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Spherically Symmetric Measures

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Title: Spherically Symmetric Measures

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Abstract: A probability distribution is called spherically symmetric if it is invariant with respect to rotations about the origin. This class includes the multivariate standard normal distribution, a multivariate extension of the t-distribution and uniform distributions inside the unit ball or the unit sphere surface. The first part of the thesis summarizes the basic properties of spherically symmetric distributions such as the form of their characteristic function and provides expressions for their moments and the density function. It turns out that spherically symmetric distributions are fully characterized by the distribution of their Euclidean norm or by any of their univariate marginal distributions. As any marginal distribution of a spherically symmetric distribution is also spherically symmetric, the aim of the second part of this thesis is to study the inverse relationship using fractional calculus. For a given n-dimensional spherically symmetric distribution we solve the problem of deciding whether there is a spherically symmetric distribution in higher dimensions whose n-dimensional marginal is as given.

Keywords: spherically symmetric distributions, normal distribution, fractional calculus, multivariate distribution

Contents

Introduction

The multivariate normal distribution is a key distribution in multivariate analysis. The class of spherically symmetric distributions maintains some of its properties and thus may be used instead of the normal distribution.

In this thesis we will cover some of the properties of spherically symmetric distributions which are defined as distributions which remain unchanged when rotated about the origin. This class includes not only the multivariate standard normal distribution or uniform distributions on some symmetric sets (such as the unit sphere surface or the unit ball) but also the multivariate generalizations of the t-distribution and the Laplace distribution.

It turns out that spherically symmetric distributions are fully described by their radial distribution (the distribution of their Euclidean norm) or the distribution of any marginal random variable. Therefore, even though it is a collection of multivariate distributions, many problems are reduced to the univariate cases.

The thesis is organized in the following way. Chapter 1 collects some preliminaries that will be useful for our treatment of spherically symmetric distributions. We introduce the Dirichlet distribution which is used to find the marginal distribution of the uniform distribution on the unit sphere surface. A brief introduction of fractional calculus is given in this chapter, the Riemann-Liouville fractional derivative and integral is defined. Fractional calculus answers the questions about differentiation of a non-integer order, for example the $\frac{1}{2}$-th derivative. The concept was famously introduced in a letter from Leibniz to l'Hopital in 1695.

The second chapter starts with the definition of spherically symmetric distributions. We show that their characteristic function is given as a scalar function of the inner product of its argument. The radial distribution is defined and used to compute the moments of spherically symmetric distributions. Elliptically symmetric distributions are briefly mentioned.

Chapter 3 covers absolutely continuous spherically symmetric distributions and connects the density function of the distribution with the density function of the radial distribution. Chapter 4 then discusses marginal distributions and their density. It is shown that spherical symmetry is preserved in marginal distributions. Thus, the question is whether a spherically symmetric distribution can be extended to higher dimensions in the same way as marginal distributions are an extension to lower dimensions. This problem is addressed in Section 4.2. Several examples are given. For example it is shown that the uniform distribution inside the unit ball in $\mathbb{R}^n$ is a marginal distribution of the uniform distribution on the unit sphere surface in $\mathbb{R}^{n+2}$. Conversely, the standard normal distribution can be extended to any dimension and the extension is also normally distributed.

The last chapter concerns statistical applications and presents several tests for spherical symmetry. We may wonder if our data sample is taken from any spherically symmetric distribution. The tests rely on the properties derived in the previous chapters. We may rely on univariate tests when testing for a given spherically symmetric distribution assuming the symmetry holds.

1. Preliminaries

The aim of this chapter is to summarize the notation, definitions and general properties of functions and distributions which are used throughout the thesis.

1.1 Symbols and Notation

In the thesis the following notation is used: vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in $\mathbb{R}^n$ are considered as column vectors $\mathbf{x} = (x_1, \dots, x_n)^\top$ and $\mathbf{0} = (0, \dots, 0)^\top$, matrices are denoted as A, B, C, ... and I is the identity matrix. The dimensions are understood from the context. A square matrix which satisfies $QQ^\top = I$ is called an *orthogonal matrix* and $\mathcal{O}(n)$ is the group of all $n \times n$ orthogonal matrices. Random variables have the same notation as matrices, yet letters from the second half of the alphabet such as R, X, Y are used. Random vectors are denoted in bold $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ and are also columns $\mathbf{X} = (X_1, \dots, X_n)^\top$. The *cumulative distribution function* (for brevity shortened as c.d.f) of a random vector is denoted by F or G. The *characteristic function* of an n-dimensional random vector $\mathbf{X}$ is given as

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \mathsf{E}\left(e^{i\mathbf{t}^\top \mathbf{X}}\right), \quad \mathbf{t} \in \mathbb{R}^n.$$

The gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad z > 0$$

and the beta function

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}, \quad \alpha, \beta > 0$$

play a key role as normalizing constants of the density functions of several distributions. Among the well-known distributions used in the thesis we have: the n-dimensional normal distributions $\mathcal{N}_n(\mathbf{b}, \Sigma)$, the t-distribution with k degrees of freedom, the χ^2-distribution with k degrees of freedom, the gamma distribution $\mathsf{Gamma}(\alpha, \beta)$, the exponential distribution $\mathsf{Exp}(\lambda)$, the beta distribution $\mathsf{Beta}(\alpha, \beta)$ and the F-distribution with k and n degrees of freedom (all as defined in Forbes et al. [2010]). If vectors $\mathbf{X}$ and $\mathbf{Y}$ have the same distribution we shall write $\mathbf{X} \overset{\mathrm{d}}{=} \mathbf{Y}$.

1.2 Dirichlet Distribution

The Dirichlet distribution is a multivariate distribution defined on a simplex in $\mathbb{R}^n$. This sections discusses its properties (as in Section 1.4 of Fang et al. [1990]) which are used to find a marginal distribution of a uniform distribution on the unit sphere surface in $\mathbb{R}^n$. The results presented in this section are further used in Sections 4.1 and 4.2.

Definition 1. *Let $\alpha_1, \ldots, \alpha_n > 0$ and $Y_1, \ldots, Y_n$ be independent random variables where Y_i has the gamma distribution with parameters (α_i, β), $\beta > 0$, and the density*

$$f_{Y_i}(y) = \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} y^{\alpha_i - 1} e^{-\beta y}$$

for $y > 0$, zero elsewhere. Then

$$(X_1, \ldots, X_n)^\top = \left(\frac{Y_1}{\sum_{i=1}^n Y_i}, \ldots, \frac{Y_n}{\sum_{i=1}^n Y_i} \right)^\top$$

has the Dirichlet distribution *with parameters $\alpha_1, \ldots, \alpha_n$. For brevity we shall write $(X_1, \ldots, X_n)^\top \sim \mathcal{D}_n(\alpha_1, \ldots, \alpha_n)$.*

Since the Dirichlet distribution as defined above is singular in $\mathbb{R}^n$, the Dirichlet distribution could be defined (as in Fang et al. [1990]) just by the first $n-1$ coordinates $(X_1, \ldots, X_{n-1})^\top$ and X_n is determined as

$$1 - \sum_{i=1}^{n-1} X_i.$$

Theorem 1. *The density function of a random vector*

$$(X_1, \ldots, X_n)^\top = \left(X_1, \ldots, X_{n-1}, 1 - \sum_{i=1}^{n-1} X_i \right)^\top \sim \mathcal{D}_n(\alpha_1, \ldots, \alpha_n),$$

for $\alpha_1, \ldots, \alpha_n > 0$, with respect to the Lebesgue measure on $\mathbb{R}^{n-1}$ is

$$f(x_1, \ldots, x_{n-1}) = \frac{\Gamma\left(\sum_{i=1}^n \alpha_i\right)}{\prod_{i=1}^n \Gamma(\alpha_i)} \left(\prod_{i=1}^{n-1} x_i^{\alpha_i - 1} \right) \left(1 - \sum_{i=1}^{n-1} x_i \right)^{\alpha_n - 1} \tag{1.1}$$

for $(x_1, \ldots, x_{n-1})^\top \in \mathbb{R}^{n-1}$, such that

$$\sum_{i=1}^{n-1} x_i < 1, \quad x_1, \ldots, x_{n-1} > 0,$$

and zero elsewhere. In particular, the Dirichlet distribution does not depend on the parameter $\beta > 0$ of the gamma distributions and is therefore defined correctly.

Proof. Denote $\sigma = \sum_{i=1}^n \alpha_i$, $\beta > 0$ and $Y_i \sim \mathsf{Gamma}(\alpha_i, \beta)$ be independent for $i \in \{1, \ldots, n\}$, then $(Y_1, \ldots, Y_n)^\top$ has the joint density

$$f(y_1, \ldots, y_n) = \frac{\beta^\sigma}{\prod_{i=1}^n \Gamma(\alpha_i)} \left(\prod_{i=1}^n y_i^{\alpha_i - 1} \right) e^{-\beta \sum_{i=1}^n y_i}, \quad y_1, \ldots, y_n > 0$$

and zero elsewhere. Let us transform the random variables: for $i \in \{1, \ldots, n-1\}$ set

$$X_i = \frac{Y_i}{\sum_{k=1}^n Y_k}, \quad X = \sum_{i=1}^n Y_i.$$

Then the Jacobian of the transform is x^{n-1} and the support changes from $(0,\infty)^n$ to $A \times (0,\infty)$ where

$$A = \left\{ x_1, \dots, x_{n-1} > 0, \quad \sum_{i=1}^{n-1} x_i < 1 \right\}.$$

Thus, the density of $(X_1, \dots, X_{n-1}, X)^\top$ is

$$\begin{aligned} f(x_1, \dots, x_{n-1}, x) &= \frac{\beta^\sigma \cdot x^{n-1}}{\prod_{i=1}^n \Gamma(\alpha_i)} \left(\prod_{i=1}^{n-1} (x \cdot x_i)^{\alpha_i - 1} \right) \left(1 - \sum_{i=1}^{n-1} x_i \right)^{\alpha_n - 1} x^{\alpha_n - 1} e^{-\beta x} \\ &= \frac{\Gamma\left(\sum_{i=1}^n \alpha_i\right)}{\prod_{i=1}^n \Gamma(\alpha_i)} \left(\prod_{i=1}^{n-1} x_i^{\alpha_i - 1} \right) \left(1 - \sum_{i=1}^{n-1} x_i \right)^{\alpha_n - 1} \frac{\beta^\sigma}{\Gamma(\sigma)} x^{\sigma - 1} e^{-\beta x} \end{aligned}$$

where we have used that $\sigma = \sum_{i=1}^n \alpha_i$. The second half of the function is a density of $\mathsf{Gamma}(\sigma, \beta)$ and hence can be integrated out over $(0,\infty)$ which completes the proof.

□

Theorem 1 and its proof are adapted and extended from Section 1.4 of Fang et al. [1990]. The Dirichlet distribution plays an important role when studying spherically symmetric distributions because it is connected to the marginal distributions of a uniform distribution on the unit sphere surface in $\mathbb{R}^n$.

The following remark is a direct consequence of a summation property of the gamma distribution (Forbes et al. [2010]). When $Y_i \sim \mathsf{Gamma}(\alpha_i, \beta)$ are independent for $i \in \{1, \dots, n\}$ and $\alpha_1, \dots, \alpha_n, \beta > 0$, then

$$\sum_{i=1}^n Y_i \sim \mathsf{Gamma}\left(\sum_{i=1}^n \alpha_i, \beta \right).$$

Remark 1. Let $(X_1, \dots, X_n)^\top \sim \mathcal{D}_n(\alpha_1, \dots, \alpha_n)$ where $\alpha_1, \dots, \alpha_n > 0$. Then for $k < n$ any k components and $1 - \sum_{i=1}^k X_i$ are distributed as

$$\left(X_1, \dots, X_k, 1 - \sum_{i=1}^k X_i \right)^\top \sim \mathcal{D}_{k+1}(\alpha_1, \dots, \alpha_k, \tilde{\alpha})$$

where $\tilde{\alpha} = \sum_{i=k+1}^n \alpha_i$.

In particular,

$$X_i \sim \mathsf{Beta}(\alpha_i, \alpha - \alpha_i)$$

where

$$\alpha = \sum_{j=1}^n \alpha_j.$$

Theorem 2. *Let $(U_1, \dots, U_n)^\top$ be a random vector uniformly distributed on the unit sphere surface in $\mathbb{R}^n$. Then for $k < n$ we have that the marginal distribution $(U_1, \dots, U_k)^\top$ has the density*

$$f(u_1, \dots, u_k) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right) \pi^{\frac{k}{2}}} \left(1 - \sum_{i=1}^k u_i^2 \right)^{\frac{n-k}{2} - 1}, \quad \sum_{i=1}^k u_i^2 < 1$$

and zero elsewhere.

Proof. Let $\mathbf{Z} = (Z_1, \dots, Z_n)^\top \sim \mathcal{N}_n(\mathbf{0}, I)$ where

$$(U_1, \dots, U_n)^\top \stackrel{\mathrm{d}}{=} \frac{\mathbf{Z}}{\|\mathbf{Z}\|}$$

which is certainly true but we will comment on this on detail also in Section 2.1. Random variables $\|\mathbf{Z}\|^2$ and Z_i^2 have χ^2-distributions with n and 1 degree of freedom, respectively. Let us use the fact that the χ^2-distribution with d degrees of freedom is also the $\mathsf{Gamma}(\frac{d}{2}, \frac{1}{2})$ distribution. Denote

$$(Y_1, \dots, Y_n)^\top = \left(\frac{Z_1^2}{\|\mathbf{Z}\|^2}, \dots, \frac{Z_n^2}{\|\mathbf{Z}\|^2}\right)^\top.$$

Thus, the random vector $(Y_1, \dots, Y_n)^\top$ has the Dirichlet distribution $\mathcal{D}_n(\frac{1}{2}, \dots, \frac{1}{2})$ and using Remark 1 the random vector

$$\left(Y_1, \dots, Y_k, 1 - \sum_{i=1}^{k} Y_i\right)^\top$$

has the distribution $\mathcal{D}_{k+1}\left(\frac{1}{2}, \dots, \frac{1}{2}, \frac{n-k}{2}\right)$ with the density function (Theorem 1)

$$f(y_1, \dots, y_k) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)\pi^{\frac{k}{2}}} \left(\prod_{i=1}^{k} y_i^{-\frac{1}{2}}\right) \left(1 - \sum_{i=1}^{k} y_i\right)^{\frac{n-k}{2}-1}$$

for $y_1, \dots, y_k > 0$ and $\sum_{i=1}^{k} y_i < 1$.

Let us transform the random vector $(Y_1, \dots, Y_k)^\top$ into

$$\left(\tilde{Y}_1, \dots, \tilde{Y}_k\right)^\top = \left(\sqrt{Y_1}, \dots, \sqrt{Y_k}\right)^\top.$$

The Jacobian of this transform is

$$2^k \prod_{i=1}^{k} \tilde{y}_i$$

which means

$$f(\tilde{y}_1, \dots, \tilde{y}_k) = \frac{\Gamma\left(\frac{n}{2}\right) 2^k}{\Gamma\left(\frac{n-k}{2}\right)\pi^{\frac{k}{2}}} \left(1 - \sum_{i=1}^{k} \tilde{y}_i^2\right)^{\frac{n-k}{2}-1}$$

for $\tilde{y}_1, \dots, \tilde{y}_k > 0$ and $\sum_{i=1}^{k} \tilde{y}_i^2 < 1$. We have derived the density function of

$$\left(\frac{|Z_1|}{\|\mathbf{Z}\|}, \dots, \frac{|Z_k|}{\|\mathbf{Z}\|}\right)^\top.$$

Let us use the symmetry of $\mathbf{Z}$ in order to remove the absolute values. The random vector $\mathbf{Z}$ is symmetric, e.g. for each $(z_1, \dots, z_n)^\top \in \mathbb{R}^n$ and $(k_1, \dots, k_n)^\top \in \{-1, 1\}^n$ we have for the density function of $\mathbf{Z}$

$$f(z_1, \dots, z_n) = f(k_1 z_1, \dots, k_n z_n).$$

Therefore, it takes the same values on all orthants of $\mathbb{R}^n$ and so does $\frac{\mathbf{Z}}{\|\mathbf{Z}\|}$. Thus, $\left(\tilde{Y}_1, \ldots, \tilde{Y}_k\right)^\top$ can be extended to

$$\left(\frac{Z_1}{\|\mathbf{Z}\|}, \ldots, \frac{Z_k}{\|\mathbf{Z}\|}\right)^\top \stackrel{\mathrm{d}}{=} (U_1, \ldots, U_k)^\top$$

with the density

$$f(u_1, \ldots, u_k) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)\pi^{\frac{k}{2}}}\left(1 - \sum_{i=1}^k u_i^2\right)^{\frac{n-k}{2}-1}, \quad \sum_{i=1}^k u_i^2 < 1.$$

□

1.3 Fractional Calculus

The results presented in Section 4.2 are typically (as in Section 2.2 of Fang et al. [1990]) presented in the language of standard differentiation and integration, yet may be generalized using fractional calculus. The aim of this section is to present the introduction to the topic and several examples.

The fundamental theorem of calculus (Folland [2002]) states that for a continuous function f and $a, x \in \mathbb{R}$

$$f(x) = \frac{\partial}{\partial x}\int_a^x f(t)dt$$

which means that the differentiation operator

$$f(x) \mapsto \frac{\partial f(x)}{\partial x}$$

is the left inverse of the integration operator

$$f(x) \mapsto \int_a^x f(t)dt$$

where the lower bound a is set arbitrarily. Fractional calculus generalizes the k-th order derivative

$$\frac{\partial^k f(x)}{\partial x^k} = \frac{\partial}{\partial x}\cdots\frac{\partial}{\partial x}f(x),$$

which is k times compound differentiation, to an integer-valued operator and then extends it to real numbers. As for integer differentiation and integration, the definition is sufficient for only a certain class of functions.

