Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

DIPLOMOVÁ PRÁCE

Bernard Lidický

Varianty problému obarvení

Variants of the coloring problem

Katedra aplikované matematiky

Vedoucí diplomové práce: RNDr. Jiří Fiala, Ph.D.

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Děkuji Jiřímu Fialovi za vedení práce a DIMACSu z Rutgers University za podporu při práci. Dále pak Martinu Tancerovi za uvedení do problematiky.

Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

V Praze dne Bernard Lidický

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Autor: Bernard Lidický
Katedra (ústav): Katedra aplikované matematiky
Vedoucí diplomové práce: RNDr. Jiří Fiala, Ph.D.
e-mail vedoucího: fiala@kam.mff.cuni.cz

Abstrakt: V předložené práci studujeme seznamové barvení rovinných grafů. Seznamové barvení je varianta problému barvení grafu, kde každý vrchol má přidělený svůj vlastní seznam možných barev. Říkáme, že graf je k-vybíravý, je-li možné nalézt dobré obarvení pokaždé, když všechny seznamy obsahují alespoň k barev.

Je známo, že každý rovinný graf bez trojúhelníků je 4-vybíravý a každý rovinný bipartitní graf (t.j. bez lichých cyklů) je 3-vybíravý. Práce ukazuje postačující podmínky pro 3-vybíravost rovinných grafů bez trojúhelníků s omezeným výskytem krátkých cyklů.

Klíčová slova: seznamové barvení, rovinné grafy, krátké cykly

Title: Variants of the coloring problem
Author: Bernard Lidický
Department: Department of Applied Mathematics
Supervisor: RNDr. Jiří Fiala, Ph.D.
Supervisor's e-mail address: fiala@kam.mff.cuni.cz

Abstract: The choice number is a graph parameter that generalizes the chromatic number. In this concept vertices are assigned lists of available colors. A graph is k-choosable if it can be colored whenever the lists are of size at least k. It is known that every planar graph without triangles is 4-choosable and there is an example of a non-3-choosable planar graph without triangles. In this work we study the choice number of planar graph without triangles and other short cycles.

Keywords: list coloring, planar graph, short cycles

Chapter 1

Introduction

Graph coloring is an assignment of colors to every vertex of a graph from a given set of possible colors. A list coloring is a generalization of coloring where every vertex has its own list of possible colors. The counterpart of the chromatic number in coloring is the choice number in the list coloring.

The concept of list coloring was invented in late 70's. The choice number of planar graphs got a great research interest in the last decade. The most important theorems on the choice number of planar graphs were discovered at that time. Several new results appeared also in last few years and the choice number is becoming an interesting topic in these years too.

In this work we survey the known results about the choice number of planar graphs, namely triangle-free graphs. We show that it is possible to get some sufficient conditions on the choice number of triangle-free graphs and exhibit an example of triangle-free graph with larger choice number. The same construction was independently found also by Montassier [6].

The work is divided in three chapters. In the first chapter we introduce some basic definitions and observations. In the second chapter we survey known results about the choice number of planar graphs. In the last chapter we show some sufficient conditions for a small choice number as well as the already mentioned construction of a graph with larger choice number.

A part of the work was at SVOČ 2007 where it was awarded by an honorable mention.

Chapter 2

Basic definitions

We start with recalling some basic definitions about graphs. Then we continue with coloring and, finally, we define the concept of list coloring and present some basic observations about the list coloring.

2.1 Graphs

Definition 2.1. A *graph* or a *simple graph* is a pair (V, E) where V is a finite set and E is a subset of $\binom{V}{2}$.

Elements of V are called *vertices* and elements of E are called *edges*. Usual letters for denoting a graph are G and H. Note that we do not allow any multiple edges, loops or oriented edges; refer to Figure 2.1.

We call a pair of vertices u and v *adjacent* if $\{u, v\}$ is an edge. All vertices adjacent to a vertex v are called the *neighbors* of v. The number

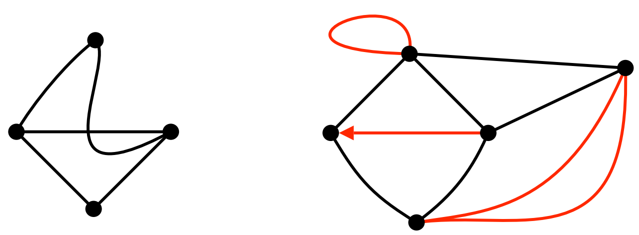

Figure 2.1: A simple graph on the left and a graph with a loop, a multi-edge and an oriented edge on the right.

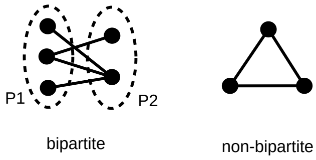

Figure 2.2: A bipartite graph with parts P_1 and P_2 and a non-bipartite graph.

of neighbors of v is called the *degree* of v. We denote the degree of v by $\deg(v)$. The maximum degree over all vertices in a graph G is denoted by $\Delta(G)$.

Definition 2.2. A graph $H = (V', E')$ is called a *subgraph* of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq \binom{V'}{2} \cap E$. The graph H is an *induced subgraph* of G if $E' = \binom{V'}{2} \cap E$.

Let $H = (V', E')$ be a subgraph of $G = (V, E)$. Then $G \setminus H$ is the graph $(V \setminus V', E \setminus \{\{u, v\} : u \in V, v \in V'\})$.

Definition 2.3. A graph $G = (V, E)$ is *bipartite* if V can be partitioned into two sets P_1 and P_2 such that there is no edge containing vertices only from one set; refer to Figure 2.2.

Note that a bipartite graph contains no odd cycle as a subgraph.

Here we define a notation used for the most common graphs:

- *cycle* $C_n = (V, E)$ where $|V| = n$ and $E = \{\{v_i, v_{i+1}\} : v_i \in V, i \in [n-1]\} \cup \{v_1, v_n\}$
- *complete graph* $K_n = (V, E)$ where $|V| = n$ and $E = \binom{V}{2}$
- *complete bipartite graph* $K_{n,m} = (V, E)$ where $V = P_1 \cup P_2, |P_1| = n, |P_2| = m$ and $E = \{\{p_1, p_2\} : p_1 \in P_1 \ \& \ p_2 \ \in P_2\}$

Refer to Figure 2.3 for an example of C_n, K_n and $K_{n,m}$.

We say that vertices u and v are in the same connected component of a graph G if there is a sequence of vertices $u = v_0, v_1, \dots v_{k-1}, v_k = v$ such that $\{v_i, v_{i+1}\} \in E$ for all $i \in \{0, 1, \dots k-1\}$. Note that being in the same connected component is an equivalence relation on vertices and

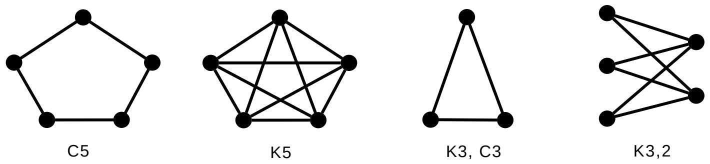

Figure 2.3: C_5, K_5, K_3, C_3 and $K_{3,2}$.

there is no edge containing vertices from different components. We say that a graph is *connected* if it has only one connected component.

A graph is *planar* if it is possible to draw it in the plane such that vertices are represented by distinct points and every edge is represented by a simple Jordan curve with two endpoints corresponding to the vertices of the edge. Moreover the edges can intersect only in their endpoints and curves for edges are not allowed to go through a vertex.

There are many possible drawings of a planar graph in the plane. A graph with a fixed drawing is called a *plane graph.* A plane graph G cuts the plane into several connected regions. These regions are called *faces* of G. A set of all faces is usually denoted by F; refer to Figure 2.4

Euler stated an important theorem about the number of faces in a planar connected graph.

Theorem 2.4. *For every connected plane graph $G = (V, E)$ with set of faces F the following formula holds*

$$|E| + 2 = |V| + |F|.$$

It is possible to prove the formula by induction[1]. An important consequence is that for a planar graph the number of faces does not depend on a particular drawing of the graph in the plane.

