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Interpolation of logarithmically convex combinations of operators

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Title: Interpolation of logarithmically convex combinations of operators

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Abstract: We study the behaviour of logarithmically convex combinations of operators given by $Tf = |S_1 f|^{\frac{1}{\theta}} |S_2 f|^{1-\frac{1}{\theta}}$, where S_1, S_2 are some (usually quasi-linear) operators acting on spaces of measurable functions and $\theta \in (1, \infty)$ is a parameter. We develop two, quite different in nature, interpolation theories, each of which enables us to obtain a rather comprehensive information about the behavior of such operators on function spaces. The first one is completely general and is based on abstract interpolation and Calderón spaces. We illustrate the theoretical results by a wide variety of examples of pairs of spaces X, Y such that $T \colon X \to Y$ is bounded, these in particular include the so-called Calderón-Lozanovskiĭ construction. The second theory departs from pointwise estimates by Calderón operators and is particularly tailored for obtaining boundedness results between Orlicz spaces given weak-type estimates that arise in applications. A common feature of both theories is an approach, apparently new, involving interpolation of four spaces. The input data in each case consists of two reasonable separate endpoint estimates for the operators S_1 and S_2.

Keywords: interpolation of operators, Orlicz space, Calderón operator, Banach function space, Calderón-Lozanovskiĭ construction, Lorentz space

Contents

Introduction

Given a pair of operators S_1 and S_2 acting on some function spaces and taking values in different ones, and an exponent $\theta \in (1, \infty)$, one may consider the operator

$$Tf(s) = |S_1 f(s)|^{\frac{1}{\theta}} |S_2 f(s)|^{\frac{1}{\theta'}},$$

where $\frac{1}{\theta} + \frac{1}{\theta'} = 1$. Even if S_1 and S_2 are linear, the operator T does not have to be linear or even quasilinear. Consequently, the classical techniques of interpolation are of no effective use for the operator T. In this thesis, we study behavior of such operators T. We call them logarithmically convex combinations of (usually) (quasi-)linear operators.

Our main objective is to develop a new comprehensive interpolation theory that would enable one to effectively handle logarithmically convex combinations of operators. The point of departure will be, as is usual in general interpolation theory, some reasonable "endpoint" estimates for each of the operators S_1, S_2, combined with an interpolation functor. The goal will be to construct, out of this data, a new interpolation method suitable for the combination T. We will do this in a completely abstract setting of interpolation functors. We obtain thereby a completely new theory, which we in turn apply to particular examples and obtain new interpolation results.

Our original motivation stems from the approach that had been taken in [15] and [14] to generalized Sobolev embeddings of the form

$$W^m X(\Omega) \hookrightarrow Y(\Omega, \mu),$$

in which X, Y are rearrangement-invariant Banach function spaces, $m \in \mathbb{N}$ is the order of the embedding (the highest order of the derivatives present), Ω is an open set in $\mathbb{R}^n$, $n \in \mathbb{N}$, and μ is a d-Frostman measure (sometimes called also a d-Ahlfors measure), governed by the condition

$$\sup_{x \in \mathbb{R}^n,\, r>0} \frac{\mu(B_r(x) \cap \Omega)}{r^d} < \infty$$

for some $d \in (0, n]$, where $B_r(x)$ denotes the ball centered at x and having radius r. Such measures enter in a number of questions in measure theory, harmonic analysis, theory of function spaces, etc. Exactly for these measures the classical Sobolev trace embedding theorems by Adams and Maz'ya ([1, 2, 29, 30]) hold. As a particular case they also allow for the ordinary Lebesgue measure, in which case $d = n$.

Now it turns out that while, for $m < n$ and $d \geq n - m$, the Sobolev embedding can be effectively reduced to the boundedness of the operator

$$t \mapsto S_1 f(t) = \int_{t^{\frac{n}{d}}}^{1} f(s) s^{-1+\frac{m}{n}} \,\mathrm{d}s,$$

acting on functions defined on an interval, on representation spaces of X and Y, the situation is dramatically different in case when $d \in (0, n - m)$. In particular, the boundedness of the operator S_1 is no longer sufficient for the embedding (although it is still necessary). The heart of the problem lies in the fact that for small values of d, in the corresponding 'endpoint' estimate, an endpoint Lorentz space has to be replaced by a (bigger) Lebesgue space. The authors of [15] and [14] overcome this obstacle in two different ways, one of which (after certain rather technical differentiation) results in replacing the operator S_1 by the combination

$$f \mapsto Tf = (S_1 f)^{\frac{m}{n-d}} (S_2 f)^{1-\frac{m}{n-d}},$$

in which

$$S_2 f(t) = t^{-\frac{m}{n-d}} \int_0^{t^{\frac{n}{d}}} f(s) s^{-1+\frac{m}{n-d}} \mathrm{d}s.$$

This way they obtain a reasonable, in some sense still best possible, sufficient condition that substitutes the reduction principle available for large values of d. Note that this T is a particular example of the operators we offer to study here.

We shall now describe our approach in detail. We start by finding a kind of a canonical target space for the operator T, given some reasonable endpoint estimates on S_1 and S_2 and some fixed domain space. The construction turns out to rely on interpolating total of 4 spaces, or rather two pairs of spaces. This is done in a given order and results in one single space, which turns out to be a target space for T. This raises an interesting question of whether the order of interpolation may be somehow reversed, that is, whether the operation in question is in some sense commutative. Therefore, we introduce the notion of *commutativity* of interpolation functors and we show rather non-trivial instances of functors which are, in this sense, commutative. We work mostly with the real method and the so-called Calderón-Lozanovskiĭ construction. The latter of these turns out to be particularly nice to work with in this setting and we provide natural, easy to check conditions under which this method possesses a nice commutativity property.

