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Gauss' calculation of Ceres' orbit

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Abstract: In 1801, a Sicilian astronomer Giuseppe Piazzi discovered a faint star-like object, which disappeared in the glare of the Sun after six weeks of observations. A year later, the object named Ceres was rediscovered thanks to calculations provided by Carl Friedrich Gauss. This thesis aims to present the full process of Gauss's contribution to the discovery of Ceres with focus on elementary and straightforward mathematical language. We introduce some basic notions and properties concerning elliptical orbits and develop a thorough account of Gauss's calculations. We also provide justification for multiple unexplained steps in the method.

Keywords: Ceres, elliptical orbits, Gauss's method

Contents

Introduction

Ceres is a dwarf planet orbiting the Sun in the asteroid belt between Mars and Jupiter. On the first day of 1801, it was discovered as an unknown faint object by Sicilian astronomer Giuseppe Piazzi. After six weeks of observations, Piazzi lost sight of the dwarf planet due to the glare of the Sun. Thus, the community of European astronomers was handed a delightful problem. Determine a future position of Ceres based on a limited series of Piazzi's observations made during 41 days. The catch was that from Piazzi's point of view, Ceres traversed an arc spanning only a few degrees on the celestial sphere during this period. At the time, there was no general method to determine an orbit from a set of observations made in such a short time interval and spanning such a short arc.

Over the summer of 1801, some of the leading mathematicians of Europe including Laplace tried to tackle the problem. Though they had calculus at their disposal, the puzzle seemed unsolvable. Unsolvable only until 24 years old Carl Friedrich Gauss decided to try his own luck. On the first day of 1802, another astronomer Franz Xaver von Zach definitely identified Ceres at the precise location predicted by Gauss [Marsden, 1977, p. 313]. This text is the story of how the young mathematician succeeded.

Gauss chose only three observations spaced out as evenly as possible and based his method upon relationships between the values attached to each of them. In this work, we aim to present his results in a hopefully straightforward and modern mathematical language. For that, we also need to introduce fundamentals of general elliptical orbits. In the first chapter, we define and describe how to determine precise parameters (called *elements*) of a planetary orbit used by astronomers to predict future positions of a planet. Throughout the chapter, we assume that we have been given two vectors completely describing the positions of the studied planet with respect to a particular coordinate system. The procedure of determining the orbit from two such vectors was fairly well-established in 1801 [Teets and Whitehead, 1999, p. 88].

The second chapter describes the main contribution of Gauss: how to use the three chosen observations to obtain the two vectors assumed to be known in the first chapter. He determined the necessary values based on nothing more than Piazzi's observations, Kepler's laws and rather elementary geometry. Though he did not use any particularly complex ideas, the complexity of the way he interconnected them can be overwhelming. To explain how he did it will take up the majority of this work.

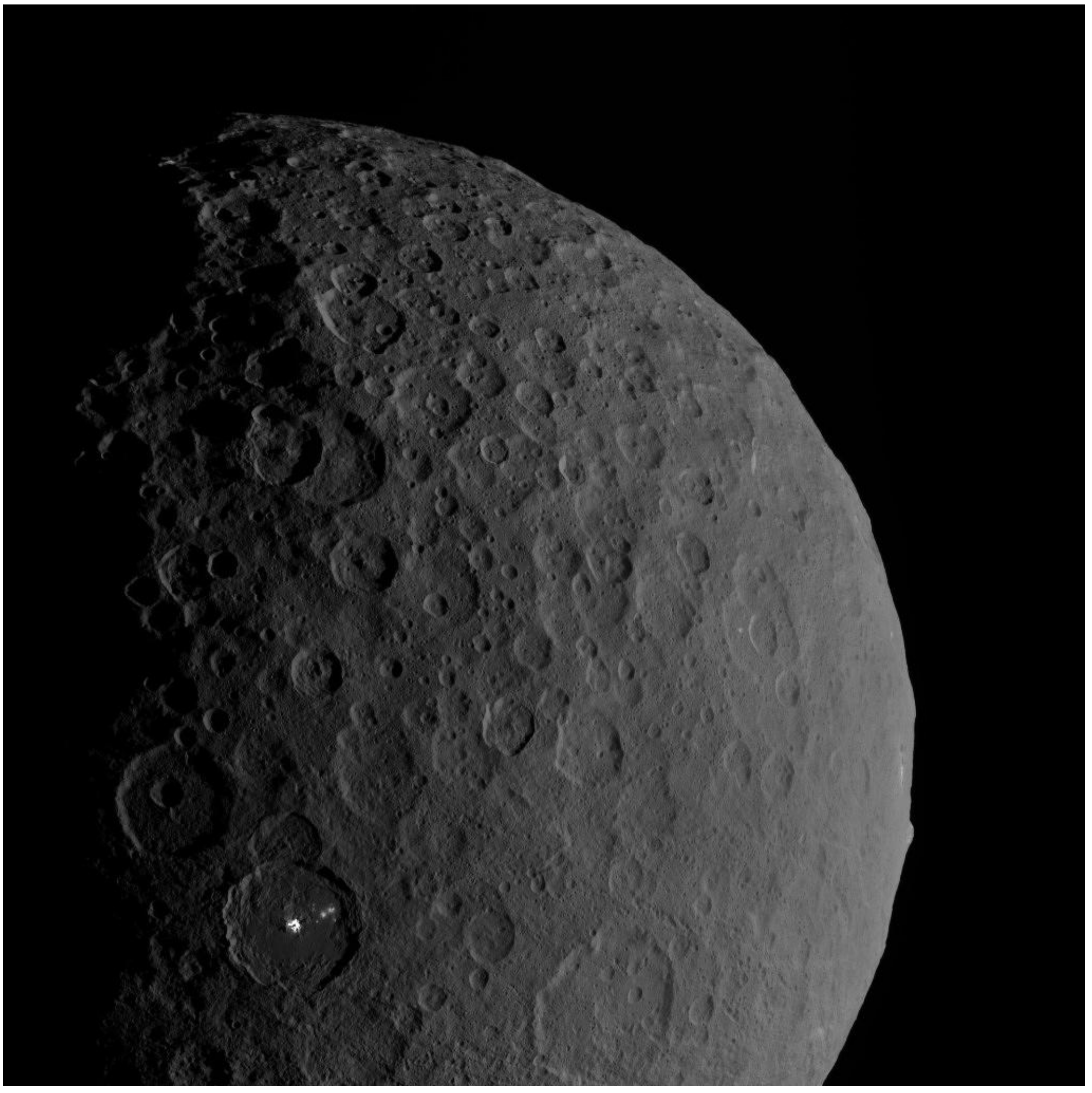

Figure 1: Ceres photographed by NASA's Dawn mission in 2017 (NASA/ JPL-Caltech/ UCLA/ MPS/ DLR/ IDA).

1. Elliptical orbits

1.1 Elements of a planetary orbit

To begin our study of planetary orbits, we need to devise some fairly robust terminology. In this chapter, we establish the basic definitions and calculations. Throughout both sections, we follow the work of Teets and Whitehead [1998]. The two authors also published an article concerning our particular problem involving the orbit of Ceres in [Teets and Whitehead, 1999], which we explore in much detail over the course of the second chapter.

First and foremost, some possibly familiar terms frequently used in astronomy. The *ecliptic* is the plane in which the Earth orbits the Sun. The term *perihelion* denotes the point in space when the studied object (Ceres or the Earth in our case) is closest to the Sun. On the other hand, *aphelion* is the point in the orbit that is furthest from the Sun. Moreover, we take the three laws of Kepler as given. For future reference, we quote a formulation of the laws from Roy [2020, p. 4]:

1. 'The orbit of each planet is an ellipse with the Sun at one focus.
2. For any planet the rate of description of area by the radius vector joining planet to Sun is constant.
3. The cubes of the semimajor axes of the planetary orbits are proportional to the squares of the planets' periods of revolution.'

The second law in other words says that for each planet, the proportion of the area swept out by the line segment joining the Sun with the planet and the time elapsed during the motion is constant for any time interval. The third law provides a straightforward way of relating the size of the orbit to the period of rotation. The laws will be mathematically reformulated each time it becomes necessary.

To be able to study orbits, we need to define a coordinate system. We denote by xyz the *heliocentric Cartesian system*. It is the orthogonal coordinate system with the origin at the center of the Sun such that the positive x-axis points towards the Earth's September equinox and the positive y-axis points towards the December solstice (see Fig.1.1). Note that this implies that the xy-plane coincides with the ecliptic. Moreover, we require the coordinate system to be right-handed, which uniquely determines the direction of the positive z-axis. A unit length in these coordinates is the length of one astronomical unit (1 AU), the length of the semi-major axis of Earth's orbit. The position vector of the planet in this coordinate system will be denoted by $\mathbf{r} = (x, y, z)^\top$.

Note that when we say *the* planet, we mean a particular celestial object orbiting the Sun since the results of this chapter apply generally to any elliptical orbit. This generality is maintained until the second chapter, when we consider the particular case of Ceres. Now we turn to describe the orbit in question. The planet's orbit is an ellipse which lies in a single plane together with the Sun. This plane (the *orbital plane*) has a normal vector denoted by $\mathbf{n} = (n_1, n_2, n_3)^\top$. We set a constraint on the choice of the normal vector: it can be any nonzero vector

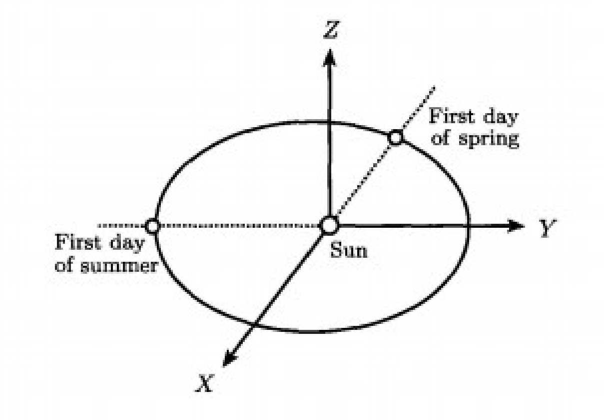

Figure 1.1: Heliocentric Cartesian coordinate system [Teets and Whitehead, 1998]. Note: in figures adopted from Teets and Whitehead [1998] (1.1 and 1.4), axes are labeled with uppercase letters X, Y, Z in contrast with our definition, which is preserved in figures adopted from Teets and Whitehead [1999] (1.2 and 2.1).

perpendicular to the orbital plane such that the angle between $\mathbf{n}$ and the positive z-axis is between 0 and 90 degrees. (Throughout the text, we measure angles in degrees until noted otherwise.)

To describe the orbit and be able to predict the planet's position at any time, we need six quantities called the *elements* of the orbit shown in Figure 1.2. The first element i is called the *inclination* of the orbit. It is the angle between 0 and 90 degrees formed by the vector $\mathbf{n}$ and the positive z-axis. Assuming that the inclination is not equal to zero, the intersection of the ecliptic and the orbital plane forms a distinctive line called the *line of nodes*. Moreover, the intersection of this line with the planet's orbit forms a set of two points. Of these two, we distinguish a single point called the *ascending node* and labeled by N as the one such that the z coordinate of the planet's position changes sign at N from negative to positive. Note that choice of N depends on the direction of the planet's rotation around the Sun. In Figure 1.2, this direction is assumed to be counterclockwise if observed from the northern hemisphere. This is the case for most objects in our solar system and from now on we assume this about our planet. Now we have the necessary tools to be able to define the second element. The *longitude of the ascending node* is the angle Ω measured at the Sun counterclockwise from the positive x-axis to the direction of the ascending node N. This angle can range from 0 to 360 degrees: it can be the case that Ω is larger than $180°$ depending on the position of N.

Before we move on to describe the elliptical orbit itself, we need one more quantity that determines the orientation of the ellipse within the orbital plane. Thus we define the third element called the *argument of perihelion* and denoted by ω as the oriented angle measured counterclockwise from the direction of the ascending node N to the direction of perihelion. We allow ω to range from 0 to 360 degrees. This angle not only describes the position of the orbit's major axis with respect to the line of nodes, but it also determines which focus is occupied by the Sun. For future reference, we also define the *true anomaly* as the oriented angle θ measured counterclockwise from the direction of perihelion

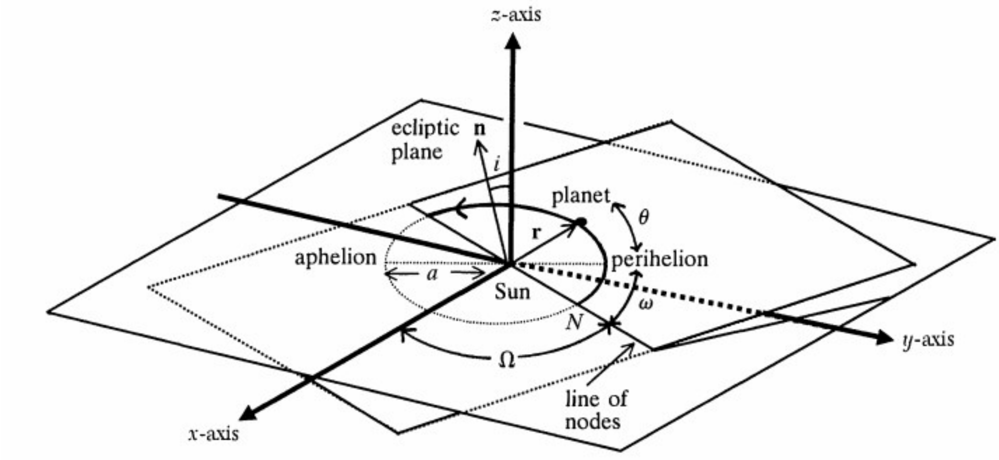

Figure 1.2: Elements of an elliptical orbit [Teets and Whitehead, 1999].

and the planet's position vector **r**. Note that unlike the values defined above, the definition of true anomaly depends on the particular position of the studied planet. This angle also occurs in the polar equation for the ellipse, which is used repeatedly later in our work.

Finally, having described the position of the orbit, we turn to describe its size and shape. The fourth element a is defined as the length of the semi-major axis of the elliptical orbit. In other words, a equals half the distance from the perihelion to the aphelion. Note that scaling this value results in scaling the orbit while keeping the other elements unchanged. The fifth element is the *eccentricity* of the ellipse denoted by e. It is defined as the ratio of the distance between the foci and the length of the major axis. The value $e \in [0, 1)$ measures how much the orbit resembles a circle. If zero, the ellipse *is* a circle. As e gets further from zero, the proportion between the major axis and the minor axis increases. Thus the two elements a and e determine the scale and the shape, respectively, of the elliptical orbit.

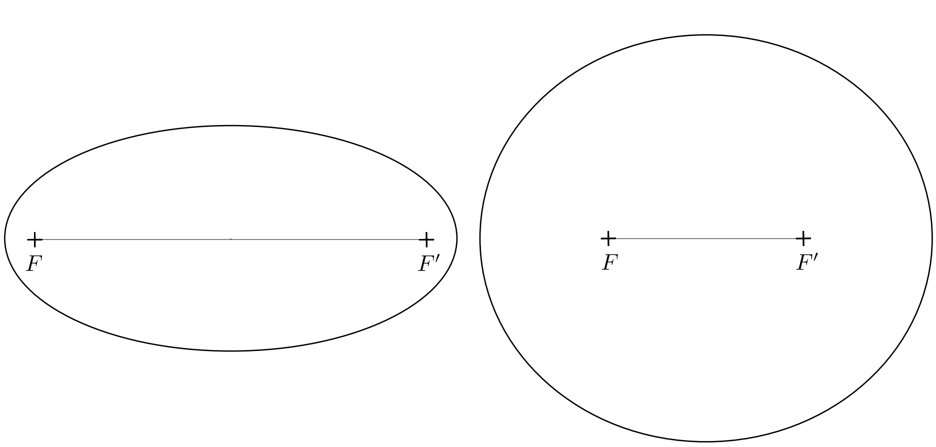

Figure 1.3: Comparing eccentricity with major axis fixed: high (left) and low (right).