2.2 Coloring

Definition 2.5. A *coloring* of a graph $G = (V, E)$ is a mapping c from V to the set C of possible colors. A coloring is *proper* if no pair of adjacent vertices have the same color; refer to Figure 2.5

[1] János Komlós told me that there were hundreds of proofs for this formula.

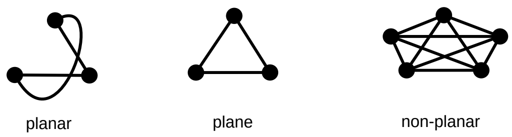

Figure 2.4: A planar graph, a plane graph and a non-planar graph. The plane graph has two faces — the inner triangle and the outer unbounded face.

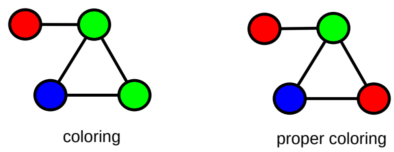

Figure 2.5: A coloring and a proper coloring of a graph.

Definition 2.6. A graph G is *k-colorable* if k colors are sufficient for the existence of a proper coloring of G.

It is obvious that every graph with n vertices is n-colorable since we can assign a unique color to every vertex and such coloring is proper. Therefore, we can define the *chromatic number* of a graph G to be the minimum k such that G is k-colorable. The chromatic number of G is denoted by $\chi(G)$.

Next we present two basic observations about an upper bound on $\chi(G)$.

Observation 2.7. *For every graph G the following formula holds*

$$\chi(G) \leq \Delta(G) + 1$$

Proof. Let G be a graph. We can find a suitable coloring using a greedy algorithm and color the vertices one by one in an arbitrary order. Observe that every vertex v has at most $\Delta(G)$ neighbors. Hence v has at most $\Delta(G)$ forbidden colors. Thus there is still at least one color left for v. □

Observation 2.8. *Every bipartite graph is 2-colorable.*

Proof. It is possible to color the first set of bipartition with the first color and the second set with the second color. There is no conflict in the coloring since edges are only between the sets. □

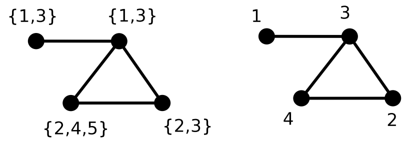

Figure 2.6: A graph with color lists and a possible list coloring.

2.3 List coloring

A one set of possible colors is common to all vertices it the concept of graph coloring. Some applications require that some vertices are not allowed to colored by every color. So rather than a set of colors common to all vertices, every vertex v has its own list of possible colors $L(v)$.

Definition 2.9. Let $G = (V, E)$ be a graph and $L(v)$ list of colors for a vertex v. A *list coloring* of G is a proper coloring c such that $c(v) \in L(v)$ for every $v \in V$.

The concept of list coloring was defined independently by Vizing [9] and Erdős, Rubin and Taylor [2].

We say that a graph G is k*-choosable* if G can be properly colored whenever $L(v) \geq k$ for every vertex v. A minimum such k is called the *choice number*of G. We denote the choice number of G by $\chi_l(G)$. Note that in some literature the choice number is denoted by $ch(G)$.

Observe that the coloring problem is a special case of the list coloring problem where all lists have the same content. Thus if a graph is k-choosable then it is also k-colorable. Therefore,

$$\chi(G) \leq \chi_l(G).$$

We show the same upper bound on $\chi_l(G)$ as the bound in Observation 2.7 on $\chi(G)$.

Observation 2.10.

$$\chi_l(G) \leq \Delta(G) + 1$$

The proof of this observation is analogous to the proof of Observation 2.7.

The next observation shows that the gap between $\chi(G)$ and $\chi_l(G)$ can be arbitrary large since $\chi(G) \leq 2$ for every bipartite graph G.

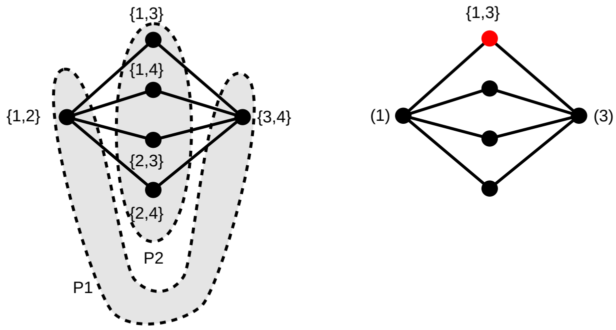

Figure 2.7: The construction of a non-2-choosable planar graph and conflict in coloring.

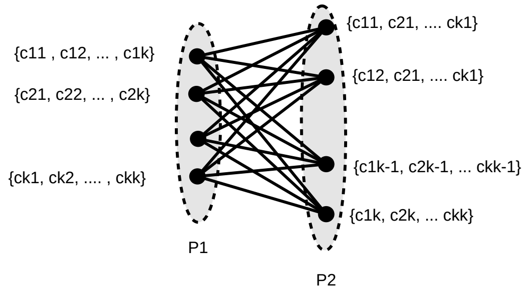

Figure 2.8: The construction of a non-k-choosable graph.

Observation 2.11. *There are bipartite graphs with arbitrary large choice number.*

Proof. First we show a construction of a non-2-choosable bipartite graph. Then we show a generalization to a non-k-choosable graph. A non-2-choosable graph is depicted in Figure 2.7. Vertices in P_1 must get some color in any coloring. For every combination of colors in P_1 there is a vertex in P_2, which is in conflict with the coloring of vertices in P_1.

For general non-k-choosable graph refer to Figure 2.8. The graph is a complete bipartite graph K_{k,k^k} with bipartition P_1 and P_2. Every vertex in P_1 gets list of unique colors. Every vertex in P_2 gets a combination of colors from lists of vertices in P_1. The vertices in P_2 get different combinations and $|P_2|$ is exactly the number of combinations of colors from P_1. Hence every possible coloring of P_1 is in a conflict with the coloring of one vertex in P_2. Thus the graph is not k-choosable.

□

Chapter 3

Known results for planar graphs

In this chapter we present the most interesting and the most important theorems and constructions for the choice number of planar graphs. We focus on planar graphs without short cycles as it is the main subject of our work.

3.1 Planar graphs

Erdős, Rubin, and Taylor [2] conjectured that there exists a planar graph which is not 4-choosable, and also that every planar graph is 5-choosable. Both conjectures appeared to be true.

First we present a theorem by Thomassen [7] that every planar graph is 5-choosable.

Theorem 3.1 (Thomassen [7]). *Every planar graph is 5-choosable.*

Proof. Assume that G is a plane connected graph with an outer face of an arbitrary size. Assume also that all inner faces are triangles. We can get such a graph just by adding chords to large inner faces if the original G is not of this kind.

Let G have at most 2 vertices. Then G is clearly 5-choosable. So assume that G has at least 3 vertices.

Assume that $|L(v)| \geq 5$ for every inner vertex v, two adjacent vertices a

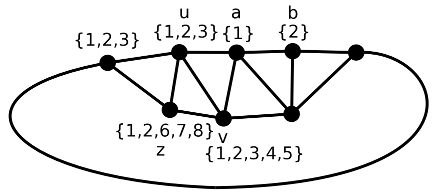

Figure 3.1: The graph G in induction of Thomassen's theorem.

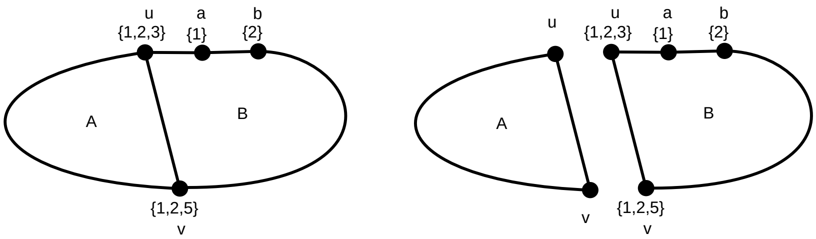

Figure 3.2: Splitting G into parts A and B along the edge $\{u, v\}$.

and b in the outer face f of G are precolored [1], and all other vertices of f have lists of size 3; refer to Figure 3.1.

