hydrodynamic properties of congruences introduced in the previous chapter represent a powerful tool for building an intuition of how geometry of a space-time affects congruences of observers. In the present chapter we will examine some hydrodynamic properties of significant congruences of observers along with other quantities, which represent characteristics of the Kerr geometry in more detail.[1]

2.1 Stationary observers

2.1.1 Gravitational field

The magnitude of 4-acceleration κ_1^2 in the Kerr space-time quantitatively represents gravitational field any observer orbiting in the proximity of the Kerr center would experience. We will find surfaces of constant κ_1^2 for stationary observers, since they play a special role in the Kerr geometry. From (1.16) we know that a_θ, a_r are only non-zero components of the 4-acceleration, κ_1^2 then reads

$$\kappa_1^2 = g_{\mu\nu} a^\mu a^\nu = g_{\theta\theta}(a^\theta)^2 + g_{rr}(a^r)^2. \tag{2.1}$$

For the contravariant components of 4-acceleration it holds

$$\begin{aligned} a^\mu = u^\mu{}_{;\nu} u^\nu = u^\nu (u^\mu{}_{,\nu} + \Gamma^\mu{}_{\nu\alpha} u^\alpha) = \\ = u^\nu u^\alpha \Gamma^\mu{}_{\nu\alpha} = (u^t)^2 (\Gamma^\mu{}_{tt} + 2\Gamma^\mu{}_{t\phi}\Omega + \Gamma^\mu{}_{\phi\phi}\Omega^2). \end{aligned} \tag{2.2}$$

Knowing non-zero components of Christoffel symbols ([9], Appendix A), we can derive a general formulae for a^r, a^θ (see also [9])

$$\begin{aligned} a^r = (u^t)^2 (\Gamma^r{}_{tt} + 2\Gamma^r{}_{t\phi}\Omega + \Gamma^r{}_{\phi\phi}\Omega^2) = \\ = (u^t)^2 \frac{\Delta}{\Sigma^3} \left[M(2r^2 - \Sigma)(1 - a\Omega \sin^2\theta)^2 - r(\Omega\Sigma \sin\theta)^2 \right], \end{aligned} \tag{2.3}$$

$$\begin{aligned} a^\theta = (u^t)^2 (\Gamma^\theta{}_{tt} + 2\Gamma^\theta{}_{t\phi}\Omega + \Gamma^\theta{}_{\phi\phi}\Omega^2) = \\ = -(u^t)^2 \frac{\sin(2\theta)}{2\Sigma^3} \left[2Mr \left(a - (r^2 + a^2)\Omega \right)^2 + \Omega^2 \Delta \Sigma^2 \right]. \end{aligned} \tag{2.4}$$

[1] The majority of relativistic calculations was performed with the help of *xAct* set of packages for Mathematica (Mathematica v.12.0.0 has been used). *xAct* represents a family of free packages that enables one to execute tensorial calculations together with index manipulation in terms of geometrical formulation of General Relativity in Mathematica. A complete documentation together with installation instructions can be found at *xAct*'s official web pages: `http://www.xact.es/index.html`. In the digital attachments to the current thesis there can be found an example Mathematica notebook *'example.nb'* from the series of lectures **xAct: tensor analysis by computer 1** presented by Alfonso Garcia Parrado Gómez-Lobo, Ph.D. at Charles University in winter term 2019/2020, that contains some commented illustrative calculations carried out in *xAct*.

Static observers ($\Omega = 0$)

$$u^t_{\rm SO} = 1/\sqrt{-g_{tt}} \;\Rightarrow\; (u^t_{\rm SO})^2 = \frac{1}{-g_{tt}}.$$

Then from (2.3), (2.4) for SO it immediately follows (see also [9])

$$\begin{aligned} a^r_{\rm SO} &= \Delta\chi^{-2}\Sigma^{-2}M(2r^2 - \Sigma),\\ a^\theta_{\rm SO} &= -Mra^2\chi^{-2}\Sigma^{-2}\sin(2\theta), \end{aligned}$$

where we denoted $\chi^2 = -\Sigma g_{tt} = \Sigma - 2Mr$, [9]. The magnitude of 4-acceleration (2.1) is then

$$\kappa_1^2 = \Sigma^{-3}\left(\Sigma - 2Mr\right)^{-2} M^2 \left[\Delta(2r^2 - \Sigma)^2 + (ra^2\ \sin 2\theta)^2\right]. \tag{2.5}$$

Carter observers ($\Omega = \frac{a}{a^2+r^2}$)

$$(u^t_{\rm CO})^2 = \frac{(r^2+a^2)^2}{\Delta\Sigma}.$$

From (2.3), (2.4) it follows (see also [9])

$$\begin{aligned} a^r_{\rm CO} &= \Sigma^{-2}\left[M(2r^2 - \Sigma) - a^2 r \sin^2\theta\right],\\ a^\theta_{\rm CO} &= -a^2\Sigma^{-2}\sin\theta\cos\theta. \end{aligned}$$

κ_1^2 then reads

$$\kappa_1^2 = \frac{M^2(2r^2-\Sigma)^2 + a^2\sin^2\theta(\Sigma a^2 - 2Mr^3)}{\Delta\Sigma^3}. \tag{2.6}$$

Zero angular momentum observers ($\Omega = \omega = \frac{2Mar}{\mathcal{A}}$)

The 4-acceleration components (2.3), (2.4) for ZAMO appear to be cumbersome, so we will employ that ZAMO's 4-acceleration can be calculated from lapse according to (1.20). For this purpose we should calculate derivatives of lapse, for the θ-component it is given by

$$N_{,\theta} = \frac{1}{2}\left(\frac{\Delta\Sigma}{\mathcal{A}}\right)^{-\frac{1}{2}} \frac{\Delta(\Sigma_{,\theta}\mathcal{A} - \Sigma\mathcal{A}_{,\theta})}{\mathcal{A}^2}.$$

Then, using (1.20) we have

$$\begin{aligned} a_\theta^{\rm ZAMO} &= \frac{1}{2}\frac{\Sigma_{,\theta}\mathcal{A} - \Sigma\mathcal{A}_{,\theta}}{\Sigma\mathcal{A}} = \frac{1}{\Sigma\mathcal{A}}\left(-\mathcal{A}a^2\cos\theta\ \sin\theta + \Sigma\Delta a^2\sin\theta\ \cos\theta\right) = \\ &= a^2\cos\theta\ \sin\theta\frac{\Delta\Sigma - \mathcal{A}}{\Sigma\mathcal{A}} = -a\cos\theta\ \sin\theta\frac{2Mar(r^2+a^2)}{\Sigma\mathcal{A}} = -a\cos\theta\ \sin\theta\frac{(r^2+a^2)}{\Sigma}\omega. \end{aligned}$$

Hence,

$$a^\theta_{\rm ZAMO} = -a\cos\theta\ \sin\theta\Sigma^{-2}(r^2+a^2)\omega.$$

Calculating a^r_{ZAMO} directly we have

$$
\begin{aligned}
a^r_{\mathrm{ZAMO}} &= \Sigma^{-2}\mathcal{A}^{-1}\left[(r-M)\mathcal{A}\Sigma + \mathcal{A}r(r^2-2Mr+a^2) - \Delta\Sigma\left(2r(r^2+a^2) - a^2r\mathrm{sin}^2\theta + Ma^2\mathrm{sin}^2\theta\right)\right] \\
&= \Sigma^{-2}\mathcal{A}^{-1}\left[\Sigma(r^2+a^2)^2(r-M) + \mathcal{A}r(r^2-2Mr+a^2) - 2r(r^2+a^2)\Delta\Sigma\right] = \\
&= \Sigma^{-2}\mathcal{A}^{-1}\left[\Sigma(r^2+a^2)^2(r-M) - r(r^2+a^2)\Delta\Sigma + 2Mr^2\Delta a^2\mathrm{sin}^2\theta\right] = \\
&= \Sigma^{-2}\mathcal{A}^{-1}\left[\Sigma(r^2+a^2)\left((r^2+a^2)(r-M) - r\Delta\right) + 2Mr^2\Delta a^2\mathrm{sin}^2\theta\right] = \\
&= \Sigma^{-2}\mathcal{A}^{-1}\left[\Sigma M(r^2+a^2)(r^2-a^2) + 2Mr^2\Delta a^2\mathrm{sin}^2\theta\right] = \\
&= M\Sigma^{-2}\mathcal{A}^{-1}\left[\Sigma(r^4-a^4) + 2r^2\Delta a^2\mathrm{sin}^2\theta\right].
\end{aligned}
$$

For κ_1^2 for ZAMO it follows

$$
\kappa_1^2 = \frac{M^2}{\Sigma^2\mathcal{A}^3}\left[\left(\Sigma(r^4-a^4) + 2\Delta a^2r^2\mathrm{sin}^2\theta\right)^2 + \Delta\left(ra^2(r^2+a^2)\mathrm{sin}2\theta\right)^2\right]. \qquad (2.7)
$$

We pictured equipotentials of gravitational field $\kappa_1^2 = \mathrm{const}$ for the three types of rotational parameter a corresponding to a generic black hole ($a = 0.99M$, two horizons r_+, r_-), an extreme black hole ($a = M$) and a naked singularity ($a > M$) for Carter observers, static observers and ZAMO in figures 2.1-2.5.[2]

For a generic black hole, CO and ZAMO exist only above the outer horizon r_+, while SO exists only above a static limit $r_0 > r_+$ (in picture 2.2 region below the static limit should not be considered).

In the case of an extreme black hole, the space between two horizons reduces to one single surface - a horizon (as it may be observed from schematical picture 1.1). In this case, the causal future of light cones points upwards in the regions on both sides of the horizon, which makes possible for observers to exist above as well as below the horizon. Note that $r_1 < r_H < r_0$, so static observer exists only above and below static limits r_1, r_0 (region between static limits in figure 2.3 should not be considered). Also note that for $a = M$ static limits r_1, r_0 join at the axis.

In the case of a naked singularity, only static limits remain and they are represented by a single surface surrounding the central singularity. As in previous cases, static observer exists only above static limits, so the region inside a toroidal surface representing static limits should not be considered in figure 2.5.

[2]We choose $M = 1$ in all figures, so M stands for a scale parameter.

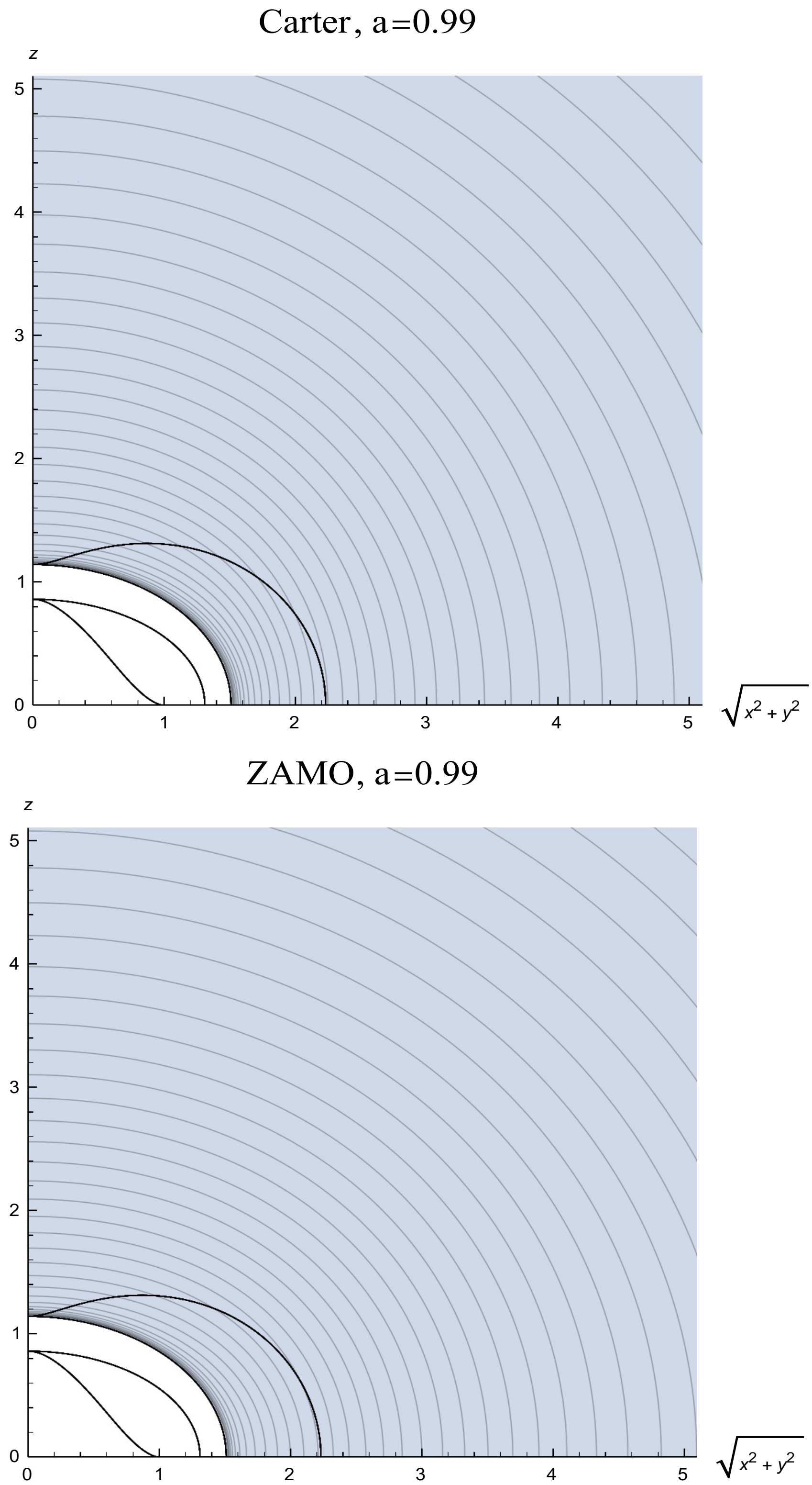

Figure 2.1: Equipotentials of gravitational field $\kappa_1^2 = \text{const}$ with rotational parameter $a = 0.99M$, $M = 1$ for CO (upper) and ZAMO (lower) in the Kerr-Schild coordinates ($\sqrt{x^2+y^2} = \sqrt{r^2+a^2}\,\sin\theta$, $z = r\,\cos\theta$). Constant values for equipotentials are chosen with the step $\exp(0.25n)$, n in $[-30, 6]$.