We have now completely described the spatial properties of the orbit. Yet we still need an additional piece of information to be able to find the planet at any *time*. To use a colloquial expression, we know the *where* but we do not know the *when*. Therefore the sixth element τ_p is defined as the time of the last passage through perihelion. We could also pick any other particular point on the orbit and consider the passage through it instead. Perihelion is picked as it allows us to study the relationship between time elapsed since τ_p and the true anomaly θ. This relationship is necessary to be able to calculate the planet's position in the orbit at any time. We will see that this element is in some sense the most difficult to determine.

Altogether, the six quantities i, Ω, ω, a, e and τ_p form the elements of the orbit. In the following sections, we show how to calculate each of them from any two known positions of the planet.

1.2 First two elements: calculating i and Ω

In the rest of this chapter, we show how to calculate the elements assuming we are given two distinct position vectors $\mathbf{r}$ and $\mathbf{r}''$ of the planet at two times τ and τ'' with $\tau < \tau''$. This is a classical astronomy problem and its solution is necessary to describe the full Gauss's method. Calculating the positions $\mathbf{r}$ and $\mathbf{r}''$ purely from the observations is the primary concern of the second chapter. For the time being, however, we assume that we have already done so.

First, we note that we assume that the oriented angle measured from $\mathbf{r}$ to $\mathbf{r}''$ is less than 180 degrees, which is the case (as is with Ceres) when the vectors come from two observations taken over a relatively short period of time. Vectors $\mathbf{r}$ and $\mathbf{r}''$ are linearly independent and hence uniquely determine the orbital plane. Their cross product $\mathbf{r} \times \mathbf{r}''$ is a straightforward choice for the normal vector $\mathbf{n}$. We observe that it satisfies our requirement for the angle between $\mathbf{n}$ and the positive z-axis to be at most 90 degrees. This is the case thanks to the assumption that the angle measured from $\mathbf{r}$ to $\mathbf{r}''$ is less than 180 degrees and also thanks to the assumption of counterclockwise rotation.

Next, note the relation $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \gamma$ between the dot product of two vectors $\mathbf{a}, \mathbf{b}$, and the angle γ formed by them. We use this fact repeatedly throughout the work and employ it here to calculate the inclination. Letting $\mathbf{e_3} = (0, 0, 1)^\top$ denote the third standard basis vector, we have

$$\cos i = \frac{\mathbf{n} \cdot \mathbf{e_3}}{\|\mathbf{n}\| \|\mathbf{e_3}\|} = \frac{n_3}{\|\mathbf{n}\|},$$

where n_3 denotes the third coordinate of the vector $\mathbf{n}$. This coordinate is a positive value thanks to the fact that the angle between $\mathbf{n}$ and the positive z-axis is less than 90 degrees. Therefore $\cos i$ is a positive value and the inclination calculated by applying the arccos function to both sides is an angle between 0 and 90 degrees as required.

To determine the longitude of the ascending node Ω, we consider the orthogonal projection $(n_1, n_2, 0)^\top$, also denoted by $(n_1, n_2)^\top$, of the normal vector $\mathbf{n}$ onto the xy-plane (see Fig.1.4). Notice that by the assumption of counterclockwise rotation, the direction of the ascending node N is exactly 90 counterclockwise degrees away from the direction of this projection. This allows us to calculate Ω directly from the slope of the vector $(n_1, n_2)^\top$. However, as shown in the figure, we need to distinguish two cases. If $n_1 > 0$, the argument of the projection (the angle between the x-axis and the vector) is

$$\arctan\left(\frac{n_2}{n_1}\right), \qquad \text{and thus} \qquad \Omega = \arctan\left(\frac{n_2}{n_1}\right) + 90°.$$

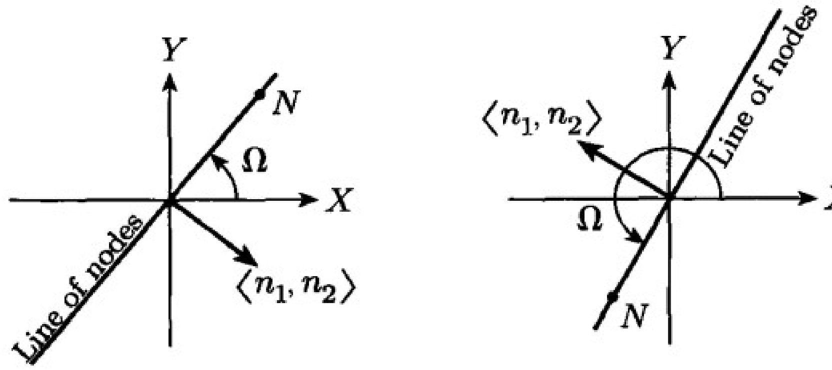

Figure 1.4: Determining Ω [Teets and Whitehead, 1998].

On the other hand, if $n_1 < 0$, then the expression

$$\arctan\left(\frac{n_2}{n_1}\right)$$

gives the argument of the flipped vector $-(n_1, n_2)^\top$. To get the argument of $(n_1, n_2)^\top$, we need to add 180 degrees to it. Therefore the required value is

$$\Omega = \arctan\left(\frac{n_2}{n_1}\right) + 270°.$$

The omitted case $n_1 = 0$ needs no calculation: it is quite straightforward to see that in this situation, Ω equals 180°.

1.3 Orientation of the orbit within its plane: calculating ω

We use our knowledge of Ω to calculate the third element: the argument of perihelion ω. From the above, we deduce that the unit vector pointing towards the ascending node N is $(\cos\Omega, \sin\Omega, 0)^\top$. As a first step, we find the value of the oriented angle α measured counterclockwise between this vector and the planet's position vector $\mathbf{r}$.

Using the fact that $\|(\cos\Omega, \sin\Omega, 0)^\top\| = 1$, we obtain

$$\cos\alpha = \frac{\mathbf{r}\cdot(\cos\Omega, \sin\Omega, 0)^\top}{\|\mathbf{r}\|}.$$

Again, we need to distinguish two cases. If the z-coordinate of $\mathbf{r}$ is positive, the angle α lies in the interval $(0°, 180°)$ and therefore we can obtain its value by taking the arccos of both sides of the equation. However, if the z-coordinate of $\mathbf{r}$ is negative, the angle α exceeds 180 degrees. In this case, $\arccos(\cos\alpha)$ is the angle complementary to the required angle α in the sense that the sum of these two angles is $360°$, the full circle. Hence, we have the required expression $\alpha = 360° - \arccos(\cos\alpha)$, where the latter term can be determined from the equation above.

Having calculated the value of α, we can now easily express ω in terms of α and the true anomaly θ corresponding to the position $\mathbf{r}$. Note that the value of θ is determined in the next section with no reference to ω and here can be assumed to be known. The particular form of the expression for ω depends on the order at which the planet crosses the ascending node N, the position $\mathbf{r}$ and the perihelion (here briefly denoted by P). If the order is $N, P, \mathbf{r}$ (as is the case in Figure 1.2), which occurs if and only if $\alpha > \theta$, we simply deduce that $\omega = \alpha - \theta$. On the other hand, if the order is $N, \mathbf{r}, P$, which occurs if and only if $\alpha < \theta$, we conclude that $\omega = 360° - \theta + \alpha$. This is due to α, ω and θ all being defined as counterclockwise oriented angles.

1.4 Size and shape of the orbit: calculating a and e

Before we move on to determine the elements, we need some more facts on ellipses. Given eccentricity e and a length of the semi-major axis a (now unknown), we define the *elliptical parameter* $k = a(1 - e^2)$. This value will serve as a bridge to obtain a and e from the given values. It also appears in the polar equation of the ellipse, which is the concern of the following lemma.

Lemma 1 (Polar equation of the ellipse)**.** *The distance $r(\theta)$ between the Sun and a point on the elliptical orbit corresponding to a true anomaly θ is*

$$r(\theta) = \frac{k}{1 + e\cos\theta}.$$

Proof. Let P be a point on the orbit. Let F denote the focus occupied by the Sun and let F' denote the other focus. Let r and r' denote the distances between F and P and between F' and P, respectively. By the defining property of ellipse, the sum of the distances $r + r'$ is constant for all points P on the ellipse. In particular, when we take P as the perihelion, we get from the horizontal symmetry of the ellipse that this constant is equal to $2a$. Therefore we have the relationship $r' = 2a - r$. Moreover, if we label the distance between the foci as d, we can use the definition of eccentricity to obtain $d = 2ae$. (Recall that eccentricity is the ratio of d and the length of the major axis, which is exactly $2a$.)

Now, looking at the triangle $F'FP$ in Figure 1.5, we can use the law of cosines to obtain

$$(r')^2 = r^2 + d^2 - 2rd\cos(180° - \theta).$$

Substituting the two expressions for r' and d and using the fact that $\cos\theta = -\cos(180° - \theta)$ yields

$$(r - 2a)^2 = r^2 + 4a^2e^2 + 4rae\cos\theta.$$

Expanding the bracket on the left hand side and cancelling out r^2, we obtain

$$-4ra + 4a^2 = 4a^2e^2 + 4rae\cos\theta.$$

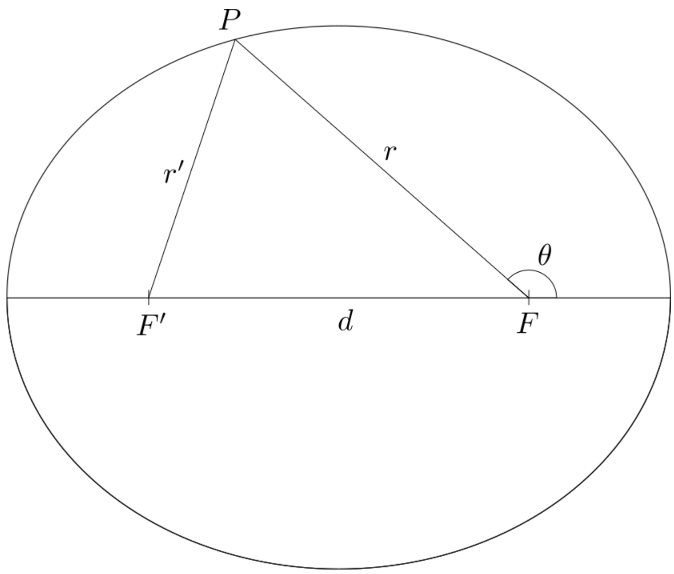

Figure 1.5: Applying the law of cosines to find the polar equation.

Dividing the equation by $4a$ and isolating r, we reach the final equation

$$r = \frac{a(1-e^2)}{1+e\cos\theta}.$$

The required form of the equation results from the definition of k. □

Now, turning to our current issue, we have two positions $\mathbf{r}$ and $\mathbf{r}''$ corresponding to two times τ and τ'' and wish to find a and e. To do so, we first estimate the parameter k and then use the polar equation to calculate the required values. Gauss himself devised a method of estimating k. Incidentally, he introduced this method in the article concerning the calculation of the orbit of Ceres [Gauss, 1809]. This method is also employed in the work [Teets and Whitehead, 1998] that we are currently following. Note that much more attention will be given to Gauss's original work in the second chapter.

To begin the calculation, we need to introduce some new notation and facts. Let $-g'$ denote the elliptical area enclosed by the vectors $\mathbf{r}$ and $\mathbf{r}''$, i.e. the area swept out by the planet between τ and τ'' (see Fig.1.6). The minus sign and the dash are to keep consistency with the notation of the second chapter, where several different areas are considered. If we denote the semi-minor axis of the orbit by b, the area enclosed by the full ellipse is equal to πab. (In case the reader is interested in a proof of this fact, it is provided as a side note in the proof of Lemma 3 below.) However, we can express b in terms of a and k using the Pythagorean theorem. Consider the special case shown in Figure 1.7. Thanks to the defining property of the ellipse and its symmetry, we can see that the segments labeled by a are in fact of length a. Moreover, recall that the distance d between the two foci is equal to $2ae$. We deduce that

$$b^2 = a^2 - \frac{d^2}{4} = a^2 - \frac{4a^2e^2}{4} = a^2(1-e^2) = ak.$$

Therefore we have an alternative expression $\pi a^{3/2}\sqrt{k}$ for the area of the ellipse. Now consider Kepler's second law, which equates ratios of swept out areas over

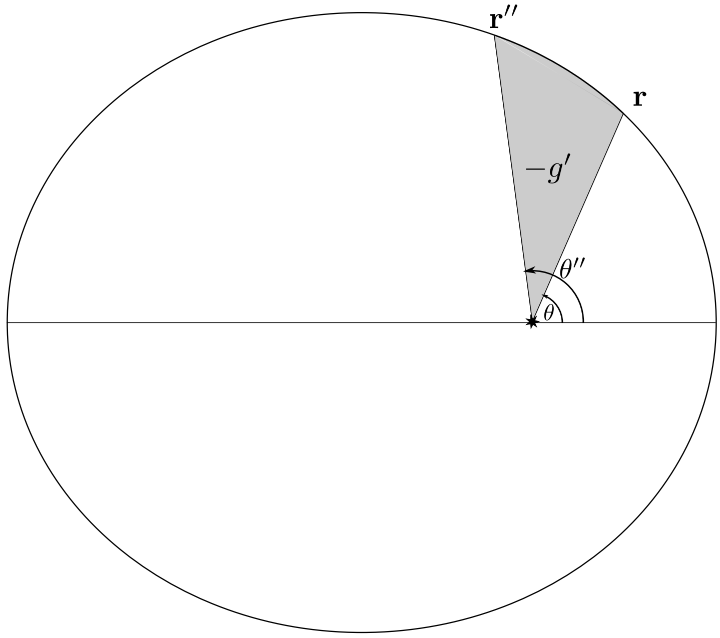

Figure 1.6: Area swept out by the planet between τ and τ''.

elapsed time intervals. We apply the law to two particular time intervals: the interval between τ and τ'', and one full period of the planet's motion. If we denote the period by t_p, we obtain

$$\frac{-g'}{\tau''-\tau}=\frac{\pi a^{3/2}\sqrt{k}}{t_p}. \tag{1.1}$$

On the other hand, Kepler's third law says that

$$\frac{a^3}{t_p^2}=\frac{A^3}{T_e^2},$$

where T_e is the Earth's period, or simply one year and $A = 1\text{AU}$ is the length of the semi-major axis of Earth's orbit. We can take the square root of the equation as it involves only positive values. Consequentially, we can substitute

$$\frac{a^{3/2}}{t_p}=\frac{1}{T_e} \qquad \text{into (1.1) and get} \qquad \frac{-g'}{\tau''-\tau}=\frac{\pi\sqrt{k}}{T_e}.$$

Isolating $\sqrt{k}$ and squaring the equation results in

$$k=\left(\frac{-g'T_e}{\pi(\tau''-\tau)}\right)^2.$$

Hence, we have expressed k in terms of known values and the area $-g'$. The next step is to estimate $-g'$ and thus directly obtain an estimate of k as well.

First, some more notation: we have already defined the true anomaly θ as corresponding to τ. Similarly, let θ'' denote the true anomaly at τ'' (see Fig.1.6 above). To find an approximation of the area, we first express it using the polar area integral formula

$$-g'=\frac{1}{2}\int_{\theta}^{\theta''} r(\alpha)^2\,d\alpha,$$

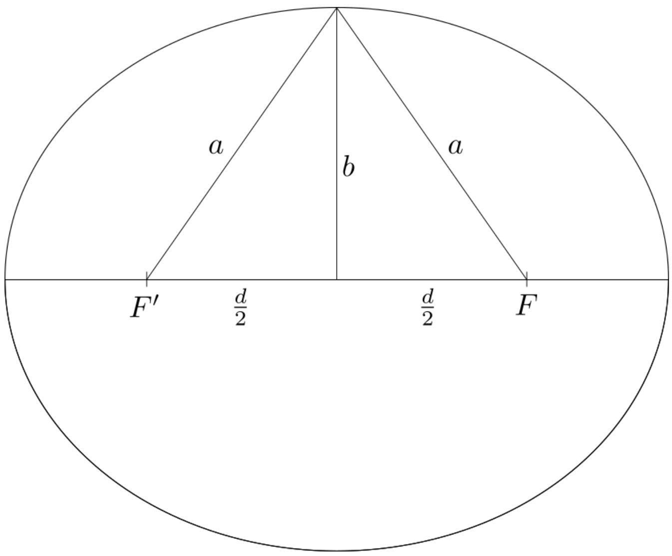

Figure 1.7: Exploring symmetry of the ellipse.

where $r(\alpha)$ denotes the distance from the Sun to the point on the ellipse corresponding to a true anomaly α. This formula comes from a simple integral trick: we assume that $d\alpha$ is a small angle inside the calculated area and estimate the area of the corresponding small elliptical sector simply by the area of the equally angled circular sector with radius $r(\alpha)$. For an angle α and its corresponding polar distance $r(\alpha)$, we calculate the circular area as shown in Figure 1.8. Summing over the full interval between θ and θ'' and taking the limit as $d\alpha$ approaches zero, we obtain the required formula.