We then apply our theory mainly to endpoint spaces of Lorentz and Orlicz type. Using the notion of commutativity we will show that many known interpolation functors for (quasi-)linear operators may be applied directly to the logarithmically convex combinations of operators. Here, it is worth noticing that there is a natural linear operator which majorizes T. Indeed, it follows immediately from Young's inequality that

$$Tf(s) = |S_1 f(s)|^{\frac{1}{\theta}} |S_2 f(s)|^{\frac{1}{\theta'}} \leq \tfrac{1}{\theta} |S_1 f(s)| + \tfrac{1}{\theta'} |S_2 f(s)|.$$

However, as simple examples show, while Young's inequality itself is in some sense sharp, its application to logarithmically convex combination of operators can result in a serious loss of regularity and therefore one should be discouraged from using it.

Finally, we consider an interpolation method for which (even the existence of) the corresponding functor is not known and obtain results for the combination T based on this method and a somewhat more concrete approach based on Calderón-type estimates. More precisely, we show that we may replace in T the operators S_1 and S_2 with particular Calderón operators. This results in a significant simplification of our efforts. In particular, we obtain thereby an Orlicz-Orlicz type boundedness for T just given weak-type estimates on S_1 and S_2. This is particularly interesting because we can get the domain Orlicz space very close to one of the weak-type endpoints.

Consider the special case when S_1 and S_2 have exactly the same endpoints. It is not complicated then to show that T in such a case may be interpolated in exactly the same way S_1 and S_2 can be. A particularly interesting case arises when S_1 and S_2 have nearly the same endpoints, but one of them is slightly worse. For example $S_1 \colon X \to L^{\theta,1}$, while $S_2 \colon X \to L^{\theta,\infty}$. It should not come as a shock that in such a case one has $T \colon X \to L^{\theta}$. If X is some Lorentz endpoint space this may be seen as either a strengthened weak-type estimate or a weakened strong-type estimate. If some other endpoint estimate is fixed and equal for S_1 and S_2, a well-known principle implies that one may obtain a reasonable interpolation target space for S_1 even when the desired domain space is very close to X. This is not the case, however, for S_2. It turns out, though, that one may get much closer to X with T than one can do with S_2.

The notion of logarithmically convex combinations is something that appears often when dealing with interpolation. This usually concerns estimates of the norms of operators acting on the interpolated spaces. This is seen for example in the classical Riesz-Thorin interpolation theorem, and up to some extent also in the Marcinkiewicz theorem (see e.g. [4]). There one works with logarithmically convex combinations of *constants*, but this is indeed intimately related to our notion of combinations of *operators*. This relation is implicitly seen in the third chapter.

Combinations of operators also appear in literature before. Aside from the motivation example above, let us recall various instances of pointwise Gagliardo-Nirenberg inequalities incorporating the Hardy maximal operator, see e.g. [31], [18] (see also [5]). The simplest example of these is the one dimensional case

$$|u'(x)| \le C|\mathbf{M}u(x)|^{\frac{1}{2}}|\mathbf{M}u''(x)|^{\frac{1}{2}},$$

where $\mathbf{M}$ stands for the Hardy–Littlewood maximal operator and C is some constant independent of u and x.

The thesis is structured as follows. The first two chapters provide preliminary results, the first one mainly on function spaces, the second one on interpolation theory and related notions. The assertions therein have been mostly known before in one form or another, but we present them in precisely the form we need and complement the known results with observations of our own. The third chapter introduces the abstract approach dealing with the general notion of an interpolation functor. It is also there that we study the notion of commutativity. The fourth chapter deals with a particular interpolation technique and applies it in some sense to the logarithmically convex combinations of operators. It is there that we obtain general Orlicz-Orlicz type boundedness results for combinations of weak-type operators.

1. Preliminaries - Function Spaces

In this chapter we recall some definitions and basic properties of Banach functions spaces and rearrangement-invariant spaces. The standard reference is [3].

Let (R, μ) be a σ-finite measure space. We say that $E \subset R$ is an *atom* if $\mu(E) > 0$ and for every measurable $F \subset E$ either $\mu(F) = 0$ or $\mu(F) = \mu(E)$. The space (R, μ) is called non-atomic, if it contains no atoms. Some results we provide are only for non-atomic measure spaces, however, they carry over almost verbatim to a more general class of so called *resonant* spaces. We recall that (R, μ) is resonant if and only if it is either non-atomic or it is a union of countably many atoms of equal finite measure.

Denote the set of extended real measurable functions on R by $\mathcal{M}(R, \mu)$. We write only $\mathcal{M}(\mu)$ or $\mathcal{M}$ if no confusion can arise. By λ we denote the one dimensional Lebesgue measure and we write $\lambda(E) = |E|$ for $E \subset \mathbb{R}$ λ-measurable.

We define

$$\mathcal{M}_+ = \{f \in \mathcal{M} : \, f \geq 0 \; a.e.\},$$

and

$$\mathcal{M}_0 = \{f \in \mathcal{M} : \, |f| < \infty \; a.e.\}$$

Notice that $\mathcal{M}_0$ is a vector space. We equip it with the topology of convergence in measure. The space then becomes a Hausdorff topological vector space.