For the repeated integration we can change the order

$$\int_a^x \int_a^t f(s)dsdt = \int_a^x \int_s^x f(s)dtds$$

and thus,

$$\int_a^x \int_a^t f(s)dsdt = \int_a^x (x-s)f(s)ds$$

and the repeated integral collapses into one integral. By induction we obtain the Cauchy formula for repeated integration (Folland [2002]). For any $k \in \mathbb{N}$ we have

$$\int_a^x \int_a^{t_1} \cdots \int_a^{t_{k-2}} \int_a^{t_{k-1}} f(s)ds\ dt_{k-1} \cdots dt_2\ dt_1 = \frac{1}{(k-1)!} \int_a^x (x-s)^{k-1} f(s)ds.$$

This formula is used to define the *Riemann-Liouville fractional integral* with the gamma function as the real extension of the factorial. There are another ways how to extend the differentiation, namely Caputo or Grünwald-Letnikov fractional derivative.[1]

Definition 2. *Let $p > 0$ and f be a continuous function. Then the* Riemann-Liouville fractional integral *of order p is*

$$D_{a,x}^{-p} f(x) = \frac{1}{\Gamma(p)} \int_a^x (x-t)^{p-1} f(t)dt$$

for $x > a$.

Conversely, the Riemann-Liouville fractional derivative *of order p is defined as*

$$D_{a,x}^{p} f(x) = \frac{1}{\Gamma(\lfloor p \rfloor + 1 - p)} \frac{\partial^{\lfloor p \rfloor + 1}}{\partial x^{\lfloor p \rfloor + 1}} \int_a^x (x-t)^{\lfloor p \rfloor - p} f(t)dt$$

for $x > a$ where $\lfloor p \rfloor$ is an integer such that $\lfloor p \rfloor \le p < \lfloor p \rfloor + 1$.[2] *Additionally, we define $D_{a,x}^0 f(x) = f(x)$.*

The Riemann-Liouville fractional derivative and integral maps functions to functions $f \mapsto D_{a,x}^p f$ such that $(D_{a,x}^p f)(x) = D_{a,x}^p f(x)$. The operator is sometimes called the left-hand in contrast to the right-hand RL (short for Riemann-Liouville) fractional integral

$$D_{x,b}^{-p} = \frac{1}{\Gamma(p)} \int_x^b (t-x)^{p-1} f(t)dt \tag{1.2}$$

and the right-hand RL fractional derivative

$$D_{x,b}^{p} f(x) = \frac{(-1)^{\lfloor p \rfloor + 1}}{\Gamma(\lfloor p \rfloor + 1 - p)} \frac{\partial^{\lfloor p \rfloor + 1}}{\partial x^{\lfloor p \rfloor + 1}} \int_x^b (t-x)^{\lfloor p \rfloor - p} f(t)dt. \tag{1.3}$$

Note the commutative property of the Riemann-Liouville fractional integral (as in Li and Zhao [2011]).

Remark 2. For $p, q > 0$ the Riemann-Liouville fractional integral satisfies

$$D_{a,x}^{-p} \circ D_{a,x}^{-q} f(x) = D_{a,x}^{-q} \circ D_{a,x}^{-p} f(x) = D_{a,x}^{-(p+q)} f(x).$$

Let us verify that the definition of the RL fractional derivative extends the standard differentiation on positive integers. Set $p \in \mathbb{N}$, $p = \lfloor p \rfloor$ then $\Gamma(\lfloor p \rfloor + 1 - p) = 1$ and

$$D_{a,x}^{p} f(x) = \frac{\partial^{p+1}}{\partial x^{p+1}} \int_a^x f(t)dt = \frac{\partial^p f(x)}{\partial x^p}.$$

[1] Thoroughly discussed in Li and Zhao [2011].

[2] The ceiling function $\lceil p \rceil$ may be used with additional definition at positive integers.

Similarly for the right-hand fractional derivative

$$\begin{aligned} D^p_{x,b} f(x) &= (-1)^{p+1} \frac{\partial^{p+1}}{\partial x^{p+1}} \int_x^b f(t)dt \\ &= (-1)^{p+1} \frac{\partial^{p+1}}{\partial x^{p+1}} \left(- \int_b^x f(t)dt \right) \\ &= (-1)^p \frac{\partial^p f(x)}{\partial x^p}. \end{aligned} \tag{1.4}$$

As for the operators of integer order, the fractional derivative is the left inverse of the fractional integral since we have

$$D^p_{a,x} \circ D^{-p}_{a,x} f(x) = D^{\lfloor p \rfloor + 1}_{a,x} \circ D^{-(\lfloor p \rfloor + 1 - p)}_{a,x} \circ D^{-p}_{a,x} f(x).$$

Using the commutative property from Remark 2

$$D^p_{a,x} \circ D^{-p}_{a,x} f(x) = D^{\lfloor p \rfloor + 1}_{a,x} \circ D^{-(\lfloor p \rfloor + 1)}_{a,x} f(x) = f(x)$$

because both operators on the right-hand side are now of integer order.

Since standard integration and differentiation is linear, for $\mu, \lambda \in \mathbb{R}$ and functions f and g we have

$$D^p(\mu f(x) + \lambda g(x)) = \mu D^p f(x) + \lambda D^p g(x).$$

Example 1. Let us compute the fractional integral and derivative of several functions.

- Let $p \in \mathbb{R}$, $\lambda > 0$, compute $D^p_{x,\infty} e^{-\lambda x}$ for $x \in (0, \infty)$. First, the fractional integral of order p is

  $$D^{-p}_{x,\infty} e^{-\lambda x} = \frac{1}{\Gamma(p)} \int_x^\infty (t-x)^{p-1} e^{-\lambda t} dt$$

  and transforming the integral $u = t - x$ yields

  $$D^{-p}_{x,\infty} e^{-\lambda x} = \frac{1}{\Gamma(p)} \int_0^\infty u^{p-1} e^{-\lambda(u+x)} du.$$

  Use the integral associated with the gamma distribution

  $$\int_0^\infty u^{p-1} e^{-\lambda u} du = \frac{\Gamma(p)}{\lambda^p}.$$

  All put together we have

  $$\begin{aligned} D^{-p}_{x,\infty} e^{-\lambda x} &= \frac{e^{-\lambda x}}{\Gamma(p)} \frac{\Gamma(p)}{\lambda^p} \\ &= \lambda^{-p} e^{-\lambda x}. \end{aligned}$$

  Since for $k \in \mathbb{N}$ following Equation (1.4)

  $$\frac{\partial^k}{\partial x^k} e^{-\lambda x} = (-\lambda)^k e^{-\lambda x}, \quad D^k_{x,\infty} e^{-\lambda x} = \lambda^k e^{-\lambda x}$$

we have

$$\begin{aligned}
D^p_{x,\infty} e^{-\lambda x} &= D^{\lfloor p \rfloor+1}_{x,\infty} \circ D^{-(\lfloor p \rfloor+1-p)}_{x,\infty} e^{-\lambda x} \\
&= D^{\lfloor p \rfloor+1}_{x,\infty} \lambda^{-(\lfloor p \rfloor+1-p)} e^{-\lambda x} \\
&= \lambda^{\lfloor p \rfloor+1-(\lfloor p \rfloor+1-p)} e^{-\lambda x} = \lambda^p e^{-\lambda x}.
\end{aligned}$$

Let us combine the results for p and $-p$ together with $p = 0$ into one formula for $p \in \mathbb{R}$

$$D^p_{x,\infty} e^{-\lambda x} = \lambda^p e^{-\lambda x}.$$

- The fractional derivatives and integrals can be extended to the complex plane. Since both integration and differentiation are linear the previous result is extended to

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad z \in \mathbb{C}.$$

From the first part of this example we obtain

$$\begin{aligned}
D^p_{x,\infty} e^{\pm ix} &= (\mp i)^p e^{\pm ix} \\
&= e^{\pm i\left(x - \frac{\pi}{2} p\right)}
\end{aligned}$$

which when plugged back inside the complex definitions for the sine and the cosine means for $p \in \mathbb{R}$

$$\begin{aligned}
D^p_{x,\infty} \sin(x) &= \sin\left(x - \frac{\pi}{2} p\right), \\
D^p_{x,\infty} \cos(x) &= \cos\left(x - \frac{\pi}{2} p\right).
\end{aligned}$$

- Compute $D^p_{x,1} \mathbf{1}_{(0,1)}(x)$, for $x \in (0,1)$ and $p \in \mathbb{R}$. First, for the fractional integral from Equation (1.2) we have for $p > 0$

$$\begin{aligned}
D^{-p}_{x,1} \mathbf{1}_{(0,1)}(x) &= \frac{1}{\Gamma(p)} \int_x^1 (t-x)^{p-1} dt \\
&= \frac{1}{\Gamma(p)} \int_0^{1-x} y^{p-1} dy \\
&= \frac{1}{\Gamma(p+1)} (1-x)^p.
\end{aligned}$$

Now, for the fractional derivative $D^p_{x,1} \mathbf{1}_{(0,1)}(x)$, $p > 0$, the integral in Equation (1.3) is

$$\int_x^1 (t-x)^{\lfloor p \rfloor - p} dt = \int_0^{1-x} y^{\lfloor p \rfloor - p} dy = \frac{(1-x)^{\lfloor p \rfloor - p + 1}}{\lfloor p \rfloor - p + 1}$$

which means that

$$D^p_{x,1} \mathbf{1}_{(0,1)}(x) = \frac{(-1)^{\lfloor p \rfloor + 1}}{\Gamma(\lfloor p \rfloor + 1 - p)} \frac{\partial^{\lfloor p \rfloor + 1}}{\partial x^{\lfloor p \rfloor + 1}} \frac{(1-x)^{\lfloor p \rfloor - p + 1}}{\lfloor p \rfloor - p + 1}.$$

Firstly, for $p \in (0,1)$ we have $\lfloor p \rfloor = 0$ which simplifies the computation

$$\begin{aligned} D^p_{x,1}\mathbf{1}_{(0,1)}(x) &= \frac{-1}{\Gamma(1-p)}\frac{\partial}{\partial x}\frac{(1-x)^{1-p}}{1-p} \\ &= \frac{1}{\Gamma(1-p)}(1-x)^{-p}. \end{aligned}$$

For $p \in \mathbb{N}$ we have $p = \lfloor p \rfloor \geq 1$ and the exponent $\lfloor p \rfloor - p + 1$ is equal to 1. Thus,

$$\begin{aligned} D^p_{x,1}\mathbf{1}_{(0,1)}(x) &= (-1)^{\lfloor p \rfloor + 1}\frac{\partial^{\lfloor p \rfloor + 1}}{\partial x^{\lfloor p \rfloor + 1}}(1-x) \\ &= (-1)^{\lfloor p \rfloor}\frac{\partial^{\lfloor p \rfloor}}{\partial x^{\lfloor p \rfloor}} 1 = 0. \end{aligned}$$

For $p > 1$ such that $p \notin \mathbb{N}$, then

$$\begin{aligned} D^p_{x,1}\mathbf{1}_{(0,1)}(x) &= \frac{(-1)^{\lfloor p \rfloor + 1}}{\Gamma(\lfloor p \rfloor + 1 - p)}\frac{\partial^{\lfloor p \rfloor + 1}}{\partial x^{\lfloor p \rfloor + 1}}\frac{(1-x)^{\lfloor p \rfloor - p + 1}}{\lfloor p \rfloor - p + 1} \\ &= \frac{(\lfloor p \rfloor - p + 1)\cdots(1-p)}{\Gamma(\lfloor p \rfloor + 2 - p)}(1-x)^{-p}. \end{aligned}$$

In conclusion, we get

$$D^p_{x,1}\mathbf{1}_{(0,1)}(x) = \begin{cases} \frac{1}{\Gamma(1-p)}(1-x)^{-p}, & p < 1, \\ 0, & p \in \mathbb{N}, \\ \frac{(\lfloor p \rfloor - p + 1)\cdots(1-p)}{\Gamma(\lfloor p \rfloor + 2 - p)}(1-x)^{-p}, & \text{otherwise}. \end{cases} \tag{1.5}$$

For the purpose of this thesis denote

$$W^p = D^p_{x,\infty} \tag{1.6}$$

which for $p < 0$ is the so called *Liouville-Weyl fractional integral* and is used in Section 4.2.

The following remark summarizes some conditions for the existence of the Riemann-Liouville fractional integral and derivative of order $p \in (0,1)$ as stated in Li and Zhao [2011].

Remark 3. If $p \geq 1$ and $f \in L^p(\mathbb{R})$ then for $q \in (0, \frac{1}{p})$ the fractional integrals $D^{-q}_{x,\infty}f$ and $D^{-q}_{-\infty,x}f(x)$ exist almost everywhere.

Let $a, b \in \mathbb{R}$, $a < b$. The fractional derivative $D^p_{a,x}f$ exists and belongs to $L^1([a,b])$ if and only if $f \in L^1([a,b])$ and $D^{p-1}_{a,x}f$ is absolutely continuous and $D^{p-1}_{a,x}f(a) = 0$.

2. Spherically Symmetric Distributions

The aim of this chapter is to summarize the basic properties of spherically symmetric distributions and follows Section 2.1 of Fang et al. [1990]. Each property shown in Section 2.1 is then elucidated using several basic examples of spherically symmetric distributions such as the multivariate standard normal distribution or uniform distributions on the unit ball or sphere surface. Section 2.2 focuses on the moments of spherically symmetric distributions and the last section briefly discusses elliptically symmetric distributions.

2.1 Definition

The class of spherically symmetric distributions is defined though its geometric property. Let $\mathbf{x} \in \mathbb{R}^n$ and Q be an $n \times n$ orthogonal matrix. Then the point $Q\mathbf{x}$ lies on the same sphere centered at the origin as $\mathbf{x}$ since $\|\mathbf{x}\| = \|Q\mathbf{x}\|$. We may call the non-empty set $A \subset \mathbb{R}^n$ *spherically symmetric* if $\mathbf{x} \in A$ implies $Q\mathbf{x} \in A$ for each orthogonal matrix Q. The spherical symmetry which defines the class of distributions is a generalization of this property up to the distributions.

Definition 3. *An n-dimensional random vector $\boldsymbol{X}$ has a* spherically symmetric *distribution if for every $Q \in \mathcal{O}(n)$ vectors $\boldsymbol{X}$ and $Q\boldsymbol{X}$ have the same distribution.*

Following Definition 3 a random vector with a spherically symmetric distribution shall be called a spherically symmetric random vector.

Theorem 3. *Let $\varphi_{\boldsymbol{X}}(\boldsymbol{t})$ be a characteristic function of an n-dimensional random vector $\boldsymbol{X}$. Then $\boldsymbol{X}$ has a spherically symmetric distribution if and only if there exists a function $\phi : [0, \infty) \to \mathbb{R}$ such that $\varphi_{\boldsymbol{X}}(\boldsymbol{t}) = \phi(\boldsymbol{t}^\top \boldsymbol{t})$ for every vector $\boldsymbol{t} \in \mathbb{R}^n$.*

Proof. Using the properties of characteristic functions (Lachout [2004]) for every orthogonal matrix Q we have $\varphi_{\mathbf{X}}(Q^\top \mathbf{t}) = \varphi_{Q\mathbf{X}}(\mathbf{t})$ which for $\mathbf{X}$ with spherically symmetric distribution means $\varphi_{\mathbf{X}}(\mathbf{t}) = \varphi_{\mathbf{X}}(Q^\top \mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^n$. Thus, the characteristic function of $\mathbf{X}$ is invariant with respect to the group of orthogonal matrices. Since $\|\mathbf{t}\| = \|Q\mathbf{t}\|$ the value of $\varphi_{\mathbf{X}}$ depends on the inner product $\mathbf{t}^\top\mathbf{t}$. Thus, it must be a scalar function of $\mathbf{t}^\top\mathbf{t}$ since for any $\mathbf{t}_1, \mathbf{t}_2 \in \mathbb{R}^n$ such that $\mathbf{t}_1^\top \mathbf{t}_1 = \mathbf{t}_2^\top \mathbf{t}_2$ there is a matrix $Q \in \mathcal{O}(n)$ such that $\mathbf{t}_1 = Q\mathbf{t}_2$.[1] Hence, $\varphi_{\mathbf{X}}(\mathbf{t}_1) = \varphi_{\mathbf{X}}(Q\mathbf{t}_2) = \varphi_{\mathbf{X}}(\mathbf{t}_2)$ and the value of $\varphi_{\mathbf{X}}(\mathbf{t})$ depends on $\mathbf{t}^\top\mathbf{t}$.

Conversely, any orthogonal matrix Q satisfies $QQ^\top = I$. Thus, $\varphi_{Q\mathbf{X}}(\mathbf{t}) = \varphi_{\mathbf{X}}(Q^\top \mathbf{t}) = \phi((Q^\top \mathbf{t})^\top Q^\top \mathbf{t}) = \phi(\mathbf{t}^\top QQ^\top \mathbf{t}) = \phi(\mathbf{t}^\top \mathbf{t}) = \varphi_{\mathbf{X}}(\mathbf{t})$ and $\mathbf{X}$ and $Q\mathbf{X}$ have the same distribution (the characteristic function determines the distribution as shown in Lachout [2004]) and $\mathbf{X}$ has a spherically symmetric distribution.