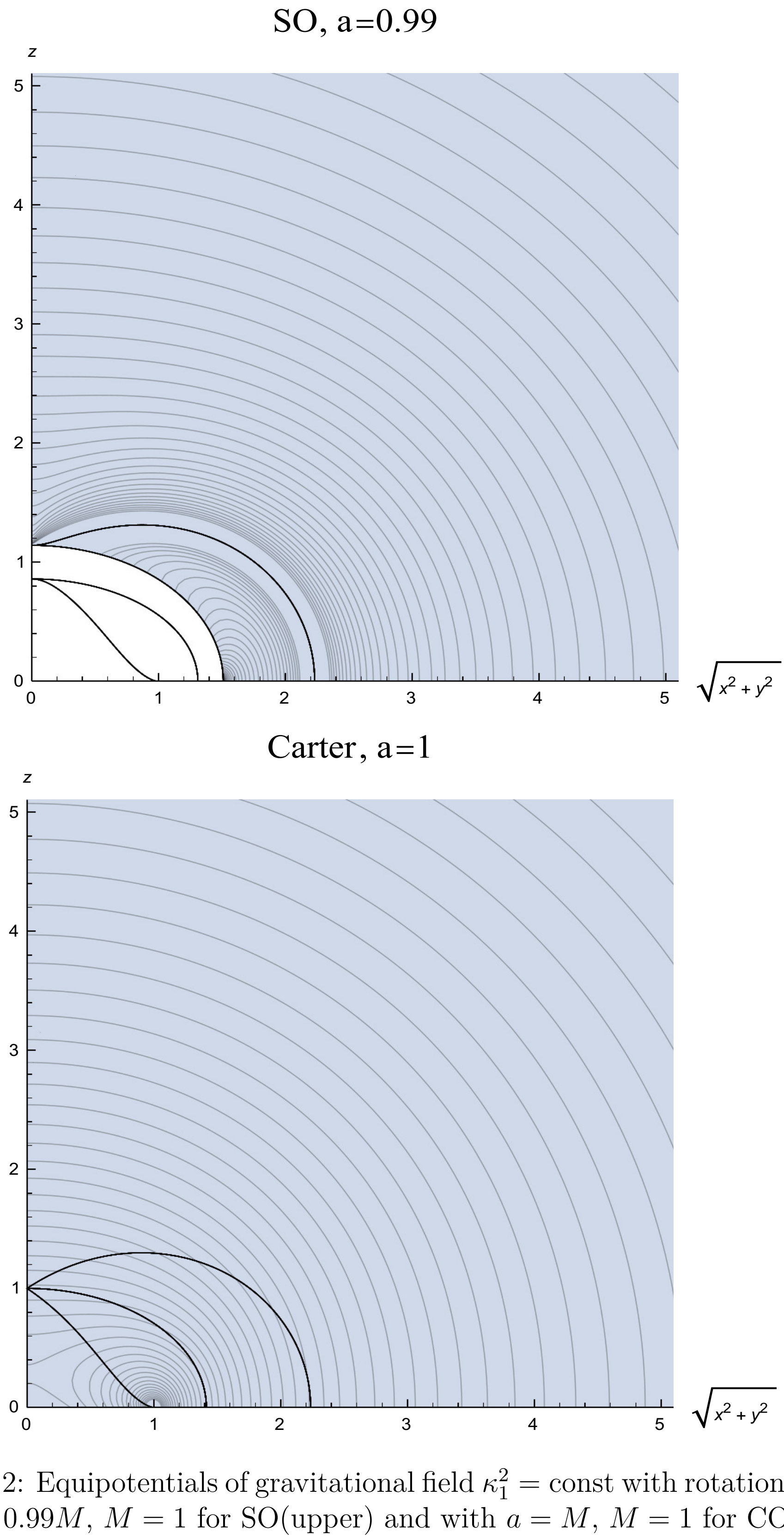

Figure 2.2: Equipotentials of gravitational field $\kappa_1^2 = \text{const}$ with rotational parameter $a = 0.99M$, $M = 1$ for SO(upper) and with $a = M$, $M = 1$ for CO(lower).

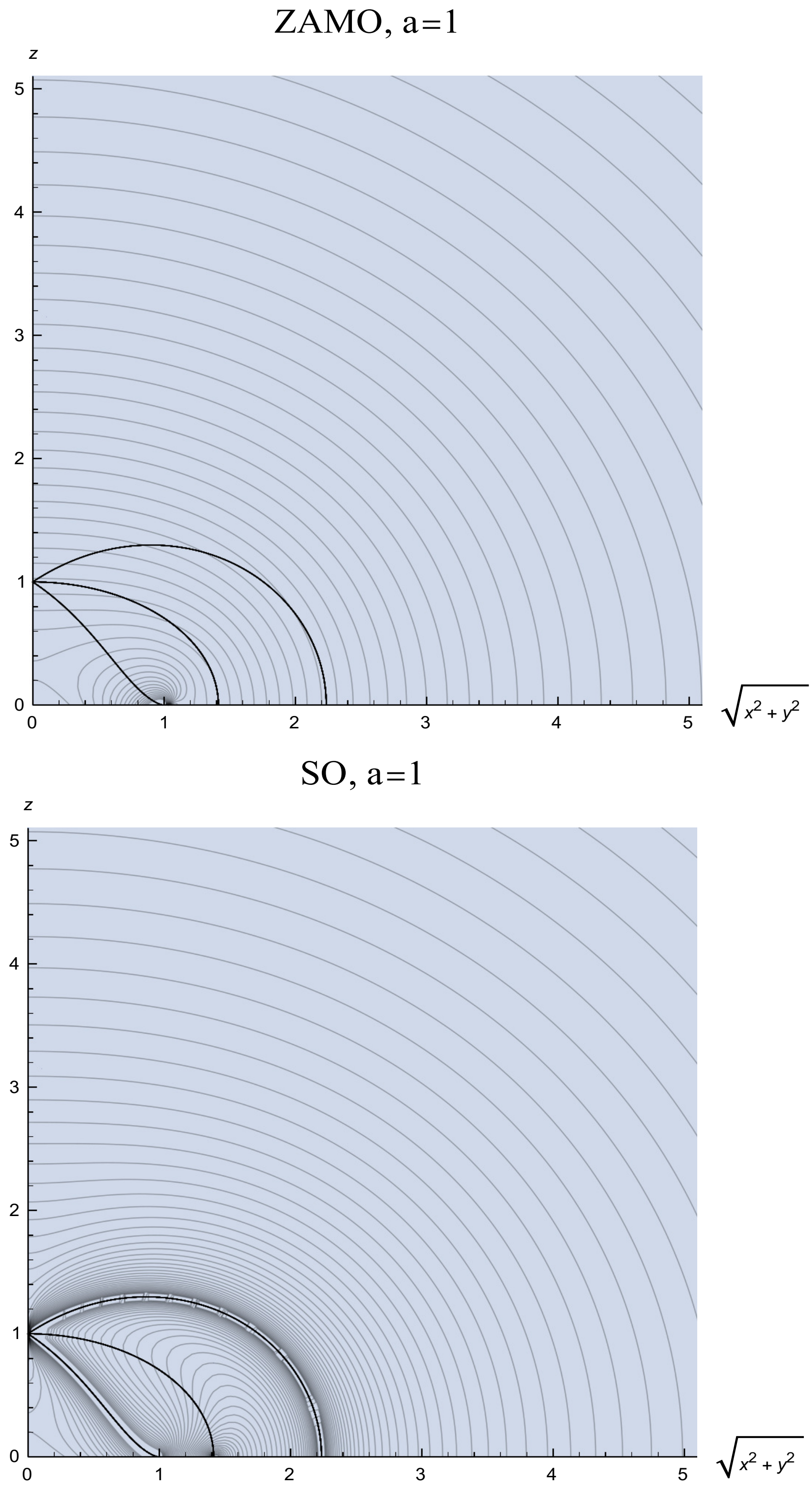

Figure 2.3: Equipotentials of gravitational field κ_1^2 = const with rotational parameter $a = M$, $M = 1$ for ZAMO(upper) and SO(lower).

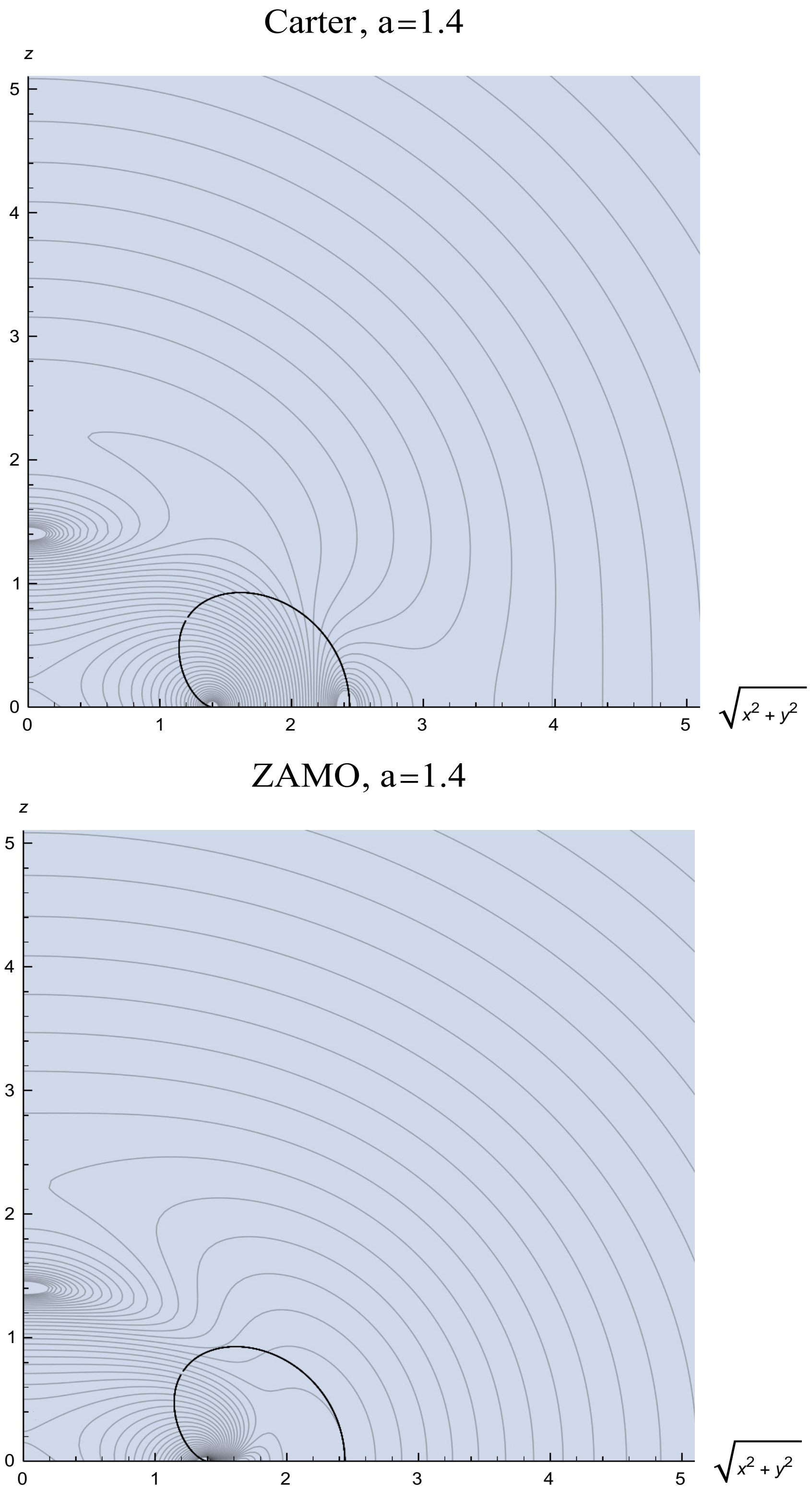

Figure 2.4: Equipotentials of gravitational field κ_1^2 = const with rotational parameter $a = 1.4M$, $M = 1$ for CO(upper) and ZAMO(lower).