The *distribution function* $f_* : (0, \infty) \to [0, \infty]$ of a function $f \in \mathcal{M}$ is defined as

$$f_*(\lambda) = \mu(\{x \in R : \, |f(x)| > \lambda\}), \; \lambda \in (0, \infty),$$

and the *non-increasing rearrangement* $f^* : (0, \infty) \to [0, \infty]$ of a function $f \in \mathcal{M}$ is defined as

$$f^*(t) = \inf\{\lambda \in (0, \infty) : \, f_*(\lambda) \leq t\}, \; t \in (0, \infty).$$

The operation $f \mapsto f^*$ is monotone in the sense that $|f| \leq |g|$ a.e. in R implies $f^* \leq g^*$. We define the *elementary maximal function* $f^{**} : (0, \infty) \to [0, \infty]$ of a function $f \in \mathcal{M}$ as

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \mathrm{d}s.$$

While the operation $f \mapsto f^{**}$ is subadditive, that is, for any $f, g \in \mathcal{M}$ and $t \in (0, \infty)$ one has

$$(f + g)^{**}(t) \leq f^{**}(t) + g^{**}(t), \tag{1.1}$$

for $f \mapsto f^*$ one only has the following property. Let $s, t \in (0, \infty)$ and $f, g \in \mathcal{M}$, then

$$(f + g)^*(s + t) \leq f^*(t) + g^*(s). \tag{1.2}$$

The *Hardy-Littlewood inequality* asserts that if $f, g \in \mathcal{M}$, then

$$\int_R |fg| \mathrm{d}\mu \leq \int_0^\infty f^*(t) g^*(t) \mathrm{d}t. \tag{1.3}$$

Hardy's inequality asserts that if $p \in (1,\infty)$ and $\alpha > -(1-\frac{1}{p})$, then

$$\int_0^\infty \left(t^{-\alpha-1}\int_0^t f(s)\mathrm{d}s\right)^p \mathrm{d}t \leq \left(\frac{1}{1+\alpha-\frac{1}{p}}\right)^p \int_0^\infty \left(f(t)t^{-\alpha}\right)^p \mathrm{d}t \quad \text{for } f \in \mathcal{M}_+.$$

We shall now consider different classes of so called function norms and the resulting function spaces. It is a matter of convention, which differs across literature, what is called a Banach function space. We present all the possible axioms of a function norm and simply consider particular combinations of these axioms when necessary. We try to use conventions which result in the least confusion possible.

Let $\rho : \mathcal{M}_+ \to [0,\infty]$, $f, g, f_n \in \mathcal{M}_+$, $n \in \mathbb{N}$, $\lambda \geq 0$, $C > 0$ and consider the following axioms.

(T_C) $\rho(f+g) \leq C(\rho(f)+\rho(g))$ *(C-triangle inequality)*

(PH) $\rho(\lambda f) = \lambda\rho(f)$; *(positive homogeneity)*

(PD) $\rho(f) = 0 \iff f = 0$ a.e.; *(positive definiteness)*

(P2) $f \leq g$ a.e. implies $\rho(f) \leq \rho(g)$; *(the lattice property)*

(P3) $f_j \nearrow f$ a.e. implies $\rho(f_j) \nearrow \rho(f)$; *(the Fatou property)*

(P4) $\rho(\chi_G) < \infty$ for every $G \subset R$ of finite measure; *(containment of simple functions)*

($\mathrm{P4}_w$) there exists $u \in \mathcal{M}_+$, $u > 0$ a.e. such that $\rho(u) < \infty$; *(existence of a weak unit)*

(P5) for every $G \subset (0,\infty)$ of finite measure there is a constant C_G such that $\int_G f(t)\mathrm{d}t \leq C_G\rho(f)$; *(local embedding into L^1)*

(P6) $\rho(f) = \rho(g)$ whenever $f^* = g^*$ *(rearrangement invariance).*

The element u defined in ($\mathrm{P4}_w$) is called a *weak unit.* Since the underlying measure space is σ-finite, it follows that (P4) implies ($\mathrm{P4}_w$). If ρ satisfies the first three properties, we say that it satisfies ($\mathrm{P1}_C$), or equivalently, that it is a quasi-norm. We denote ($\mathrm{P1}_1$) as (P1) and call ρ a norm if it satisfies (P1). For convenience reasons, we call ρ which satisfies (T_C) and (PH) a C-pseudonorm.

If ρ is a C-pseudonorm, then we define the space

$$X = X(\rho) = \{f \in \mathcal{M} :\ \rho(|f|) < \infty\}.$$

We call this X a C-pseudonormed space and denote $\|\cdot\|_X = \rho(|\cdot|)$.

If ρ satisfies properties (P1)-(P5) we call it a Banach function norm. If it satisfies (P6) we call it *rearrangement-invariant* (r.i. for short). If ρ satisfies properties ($\mathrm{P1}_C$), (P2), (P4) and (P5) we call it a *Fatou-convexifiable* quasi-norm. We notice that if ρ is a C-pseudonorm and $X(\rho) \subset \mathcal{M}_0$, then $X(\rho)$ is linear.