The proof is done by induction. For the first step consider a graph with 3 vertices. In this case we have two precolored vertices a and b and one more vertex v with $|L(v)| = 3$. The vertices form a triangle and there is at least one possible color for v, thus we can properly color the whole graph G.

In the induction step we try to split G into two smaller graphs or to remove a neighbor of the vertex a.

First we examine the splitting case. Let $\{u, v\}$ be an edge where u and v are vertices of the outer face; refer to Figure 3.2. Then we can split G into two parts A and B along $\{u, v\}$. By induction we find a coloring of the part containing the vertices a and b (part B). We precolor u and v in A by colors from B and use induction on A. Observe that the colorings of A and B can be combined together to obtain a coloring of G since u and v get the same colors in both parts.

For the other case assume that no such edge $\{u, v\}$; refer to Figure 3.1. Let u be a vertex of the outer face adjacent to a with $|L(u)| = 3$. There are at least two different colors available for u since a is precolored.

[1] The vertices a, b have lists of size 1. The lists must be different.

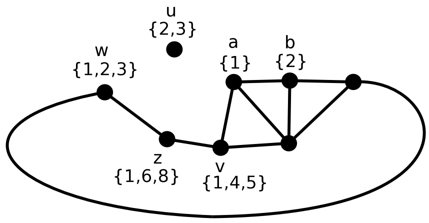

Figure 3.3: G' with new colors lists and the separated vertex u.

Reserve these two colors for u and remove them from color lists of all neighbors of u that are not in the outer face. We get a graph $G' = G \backslash \{u\}$. There may be some vertex z in the outer face of G' with more than 3 colors in its color list when $|L(z) \cap L(u)|$ is of size at most 1. In such case we simply reduce the size of $L(z)$ to 3 arbitrarily; refer to Figure 3.3.

We color G' by induction. Then we extend a coloring of G' to G. The only uncolored vertex is u with two reserved colors. These two colors do not conflict with coloring of a or with any inner vertex. The only possible conflict is with the other neighbor w in the outer face. The vertex w has assigned one color and thus at least one color remains for u and the coloring can be extended.

□

The other conjecture about the existence of a non-4-choosable graph was answered by Voigt [10]. She constructed a non-4-choosable planar graph on 238 vertices.

Theorem 3.2 (Voigt [10]). *A planar non-4-choosable graph exists.*

Proof. Our goal is to present a planar graph G with a list assignment where every list is of size 4 and show that the graph cannot be properly colored. We start with two vertices u and v and give them disjoint lists of colors. For example $L(u) = \{1, 2, 3, 4\}$ and $L(v) = \{5, 6, 7, 8\}$. In every coloring c the vertices u and v must get the some color. We connect u and v by 16 copies of graph H where for every coloring of u and v some copy of H fails to be colorable; refer to Figure 3.4.

Next we show the graph H. Assume that $c(u)$ is 1 and $c(v)$ is 4; refer to Figure 3.5. The side vertices x and y have possible colors $\{2, 3\}$. Assume that $c(x) = 2$ and $c(y) = 3$. Then it is not possible to extend c to the triangle A. The other case fails at the triangle B.

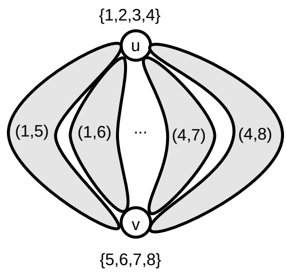

Figure 3.4: G created from vertices u, v and 16 copies of H.

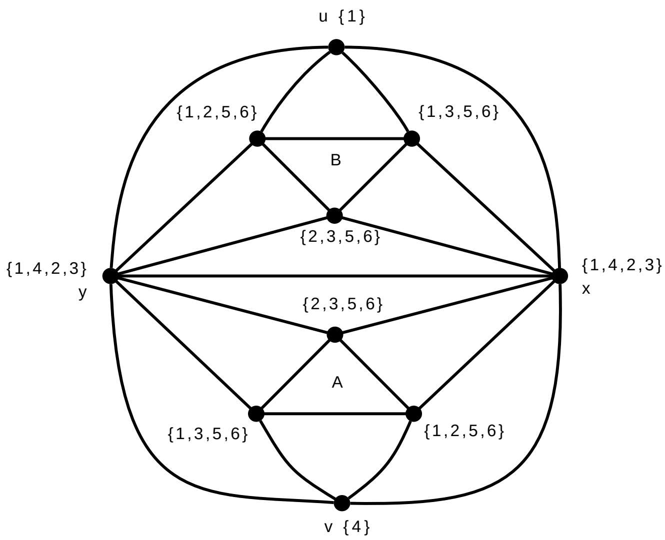

Figure 3.5: The graph H with precolored vertices u and v.

Hence the graph G is not colorable by using the given lists.

□

3.2 Planar graphs without triangles

Recall Observation 2.8 where we show that every bipartite graph is 2-colorable. On the other hand, Observation 2.11 shows that the choice number is unbounded on the class of bipartite graphs. The question about the choice number of planar bipartite graphs was answered by Alon and Tarsi. They proved that every planar bipartite graph is 3-choosable. It is the best possible bound since non-2-choosable planar bipartite graphs are known; refer to Figure 2.7.

Theorem 3.3 (Alon and Tarsi [1]). *Every planar bipartite graph is 3-choosable.*

The proof is by an algebraic method. We omit details of the proof and mention only the main idea. With every graph G we associate a polynomial on $|V|$ variables where every coloring corresponds to an evaluation of the polynomial. The polynomial is nonzero for a proper coloring of G and zero otherwise. It is possible to show that the polynomial has a nonzero solution by using some suitable orientation of the planar bipartite G.

Theorem 3.4 (Kratochvíl and Tuza [4]). *Every planar graph without triangles is 4-choosbale.*

Proof. Let G be a planar triangle-free graph. The proof is done by induction on the number of vertices of G and uses a degeneracy argument[2].

If $|V| \leq 3$ then G can be colored.

For the induction step consider the Euler's formula for triangle-free graphs. From the formula we get that $|E| \leq 2|V| - 4$. Hence G has a vertex v of degree at most 3. Thus every coloring of neighbors of v can be extended to v since $|L(v)| = 4$ and only 3 colors are forbidden. So we can remove v from G and apply induction on $G \setminus \{v\}$. Then we extend the coloring of $G \setminus \{v\}$ to G and we are done; refer to Figure 3.6. □

[2]A graph G is k-degenerated if every subgraph H of G contains a vertex of degree at most k. In this case we use the fact that G is 3-degenerated.

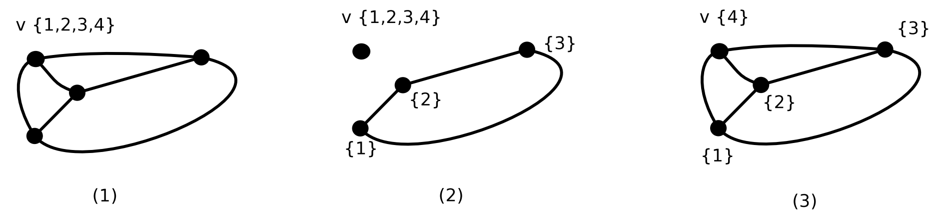

Figure 3.6: Removing of v and extending the coloring of $G \setminus \{v\}$ to G.

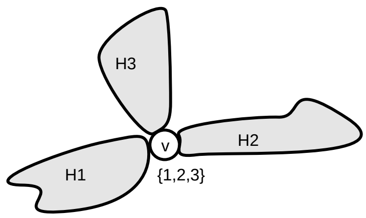

Figure 3.7: The central vertex v surrounded by 3 copies of H.

A question whether every triangle-free graph is 3-choosable arises from the previous theorem. The negative answer was given by Voigt [11] by showing a graph on 166 vertices.