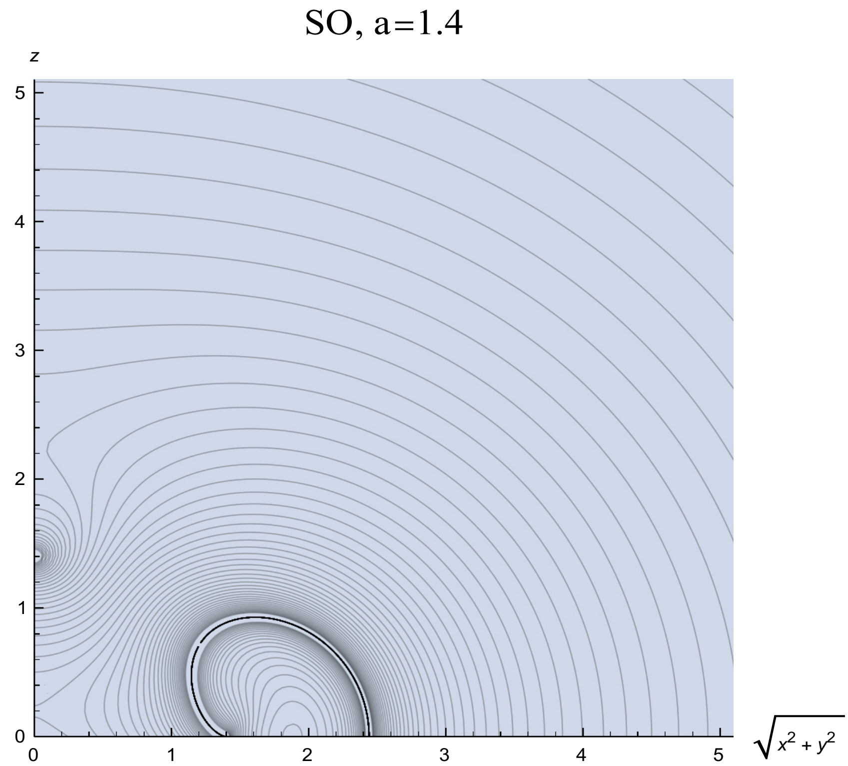

Figure 2.5: Equipotentials of gravitational field $\kappa_1^2 = \text{const}$ with rotational parameter $a = 1.4M$, $M = 1$ for SO.

2.1.2 Energy at infinity, redshift

Specific energy at infinity is given by $\mathcal{E} = -u_t = -u^t(g_{tt} + g_{t\phi}\Omega)$. Let us find surfaces of constant $\mathcal{E}$ for stationary observers. Specific energy at infinity for stationary observers is

$$\mathcal{E}_{\text{SO}} = -g_{tt}u^t = \sqrt{-g_{tt}} = \left(1 - \frac{2Mr}{\Sigma}\right)^{\frac{1}{2}} = \frac{1}{u^t_{\text{SO}}}. \tag{2.8}$$

$$\mathcal{E}_{\text{ZAMO}} = N^2 u^t = N = \sqrt{\frac{\Sigma\Delta}{\mathcal{A}}} = \frac{1}{u^t_{\text{ZAMO}}}. \tag{2.9}$$

$$\mathcal{E}_{\text{CO}} = \sqrt{\frac{\Delta}{\Sigma}}. \tag{2.10}$$

Surfaces $\mathcal{E} = \text{const}$ for stationary observers given by (2.8), (2.9), (2.10) are pictured in figures 2.6 - 2.8.

Following [10], we also can introduce the redshift potential ϕ by

$$e^{2\phi} = 1/(u^t)^2,$$

where for stationary observers it holds $1/(u^t)^2 = -g_{tt} - 2\Omega g_{t\phi} - \Omega^2 g_{\phi\phi}$. Hence, using (2.8), (2.9) for SO and ZAMO and (1.22) for CO, we have

$$\begin{aligned} e^{2\phi}_{\text{SO}} &= -g_{tt} = \left(1 - \frac{2Mr}{\Sigma}\right), \\ e^{2\phi}_{\text{ZAMO}} &= \frac{\Sigma\Delta}{\mathcal{A}}, \\ e^{2\phi}_{\text{CO}} &= \frac{\Sigma\Delta}{(r^2+a^2)^2}. \end{aligned}$$

Equipotential surfaces $e^{2\phi} = \text{const}$ for CO, ZAMO and SO are pictured in figures 2.9 - 2.11. Surfaces $e^{2\phi} = \text{const}$ for CO, ZAMO and SO coincide with the level sets of the relativistic potential for static congruences defined in terms of [10] pictured in figure 3(c) [10] for the Kerr space-time. This supports the suggestion that Carter observers, static observers and ZAMO represent different concepts of standing in the Kerr space-time.

Relativistic redshift between any two observers is defined by

$$\frac{\nu}{\tilde{\nu}} = \frac{e^{\phi(\tilde{r},\tilde{\theta})}}{e^{\phi(r,\theta)}}. \tag{2.11}$$

Redshift between two stationary observers, one of which stands, i.e. is static, at spatial infinity ($-g_{tt}\mid_\infty = 1$) is given by

$$\frac{\nu_\infty}{\nu_r} = \frac{e^\phi}{e^\phi_{\text{SO}}\mid_\infty} = \frac{1/u^t}{\sqrt{-g_{tt}}\mid_\infty} = \frac{1}{u^t}, \tag{2.12}$$

Redshift at infinity (2.12) for stationary observers ZAMO and SO coincides with their specific energy at infinity (2.8), (2.9). Therefore, pictures 2.7, 2.8 also describe surfaces of constant redshift at infinity for ZAMO and SO.

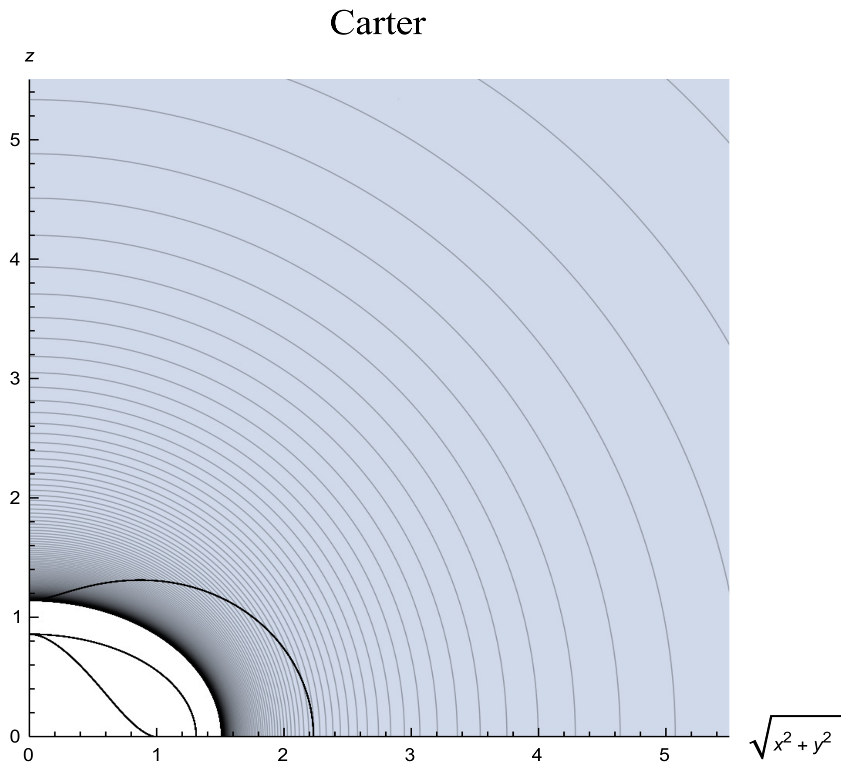

Figure 2.6: Surfaces $\mathcal{E} = \text{const}$ for Carter observer (CO), $a = 0.99$.