We will often talk about function spaces without explicitly defining ρ. That is, when we say that X is a function space satisfying some specific properties (a subset of those listed above), we mean there is a σ-finite measure space (R,μ) and a mapping $\rho\colon \mathcal{M}_+(R,\mu) \to [0,\infty]$ satisfying stated properties such that $X = X(\rho)$. In that case we write $\|\cdot\|_X = \rho(\cdot)$. Similarly, we say that X is a *Banach function space* (BFS for short) if there is a Banach function norm ρ such that $X = X(\rho)$. Analogously, X is called a *Fatou-convexifiable lattice* if there is ρ a Fatou-convexifiable quasi-norm, such that $X = X(\rho)$. Sometimes it will be convenient to stress what specific underlying measure space (R,μ) a given space of functions X is defined over, we write $X(R,\mu)$ or $X(\mu)$ in such a case.

The term Banach function space is defined in the same way as in [3]. However, in some literature, this term is sometimes used for spaces satisfying (P1) and (P2) which are also complete. We recall that it follows from the axioms that a Banach function space is always complete.

Notice also that we have a one-to-one correspondence between C-pseudonorms and respective spaces, that is the equivalence

$$f \in X = X(\rho) \quad \Leftrightarrow \quad \rho(|f|) < \infty.$$

If X and Y are Banach function spaces over the same measure space, one has $X \subset Y$ if and only if $X \hookrightarrow Y$. We shall denote by B_X or by B_ρ the "closed" unit ball in a C-pseudonormed space $X(\rho)$, that is $B_X = B_\rho = \{f \in X, \|f\|_X \leq 1\}$.

Given a C-pseudonorm ρ over some σ-finite measure space (R,μ), we define its *associate norm* by

$$\rho'(g) = \sup\left\{\int_R fg\,\mathrm{d}\mu : \, f \in \mathcal{M}_+(\mu),\, \rho(f) \leq 1\right\} \quad \text{for } g \in \mathcal{M}_+(\mu).$$

As per usual, we use the convention that $0 \cdot \infty = 0$. If ρ is a Banach function norm, then ρ' is also a Banach function norm. If ρ is an r.i. Banach function norm, so is ρ'. Furthermore, for a Banach function norm ρ it also holds that $\rho'' = \rho$. If $X = X(\rho)$ is a C-pseudonormed space and ρ' is the norm associate to ρ, then $X(\rho')$ is the *associate space* of X and is denoted by X'.

If X, Y are C-pseudonormed spaces, we denote by $T\colon X \to Y$ the boundedness of an operator T from X to Y. An operator in our case stands for an arbitrary mapping and by boundedness we mean that T maps bounded sets in X into bounded sets in Y. If $\mathrm{Id}\colon X \to Y$ (i.e. bounded sets in X are also bounded in Y), we write $X \hookrightarrow Y$. Note that if these spaces are normed and T is linear, then $T\colon X \to Y$ is equivalent to T being continuous from X to Y.

If (R,μ) is a σ-finite measure space and X, Y are C-pseudonormed spaces over this measure space, we say that X is *locally embedded* into Y, denoted $X \overset{\text{loc}}{\hookrightarrow} Y$ if for any $E \subset R$ with $0 < \mu(E) < \infty$ there is a constant $C > 0$ such that for all $f \in \mathcal{M}$ we have $\|f\chi_E\|_X \leq C\|f\chi_E\|_Y$.

For two locally integrable functions f and g we write $f \prec g$ if $f^{**}(t) \leq g^{**}(t)$ for all $t \in (0,\infty)$. This relation is called the *Hardy-Littlewood-Pólya* (*HLP* for short) *relation*. The following so called *majorant property* holds for all locally integrable functions f, g:

$$f \prec g \implies \|f\|_X \leq \|g\|_X \quad \text{for all r.i. BFS } X.$$

Let (R,μ) be a σ-finite resonant measure space and X some rearrangement-invariant Banach function space over it. Then the *Luxemburg representation theorem* states that there is a rearrangement-invariant Banach function space $\tilde{X}$ over $(0,\infty)$ such that

$$\|f\|_X = \|f^*\|_{\tilde{X}}$$

for every $f \in \mathcal{M}(R,\mu)$. This space is not necessarily unique. Any such space is called a *Luxemburg representation space* and in literature is usually denoted by $\overline{X}$. We need this notation for another notion (see Chapter 2) which is usually denoted the same way, hence the change.

We define the *fundamental function*, φ_X, of a given an r.i. Banach function space X by $\varphi_X(t) = \|\chi_E\|_X$, $E \subset R$, $\mu(E) = t$, for t in range of μ. Recall that the definition is valid as it does not depend on the choice of particular E. We say that a function $\varphi\colon (0,\infty) \to (0,\infty)$ is *quasiconcave* if it is non-decreasing and $\frac{t}{\varphi(t)}$ is non-decreasing. We say that the function $\varphi\colon (0,\infty) \to (0,\infty)$ satisfies the Δ_2 condition if it is non-decreasing and there exists a constant $C > 0$ such that $\varphi(2t) \leq C\varphi(t)$ for all $t > 0$.

Assume now that (R,μ) is a σ-finite non-atomic measure space and recall that by the Sierpinski theorem the range of μ is the interval $[0,\mu(R)]$. If $\mu(R) = \infty$, the fundamental function φ_X of any rearrangement-invariant space $X(\mu,R)$ is quasiconcave. If $\mu(R)$ is finite and we define $\varphi_X = \varphi_X(\mu(R))$ on $(\mu(R),\infty)$, then $\varphi_X\colon (0,\infty) \to (0,\infty)$ is quasiconcave.