□

A function ϕ which satisfies the previous theorem for some random vector is called a *characteristic generator*.

[1]The matrix Q is the Householder matrix $Q = I - 2\mathbf{q}\mathbf{q}^\top$ where $\mathbf{q} = \frac{\mathbf{t}_2 - \mathbf{t}_1}{\|\mathbf{t}_2 - \mathbf{t}_1\|}$ is a unit vector. Since $Q^2 = I^2 - 4\mathbf{q}\mathbf{q}^\top + 4\mathbf{q}\mathbf{q}^\top\mathbf{q}\mathbf{q}^\top = I - 4\mathbf{q}\mathbf{q}^\top + 4\mathbf{q}\mathbf{q}^\top = I$, the matrix is orthogonal because $|\det Q| = 1$ (The properties of the Householder reflection are discussed in Kerl [2008]).

Example 2. The class of spherically symmetric distributions includes:

- *Uniform distribution on the unit sphere surface.* By $\mathbf{u}_n$ let us denote the uniform distribution on the unit sphere surface in $\mathbb{R}^n$. For $X \sim \mathbf{u}_1$ we have $\mathsf{P}(X = 1) = \mathsf{P}(X = -1) = \frac{1}{2}$.

- *Uniform distribution inside the unit ball.* By $\mathbf{s}_n$ let us denote the uniform distribution inside the unit ball in $\mathbb{R}^n$. For $n = 1$ the distribution $\mathbf{s}_1$ is a uniform distribution on the interval $(-1, 1)$.

- *Normal distribution.* An n-dimensional vector with a multivariate normal distribution $\mathcal{N}_n(\mathbf{0}, I)$ is spherically symmetric and its characteristic function is $e^{-\mathbf{t}^\top \mathbf{t}/2}$, therefore the characteristic generator is $e^{-y/2}$ for $y \geq 0$.

- *Student's t-distribution.* Let $\mathbf{Z} \sim \mathcal{N}_n(\mathbf{0}, I)$ and Y be a random variable with a χ-distribution[2] with m degrees of freedom where $\mathbf{Z}$ and Y are independent. Then $\mathbf{T} = \frac{\sqrt{m}\mathbf{Z}}{Y}$ has a symmetric *multivariate t-distribution* with m degrees of freedom as an extension of the univariate t-distribution. For $m = 1$ it becomes a symmetric *multivariate Cauchy distribution.*

- *Laplace distribution.* Let $\mathbf{Z} \sim \mathcal{N}_n(\mathbf{0}, I)$ and $W \sim \mathsf{Exp}(1)$ where $\mathbf{Z}$ and W are independent. Then $Y = \sqrt{W}\mathbf{Z}$ has a symmetric *multivariate Laplace distribution* and its characteristic function is $\frac{1}{1+\mathbf{t}^\top \mathbf{t}/2}$ (further discussed in Kozubowski and Podgórski [2000]).

- *Sum of random vectors.* Generally, for k independent n-dimensional spherically symmetric random vectors $\mathbf{X}_1, \ldots, \mathbf{X}_k$ with characteristic generators $\phi_1, \ldots, \phi_k$, the sum
$$\mathbf{Y} = \sum_{i=1}^{k} \mathbf{X}_i$$
is an n-dimensional spherically symmetric random vector with the characteristic generator
$$\phi(u) = \prod_{i=1}^{k} \phi_i(u), \quad u \geq 0,$$
since the characteristic function (Lachout [2004]) is
$$\varphi_{\mathbf{Y}}(\mathbf{t}) = \prod_{i=1}^{n} \varphi_{\mathbf{X}_i}(\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}^n.$$

- *Mixture of random vectors.* Let $\{\mathbf{X}_k\}_{k=1}^{\infty}$ be a sequence of n-dimensional spherically symmetric random vectors with characteristic generators ϕ_1, $\phi_2, \ldots$ and $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence of non-negative real numbers such that $\sum_{k=1}^{\infty} \alpha_k = 1$. Then the mixture of $\{\mathbf{X}_k\}_{k=1}^{\infty}$ with weights $\{\alpha_k\}_{k=1}^{\infty}$ is spherically symmetric with the characteristic generator
$$\phi(u) = \sum_{k=1}^{\infty} \alpha_k \phi_k(u), \quad u \geq 0.$$

[2]See p. 25.

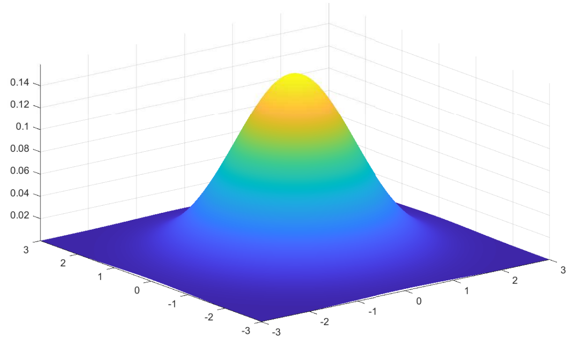

Figure 2.1: The density function of $\mathcal{N}_2(\mathbf{0}, I)$.

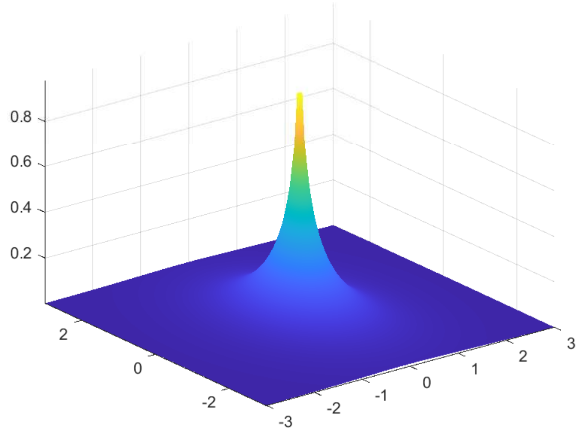

Figure 2.2: The density function of the bivariate Laplace distribution.

Remark 4. Let us look closely at the scalar functions that could serve as a characteristic generator of a random vector. Suppose for $n \in \mathbb{N}$ we have that $\phi(t_1^2 + \cdots + t_n^2)$ is a characteristic function of some n-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_n)^\top$ where ϕ is the generator. Then for $1 \leq m \leq n$ we have that $\phi(t_1^2 + \cdots + t_m^2)$ is a characteristic function of the marginal random vector $(X_1, \ldots, X_m)^\top$. That follows using the formula $\varphi_{A\mathbf{X}}(\mathbf{t}) = \varphi_{\mathbf{X}}(A^\top \mathbf{t})$ for any $\mathbf{t} \in \mathbb{R}^m$ for an $m \times n$-matrix

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}$$

which is the matrix of projection to the first m components.

Theorem 4. *A function ϕ is a characteristic generator of an n-dimensional spherically symmetric random vector if and only if there exists a c.d.f. $F : \mathbb{R} \to [0, 1]$ such that $F(r) = 0$ for $r < 0$ and*

$$\phi(x) = \int_0^\infty \phi_{\boldsymbol{u}_n}(xr^2) dF(r) \tag{2.1}$$

where $\phi_{\mathbf{u}_n}$ is the characteristic generator of $\mathbf{u}_n$, the random vector uniformly distributed on the unit sphere surface.

Proof. Let us define a measure μ on the unit sphere where $\mu(B)$ is equal to the surface area of B for any Borel subset B of the unit sphere surface, then $\mathsf{P}(\mathbf{u}_n \in B) = \frac{\mu(B)}{S_n}$ where S_n is a surface area of the unit sphere. From the definition of the characteristic function and the characteristic generator we have

$$\phi_{\mathbf{u}_n}(\mathbf{t}^\top \mathbf{t}) = \mathsf{E}\left(e^{i\mathbf{t}^\top \mathbf{u}_n}\right) = \int_{\|\mathbf{x}\|=1} \frac{1}{S_n} e^{i\mathbf{t}^\top \mathbf{x}} d\mu(\mathbf{x}).$$

Firstly, for simplicity let us use the notation $\phi(\mathbf{t}^\top \mathbf{t}) = \varphi_{\mathbf{Y}}(t_1, \ldots, t_n)$ for a characteristic generator and a char. function of an n-dimensional random vector $\mathbf{Y}$ with a c.d.f $G(\mathbf{y})$. Denote ν the measure corresponding to G. For $\mathbf{t} \in \mathbb{R}^n$, $\mathbf{t} \neq \mathbf{0}$, denote $\mathbf{v} = (v_1, \ldots, v_n)^\top = \frac{\mathbf{t}}{\|\mathbf{t}\|}$, then using Fubini's theorem (Folland [2002])

$$\begin{aligned}
\phi(\mathbf{t}^\top \mathbf{t}) &= \varphi_{\mathbf{Y}}(\|\mathbf{t}\|v_1, \ldots, \|\mathbf{t}\|v_n) \\
&= \int_{\|\mathbf{v}\|=1} \frac{1}{S_n} \varphi_{\mathbf{Y}}(\|\mathbf{t}\|v_1, \ldots, \|\mathbf{t}\|v_n) d\mu(\mathbf{v}) \\
&= \int_{\|\mathbf{v}\|=1} \frac{1}{S_n} \int_{\mathbb{R}^n} e^{i\|\mathbf{t}\|\mathbf{v}^\top \mathbf{y}} d\nu(\mathbf{y}) d\mu(\mathbf{v}) \\
&= \int_{\mathbb{R}^n} \frac{1}{S_n} \int_{\|\mathbf{v}\|=1} e^{i\|\mathbf{t}\|\mathbf{v}^\top \mathbf{y}} d\mu(\mathbf{v}) d\nu(\mathbf{y}) \\
&= \int_{\mathbb{R}^n} \phi_{\mathbf{u}_n}(\mathbf{t}^\top \mathbf{t}\ \mathbf{y}^\top \mathbf{y}) d\nu(\mathbf{y}).
\end{aligned}$$

And since the characteristic function is continuous

$$\phi(\mathbf{t}^\top \mathbf{t}) = \int_{\mathbb{R}^n} \phi_{\mathbf{u}_n}(\mathbf{t}^\top \mathbf{t}\ \mathbf{y}^\top \mathbf{y}) d\nu(\mathbf{y}) \tag{2.2}$$

holds for all $\mathbf{t} \in \mathbb{R}^n$. Let

$$F(r) = \int_{\|\mathbf{y}\| \le r} d\nu(\mathbf{y}) = \mathsf{E}\left(\mathbf{1}_{\{\|\mathbf{Y}\| \le r\}}\right) = \mathsf{P}(\|\mathbf{Y}\| \le r).$$

Then $F : \mathbb{R} \to [0, 1]$ satisfies the conditions to be a c.d.f. of a random variable. For any $r < 0$ we have $F(r) = \int_{\|\mathbf{y}\| \le r} d\nu(\mathbf{y}) = \int_\emptyset d\nu(\mathbf{y}) = 0$. Function F is right-continuous since for any $r \in \mathbb{R}$ and a sequence $\{r_n\}_{n=1}^\infty$, $r_n > r$, $r_n \to r$, we have

$$\lim_{n\to\infty} F(r_n) = \lim_{n\to\infty} \int_{\|\mathbf{y}\| \le r_n} d\nu(\mathbf{y}) = \int_{\mathbb{R}^n} \lim_{n\to\infty} \mathbf{1}_{\{\|\mathbf{y}\| \le r_n\}} d\nu(\mathbf{y}) = \int_{\|\mathbf{y}\| \le r} d\nu(\mathbf{y}) = F(r)$$

using the dominated convergence theorem (Folland [2002]) since $F(r_n) \le 1$ for $n \in \mathbb{N}$. Limits to $-\infty$ and $+\infty$ can be proven analogously. For any $r_1, r_2 \in \mathbb{R}$ such that $r_1 < r_2$ we have that

$$F(r_2) - F(r_1) = \int_{\|\mathbf{y}\| \le r_2} d\nu(\mathbf{y}) - \int_{\|\mathbf{y}\| \le r_1} d\nu(\mathbf{y}) = \int_{r_1 < \|\mathbf{y}\| \le r_2} d\nu(\mathbf{y}) \ge 0.$$

We can also observe that for the random variable R corresponding to $F(r)$ we have $F(r) = \mathsf{P}(R \le r) = \mathsf{P}(\|\mathbf{Y}\| \le r)$ which means that when integrating with respect to $F(r)$ instead of $G(\mathbf{y})$ or $\nu(\mathbf{y})$ we replace $\|\mathbf{y}\|$ with r. In Equation (2.2) set $x = \mathbf{t}^\top\mathbf{t}$ and change the integration, then

$$\phi(x) = \int_0^\infty \phi_{\mathbf{u}_n}(xr^2)dF(r).$$

For the second implication let us assume Equation (2.1) holds for some c.d.f. F. Denote R the random variable corresponding to the c.d.f. F. The proof is completed if we find an n-dimensional spherically symmetric random vector such that ϕ is its characteristic generator. Set $\mathbf{X} = R\mathbf{u}_n$ where R and $\mathbf{u}_n$ are independent which means the c.d.f. of $\mathbf{X}$ is $F(r) \cdot G(\mathbf{u})$ as $G(u_1, \ldots, u_n)$ is the c.d.f. of $\mathbf{u}_n$. The characteristic function of $\mathbf{X}$ is

$$\begin{aligned}
\varphi_{\mathbf{X}}(\mathbf{t}) &= \mathsf{E}\left(e^{i\mathbf{t}^\top R\mathbf{u}_n}\right) \\
&= \int_0^\infty \int_{\|\mathbf{u}\|=1} e^{i\mathbf{t}^\top r\mathbf{u}}\, dG(\mathbf{u})\, dF(r) \\
&= \int_0^\infty \varphi_{\mathbf{u}_n}(r\mathbf{t})\, dF(r) \\
&= \int_0^\infty \phi_{\mathbf{u}_n}(r^2\mathbf{t}^\top\mathbf{t})\, dF(r) \\
&= \phi(\mathbf{t}^\top\mathbf{t})
\end{aligned}$$

which means the random vector $\mathbf{X}$ is spherically symmetric (Theorem 3) and ϕ is its characteristic generator.

□

Theorem 4 is due to Schoenberg [1938] and the proof is adapted from Fang et al. [1990]. The second part gives us a *stochastic representation* for an n-dimensional random vector $\mathbf{X}$ with characteristic function $\phi(\mathbf{t}^\top\mathbf{t})$ through a non-negative random variable R with a c.d.f. F. From now on for a spherically symmetric random vector $\mathbf{X}$ we shall denote R the random variable which satisfies Theorem 4 and the distribution of R shall be called the *radial distribution* of $\mathbf{X}$. In conclusion, when $\mathbf{X}$ is an n-dimensional spherically symmetric random vector with a characteristic generator ϕ we shall write $\mathbf{X} \stackrel{\mathrm{d}}{=} R\mathbf{u}_n$ where R and $\mathbf{u}_n$ are independent and ϕ and R are connected though Theorem 4.

The following theorems describe an easier way how to derive the radial distribution of $\mathbf{X}$ and that the radial distribution is unique.

Theorem 5. *Suppose for a spherically symmetric random vector $\boldsymbol{X} \stackrel{\mathrm{d}}{=} R\boldsymbol{u}_n$ and $\mathsf{P}(\boldsymbol{X} = \boldsymbol{0}) = 0$, then*

$$\|\boldsymbol{X}\| \stackrel{\mathrm{d}}{=} R \quad \textit{and} \quad \frac{\boldsymbol{X}}{\|\boldsymbol{X}\|} \stackrel{\mathrm{d}}{=} \boldsymbol{u}_n.$$

Proof. From Anderson and Fang [1982] if $\mathbf{Y}_1 \stackrel{\mathrm{d}}{=} \mathbf{Y}_2$ are random vectors and f_j, $j \in \{1, \ldots, m\}$ are measurable functions, then

$$\left(f_1(\mathbf{Y}_1), \ldots, f_m(\mathbf{Y}_1)\right)^\top \stackrel{\mathrm{d}}{=} \left(f_1(\mathbf{Y}_2), \ldots, f_m(\mathbf{Y}_2)\right)^\top.$$

The functions $f_1(\mathbf{Y}) = \|\mathbf{Y}\|$ and $f_2(\mathbf{Y}) = \frac{\mathbf{Y}}{\|\mathbf{Y}\|}$ are used in order to prove the theorem. Since $\mathsf{P}(\mathbf{X} = \mathbf{0}) = 0$, these functions are measurable. First using $\mathsf{P}(\|\mathbf{u}_n\| = 1) = 1$ we get

$$\|\mathbf{X}\| \stackrel{\mathrm{d}}{=} \|R\mathbf{u}_n\| = \|R\| = R.$$

This also means that $\mathsf{P}(R=0)=\mathsf{P}(\|\mathbf{X}\|=\mathbf{0})=\mathsf{P}(\mathbf{X}=\mathbf{0})=0$. Thus, the distribution of $\mathbf{X}/\|\mathbf{X}\|$ is

$$\frac{\mathbf{X}}{\|\mathbf{X}\|}\overset{\mathrm{d}}{=}\frac{R\mathbf{u}_n}{\|R\mathbf{u}_n\|}=\mathbf{u}_n.$$

□

Theorem 6. *If $\mathsf{P}(\boldsymbol{X}=\boldsymbol{0})=0$ and $\boldsymbol{X}$ has two different stochastic representations, e. g. $\boldsymbol{X}\overset{\mathrm{d}}{=}R_1\boldsymbol{u}_n\overset{\mathrm{d}}{=}R_2\boldsymbol{u}_n$, then $R_1\overset{\mathrm{d}}{=}R_2$.*

Proof. Since $\mathbf{X}^\top\mathbf{X}\overset{\mathrm{d}}{=}(R_1\mathbf{u}_n)^\top R_1\mathbf{u}_n\overset{\mathrm{d}}{=}R_1^2\mathbf{u}_n^\top\mathbf{u}_n\overset{\mathrm{d}}{=}R_1^2$ and similarly $\mathbf{X}^\top\mathbf{X}\overset{\mathrm{d}}{=}R_2^2$. Random variables R_1, R_2 are non-negative which concludes $R_1\overset{\mathrm{d}}{=}R_2$.