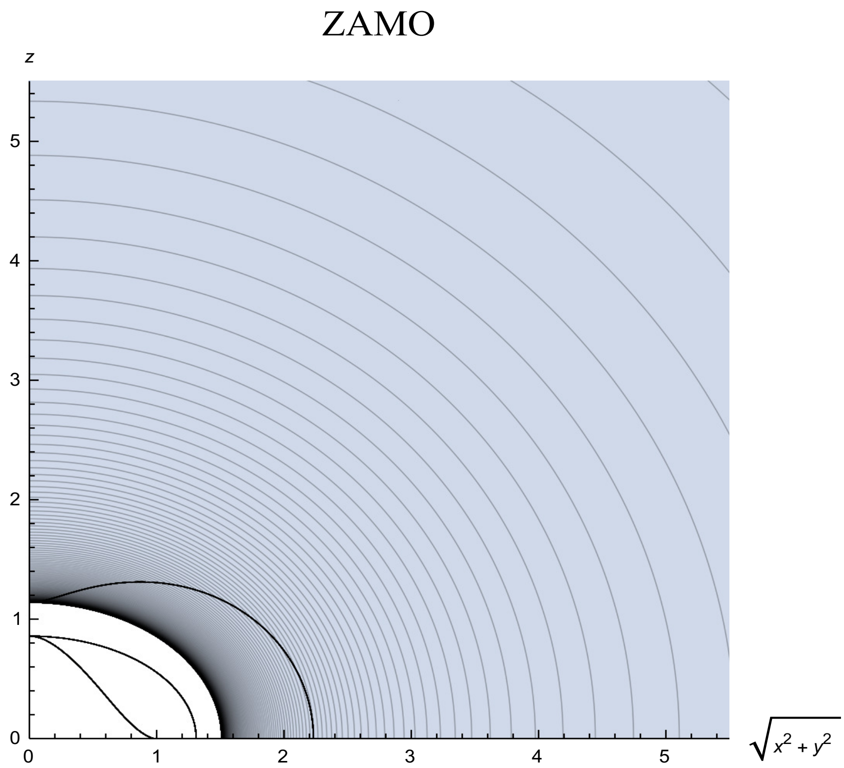

Figure 2.7: Surfaces of $\mathcal{E} = \text{const}$ for ZAMO, $a = 0.99$

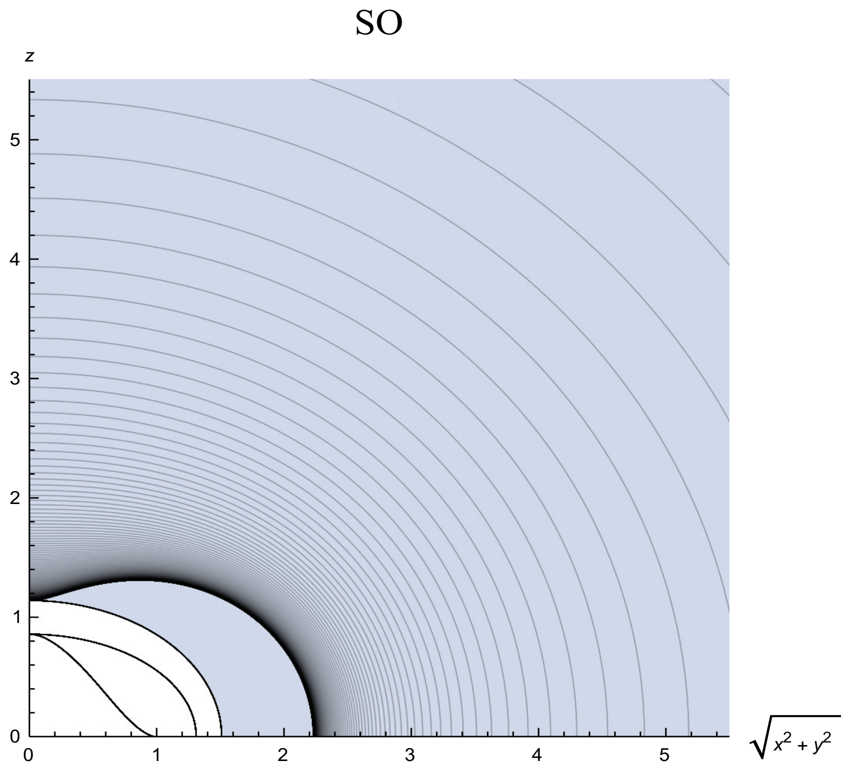

Figure 2.8: Surfaces of $\mathcal{E} = \mathrm{const}$ for SO, $a = 0.99$

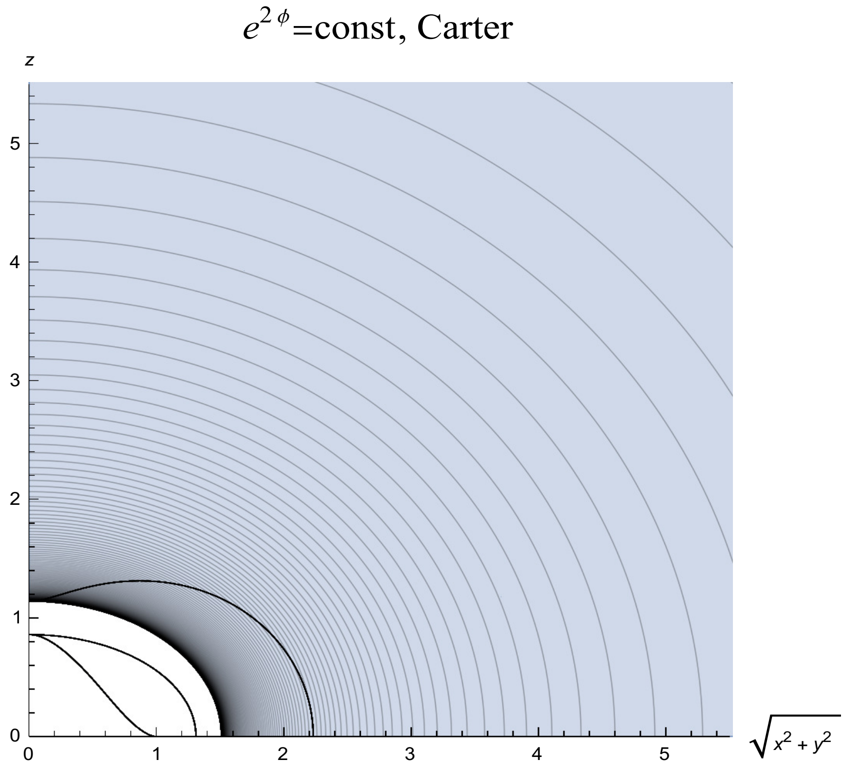

Figure 2.9: Surfaces $e^{2\phi} = \mathrm{const}$ for CO, $a = 0.99$

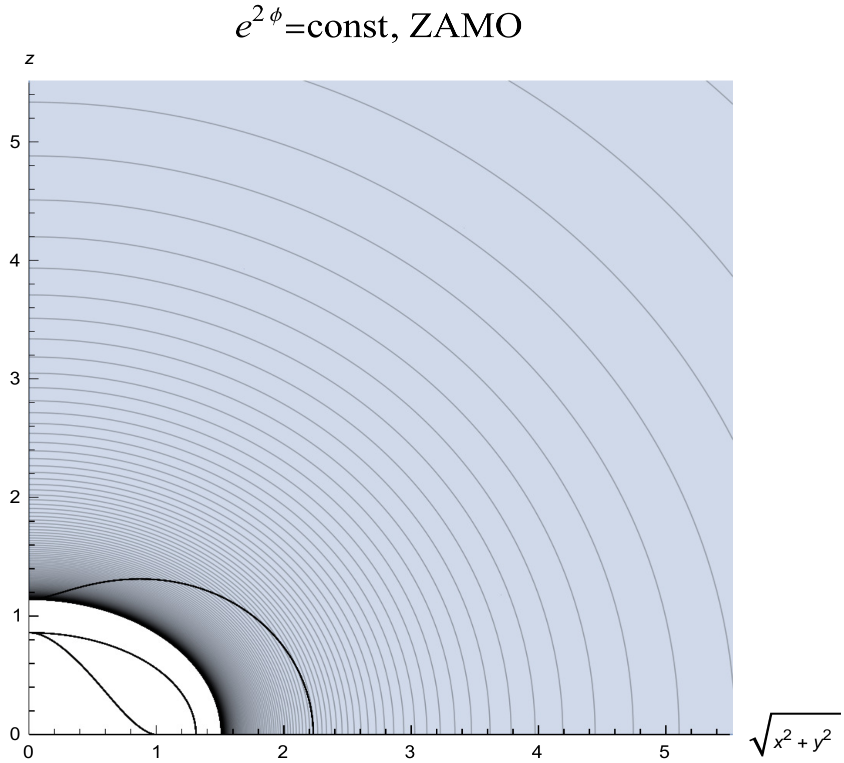

Figure 2.10: Surfaces $e^{2\phi} = \text{const}$ for ZAMO, $a = 0.99$

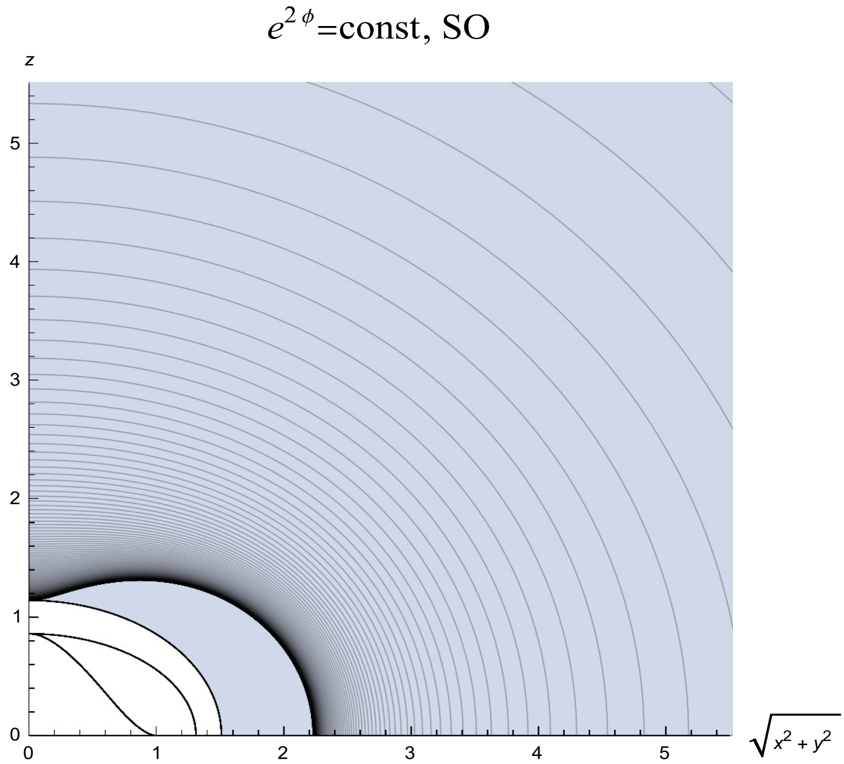

Figure 2.11: Surfaces $e^{2\phi} = \text{const}$ for SO, $a = 0.99$

2.1.3 The expansion, twist and shear scalars

Let us examine other scalars that describe hydrodynamic properties of congruences for CO, ZAMO and SO. The expansion scalar is zero for all stationary observers (as it may be observed from (1.35) and the components of 4-velocity (1.11)). For the purpose of calculation of the remaining scalars, we will employ *xAct* package for Mathematica in order to perform corresponding covariant derivatives from (1.34), (1.36). Mathematica notebooks containing calculations can be found in the digital attachment to the current thesis.

Carter observers (CO)

The shear scalar for CO reads

$$\sigma^2_{\rm CO} = \frac{a^2\ r^2\ \sin^2\theta}{\Sigma^3}.$$

The twist scalar for CO reads

$$\omega^2_{\rm CO} = \frac{\Delta\ a^2\ \cos^2\theta}{\Sigma^3}.$$

Surfaces of a constant shear and twist scalars $\sigma^2_{\rm CO} = \text{const}$, $\omega^2_{\rm CO} = \text{const}$ for CO are pictured in figures 2.12, 2.13.

Zero angular momentum observers (ZAMO)

The twist scalar came out zero, $\omega^2_{\rm ZAMO} = 0$, as it was expected for ZAMO congruence, since ZAMO are orthogonal to hypersurfaces $t = \text{const}$. The shear scalar for ZAMO reads

$$\sigma^2_{\rm ZAMO} = \frac{a^2M^2\sin^2\theta\left[18r^6 + 2a^4\Sigma - 12a^2r^2\Sigma - 3a^2r^4(\cos2\theta - 7) - 4a^4Mr^3\Sigma^{-1}\sin^2(2\theta)\right]}{2(\Sigma\mathcal{A})^2}.$$

Surfaces of a constant shear scalar $\sigma^2_{\rm ZAMO} = \text{const}$ are pictured in figure 2.14.

Static observers (SO)

Static observers have no shear scalar, $\sigma^2_{\rm SO} = 0$, but a non-zero scalar for twist. The twist scalar reads

$$\omega^2_{\rm SO} = \frac{a^2M^2\left[\Sigma^2\sin^2\theta + 4r^2\cos^2\theta(\Delta - a^2\sin^2\theta)\right]}{\Sigma^3(\Delta - a^2\sin^2\theta)^2}.$$

Surfaces of a constant twist scalar, $\omega^2_{\rm SO} = \text{const}$, are pictured in figure 2.15.

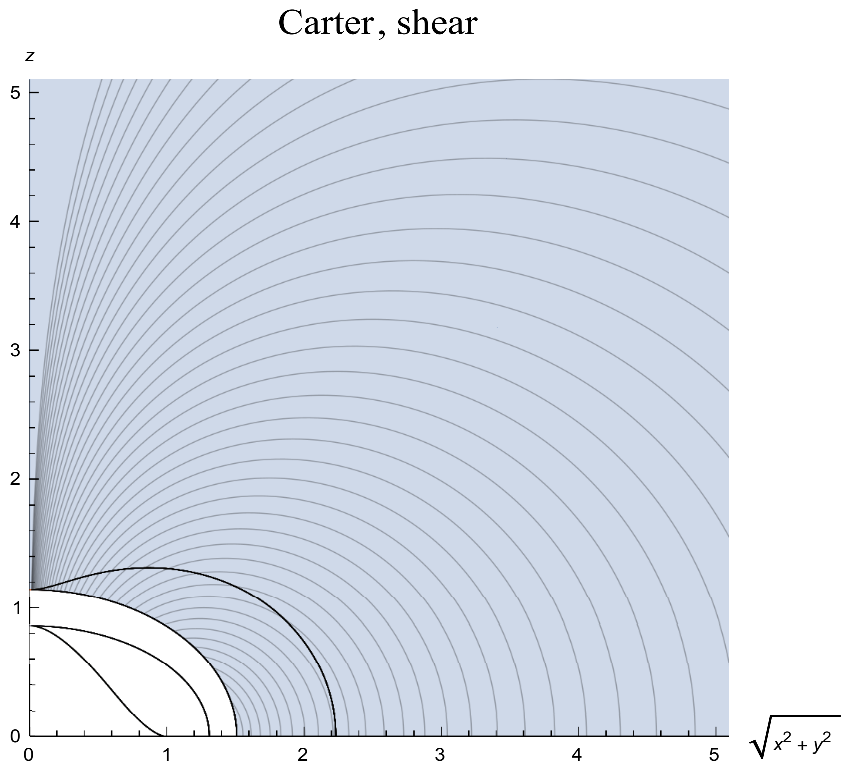

Figure 2.12: Surfaces $\sigma^2_{\mathrm{CO}} = \mathrm{const}$, $a = 0,99$.