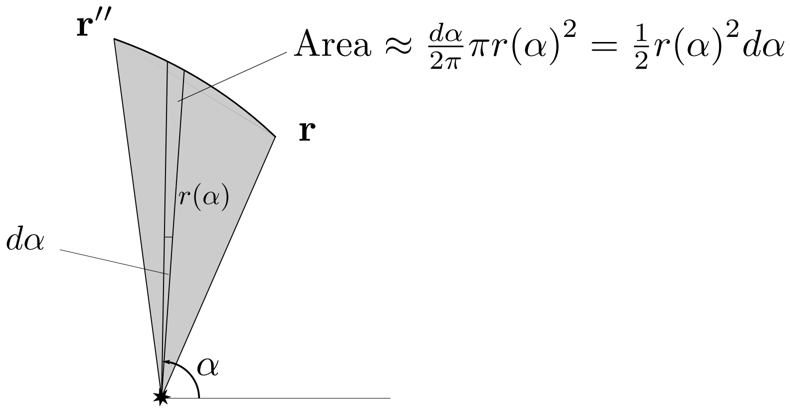

Figure 1.8: Explaining the polar area integral.

The borderline distances $r(\theta)$ and $r(\theta'')$ are exactly the norms of the vectors $\mathbf{r}$ and $\mathbf{r}''$ and will be further denoted simply by r and r''. Approximating the integral as area of trapezoid with bases of lengths r^2 and $(r'')^2$ and a distance $\theta'' - \theta$ between them measured in radians (see Fig.1.9), we obtain

$$-g' \approx \frac{1}{2}\frac{(r^2 + (r'')^2)(\theta'' - \theta)}{2}.$$

But the angle $\theta'' - \theta$ is exactly the angle formed by $\mathbf{r}$ and $\mathbf{r}''$ (measured in a

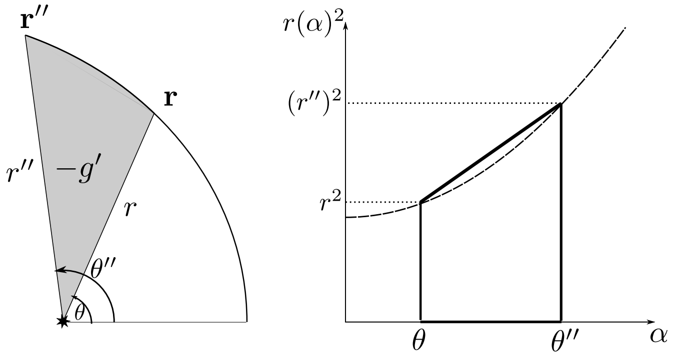

Figure 1.9: Approximating $-g'$.

counterclockwise direction) and therefore we have

$$\cos(\theta'' - \theta) = \frac{\mathbf{r} \cdot \mathbf{r}''}{\|\mathbf{r}\| \|\mathbf{r}''\|}.$$

Recall that we are assuming that $\theta'' - \theta$ is less than 180 degrees. Hence we can directly calculate $\theta'' - \theta$ necessary to estimate $-g'$. Altogether, we obtain the required estimate of k.

To conclude the calculation of a and e, we use two particular instances of the polar equation considered at τ and τ'':

$$r = \frac{k}{1 + e\cos\theta} \qquad \text{and} \qquad r'' = \frac{k}{1 + e\cos\theta''}.$$

Rewriting the equations so that the right hand sides are known, we get

$$e\cos\theta = \frac{k}{r} - 1 \qquad \text{and} \qquad e\cos\theta'' = \frac{k}{r''} - 1. \tag{1.2}$$

Dividing the second equation by the first (assuming that $\cos\theta$ is nonzero) and using the cosine addition formula yields

$$\begin{aligned} \frac{\frac{k}{r''} - 1}{\frac{k}{r} - 1} = \frac{\cos\theta''}{\cos\theta} = \frac{\cos(\theta + (\theta'' - \theta))}{\cos\theta} &= \frac{\cos\theta\cos(\theta'' - \theta) - \sin\theta\sin(\theta'' - \theta)}{\cos\theta} \\ &= \cos(\theta'' - \theta) - \tan\theta\sin(\theta'' - \theta). \end{aligned}$$

But recall that we have already determined $\theta'' - \theta$ and thus the only unknown in the equation above is $\tan\theta$. Hence, $\tan\theta$ can be directly expressed in terms of the known values k, r, r'', $\sin(\theta'' - \theta)$ and $\cos(\theta'' - \theta)$. To obtain θ itself, we first calculate the angle $\bar{\theta} = \arctan(\tan\theta)$. This angle might not equal the true value of θ and we need to place it into the correct quadrant by checking whether

$$\frac{k}{r} - 1 > 0, \qquad \text{or} \qquad \frac{k}{r} - 1 < 0.$$

Because e is non-negative, the former case together with (1.2) implies $\cos\theta > 0$ and thus the correct angle θ belongs to the first or the fourth quadrant. If $\tan\theta > 0$, we have $\theta = \bar{\theta}$ as $\bar{\theta}$ already is in the first quadrant. If $\tan\theta < 0$, then $\bar{\theta} < 0°$ and we let $\theta = \bar{\theta} + 360°$ to place it into the fourth quadrant. On the other hand, the latter inequality above implies $\cos\theta < 0$ and thus the true value of θ belongs to the second or the third quadrant. Regardless of whether $\tan\theta > 0$ or $\tan\theta < 0$, in both cases letting $\theta = \bar{\theta} + 180°$ places θ into the required quadrant.

Note that only now we have effectively completed the calculation of ω from section 1.3 where the value of θ was left to be determined. Now the only unknown in the rewritten polar equations (1.2) is e and therefore can be easily calculated. With k and e at hand, it is straightforward to conclude the section by finding the value of the fifth element

$$a = \frac{k}{1 - e^2}.$$

1.5 The sixth element: calculating τ_p

To wrap up the calculations, we need to determine the time τ_p of the last perihelion passage. This requires a new definition and some more geometric results. First, we define the *eccentric anomaly* E of a point on the orbit. Let us assume we have a circle of radius a whose center coincides with the center of the studied ellipse (see Fig.1.10). We have defined the true anomaly θ of a particular point P on the ellipse. Let us construct a line perpendicular to the major axis through this point. Then the eccentric anomaly E is defined as the angle between the direction of the perihelion and the direction of the point Q at which the vertical line crosses the circle (above the major axis if P is above it and below the major axis if P is below it). Same as the true anomaly, E admits values between 0 and 360 degrees.

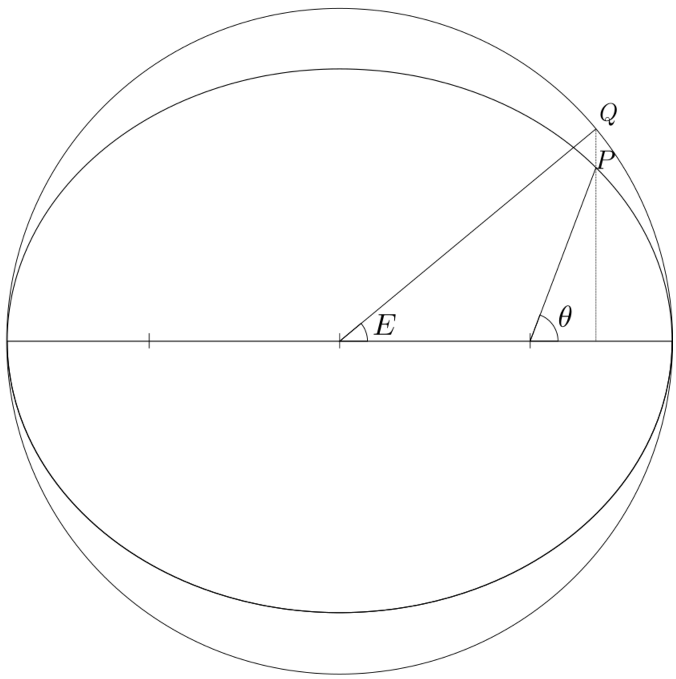

Figure 1.10: Definition of eccentric anomaly.

Next, we need a couple of properties of the eccentric anomaly that will allow us to calculate τ_p. We begin by deriving its relation to the true anomaly.

Lemma 2. *The eccentric and the true anomaly at any point on the ellipse satisfy*

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}.$$

Proof. We prove the case when θ is in the first quadrant. The remaining cases can be proved similarly or using the symmetry of the ellipse. We employ the following half-angle formula

$$\tan^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{1+\cos\alpha},$$

which can be derived from the identities $\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} = 2\cos^2\frac{\alpha}{2} - 1$ by isolating the squared sine and cosine and taking their ratio.

Let R denote the point where the vertical line from the definition of E intersects the major axis and let C denote the center of the ellipse. Looking at Figure 1.11, we can decompose the length of the segment CR as $|CR| = |CF| + |FR|$. Recall that the distance between the foci d is equal to $2ae$. Thus we have $|CR| = a\cos E$, $|CF| = ae$ and $|FR| = r\cos\theta$.

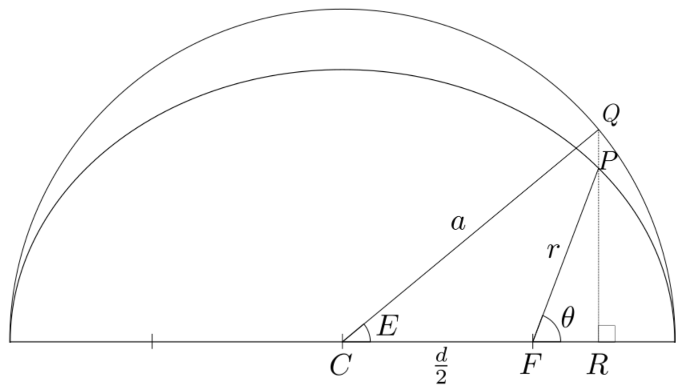

Figure 1.11: Closer look at the anomalies.

Altogether, we obtain the equality

$$a\cos E = ae + r\cos\theta,$$

which can be further rewritten using the polar equation of the ellipse as

$$a\cos E = ae + \frac{a(1-e^2)}{1+e\cos\theta}\cos\theta.$$

Dividing by a yields an expression for $\cos E$ that can be substituted into the tangent half-angle formula applied to the angle E. As a result, we have

$$\begin{aligned}\tan^2\frac{E}{2} &= \frac{1-(e+\frac{1-e^2}{1+e\cos\theta}\cos\theta)}{1+e+\frac{1-e^2}{1+e\cos\theta}\cos\theta} = \frac{(1-e)(1+e\cos\theta)-(1-e^2)\cos\theta}{(1+e)(1+e\cos\theta)+(1-e^2)\cos\theta}\\ &= \frac{(1-e)(1+e\cos\theta-(1+e)\cos\theta)}{(1+e)(1+e\cos\theta+(1-e)\cos\theta)} = \frac{1-e}{1+e}\cdot\frac{1-\cos\theta}{1+\cos\theta}.\end{aligned}$$

But by applying the same half-angle formula to θ, we can rewrite the latter fraction and obtain

$$\tan^2 \frac{E}{2} = \frac{1-e}{1+e} \tan^2 \frac{\theta}{2}.$$

Noting that $\theta > 180°$ if and only if $E > 180°$, we deduce that the tangents of $\theta/2$ and $E/2$ always share a sign. This proves the required result. □

Because we have determined θ and e in the previous section, we can further assume thanks to the lemma that the eccentric anomaly E corresponding to the position $\mathbf{r}$ at time τ is also known. The following result provides the last missing link in the calculation of the sixth element: the relation between E and τ_p.

Lemma 3 (Kepler's equation)**.** *At each time τ, we have*

$$E - e \sin E = \frac{2\pi}{t_p}(\tau - \tau_p),$$

where E is the eccentric anomaly corresponding to τ measured in radians, t_p is the period of planet's motion and τ_p is time of the last perihelion passage.

Proof. Let S denote the area of the elliptical sector swept out by the planet between τ_p and τ. Recall that the area of the ellipse is πab, where b denotes the length of the semi-minor axis. Then Kepler's second law says that

$$\frac{S}{\tau - \tau_p} = \frac{\pi ab}{t_p}.$$

What we need to do is express the area S in terms of E. The following derivation adapted from Roy [2020] was chosen because of its elementary and geometric nature. Note that we consider the case when θ is less than 180 degrees. The symmetry of the ellipse implies the same results for the values between 180 and 360 degrees.

Consider Figure 1.12. We temporarily introduce a uv coordinate system with origin at the center of the ellipse and the axes identical with the axes of the ellipse. This is to be able to use the algebraic equation of the ellipse in the form

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1.$$

Note that the circumscribed circle has equation $u^2 + v^2 = a^2$. Restricting our focus on the upper branches of the ellipse and the circle allows us to express the v coordinate as a function of u. If we denote the v coordinate of the upper semi-ellipse by v_e and the v coordinate of the upper semicircle by v_c, we have

$$v_e(u) = b\sqrt{1 - \frac{u^2}{a^2}} = \frac{b}{a}\sqrt{a^2 - u^2}$$

and

$$v_c(u) = \sqrt{a^2 - u^2}$$

for all $u \in [-a, a]$. We conclude that for each such u,

$$v_e(u) = \frac{b}{a} v_c(u). \tag{1.3}$$

Therefore we can formulate three particularly useful observations:

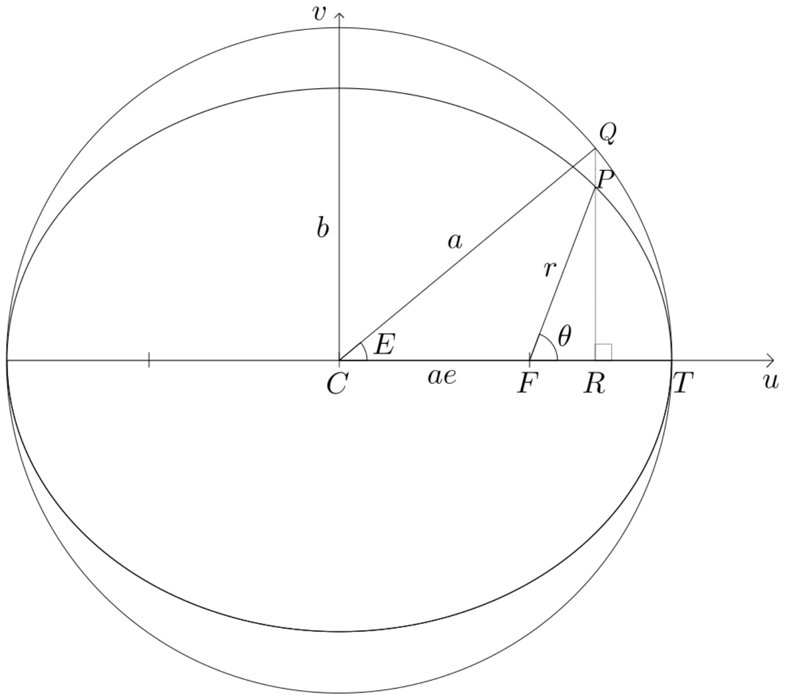

Figure 1.12: Triangles, elliptical and circular regions.

1. $|RP| = (b/a)|RQ|$. Therefore the area of triangle FRP is equal to the area of triangle FRQ scaled down by the factor b/a.

2. The area of the elliptical region RTP is equal to the area of the circular region RTQ scaled down by the factor b/a.

3. The area of the elliptical region FTP (also denoted by S) is equal to the area of the circular region FTQ scaled down by the factor b/a.