Theorem 3.5 (Voigt [11]). *A planar non-3-choosable triangle-free graph exists.*

Proof. The construction of a non-2-choosable triangle-free graph G starts with one central vertex v. We assign a list of colors $\{1, 2, 3\}$ to v. We add 3 copies of some graph H for every possible coloring of v; refer to Figure 3.7. The graph H is depicted in Figure 3.8 where the color a is 1,2 or 3 depending on a copy of H.

Observe that if v has been assigned color a then one of the following colorings must be present:

- v, u, x, z: $a, 5, 6, 5$
- v, u, x, y: $a, 6, 4, 6$
- v, u, r, s: $a, 7, 5, 7$
- v, u, r, t: $a, 7, 4, 7$

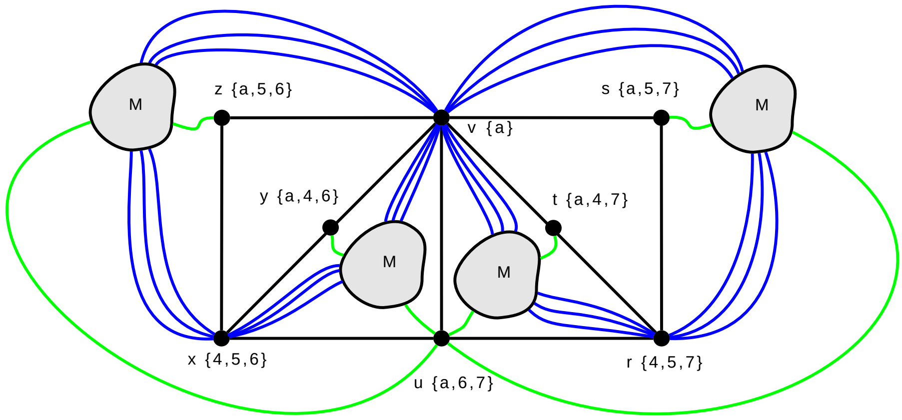

Figure 3.8: The graph Ha. The colors of edges are only of better visual connection with Figure 3.9.

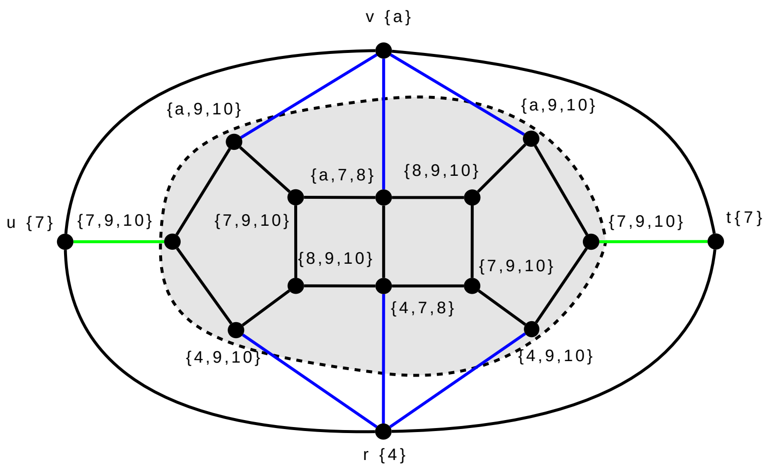

Figure 3.9: The graph M, the basic building block of G. The colors of edges are only of better visual connection with Figure 3.8.

Thus we get a 4-cycle with known colors. So we use it and we insert the graph M from Figure 3.9 to the 4-cycles as indicated in Figure 3.8.

The graph M is the basic building block of all know non-3-choosable planar graphs without triangles. The argument why M is not 3-choosable is discussed in detail in Section 4.5.

□

We are interested in planar triangle-free graphs and we explore some sufficient conditions for a triangle-free graph to be 3-choosable.

Theorem 3.6 (Thomassen [8]). *Every planar graph without C_3 and C_4 is 3-choosable.*

The proof proceeds by induction on the number of vertices. It is based on an exhaustive case study, so we omit it here.

3.3 Latest results

A different sufficient condition was given lately by Lam, Shui and Song [5]. They proved that every planar triangle-free graph without 5-cycles and 6-cycles is 3-choosable.

Theorem 3.7 (Lam, Shui and Song [5]). *Every planar graph without C_3, C_5 and C_6 is 3-choosable.*

The proof is done by the discharging method. The real result is some condition on adjacent faces. Our result can be stated in the same way but particular conditions on faces are different when compared to this result.

A construction of a smaller planar non-3-choosable triangle-free graph was given by Montassier [6]. We found the same construction independently. We describe the construction in detail in Section 4.5.

Theorem 3.8 (Montassier [6]). *There exist a planar non-3-choosable triangle-free graph on 128 vertices.*

Even smaller but less clear construction is by Glebov, Kostochka and Tashkinov [3]. The basic building parts are same as in previous result but they are put together in a more clever way.

Theorem 3.9 (Glebov, Kostochka and Tashkinov [3]). *There exist a planar non-3-choosable triangle-free graph on 97 vertices.*

Now we present several results on sufficient conditions for 3-choosability based on forbidding small cycles. For convenience we also include already mentioned results. Note that the absence of larger cycles is not always stated explicitly but it implies absence of some constellations of smaller cycles. For example forbidding C_6 implies that no two C_4 may share an edge; refer to Figure 4.2.

Theorem 3.10. *Every planar graph without*

- C_3 *and* C_4 *(Thomassen [8])*
- C_3, C_5 *and* C_6 *(Lam, Shui and Song [5]) or*
- C_3, C_6, C_7 *and* C_9 *(Zhang and Xu [14]) or*
- C_3, C_5, C_8 *and* C_9 *(Zhang and Haihui [13]) or*
- C_4, C_5, C_6 *and* C_9 *(Zhang and Wu [16]) or*
- C_4, C_5, C_7 *and* C_9 *(Zhang and Wu [15])*

is 3-choosable.

Some researchers studied also the case with allowed triangles. As their results are based on conditions of different kind[3] we do not include them in our survey.

[3]The conditions are based on restricting short cycles and restricting the distance of the triangles in a graph.

Chapter 4

New results

4.1 Preliminaries

Definition 4.1. Let G be a non-3-choosable graph. We say that an induced subgraph S of G is *reducible* if $G \setminus S$ is also non-3-choosable.

In every non-3-choosable graph we may sequentially remove reducible subgraphs and end up with a graph without any reducible subgraph.

Lemma 4.2. *All vertices of degree 1 and 2 are reducible.*

Proof. Let G be a graph. Let every vertex have a color list of size 3. Let v be a vertex of graph G of degree 1 or 2.

We show that every list coloring c of $G \setminus v$ can be extended into a list coloring of G.

The vertex v has at most two neighbors. Thus at most two different colors are forbidden for v. Hence at least one color for v remains and c can be extended. □

Definition 4.3. The *degree of a face* f is number of incident edge sides. It is denoted by $\deg(f)$; refer to Figure 4.1

Observe that $\sum_{f \in F} \deg(f) = 2e$ since every edge is counted from both sides. Also for vertices holds a similar formula: $\sum_{v \in V} \deg(v) = 2e$.

Definition 4.4. The initial charge of a face f is defined by $w(f) = \deg(f) - 6$. The initial charge of a vertex v is defined by $w(v) = 2\deg(v) - 6$.

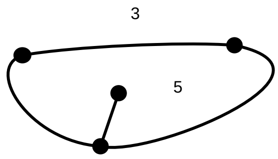

Figure 4.1: A graph with the inner face of degree 5 and the outer face of degree 3.

deg	$w(f)$	$w(v)$
1	-	-4
2	-	-2
3	-3	0
4	-2	2
5	-1	4
6	0	6
7	1	8
8	2	10

Table 4.1: Initial charges of faces and vertices depending on their degree.