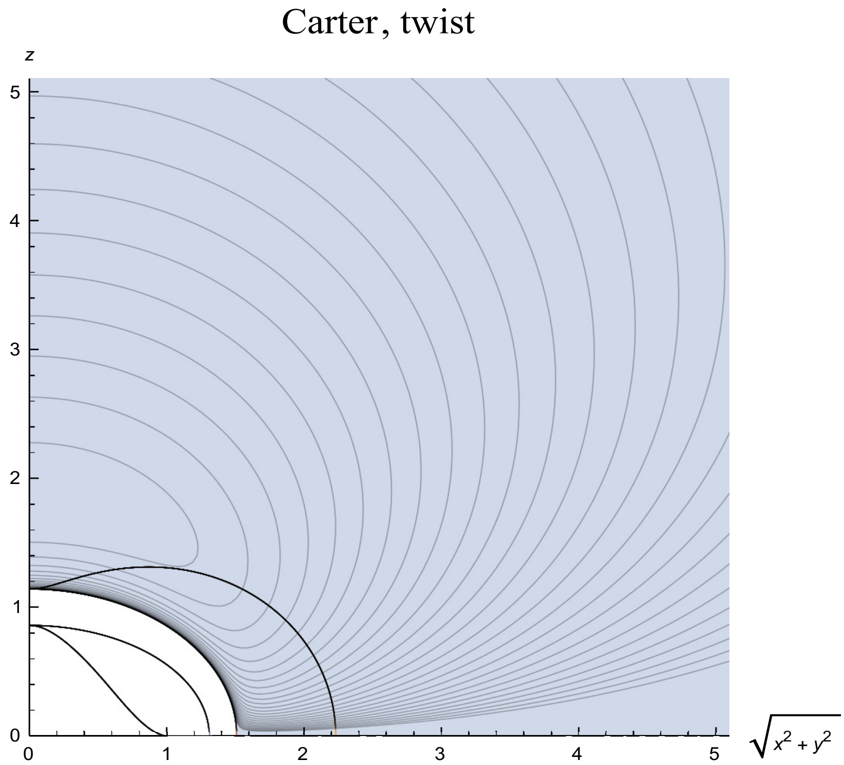

Figure 2.13: Surfaces $\omega^2_{\mathrm{CO}} = \mathrm{const}$, $a = 0,99$.

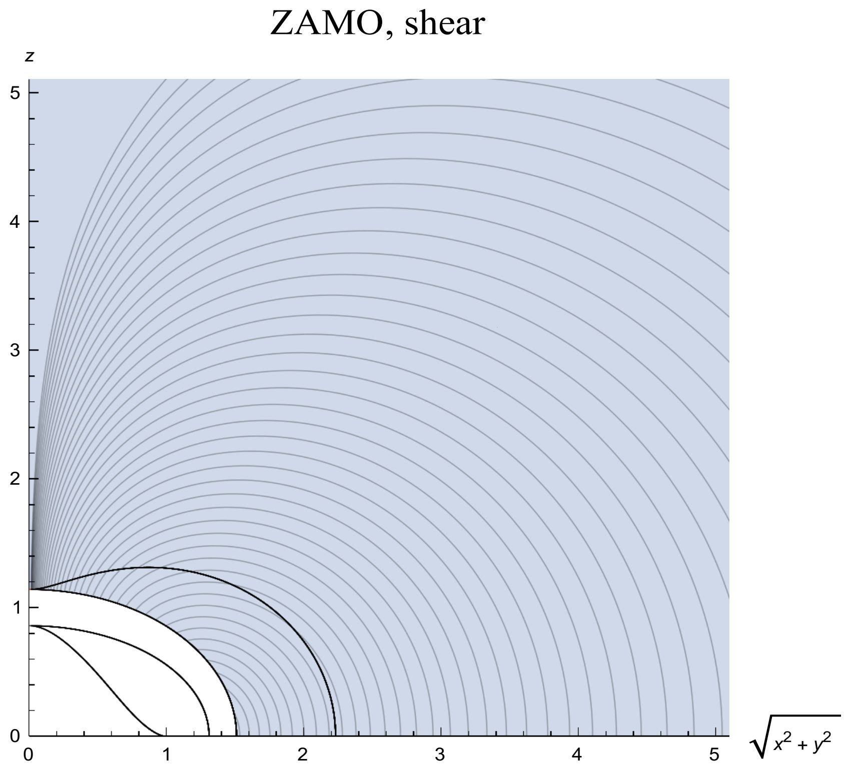

Figure 2.14: Surfaces $\sigma^2_{\mathrm{ZAMO}} = \mathrm{const}$, $a = 0,99$.

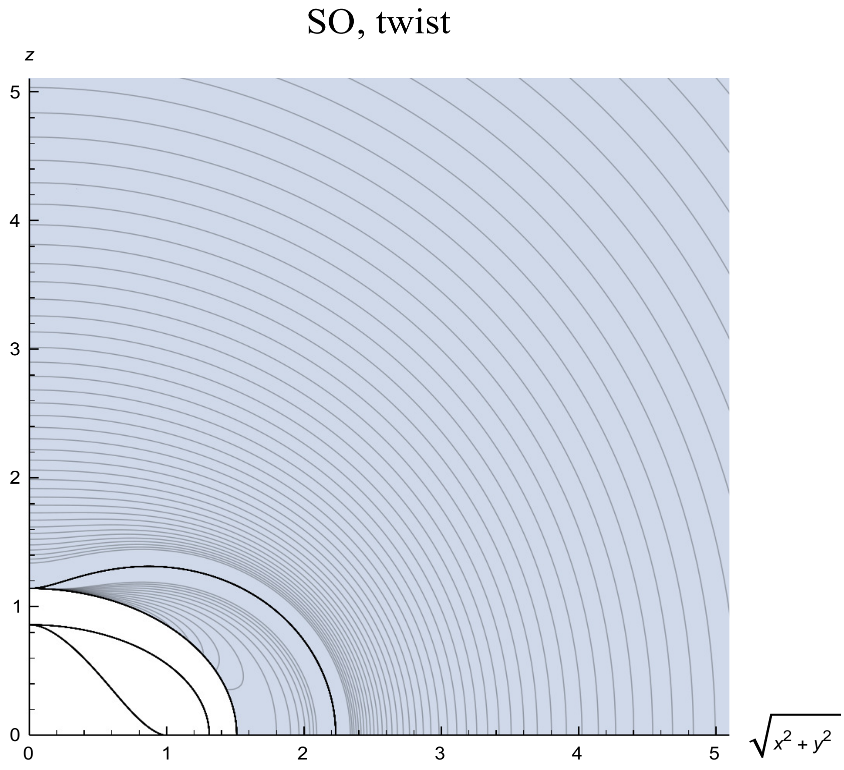

Figure 2.15: Surfaces $\omega^2_{\mathrm{SO}} = \mathrm{const}$, $a = 0,99$.

2.2 Surfaces equidistant from the horizon. Killing(-Yano) tensor invariants

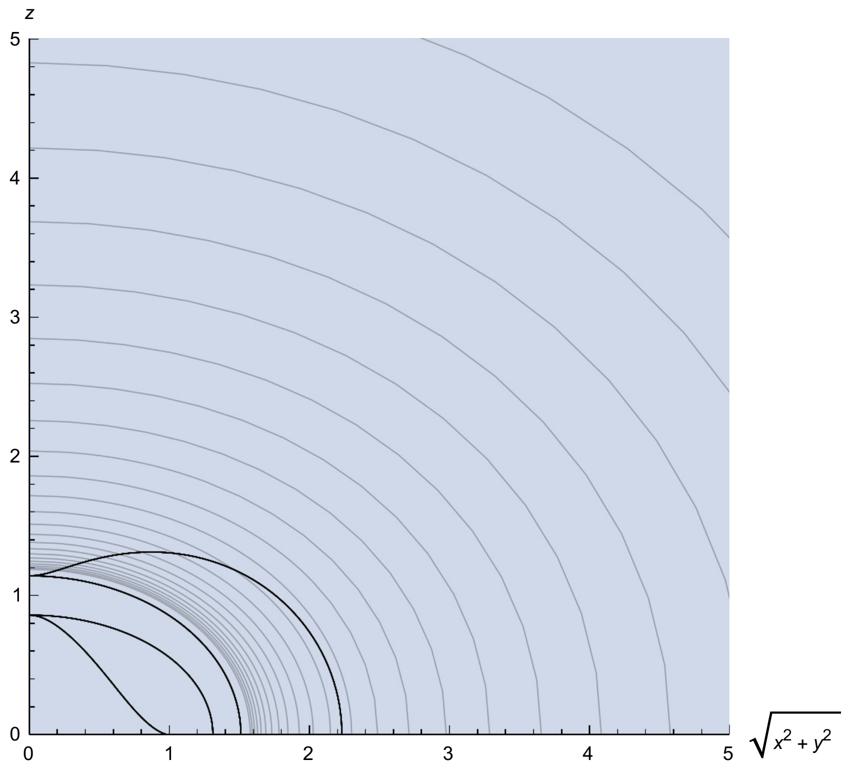

Figure 2.16: Surfaces $l(r, r_+) = \text{const}$, $a = 0.99$.

Let us find surfaces of equal distance from the outer horizon r_+. For this purpose we will calculate a proper distance from the horizon to some reference r and then numerically find surfaces, which are equally distant from the horizon. A proper distance to the outer horizon from some radius r is given by

$$l(r, r_+) = \int_{r_+}^{r} \sqrt{g_{rr}} \mathrm{d}r' = \int_{r_+}^{r} \sqrt{\frac{r'^2 + a^2 \cos^2\theta}{r'^2 - 2Mr' + a^2}} \mathrm{d}r'. \tag{2.13}$$

In the special case of equatorial plane $\theta = \frac{\pi}{2}$, the result of the integration given by (2.13) can be written relatively simple

$$l(r, r_+) \mid_{\theta=\frac{\pi}{2}} = \int_{r_+}^{r} \frac{r'}{\sqrt{r'^2 - 2Mr' + a^2}} \mathrm{d}r' = \left[M \ln(\sqrt{\Delta} + r' - M) + \sqrt{\Delta} \right] \Big|_{r_+}^{r} =$$

$$= M \ln\left(\frac{\sqrt{\Delta} + r - M}{\sqrt{M^2 - a^2}} \right) + \sqrt{\Delta}.$$

Equidistant surfaces from the outer horizon $l(r, r_+) = \text{const}$ are pictured in figure 2.16.

As we have mentioned in the previous chapter, the Kerr space-time possesses Killing-Yano and Killing tensors of the second rank, from which we can built invariants. Invariants from Killing tensor $\xi_{\mu\nu} = Y_{\mu\alpha} Y_{\nu}{}^{\alpha}$ are

$$g^{\mu\nu}\xi_{\mu\nu} = 2(r^2 - a^2 \cos^2\theta), \quad \xi^{\mu\nu}\xi_{\mu\nu} = 2(r^4 + a^4 \cos^4\theta).$$

Invariants for Killing-Yano tensor $Y_{\mu\nu}$ are[3]

$$Y^{\mu\nu}Y_{\mu\nu} = 2(r^2 - a^2\cos^2\theta), \quad {}^*Y^{\mu\nu}Y_{\mu\nu} = 4ar\cos\theta.$$

2.3 Optical scalars

In the case of light-like congruences, scalars for the expansion, twist and shear are called *optical scalars*. In this section we will explore optical scalars on example of two light-like congruences.

2.3.1 Principal null congruence (PNC)

First we choose principal null congruence (PNC) given by (1.27), (1.28). PNC has zero scalar for shear $\sigma^2_{\mathrm{PNC}} = 0$ (due to Goldberg-Sachs theorem). The expansion Θ_{PNC} and twist ω^2_{PNC} scalars calculated from (1.38), (1.39) with the help of Mathematica are (see also [7])

$$\Theta_{\mathrm{PNC}} = \frac{2r}{\Sigma}, \quad \omega^2_{\mathrm{PNC}} = \frac{a^2\cos^2\theta}{\Sigma^2}.$$

Surfaces $\Theta_{\mathrm{PNC}} = \mathrm{const}$, $\omega^2_{\mathrm{PNC}} = \mathrm{const}$ are pictured in figures 2.17, 2.18.

2.3.2 Twist-free congruence

For the second example of light-like congruence, we consider a congruence defined by Carter constant $\mathcal{K} = a^2E^2$ and $L = 0$, which according to [11] defines twist-free null geodesic congruence. From Carter equations we can determine its exact components in the same way as it has been done for PNC earlier from (1.25), (1.26). Carter equations (1.24) for the twist-free congruence read

$$\Delta\Sigma p^t = \mathcal{A}E, \tag{2.14a}$$
$$(\Sigma p^r)^2 = (r^2 + a^2)^2\, E^2 - \Delta a^2E^2, \tag{2.14b}$$
$$(\Sigma p^\theta)^2 = a^2E^2 - a^2E^2\sin^2\theta = a^2E^2\cos^2\theta, \tag{2.14c}$$
$$\Delta\Sigma p^\phi = 2MraE. \tag{2.14d}$$

By solving these equations we have the following relations for the components of the congruence

$$p^t = \frac{\mathcal{A}}{\Delta\Sigma}\, E, \tag{2.15a}$$
$$p^r = \pm\frac{\sqrt{(r^2+a^2)^2 - \Delta a^2}}{\Sigma}\, E, \tag{2.15b}$$
$$p^\theta = \pm\frac{a\cos\theta}{\Sigma}\, E, \tag{2.15c}$$
$$p^\phi = \frac{2Mra}{\Delta\Sigma}\, E. \tag{2.15d}$$

[3]Having a totally antisymmetric tensor $T_{\mu_1\ldots\mu_k}$ $(2\le k\le 4)$. The (pseudo-)tensor ${}^*T^{\alpha_1\ldots\alpha_{4-k}} := \frac{1}{k!}\epsilon^{\alpha_1\ldots\alpha_{4-k}\mu_1\ldots\mu_k}T_{\mu_1\ldots\mu_k}$ is called its Hodge dual [2].