The first observation is a special case of (1.3) when $u = R_1$, the first coordinate of the point R, and a consequence of the triangle area formula. The second observation is due to linearity of integration. In particular, we express the area of RTP as an integral and use (1.3) to obtain

$$\int_{R_1}^{a} v_e(u)\,du = \int_{R_1}^{a} \frac{b}{a} v_c(u)\,du = \frac{b}{a}\int_{R_1}^{a} v_c(u)\,du,$$

where the rightmost integral represents the area of the circular region RTQ. That proves the second observation. (Note that using the same procedure, we can also express the area enclosed by the semi-ellipse and the major axis as one half of πa^2 scaled down by b/a. Doubling back proves that the area enclosed by the full ellipse is equal to πab.) The third observation is a simple corollary to the first two.

Now we can finally turn to calculate the studied area S. In the figure, it is the area bounded by the region FTP. Thanks to the third observation, we only need to calculate the area of the circular region FTQ and then scale it down by the factor b/a.

Our strategy to determine the area FTQ is to subtract the area of the triangle CFQ from the area of the circular sector CTQ. The latter area CTQ is calculated as a fraction of the full-circle area πa^2 corresponding to the angle E and hence equal to

$$\frac{E}{2\pi}\pi a^2 = \frac{Ea^2}{2}.$$

The former area CFQ concerns a triangle whose base is of length ae and whose height can be easily seen to be of length $a \sin E$. Therefore we conclude that the triangle CFQ has area

$$\frac{a^2 e \sin E}{2}.$$

Subtracting the two yields the area of the circular region FTQ in the form

$$\frac{a^2(E - e\sin E)}{2}$$

and using the third observation, we obtain the required result

$$S = \frac{ab(E - e\sin E)}{2}.$$

To complete the proof, we recall the relationship

$$\frac{S}{\tau - \tau_p} = \frac{\pi ab}{t_p}$$

implied by Kepler's second law. We substitute the calculated value of S, cancel ab out and obtain

$$\frac{E - e\sin E}{2(\tau - \tau_p)} = \frac{\pi}{t_p}.$$

Multiplying the equation by $2(\tau - \tau_p)$ finishes the proof of the lemma. □

To conclude the calculations, note what is known and what is unknown in our particular case of Kepler's equation when τ is given. We have already determined e and E. Kepler's third law

$$\frac{a^3}{t_p^2} = \frac{A^3}{T_e^2},$$

allows us to calculate the period t_p solely from the value of a and Earth's known values. Thus the only unknown in Kepler's equation is the sixth element τ_p. Since the equation is linear in τ_p, we are directly able to calculate it and thus finish determining the elements.

We conclude the chapter by sketching out how to calculate the position of the planet at any given time t using the six elements of its orbit. First, we solve Kepler's equation with the now known τ_p and involving t instead of τ to find the eccentric anomaly E corresponding to t. Note that when the unknown is E, the equation is nonlinear and needs to be solved numerically (see e.g. Roy [2020] for a method). Then we determine the true anomaly θ corresponding to t from E using Lemma 2. Finally, we use the polar equation to determine the distance between the planet and the Sun at t. Thus, using all the known angles i, Ω, ω and θ and this distance, we are able to obtain the xyz coordinates of the planet's location at t. As we will see in the next chapter, these can be easily translated to coordinates centered at the Earth that are used in observations. That allows us to look at the particular region in the sky at time t and search for the planet.

2. Gauss's method

2.1 Setting up the equations

In this section, we derive equations that provide the basis of Gauss's method. Throughout the chapter, we follow the work of Teets and Whitehead [1999], which introduces the calculation described in Gauss's original article [Gauss, 1809] in a more accessible and modern mathematical language. The authors preserve the same notation as Gauss and until noted otherwise, we also keep the former notation unchanged. Building on the work of Teets and Whitehead, we aim to provide a more thorough version of the briefly presented derivations.

First, we have to note a crucial piece of notation used throughout the text. Because Gauss worked with three sets of observed values, we will use τ, τ' and τ'' to denote the respective times at which Giuseppe Piazzi recorded his observations. The following list of variables is explicitly defined as corresponding to the time τ. However, we also implicitly define the same variables for the times τ' and τ''. We refer to them by adding a single dash or a double dash to the particular letters.

Now we turn to define the necessary coordinate systems and describe their relations (see Fig.2.1). We have already defined the heliocentric Cartesian system xyz in the first chapter. Recall that we express the corresponding positions of the studied planet (from now on, this is Ceres) as $\mathbf{r} = (x, y, z)^\top$. Here, we add that we express the heliocentric positions of the Earth using uppercase letters $(X, Y, Z)^\top$. This convention is kept throughout the text. Variables denoted by lowercase letters consistently correspond to Ceres while variables denoted by uppercase letters consistently correspond to the Earth.

Let $\xi\eta\zeta$ denote the *geocentric Cartesian system.* It is the orthogonal coordinate system centered at the center of the Earth such that each of the ξ, η and ζ-axis is parallel to the x, y and z-axis, in the respective order and with the same direction. Length is again measured in astronomical units. We denote the corresponding position of Ceres by $(\xi, \eta, \zeta)^\top$.

Having defined the Cartesian coordinate systems, we now turn to the spherical coordinates. Let L be the *heliocentric longitude* and B the *heliocentric latitude of the Earth.* This means that L is the counterclockwise angular distance between the x-axis and the Earth's orthogonal projection onto the xy-plane measured from the Sun. The latitude B is the northward angular distance between the xy-plane and the Earth measured from the Sun. To define the last coordinate, let R be the distance from the Sun to the Earth. Together, (L, B, R) form the *heliocentric spherical coordinates* of the Earth. Moreover, let $D = R\cos B$ denote the length of the orthogonal projection of R onto the xy-plane. Note that we have already chosen the xy-plane to be the plane of the Earth's ecliptic and thus, effectively, $B = 0$ and $D = R$. However, the letters B and D remain in use throughout the text to keep the notation dual to the following notation for Ceres. Also note that all of the values defined above are considered to be known at all times as they result from Earth's well documented motion.

Now we define λ to be the *geocentric longitude* and β to be the *geocentric latitude of Ceres.* The longitude λ is the counterclockwise angular distance between the ξ-axis and the orthogonal projection of Ceres onto the $\xi\eta$-plane-plane

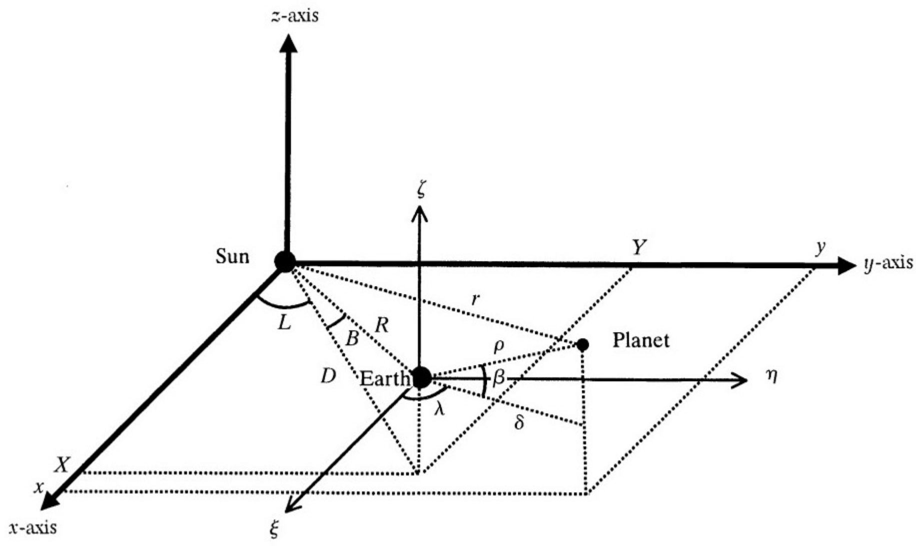

Figure 2.1: Cartesian and spherical coordinate systems [Teets and Whitehead, 1999].

measured from the Earth. The latitude β is the northward angular distance between the $\xi\eta$-plane and Ceres measured from the Earth. These two values are the measurements made by Giuseppe Piazzi in his Sicilian observatory. For the last coordinate, let ρ be the distance from the Earth to Ceres and let $\delta = \rho \cos \beta$ be the length of its orthogonal projection onto the $\xi\eta$-plane. These values are unknown and to determine them is the main concern of the method. Note that the values of λ and L can range from 0 to 360 degrees, while the values of β (and theoretically also B) can range from -90 to 90 degrees. See Figure 2.1 for illustration of the longitudes and latitudes.

Recall that $\mathbf{r}$ denotes the heliocentric position vector of Ceres and r denotes its norm, which equals the distance from the Sun to Ceres. As a reminder of the notation for the other two times of observations, we have $\mathbf{r}' = (x', y', z')^\top$ denoting the position vector of Ceres at τ'. Similarly, $\mathbf{r}''$ denotes the position of Ceres at τ''. This just repeats the definition of $\mathbf{r}''$ from the first chapter in a slightly different context. Moreover, r' and r'' denote the respective distances.

Relations between the Cartesian and the spherical coordinate systems are canonical. For example, the geocentric coordinates of Ceres by definition satisfy $\xi = \rho \cos \lambda \cos \beta$, $\eta = \rho \sin \lambda \cos \beta$ and $\zeta = \rho \sin \beta$. Similar relations are satisfied by the heliocentric coordinates of the Earth. Furthermore, the two Cartesian coordinate systems satisfy simple relations $\xi = x - X$, $\eta = y - Y$ and $\zeta = z - Z$ by their constructions.

For the purposes of the method, we need to work with areas bounded by the involved celestial objects. Let f denote the area of the triangle formed by the Sun and the two positions of Ceres at τ' and τ'' (see Fig.2.2). Similarly, let $-f'$ denote the triangular area corresponding to the times τ and τ''. Analogously, we define the area f'' of the triangle formed between τ and τ'. We choose to involve the negative sign in the definition of $-f'$ as then the sum $f + f' + f''$ denotes the area of the small triangle formed by the three positions of Ceres at τ, τ' and τ''. To be able to transform the equations below into matrix notation, we define the vector $\mathbf{f} = (f, f', f'')^\top$.

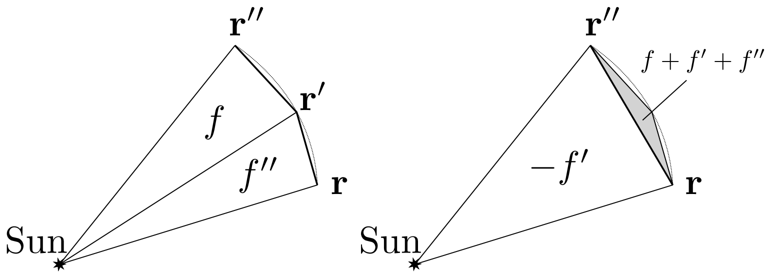

Figure 2.2: Areas f, $-f'$, f'' and $f + f' + f''$.

Similarly, let F, $-F'$ and F'' represent the relevant triangular areas formed by the positions of the Earth. We also define $\mathbf{F} = (F, F', F'')^\top$.

Next, we consider elliptical sectors instead of triangles. Let g denote the area of the elliptical sector swept out by Ceres between τ' and τ''. The areas $-g'$ and g'' are then defined similarly (see Fig.2.3).

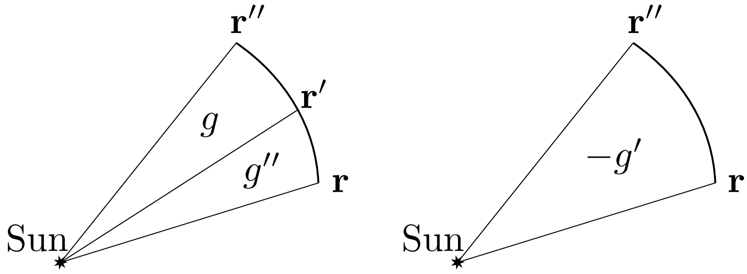

Figure 2.3: Areas g, $-g'$ and g''.

At this point, we begin deriving the necessary equations. According to Kepler's first law, the orbit of Ceres is an ellipse with the Sun occupying one of the foci. Particularly the vectors $\mathbf{r}$, $\mathbf{r}'$ and $\mathbf{r}''$ lie in a single plane determined by any pair of them. Therefore there are real coefficients c_1 and c_2 such that

$$\mathbf{r} = c_1\mathbf{r}' + c_2\mathbf{r}''. \tag{2.1}$$

We can take the cross product of each side of (2.1) with $\mathbf{r}'$ and also with $\mathbf{r}''$ to get

$$\mathbf{r} \times \mathbf{r}' = c_2(\mathbf{r}'' \times \mathbf{r}') \qquad \text{and} \qquad \mathbf{r} \times \mathbf{r}'' = c_1(\mathbf{r}' \times \mathbf{r}''). \tag{2.2}$$

The cross product has a nice geometric property: $\|\mathbf{a} \times \mathbf{b}\|$ equals the area of the parallelogram formed by the vectors $\mathbf{a}$ and $\mathbf{b}$. Moreover, $2f$ is exactly equal to the area of the parallelogram formed by the vectors $\mathbf{r}'$ and $\mathbf{r}''$. Therefore we have the relations $2f = \|\mathbf{r}' \times \mathbf{r}''\|$, $-2f' = \|\mathbf{r} \times \mathbf{r}''\|$ and $2f'' = \|\mathbf{r} \times \mathbf{r}'\|$. Together with equations (2.2), they imply $f'' = c_2 f$ and $-f' = c_1 f$. Dividing by f, substituting the coefficients c_1 and c_2 into (2.1) and then multiplying by f, we get

$$f\mathbf{r} = -f'\mathbf{r}' - f''\mathbf{r}'' \qquad \text{or, equivalently,} \qquad f\mathbf{r} + f'\mathbf{r}' + f''\mathbf{r}'' = 0. \tag{2.3}$$

This equation can be rewritten using matrix multiplication. The relevant matrices will be defined as

$$\phi = \begin{pmatrix} x & x' & x'' \\ y & y' & y'' \\ z & z' & z'' \end{pmatrix} = (\mathbf{r}|\mathbf{r}'|\mathbf{r}'') \qquad \text{and} \qquad \Phi = \begin{pmatrix} X & X' & X'' \\ Y & Y' & Y'' \\ Z & Z' & Z'' \end{pmatrix}.$$

With the help of matrix notation, we can rewrite (2.3) as $\phi\mathbf{f} = \mathbf{0}$. In a completely analogous way, one can obtain another crucial equation $\Phi\mathbf{F} = \mathbf{0}$, which involves the positions and areas corresponding to the Earth. Subtracting $\Phi\mathbf{f}$ from the equation $\phi\mathbf{f} = \mathbf{0}$ and then multiplying by $(F + F'')$ yields

$$(F + F'')(\phi\mathbf{f} - \Phi\mathbf{f}) = (F + F'')(-\Phi\mathbf{f})$$

and therefore we have

$$(F + F'')(\phi - \Phi)\mathbf{f} = -\Phi(F + F'')\mathbf{f}. \tag{2.4}$$

Because $\Phi\mathbf{F} = \mathbf{0}$, it also holds that

$$\Phi(f + f'')\mathbf{F} = (f + f'')\Phi\mathbf{F} = \mathbf{0}.$$

Hence we can add this zero term to the right hand side of (2.4) and obtain

$$(F + F'')(\phi - \Phi)\mathbf{f} = -\Phi(F + F'')\mathbf{f} + \Phi(f + f'')\mathbf{F}.$$

Factoring Φ out yields a crucial equation

$$(F + F'')(\phi - \Phi)\mathbf{f} = \Phi((f + f'')\mathbf{F} - (F + F'')\mathbf{f}). \tag{2.5}$$

In the next step, we transform (2.5) from the Cartesian into the spherical coordinates. It will be useful to define the vectors

$$\mathbf{w} = (\cos\lambda, \sin\lambda, \tan\beta)^\top \qquad \text{and} \qquad \mathbf{W} = (\cos L, \sin L, \tan B)^\top.$$

Recall that by doing this, we have also implicitly defined $\mathbf{w}'$, $\mathbf{w}''$, $\mathbf{W}'$ and $\mathbf{W}''$ for the other times τ' and τ''. At this point, we alter the original notation to avoid confusion: instead of $\mathbf{w}$, Gauss used π, which appears repeatedly throughout this work as the circular constant.