We now introduce Lorentz and Marcinkiewicz endpoint spaces, Lorentz $L^{p,q}$ spaces and Orlicz spaces. Given a quasiconcave function φ, we define the rearrangement-invariant spaces $M_\varphi, \Lambda_\varphi$ with the rearrangement-invariant norms given by

$$\|f\|_{M_\varphi} = \sup_{t \in (0,\mu(R))} \varphi(t) f^{**}(t),\ f \in \mathcal{M}(\mu), \tag{1.4}$$

and

$$\|f\|_{\Lambda_\varphi} = \int_0^{\mu(R)} f^*(t)\mathrm{d}\varphi(t),\ f \in \mathcal{M}(\mu). \tag{1.5}$$

These are indeed r.i. Banach function norms and the spaces are called the *Marcinkiewicz endpoint space* and the *Lorentz endpoint space* respectively. It is also known that both M_φ and Λ_φ have a common fundamental function which is equal to φ. If X is an r.i. Banach function space and φ_X is the fundamental function of X, we define

$$M(X) = M_{\varphi_X}, \quad \Lambda(X) = \Lambda_{\varphi_X}.$$

We recall that

$$\Lambda(X) \hookrightarrow X \hookrightarrow M(X).$$

That is, the spaces $M_\varphi, \Lambda_\varphi$ are respectively the largest and the smallest rearrangement-invariant space with the fixed fundamental function equal to φ.

One of the basic examples of an important class of rearrangement-invariant spaces can be obtained by considering general Lorentz $L^{p,q}$ spaces with $p, q \in (0, \infty]$, governed for $f \in \mathcal{M}(\mu)$ by the functional

$$\rho_{p,q}(f) = \begin{cases} \left(\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \, \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} & \text{if } 0 < q < \infty, \\ \sup_{t \in (0,\infty)} t^{\frac{1}{p}} f^*(t) & \text{if } q = \infty. \end{cases}$$

We recall that these are equivalent to rearrangement-invariant Banach function norms in cases when either $p \in (1, \infty)$ and $q \in [1, \infty]$ or $p = q = 1$ or $p = q = \infty$. This means that in such a case there is a rearrangement-invariant Banach function space X such that $X = L^{p,q}$ with equivalent norms.

In case $p = \infty$ and $q < \infty$ the resulting space is a trivial set containing only the zero function. Furthermore, note that $L^{p,p} = L^p$, where L^p is the classical Lebesgue space. For $p \in [1, \infty]$ we define the associated exponent p' by $\frac{1}{p} + \frac{1}{p'} = 1$. If the underlying measure space is non-atomic, the following equalities hold up to an equivalence of norms:

$$\begin{aligned} &L^{p,1} = \Lambda(L^p) \quad \text{for } p \in [1, \infty), \\ &L^{p,\infty} = M(L^p) \quad \text{for } p \in (1, \infty], \\ &(L^{p,q})' = L^{p',q'} \quad \text{for } p \in (1, \infty),\ q \in [1, \infty] \text{ or } p = q = \infty \text{ or } p = q = 1. \end{aligned}$$

We may now drop the assumption of non-atomicity and continue with a general σ-finite measure space (R, μ). We introduce the so-called Orlicz space. We say that $A\colon [0, \infty] \to [0, \infty]$ is a Young function if it is convex, non-decreasing, left-continuous, $A(0) = 0$ and A is not identically zero. We define the Orlicz space L^A as the collection of all measurable functions for which there is $\lambda > 0$ such that

$$\int_R A\left(\frac{|f|}{\lambda}\right) \mathrm{d}\mu < \infty. \tag{1.6}$$

When equipped with the Luxemburg norm

$$\|f\|_{L^A} = \inf\{\lambda > 0,\ \int_R A\left(\frac{|f|}{\lambda}\right) \mathrm{d}\mu \leq 1\},$$

L^A is a rearrangement-invariant Banach function space. The function

$$B(y) = \sup\{xy - A(x),\ x \in [0, \infty]\},\ y \in [0, \infty],$$

is called the complementary function to A and satisfies the so-called Young's inequality

$$xy \leq A(x) + B(y) \quad \text{for all } x, y \in [0, \infty]. \tag{1.7}$$

Furthermore, B is a Young function and $(L^A)' = L^B$, up to equivalence of norms. Recall that to each Young function A there corresponds a uniquely determined left continuous non-decreasing function a such that

$$A(t) = \int_0^t a(s) \mathrm{d}s \quad \text{for } t \in [0, \infty].$$

Furthermore A' exists a.e. on the set where $A < \infty$ and $A' = a$ in all points where A' exists. It immediately follows that one has

$$\frac{A(t)}{t} \leq a(t) \quad \text{for } t \in (0, \infty).$$

It will be useful to invert Young functions even if they are not invertible. We realize that a Young function is not invertible if and only if it is identically zero on some right neighbourhood of 0 or identically infinite on some left neighbourhood of infinity. We define

$$A^{-1}(s) = \sup\{t \in [0, \infty] : A(t) \leq s\} \quad \text{for } s \in [0, \infty].$$

Note that since A is a Young function, A^{-1} needs to be continuous except perhaps at 0, hence it is trivially left-continuous. Moreover, it holds that $A^{-1}(0) = \max_{A(t)=0}\{t\}$ and if A is invertible, then A^{-1} is the classical inverse.