□

Example 3. Let us consider a spherically symmetric random vector $\mathbf{X}$ and find its radial distribution.

- *Uniform distribution on the unit sphere surface.* If $\mathbf{X}\sim\mathbf{u}_n$, then R has a degenerate distribution with $\mathsf{P}(R=1)=1$.
- *Normal distribution.* For a random vector $\mathbf{X}$ with a multivariate normal distribution $\mathcal{N}_n(\mathbf{0},I)$, $\|\mathbf{X}\|$ has a χ-distribution with n degrees of freedom.[3]
- *Uniform distribution inside the unit ball.* If a random variable $\mathbf{X}\sim\mathbf{s}_n$, then $\|\mathbf{X}\|$ has $\mathsf{Beta}(n,1)$ distribution.[4]

A spherically symmetric distribution can be further characterized by a linear combination of its components.

Theorem 7. *A random vector $\boldsymbol{X}=(X_1,\dots,X_n)^\top$ is spherically symmetric if and only if for all $\boldsymbol{a}\in\mathbb{R}^n$*

$$\boldsymbol{a}^\top\boldsymbol{X}\overset{\mathrm{d}}{=}\|\boldsymbol{a}\|X_1.$$

Proof. Following Remark 4 if ϕ is a characteristic generator of a spherically symmetric random vector $\mathbf{X}$ then it is also a characteristic generator of X_1, thus for the characteristic function $\varphi_{\mathbf{a}^\top\mathbf{X}}$ of the random variable $\mathbf{a}^\top\mathbf{X}$ and $t\in\mathbb{R}$

$$\varphi_{\mathbf{a}^\top\mathbf{X}}(t)=\mathsf{E}\left(e^{it\ \mathbf{a}^\top\mathbf{X}}\right)=\phi((t\mathbf{a}^\top)t\mathbf{a})=\phi(t^2\|\mathbf{a}\|^2)=\varphi_{\|\mathbf{a}\|X_1}(t).$$

Therefore, $\mathbf{a}^\top\mathbf{X}$ and $\|\mathbf{a}\|X_1$ have the same characteristic function which implies equality in distribution.

Conversely, let $\mathbf{a}^\top\mathbf{X}\overset{\mathrm{d}}{=}\|\mathbf{a}\|X_1$ for all $\mathbf{a}\in\mathbb{R}^n$. Then $\varphi_{X_1}(\|\mathbf{a}\|)=\varphi_{\|\mathbf{a}\|X_1}(1)$ and since the distributions are the same, the characteristic functions are the same. We obtain $\varphi_{\|\mathbf{a}\|X_1}(1)=\varphi_{\mathbf{a}^\top\mathbf{X}}(1)=\varphi_{\mathbf{X}}(\mathbf{a})$ thus the characteristic function of $\mathbf{X}$ satisfies $\varphi_{\mathbf{X}}(\mathbf{a})=\varphi_{X_1}(\|\mathbf{a}\|)=\varphi_{X_1}(\sqrt{\mathbf{a}^\top\mathbf{a}})$ for all $\mathbf{a}\in\mathbb{R}^n$. Using Theorem 3 we have that $\mathbf{X}$ is a spherically symmetric random vector.

[3]As used in Theorem 2 and with the density derived on p. 25.

[4]See p. 25.

□

Following Theorem 7 all projections of a spherically symmetric random vector to a line through the origin[5] have the same distribution (same as the distribution of any component of the random vector). Theorems 3, 4 and 7 are formulated as equivalences, thus any of these characteristics can be used to define spherically symmetric random vectors.

2.2 Moments

As presented in the previous section a spherically symmetric random vector $\mathbf{X}$ is distributed as $R\mathbf{u}_n$ where R is some non-negative random variable and $\mathbf{u}_n$ is uniformly distributed on the unit sphere surface and R and $\mathbf{u}_n$ are independent. The results presented in this section are taken from Section 2.2 of Fang et al. [1990].

Theorem 8. *Let $\boldsymbol{X}$ be an n-dimensional spherically symmetric random vector. If $\mathsf{E}(\boldsymbol{X})$ exists, then $\mathsf{E}(\boldsymbol{X}) = \boldsymbol{0}$.*

Proof. By Theorem 4 we have that $\mathbf{X} \overset{\mathrm{d}}{=} R\mathbf{u}_n$ where R and $\mathbf{u}_n$ are independent, thus $\mathsf{E}(\mathbf{X}) = \mathsf{E}(R)\,\mathsf{E}(\mathbf{u}_n)$.

Let us find $\mathsf{E}(\mathbf{u}_n)$. Denote $\mathbf{Z} \sim \mathcal{N}_n(\mathbf{0}, I)$, then $\mathbf{Z} \overset{\mathrm{d}}{=} \|\mathbf{Z}\|\mathbf{u}_n$. We know $\mathsf{E}(\mathbf{Z}) = \mathbf{0}$ and $\mathsf{E}(\|\mathbf{Z}\|) < \infty$ since $\mathsf{E}(\|\mathbf{Z}\|^2) = n$ which is the expected value of the χ^2-distribution with n degrees of freedom. This means $\mathsf{E}(\mathbf{u}_n) = \mathbf{0}$.

Assuming $\mathsf{E}(\mathbf{X})$ exists, it necessarily means $\mathsf{E}(R) < \infty$ and $\mathsf{E}(\mathbf{X}) = \mathbf{0}$.

□

The assumption of the existence of $\mathsf{E}(\mathbf{X})$ is necessary, as in the case of a univariate Cauchy distribution the expected value is undefined.

Using a similar procedure for $\mathbf{X} \overset{\mathrm{d}}{=} R\mathbf{u}_n$ where $\mathsf{E}(R^2) < \infty$ the covariance matrix of $\mathbf{X}$ if exists is

$$\mathsf{Cov}(\mathbf{X}) = \frac{\mathsf{E}(R^2)}{n} I.$$

Thus, if it exists the correlation matrix is the same for all n-dimensional spherically symmetric distributions since the constant

$$\frac{\mathsf{E}(R^2)}{n}$$

cancels out and $\mathsf{Corr}(\mathbf{X}) = I$.

The following theorem provides a generalization for higher moments.

Theorem 9. *Let $\boldsymbol{X} \overset{\mathrm{d}}{=} R\boldsymbol{u}_n$ be a spherically symmetric random vector, $\boldsymbol{X} = (X_1, \dots, X_n)^\top$ and $m_1, \dots, m_n$ are non-negative integers. Provided the moments of $\boldsymbol{X}$ exist, if at least one of m_i is odd, then*

$$\mathsf{E}\left(\prod_{i=1}^{n} X_i^{m_i}\right) = 0.$$

[5] Such a projection can be computed using a dot product with the unit direction vector of the line.

Otherwise, let us denote $m_i = 2l_i$ and $m_1 + \ldots + m_n = m = 2l$, then

$$\mathsf{E}\left(\prod_{i=1}^{n} X_i^{m_i}\right) = \mathsf{E}(R^{2l})\frac{\Gamma(\frac{n}{2})}{4^l\ \Gamma(\frac{n}{2}+l)}\prod_{i=1}^{n}\frac{(2l_i)!}{(l_i)!}.$$

Proof. Let $\mathbf{Z} \sim \mathcal{N}_n(0, I)$, then $\mathbf{Z} \stackrel{\mathrm{d}}{=} Y\mathbf{u}_n$ where Y has a χ-distribution with n degrees of freedom and Y, $\mathbf{u}_n = (U_1, \ldots, U_n)^\top$ are independent. Thus, $\mathsf{E}(\mathbf{Z}) = \mathsf{E}(Y\mathbf{u}_n) = \mathsf{E}(Y)\,\mathsf{E}(\mathbf{u}_n)$. Now using independence of uncorrelated normally distributed random variables

$$\prod_{i=1}^{n}\mathsf{E}(Z_i^{m_i}) = \mathsf{E}\left(\prod_{i=1}^{n} Z_i^{m_i}\right) = \mathsf{E}\left(\prod_{i=1}^{n}(YU_i)^{m_i}\right)$$
$$= \mathsf{E}\left(Y^m\prod_{i=1}^{n} U_i^{m_i}\right) = \mathsf{E}(Y^m)\,\mathsf{E}\left(\prod_{i=1}^{n} U_i^{m_i}\right).$$

If any m_i is odd, then from $\mathsf{E}(Z_i^{m_i}) = 0$. For even m_i we have that $\mathsf{E}(Z_i^{m_i}) = 1\cdot 3\cdots(m_i-1) = (m_i-1)!!$ and from Forbes et al. [2010]

$$\mathsf{E}(Y^m) = \frac{2^{m/2}\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \tag{2.3}$$

We have that

$$\mathsf{E}\left(\prod_{i=1}^{n} X_i^{m_i}\right) = \mathsf{E}\left(\prod_{i=1}^{n}(RU_i)^{m_i}\right) = \mathsf{E}(R^m)\,\mathsf{E}\left(\prod_{i=1}^{n} U_i^{m_i}\right) = \mathsf{E}(R^m)\frac{\prod_{i=1}^{n}\mathsf{E}(Z_i^{m_i})}{\mathsf{E}(Y^m)}. \tag{2.4}$$

If any m_i is odd, then $\mathsf{E}(Z_i^{m_i}) = 0$ which gives us the first result. In the second case $\mathsf{E}(Z_i^{2l_i}) = (2l_i-1)!!$ can expressed using the factorial as

$$\mathsf{E}(Z_i^{2l_i}) = (2l_i-1)!! = \frac{(2l_i)!}{2^{l_i}(l_i)!}. \tag{2.5}$$

Since for any $k \in \mathbb{N}$

$$(2k)! = \prod_{j=1}^{k}(2j)\prod_{i=1}^{k}(2l-1) = 2^k k!(2k-1)!!$$

which when combining Equations (2.3), (2.4) and (2.5) gives us

$$\mathsf{E}\left(\prod_{i=1}^{n} X_i^{m_i}\right) = \mathsf{E}(R^{2l})\frac{\Gamma(\frac{n}{2})}{2^l\Gamma(\frac{n}{2}+l)}\prod_{i=1}^{n}\frac{(2l_i)!}{2^{l_i}(l_i)!}$$

and the proof is completed.

□

Corollary 1. Fang et al. [1990] also present another result

$$\mathsf{E}\left(\prod_{i=1}^{n} X^{m_i}\right) = \mathsf{E}(R^{2l})\frac{\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}}\Gamma(\frac{n}{2}+l)}\prod_{i=1}^{n}\Gamma\left(l_i+\frac{1}{2}\right)$$

which can be obtained from Theorem 9 using the Legendre duplication formula

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$$

for $z = l_i + \frac{1}{2}$ for $i \in \{1, \ldots, n\}$.

2.3 Elliptically Symmetric Distributions

The class of elliptically symmetric distributions is defined through the spherically symmetric distribution as an affine transformation. Since many properties of spherically symmetric distributions are easily applicable to this class, this chapter is only a brief introduction. Anderson and Fang [1990] further focus on the class of matrix symmetric distributions which are not discussed in this thesis.

Definition 4. *Let $\boldsymbol{X}$ be a k-dimensional spherically symmetric random vector, A be a $k \times n$-matrix with rank k, $A^\top A = \Sigma$, and $\boldsymbol{b} \in \mathbb{R}^n$. Then the random vector*

$$\boldsymbol{b} + A^\top \boldsymbol{X}$$

is said to have an elliptically symmetric distribution *with parameters $\boldsymbol{b}$ and Σ.*

If $\mathbf{b} = \mathbf{0}$ and $\Sigma = I$, then an elliptically symmetric distribution becomes spherically symmetric. The elliptically symmetric distribution is fully characterized by a positive semidefinite matrix Σ regardless of the matrix A as we show in the following remark.

Remark 5. As for spherically symmetric distributions, the characteristic function of an elliptically symmetric random vector takes a special form. For an elliptically symmetric random vector $\mathbf{Y}$ with parameters $\mathbf{b} \in \mathbb{R}^n$ and an $n \times n$ positive definite matrix Σ the characteristic function is

$$\mathsf{E}\left(e^{i\mathbf{t}^\top \mathbf{Y}}\right) = \mathsf{E}\left(e^{i\mathbf{t}^\top(\mathbf{b}+A^\top\mathbf{X})}\right) = e^{i\mathbf{t}^\top\mathbf{b}}\, \mathsf{E}\left(e^{i(A\mathbf{t})^\top\mathbf{X}}\right) = e^{i\mathbf{t}^\top\mathbf{b}}\phi(\mathbf{t}^\top\Sigma\mathbf{t}) \qquad (2.6)$$

where $\phi : [0,\infty) \to \mathbb{R}$ is a scalar function. Because the char. function of $\mathbf{Y}$ depends on A only via Σ, we have shown that the distribution indeed depends only on $\mathbf{b}$, Σ and the scalar function. The formula for the characteristic function is used in Kelker [1970] and Fang et al. [1990] to define the class of elliptically symmetric distributions.

Remark 6. For an n-dimensional elliptically symmetric random vector the parameter $\mathbf{b} \in \mathbb{R}^n$ is unique but Σ and ϕ are only unique up to a constant $c > 0$ since using $c\Sigma$ and $\phi(\frac{\cdot}{c})$ instead of Σ and $\phi(\cdot)$ will result in the same distribution.

Example 4. Denote $\mathbf{X} \sim \mathcal{N}_k(\mathbf{0}, I)$, then

$$\mathbf{Y} = \mathbf{b} + A^\top \mathbf{X}$$

for a vector $\mathbf{b} \in \mathbb{R}^n$ and $k \times n$-matrix A with rank k has a normal distribution $\mathcal{N}_n(\mathbf{b}, \Sigma)$ where $\Sigma = A^\top A$ is its covariance matrix.

The Laplace and the multivariate t-distribution can be also extended into elliptically symmetric distributions alternating the characteristic function (or the density) as in Equation (2.6).

Theorem 10. *Suppose $\boldsymbol{X}$ is an n-dimensional elliptically symmetric random vector with parameters $\boldsymbol{b} \in \mathbb{R}^n$ and Σ, a positive definite $n \times n$-matrix. Let $\boldsymbol{c} \in \mathbb{R}^m$ and D be an $m \times n$-matrix of rank m, $m \leq n$. Then $\boldsymbol{Y} = \boldsymbol{c} + D\boldsymbol{X}$ has an elliptically symmetric distribution with parameters $\boldsymbol{c} + D\boldsymbol{b}$ and $D\Sigma D^\top$.*

Proof. From the definition of elliptical symmetry there is a k-dimensional spherically symmetric random vector $\tilde{\mathbf{X}}$, $\mathbf{b} \in \mathbb{R}^n$ and a $k \times n$-matrix A such that $A^\top A = \Sigma$

$$\mathbf{X} = \mathbf{b} + A^\top \tilde{\mathbf{X}}.$$

Denote ϕ the characteristic generator of $\tilde{\mathbf{X}}$. For $\mathbf{s} \in \mathbb{R}^n$ we have

$$\varphi_{\mathbf{X}}(\mathbf{s}) = \mathsf{E}\left(e^{i\mathbf{s}^\top \mathbf{X}}\right) = \mathsf{E}\left(e^{i\mathbf{s}^\top (\mathbf{b}+A^\top \tilde{\mathbf{X}})}\right) = e^{i\mathbf{s}^\top \mathbf{b}}\,\mathsf{E}\left(e^{i(A\mathbf{s})^\top \tilde{\mathbf{X}}}\right) = e^{i\mathbf{s}^\top \mathbf{b}}\phi(\mathbf{s}^\top \Sigma \mathbf{s}).$$

Thus, for $\mathbf{Y} = \mathbf{c} + D\mathbf{X}$ and $\mathbf{u} \in \mathbb{R}^m$ we have

$$\varphi_{\mathbf{Y}}(\mathbf{u}) = e^{i\mathbf{u}^\top \mathbf{c}}\varphi_{D\mathbf{X}}(\mathbf{u})$$

and

$$\varphi_{D\mathbf{X}}(\mathbf{u}) = \varphi_{\mathbf{X}}(D^\top \mathbf{u}) = e^{i(D^\top \mathbf{u})^\top \mathbf{b}}\phi((D^\top \mathbf{u})^\top \Sigma D^\top \mathbf{u})) = e^{i\mathbf{u}^\top D\mathbf{b}}\phi(\mathbf{u}^\top D\Sigma D^\top \mathbf{u})$$

which means for $\mathbf{u} \in \mathbb{R}^m$

$$\varphi_{\mathbf{Y}}(\mathbf{u}) = e^{i\mathbf{u}^\top (\mathbf{c}+D\mathbf{b})}\phi(\mathbf{u}^\top D\Sigma D^\top \mathbf{u})$$

and $\mathbf{Y}$ has an elliptically symmetric distribution with parameters $\mathbf{c} + D\mathbf{b}$ and $D\Sigma D^\top$.

□

Theorem 10 is a generalization of Theorem 7 and is mentioned in Kelker [1970].

3. Density

This chapter presents the properties of the density of a spherically symmetric distribution from Kelker [1970] and Section 2.2 of Fang et al. [1990]. It turns out that the density of the distribution is closely connected with the density of its radial distribution.