Note that all values involved in the definitions of $\mathbf{w}$ and $\mathbf{W}$ are known. Also note that multiplying $\mathbf{w}$ by the unknown $\delta = \rho\cos\beta$ yields

$$\delta\mathbf{w} = (\rho\cos\lambda\cos\beta, \rho\sin\lambda\cos\beta, \rho\sin\beta)^\top = (\xi, \eta, \zeta)^\top$$

by the relationship between spherical and Cartesian coordinates. Similarly, we have $D\mathbf{W} = (X, Y, Z)^\top$ for the Earth's known position. The same holds for the values corresponding to τ' and τ''.

Also recall the relation between the geocentric and the heliocentric Cartesian system $\xi = x - X$, $\eta = y - Y$ and $\zeta = z - Z$ and similarly for τ' and τ''. Translating these relations into matrix notation, we have

$$\phi - \Phi = \begin{pmatrix} \xi & \xi' & \xi'' \\ \eta & \eta' & \eta'' \\ \zeta & \zeta' & \zeta'' \end{pmatrix}$$

and thus we obtain $\phi - \Phi = (\delta\mathbf{w}|\delta'\mathbf{w}'|\delta''\mathbf{w}'')$ and $\Phi = (D\mathbf{W}|D'\mathbf{W}'|D''\mathbf{W}'')$. Using these relationships together with the triple scalar product identity $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \det(\mathbf{a}|\mathbf{b}|\mathbf{c})$ allows us to transform equation (2.5) into three new equations which constitute the starting point of approximations that will later allow us to estimate the unknown values. We will also make use of a few properties of determinants, namely linearity in each column, the fact that $\det(\mathbf{a}|\mathbf{b}|\mathbf{b}) = 0$ and invariance to even permutations of the columns.

To derive the first equation, we take the dot product of each side of (2.5) with $(\mathbf{w}'' \times \mathbf{w})$. Thus the left hand side equals

$$\begin{aligned}
&(\mathbf{w}'' \times \mathbf{w}) \cdot (F + F'')(\phi - \Phi)\mathbf{f} \\
&\quad = (F + F'')(\mathbf{w}'' \times \mathbf{w}) \cdot (\phi - \Phi)\mathbf{f} \\
&\quad = (F + F'')\det(\mathbf{w}''|\mathbf{w}|(\phi - \Phi)\mathbf{f}) \\
&\quad = (F + F'')\det(\mathbf{w}''|\mathbf{w}|f\delta\mathbf{w} + f'\delta'\mathbf{w}' + f''\delta''\mathbf{w}'') \\
&\quad = (F + F'')(\det(\mathbf{w}''|\mathbf{w}|f\delta\mathbf{w}) + \det(\mathbf{w}''|\mathbf{w}|f'\delta'\mathbf{w}') + \det(\mathbf{w}''|\mathbf{w}|f''\delta''\mathbf{w}'')) \\
&\quad = (F + F'')(f\delta\det(\mathbf{w}''|\mathbf{w}|\mathbf{w}) + f'\delta'\det(\mathbf{w}''|\mathbf{w}|\mathbf{w}') + f''\delta''\det(\mathbf{w}''|\mathbf{w}|\mathbf{w}'')) \\
&\quad = (F + F'')f'\delta'\det(\mathbf{w}|\mathbf{w}'|\mathbf{w}'')
\end{aligned}$$

while the right hand side is equal to

$$\begin{aligned}
&(\mathbf{w}'' \times \mathbf{w}) \cdot \Phi((f + f'')\mathbf{F} - (F + F'')\mathbf{f}) = \\
&= (f + f'')(\mathbf{w}'' \times \mathbf{w}) \cdot \Phi\mathbf{F} - (F + F'')(\mathbf{w}'' \times \mathbf{w}) \cdot \Phi\mathbf{f} \\
&= (f + f'')\det(\mathbf{w}''|\mathbf{w}|\Phi\mathbf{F}) - (F + F'')\det(\mathbf{w}''|\mathbf{w}|\Phi\mathbf{f}) \\
&= (f + f'')\det(\mathbf{w}''|\mathbf{w}|FD\mathbf{W} + F'D'\mathbf{W}' + F''D''\mathbf{W}'') - \\
&- (F + F'')\det(\mathbf{w}''|\mathbf{w}|fD\mathbf{W} + f'D'\mathbf{W}' + f''D''\mathbf{W}'') \\
&= (f + f'')(FD\det(\mathbf{w}''|\mathbf{w}|\mathbf{W}) + F'D'\det(\mathbf{w}''|\mathbf{w}|\mathbf{W}') + F''D''\det(\mathbf{w}''|\mathbf{w}|\mathbf{W}'')) - \\
&- (F + F'')(fD\det(\mathbf{w}''|\mathbf{w}|\mathbf{W}) + f'D'\det(\mathbf{w}''|\mathbf{w}|\mathbf{W}') + f''D''\det(\mathbf{w}''|\mathbf{w}|\mathbf{W}'')) \\
&= (f''F - fF'')(D\det(\mathbf{w}''|\mathbf{w}|\mathbf{W}) - D''\det(\mathbf{w}''|\mathbf{w}|\mathbf{W}'')) + \\
&+ ((f + f'')F' - (F + F'')f')D'\det(\mathbf{w}''|\mathbf{w}|\mathbf{W}') \\
&= (f''F - fF'')(D\det(\mathbf{w}|\mathbf{W}|\mathbf{w}'') - D''\det(\mathbf{w}|\mathbf{W}''|\mathbf{w}'')) + \\
&+ ((f + f'')F' - (F + F'')f')D'\det(\mathbf{w}|\mathbf{W}'|\mathbf{w}'').
\end{aligned}$$

The remaining two equations arise similarly from (2.5) after taking the dot product of each side with $(\mathbf{W}' \times \mathbf{w})$ and $(\mathbf{W}' \times \mathbf{w}'')$, respectively. To save space and spare the reader of the lengthy calculations, we omit the step by step derivations, which make use of the same properties and identities as the one above. Altogether, we have three central equations

$$\begin{aligned}
&(F + F'')f'\delta'\det(\mathbf{w}|\mathbf{w}'|\mathbf{w}'') = \\
&\qquad = (f''F - fF'')(D\det(\mathbf{w}|\mathbf{W}|\mathbf{w}'') - D''\det(\mathbf{w}|\mathbf{W}''|\mathbf{w}'')) + \\
&\qquad + ((f + f'')F' - (F + F'')f')D'\det(\mathbf{w}|\mathbf{W}'|\mathbf{w}'')
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
&(F + F'')(f'\delta'\det(\mathbf{w}|\mathbf{w}'|\mathbf{W}') + f''\delta''\det(\mathbf{w}|\mathbf{w}''|\mathbf{W}')) = \\
&\qquad = (f''F - fF'')(D\det(\mathbf{w}|\mathbf{W}|\mathbf{W}') - D''\det(\mathbf{w}|\mathbf{W}''|\mathbf{W}'))
\end{aligned} \tag{2.7}$$

$$(F+F'')(f\delta \det(\mathbf{w}''|\mathbf{w}|\mathbf{W}') + f'\delta' \det(\mathbf{w}''|\mathbf{w}'|\mathbf{W}')) = \\ = (f''F - fF'')(D\det(\mathbf{w}''|\mathbf{W}|\mathbf{W}') - D''\det(\mathbf{w}''|\mathbf{W}''|\mathbf{W}')). \tag{2.8}$$

We will use the second and the third equation to base an estimate of δ and δ'' on the value of δ' in section 2.3 and then estimate δ' itself using the first equation in section 2.4. Before that, however, we derive their approximated versions.

2.2 Justifying the approximations

In this section, we simplify the equations by introducing approximations. The results of this section are presented in both [Gauss, 1809] and [Teets and Whitehead, 1999] without much justification. Here, we present our own derivation of the results.

Because Ceres traversed a fairly short arc during Piazzi's observations, Gauss noted that we can consider the time differences $\tau' - \tau$, $\tau'' - \tau'$ and $\tau'' - \tau$ as infinitesimal and use this fact to discard some of the terms in the equations as close enough to zero. With the modern notion of limit, we can analyse behaviour of the terms if the time differences between the observations approached zero. Particularly, we will consider the hypothetical situation in which both τ and τ'' *simultaneously* approach τ'. We define a shorthand version of this statement: instead of $\tau \to \tau'$ *and* $\tau'' \to \tau'$, we simply write $t \to 0$. In this section, t denotes a time difference between any two observations and is always understood as approaching zero.

Note that what we understand throughout the text as three distinct rigid states of the Sun-Earth-Ceres system becomes dynamic for this particular section. We will keep the notation unchanged, but bear in mind that we have switched the way of looking at the variables and are actually working with functions of time instead of static concrete values.

With this temporary shift in perspective, it makes sense to say that for example

$$\lim_{t\to 0} \mathbf{r} = \lim_{t\to 0} \mathbf{r}'' = \mathbf{r}' \qquad \text{or} \qquad \lim_{t\to 0} D = \lim_{t\to 0} D'' = D'.$$

We can interpret the situation as if the first and the third observations made at τ and τ'' together with all the values associated to them approached the middle observation made at τ' with its respective values.

To analyse the equations, we use the notion of *order*, denoted by o. To say that a function $h(t)$ *is of order* n (as t approaches zero) means that the limit

$$\lim_{t\to 0} \frac{h(t)}{t^n}$$

is finite. For short, we may say that $h(t)$ *is* $o(t^n)$. This definition is equivalent to the fact that the Taylor expansion of the function expressed in terms of the time difference t contains only terms of order n or higher. The only Taylor expansions repeatedly used in the section are the standard sine and cosine series

$$\sin\alpha = \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n+1}}{(2n+1)!} = \alpha - \frac{\alpha^3}{6} + \dots$$

and

$$\cos\alpha = \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n}}{(2n)!} = 1 - \frac{\alpha^2}{2} + \dots$$

We present a set of lemmas which will ultimately justify the required approximations.

Lemma 4. *Each of the triangular areas f, $-f'$, f'', F, $-F'$ and F'' is of order one.*

Proof. Let γ denote the smaller angle formed by the vectors $\mathbf{r}'$ and $\mathbf{r}''$ (see Fig.2.4). Then the length of the height perpendicular to $\mathbf{r}'$ in the studied triangle equals $r'' \sin\gamma$.

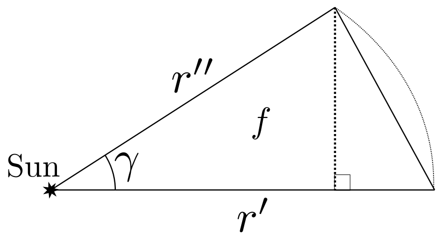

Figure 2.4: Calculating f.

Therefore

$$f = \frac{1}{2} r' r'' \sin\gamma$$

and

$$\lim_{t\to 0} \frac{f}{t} = \frac{1}{2} \lim_{t\to 0} r' r'' \frac{\sin\gamma}{t} = \frac{1}{2}(r')^2 \lim_{t\to 0} \frac{\gamma}{t}\frac{\sin\gamma}{\gamma}.$$

Note that we have

$$\lim_{t\to 0} \gamma = 0, \qquad \text{and hence} \qquad \lim_{t\to 0} \frac{\sin\gamma}{\gamma} = 1.$$

Moreover, the value of γ divided by t expresses the average angular velocity of Ceres during the time interval t measured with respect to the Sun. Although this velocity depends on the position of Ceres and is not constant, it is certainly bounded for all time intervals t. Hence the instantaneous heliocentric angular velocity of Ceres at τ' equal to

$$\lim_{t\to 0} \frac{\gamma}{t}$$

is finite. In other words, the angle γ is $o(t)$. Thus, the limit

$$\lim_{t\to 0} \frac{f}{t} = \frac{1}{2}(r')^2 \lim_{t\to 0} \frac{\gamma}{t}$$

is finite as well. Therefore we can say that f is of order one. The order of the remaining areas can be justified in a completely analogous way. □

Lemma 5. *Both terms $f''F - fF''$ and $(f + f'')F' - (F + F'')f'$ are of order four.*

Proof. Exactly as in the previous proof, let γ denote the smaller angle formed by the vectors $\mathbf{r}'$ and $\mathbf{r}''$. Similarly, γ'' will denote the smaller angle formed by $\mathbf{r}$ and $\mathbf{r}'$. For the Earth related variables, we analogously define angles Γ and Γ'' as those formed by the pairs of vectors corresponding to the areas F and F'', respectively.

Let v and V denote the instantaneous heliocentric angular velocity at τ' of Ceres and the Earth, respectively. In other words,

$$v = \lim_{\tau'' \to \tau'} \frac{\gamma}{\tau'' - \tau'} = \lim_{\tau \to \tau'} \frac{\gamma''}{\tau' - \tau} = \lim_{t \to 0} \frac{\gamma}{t} = \lim_{t \to 0} \frac{\gamma''}{t}$$

and

$$V = \lim_{\tau'' \to \tau'} \frac{\Gamma}{\tau'' - \tau'} = \lim_{\tau \to \tau'} \frac{\Gamma''}{\tau' - \tau} = \lim_{t \to 0} \frac{\Gamma}{t} = \lim_{t \to 0} \frac{\Gamma''}{t}.$$

Therefore, we can replace both γ angles by vt and the Γ angles by Vt when considering the limit as t approaches zero. Now we have the setup necessary to rewrite the analysed expressions in terms of t.

First, using the same geometric argument as in the proof above, we express the areas explicitly as

$$f''F - fF'' = \frac{rr' \sin\gamma''}{2} \frac{R'R'' \sin\Gamma}{2} - \frac{r'r'' \sin\gamma}{2} \frac{RR' \sin\Gamma''}{2}.$$

We claim that the expression $\sin\gamma'' \sin\Gamma - \sin\gamma \sin\Gamma''$ is $o(t^4)$. Indeed, we can expand each sine into its Taylor series and as $t \to 0$, we obtain

$$\begin{aligned} &(\gamma'' - \frac{(\gamma'')^3}{6} + \dots)(\Gamma - \frac{\Gamma^3}{6} + \dots) - (\gamma - \frac{\gamma^3}{6} + \dots)(\Gamma'' - \frac{(\Gamma'')^3}{6} + \dots) = \\ &= (vt - \frac{(vt)^3}{6} + \dots)(Vt - \frac{(Vt)^3}{6} + \dots) - (vt - \frac{(vt)^3}{6} + \dots)(Vt - \frac{(Vt)^3}{6} + \dots). \end{aligned}$$

The quadratic terms vVt^2 cancel each other out and all the remaining terms contain t in order four or higher than four.

Altogether,

$$\begin{aligned} \lim_{t \to 0} \frac{f''F - fF''}{t^4} &= \lim_{t \to 0} \frac{rr'R'R'' \sin\gamma'' \sin\Gamma}{4t^4} - \lim_{t \to 0} \frac{r'r''RR' \sin\gamma \sin\Gamma''}{4t^4} \\ &= \frac{(r'R')^2}{4} \lim_{t \to 0} \frac{\sin\gamma'' \sin\Gamma}{t^4} - \frac{(r'R')^2}{4} \lim_{t \to 0} \frac{\sin\gamma \sin\Gamma''}{t^4} \\ &= \frac{(r'R')^2}{4} \lim_{t \to 0} \frac{\sin\gamma'' \sin\Gamma - \sin\gamma \sin\Gamma''}{t^4}, \end{aligned}$$

which is a finite value thanks to the proposition above. Therefore $f''F - fF''$ is of order four. The order of the second expression can be obtained by following an analogous argument.