We recall that if (R, μ) is resonant and $X(R, \mu)$ is any of the introduced spaces, that is a Lorentz endpoint space, Marcinkiewicz endpoint space, Lorentz space, or Orlicz space, then $X(0, \mu(R))$ is a Luxemburg representation space $\tilde{X}$ (recall that such space needs not be unique).

Consider a couple of C-pseudonormed spaces X and Y. We define two new spaces $X^{\mathrm{asc}(Y)}$ and $X \cdot Y$ with the following formulae

$$\begin{aligned} &X^{\mathrm{asc}(Y)} = \{f \in \mathcal{M},\ \|fg\|_Y < \infty \text{ for all } g \in X\}, \\ &X \cdot Y = \{fg,\ f \in X \text{ and } g \in Y\}. \end{aligned} \tag{1.8}$$

We call the space $X^{\mathrm{asc}(Y)}$ the Y-*dual*, or the Y-*associate space*, of X. We call the space $X \cdot Y$ the *pointwise product* of X and Y. Note that if X and Y are subsets of $\mathcal{M}_0$, then X and Y need to be linear. If that is the case, then $X \cdot Y$ is also a linear subspace of $\mathcal{M}_0$. We equip these spaces with functionals

$$\begin{aligned} \|f\|_{X^{\mathrm{asc}(Y)}} &= \sup_{g \in B_X} \|fg\|_Y \quad \text{for } f \in \mathcal{M}, \\ \|f\|_{X \cdot Y} &= \inf_{f = gh} \|g\|_X \|h\|_Y \quad \text{for } f \in \mathcal{M}, \end{aligned}$$

where the infimum is taken over all such representations of f. It is matter of a simple check to show that $X^{\mathrm{asc}(Y)}$ is now a C-pseudonormed space (the constant of the triangle inequality is equal to that of Y and does not depend at all on the constant in X). It is however entirely possible that $X^{\mathrm{asc}(Y)}$ is either one of the trivial spaces $\{0\}$ and $\mathcal{M}$. It is even possible that if X and Y are both rearrangement-invariant Banach function spaces (they satisfy all the axioms (P1)-(P6)), the space $X^{\mathrm{asc}(Y)}$ is the trivial space containing only the zero function. One can take for example $X = L^p$ and $Y = L^q$, $q > p$, and if (R, μ) is non-atomic, then $X^{\mathrm{asc}(Y)} = \{0\}$. This result is very easy to show and may be found in [27].

We remark that $X^{\mathrm{asc}(L^1)} = X'$ and that $X \hookrightarrow Z$ implies $Z^{\mathrm{asc}(Y)} \hookrightarrow X^{\mathrm{asc}(Y)}$.

Let us remark that if V is a vector space and $U \subset V$ is convex, one can define the so called *Minkowski functional*

$$\mu_U(x) = \inf\{\lambda > 0,\ \tfrac{x}{\lambda} \in U\} \quad \text{for } x \in V. \tag{1.9}$$

The possible infimum of an empty set is defined as ∞ and if μ_U is finite on V, we call U *absorbing*. Recall that if U is *balanced* (i.e. $x \in U$ implies $-x \in U$), absorbing and convex, then μ_U defines a pseudo-norm on V. If it is a norm, then it is sometimes also called the *gauge* norm of a given set. It is a common practice in theory of function spaces when defining new spaces, to simply define the unit ball and then the norm as the Minkowski functional of that ball. It is immediately clear that if X is a normed space then $\mu_{B_X}(\cdot) = \|\cdot\|_X$. The example of this is the functional $\|\cdot\|_{X \cdot Y}$, which is merely the Minkowski functional of the set $B_X \cdot B_Y = \{gh,\ g \in B_X, h \in B_Y\}$. Here we assume additionally that $X, Y \subset \mathcal{M}_0$. The Luxemburg norm in any Orlicz space is also defined as a Minkowski functional.

We present here in short some interesting results and open questions regarding the generalized associate and product spaces.

Of particular interest is the situation when the Y-dual and the pointwise product commute in the sense that $X^{\mathrm{asc}(Y)} \cdot X = Y$. There is no known characterization of this identity and in the general setting of Banach function spaces even sufficient conditions are not known. In case of the so called Calderón-Lozanovskiĭ spaces, sufficient conditions are provided in [20, Theorem 9]. The definition of these spaces will follow later.

For the particular case of $Y = L^1$ it is the well known factorization theorem of Lozanovskiĭ which asserts that if X is a Banach function space, then one has $L^1 = X \cdot X'$ with equality of norms, in particular $\|\cdot\|_{X \cdot X'}$ is a norm. See [23, Theorem 6]. Some of the conditions on X can even be relaxed.

There are some results concerning the operation $X \mapsto X^{\mathrm{asc}(Y)}$ under so called p-convexity and p-concavity assumptions. Let us define what this means.