Suppose a spherically symmetric random vector $\mathbf{X}$ possesses a density $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, then the density can be also expressed as a scalar function $g : [0, \infty) \to [0, \infty)$ where for $\mathbf{x} \in \mathbb{R}^n$

$$g(\mathbf{x}^\top \mathbf{x}) = f(\mathbf{x}).$$

Such a scalar function g shall be called a *density generator* of $\mathbf{X}$. The following lemma presents a method how to integrate a spherically symmetric function.

Lemma 11. *Let $w : \mathbb{R}^n \to [0, \infty)$ be a function such that there exists a function $\tilde{w} : [0, \infty) \to [0, \infty)$ where $w(\boldsymbol{x}) = \tilde{w}(\boldsymbol{x}^\top \boldsymbol{x})$ for each $\boldsymbol{x} \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} w(\boldsymbol{x}) d\boldsymbol{x} = \int_{\mathbb{R}^n} \tilde{w}(\boldsymbol{x}^\top \boldsymbol{x}) d\boldsymbol{x} = \int_0^\infty \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} y^{\frac{n}{2}-1} \tilde{w}(y) dy$$

provided the integrals exist.

Proof. Since the function $\tilde{w}(\mathbf{x}^\top \mathbf{x})$ is the same on all 2^n orthants of $\mathbb{R}^n$ we can restrict the integral to one orthant

$$\int_{\mathbb{R}^n} \tilde{w}\left(\mathbf{x}^\top \mathbf{x}\right) d\mathbf{x} = 2^n \int_{(0,\infty)^n} \tilde{w}(x_1^2 + \cdots + x_n^2) d\mathbf{x}.$$

We will now use a transformation to remove the squared terms: for each $i \in \{1, \ldots, n\}$ set $u_i = x_i^2$, then the Jacobian of this transformation is

$$\frac{1}{2^n \prod_{i=1}^n \sqrt{u_i}}$$

and the boundaries remain unchanged. Hence

$$2^n \int_{(0,\infty)^n} \tilde{w}(x_1^2 + \cdots + x_n^2) d\mathbf{x} = \int_{(0,\infty)^n} \tilde{w}(u_1 + \cdots + u_n) \prod_{i=1}^n u_i^{-\frac{1}{2}} d\mathbf{u}.$$

Let us set $y_i = u_i$ for $i \in \{1, \ldots, n-1\}$ and $y_n = u_1 + \cdots + u_n$, then the Jacobian of the transformation is 1 but now we are integrating over

$$A = \left\{ y_i \in (0, \infty), y_n \geq \sum_{i=1}^{n-1} y_i, i \in \{1, \ldots, n\} \right\}.$$

Then

$$\int_{(0,\infty)^n} \tilde{w}(u_1 + \cdots + u_n) \prod_{i=1}^n u_i^{-\frac{1}{2}} d\mathbf{u} = \int_A \tilde{w}(y_n) \prod_{i=1}^{n-1} y_i^{-\frac{1}{2}} \left(y_n - \sum_{i=1}^{n-1} y_i \right)^{-\frac{1}{2}} d\mathbf{y}.$$

Next let $z_i = y_i/y_n$ and $z_n = y_n$ which changes the boundaries from A to

$$B = \left\{ z_i \in (0,\infty), 1 \geq \sum_{i=1}^{n-1} z_i, i \in \{1,\ldots,n\} \right\}.$$

The Jacobian of the transformation is z_n^{n-1}. The integrated function and the set can be split into two parts: $B = B' \times (0,\infty)$ where

$$B' = \left\{ z_i \in (0,\infty), 1 \geq \sum_{i=1}^{n-1} z_i, i \in \{1,\ldots,n-1\} \right\},$$

thus the integral can now be split into two parts:

$$\begin{aligned}
&\int_A \tilde{w}(y_n) \prod_{i=1}^{n-1} y_i^{-\frac{1}{2}} \left(y_n - \sum_{i=1}^{n-1} y_i \right)^{-\frac{1}{2}} d\mathbf{y} \\
&= \int_B \tilde{w}(z_n) \prod_{i=1}^{n-1} (z_i z_n)^{-\frac{1}{2}} \left(z_n - \sum_{i=1}^{n-1} z_i z_n \right)^{-\frac{1}{2}} z_n^{n-1} d\mathbf{z} \\
&= \int_B \tilde{w}(z_n) z_n^{-\frac{n-1}{2}} \prod_{i=1}^{n-1} z_i^{-\frac{1}{2}} \left(1 - \sum_{i=1}^{n-1} z_i \right)^{-\frac{1}{2}} z_n^{-\frac{1}{2}} z_n^{n-1} d\mathbf{z} \\
&= \int_{(0,\infty)} \tilde{w}(z_n) z_n^{\frac{n}{2}-1} dz_n \int_{B'} \prod_{i=1}^{n-1} z_i^{-\frac{1}{2}} \left(1 - \sum_{i=1}^{n-1} z_i \right)^{-\frac{1}{2}} dz_1 \cdots dz_{n-1}.
\end{aligned}$$

The second integral on the right hand side is closely related to the Dirichlet distribution

$$\mathcal{D}_n \left(\frac{1}{2}, \ldots, \frac{1}{2} \right)$$

since we are integrating its density function (from Theorem 1) without the normalizing constant over its support. The second integral is equal to one over the normalizing constant:

$$\frac{\prod_{i=1}^n \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.$$

Together the n-dimensional integral is transformed into a one-dimensional one

$$\int_{\mathbb{R}^n} w(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^n} \tilde{w}(\mathbf{x}^\top \mathbf{x}) d\mathbf{x} = \int_0^\infty \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} y^{\frac{n}{2}-1} \tilde{w}(y) dy.$$

□

Following Lemma 11 with $w = f$ and $\tilde{w} = g$ as the density and the density generator. For $g : [0,\infty) \to [0,\infty)$ in order to be a density generator of an n-dimensional spherically symmetric random vector we need

$$\int_0^\infty y^{\frac{n}{2}-1} g(y) dy < \infty, \tag{3.1}$$

and any function $g : [0,\infty) \to [0,\infty)$ satisfying this condition is, up to a constant, a density generator of some spherically symmetric random vector. If $\mathbf{X}$ has k finite moments we need $y^{k+\frac{n}{2}-1} g(y)$ to be integrable over $(0,\infty)$ which follows by a similar computation as in Lemma 11.

3.1 Density of the Radial Distribution

The aim of this section is to derive the density of the radial distribution. It follows mostly Section 2.2 of Fang et al. [1990]. The proofs are extended and examples are added.

Theorem 12. *Let $\boldsymbol{X}$ be an n-dimensional spherically symmetric random vector with the radial distribution R. Then $\boldsymbol{X}$ possesses a density $f(\boldsymbol{x}) = g(\boldsymbol{x}^\top \boldsymbol{x})$ if and only if R has a density h and*

$$h(r) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} r^{n-1} g(r^2), \quad r > 0.$$

Proof. Let $w : [0, \infty) \to [0, \infty)$ be a measurable function. Denoting $r^2 = \mathbf{x}^\top \mathbf{x}$ and $f(\mathbf{x}) = g(\mathbf{x}^\top \mathbf{x}) = g(r^2)$, then

$$\begin{aligned} \mathsf{E}\,(w(R)) &= \mathsf{E}\left(w\left(\sqrt{\mathbf{X}^\top \mathbf{X}}\right)\right) \\ &= \int_{\mathbb{R}^n} w\left(\sqrt{\mathbf{x}^\top \mathbf{x}}\right) f(\mathbf{x}) d\mathbf{x} \\ &= \int_{\mathbb{R}^n} w\left(\sqrt{\mathbf{x}^\top \mathbf{x}}\right) g(\mathbf{x}^\top \mathbf{x}) d\mathbf{x}. \end{aligned}$$

Denote $w\left(\sqrt{\mathbf{x}^\top \mathbf{x}}\right) g(\mathbf{x}^\top \mathbf{x}) = \tilde{w}(\mathbf{x}^\top \mathbf{x})$ for brevity and let us use Lemma 11. The n-dimensional integral is transformed into one-dimensional

$$\int_{\mathbb{R}^n} w\left(\sqrt{\mathbf{x}^\top \mathbf{x}}\right) g(\mathbf{x}^\top \mathbf{x}) d\mathbf{x} = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty w\left(\sqrt{z_n}\right) g(z_n) z_n^{\frac{n}{2}-1} dz_n.$$

Finally, set $r = \sqrt{z_n}$, the boundaries remain unchanged and the Jacobian is $2r$, thus

$$\mathsf{E}\,(w(R)) = \int_0^\infty w(r) \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} r^{n-1} g(r^2) dr$$

and for $r > 0$

$$h(r) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} r^{n-1} g(r^2)$$

and if one of the densities exist, the other one can be found using the derived formula.

□

Remark 7. Theorem 12 can be used to derive the density of $Q = \mathbf{X}^\top \mathbf{X}$ which is

$$\tilde{h}(q) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} q^{\frac{n}{2}-1} g(q), \quad q > 0.$$

where g is the density generator of a spherically symmetric vector $\mathbf{X}$.

Example 5. Let us derive the density of the radial distribution for several examples of spherically symmetric random vectors.

- *Uniform distribution inside the unit ball.* For $\mathbf{X} \sim \mathbf{s}_n$, the density of $\mathbf{X}$ is $1/V_n$ inside the unit ball and zero elsewhere, where V_n is the volume of an n-dimensional unit ball,

$$V_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)},$$

which means that the density of $\|\mathbf{X}\|$ is

$$\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} r^{n-1} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\pi^{\frac{n}{2}}} = \frac{2n}{2} r^{n-1} = n r^{n-1}$$

for $r \in (0,1)$ and $\|\mathbf{X}\| \sim \mathsf{Beta}(n,1)$.

- *Normal distribution.* Theorem 12 can be used to derive the density of the χ-distribution using the normal distribution $\mathcal{N}_n(\mathbf{0}, I)$ with the density

$$f(\mathbf{x}) = (2\pi)^{-\frac{n}{2}} e^{-\mathbf{x}^\top \mathbf{x}/2} \quad \mathbf{x} \in \mathbb{R}^n.$$

Thus, the density of a χ-distribution with n degrees of freedom is

$$h_n(r) = \frac{r^{n-1} e^{-r^2/2}}{2^{n/2-1}\Gamma\left(\frac{n}{2}\right)}, \quad r > 0.$$

- *Student's t-distribution.* For a random vector $\mathbf{X}$ with an n-dimensional symmetric t-distribution (Fang et al. [1990]) with k degrees of freedom with the density

$$f(\mathbf{x}) = \frac{\Gamma\left(\frac{n+k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)(\pi k)^{\frac{n}{2}}} \left(1 + \frac{\mathbf{x}^\top \mathbf{x}}{k}\right)^{-\frac{n+k}{2}}, \quad \mathbf{x} \in \mathbb{R}^n, \tag{3.2}$$

the density of $\|\mathbf{X}\|$ is

$$h(r) = \frac{\Gamma\left(\frac{n+k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{n}{2}\right)} \frac{2r^{n-1}}{k^{\frac{n}{2}}} \left(1 + \frac{r^2}{k}\right)^{-\frac{n+k}{2}}, \quad r > 0.$$

That means a random variable $Q = \frac{R^2}{k}$ has an inverted beta distribution (Forbes et al. [2010]) with parameters $\frac{k}{2}$ and $\frac{n}{2}$ with the density

$$f_Q(q) = \frac{q^{\frac{n}{2}-1}(1+q)^{-\frac{n+k}{2}}}{B\left(\frac{n}{2}, \frac{k}{2}\right)}, \quad q > 0$$

where $B(\cdot,\cdot)$ is the Beta function as defined in Section 1.1.

4. Marginal Distributions

As mentioned in Remark 4, for a spherically symmetric random vector

$$\mathbf{X} = (X_1, \ldots, X_n)^\top$$

and $m < n$ the random vector $(X_1, \ldots, X_m)^\top$ is also spherically symmetric with the same characteristic generator function. In this chapter we firstly derive the distribution and the density of marginal distributions. The theorems are taken from Fang et al. [1990] and Kelker [1970]. Sections 4.2 and 4.3 then generalize the relationship between a random vector and its marginal.

Theorem 13. *Denote $\boldsymbol{X}$ an n-dimensional spherically symmetric random vector with the radial distribution R partitioned into k parts, $\boldsymbol{X} = (\boldsymbol{X}^{(1)}, \ldots, \boldsymbol{X}^{(k)})^\top$ where each $\boldsymbol{X}^{(i)}$ has n_i components. Then for each $i \in \{1, \ldots, k\}$ we have*

$$\boldsymbol{X}^{(i)} \overset{\mathrm{d}}{=} RD_i\boldsymbol{u}_{n_i}$$

where $\boldsymbol{u}_{n_i}$ is uniformly distributed on the unit sphere surface in $\mathbb{R}^{n_i}$ and

$$(D_1^2, \ldots, D_k^2)^\top$$

has a Dirichlet distribution

$$\mathcal{D}_k\left(\frac{n_1}{2}, \ldots, \frac{n_k}{2}\right).$$

Moreover, $\boldsymbol{u}_{n_1}, \ldots, \boldsymbol{u}_{n_k}$ and $(D_1^2, \ldots, D_k^2)^\top$ are independent.

Proof. Denote $\mathbf{X} \overset{\mathrm{d}}{=} R\mathbf{u}_n$ where R and $\mathbf{u}_n$ independent and we are only interested in the marginal distribution of $\mathbf{u}_n$.

Let $\mathbf{Z} \sim \mathcal{N}_n(\mathbf{0}, I)$ and

$$\frac{\mathbf{Z}}{\|\mathbf{Z}\|} \overset{\mathrm{d}}{=} \mathbf{u}_n.$$

When $\mathbf{Z}$ is partitioned similarly as $\mathbf{X}$ into $(\mathbf{Z}^{(1)}, \ldots, \mathbf{Z}^{(k)})^\top$ where each $\mathbf{Z}^{(i)}$ has a standard normal distribution and $\mathbf{Z}^{(i)}$ are independent of each other. Thus, for each $i \in \{1, \ldots k\}$

$$\mathbf{u}_{(i)} \overset{\mathrm{d}}{=} \frac{\mathbf{Z}^{(i)}}{\|\mathbf{Z}\|} = \frac{\mathbf{Z}^{(i)}}{\|\mathbf{Z}^{(i)}\|}\frac{\|\mathbf{Z}^{(i)}\|}{\|\mathbf{Z}\|}.$$

The first fraction on the right-hand side has a uniform distribution on the n_i-dimensional unit sphere, $\|\mathbf{Z}^{(i)}\|^2$ has a χ^2-distribution with n_i degrees of freedom and $\|\mathbf{Z}\|^2$ has a χ^2-distribution with n degrees of freedom which (as in Theorem 2 and Remark 1) is also the gamma distribution. We conclude that for

$$D_i = \frac{\|\mathbf{Z}^{(i)}\|}{\|\mathbf{Z}\|}$$

the random vector $(D_1^2, \ldots, D_k^2)^\top$ has the Dirichlet distribution $\mathcal{D}_k\left(\frac{n_1}{2}, \ldots, \frac{n_k}{2}\right)$ as in the proof of Theorem 2. The independence follows Theorem 5.

□

The theorem is adapted from Fang et al. [1990]. The components of $\mathbf{X}$ may generally be dependent but the dependency is given by the Dirichlet distribution and the radial distribution as in the previous theorem.

4.1 Marginal Density

Denote $\mathbf{u}_{n,r}$ a random vector uniformly distributed on the n-dimensional sphere surface with radius r. Both Kelker [1970] and Fang et al. [1990] then look at a spherically symmetric distribution as a mixture of $\mathbf{u}_{n,r}$ over $[0, \infty)$ with the distribution R. For any $r > 0$ the marginal densities of $\mathbf{u}_{n,r}$ exist,[1] they can be used to find the marginal densities of $R\mathbf{u}_n$.

Theorem 14. *Let $\boldsymbol{X} = (X_1, \ldots, X_n)^\top$ be an n-dimensional spherically symmetric random vector with a radial distribution R with the c.d.f F and $\mathsf{P}(\boldsymbol{X} = \boldsymbol{0}) = 0$. Then all marginal distributions have densities, for $k < n$ the marginal density[2] of $(X_1, \ldots, X_k)^\top$ is for $\boldsymbol{x} = (x_1, \ldots, x_k)^\top \in \mathbb{R}^k$*

$$f_{(X_1,\ldots,X_k)^\top}(\boldsymbol{x}) = f(x_1, \ldots, x_k) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)\pi^{\frac{k}{2}}} \int_{\|\boldsymbol{x}\|}^{\infty} r^{2-n} \left(r^2 - \sum_{i=1}^{k} x_i^2\right)^{\frac{n-k}{2}-1} dF(r).$$

Proof. Denote $\mathbf{u}_n = (U_1, \ldots, U_n)^\top$, then from Theorem 2 the marginal density of its first k components is

$$f_{(U_1,\ldots,U_k)^\top}(u_1, \ldots, u_k) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)\pi^{\frac{k}{2}}} \left(1 - \sum_{i=1}^{k} u_i^2\right)^{\frac{n-k}{2}-1}, \quad \sum_{i=1}^{k} u_i^2 < 1.$$

Let us find the c.d.f of $(X_1, \ldots, X_k)^\top = (RU_1, \ldots, RU_k)^\top$. The function $F(r)$ is the c.d.f. of R and denote $G(u_1, \ldots, u_k)$ the c.d.f of $(U_1, \ldots, U_k)^\top$. Then the joint random vector

$$(U_1, \ldots, U_k, R)^\top$$

has the c.d.f

$$G(u_1, \ldots, u_k) \cdot F(r)$$

because R and $(U_1, \ldots, U_k)^\top$ are independent. Hence, for $(x_1, \ldots, x_k)^\top \in \mathbb{R}^k$

$$\begin{aligned} \mathsf{P}(X_1 \leq x_1, \ldots, X_k \leq x_k) &= \mathsf{P}\left(U_1 \leq \frac{x_1}{R}, \ldots, U_k \leq \frac{x_k}{R}\right) \\ &= \int_0^\infty \int_{-1}^{\frac{x_1}{r}} \cdots \int_{-1}^{\frac{x_k}{r}} 1 \; dG(u_1, \ldots, u_k) \; dF(r). \end{aligned}$$

Let us plug in the density function of $(U_1, \ldots, U_k)^\top$:

$$\int_0^\infty \int_{-1}^{\frac{x_1}{r}} \cdots \int_{-1}^{\frac{x_k}{r}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)\pi^{\frac{k}{2}}} \left(1 - \sum_{i=1}^{k} u_i^2\right)^{\frac{n-k}{2}-1} \mathbf{1}_{\{\sum_{i=1}^k u_i^2 < 1\}} du_k \cdots du_1 \; dF(r).$$

The density of $(X_1, \ldots, X_k)^\top$ is obtained by taking derivatives of the c.d.f. with respect to all variables $x_1, \ldots, x_k$ using the Leibniz integral rule (as in Folland [2002]). The boundaries of the outer integral do not depend on $x_1, \ldots, x_k$, so we are interested in the derivative of the inner integrals where the upper bound is

[1] See Theorem 2.