Lemma 4.5. *If a graph G is planar, then the sum of all initial charges is negative.*

Proof. The idea of the proof is based on counting with Euler's formula. Let $G = (V, E)$ be a planar graph. Recall Euler's formula, which says that $|E| + 2 = |V| + |F|$ where F is the set of faces. Counting with the formula gives:

$$|E| + 2 = |F| + |V| \tag{4.1}$$

$$6|E| + 12 = 6|F| + 6|V| \tag{4.2}$$

$$2|E| - 6|F| + 4|E| - 6|V| = -12 \tag{4.3}$$

Recall that $\sum_{f \in F} \deg(f) = 2|E|$ and $\sum_{v \in V} \deg(v) = 2|E|$.

$$\sum_{f \in F} \deg(f) - 6|F| + \sum_{v \in V} 2\deg(v) - 6|V| = -12 \tag{4.4}$$

$$\sum_{f \in F} (\deg(f) - 6) + \sum_{v \in V} (2\deg(v) - 6) = -12 \tag{4.5}$$

By using the previous definition of initial charges we get

$$\sum_{f\in F} w(f) + \sum_{v\in V} w(v) \quad = \quad -12. \tag{4.6}$$

□

Even though Thomassen [8] proved a stronger result that every planar graph without C_3 and C_4 is 3-choosable, we present here the following theorem for a demonstration of the discharging method.

Theorem 4.6. *Every planar graph without C_3, C_4, and C_5 is 3-choosable.*

Proof. Assume for a contradiction that G' is a counterexample. First we remove vertices of degree 1 and 2 from G'. These vertices are reducible by Lemma 4.2. The resulting graph G is still a counterexample since removing reducible subgraphs does not change planarity or 3-choosability. The reduction of the counterexample is the only part of the proof where we involve the list coloring.

Next we argue that G is not a planar graph since the condition from Lemma 4.5 does not hold for G.

deg	$w(f)$	$w(v)$
3	-	0
4	-	2
5	-	4
6	0	6

Table 4.2: Charges in a graph without C_3, C_4, and C_5.

We take some drawing of G in the plane. The smallest face in a plane graph G may have degree 6 since smaller cycles are forbidden. The smallest degree of a vertex is 3 since vertices of degree 1 and 2 are reduced. Thus all faces and vertices in G have nonnegative initial charges; refer to Table 4.2. Hence the following formula holds.

$$0 \leq \sum_{f\in F} w(f) + \sum_{v\in V} w(v)$$

Therefore, the counterexample is not planar and the proof is finished. □

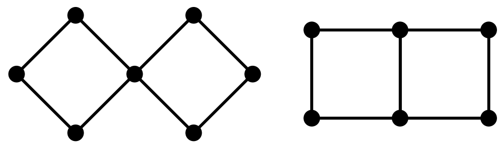

Figure 4.2: $C_{4\bullet 4}$ and $C_{4|4}$

Actually we do not need discharging to prove the previous result. Let G' be a dual[1] graph to G. The smallest face of G is at least 6-face thus the smallest degree of vertex in G' is 6. Hence G' is not planar since every planar graph has a vertex of degree at most 5.

In the next section we show how to allow some 4-cycles in the family of 3-choosable graphs.

We denote the graph consisting of two cycles sharing exactly one edge by $C_{x|y}$ where x and y are length of those two cycles. By $C_{x\bullet y}$ we denote two cycles which share at least one vertex; refer to Figure 4.2. We say that cycles are *touching* if they share exactly one vertex.

4.2 The 4-cycles

Lemma 4.7. *Every induced 4-cycle of graph G with all vertices of degree 3 in G is reducible.*

Proof. Let G be a graph where every vertex has a color list of size 3. Let C_4 be an induced 4-cycle where all vertices have degree 3.

We show that every list coloring c of $G \setminus C_4$ can be extended into a list coloring of G.

Every vertex in C_4 has one neighbor colored by c. Thus there may be one forbidden color at each vertex of C_4. Thus it is still possible to choose from two at least colors at each vertex of C_4.

Since C_4 is 2-choosable the list coloring c of $G \setminus C_4$ can be extended into a list coloring of G. □

Theorem 4.8. *Every planar graph without C_3, $C_{4\bullet 4}$, and C_5 is 3-choosable.*

[1]Vertices of G' corresponds to the faces of G and every edge $\{a, b\}$ in G' corresponds to a common edge of faces a and b in G. Note that G' is a multigraph.

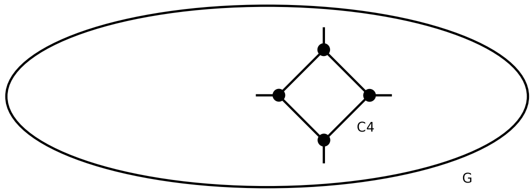

Figure 4.3: A 4-cycle with vertices of degree 3 in a graph G

Proof. Assume for a contradiction that G' is a counterexample. First we remove reducible subgraphs exposed in Lemma 4.2 and Lemma 4.7 from the graph G'. The resulting graph G is still a counterexample since removing reducible subgraphs does not change planarity or 3-choosability.

We show that G is not planar because the sum of all charges in G is nonnegative. As Lemma 4.5 claims that the sum is negative for every planar graph and we get a contradiction.

To show that the sum of all charges is nonnegative we discharge the negative charges of faces by using charges of vertices while keeping the charges of vertices nonnegative. We also show that the sum of all charges does not change during the discharging procedure.

deg	$w(f)$	$w(v)$
3	-	0
4	-2	2
5	-	4
6	0	6

Table 4.3: Initial charges without forbidden parts.

The only pieces with negative charges in G are 4-faces. All other cases with negative charges like 5-faces or vertices of degree 2 are already reduced or forbidden by the statement of the theorem.

We can assume that every 4-face f has a vertex v of degree at least 4 since 4-face without such a vertex v is reducible. Thus the charge of the vertex v is at least two. This charge at the vertex v can be used to eliminate the negative charge of f.

The description of the discharging procedure follows.

$$w'(f) = w(f) + 2$$

$$w'(v) = w(v) - 2$$

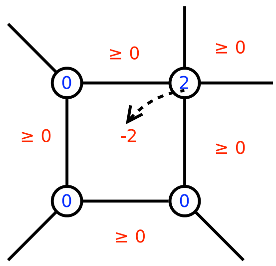

Figure 4.4: A 4-cycle and moving charges

Since the charge of every face and every vertex is nonnegative, the following formula holds:

$$0 \leq \sum_{f \in F} w(f) + \sum_{v \in V} w(v)$$

Therefore, the counterexample is not planar and the proof is finished. □

The previous theorem allows 4-cycles but the 4-cycles are separated. In the next part we show that it is possible to allow 4-cycles to share a vertex.

4.3 The 4-cycles and shared vertices

In this section we extend the class of 3-choosable planar graphs from the previous section by allowing 4-faces to share a vertex.

For the discharging process we can assume that every 4-face still has a vertex of degree at least 4 but unfortunately one such vertex may be shared by more 4-faces. Hence it is not possible to shift the charges as in the previous section and we need to exhibit some new discharging rules and reducible graphs.

First we exhibit the discharging rules. Then we argue that configurations not covered by the discharging rules are actually reducible.

A vertex v of degree at least 4 is called *non-shared* if it is a part of only one 4-face with a negative charge and $w(v) \geq 2$. We say that vertex v is *shared* if it is a part of at least two faces with negative charges.

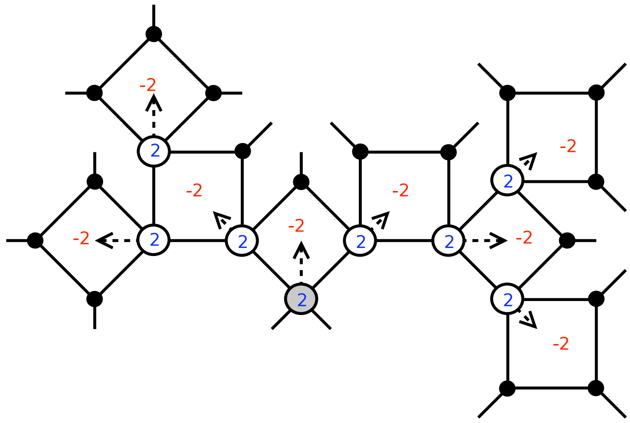

Figure 4.5: The grey non-shared vertex eliminate the negative charge of an adjacent 4-face. The other white shared vertices then become also non-shared.