Let $p \in (0, \infty)$. A functional $\rho\colon \mathcal{M}_+ \to [0, \infty]$ is said to be *p-convex* if there exists $K > 0$ such that for any finite sequence $(f_i)_{i=1}^n \in \mathcal{M}_+$ it holds that

$$\rho\left(\left(\sum_{i=1}^n f_i^p\right)^{\frac{1}{p}}\right) \leq K \left(\sum_{i=1}^n \rho^p(f_i)\right)^{\frac{1}{p}}. \tag{1.10}$$

The least such K is called the *p-convexity constant* of ρ. Similarly, ρ is said to be *p-concave* if there exists $K > 0$ such that for any finite sequence $(f_i)_{i=1}^n \in \mathcal{M}_+$ it holds that

$$\left(\sum_{i=1}^n \rho^p(f_i)\right)^{\frac{1}{p}} \leq K \rho\left(\left(\sum_{i=1}^n f_i^p\right)^{\frac{1}{p}}\right). \tag{1.11}$$

The least such K is called the *p-concavity constant* of ρ. We say that a C-pseudonormed space is p-convex or p-concave if the underlying C-pseudonorm is.

It is possible to show directly that (on a sufficiently rich measurable space) the only Banach function space which is both p-concave and p-convex is L^p.

Given a function space X satisfying at least (P1$_C$) and (P2), we define the space X^p as the set of all $f \in \mathcal{M}$ such that

$$\|f\|_{X^p} = \|f^p\|_X^{\frac{1}{p}} < \infty.$$

Notice that $(L^1)^p = L^p$ and recall that if X is q-convex, resp. q-concave, then X^p is qp-convex, resp. pq-concave. The space X^p is sometimes called the *p-convexification* of the space X. The reason for this is that if X is normed, then X^p is p-convex with constant 1.

Of interest is the so called *Y-perfectness* of X, which is defined to mean that

$$(X^{\operatorname{asc}(Y)})^{\operatorname{asc}(Y)} = X.$$

It is known that if X and Y are Banach function spaces, then $X^{\operatorname{asc}(Y)} \cdot X = Y$ implies $(X^{\operatorname{asc}(Y)})^{\operatorname{asc}(Y)} = X$, see [20]. However, in the general setting of Banach function spaces no reasonable characterization of when X is Y-perfect is known.

In [9] (and independently in [34]) one can find a proof of a slight modification of the following

Proposition 1.1. *Let X, Y be spaces of functions satisfying (P1), (P2), (P4$_w$) which are complete, and let $1 \leq p < \infty$. Then*

- *(i) X is L^p-perfect if and only if X is p-convex and satisfies the Fatou property (P3).*
- *(ii) If Y is a p-concave Banach function space, X is a p-convex Banach function space and $X \overset{loc}{\hookrightarrow} Y$, then X is Y-perfect.*

Proof. Statement *(i)* is proven in exactly the same form in [9, Proposition 5.3]. We establish *(ii)*. From that fact, that $X \overset{\text{loc}}{\hookrightarrow} Y$, it follows that whenever $E \subset R$ has finite measure, one has $\|\chi_E\|_{X^{\operatorname{asc}(Y)}} = \sup_{f \in B_X} \|\chi_E f\|_Y \leq C_E$, where C_E is the constant of the local embedding $X \overset{\text{loc}}{\hookrightarrow} Y$ with respect to E. It follows that $X^{\operatorname{asc}(Y)}$ satisfies (P4), in particular it is saturated (satisfies (P4$_w$)) and so [9, Theorem 5.7] gives the result. $\square$

Note that the so-called *generalized Hölder inequality*

$$\|fg\|_Y \leq \|f\|_X \|g\|_{X^{\operatorname{asc}(Y)}} \quad \text{for } f \in \mathcal{M}, \tag{1.12}$$

holds whenever X and Y are C-pseudonormed spaces. If, in addition, Y is normed, it holds that $X^{\operatorname{asc}(Y)}$ is also normed. Indeed this follows immediately from the definition.

The following proposition asserts that under some assumptions some nice properties of Y are also enjoyed by $X^{\operatorname{asc}(Y)}$ and in extension also by $(X^{\operatorname{asc}(Y)})^{\operatorname{asc}(Y)}$. This can be used together with the embedding $X \hookrightarrow (X^{\operatorname{asc}(Y)})^{\operatorname{asc}(Y)}$, allowing us to obtain a space which contains X and has useful properties that X itself may lack. There are some results in literature (e.g. [17], [33]), we failed to find them in this precise form and so we prove the following result for the reader's convenience.

Proposition 1.2. *Let X be a Fatou-convexifiable lattice and let Y be a Banach function space such that $X \overset{loc}{\hookrightarrow} Y$. Then $X^{asc(Y)}$ is a Banach function space. In particular, X'' is the smallest Banach function space containing X.*

Proof. Denote $\rho = \|\cdot\|_{X^{\mathrm{asc}(Y)}}$. Then it is clear that ρ satisfies (T$_1$), (PH) and (P2). We prove the remaining properties.

Assume that $f \in \mathcal{M}$ satisfies $\rho(f) = 0$, then we have for every $g \in X$ that $0 = \|fg\|_Y$, this implies that $fg = 0$ a.e. Since X satisfies (P4$_w$) it has a weak unit u and so $fu = 0$ a.e. This implies that $f = 0$ a.e., therefore (PD) holds.

Choose a measurable set $E \subset R$, $\mu(E) < \infty$. Then since X has (P4), $\chi_E \in X$ and so for any $f \in \mathcal{M}$ we have $\rho(f) \geq \frac{1}{\|\chi_E\|_X}\|f\chi_E\|_Y \geq \frac{C_E}{\|\chi_E\|_X}\|f\|_{L^1}$. Here C_E stands for the constant of the local embedding of Y into L^1 with respect to E. Hence, ρ satisfies (P5).