[2] Since all permutations of the components of a spherically symmetric random vector have the same distribution, we can use any k-dimensional marginal density.

x_i/r and the lower bound as well as the integrated function does not depend on $x_1, \ldots, x_k$. For every inner integral we add $\frac{1}{r}$ and the integrated function is only evaluated in the upper bound. After taking the derivatives we obtain

$$\begin{aligned} f(x_1, \ldots, x_k) &= \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)\pi^{\frac{k}{2}}} \int_0^\infty \left(\frac{1}{r}\right)^k \left(1 - \sum_{i=1}^k \left(\frac{x_i}{r}\right)^2\right)^{\frac{n-k}{2}-1} \mathbf{1}_{\{\sum_{i=1}^k x_i^2 < r^2\}} dF(r) \\ &= \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)\pi^{\frac{k}{2}}} \int_{\|\mathbf{x}\|}^\infty r^{2-n} \left(r^2 - \sum_{i=1}^k x_i^2\right)^{\frac{n-k}{2}-1} dF(r). \end{aligned}$$

We have used that the indicator $\sum_{i=1}^k u_i^2 < 1$ transforms into

$$\sum_{i=1}^k \left(\frac{x_i}{r}\right)^2 < 1$$

or $\sum_{i=1}^k x_i^2 < r^2$ which changes the lower bound in the right-hand side. This concludes the proof of Theorem 14.

□

Theorem 14 is due to Fang et al. [1990] and their proof is further expanded.

Example 6. Let us verify Theorem 14 with the already known distributions.

- *Normal Distribution.* For $\mathbf{Z} \sim \mathcal{N}_n(\mathbf{0}, I)$ the density function of the radial distribution is known[3] and any marginal distribution is also standard normal: for $k < n$ and $(z_1, \ldots, z_k)^\top = \mathbf{z} \in \mathbb{R}^k$ we have

$$\begin{aligned} f(z_1, \ldots, z_k) &= \int_{\|\mathbf{z}\|}^\infty \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)\pi^{\frac{k}{2}}} r^{2-n} \left(r^2 - \sum_{i=1}^k z_i^2\right)^{\frac{n-k}{2}-1} h(r)dr \\ &= \int_{\|\mathbf{z}\|}^\infty \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)\pi^{\frac{k}{2}}} r^{2-n} \left(r^2 - \sum_{i=1}^k z_i^2\right)^{\frac{n-k}{2}-1} \frac{r^{n-1}e^{-\frac{r^2}{2}}}{2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)} dr \\ &= \frac{2^{-\frac{n}{2}}}{\Gamma\left(\frac{n-k}{2}\right)\pi^{\frac{k}{2}}} \int_{\|\mathbf{z}\|}^\infty 2r \left(r^2 - \sum_{i=1}^k z_i^2\right)^{\frac{n-k}{2}-1} e^{-\frac{r^2}{2}} dr. \end{aligned}$$

  Set $u = r^2 - \|\mathbf{z}\|^2$. The integral is then the density of the $\mathsf{Gamma}(\frac{n-k}{2}, \frac{1}{2})$ distribution integrated over $(0, \infty)$ and thus, equal to 1. We obtain

$$\begin{aligned} f(z_1, \ldots, z_k) &= \frac{2^{-\frac{n}{2}}}{\Gamma\left(\frac{n-k}{2}\right)\pi^{\frac{k}{2}}} \int_0^\infty u^{\frac{n-k}{2}-1} e^{-\frac{1}{2}(u+\|\mathbf{z}\|^2)} du \\ &= \frac{2^{-\frac{k}{2}}}{\pi^{\frac{k}{2}}} e^{-\frac{1}{2}\|\mathbf{z}\|^2} \frac{2^{-\frac{n-k}{2}}}{\Gamma\left(\frac{n-k}{2}\right)} \int_0^\infty u^{\frac{n-k}{2}-1} e^{-\frac{1}{2}u} du \\ &= (2\pi)^{-\frac{k}{2}} e^{-\frac{1}{2}\|\mathbf{z}\|^2} \end{aligned}$$

  which is the density of the standard normal distribution $\mathcal{N}_k(\mathbf{0}, I)$.

[3]See p. 25.

- *Uniform distribution inside the unit ball.* For $\mathbf{X} \sim \mathbf{s}_n$ the density of $\|\mathbf{X}\|$ is $h(r) = nr^{n-1}$, $r \in (0,1)$, as derived in Example 5. Hence, for $\mathbf{x} \in \mathbb{R}^k$ such that $\|\mathbf{x}\| \leq 1$ the marginal density of $\mathbf{X}$ is

$$\begin{aligned} f(\mathbf{x}) = f(x_1,\ldots,x_k) &= \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)\pi^{\frac{k}{2}}} \int_{\|\mathbf{x}\|}^{1} r^{2-n}\left(r^2 - \sum_{i=1}^{k} x_i^2\right)^{\frac{n-k}{2}-1} nr^{n-1}dr \\ &= \frac{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)\pi^{\frac{k}{2}}} \int_{\|\mathbf{x}\|}^{1} 2r\left(r^2 - \sum_{i=1}^{k} x_i^2\right)^{\frac{n-k}{2}-1} dr. \end{aligned}$$

Again, set $u = r^2 - \|\mathbf{x}\|^2$, the Jacobian is $2r$ and the boundaries change from $(\|\mathbf{x}\|, 1)$ to $(0, 1-\|\mathbf{x}\|^2)$. For $\sum_{i=1}^{k} x_i^2 < 1$ the marginal density is

$$\begin{aligned} f(x_1,\ldots,x_k) &= \frac{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)\pi^{\frac{k}{2}}} \int_{0}^{1-\|\mathbf{x}\|^2} u^{\frac{n-k}{2}-1} du \\ &= \frac{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)\pi^{\frac{k}{2}}} \frac{(1-\|\mathbf{x}\|^2)^{\frac{n-k}{2}}}{\frac{n-k}{2}} \\ &= \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n-k}{2}+1\right)\pi^{\frac{k}{2}}} \left(1 - \sum_{i=1}^{k} x_i^2\right)^{\frac{n-k}{2}}. \end{aligned}$$

- *Mixture of random vectors.* For $p \in (0,1)$ and two n-dimensional spherically symmetric random vectors $\mathbf{X}_1$, $\mathbf{X}_2$ set $\mathbf{Y}$ as a mixture of $\mathbf{X}_1$ and $\mathbf{X}_2$ with weights p and $1-p$. Denote the k-dimensional marginal random vectors $\mathbf{X}_{1k}$, $\mathbf{X}_{2k}$ and $\mathbf{Y}_k$. Thus, their density generators $g_{\mathbf{Y}_k}$, $g_{\mathbf{X}_{1k}}$ and $g_{\mathbf{X}_{2k}}$, if exist (see Theorem 14), satisfy for all $y > 0$

$$g_{\mathbf{Y}_k}(y) = p \cdot g_{\mathbf{X}_{1k}}(y) + (1-p) \cdot g_{\mathbf{X}_{2k}}(y).$$

4.2 Projections and Antiprojections

Since every marginal distribution of a spherically symmetric random vector is also spherically symmetric, the aim of this section is to generalize this relationship in the opposite direction. Theorems that we list here are derived in Kelker [1970] and Section 2.2 of Fang et al. [1990]. Fractional calculus is used as in Laurent [1975]. As shown in Remark 4 and further discussed at the beginning of this chapter for an N-dimensional spherically symmetric random vector any k-dimensional marginal vector is also spherically symmetric. Thus, we can take the random vector and all its marginals as a sequence of spherically symmetric random vectors.

For the purpose of this section denote a sequence of spherically symmetric random vectors ending with $\mathbf{X}^{(N)}$ as $\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(N)}$, where $N \in \mathbb{N}$, and $\mathbf{X}^{(n)}$ is an n-dimensional random vector, $n \in \{1, \ldots, N\}$, and where for $k < n \leq N$ the random vector $\mathbf{X}^{(k)}$ has the k-dimensional marginal distribution of $\mathbf{X}^{(n)}$. For each $\mathbf{X}^{(k)}$ denote R_k its radial distribution, f_k its density and g_k its density generator.[4]

Finally, denote by ϕ the common characteristic generator of this sequence of distributions. From Remark 4 all $\mathbf{X}^{(n)}$, $n \leq N$, possess the same characteristic generator as $\mathbf{X}^{(N)}$. Conversely, if we only know the characteristic generator of $\mathbf{X}^{(k)}$ and there is a spherically symmetric random vector

$$\mathbf{X}^{(n)} = \begin{pmatrix} \mathbf{X}^{(k)} \\ \mathbf{X}^{(n-k)} \end{pmatrix},$$

the joint vector in $\mathbb{R}^n$ with the marginal vectors $\mathbf{X}^{(k)}$ in $\mathbb{R}^k$ and $\mathbf{X}^{(n-k)}$ in $\mathbb{R}^{n-k}$, and $\mathbf{X}^{(n)}$ has the characteristic generator $\tilde{\phi}$. Then $\tilde{\phi} = \phi$ using the same remark.

For $\mathbf{X}^{(n)}$ we shall call the random vector $\mathbf{X}^{(k)}$ its *projection* if $k < n$ and *antiprojection* if $N \geq k > n$. For any $\mathbf{X}^{(k)}$ there is at most one (anti)projection in each dimension, given by the shared characteristic generator.

Thus, the aim of this section is to find the necessary and sufficient conditions for the existence of antiprojections through the density generators of the spherically symmetric random variables. The focus is put on distributions without an atom at the origin because in that case we have the existence of marginal densities as shown in Theorem 14. In other words, for a k-dimensional spherically symmetric random vector $\mathbf{X}^{(k)}$ we are trying to find the highest $N \in \mathbb{N}$ such that $\mathbf{X}^{(N)}$ is the N-dimensional antiprojection of $\mathbf{X}^{(k)}$ and $\mathbf{X}^{(N+1)}$ does not exist or to show that such $N \in \mathbb{N}$ cannot be found and any antiprojection of $\mathbf{X}^{(k)}$ exists.

Corollary 2. Consider Theorem 13 applied on an n-dimensional random vector $\mathbf{X}^{(n)}$ with radial distribution R_n partitioned into two parts

$$\mathbf{X}^{(n)} = \begin{pmatrix} \mathbf{X}^{(k)} \\ \mathbf{X}^{(n-k)} \end{pmatrix}.$$

Then the marginal vector is distributed as

$$\mathbf{X}^{(k)} \stackrel{\mathrm{d}}{=} R_n B \mathbf{u}_k$$

where $(B^2, 1 - B^2)^\top$ has the Dirichlet distribution $\mathcal{D}_2(\frac{k}{2}, \frac{n-k}{2})$ and R_n, B and $\mathbf{u}_k$ are independent. Which as in Remark 1 means $B^2 \sim \mathsf{Beta}(\frac{k}{2}, \frac{n-k}{2})$. Since $\mathbf{X}^{(k)}$ is also spherically symmetric we shall denote R_k its radial distribution, thus $\mathbf{X}^{(k)} \stackrel{\mathrm{d}}{=} R_k \mathbf{u}_k$ where R_k and $\mathbf{u}_k$ are independent. When combined

$$R_k \stackrel{\mathrm{d}}{=} B R_n$$

where $B^2 \sim \mathsf{Beta}(\frac{k}{2}, \frac{n-k}{2})$ is independent of R_n.

Example 7. First, let us look at two examples:

[4]Up to a constant c_k given by

$$\int_{\mathbb{R}^k} f_k(\mathbf{x}) d\mathbf{x} = \int_0^\infty c_k \ y^{\frac{k}{2}-1} g_k(y) dy = 1$$

as in Lemma 11 and Equation (3.1).

- *Normal distribution.* For $\mathbf{Z}^{(n)} \sim \mathcal{N}_n(\mathbf{0}, I)$ we know that the characteristic generator is $\phi(u) = e^{-u/2}$ for any $n \in \mathbb{N}$ and any projection has also a standard normal distribution. In conclusion, the (anti)projection of $\mathbf{Z}^{(n)}$ is $\mathbf{Z}^{(k)} \sim \mathcal{N}_k(\mathbf{0}, I)$, $k \in \mathbb{N}$.

- *Uniform distribution on the unit sphere surface.* Let us show that for $\mathbf{X}^{(n)} \sim \mathbf{u}_n$ we have $N = n$. For contradiction denote

  $$\mathbf{u}_{n+1} = \begin{pmatrix} \mathbf{Y} \\ Y_{n+1} \end{pmatrix},$$

  where $\mathbf{Y}$ is a random vector of its first n components. If the antiprojection $\mathbf{X}^{(n+1)}$ existed we would have

  $$\mathbf{X}^{(n+1)} \overset{\mathrm{d}}{=} R_{n+1}\mathbf{u}_{n+1}$$

  with the radial distribution R_{n+1}. For the first n components we would have that $\mathbf{X}^{(n)} \overset{\mathrm{d}}{=} R_{n+1}\mathbf{Y}$. But since $\mathbf{Y}$ is a marginal vector of $\mathbf{u}_{n+1}$ it possesses a density as shown in Example 6. Moreover, $\mathsf{P}(R_{n+1} = 0) = \mathsf{P}(\mathbf{X}^{(n+1)} = \mathbf{0}) \leq \mathsf{P}(\mathbf{X}^{(n)} = \mathbf{0}) = 0$ which means that $\mathbf{X}^{(n)} \sim \mathbf{u}_n$ also possesses a density which is a contradiction.

The second example above can be extended to a situation where the radial distribution R_n is not absolutely continuous. In this case neither the random vector $\mathbf{X}^{(n)}$ nor its radial distribution R_n possess densities.

Theorem 15. *Let $\boldsymbol{X}^{(n)}$ be an n-dimensional spherically symmetric random vector such that it does not possess a density and $\mathsf{P}(\boldsymbol{X}^{(n)} = \boldsymbol{0}) = 0$. Then its antiprojection does not exist.*

Proof. Suppose for contradiction that $\mathbf{X}^{(n+1)}$ exists, thus $\mathbf{X}^{(n+1)} \overset{\mathrm{d}}{=} R_{n+1}\mathbf{u}_{n+1}$ where R_{n+1} is its radial distribution. As in Example 7 denote $\mathbf{Y}$ the n-dimensional marginal distribution of $\mathbf{u}_{n+1}$, thus $\mathbf{X}^{(n)} \overset{\mathrm{d}}{=} R_{n+1}\mathbf{Y}$. Since $\mathsf{P}(\mathbf{X}^{(n)} = \mathbf{0}) = 0$, we have that $\mathsf{P}(\mathbf{X}^{(n+1)} = \mathbf{0}) = 0$. Thus, the assumptions of Theorem 14 are met and all marginal random vectors of $\mathbf{X}^{(n+1)}$ including $\mathbf{X}^{(n)}$ possess a density. We have found a contradiction, therefore an $(n+1)$-dimensional spherically symmetric random vector $\mathbf{X}^{(n+1)}$ such that its first n components have $\mathbf{X}^{(n)}$ distribution does not exist.

□

Equation (3.1) gives us an integrability condition on the density generator

$$\int_0^\infty y^{\frac{n}{2}-1} g_n(y) dy < \infty$$

which excludes some functions. For example

$$g(y) = \frac{c}{y+1}$$

is a density generator only for $n = 1$.[5]

Theorem 14 provides a marginal density of a spherically symmetric random vector and the following theorem connects their density generators. The theorem and its proof are due to Fang et al. [1990].

[5]It corresponds to the univariate symmetric Cauchy distribution. For the density of the multivariate Cauchy or the t-distribution see Example 5.