Lemma 4.9. *If some 4-face f with a negative charge has a non-shared vertex v, then it is possible to discharge the charge of f by using the charge of v.*

Proof. Recall that the 4-face has charge -2. Hence, the charge of a non-shared vertex is large enough. □

Observe what happens if few 4-faces share some vertices and one of the faces contains a non-shared vertex. We can use the previous lemma several times and eliminate the negative charges of many 4-faces since some shared vertices are becoming non-shared as we eliminate the negative charges of some 4-faces; refer to Figure 4.5.

Lemma 4.10. *If some 4-faces share a vertex of degree at least 5, then it is possible to eliminate the negative charge of the 4-faces using the charge of the vertex.*

Proof. Observe that every 4-face adjacent to a shared vertex v increases the degree of v by 2 because the 4-face cannot share any edge with other 4-face. Thus every vertex shared by at least two 4-faces has degree at least 4; refer to Figure 4.6.

We show that if a vertex v is adjacent to at least three 4-faces, then its charge is large enough to eliminate the negative charges of all adjacent 4-faces. Assume that there are k touching 4-faces. Then the sum of their

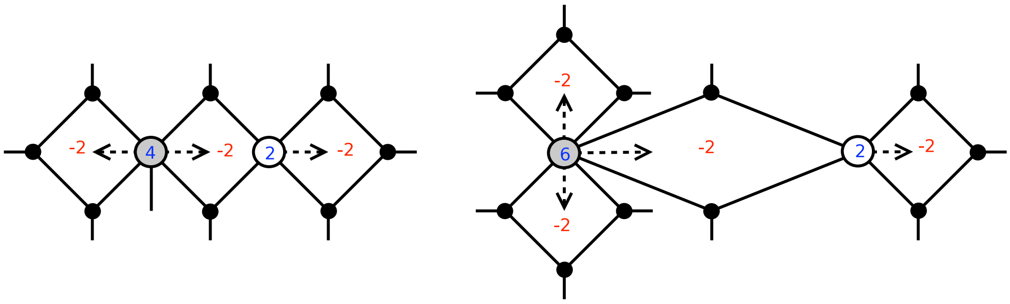

Figure 4.6: The grey shared vertices have sufficient charge to eliminate the negative charges of the adjacent 4-faces.

charges is $-2k$. The degree of v is at least $2k$. Thus the charge of v is at least $4k - 6$. Finally, for every k greater then 3 holds that $2k \leq 4k - 6$ and we can discharge the negative charges.

If there are only two 4-faces adjacent to a vertex v of degree at least 5, then it is also possible to discharge the negative charges. It follows from the fact that the two 4-faces have together charge -4 and the charge of the vertex v is at least 4. □

Lemma 4.11. *If 4-faces create a cycle, then it is possible to eliminate their negative charges.*

Proof. If the 4-faces are in a cycle, then it is possible to discharge clockwise around the cycle; refer to Figure 4.7; □

It can happen that none of the previous lemmas apply. In that case we show that the configuration of 4-faces is reducible. Observe that non-shared vertices are not present and every shared vertex has degree exactly 4. Moreover the 4-faces create a forest structure.

Lemma 4.12. *Let G be a graph where every vertex is in some 4-face, shared vertices have degree 4 and all other vertices have degrees 2 or 3. Moreover the 4-faces create no cycle and no two 4-faces share an edge; refer to Figure 4.8. If vertices of degree 2 have color lists of size at least 2 and other vertices have color lists of size at least 3, then a list coloring of G exists.*

Proof. The proof is done by induction on the number of 4-faces.

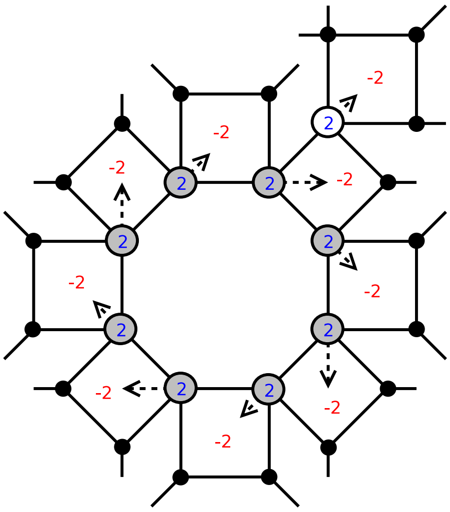

Figure 4.7: A cycle of 4-cycles and moving charges

The case of only one 4-face was already solved by Lemma 4.7.

If all 4-faces have at least two shared vertices, then we can find a cycle of 4-faces which is forbidden by the statement of the lemma. Thus there is a 4-face with at most one shared vertex. If the 4-face has no shared vertex, it may be colored as in the previous case.

Otherwise, the 4-face has exactly one shared vertex. We color the other vertices of the face such that there will remain two possible colors for the shared vertex. Then we may remove the other vertices from the graph. This decreases the number of 4-faces and the shared vertex becomes

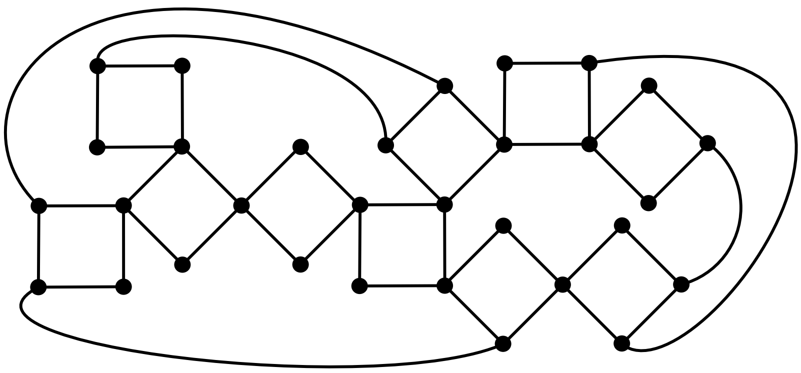

Figure 4.8: The graph from Lemma 4.12.

a vertex of degree 2 with a list of two colors. Hence we may use the induction.

Description of the coloring of non-shared vertices follows. Let the shared vertex be x, vertices adjacent to x be u and v, and the last vertex be w; refer to Figure 4.9. We distinguish two cases.

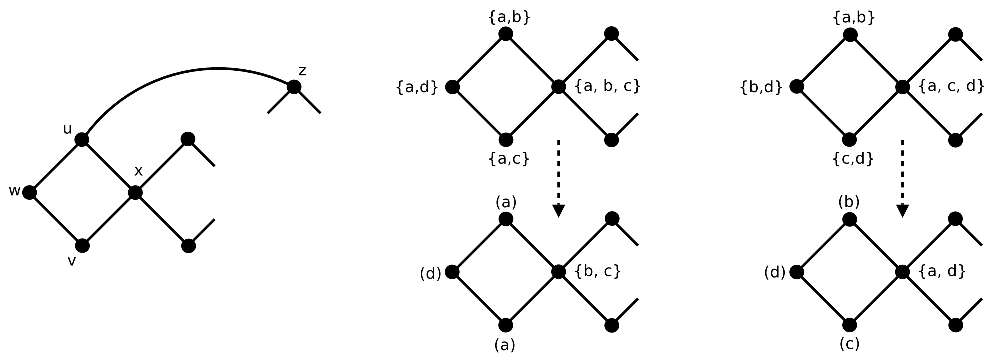

Figure 4.9: The 4-face in the induction step.

If a is a common color from the lists of u and v the we color both u and v by a. Since u and v have the same color a, there are still at least two colors left in color list of x and at least one color is left in color list of w. Thus we may assign a color to w and preserve a color list of size 2 for x.