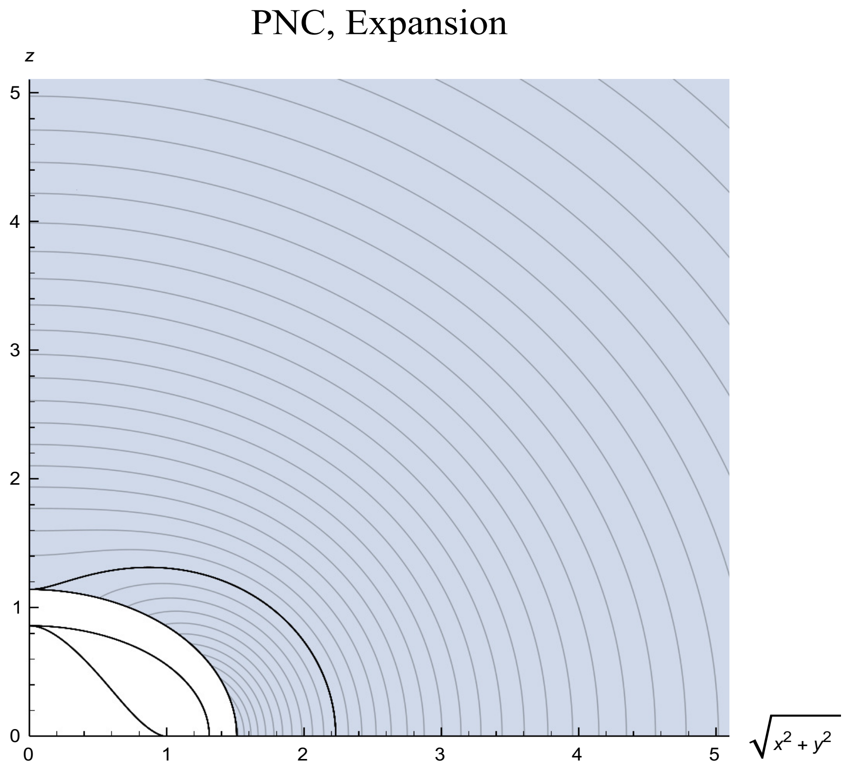

Figure 2.17: Surfaces $\Theta_{\mathrm{PNC}} = \mathrm{const}$, $a = 0.99$.

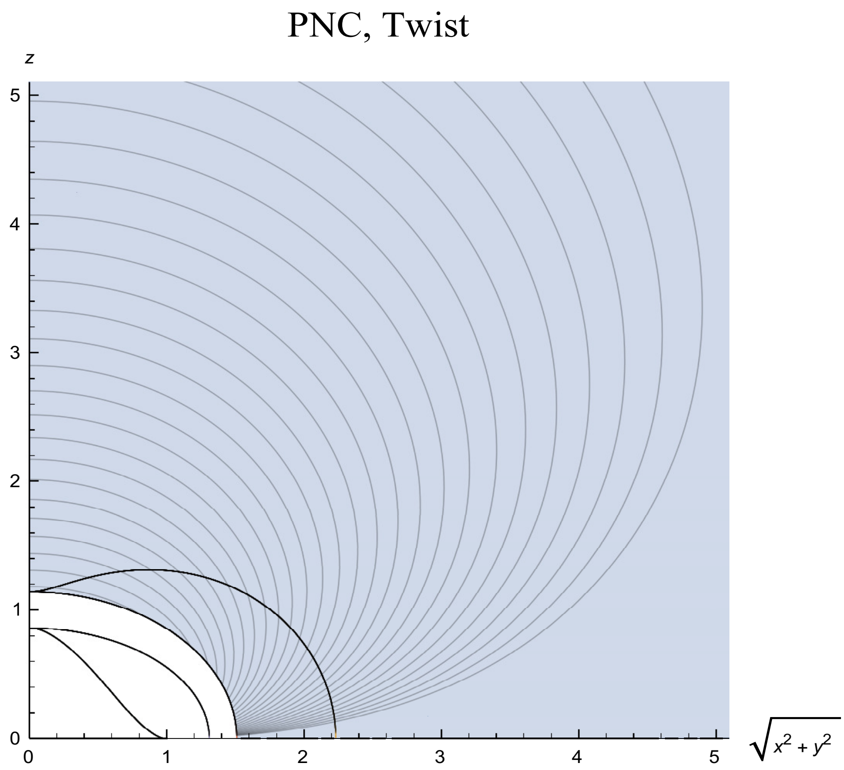

Figure 2.18: Surfaces $\omega^2_{\mathrm{PNC}} = \mathrm{const}$, $a = 0.99$.

Note how both the θ and r-components in (2.15) have two solutions. Two solutions for the radial component define outgoing k^μ and ingoing l^μ congruences in the radial direction, same as it was for PNC. With double solution for the θ-component, however, we have two versions of this congruence that are equally valid - one with positive θ-component and one with negative θ-component for the definite choice of sign r.

In order to use relations (1.38), (1.40) for the calculation of the expansion and shear for the twist-free congruence, we will assume normalization condition $k^\mu l_\mu = -1$. From the normalization it follows for energy $E = \frac{\Delta\Sigma}{2\mathcal{A}}$. In the same way as it has been done for PNC in (1.26), we will multiply the ingoing congruence l^μ with the factor $E = \frac{\Delta\Sigma}{2\mathcal{A}}$ and use factor $E = 1$ for the outgoing congruence k^μ. In total, the twist-free congruence reads

$$
\begin{aligned}
k^\mu &= \frac{1}{\Sigma}\left(\frac{\mathcal{A}}{\Delta}, \sqrt{(r^2+a^2)^2-\Delta a^2}, \pm a\ \cos\theta,\ \frac{2Mra}{\Delta}\right), \\
k_\mu &= \left(-1, \frac{1}{\Delta}\sqrt{(r^2+a^2)^2-\Delta a^2}, \pm a\ \cos\theta,\ 0\right), \\
l^\mu &= \frac{1}{2}\left(1, -\frac{\Delta}{\mathcal{A}}\sqrt{(r^2+a^2)^2-\Delta a^2}, \pm\frac{\Delta}{\mathcal{A}}\ a\cos\theta,\ \omega\right), \\
l_\mu &= \frac{\Sigma\Delta}{2\mathcal{A}}\left(-1, -\frac{\sqrt{(r^2+a^2)^2-\Delta a^2}}{\Delta}, \pm a\ \cos\theta,\ 0\right),
\end{aligned}
\tag{2.16}
$$

with $\omega = \frac{2Mra}{\mathcal{A}}$.

Two equivalent options for the outgoing twist-free congruence are

$$
\begin{aligned}
k^\mu_{\theta_+} &= \frac{1}{\Sigma}\left(\frac{\mathcal{A}}{\Delta}, \sqrt{(r^2+a^2)^2-\Delta a^2}, a\ \cos\theta,\ \frac{2Mra}{\Delta}\right), \\
k^\mu_{\theta_-} &= \frac{1}{\Sigma}\left(\frac{\mathcal{A}}{\Delta}, \sqrt{(r^2+a^2)^2-\Delta a^2}, -a\ \cos\theta,\ \frac{2Mra}{\Delta}\right).
\end{aligned}
\tag{2.17}
$$

Analogously it would be for the ingoing congruence.

Let us picture $k^\mu_{\theta_+}$, $k^\mu_{\theta_-}$ congruences. For this purpose we will need to find their integral curves. Within the (r, θ) meridional plane, integral curves for $k^\mu_{\theta_+}$, $k^\mu_{\theta_-}$ are given by the equation

$$
\frac{\mathrm{d}r}{\mathrm{d}\theta} = \pm\frac{\sqrt{(r^2+a^2)^2-\Delta a^2}}{a\ \cos\theta},
\tag{2.18}
$$

with plus or minus sign corresponding to the plus or minus version of the congruence. By numerical integration of (2.18) we obtain integral curves for $k^\mu_{\theta_+}$, $k^\mu_{\theta_-}$ congruences. Integral curves for congruences $k^\mu_{\theta_+}$, $k^\mu_{\theta_-}$ are pictured in figures 2.19, 2.20. Note how the outgoing $k^\mu_{\theta_+}$ congruence starts at the axis and the outgoing $k^\mu_{\theta_-}$ congruence starts at the singularity.

From (1.38), (1.40) we find the expansion and shear optical scalars for both versions of the congruence. The expansion scalar Θ_{TF} for the twist-free (TF) congruence is given by

$$
\Theta_{\mathrm{TF}} = \frac{(Ma^2 + a^2 r + 2r^3) \pm a\ \cos2\theta\ \sin\theta^{-1}\sqrt{(r^2+a^2)^2-\Delta a^2}}{\Sigma\sqrt{(r^2+a^2)^2-\Delta a^2}}.
\tag{2.19}
$$