□

Lemma 6. *The term* $\det(\mathbf{w}|\mathbf{w}'|\mathbf{w}'')$ *is of order three.*

Proof. First of all, we find a suitable expression for the determinant which will also be useful later. Using the cofactor expansion with respect to the third row and then the sine addition formula yields

$$\det(\mathbf{w}|\mathbf{w}'|\mathbf{w}'') = \begin{vmatrix} \cos\lambda & \cos\lambda' & \cos\lambda'' \\ \sin\lambda & \sin\lambda' & \sin\lambda'' \\ \tan\beta & \tan\beta' & \tan\beta'' \end{vmatrix} = \tan\beta(\cos\lambda'\sin\lambda'' - \cos\lambda''\sin\lambda') -$$
$$- \tan\beta'(\cos\lambda\sin\lambda'' - \cos\lambda''\sin\lambda) + \tan\beta''(\cos\lambda\sin\lambda' - \cos\lambda'\sin\lambda) =$$
$$= \tan\beta\sin(\lambda'' - \lambda') - \tan\beta'\sin(\lambda'' - \lambda) + \tan\beta''\sin(\lambda' - \lambda).$$

The crucial expression here will be $\sin(\lambda'' - \lambda') - \sin(\lambda'' - \lambda) + \sin(\lambda' - \lambda)$. Using Taylor series, we can rewrite it as

$$\lambda'' - \lambda' - \frac{(\lambda'' - \lambda')^3}{6} + \ldots - (\lambda'' - \lambda - \frac{(\lambda'' - \lambda)^3}{6} + \ldots) + \lambda' - \lambda - \frac{(\lambda' - \lambda)^3}{6} + \ldots$$

We can see that the linear terms cancel each other out and what remains is of order three or higher in terms of the respective angles. However, each of the values $\lambda'' - \lambda'$, $\lambda'' - \lambda$ and $\lambda' - \lambda$ is $o(t)$. This is due to the fact that these angles divided by the corresponding time intervals represent the mean angular velocity of the orthogonal projection of Ceres onto the ecliptic relative to the Earth during each interval. Although the relative motion of Ceres with respect to the Earth is rather complicated compared to the motion relative to the Sun, its velocity is nonetheless bounded and thus the limit as t approaches zero is finite. Therefore the expression $\sin(\lambda'' - \lambda') - \sin(\lambda'' - \lambda) + \sin(\lambda' - \lambda)$ is of order three not only with respect to the angular differences, but also with respect to t. Hence the limit

$$\lim_{t\to 0} \frac{\det(\mathbf{w}|\mathbf{w}'|\mathbf{w}'')}{t^3} =$$
$$= \lim_{t\to 0} \frac{\tan\beta\sin(\lambda'' - \lambda')}{t^3} - \lim_{t\to 0} \frac{\tan\beta'\sin(\lambda'' - \lambda)}{t^3} + \lim_{t\to 0} \frac{\tan\beta''\sin(\lambda' - \lambda)}{t^3}$$
$$= \tan\beta' \lim_{t\to 0} \frac{\sin(\lambda'' - \lambda')}{t^3} - \tan\beta' \lim_{t\to 0} \frac{\sin(\lambda'' - \lambda)}{t^3} + \tan\beta' \lim_{t\to 0} \frac{\sin(\lambda' - \lambda)}{t^3}$$
$$= \tan\beta' \lim_{t\to 0} \frac{\sin(\lambda'' - \lambda') - \sin(\lambda'' - \lambda) + \sin(\lambda' - \lambda)}{t^3}$$

is a finite value. That proves the statement of the lemma. □

Lemma 7. *Each of the terms* $\det(\mathbf{w}|\mathbf{w}'|\mathbf{W}')$, $\det(\mathbf{w}|\mathbf{w}''|\mathbf{W}')$, $\det(\mathbf{w}''|\mathbf{w}|\mathbf{W}')$, $\det(\mathbf{w}''|\mathbf{w}'|\mathbf{W}')$ *and* $\det(\mathbf{w}|\mathbf{W}'|\mathbf{w}'')$ *is of order one.*

Proof. First of all, recall that by definition, we have identified the ecliptic with the xy-plane and therefore the Earth's heliocentric latitudes B, B' and B'' are all zero. Hence the term involving $\tan B'$ in the following expression is also zero. Similarly to the previous proof, we begin by explicitly calculating the determinant:

$$\det(\mathbf{w}|\mathbf{w}'|\mathbf{W}') = \begin{vmatrix} \cos\lambda & \cos\lambda' & \cos L' \\ \sin\lambda & \sin\lambda' & \sin L' \\ \tan\beta & \tan\beta' & \tan B' \end{vmatrix} = \tan\beta(\cos\lambda'\sin L' - \cos L'\sin\lambda') -$$
$$- \tan\beta'(\cos\lambda\sin L' - \cos L'\sin\lambda).$$

As we will see, it is sufficient to show that both $\cos\lambda' - \cos\lambda$ and $\sin\lambda' - \sin\lambda$ are $o(t)$. This follows from the full Taylor expansions. Indeed, we have

$$\cos\lambda' - \cos\lambda = \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda')^{2n}}{(2n)!} - \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^{2n}}{(2n)!} = \sum_{n=1}^{\infty} (-1)^n \frac{(\lambda')^{2n} - \lambda^{2n}}{(2n)!}$$

and similarly

$$\sin\lambda' - \sin\lambda = \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda')^{2n+1} - \lambda^{2n+1}}{(2n+1)!}.$$

Now, because $u-v$ divides $u^n - v^n$ for all $u, v \in \mathbb{R}$ and $n \in \mathbb{N}$, we can see that $\lambda' - \lambda$ divides each term of the Taylor series of $\cos\lambda' - \cos\lambda$ and of $\sin\lambda' - \sin\lambda$ as well. Using the fact that $\lambda' - \lambda$ is $o(t)$ (see the proof of Lemma 6), we conclude that the studied expressions are also $o(t)$. This enables us to determine the order of the determinant in question.

Particularly, we have

$$\begin{aligned}
&\lim_{t\to 0} \frac{\det(\mathbf{w}|\mathbf{w}'|\mathbf{W}')}{t} = \\
&\quad = \tan\beta' \lim_{t\to 0} \frac{\cos\lambda' \sin L' - \cos L' \sin\lambda' - (\cos\lambda \sin L' - \cos L' \sin\lambda)}{t} \\
&\quad = \tan\beta' \lim_{t\to 0} \frac{\sin L'(\cos\lambda' - \cos\lambda) - \cos L'(\sin\lambda' - \sin\lambda)}{t} \\
&\quad = \tan\beta' \left(\sin L' \lim_{t\to 0} \frac{\cos\lambda' - \cos\lambda}{t} - \cos L' \lim_{t\to 0} \frac{\sin\lambda' - \sin\lambda}{t} \right)
\end{aligned}$$

with both latter limits being finite. So the determinant is of order one as required. The order of the remaining determinants can be determined analogously. □

Lemma 8. *Each of the expressions*

- $D \det(\mathbf{w}|\mathbf{W}|\mathbf{w}'') - D'' \det(\mathbf{w}|\mathbf{W}''|\mathbf{w}'')$,
- $D \det(\mathbf{w}|\mathbf{W}|\mathbf{W}') - D'' \det(\mathbf{w}|\mathbf{W}''|\mathbf{W}')$,
- $D \det(\mathbf{w}''|\mathbf{W}|\mathbf{W}') - D'' \det(\mathbf{w}''|\mathbf{W}''|\mathbf{W}')$

is of order two.

Proof. We will determine the order of the first expression. Again, recall that $\tan B = \tan B'' = 0$. We begin by rewriting the determinants

$$\det(\mathbf{w}|\mathbf{W}|\mathbf{w}'') = \begin{vmatrix} \cos\lambda & \cos L & \cos\lambda'' \\ \sin\lambda & \sin L & \sin\lambda'' \\ \tan\beta & \tan B & \tan\beta'' \end{vmatrix} = \tan\beta(\cos L \sin\lambda'' - \cos\lambda'' \sin L) + \\ + \tan\beta''(\cos\lambda \sin L - \cos L \sin\lambda)$$

and

$$\det(\mathbf{w}|\mathbf{W}|\mathbf{w}'') = \begin{vmatrix} \cos\lambda & \cos L'' & \cos\lambda'' \\ \sin\lambda & \sin L'' & \sin\lambda'' \\ \tan\beta & \tan B'' & \tan\beta'' \end{vmatrix} = \tan\beta(\cos L'' \sin\lambda'' - \cos\lambda'' \sin L'') + \\ + \tan\beta''(\cos\lambda \sin L'' - \cos L'' \sin\lambda).$$

Next, we factorize

$$\begin{aligned}
&\cos L \sin\lambda'' - \cos\lambda'' \sin L + \cos\lambda \sin L - \cos L \sin\lambda - \\
&\quad - (\cos L'' \sin\lambda'' - \cos\lambda'' \sin L'' + \cos\lambda \sin L'' - \cos L'' \sin\lambda) = \\
&\quad = \cos L(\sin\lambda'' - \sin\lambda) - \sin L(\cos\lambda'' - \cos\lambda) - \\
&\quad - \cos L''(\sin\lambda'' - \sin\lambda) + \sin L''(\cos\lambda'' - \cos\lambda) = \\
&\quad = (\sin L'' - \sin L)(\cos\lambda'' - \cos\lambda) - (\cos L'' - \cos L)(\sin\lambda'' - \sin\lambda).
\end{aligned}$$

In the proof of the previous lemma, we have seen that $\cos\lambda' - \cos\lambda$ and $\sin\lambda' - \sin\lambda$ are expressions of order one. By a completely analogous argument, we obtain that each of the expressions $(\sin L'' - \sin L)$, $(\cos\lambda'' - \cos\lambda)$, $(\cos L'' - \cos L)$ and $(\sin\lambda'' - \sin\lambda)$ is also $o(t)$. Therefore the products of their pairs are $o(t^2)$ and the whole expression is of order two as well. Now, because

$$\lim_{t\to 0} D = \lim_{t\to 0} D'' = D'$$

and

$$\lim_{t\to 0} \tan\beta = \lim_{t\to 0} \tan\beta'' = \tan\beta',$$

we can factor these constants out of the studied limit

$$\lim_{t\to 0} \frac{D \det(\mathbf{w}|\mathbf{W}|\mathbf{w}'') - D'' \det(\mathbf{w}|\mathbf{W}''|\mathbf{w}'')}{t^2}$$

as we did with other constants a few times already in the previous proofs. What remains after this factorization is equal to

$$D' \tan\beta' \lim_{t\to 0} \frac{(\sin L'' - \sin L)(\cos\lambda'' - \cos\lambda) - (\cos L'' - \cos L)(\sin\lambda'' - \sin\lambda)}{t^2}.$$

But we know that the numerator is $o(t^2)$ and hence the limit is finite. Therefore the studied expression is of order two. The order of the remaining expressions can be obtained similarly. □

Corollary. 1. The left hand side of equation (2.6) is $o(t^5)$. The first term $(f''F - fF'')(D\det(\mathbf{w}|\mathbf{W}|\mathbf{w}'') - D''\det(\mathbf{w}|\mathbf{W}''|\mathbf{w}''))$ on the right hand side is $o(t^6)$. The second term $((f + f'')F' - (F + F'')f')D'\det(\mathbf{w}|\mathbf{W}'|\mathbf{w}'')$ on the right hand side is $o(t^5)$. Therefore we can approximate the first term on the right hand side by zero.

2. The left hand side of equation (2.7) is of order three, while the right hand side is of order six. Therefore we can approximate the right hand side by zero.

3. The left hand side of equation (2.8) is of order three, while the right hand side is of order six. Therefore we can approximate the right hand side by zero.

Proof. The corollary follows from the five lemmas above and the small o arithmetic: the sum of two functions of order n is again of order n and the product of two functions of orders n and m is of order $n + m$. □

2.3 First results

In this section, we return to follow the works [Gauss, 1809] and [Teets and Whitehead, 1999] and base estimates of the values δ and δ'' on the value of δ'. To that end, we use the second and the third statements in the Corollary above, which allow us to approximate equations (2.7) and (2.8). In particular, we have equations (we will often refer to approximately correct equations still as equations)

$$(F + F'')(f'\delta' \det(\mathbf{w}|\mathbf{w}'|\mathbf{W}') + f''\delta'' \det(\mathbf{w}|\mathbf{w}''|\mathbf{W}')) \approx 0$$

and

$$(F + F'')(f\delta \det(\mathbf{w}''|\mathbf{w}|\mathbf{W}') + f'\delta' \det(\mathbf{w}''|\mathbf{w}'|\mathbf{W}')) \approx 0.$$

We can divide them by $(F + F'')$ and move some terms to the other side to get

$$f''\delta'' \det(\mathbf{w}|\mathbf{w}''|\mathbf{W}') \approx -f'\delta' \det(\mathbf{w}|\mathbf{w}'|\mathbf{W}')$$

and

$$f\delta \det(\mathbf{w}''|\mathbf{w}|\mathbf{W}') \approx -f'\delta' \det(\mathbf{w}''|\mathbf{w}'|\mathbf{W}').$$

Isolating the delta values that we seek yields

$$\delta'' \approx \frac{-f' \det(\mathbf{w}|\mathbf{w}'|\mathbf{W}')}{f'' \det(\mathbf{w}|\mathbf{w}''|\mathbf{W}')}\delta' \qquad \text{and} \qquad \delta \approx \frac{-f' \det(\mathbf{w}''|\mathbf{w}'|\mathbf{W}')}{f \det(\mathbf{w}''|\mathbf{w}|\mathbf{W}')}\delta'. \tag{2.9}$$

Though we have found expressions in terms of δ' and all the involved determinants can be calculated from the observed values, there are still unknown variables on the right hand sides: areas f, $-f'$ and f''. However, they only occur in ratios, which enables us to make further approximations based on Kepler's second law. Recall that the law says that the proportion of the area of the elliptical sector swept out by Ceres and the time interval during which the motion occurs is constant for any time interval. Applying the law to our situation, we obtain

$$\frac{g}{\tau'' - \tau'} = \frac{-g'}{\tau'' - \tau} = \frac{g''}{\tau' - \tau}.$$

Yet more particularly, this implies that

$$\frac{g''}{-g'} \cdot \frac{\tau'' - \tau}{\tau' - \tau} = \frac{g}{-g'} \cdot \frac{\tau'' - \tau}{\tau'' - \tau'} = 1.$$

Therefore we can multiply each right hand side of equations (2.9) by each of these expressions of one, respectively. The minus signs cancel each other out and after rearranging the terms according to the dashes, we get the final formulations

$$\delta'' \approx \frac{f'}{g'} \cdot \frac{g''}{f''} \cdot \frac{\tau'' - \tau}{\tau' - \tau} \cdot \frac{\det(\mathbf{w}|\mathbf{w}'|\mathbf{W}')}{\det(\mathbf{w}|\mathbf{w}''|\mathbf{W}')}\delta'$$

and

$$\delta \approx \frac{g}{f} \cdot \frac{f'}{g'} \cdot \frac{\tau'' - \tau}{\tau'' - \tau'} \cdot \frac{\det(\mathbf{w}''|\mathbf{w}'|\mathbf{W}')}{\det(\mathbf{w}''|\mathbf{w}|\mathbf{W}')}\delta'.$$

Once we approximate the ratios

$$\frac{g}{f}, \frac{f'}{g'} \text{ and } \frac{g''}{f''}$$

of triangular and elliptical areas simply by one, we obtain explicit estimates of δ'' and δ based solely on the observations and δ'. The rationale is that when the angle is small, the elliptical arc resembles a line and the difference between the areas is tiny. Moreover, because

$$\frac{g}{f} > 1, \qquad \frac{f'}{g'} < 1 \qquad \text{and} \qquad \frac{g''}{f''} > 1,$$

the errors tend to cancel each other out. This simple approximation is presented in [Teets and Whitehead, 1999] as the one Gauss used in his earliest works. As we will see in the end, Gauss devised a smart way of refining the results which handles the case if the errors of the approximations are too large.