In the other case the color lists of u and v are disjoint. Their color lists have 4 different colors together while the color list for x has 3 different colors. Hence there is a color b which is not in the color list of x. Without loss of generality assume that b is in the color list of u. We color u by the color b. Then we color w by a color distinct from b. Finally we color v by a color different from the color of w. The color of v may be present in the color list of x, but there are still at least 2 colors left in the color list of x.

We also need to deal with vertices connected to vertices u, v, and w, if there are any. Let z be a vertex connected to vertex u. The vertex z has a color list of size 3. So we can remove the color of u from the color list of z to avoid color conflict. The vertex z is then treated as a vertex of degree 2. □

Theorem 4.13. *Every planar graph without C_3, $C_{4|4}$, and C_5 is 3-choosable.*

Proof. Observe that the previous lemmas were allowing 4-cycles to share a vertex whereas the statement of this theorem allows them also to share two edges. A possible configuration is depicted in Figure 4.10. We claim that two 4-cycles with two edges in common are not present it the reduced graph in a form of two 4-faces since there would be a vertex of degree 2.

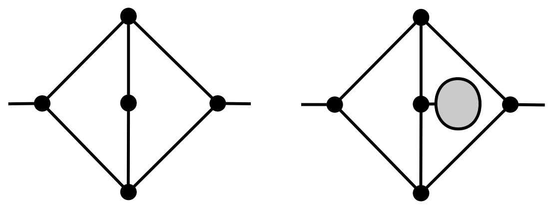

Figure 4.10: Two 4-cycles share a vertex of degree two or they are not both 4-faces.

The rest of the proof is done by a contradiction. Let there be a counterexample G. If G contains some reducible subgraphs, we can remove them while G is still a planar counterexample.

The negative charges are present only at 4-faces. Due to previous lemmas we shift the charges and eliminate all the negative charges to nonnegative.

Therefore we have a contradiction with the fact that sum of all charges is negative for every planar graph. □

Observe that reduced graph without C_6 does not contain $C_{4|4}$ in form where the both cycles can be faces. Thus they do not affect counting charges and we say that planar graph without C_3, C_5, and C_6 is 3-choosable as a consequence.

4.4 Allowing 5-cycles

We would like to extend the previous class of 3-choosable graphs such that it would allow also graphs with some 5-cycles. We are not going to show any reducible subgraphs containing 5-cycles. Hence we have to add more discharging rules.

The 4-faces use charges from vertices to eliminate their own negative charge. Thus we must shift the charges from vertices to 5-faces carefully. On the other hand, 4-faces do not use charges from faces. Hence we may use the charges from large faces for the 5-faces exclusively.

A problematic configuration would be a 5-face surrounded with 4-faces, 5-faces or 6-faces since there are no positive charges around the 5-face and the 5-face with all vertices of degree 3 is not reducible.

To avoid the problematic configuration we place some conditions on neighbors of 5-cycles. We forbid $C_{4|5}$, $C_{5|5}$, and $C_{5|6}$.

Theorem 4.14. *Every planar graph without* C_3*,* $C_{4|4}$*,* $C_{4|5}$*,* $C_{5|5}$*, and* $C_{5|6}$ *is 3-choosable.*

Proof. Assume for a contradiction that G' is a counterexample. Recall Lemmas 4.2 and 4.7 about reducibility of vertices of a small degree and reducibility of configuration of 4-cycles. Apply these Lemmas on the graph G'. The resulting reduced graph G is still a counterexample since removing reducible subgraphs does not change planarity or 3-choosability.

We claim that there are no two 4-faces, 4-face and 5-face, two 5-faces and 5-face and 6-face that would share two edges because G does not contain any vertices of degree two or triangles; refer to Figure 4.11.

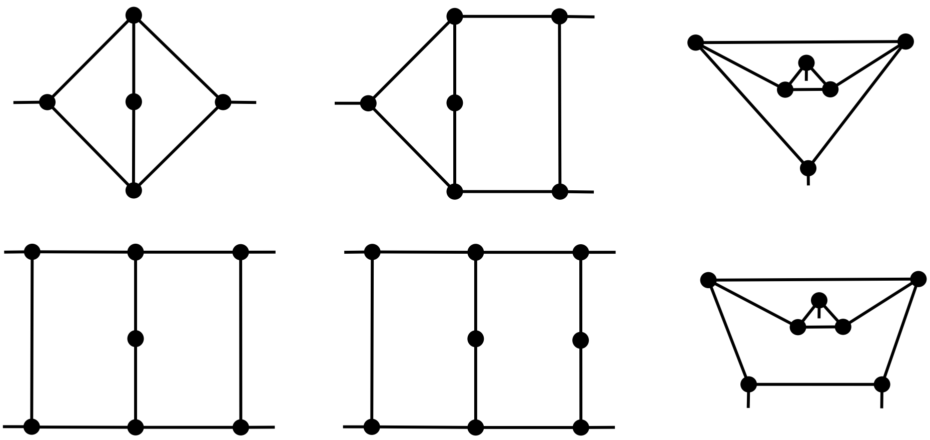

Figure 4.11: Two small cycles drawn as a faces that share two edges require a vertex of degree two or triangles.

We assume that 4-faces have nonnegative charges since we can apply Lemma 4.9 - 4.11 and eliminate the negative charges of all 4-faces.

Next we show that we can eliminate the negative charges of 5-faces while maintaining nonnegative charges of 4-faces.

We say that a 5-face is isolated if it share no vertex with any other 5-face.

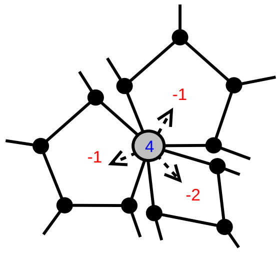

Figure 4.12: A vertex shared by two 5-faces and a 4-face.

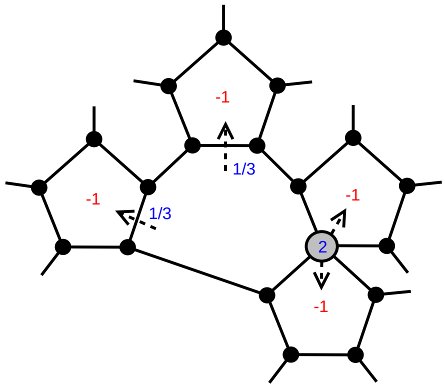

Figure 4.13: A vertex shared by two 5-faces can eliminate their negative charge.

First we discuss the case of non-isolated faces. Let some l 5-faces share a vertex v. Note that v can belong also to some k 4-faces. Since $C_{4|4}$, $C_{4|5}$ and $C_{5|5}$ are forbidden every 4- and 5-face contributes two edges to the degree of v. Hence $\deg(v) \geq 2k + l$ and the initial charge of v is at least $4k + 4l - 6$. The negative charge of 4-faces and 5-faces can be eliminated whenever the charge of v is at least $k + 2l$. Observe that $4k + 4l - 6 \geq k + 2l$ whenever $k \geq 2$. Refer to Figure 4.12. Hence we may assume that the charge of all non-isolated 5-faces is nonnegative.

A negative charge of an isolated 5-face can be eliminated by using the charge of surrounding large faces. Assume that some k-face share some edges with some isolated 5-faces. Note that the number of isolated 5-faces is at most $\lfloor k/2 \rfloor$ since the 5-faces share no vertices; refer to Figure 4.13.

Recall that the initial charge of any k-face is $k - 6$. We redistribute the charge from f to 5-faces and each 5-face receives at least $(k - 6)/\lfloor k/2 \rfloor$.

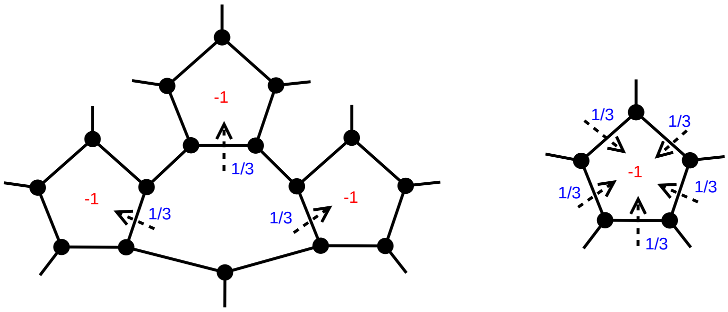

Figure 4.14: Distributing charge 1 from a 7-face to surrounding 5-faces and incoming charges to a 5-face.