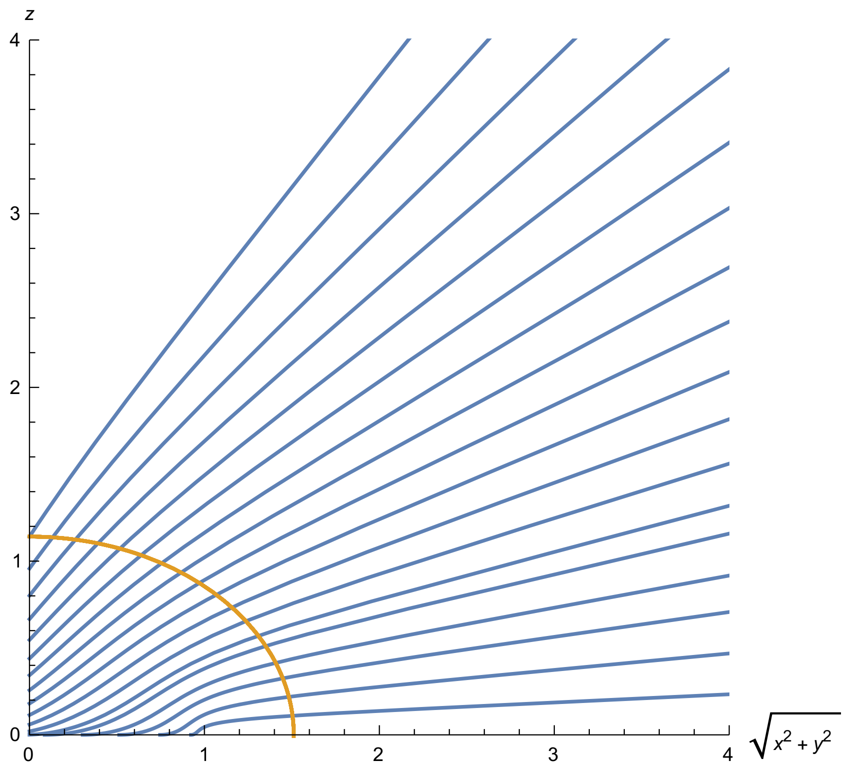

Figure 2.19: Integral curves for $k^{\mu}_{\theta_+}$ congruence, $a = 0.99$

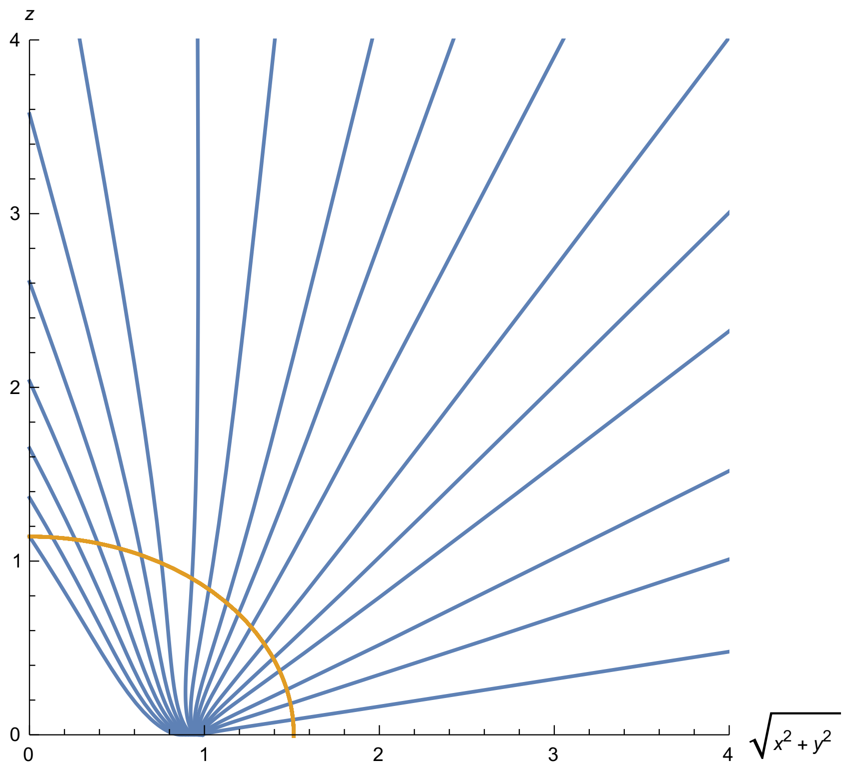

Figure 2.20: Integral curves for $k^{\mu}_{\theta_-}$ congruence, $a = 0.99$

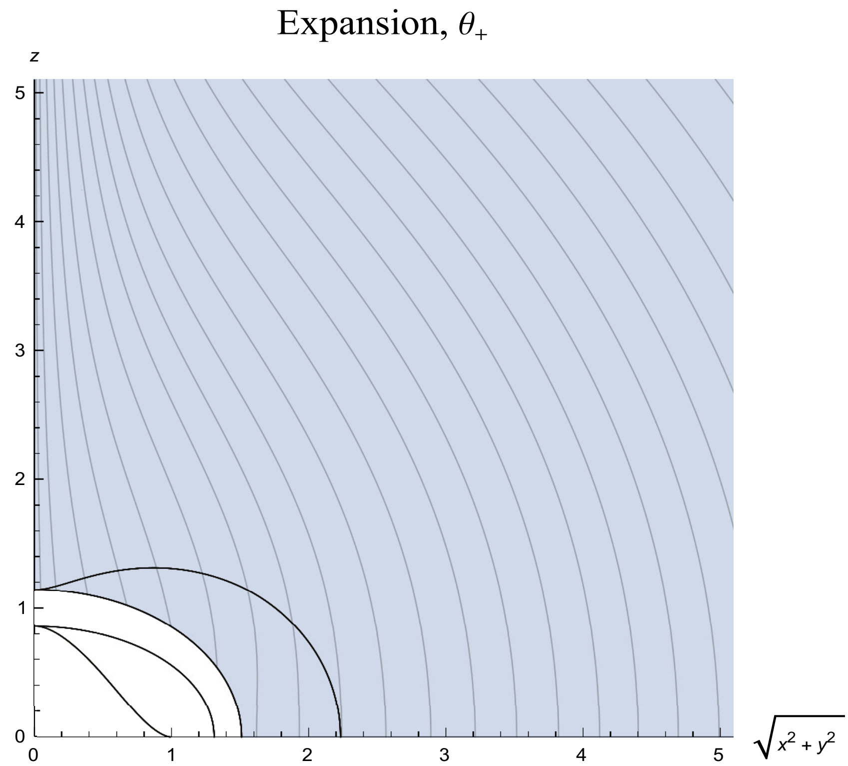

Figure 2.21: Surfaces $\Theta_{\mathrm{TF}} = \mathrm{const}$ for $k^{\mu}_{\theta_+}$-congruence, $a = 0.99$.

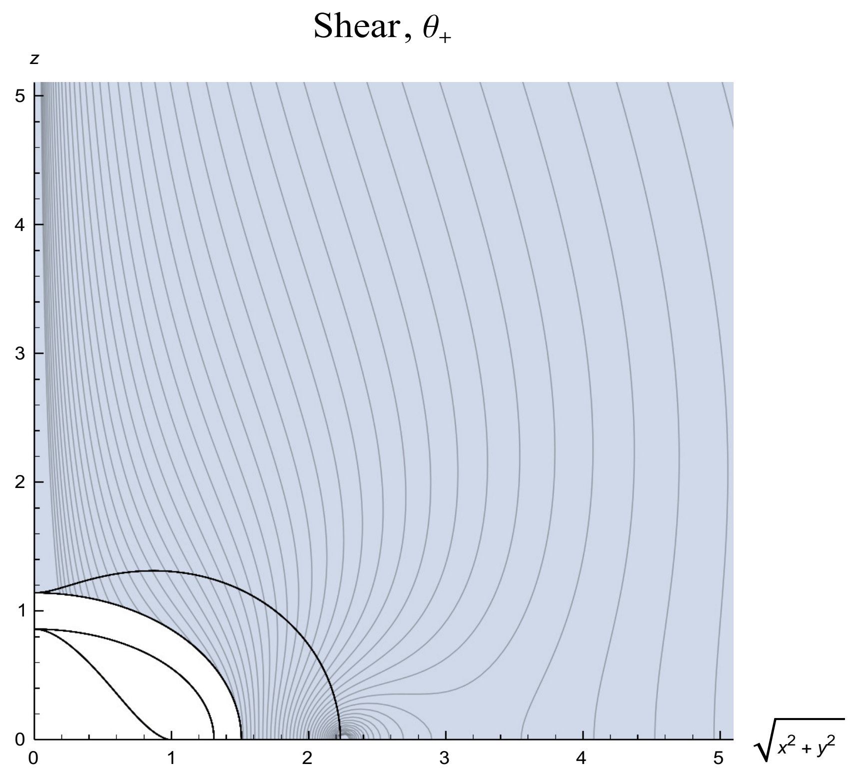

Figure 2.22: Surfaces $\sigma^2_{\mathrm{TF}} = \mathrm{const}$ for $k^{\mu}_{\theta_+}$-congruence, $a = 0.99$.

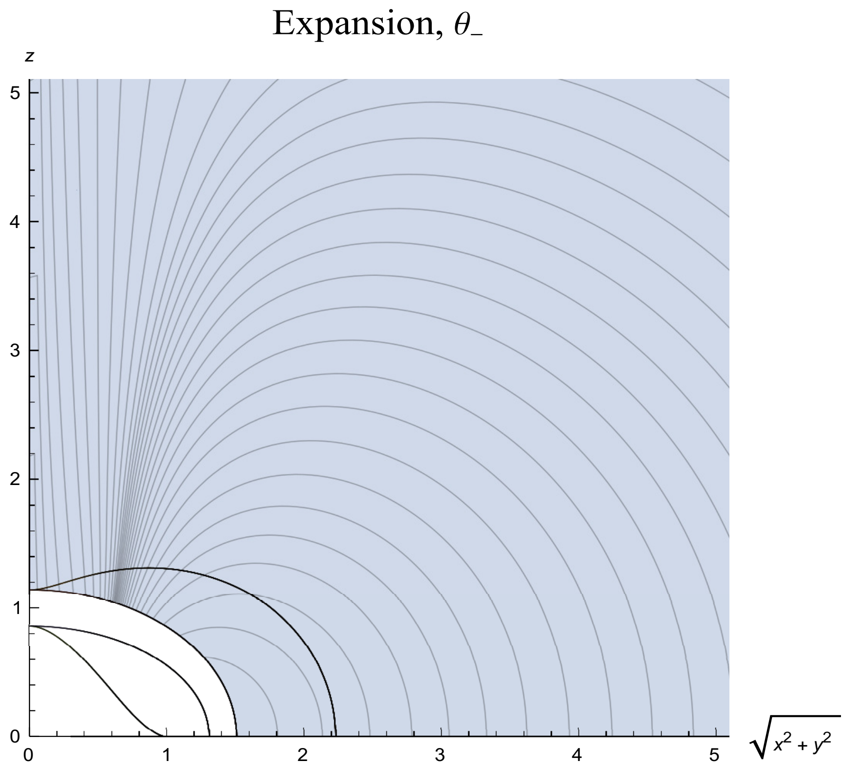

Figure 2.23: Surfaces $\Theta_{\mathrm{TF}} = \mathrm{const}$ for $k^{\mu}_{\theta_-}$-congruence, $a = 0.99$.

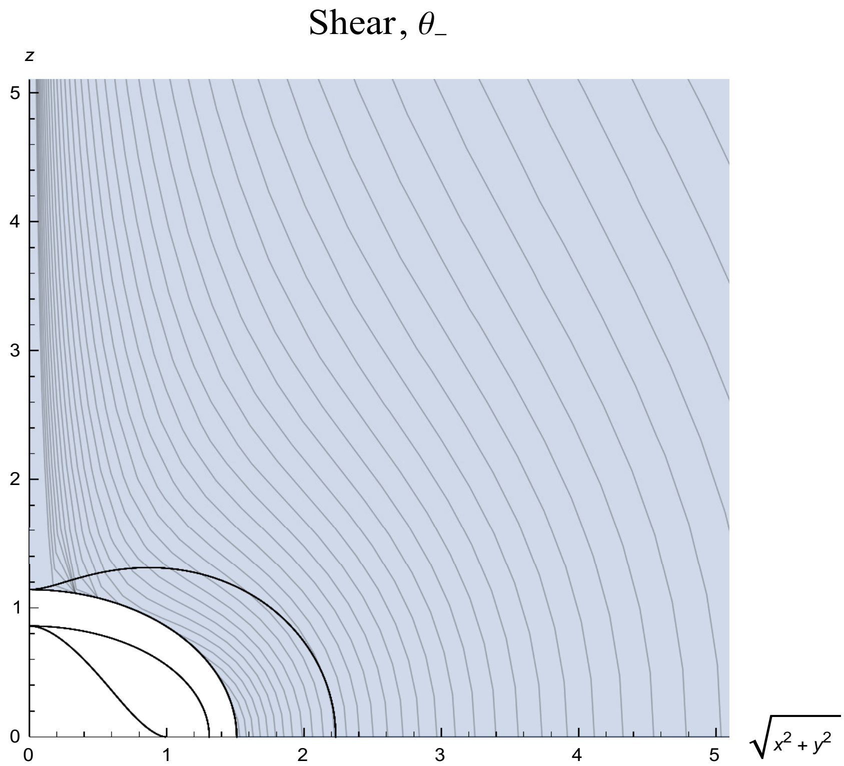

Figure 2.24: Surfaces $\sigma^2_{\mathrm{TF}} = \mathrm{const}$ for $k^{\mu}_{\theta_-}$-congruence, $a = 0.99$.

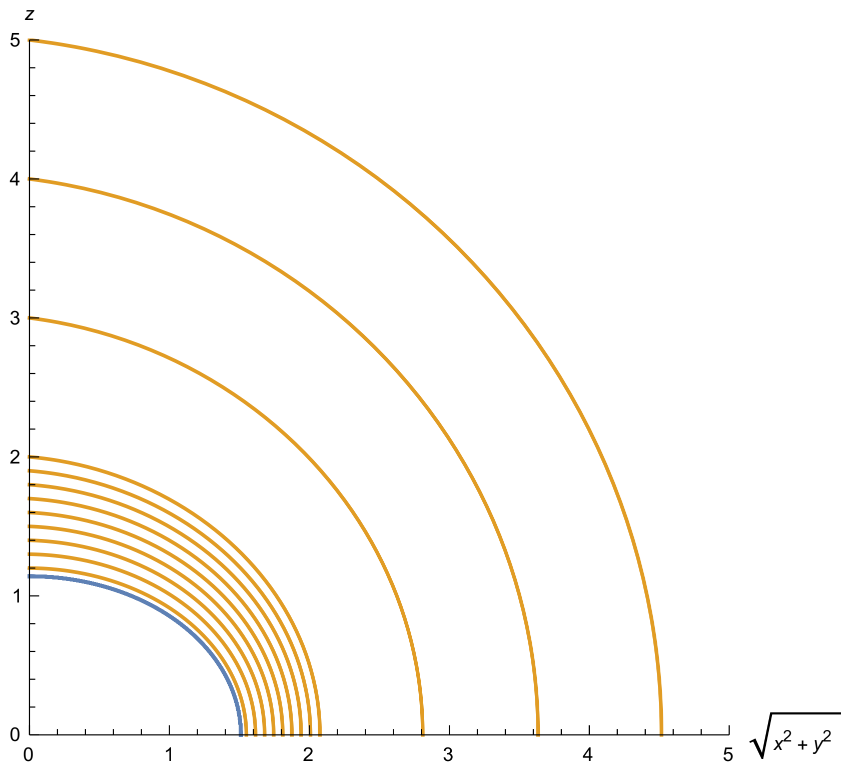

Figure 2.25: Globally orthogonal surfaces to the twist-free $k^{\mu}_{\theta_+}$ congruence (blue line marks the horizon), $a = 0.99$.