Theorem 16. *Provided the spherically symmetric random vector $\boldsymbol{X}^{(n)}$ possesses density $g_n(\boldsymbol{x}^\top \boldsymbol{x})$, for $k < n$ we have*

$$g_k(y) = \frac{\pi^{\frac{n-k}{2}}}{\Gamma\left(\frac{n-k}{2}\right)} \int_y^\infty (z-y)^{\frac{n-k}{2}-1} g_n(z)dz, \quad y \geq 0.$$

Proof. Denote h the density of the radial distribution R_n whose existence stems from Theorem 12. Furthermore, for $r \geq 0$ we know that

$$h(r) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} r^{n-1} g_n(r^2).$$

From Theorem 14 we have for $\mathbf{x} = (x_1, \ldots, x_k)^\top \in \mathbb{R}^k$

$$f_k(x_1, \ldots, x_k) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)\pi^{\frac{k}{2}}} \int_{\|\mathbf{x}\|}^\infty r^{2-n} \left(r^2 - \sum_{i=1}^k x_i^2\right)^{\frac{n-k}{2}-1} h(r)dr.$$

Set $g_k(y) = g_k(\mathbf{x}^\top \mathbf{x}) = f_k(x_1, \ldots, x_k)$ and plug in the formula for h

$$\begin{aligned} g_k(y) &= \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)\pi^{\frac{k}{2}}} \int_{\sqrt{y}}^\infty r^{2-n} \left(r^2 - y\right)^{\frac{n-k}{2}-1} \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} r^{n-1} g_n(r^2)dr \\ &= \frac{\pi^{\frac{n-k}{2}}}{\Gamma\left(\frac{n-k}{2}\right)} \int_{\sqrt{y}}^\infty 2r \left(r^2 - y\right)^{\frac{n-k}{2}-1} g_n(r^2)dr. \end{aligned}$$

Transforming $z = r^2$ with Jacobian $2r$ which changes the lower bound to y and completes the proof as for $y \geq 0$

$$g_k(y) = \frac{\pi^{\frac{n-k}{2}}}{\Gamma\left(\frac{n-k}{2}\right)} \int_y^\infty (z-y)^{\frac{n-k}{2}-1} g_n(z)dz.$$

□

Setting $k = n-1$ and $k = n-2$, Theorem 16 is simplified as below, the results are adapted from Fang et al. [1990] and Uchaikin and Zolotarev [1999].

Remark 8. Under the assumptions from Theorem 16 we have

$$g_{n-1}(y) = \int_y^\infty \frac{g_n(z)}{\sqrt{z-y}} dz, \tag{4.1}$$

$$g_{n-2}(y) = \pi \int_y^\infty g_n(z)dz, \tag{4.2}$$

and conversely for almost all $y > 0$

$$g_n(y) = -\frac{1}{\pi} g'_{n-2}(y). \tag{4.3}$$

Generally for $k < n$, we can apply the formulas derived in Section 1.3 to the result of Theorem 16,

$$g_k(y) = \pi^{\frac{n-k}{2}} W^{-\frac{n-k}{2}} g_n(y) \tag{4.4}$$

which means

$$g_n(y) = \pi^{-\frac{n-k}{2}} W^{\frac{n-k}{2}} g_k(y) \tag{4.5}$$

where W^p is the fractional integral and derivative defined in Equation (1.6).

The integrand in Equation (4.1) is non-negative which shows that for a spherically symmetric random vector $\mathbf{X}^{(n)}$ there is $a \in (0, \infty]$ such that $g_k(y) > 0$, $k < n$, if and only if $y \in [0, a)$, thus the support of all projections of $\mathbf{X}^{(n)}$ is a ball with some radius a

$$S_a = \left\{ \mathbf{x} = (x_1, \ldots, x_k)^\top \in \mathbb{R}^k, \|\mathbf{x}\| < a \right\}.$$

Therefore, if g_n does not satisfy this property, $\mathbf{X}^{(n)}$ does not have any antiprojection with a density, and thus $\mathbf{X}^{(n+2)}$ does not exist.

Let us examine again Equation (4.1) in terms of continuity using a theorem presented in Kelker [1970].

Theorem 17. *Let $g_n(\boldsymbol{x}^\top \boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^n$, be a density of a spherically symmetric random vector $\boldsymbol{X}^{(n)}$ and $\boldsymbol{X}^{(k)}$ its projection with density $g_k(\boldsymbol{x}^\top \boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^k$. If g_n is bounded in a neighborhood of $y \in (0, \infty)$, then g_k is continuous at y.*

Proof. It is sufficient to prove the theorem only for g_{n-1} since if g_{n-1} is continuous at y, it is also bounded in some neighborhood of y and induction may be applied.

Assume there is a $\delta > 0$ and $K > 0$ such that $0 \leq g_n(x) < K$ for $x \in (y - \delta, y + \delta)$. For $\nu \in (0, \delta)$ let us bound from above $|g_{n-1}(y + \nu) - g_{n-1}(y)|$ as $\nu \to 0^+$ using

$$g_{n-1}(y) = \int_y^\infty \frac{g_n(t)}{\sqrt{t-y}} dt.$$

Both integrals in

$$|g_{n-1}(y+\nu) - g_{n-1}(y)| = \left| \int_{y+\nu}^\infty \frac{g_n(t)}{\sqrt{t-y-\nu}} dt - \int_y^\infty \frac{g_n(t)}{\sqrt{t-y}} dt \right|$$

can be split to three integrals over intervals $(y, y+\nu)$, $(y+\nu, y+\delta)$ and $(y+\delta, \infty)$ and regrouped

$$\begin{aligned} |g_{n-1}(y+\nu) - g_{n-1}(y)| \leq & \int_y^{y+\nu} \frac{g_n(t)}{\sqrt{t-y}} dt \\ & + \int_{y+\nu}^{y+\delta} g_n(t) \left(\frac{1}{\sqrt{t-y-\nu}} - \frac{1}{\sqrt{t-y}} \right) dt \\ & + \int_{y+\delta}^\infty g_n(t) \left(\frac{1}{\sqrt{t-y-\nu}} - \frac{1}{\sqrt{t-y}} \right) dt. \end{aligned}$$

The first two integrals are

$$\int_y^{y+\nu} \frac{g_n(t)}{\sqrt{t-y}} dt < K \int_0^\nu \frac{1}{\sqrt{t}} dt = 2K\sqrt{\nu} \xrightarrow[\nu \to 0^+]{} 0,$$

$$\int_{y+\nu}^{y+\delta} g_n(t)\left(\frac{1}{\sqrt{t-y-\nu}} - \frac{1}{\sqrt{t-y}}\right)dt < K\int_0^{\delta-\nu}\left(\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t+\nu}}\right)dt$$
$$= 2K(\sqrt{\delta-\nu} - \sqrt{\delta} + \sqrt{\nu}) \xrightarrow[\nu\to 0^+]{} 0.$$

As for the third integral

$$\int_{y+\delta}^{\infty} g_n(t)\left(\frac{1}{\sqrt{t-y-\nu}} - \frac{1}{\sqrt{t-y}}\right)dt = \int_{y+\delta}^{\infty} g_n(t)\frac{\sqrt{t-y} - \sqrt{t-y-\nu}}{\sqrt{(t-y-\nu)(t-y)}}dt$$

where

$$\sqrt{t-y} - \sqrt{t-y-\nu} \leq \sqrt{\nu}$$

since the left-hand side is decreasing in $t-y$ and for $t-y=\nu$ we have that $\sqrt{t-y} - \sqrt{t-y-\nu} = \sqrt{\nu}$. Thus,

$$\int_{y+\delta}^{\infty} g_n(t)\frac{\sqrt{t-y} - \sqrt{t-y-\nu}}{\sqrt{(t-y-\nu)(t-y)}}dt \leq \int_{y+\delta}^{\infty} g_n(t)\frac{\sqrt{\nu}}{\sqrt{(t-y-\nu)(t-y)}}dt.$$

Let us use Equation (3.1) and since $n > 1$ and the integrated function is non-negative

$$\int_{y+\delta}^{\infty} g_n(t)dt < \infty.$$

Thus, since for $t \in (y+\delta, \infty)$ we have that

$$\frac{1}{\sqrt{(t-y-\nu)(t-y)}} < \infty$$

and

$$\frac{1}{\sqrt{(t-y-\nu)(t-y)}} \xrightarrow[t\to\infty]{} 0$$

which assures the convergence of the integral

$$\int_{y+\delta}^{\infty} g_n(t)\frac{1}{\sqrt{(t-y-\nu)(t-y)}}dt < \infty.$$

In the neighborhood of $y+\delta$ the integrand is finite and the density generator g_n ensures the convergence of the integral in the neighborhood of ∞. With $\sqrt{\nu}$ we obtain

$$\sqrt{\nu}\int_{y+\delta}^{\infty} g_n(t)\frac{1}{\sqrt{(t-y-\nu)(t-y)}}dt \xrightarrow[\nu\to 0^+]{} 0$$

which means $|g_{n-1}(y+\nu) - g_{n-1}(y)| \xrightarrow[\nu\to 0^+]{} 0$ and g_{n-1} is continuous from the right at y. Similarly, g_{n-1} is continuous from the left at y and thus, is continuous at y.

□

If for $n \geq 4$ we know that g_n is bounded, then g_{n-1} is continuous and using Equation (4.2) and the fundamental theorem of calculus we obtain that g_{n-3} is differentiable for $y > 0$. Even if there is some $y_0 > 0$ such that g_n is unbounded at some neighborhood $(y_0 - \delta, y_0 + \delta)$, the integrability condition from Equation (3.1) holds and for all $y > 0$

$$g_{n-2}(y) = \int_y^\infty g_n(t)dt < \infty$$

which means that g_{n-2} is bounded and also continuous because g_n is integrable. Thus, for any g_n, $n > 5$, the function g_{n-2} is continuous and g_{n-4} is differentiable (Kelker [1970]).

For the construction and existence of antiprojection, Equation (4.2)

$$g_n(y) = -\frac{1}{\pi} g'_{n-2}(y)$$

enables to construct all densities of possible antiprojections if the univariate density is known. Fractional derivatives of necessary orders may be taken. Alternatively, for odd dimensions use standard differentiation and for even dimensions we could use one integration to reduce the dimension by one in order to reach an even dimension and then differentiation again in even dimensions.

Conversely, we can reach some conclusions about the existence of antiprojections.

- Theorem 17 states that if g_k is not continuous, then g_{k+1} is not bounded.
- As from Equation (4.3) for an increasing function g_n the derivative of g_n is positive and g_{n+2} is negative which means $\mathbf{X}^{(n+2)}$ does not exist.
- If g_1 is not differentiable up to the order $\lfloor \frac{k+1}{2} \rfloor$, its antiprojection in the k-th dimension does not possess a density (Kelker [1970]).

Example 8. Let us find all possible antiprojections of two simple distributions with bounded support in $\mathbb{R}^n$:

- For $\mathbf{X}^{(n)} \sim \mathbf{s}_n$, the density generator is $g_n(y) = \mathbf{1}_{(0,1)}(y)$ and using Example 1 we have that

  $$W^p \mathbf{1}_{(0,1)}(y) = D^p_{x,1} 1 = c \cdot (1-x)^{-p}, \quad p < 1$$

  and $W^p \mathbf{1}_{(0,1)} = 0$ for $p = 1$ which cannot be a density generator, thus we are not interested in higher fractional derivatives and higher antiprojections than $n + 2$.

  From Equation (4.5) the random vector $\mathbf{X}^{(n+1)}$ possesses a density and $\mathbf{X}^{(n+2)}$ does not. Using Theorem 15 we have $N \leq n + 2$. The marginal density generators in the dimension $l < k$ of both $\mathbf{s}_k$ and $\mathbf{u}_k$ as in Theorem 2 and Example 6 are

  $$g_{\mathbf{u}_k}(y) = \frac{\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k-l}{2}\right)\pi^{\frac{l}{2}}} (1-y)^{\frac{k-l}{2}-1}, \quad y \in (0,1),$$

  $$g_{\mathbf{s}_k}(y) = \frac{\Gamma\left(\frac{k}{2}+1\right)}{\Gamma\left(\frac{k-l}{2}+1\right)\pi^{\frac{l}{2}}} (1-y)^{\frac{k-l}{2}}, \quad y \in (0,1).$$

Setting $k = n$ and $l = n - 2$ in the first equation gives us that the marginal distribution of $\mathbf{u}_n$ of dimension $n - 2$ is $\mathbf{s}_{n-2}$. Conversely, for $\mathbf{s}_n$ its antiprojection of dimension $n + 2$ is $\mathbf{u}_{n+2}$ and higher antiprojections of $\mathbf{X}^{(n)} \sim \mathbf{s}_n$ than $\mathbf{X}^{(n+2)} \sim \mathbf{u}_{n+2}$ do not exist.

- If $g_n(y) = (1 - \sqrt{y})\mathbf{1}_{(0,1)}(y)$, then we shall call the distribution of $\mathbf{X}^{(n)}$ *generalized triangular.*[6] As g_n is continuous but not differentiable for all $y \in (0, \infty)$ which means $\mathbf{X}^{(n+4)}$ does not possess a density and $\mathbf{X}^{(n+5)}$ does not exist from Theorem 15.

  The density generator of $\mathbf{X}^{(n+3)}$ is from Equation (4.5) for some $c > 0$

$$g_{n+3}(y) = \pi^{-\frac{3}{2}} W^{\frac{3}{2}} g_n(y) = \frac{c}{y\sqrt{1-y}}, \quad y \in (0,1).$$

  As said above $\mathbf{X}^{(n+4)}$ is not absolutely continuous, yet the function g_{n+3} may indicate how is $\mathbf{X}^{(n+4)}$ distributed.

  The function g_{n+3} may be rewritten as in Example 6

$$g_{n+3}(y) = \frac{c}{y\sqrt{1-y}} = \frac{c}{\sqrt{1-y}}\left(1 + \frac{1-y}{y}\right) = \frac{c_1}{\sqrt{1-y}} + \frac{c_2\sqrt{1-y}}{y}$$

  for $c_1 + c_2 = c$. The first fraction on the right-hand side is the marginal density generator of $\mathbf{u}_{n+4}$ in the dimension $n + 3$ (Theorem 2). For the second part we have up to a constant

$$W^{\frac{1}{2}} \frac{c_2\sqrt{1-y}}{y} = \frac{c_2}{2y^{\frac{3}{2}}}.$$

  Set $\tilde{g}_{n+3}(y) = y^{-\frac{3}{2}}$ which is a density generator of a spherically symmetric random vector (satisfies the integrability condition from Equation (3.1) and non-negativity). The random vector $\mathbf{X}^{(n+4)}$ is a mixture of $\mathbf{u}_{n+4}$ and some absolutely continuous distribution.

  In conclusion, for $\mathbf{X}^{(n)}$ with the generalized triangular distribution the antiprojection $\mathbf{X}^{(n+4)}$ is a mixture of $\mathbf{u}_{n+4}$ and the absolutely continuous distribution with the density generator $\tilde{g}_{n+3}(y) = y^{-\frac{3}{2}}$. From Theorem 15 the antiprojection $\mathbf{X}^{(n+5)}$ does not exist, thus $N = n + 4$.

- Theorem 14 is applicable on any distribution, denote R a random variable with the Cantor distribution and $\mathbf{X}^{(2)} \stackrel{d}{=} R\mathbf{u}_2$. The Cantor distribution (Lad and Taylor [1992]) is singular and does not posses a density which means $\mathbf{X}^{(n)}$ does not have any antiprojections from Theorem 15. But we may find its projection $\mathbf{X}^{(1)}$ by its density (Theorem 14)

$$f(x) = \int_{|x|}^{1} (r^2 - x^2)^{-\frac{1}{2}} dF(r)$$

  where F is the c.d.f. of R. Section 2.2 provides the moments of $\mathbf{X}^{(2)}$. Since $\mathsf{E}\,(R^2) = \frac{3}{8}$ the covariance matrix is $\mathsf{Cov}(\mathbf{X}^{(2)}) = \frac{3}{16}I$.

As seen above in the examples the problem of finding projections and antiprojections is not a straightforward problem and its complexity depends on the properties of the distribution.

[6]For $n = 1$ it is the univariate triangular distribution.

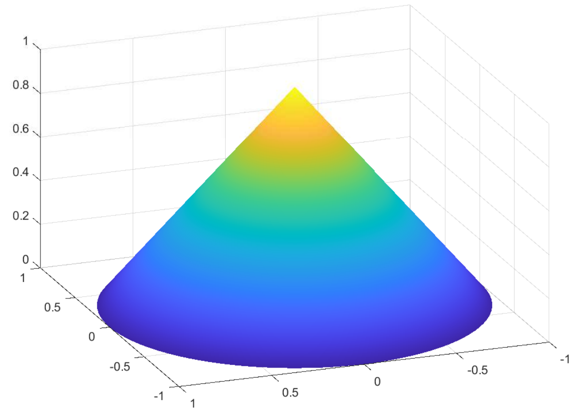

Figure 4.1: The density function of the bivariate triangular distribution.

4.3 Mixtures of Normal Distributions

The aim of this section is to answer the question for which spherically symmetric distribution it is possible to construct antiprojecitons in any dimension. Kelker [1970] focuses on a specific subclass of elliptically symmetric random vectors. For the purpose of this thesis let us focus only on spherically symmetric random vectors.

Definition 5. *Let W be a non-negative random variable. Then the random vector $\boldsymbol{X}$ is a* variance mixture of normal distributions *if, conditionally given $W = w$, we have $\boldsymbol{X} \sim \mathcal{N}_n(\boldsymbol{0}, wI)$. Thus, the conditional density of $\boldsymbol{X}$ is for $\boldsymbol{x} \in \mathbb{R}^n$ and $w > 0$*

$$f_{\mathbf{X}|W}(\boldsymbol{x} \mid w) = \frac{1}{(2w\pi)^{\frac{n}{2}}} e^{-\frac{\boldsymbol{x}^\top \boldsymbol{x}}{2w}}.$$

For $w = 0$ we have $\boldsymbol{X} = \boldsymbol{0}$ a.s.