The degree k of the face f is at least 7 since lower values are forbidden by the statement of the theorem. Thus each 5-face receives charge at least 1/3; refer to Figure 4.14.

Every isolated 5-face is adjacent to five large faces which are sending some charges to the 5-face. Hence the sum of all received charges by the 5-face is at least 5/3. Recall that the initial charge of every 5-face is -1. Thus the resulting charge of the 5-face is at least 2/3.

Therefore after the discharging every face will receive a nonnegative charge and the proof is finished. □

Note that there is still a possibility to extend the proof. You may notice that we have changed the negative charges of some 5-faces to a positive value instead of only nonnegative value. It would be suffice if a 5-face receives charges only from three adjacent faces since every transfer is at least 1/3.

We cannot allow $C_{4|5}$ or $C_{5|5}$ since we use the absence of these subgraphs in the first part of the proof where we use charges from vertices. On the other hand, $C_{5|6}$ is forbidden only because we need to transfer charges to 5-faces from the surrounding faces. Thus the assumption we really need instead of forbidding $C_{5|6}$ is that a C_5 can have at most two common edges with 6-faces.

4.5 Smaller construction

Voigt [11] gave a construction of a non-3-choosable planar graph of girth 4. We present the construction in Theorem 3.5. We found a similar construction of non-3-choosable planar graph of girth 4 but our construction has fewer vertices. Such a graph is a counterexample for the conjecture that every triangle free graph is 3-choosable. The original construction has 166 vertices and our construction has only 128 vertices. The same construction as our was independently found by Montassier [6].

When one is trying to find a construction for some coloring problem, it is possible to start with a constructions where one vertex v is precolored. If you manage to find such a construction, then you can take a few copies of the construction and glue them together by the vertex v. Every copy excludes a different case of a possible coloring of v. This is approach used by Voigt.

An other possibility is to create constructions with two precolored vertices. Then glue the constructions together by two vertices. Generally it is easier to design a gadget if you have two precolored vertices instead of only one precolored vertex. Since we want a planar graph we require that both precolored vertices are in the outer face of the gadget. Otherwise the gadgets cannot be glued together.

We use the second possibility for our construction.

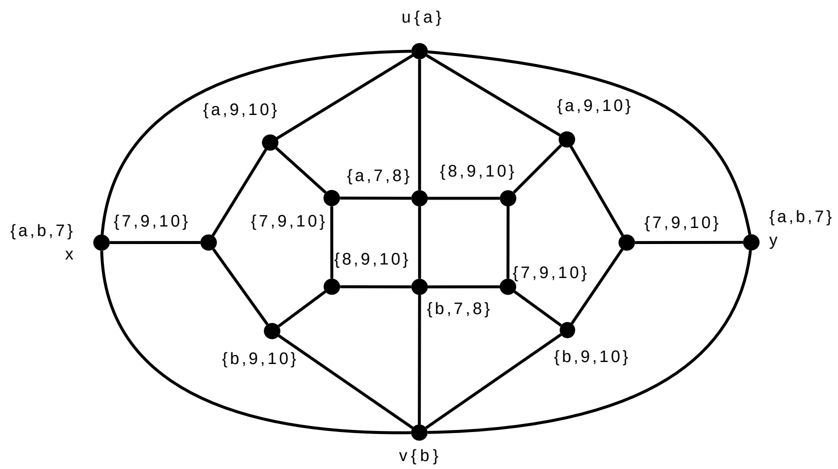

Figure 4.15: The gadget where vertices u and v are precolored.

The gadget with two precolored vertices u and v is depicted in Figure

4.15. We will argue that the gadget cannot be properly colored using given lists of colors.

Vertices x and y are both connected to vertices u and v. Thus colors a and b cannot be used on x and y. Hence the only possible color for x and y is 7 and we may consider them as precolored vertices too.

Next we can remove colors from list of vertices adjacent to precolored vertices. The graph after clearing the color lists is depicted in Figure 4.16.

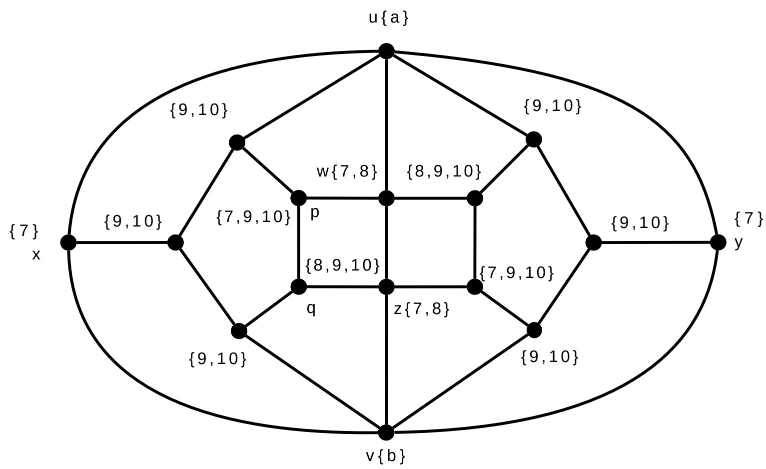

Figure 4.16: The construction with cleared color lists.

Every list coloring c assigns some color to vertices w and z.

Suppose that $c(w) = 7$ and $c(z) = 8$. Then colors 7 and 8 are forbidden for vertices p and q. Thus we get an odd cycle in the left part where the color list at each vertex is $\{9, 10\}$. Therefore we get a contradiction since the cycle cannot be properly colored.

The argument for the other case where $c(w) = 8$ and $c(z) = 7$ is analogous.

Finally we glue the gadgets together. We take 9 copies of the graph in Figure 4.15: $G_{(1,4)}, G_{(1,5)}, \ldots G_{(3,4)}, G_{(3,5)}$. In a copy $G_{(i,j)}$ we replace colors a and b by i and j. Then we identify vertices u and v of these copies and assign color lists $\{1, 2, 3\}$ to u and $\{4, 5, 6\}$ to v; refer to Figure 4.17. The whole construction of a planar non-3-choosable graph without triangles is finished.

Therefore we have proved the following theorem.

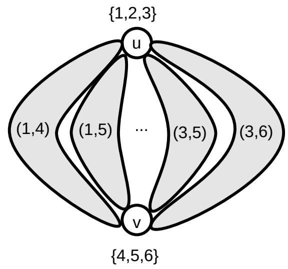

Figure 4.17: Gluing 9 smaller constructions together and creating the resulting graph.

Theorem 4.15. *A planar non-3-choosable triangle-free graph on 128 vertices exists.*

Some authors consider a variant of the list coloring problem called (p, q, r)-choosability. The parameter p is the size of every color list, q is the number of colors assigned to each vertex and r is the size of the intersection of two list if the corresponding vertices are adjacent.

The graph from Voigt [11] can be slightly modified to become an example of planar triangle-free graph which is not $(3, 1, 2)$-choosable [12, pages 67-76].

The graph from the previous theorem is also an example of a non-$(3, 1, 2)$-choosable planar triangle-free graph.

Chapter 5

Conclusion

We have studied the choice number of planar graphs without triangles and other small cycles.

Most of the latest sufficient conditions for 3-choosability were obtained by using the discharging method where negative initial charges were on vertices. We have showed that it is possible to get interesting results also with negative initial charges only on faces. We also independently found a construction of a planar non-3-choosable graph.

As an open problem we have the question whether a planar triangle-free graph without 5-cycles which is not 3-choosable exists or whether all these graphs are 3-choosable.

We are interested if there is a planar triangle-free graph without 5-cycles which is not 3-choosable or wether such a graph is always 3-choosable.

Several conditions for a planar planar graph without short cycles to be 3-colorbale were exposed in other works. Maybe some of them could be extended to 3-choosability.

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