Alternatively, Gauss [1809] also offers a slightly better approximation. Let us denote by p, p' and p'' the positions of Ceres at τ, τ' and τ''. We begin with the intuitive approximation which says that the area of the small triangle $pp'p''$ is roughly equal to three quarters of the area of the elliptical region between the arc and the chord pp'' (see Fig.2.5). In our notation, this means that

$$f + f' + f'' \approx \frac{3}{4}(f' - g') \text{ or, equivalently, } f' - g' \approx \frac{4}{3}(f + f' + f'').$$

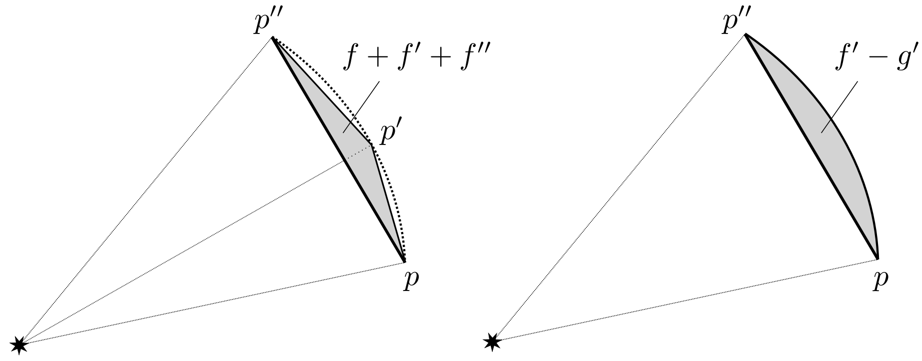

Figure 2.5: Approximating $f' - g'$.

Dividing the latter approximation by $-f'$ and adding 1 to the right hand side, we get

$$\frac{g'}{f'} \approx 1 - \frac{4}{3}\frac{f + f' + f''}{f'}.$$

The ratio on the right hand side is still unknown. However, in the next section, we find an estimate for this value expressed in terms of known values and r' only. (Note that r' can be easily calculated from δ', which is considered as given in this section.) Thus we will be able to substitute this estimate into the approximation above. Inverting the value then yields the required result. Note that the other two ratios, namely of g and f, and of g'' and f'', involve areas that are significantly smaller than $-f'$ and $-g'$. Hence we deem the approximation by one sufficient enough.

Regardless of the choice of final approximations, we have found estimates for δ and δ''. Multiplying each of these by the respective vectors $\mathbf{w}$ and $\mathbf{w}''$, we obtain the Ceres' geocentric position vectors $\delta\mathbf{w} = (\xi, \eta, \zeta)^\top$ and $\delta''\mathbf{w}'' = (\xi'', \eta'', \zeta'')^\top$. Finally, we can add $(\xi, \eta, \zeta)^\top$ and $(\xi'', \eta'', \zeta'')^\top$ to the Earth's heliocentric position vectors $(X, Y, Z)^\top$ and $(X'', Y'', Z'')^\top$ to get the required Ceres' heliocentric position vectors $\mathbf{r}$ and $\mathbf{r}''$ used in the first chapter to calculate the elements of the orbit.

2.4 The main result

The final task is to determine the distance δ' between the Sun and Ceres at τ' assumed as given in the previous section. Again, we follow the work of Teets and Whitehead [1999] based on [Gauss, 1809] with the aim of properly justifying each step and also clarifying some of the calculations. We use many of the tools introduced throughout this work such as the polar equation, Kepler's second and third law and the approximated version of equation (2.6) from section 2.1.

Recall that the true anomaly θ is defined as the counterclockwise angular displacement of Ceres from the perihelion measured with respect to the Sun. Also recall that the polar equation of ellipse in terms of θ (see Lemma 1) is

$$r(\theta) = \frac{k}{1 + e\cos\theta},$$

where $r(\theta)$ is the distance between the Sun and Ceres corresponding to θ, e is the eccentricity and k is the elliptical parameter of the orbit, both defined in the first chapter. For the three particular times τ, τ' and τ'' and the respective true anomalies θ, θ' and θ'', we obtain three particular polar equations

$$r = \frac{k}{1 + e\cos\theta}, \qquad r' = \frac{k}{1 + e\cos\theta'}, \qquad r'' = \frac{k}{1 + e\cos\theta''}.$$

Inverting each of them yields

$$\frac{1}{r} = \frac{1 + e\cos\theta}{k}, \qquad \frac{1}{r'} = \frac{1 + e\cos\theta'}{k}, \qquad \frac{1}{r''} = \frac{1 + e\cos\theta''}{k}. \tag{2.10}$$

Now we multiply the equations above by $\sin(\theta'' - \theta')$, $\sin(\theta - \theta'')$ and $\sin(\theta' - \theta)$, respectively, and add the new equations together to create a new equation (2.14). Thus the left hand side is

$$\frac{\sin(\theta'' - \theta')}{r} + \frac{\sin(\theta - \theta'')}{r'} + \frac{\sin(\theta' - \theta)}{r''} = \frac{2f + 2f' + 2f''}{rr'r''}, \tag{2.11}$$

which is implied by the fact that $2f = r'r''\sin(\theta'' - \theta')$, $2f' = rr''\sin(\theta - \theta'')$ and $2f'' = rr'\sin(\theta' - \theta)$. The case of f was already proved in Lemma 4 and the remaining relations are derived similarly. On the right hand side of the new equation, we get

$$\begin{aligned}\frac{1 + e\cos\theta}{k}\sin(\theta'' - \theta') + \frac{1 + e\cos\theta'}{k}\sin(\theta - \theta'') + \frac{1 + e\cos\theta''}{k}\sin(\theta' - \theta) = \\ = \frac{1}{k}[\sin(\theta'' - \theta') + \sin(\theta - \theta'') + \sin(\theta' - \theta) + \\ + e(\cos\theta\sin(\theta'' - \theta') + \cos\theta'\sin(\theta - \theta'') + \cos\theta''\sin(\theta' - \theta))].\end{aligned}$$

But the bracket multiplied by e can be shown using the sine addition formula to be zero:

$$\begin{aligned}&\cos\theta\sin(\theta'' - \theta') + \cos\theta'\sin(\theta - \theta'') + \cos\theta''\sin(\theta' - \theta) = \\ &= \cos\theta(\sin\theta''\cos\theta' - \sin\theta'\cos\theta'') + \cos\theta'(\sin\theta\cos\theta'' - \sin\theta''\cos\theta) + \\ &+ \cos\theta''(\sin\theta'\cos\theta - \sin\theta\cos\theta') = 0.\end{aligned}$$

Therefore the right hand side equals

$$\frac{\sin(\theta'' - \theta') + \sin(\theta - \theta'') + \sin(\theta' - \theta)}{k}. \tag{2.12}$$

This expression can be further manipulated to serve our purposes. For that, we need a rather obscure trigonometric identity.

Lemma 9. *For all $U, V \in \mathbb{R}$, it holds that*

$$\sin U + \sin V - \sin(U+V) = 4\sin\left(\frac{U}{2}\right)\sin\left(\frac{V}{2}\right)\sin\left(\frac{U+V}{2}\right).$$

Proof. To simplify the notation, let $u = U/2$ and $v = V/2$. We will use the sine addition formula and the double angle identities: $\cos U = 1 - 2\sin^2 u$ in the form $1 - \cos U = 2\sin^2 u$ and $\sin U = 2\sin u \cos u$ for both U and V. Altogether, we have

$$\begin{aligned}
\sin U + \sin V - \sin(U+V) &= \sin U + \sin V - (\sin U \cos V + \sin V \cos U) \\
&= \sin U(1-\cos V) + \sin V(1-\cos U) \\
&= 2\sin u \cos u \cdot 2\sin^2 v + 2\sin v \cos v \cdot 2\sin^2 u \\
&= 4\sin u \sin v(\sin v \cos u + \sin u \cos v) \\
&= 4\sin u \sin v \sin(u+v).
\end{aligned}$$

□

Consider the case $U = \theta'' - \theta'$ and $V = \theta' - \theta$. Then $U + V = \theta'' - \theta$ and, noting that $\sin(\theta - \theta'') = -\sin(\theta'' - \theta)$, we can use the lemma to rewrite expression (2.12) as

$$\frac{4}{k}\sin\left(\frac{\theta''-\theta'}{2}\right)\sin\left(\frac{\theta'-\theta}{2}\right)\sin\left(\frac{\theta''-\theta}{2}\right). \tag{2.13}$$

Overall, we can equate expression (2.11) derived from the left hand sides of (2.10) with expression (2.13) derived from the right hand sides of (2.10) to obtain

$$\frac{2f+2f'+2f''}{rr'r''} = \frac{4}{k}\sin\left(\frac{\theta''-\theta'}{2}\right)\sin\left(\frac{\theta'-\theta}{2}\right)\sin\left(\frac{\theta''-\theta}{2}\right) \tag{2.14}$$

Now recall the area relationship $-2f' = rr''\sin(\theta''-\theta)$. We use it together with the double angle identity

$$\sin(\theta''-\theta) = 2\sin\left(\frac{\theta''-\theta}{2}\right)\cos\left(\frac{\theta''-\theta}{2}\right)$$

to obtain an expression for one of the sines in (2.14):

$$\sin\left(\frac{\theta''-\theta}{2}\right) = \frac{\sin(\theta''-\theta)}{2\cos\left(\frac{\theta''-\theta}{2}\right)} = \frac{-2f'}{2rr''\cos\left(\frac{\theta''-\theta}{2}\right)} = \frac{-f'}{rr''\cos\left(\frac{\theta''-\theta}{2}\right)}.$$

Substituting this fraction into equation (2.14) and cancelling the common factors 2 and rr'' yields

$$\frac{f+f'+f''}{r'} = \frac{2}{k}\sin\left(\frac{\theta''-\theta'}{2}\right)\sin\left(\frac{\theta'-\theta}{2}\right)\frac{-f'}{\cos\left(\frac{\theta''-\theta}{2}\right)}.$$

Multiplying by r' and dividing by f' results in

$$\frac{f+f'+f''}{f'} = -\frac{2r'}{k}\sin\left(\frac{\theta''-\theta'}{2}\right)\sin\left(\frac{\theta'-\theta}{2}\right)\frac{1}{\cos\left(\frac{\theta''-\theta}{2}\right)}. \tag{2.15}$$

Now we turn our attention to the elliptical parameter k. In section 1.4, we expressed the area of the ellipse as $\pi a^{3/2}\sqrt{k}$, where a denotes the length of the semi-major axis of the orbit and is one of the six elements. We can use Kepler's second law to equate the ratios of the elliptical areas and the corresponding time intervals as

$$\frac{\pi a^{3/2}\sqrt{k}}{t_p} = \frac{g}{\tau'' - \tau'} = \frac{g''}{\tau' - \tau},$$

where t_p denotes the period of the motion of Ceres. Therefore, we also have

$$\frac{\pi^2 a^3 k}{t_p^2} = \frac{gg''}{(\tau'' - \tau')(\tau' - \tau)}.$$

Recall that Kepler's third law says that

$$\frac{A^3}{T_e^2} = \frac{a^3}{t_p^2},$$

where A is the length of the semi-major axis of the Earth's orbit and T_e is one year. Combining the law with the equation above yields

$$\frac{\pi^2 A^3 k}{T_e^2} = \frac{gg''}{(\tau'' - \tau')(\tau' - \tau)}$$

and thus

$$k = \frac{T_e^2 gg''}{\pi^2 A^3 (\tau'' - \tau')(\tau' - \tau)}. \tag{2.16}$$

To proceed, we define new variables M, M' and M'' at times τ, τ' and τ'', respectively, by the formulas

$$M = \frac{2\pi}{T_e}(\tau - \tau_e), \qquad M' = \frac{2\pi}{T_e}(\tau' - \tau_e), \qquad M'' = \frac{2\pi}{T_e}(\tau'' - \tau_e),$$

where τ_e denotes the time of the last Earth's passage through perihelion. We can interpret M as a hypothetical angle which would measure Earth's angular displacement from perihelion at τ on a hypothetical orbit if the orbit was circular with the same period as the actual one and the Earth moved along it with constant angular velocity. (In fact, M is often used in astronomy and called the *mean anomaly*.)

Note that all three values of M are known by definition: the Earth's period is one year, the Earth's perihelion passage is also well documented and the times τ, τ' and τ'' come together with the Piazzi's observations. Moreover, we have that

$$\frac{M'' - M'}{\tau'' - \tau'} = \frac{2\pi}{T_e(\tau'' - \tau')}(\tau'' - \tau_e - (\tau' - \tau_e)) = \frac{2\pi}{T_e},$$

which agrees with the interpretation that the angular velocity in the hypothetical orbit is constant and this constant can be expressed simply as one full rotation over one full period. Similarly, we also have

$$\frac{M' - M}{\tau' - \tau} = \frac{2\pi}{T_e}.$$

Inverting these two equations and multiplying them together yields

$$\frac{T_e^2}{4\pi^2} = \frac{(\tau'' - \tau')(\tau' - \tau)}{(M'' - M')(M' - M)}.$$

Substituting this expression into equation (2.16) and cancelling the time differences out, we obtain

$$k = \frac{4gg''}{A^3(M'' - M')(M' - M)}. \tag{2.17}$$

In the next step, we need to introduce small angle approximations. In particular,

$$\cos\left(\frac{\theta'' - \theta}{2}\right) \approx 1, \qquad rr'' \approx (r')^2,$$

which are rather intuitive: the first one can be justified using Taylor series and the second one is due to the fact that as the angle between $\mathbf{r}$ and $\mathbf{r}''$ approaches zero, the distances r and r'' both approach a single distance r'. Also, in case the observed short arc between $\mathbf{r}$ and $\mathbf{r}''$ does not involve the perihelion or the aphelion it holds that $r < r' < r''$ or $r > r' > r''$ and hence the errors tend to cancel each other out when multiplied. Further, we approximate the elliptical areas as

$$g \approx r'r'' \sin\left(\frac{\theta'' - \theta'}{2}\right), \qquad g'' \approx rr' \sin\left(\frac{\theta' - \theta}{2}\right).$$

This makes geometrical sense. Consider for example the first approximation (see Fig.2.6). The right hand side expression is equal to twice the area of a right triangle with a base of length r' and a height of length

$$r'' \sin\left(\frac{\theta'' - \theta'}{2}\right).$$

However, a right triangle with a base of length r'' and a height of length

$$r' \sin\left(\frac{\theta'' - \theta'}{2}\right)$$

has the same area. Each of these two right triangles approximates the corresponding part of the g area. Thus their sum approximates g.

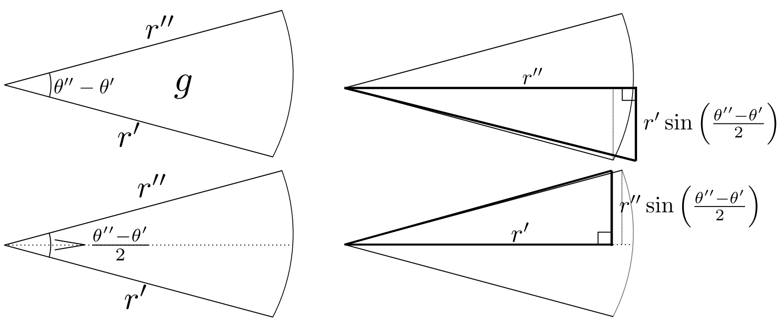

Figure 2.6: Approximating g.