If W has the c.d.f. $G(w)$ we can write the unconditional density (Lachout [2004]) as

$$f(\mathbf{x}) = \int_0^\infty f_{\mathbf{X}|W}(\mathbf{x} \mid w) dG(w) = \int_0^\infty \frac{1}{(2w\pi)^{\frac{n}{2}}} e^{\frac{\mathbf{x}^\top \mathbf{x}}{2w}} dG(w)$$

or since given $W = w$ the distribution of $\mathbf{X}$ is the normal distribution with $\text{var}(\mathbf{X}) = wI$, then $\mathbf{X} = (X_1, \ldots, X_n)^\top$ given $W = w$ is distributed as

$$(X_1, \ldots, X_n)^\top \stackrel{\text{d}}{=} (\sqrt{w} Z_1, \ldots, \sqrt{w} Z_n)^\top$$

for $(Z_1, \ldots, Z_n)^\top \sim \mathcal{N}_n(\mathbf{0}, I)$. That means the unconditional distribution of $\mathbf{X}$ is given as

$$\mathbf{X} \stackrel{\text{d}}{=} \sqrt{W} \mathbf{Z}$$

where W and $\mathbf{Z} \sim \mathcal{N}_n(\mathbf{0}, I)$ are independent. Moreover, the variance mixture of normal distributions is spherically symmetric.

Theorem 18. *Let $\{X_n\}_{n=1}^\infty$ be a sequence of random variables and denote the joint random vector of the first n terms as $\boldsymbol{X}^{(n)} = (X_1, \ldots, X_n)^\top$. Then $\boldsymbol{X}^{(n)}$*

is a spherically symmetric random vector for all $n \in \mathbb{N}$ if and only if there is a non-negative random variable W such that for all $n \in \mathbb{N}$ we can decompose $\boldsymbol{X}^{(n)} \overset{\mathrm{d}}{=} \sqrt{W}\boldsymbol{Z}_n$ where $\boldsymbol{Z}_n \sim \mathcal{N}_n(\boldsymbol{0}, I)$ is independent of W.

The proof of Theorem 18 can be found in Fang et al. [1990] and is based on elaborate results on positive definite functions from Schoenberg [1938]. Denote Φ the c.d.f. of the n-dimensional standard normal distribution. Since $\mathbf{Z}_n$ and W are independent, the characteristic function of $\mathbf{X}^{(n)}$ is for $\mathbf{t} \in \mathbb{R}^n$

$$\begin{aligned} \mathsf{E}\left(e^{i\mathbf{t}^\top \mathbf{X}^{(n)}}\right) &= \mathsf{E}\left(e^{i\mathbf{t}^\top (\sqrt{W}\mathbf{Z}_n)}\right) \\ &= \int_0^\infty \int_{\mathbb{R}^n} e^{i\mathbf{t}^\top (\sqrt{w}\mathbf{z})}\, d\Phi(\mathbf{z})\, dG(w) \\ &= \int_0^\infty e^{-\frac{\mathbf{t}^\top \mathbf{t}}{2w}} dG(w), \end{aligned}$$

the common characteristic generator for $\mathbf{X}^{(n)}$ is now

$$\phi(u) = \int_0^\infty e^{-\frac{u}{2w}} dG(w)$$

as a mixture of $e^{-\frac{u}{2w}}$, the characteristic generator of the normal distribution $\mathcal{N}(0, w)$.

The class of variance mixtures of normal distributions includes not only all normal distributions $\mathcal{N}(\mathbf{0}, cI)$, $c > 0$, but also the t-distributions, the Cauchy distribution and the Laplace distribution (Gneiting [1997]).

Example 9. Let us show that the antiprojections of $\mathbf{X}^{(1)}$ with the Cauchy distribution have the multivariate Cauchy distribution.

The density generator of $\mathbf{X}^{(1)}$ is

$$g_1(y) = \frac{1}{\pi(1+y)},$$

thus the density generators of all odd dimensions $n = 2k-1$ are up to a normalizing constant

$$g_{2k-1}(y) = (1+y)^{-k} = (1+y)^{-\frac{2k-1+1}{2}} = (1+y)^{-\frac{n+1}{2}}, \quad y > 0$$

following Equation (4.3). Let us find $W^{\frac{1}{2}} g_1$ to obtain the density generator of $\mathbf{X}^{(2)}$. From Equation (1.6) we use the definition of the right-hand RL fractional derivative (Equation (1.3))

$$g_2(y) = \frac{-1}{\Gamma(\frac{1}{2})} \frac{\partial}{\partial y} \int_y^\infty \frac{1}{\pi(1+t)\sqrt{t-y}} dt.$$

Set $u = \sqrt{t-y}$ which transforms the integral

$$\int_y^\infty \frac{1}{\pi(1+t)\sqrt{t-y}} dt = \int_0^\infty \frac{\pi}{u^2+y+1} du = \frac{1}{\sqrt{1+y}}.$$

The derivative is $-(1+y)^{-\frac{3}{2}}$ which means

$$g_2(y) = (1+y)^{-\frac{1+2}{2}}$$

and $\mathbf{X}^{(2)}$ also has the multivariate Cauchy distribution. Higher dimensions are again obtained using standard differentiation. In conclusion, all antiprojections of the Cauchy distribution are as well the multivariate Cauchy distribution with the density generator

$$g_n(y) = (1+y)^{-\frac{1+n}{2}}, \quad y > 0.$$

The multivariate Cauchy is the multivariate t-distribution with 1 degree of freedom with the density in Equation (3.2).

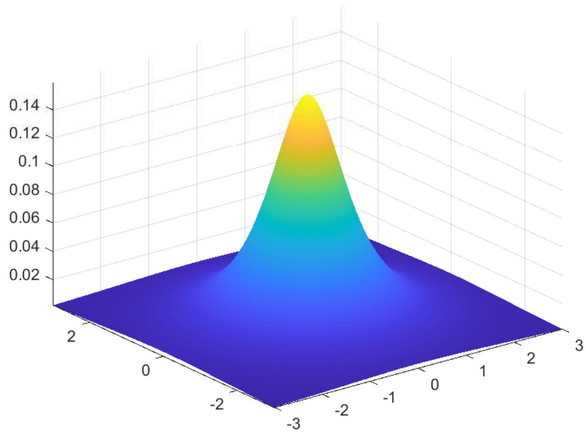

Figure 4.2: The density function of the bivariate Cauchy distribution.

In conclusion, only variance mixtures of normal distributions have infinitely many antiprojections. If W is a random variable such that $\mathsf{P}(W = 0) = 1$ then

$$\mathsf{P}(\mathbf{X}^{(n)} = \mathbf{0}) = 1$$

and the degenerate sequence $\{\mathbf{X}^{(n)}\}_{n=1}^{\infty}$ of n-dimensional random vectors distributed as Dirac measures at 0 satisfies Theorem 18. For any random variable W such that $\mathsf{P}(W = 0) \neq 1$ the support of $\mathbf{X}^{(n)}$ defined as in Theorem 18 is unbounded for all $n \in \mathbb{N}$. That means that for any n-dimensional spherically symmetric random vector $\mathbf{X}^{(n)}$ with bounded support such that $\mathsf{P}(\mathbf{X}^{(n)} = \mathbf{0}) \neq 1$ there are only finitely many antiprojections.

5. Inference

The aim of this chapter is to review several methods of estimation and testing involving spherically symmetric distributions. The topic is split into two parts. In the first section the test involves a given spherically symmetric distribution. The multivariate problem is reduced to a univariate problem using the results derived in the previous chapters. The second part tackles a more general problem. Testing whether the random sample comes from a spherically symmetric distribution has been approached from numerous ways as reviewed in Chmielewski [1981] and Serfling [2006]. Section 5.2 presents three basic approaches from Fang et al. [1993], Henze et al. [2014] and Li et al. [1997].

For the purpose of this chapter let $\mathbf{X}_1, \ldots, \mathbf{X}_m$ be random sample of size m from some n-dimensional distribution $\mathbf{X}$.

5.1 Testing for a Reference Distribution

Firstly, suppose we want to test the following hypothesis

$$H_0 : \mathbf{X}_i \sim \mathcal{G}$$

against

$$H_1 : \mathbf{X}_i \nsim \mathcal{G}$$

for a given spherically symmetric distribution $\mathcal{G}$ when $\mathbf{X}$ is a spherically symmetric random vector.

All spherically symmetric distributions are fully described by univariate distributions. Theorems 4 and 6 state that an n-dimensional random vector $\mathbf{X}$ has the same distribution as $R\mathbf{u}_n$ for independent R and $\mathbf{u}_n$ which means all the information about the distribution is in the univariate distribution of $R = \|\mathbf{X}\|$.

From Theorem 7 under the assumptions of spherical symmetry all univariate marginal distributions $X_1, \ldots, X_n$ are the same and uniquely determine the multivariate distribution.

Thus, the testing whether a random sample is drawn from a particular multivariate spherically symmetric distribution $\mathcal{G}$ is reduced to a univariate test for R, R^2 or X_i with the possible density derived in Sections 3.1 and 4.1. For example we may use the Kolmogorov-Smirnov test (Hollander and Wolfe [2013]) whether R or the marginal random variable is drawn from $\mathcal{G}'$, the radial (or marginal) distribution of $\mathcal{G}$, using the empirical distribution function

$$\hat{F}(y) = \frac{1}{m} \sum_{i=1}^{m} \mathbf{1}(Y_i \leq y)$$

where Y is either the marginal or the radial distribution of $\mathbf{X}$. This approach is heavily sensitive to non-symmetry.

5.2 Testing for Spherical Symmetry

Generally, we may test

$$H_0 : \mathbf{X}_i \text{ is a spherically symmetric random vector}$$

against the alternative where the distribution is not spherically symmetric.

5.2.1 Testing via Comparing Projections

Fang et al. [1993] suggest using the property from Theorem 7 and change the null hypothesis into

$$H_0' : \forall \mathbf{a} \in \mathbb{R}^n, \|\mathbf{a}\| = 1 : \mathbf{a}^\top \mathbf{X} \text{ have the same distribution.}$$

Then H_0' can be tested by using a finite set of unit vectors. The test statistic of the problem is similar to the one used in the Wilcoxon test.

Firstly, independently sample N unit vectors $\mathbf{a}_1, \ldots, \mathbf{a}_N$ as N points from the uniform distribution on the unit sphere surface.

For $\mathbf{a}_i$, $\mathbf{a}_j$ set

$$\rho(\mathbf{a}_i, \mathbf{a}_j) = \frac{1}{m(m-1)} \sum_{k=1}^{m} \sum_{\substack{l=1 \\ l \neq k}}^{m} \mathbf{1}(\mathbf{a}_i^\top \mathbf{X}_k < \mathbf{a}_j^\top \mathbf{X}_l)$$

and the test statistic is

$$W = \min \left\{ \rho(\mathbf{a}_i, \mathbf{a}_j),\ 1 \leq i, j \leq N, i \neq j \right\}.$$

Asymptotic properties of

$$\sqrt{m} \left(W - \frac{1}{2} \right)$$

are derived in Fang et al. [1993]. The null hypothesis is rejected for low values of W, when there is at least one significantly different distribution of $\mathbf{a}_i^\top \mathbf{X}_k$.

5.2.2 Testing via Empirical Characteristic Function

Henze et al. [2014] test the same hypothesis using the property of the characteristic function derived in Theorem 3. For $\mathbf{t} \in \mathbb{R}^n$ denote

$$\hat{\varphi}(\mathbf{t}) = \frac{1}{m} \sum_{k=1}^{m} e^{i \mathbf{t}^\top \mathbf{X}_k}$$

the empirical characteristic function of the random sample $\mathbf{X}_1, \ldots, \mathbf{X}_m$. Then for $\mathbf{t}_1, \mathbf{t}_2 \in \mathbb{R}^n$ such that $\|\mathbf{t}_1\| = \|\mathbf{t}_2\|$ set

$$d(\mathbf{t}_1, \mathbf{t}_2) = \hat{\varphi}(\mathbf{t}_1) - \hat{\varphi}(\mathbf{t}_2).$$

The test statistic

$$K = \sqrt{m} \sup \left\{ |d(\mathbf{t}_1, \mathbf{t}_2)|,\ \mathbf{t}_1, \mathbf{t}_2 \in \mathbb{R}^n, \|\mathbf{t}_1\| = \|\mathbf{t}_2\| \right\}$$

is then similar to the statistic used in the Kolmogorov-Smirnov test. The null hypothesis is rejected for high values of K since under the null hypothesis and for φ instead of the empirical characteristic function we have $\varphi(\mathbf{t}_1) = \varphi(\mathbf{t}_2)$ whenever $\|\mathbf{t}_1\| = \|\mathbf{t}_2\|$.

5.2.3 Testing via QQ-plots

A visual test is proposed in Li et al. [1997] based on the following remark from Section 2.7 of Fang et al. [1990]. The test derives the distribution of some statistic t applied on $\mathbf{X}_i$ for $i \in \{1, \dots, m\}$ and constructs a QQ-plot which compares the empirical quantiles of $t(\mathbf{X}_i)$ with the theoretical quantiles.

Remark 9. Let $\mathbf{X}$ be an n-dimensional spherically symmetric random vector, $\mathsf{P}(\mathbf{X} = \mathbf{0}) = 0$, and t be a statistic such that for any $c > 0$ the distributions of $t(\mathbf{X})$ and $t(c\mathbf{X})$ are the same. Then

$$t(\mathbf{X}) \stackrel{\mathrm{d}}{=} t(\mathbf{Z})$$

where $\mathbf{Z} \sim \mathcal{N}_n(\mathbf{0}, I)$.

Let us use two well-known statistics that satisfy the condition that $t(\mathbf{X})$ and $t(c\mathbf{X})$ have the same distribution, the standard t-statistic and the F-statistic. The distribution of the standard t-statistic and F-statistic is well known under the standard normal distribution (Wilks [1947]).

Therefore, for a spherically symmetric random vector $\mathbf{X}_i = (X_{i1}, \dots, X_{in})^\top$ the t-statistic

$$T(\mathbf{X}_i) = \sqrt{n}\frac{\bar{X}_i}{S_i}$$

is constructed using the sample mean and the sample standard deviation applied to the elements of $\mathbf{X}_i$

$$\bar{X}_i = \frac{1}{n}\sum_{j=1}^{n} X_{ij}, \quad S_i = \frac{1}{n-1}\sqrt{\sum_{j=1}^{n}(X_{ij} - \bar{X}_i)^2}.$$

The t-statistic $T(\mathbf{X}_i)$ has the standard univariate t-distribution with $n-1$ degrees of freedom. For each realization $\mathbf{X}_i$ from the random sample $\mathbf{X}_1, \dots, \mathbf{X}_m$ let us find $T(\mathbf{X}_i)$. Since $\mathbf{X}_1, \dots, \mathbf{X}_m$ are independent and identically distributed, the same holds for $T(\mathbf{X}_1), \dots, T(\mathbf{X}_m)$ which are independent and identically distributed according to the t-distribution with $n-1$ degrees of freedom.

The F-statistic is constructed for a given k as

$$F_{k,n}(\mathbf{X}_i) = \frac{n-k}{k}\frac{\sum_{j=1}^{k} X_{ij}^2}{\sum_{j=k+1}^{n} X_{ij}^2}$$

and is distributed according to the F-distribution with k and $n-k$ degrees of freedom.

The QQ-plot graphically compares the quantiles of the constructed sample $T(\mathbf{X}_1), \dots, T(\mathbf{X}_m)$ with the quantiles of the t-distribution (and similarly for the F-statistic). The QQ-plot is easily constructed by any statistical software. However, distributions which are not spherically symmetric may also pass this test, namely when $\mathbf{X} \stackrel{\mathrm{d}}{=} R\mathbf{u}_n$ as in Section 2.1 but for dependent R and $\mathbf{u}_n$ (Li et al. [1997]).

Conclusion

In this thesis, spherically symmetric distributions were introduced as distributions which remain unchanged under rotations about the origin.

Section 2.1 showed that spherically symmetric distributions can be decomposed into two independent factors – a non-negative random variable (called the radial distribution) and the uniform distribution on the unit sphere surface. This property was further used in Section 2.2 to find marginal distributions and moments of spherically symmetric distributions. Marginal distributions of a spherically symmetric distribution are also spherically symmetric. Even if the distribution is not absolutely continuous all marginal distributions possess a density if the distribution does not have an atom at the origin (as shown in Section 4.1).

Section 4.2 extended the relationship between the spherically symmetric distribution and its marginal distribution in the opposite direction, from a lower dimension to a higher. Let $\mathbf{X}$ be a spherically symmetric random vector without an atom at the origin. If $\mathbf{X}$ does not possess a density, there are no extensions into higher dimensions. The number of dimensions generally depends on the differentiability of the density function. Fractional calculus happens to be a useful tool when looking for density functions in higher dimensions. The class of spherically symmetric distributions which can be extended up to any dimension is described in Section 4.3.

The last chapter focused on estimation of spherically symmetric distributions and presents three tests for spherical symmetry.

Spherically symmetric distributions may be generalized into elliptically symmetric distributions as briefly discussed in Section 2.3. All properties of spherically symmetric distributions may be extended to elliptically symmetric distributions since the extension is done via affine transformations as shown in Section 2.3. Fang et al. [1990] present multivariate log-elliptical distributions (the class includes the log-normal distribution) and symmetric distributions with respect to other norms than the Euclidean Gupta and Song [1997]. Anderson and Fang [1982] extend random vectors with spherically symmetric distributions into random matrices.

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