Sign option in (2.19) corresponds to plus $k^{\mu}_{\theta_+}$ or minus $k^{\mu}_{\theta_-}$ version of the twist-free congruence.

An expression for the shear scalar calculated with *xAct* is rather long and complicated, so we will not quote a formula for it here, but plot it directly. Surfaces of the constant optical scalars $\Theta_{\text{TF}} = \text{const}$, $\sigma^2_{\text{TF}} = \text{const}$ for the plus/minus versions of the congruence are pictured in figures 2.21, 2.22, 2.23, 2.24.

2.4 Globally orthogonal surfaces to the light-like congruences

2.4.1 PNC

From the previous section we learned that PNC has a non-zero twist and, thus, according to Frobenius theorem, there do not exist $3D$ hypersurfaces globally orthogonal to that congruence. Nevertheless, there may exist $2D$ surfaces globally orthogonal to PNC. Indeed, (1.37) defines a $2D$ metric on a surface orthogonal to the congruence, our task will be to find a surface $r = r(\theta)$ corresponding to that particular $2D$ metric.

The $2D$ metric orthogonal to PNC defined by (1.27), (1.28) reads

$$\mathrm{d}S^2 = \frac{\sin^2\theta}{\Sigma}\left[(r^2 + a^2)\mathrm{d}\phi - a\ \mathrm{d}t\right]^2 + \Sigma \mathrm{d}\theta^2. \tag{2.20}$$

We wish to embed this surface in the hypersurface $t = \text{const}$ of the Kerr space-time. The $t = \text{const}$ space is decribed by the metric

$$\mathrm{d}\sigma^2 = g_{rr}\mathrm{d}r^2 + g_{\theta\theta}\mathrm{d}\theta^2 + g_{\phi\phi}\mathrm{d}\phi^2, \tag{2.21}$$

and restriction to any given axisymmetric surface $r = r(\theta)$ means that $\mathrm{d}r$ in the metric should be expressed as follows

$$\mathrm{d}r \equiv \mathrm{d}r(\theta) = \frac{\mathrm{d}r}{\mathrm{d}\theta}\mathrm{d}\theta.$$

Hence, the metric given by (2.21) becomes

$$\mathrm{d}\sigma^2 = g_{rr}\left(\tfrac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2\mathrm{d}\theta^2 + g_{\theta\theta}\mathrm{d}\theta^2 + g_{\phi\phi}\mathrm{d}\phi^2 = \left[g_{rr}\left(\tfrac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + g_{\theta\theta}\right]\mathrm{d}\theta^2 + g_{\phi\phi}\mathrm{d}\phi^2. \tag{2.22}$$

A surface $r = r(\theta)$ is the surface in question. Hence, knowing the exact metric on the two dimensional surface, we can demand metrics (2.20) and (2.22) to equal, equate them, and then, by comparing coefficients standing at $\mathrm{d}\theta^2$ and integrating the resulting equation for $\frac{\mathrm{d}r}{\mathrm{d}\theta}$, we can obtain a surface $r = r(\theta)$.

By comparing coefficients standing at $\mathrm{d}\theta^2$ in metrics (2.20) and (2.22), we can easily observe that $\frac{\mathrm{d}r}{\mathrm{d}\theta} = 0$, which means that the two dimensional surface orthogonal to PNC is $r(\theta) = \text{const}$. Surfaces $r = \text{const}$ in Kerr-Schild coordinates are plotted as ellipsoids $r = 0.25, 0.5, 0.75, ..$ in the lower panel of figure 1.2.

2.4.2 Twist-free congruence

In the same way we can find a two dimensional surface globally orthogonal to the twist-free congruence. First, we should derive a $2D$ metric given by (1.37), orthogonal to the congruence. Since the twist-free outgoing congruence can be found in two variations $k^\mu_{\theta_+}$ and $k^\mu_{\theta_-}$, an ingoing version of each congruence should be its opposite not only in r, but also in θ-direction. Hence, for the outgoing congruence $k^\mu_{\theta_+}$ its ingoing counterpart would be $l^\mu_{\theta_-}$, and for the outgoing congruence $k^\mu_{\theta_-}$ its ingoing counterpart would be $l^\mu_{\theta_+}$. From (1.37) for the $2D$ metric orthogonal to each version of the twist-free congruence it follows

$$\mathrm{d}S^2 = \frac{\mathcal{A}\sin^2\theta}{\Sigma}(\mathrm{d}\phi - \omega\mathrm{d}t)^2 + \frac{\Sigma}{\mathcal{A}}\left[a\cos\theta\ \mathrm{d}r \mp \sqrt{(r^2+a^2)^2 - \Delta a^2}\ \mathrm{d}\theta\right]^2, \tag{2.23}$$

where minus/plus sign in square brackets belongs to $k^\mu_{\theta_+}/k^\mu_{\theta_-}$ version of the congruence. Equating metrics (2.22) and (2.23), for the coefficients standing at $\mathrm{d}\theta^2$ we have

$$\begin{aligned}
\left[g_{rr}\left(\tfrac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + g_{\theta\theta}\right]\mathrm{d}\theta^2 &= \frac{\Sigma}{\mathcal{A}}\left[a\cos\theta\ \mathrm{d}r \mp \sqrt{(r^2+a^2)^2 - \Delta a^2}\ \mathrm{d}\theta\right]^2,\\
\left[g_{rr}\left(\tfrac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + g_{\theta\theta}\right]\mathrm{d}\theta^2 &= \frac{\Sigma}{\mathcal{A}}\left[a\cos\theta\ \tfrac{\mathrm{d}r}{\mathrm{d}\theta} \mp \sqrt{(r^2+a^2)^2 - \Delta a^2}\right]^2\ \mathrm{d}\theta^2,\\
g_{rr}\left(\tfrac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + g_{\theta\theta} &= \frac{\Sigma}{\mathcal{A}}\left[a\cos\theta\tfrac{\mathrm{d}r}{\mathrm{d}\theta} \mp \sqrt{(r^2+a^2)^2 - \Delta a^2}\right]^2,
\end{aligned}$$

$$\begin{aligned}
&\frac{\Sigma}{\Delta\mathcal{A}}\left[(r^2+a^2)^2 - \Delta a^2\right]\left(\tfrac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \pm \frac{2\Sigma}{\mathcal{A}}\ a\cos\theta\sqrt{(r^2+a^2)^2 - \Delta a^2}\left(\tfrac{\mathrm{d}r}{\mathrm{d}\theta}\right) + \frac{\Sigma\Delta}{\mathcal{A}}\ a^2\cos^2\theta = 0,\\
&\left[\sqrt{\frac{\Sigma}{\Delta\mathcal{A}}}\sqrt{(r^2+a^2)^2 - \Delta a^2}\left(\tfrac{\mathrm{d}r}{\mathrm{d}\theta}\right) \pm a\cos\theta\sqrt{\frac{\Sigma\Delta}{\mathcal{A}}}\right]^2 = 0.
\end{aligned} \tag{2.24}$$

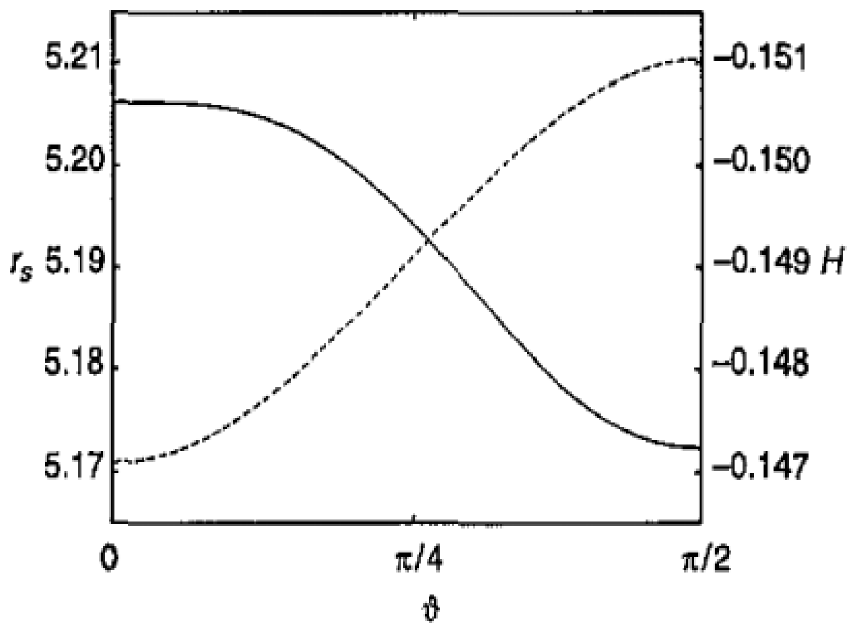

Figure 2.26: The dependence of the radial coordinate r_S on θ for a surface with constant mean curvature $H = -0.150$ (bold line) along with the plot of the mean curvature H of a surface with constant $r = 5.190$ (dotted line). Adopted from [12].

Solution for $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ is

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \mp \frac{\Delta\ a\ \cos\theta}{\sqrt{(r^2 + a^2)^2 - \Delta a^2}}, \tag{2.25}$$

where minus/plus sign belongs to $k^{\mu}_{\theta_+}/k^{\mu}_{\theta_-}$ version of the congruence.

By numerical integration of (2.25), we can obtain orthogonal surfaces to both versions of the twist-free congruence. Globally orthogonal surfaces to the $k^{\mu}_{\theta_+}$ version of the twist-free congruence are plotted in figure 2.25.

2.5 Remark on the radial coordinate

At the end we would like to mention the article [12], where authors have studied mean curvature H of surfaces $r = \text{const}$ and surfaces defined by constant mean curvature $H = \text{const}$. In their research, authors proved that surfaces $H = \text{const}$ given by constant mean curvature are only slightly different from surfaces $r = \text{const}$, which is demonstrated in figure 2.26 reproduced from [12].

This is an important result in terms of our understanding of meaning of the r-coordinate in Boyer-Lindquist coordinates. As it follows from integral (1.7) calculated as a surface integral at $t = \text{const}$, $r = \text{const}$, r no longer stays for surface radius as it was in spherically symmetric Schwarzschild. Nevertheless, we still understand the Boyer-Lindquist radius r as defining ”spherical” surfaces in a reasonable approximation. The observation that the surfaces $H = \text{const}$ are very close to $r = \text{const}$ supports such a view.

Conclusion

In the current thesis we have studied different aspects of the Kerr geometry in Boyer-Linquist coordinates. In the first chapter we reviewed fundamental properties of the Kerr metric, which describes vacuum stationary axisymmetric solution to Einstein equations, such as symmetries, metric singularities, static limits. The rotation of the center introduces the phenomenon of dragging of inertial systems by the geometry, which forces us to introduce different possibilities for observers to be non-orbiting in such a situation. We reviewed special congruences of observers, i.e. Carter observers, static observers and zero angular momentum observers (ZAMO), which represent different notions of standing in the Kerr geometry. In the remaining of the first chapter we also mentioned geodesic motion in the Kerr space-time, defined principal null congruence (PNC) and introduced hydrodynamic characteristics of time-like and light-like congruences.

In the second chapter we have calculated and verified formulae of the 4-acceleration components for stationary observers given in [9]. With these formulae in hand, we have calculated and plotted surfaces of the constant magnitude of 4-acceleration for stationary observers, which gave us an idea of how these observers are attracted by the center. We have plotted equipotentials of the magnitude of a^μ for the three types of black holes - a generic black hole, an extreme black hole and a naked singularity. We have continued by evaluating equipotential surfaces of hydrodynamic scalars for stationary observers and verified that e.g. ZAMO have zero vorticity as it has been expected, since this congruence is orthogonal to hypersurfaces of $t = \text{const}$. In what follows we have also calculated invariants given by Killing and Killing-Yano tensors, plotted surfaces of constant energy with respect to infinity and surfaces of the same radial distance from the horizon.

In the last sections we have probed optical scalars for two light-like congruences, one of which was principal null congruence and the other was twist-free congruence given by [11]. The twist-free congruence demonstrated an interesting pattern, due to its non-zero θ-component, this congruence is split into two congruences by the sign of θ, i.e. the congruence with positive sign of θ and the congruence with negative sign of θ. We have found that integral curves for the twist-free congruence with positive sign of θ start at the axis and integral curves of the twist-free congruence of negative sign start at the singularity. We have also found two dimensional orthogonal surfaces for the null congruence and for the twist-free congruence. In the final remark we have pointed out that the r-coordinate in Boyer-Lindquist coordinates is a meaningful choice of a radial coordinate.

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List of Figures