We substitute these approximations in the form

$$\sin\left(\frac{\theta''-\theta'}{2}\right)\approx\frac{g}{r'r''},\qquad \sin\left(\frac{\theta'-\theta}{2}\right)\approx\frac{g''}{rr'}$$

together with the ones above into equation (2.15). Thus we simply ignore the cosine and further have

$$\frac{f+f'+f''}{f'}\approx-\frac{2r'}{k}\frac{g}{r'r''}\frac{g''}{rr'}\approx-\frac{2gg''}{k(r')^3}$$

and substituting k from (2.17) further yields

$$\frac{f+f'+f''}{f'}\approx-\frac{A^3(M''-M')(M'-M)}{4gg''}\cdot\frac{2gg''}{(r')^3}.$$

We can cancel out the areas to obtain the final approximation

$$\frac{f+f'+f''}{f'}\approx-\frac{A^3(M''-M')(M'-M)}{2(r')^3}. \tag{2.18}$$

This is the estimate expressed in terms of r' and known values required in the end of the previous section. By a completely analogous argument involving the variables concerning the Earth, we obtain a similar approximation

$$\frac{F+F'+F''}{F'}\approx-\frac{A^3(M''-M')(M'-M)}{2(R')^3}. \tag{2.19}$$

Combining the two approximate equations, we infer that

$$\frac{f+f'+f''}{f'}-\frac{F+F'+F''}{F'}\approx\frac{A^3}{2}(M''-M')(M'-M)\left(\frac{1}{(R')^3}-\frac{1}{(r')^3}\right).$$

Multiplying both sides by $f'F'$ yields

$$(f+f'+f'')F'-(F+F'+F'')f'\approx f'F'\frac{A^3}{2}(M''-M')(M'-M)\left(\frac{1}{(R')^3}-\frac{1}{(r')^3}\right)$$

and after simplification on the left hand side

$$(f+f'')F'-(F+F'')f'\approx f'F'\frac{A^3}{2}(M''-M')(M'-M)\left(\frac{1}{(R')^3}-\frac{1}{(r')^3}\right). \tag{2.20}$$

Now we finally return to the first central equation (2.6) from the end of section 2.1. First, we repeat it in full:

$$\begin{aligned}(F+F'')f'\delta'\det(\mathbf{w}|\mathbf{w}'|\mathbf{w}'')=&\\ &=(f''F-fF'')(D\det(\mathbf{w}|\mathbf{W}|\mathbf{w}'')-D''\det(\mathbf{w}|\mathbf{W}''|\mathbf{w}''))+\\ &+((f+f'')F'-(F+F'')f')D'\det(\mathbf{w}|\mathbf{W}'|\mathbf{w}'').\end{aligned}$$

However, recall that in the Corollary of section 2.2, we concluded that the first term on the right hand side can be approximated by zero. Therefore, we have the approximate version

$$(F+F'')f'\delta'\det(\mathbf{w}|\mathbf{w}'|\mathbf{w}'')\approx((f+f'')F'-(F+F'')f')D'\det(\mathbf{w}|\mathbf{W}'|\mathbf{w}'').$$

and we finally substitute the tediously derived estimate (2.20) into this equation, obtaining

$$(F+F'')f'\delta' \det(\mathbf{w}|\mathbf{w}'|\mathbf{w}'') \approx$$
$$\approx f'F'\frac{A^3}{2}(M''-M')(M'-M)\left(\frac{1}{(R')^3}-\frac{1}{(r')^3}\right)D'\det(\mathbf{w}|\mathbf{W}'|\mathbf{w}'').$$

Notice that we can cancel f' out. Next, we turn our attention to F, F' and F''. We could calculate the precise values of each of the areas, but for simplicity, Gauss uses another approximation $F+F'' \approx -F'$. This means that we choose to ignore the small triangle of area $F+F'+F''$ formed by the three positions of the Earth at τ, τ' and τ'' labeled in Figure 2.7 as P, P' and P''. Hence if we substitute $-F'$ in for the factor $F+F''$ on the left hand side, we can cancel F' out as well and what remains is

$$-\delta'\det(\mathbf{w}|\mathbf{w}'|\mathbf{w}'') \approx \frac{A^3}{2}(M''-M')(M'-M)\left(\frac{1}{(R')^3}-\frac{1}{(r')^3}\right)D'\det(\mathbf{w}|\mathbf{W}'|\mathbf{w}'').$$

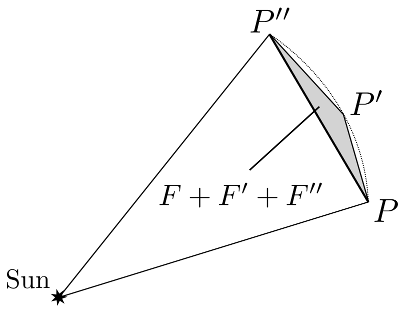

Figure 2.7: Area $F+F'+F''$.

Now we recall that by definition of the coordinate systems and particularly the xy-plane, we have $D'=R'$. We also move some of the factors to obtain

$$-\frac{\det(\mathbf{w}|\mathbf{w}'|\mathbf{w}'')}{\det(\mathbf{w}|\mathbf{W}'|\mathbf{w}'')}\cdot\frac{2}{(M''-M')(M'-M)} \approx A^3\left(\frac{1}{(R')^3}-\frac{1}{(r')^3}\right)\frac{R'}{\delta'}.$$

Because Earth's orbit is almost circular and the distance between the Sun and the Earth is always very close to the mean distance A, we can use a rather unusual trick: we substitute R' in for the simpler value of A. This will perhaps surprisingly allow us to finish the calculations. Thus we have

$$-\frac{\det(\mathbf{w}|\mathbf{w}'|\mathbf{w}'')}{\det(\mathbf{w}|\mathbf{W}'|\mathbf{w}'')}\cdot\frac{2}{(M''-M')(M'-M)} \approx \left(1-\frac{(R')^3}{(r')^3}\right)\frac{R'}{\delta'}. \tag{2.21}$$

We are now very close to the finish. However, before we reach the climax, we need a final auxiliary lemma.

Lemma 10. *The following relationship between R', r', δ' and the observed values holds:*

$$\frac{R'}{r'} = \frac{R'}{\delta'}\left(1+\tan^2\beta' + \left(\frac{R'}{\delta'}\right)^2 + 2\frac{R'}{\delta'}\cos(\lambda'-L')\right)^{-1/2}.$$

Proof. The equality is a result of the definitions and a few trigonometric observations. Before we begin with some necessary notation, we note that each value in the statement of the lemma belongs to the time τ'. Each of the following values, positions and figures is therefore considered to be static and taken at τ'. To avoid confusion, we keep labeling the new variables with a single dash even though the other two versions will not be needed.

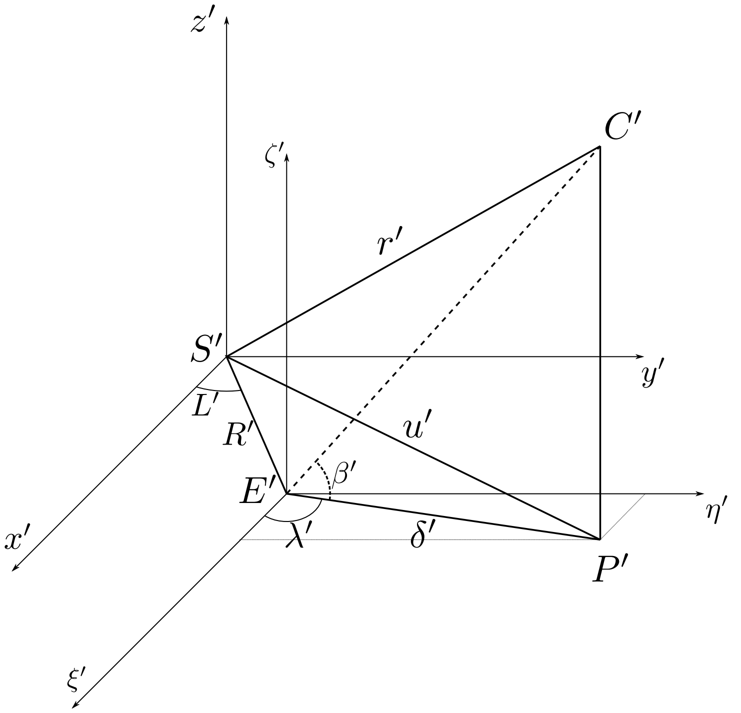

Figure 2.8: Situation at τ'.

First of all, let C',E' and S' denote the positions of Ceres, the Earth and the Sun at τ', respectively (see Fig.2.8). Moreover, let P' denote the orthogonal projection of C' onto the $x'y'$-plane and let u' denote the distance $|S'P'|$. We have two triangles $E'P'C'$ and $S'P'C'$, both with a right angle at P' and the same height $P'C'$ (see Fig.2.9). From the first triangle, we deduce that $|P'C'| = \delta' \tan \beta'$ by definition of tangent. Looking at the second triangle and using the Pythagorean theorem, we observe that

$$(u')^2 = (r')^2 - (\delta')^2 \tan^2 \beta'. \tag{2.22}$$

Finally, we consider the triangle $S'E'P'$ that lies within the $x'y'$-plane (see Fig.2.9). The angle at E' is equal to $180° - \alpha$, where $\alpha = |\lambda' - L'|$ if $|\lambda' - L'| < 180°$, and $\alpha = 360° - |\lambda' - L'|$ otherwise. Either way, we have that $\cos \alpha = \cos |\lambda' - L'| = \cos (\lambda' - L')$ and we can use the law of cosines to deduce that

$$\begin{aligned}(u')^2 &= (R')^2 + (\delta')^2 - 2R'\delta' \cos (180° - \alpha) \\ &= (R')^2 + (\delta')^2 + 2R'\delta' \cos (\lambda' - L').\end{aligned}$$

Substituting the value of $(u')^2$ from (2.22) and adding $(\delta')^2 \tan^2 \beta'$ to both sides of the equation yields

$$(r')^2 = (R')^2 + (\delta')^2 + (\delta')^2 \tan^2 \beta' + 2R'\delta' \cos (\lambda' - L').$$

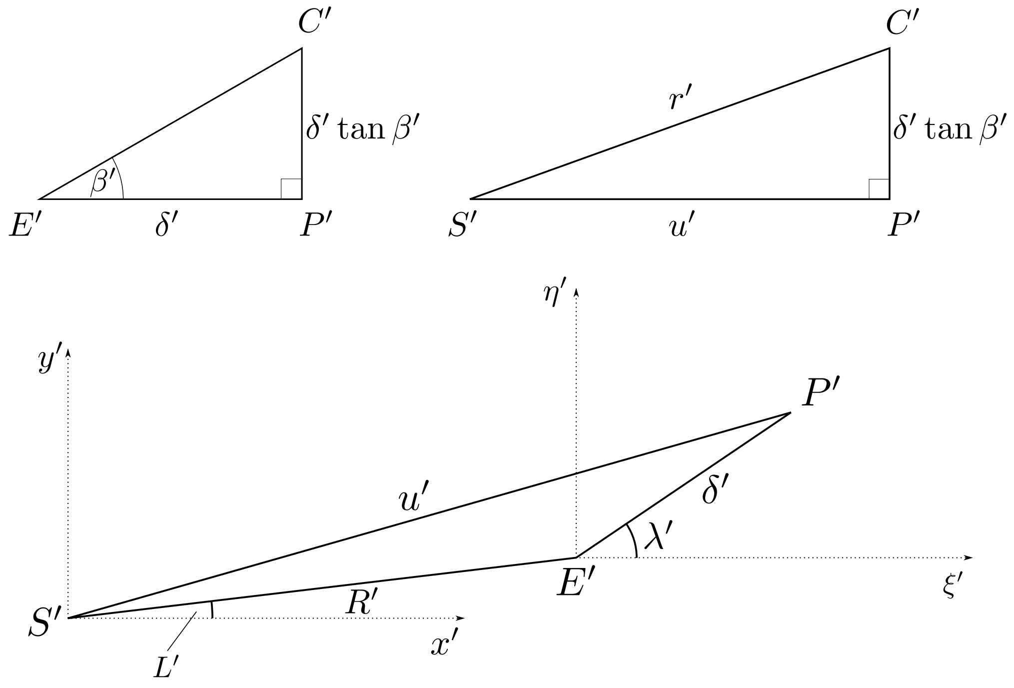

Figure 2.9: A closer look at the triangles.

To conclude the calculation, we simultaneously multiply both sides by $(R')^2$ and divide them by $(r')^2(\delta')^2$ to obtain

$$\left(\frac{R'}{\delta'}\right)^2 = \left(\frac{R'}{r'}\right)^2 \left(\left(\frac{R'}{\delta'}\right)^2 + 1 + \tan^2\beta' + 2\frac{R'}{\delta'}\cos(\lambda' - L')\right).$$

Dividing by the expression in the large bracket on the right hand side and taking the square root of both sides results in the required equality. □

Now we have almost all we need to conclude the method. What remains is expanding the determinants in equation (2.21). We have already calculated $\det(\mathbf{w}|\mathbf{w}'|\mathbf{w}'')$ in the proof of Lemma 6:

$$\det(\mathbf{w}|\mathbf{w}'|\mathbf{w}'') = \tan\beta\sin(\lambda'' - \lambda') - \tan\beta'\sin(\lambda'' - \lambda) + \tan\beta''\sin(\lambda' - \lambda).$$

Recalling one more time that $B' = 0$ and hence $\tan B' = 0$, the second determinant involved is equal to

$$\det(\mathbf{w}|\mathbf{W}'|\mathbf{w}'') = \begin{vmatrix} \cos\lambda & \cos L' & \cos\lambda'' \\ \sin\lambda & \sin L' & \sin\lambda'' \\ \tan\beta & \tan B' & \tan\beta'' \end{vmatrix} = \tan\beta(\cos L'\sin\lambda'' - \cos\lambda''\sin L') +$$
$$+ \tan\beta''(\cos\lambda\sin L' - \cos L'\sin\lambda) = \tan\beta\sin(\lambda'' - L') + \tan\beta''\sin(L' - \lambda)$$

by the sine addition formula.

Substituting these expressions into (2.21), we obtain the final approximate equality

$$\left(1-\left(\frac{R'}{r'}\right)^3\right)\frac{R'}{\delta'}\approx$$
$$\approx -2\frac{\tan\beta\sin(\lambda''-\lambda')-\tan\beta'\sin(\lambda''-\lambda)+\tan\beta''\sin(\lambda'-\lambda)}{(M''-M')(M'-M)(\tan\beta\sin(\lambda''-L')+\tan\beta''\sin(L'-\lambda))}.$$

Notice that the variables on the right hand side of the equation either come directly from Piazzi's observations or can be determined from Earth's well documented motion. Using the substitution from Lemma 10, we obtain a transformed equation with a single unknown R'/δ'. This equation is fairly complicated but can be solved numerically. Because the value of R' is known, we can determine δ' directly from this numerical solution. Thus we have what we needed in the previous section to find $\mathbf{r}$ and $\mathbf{r}''$. If necessary, we can calculate r' using the relationship $\mathbf{r}' = (X', Y', Z')^\top + \delta'\mathbf{w}'$. With that, we consider the calculations finished.

2.5 Adjustments

The work of Teets and Whitehead [1999] ends at this point. However, the original article [Gauss, 1809] offers some comments on how to improve the estimates. He gives two alternatives: one can either use more precise approximations, which he finds far more difficult and immediately discards, or repeat the method with a slightly altered input. As Abdulle and Wanner [2003] noted, this adjustment process might be in fact an early instance of the least squares method introduced by Legendre four years later in 1805.

In particular, we are advised to calculate the elements of the orbit of Ceres using the estimates of $\mathbf{r}$ and $\mathbf{r}''$ as described in the first chapter. From the elements, we can calculate a hypothetical position of Ceres at τ'. This position comes with alternative values of λ' and β', say $\tilde{\lambda}$ and $\tilde{\beta}$. Further, we let $\mathcal{L} = |\lambda' - \tilde{\lambda}|$ and $\mathcal{B} = |\beta' - \tilde{\beta}|$ denote the discrepancies between the observed and the calculated positions at τ'. Finally, we repeat the Gauss's method with the alternative input values λ,$\lambda' - \mathcal{L}$,λ'' and β,$\beta' - \mathcal{B}$, β''. If necessary, we can iterate until the error is low enough. This is the reason for allowing some perhaps crude approximations throughout the method.

Conclusion

To conclude our work, we sketch out the overall process between the first discovery of Ceres and its definite rediscovery a year later.

On the first day of 1801, Piazzi discovered Ceres and made a number of observations before Ceres got lost in the glare of the Sun. Gauss picked three observations made at τ, τ' and τ''. He estimated the value of δ' only from these observations and Earth's known motion (Section 2.4). Next, he estimated δ and δ'' from the observations, Earth's motion and the value of δ' and thus obtained $\mathbf{r}$ and $\mathbf{r}''$ (Section 2.3). Then Gauss calculated the elements of the hypothetical orbit if $\mathbf{r}$ and $\mathbf{r}''$ were correct (Chapter 1). He checked if this orbit agreed with the observation made at τ' and adjusted the calculations by iterating the method (Section 2.5). Once he had the elements of an orbit that fits all observations within a satisfactory margin for error, he calculated some future positions of Ceres eagerly expected by his fellow astronomers. Thus, on the first day of 1802, Ceres was definitely rediscovered.

Though the strategy can be simply sketched out, the major steps of the method are highly sophisticated. We find it rather difficult to understand how Gauss managed to connect the necessary equations with such extraordinary foresight. The aim of this work was to present the full process in an accessible manner. We hope we have succeeded in explaining each step carefully and managed to leave the reader with some insight into the secrets of astronomy and perhaps also the great mind of Carl Friedrich Gauss.

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