Now with E as before, $\rho(\chi_E) = \sup_{g \in B_X}\|\chi_E g\|_Y \leq C_E$, where C_E is the constant of the local embedding $X \overset{\mathrm{loc}}{\hookrightarrow} Y$ with respect to E. Hence (P4) holds.

Let now $0 \leq f_n \nearrow f$ be measurable functions and assume $\rho(f) > 0$. We have by the (P2) property of $X^{\mathrm{asc}(Y)}$ that $\rho(f_n)$ increases and $\rho(f_n) \leq \rho(f)$ for all $n \in \mathbb{N}$, hence if $\rho(f_n) = \infty$ for some $n \in \mathbb{N}$, there is nothing to prove. Assume therefore that $\rho(f_n) < \infty$ for all $n \in \mathbb{N}$. Let $0 < \varepsilon < \rho(f) < \infty$. We find a function $0 \leq g \in B_X$ such that $\rho(f) - \varepsilon < \|fg\|_Y$. We have $f_n g \nearrow fg$ and so (P3) in Y dictates that $\|f_n g\|_Y \nearrow \|fg\|_Y$. In other words, there is $N \in \mathbb{N}$ such that $n \geq N$ implies $\|f_n g\|_Y \geq \rho(f) - \varepsilon$, and therefore also $\rho(f_n) \geq \rho(f) - \varepsilon$. If $\rho(f) = \infty$, the proof is analogous. Hence (P3) holds in $X^{\mathrm{asc}(Y)}$.

Since X satisfies (P5), that is $X \overset{\mathrm{loc}}{\hookrightarrow} L^1$, we know by the above argument that X' and hence also X'' are Banach function spaces. It is obvious that $X \hookrightarrow X''$. If Z is a Banach function space satisfying $X \hookrightarrow Z$, then $Z' \hookrightarrow X'$ and so $X'' \hookrightarrow Z'' = Z$. The statement follows. $\square$

We remark here that we have actually shown that $X^{\mathrm{asc}(Y)} \overset{\mathrm{loc}}{\hookrightarrow} Y$, which is stronger than $X^{\mathrm{asc}(Y)} \overset{\mathrm{loc}}{\hookrightarrow} L^1$ as $Y \overset{\mathrm{loc}}{\hookrightarrow} L^1$.

We can now make the following observation.

Proposition 1.3. *Let X be a Fatou-convexifiable lattice which is locally embedded into L^p, $p \in [1, \infty)$. Then $(X^{asc(L^p)})^{asc(L^p)}$ is the smallest among all Banach function spaces which are p-convex and contain X.*

Proof. It follows from the generalized Hölder inequality (1.12) that $X \hookrightarrow (X^{\mathrm{asc}(L^p)})^{\mathrm{asc}(L^p)}$, while the p-convexity of $(X^{\mathrm{asc}(L^p)})^{\mathrm{asc}(L^p)}$ is easily verified. Assume Y is a p-convex Banach function space satisfying $X \hookrightarrow Y$. Then it follows that $Y^{\mathrm{asc}(L^p)} \hookrightarrow X^{\mathrm{asc}(L^p)}$, therefore, due to L^p-perfectness of Y, it must be the case that $(X^{\mathrm{asc}(L^p)})^{\mathrm{asc}(L^p)} \hookrightarrow Y$. It remains to show that $(X^{\mathrm{asc}(L^p)})^{\mathrm{asc}(L^p)}$ is a Banach function space, but that follows from Proposition 1.2. $\square$

As for the pointwise product we are interested mainly in the result of [20, Corollary 1] which asserts that if X and Y are complete spaces of functions satisfying (P1) and (P2), then $X \cdot Y$ is a complete function space satisfying (P1$_2$) and (P2). Furthermore, if X

and Y have the Fatou property (P3), then $X \cdot Y$ also has the Fatou property (P3). We complement this result with an observation of our own. Recall that all Fatou-convexifiable lattices are continuously embedded into $\mathcal{M}_0$ since L^1 convergence implies convergence in measure.

Proposition 1.4. *Let X, Y be Fatou-convexifiable lattices with $Y \overset{loc}{\hookrightarrow} X'$. Then $Z = (X \cdot Y)''$ is a Banach function space satisfying*

$$P \colon X \times Y \to Z, \tag{1.13}$$

where P is the product operator defined as $P(f,g) = fg$, $f, g \in \mathcal{M}_0$. Furthermore, the space Z is the smallest Banach function space for which (1.13) *holds.*

Proof. We know that $X \cdot X' = L^1$ and so it follows immediately that $X \cdot Y$ is a Fatou-convexifiable lattice satisfying (P5) and (P4). Which implies that $X \cdot Y \hookrightarrow Z$ and Z is a Banach function space (due to Proposition 1.2). Boundedness of the product operator is now obvious. Optimality follows from the fact that if W is a Banach function space satisfying $P \colon X \times Y \to W$, then $X \cdot Y \hookrightarrow W$ and therefore $W' \hookrightarrow (X \cdot Y)'$, which implies $Z \hookrightarrow W'' = W$. □

The properties of normability of $X \cdot Y$ and of Y-perfectness of X are somehow related to separating the said spaces sufficiently. This can be seen for instance in the preceding proposition, where the condition $Y \overset{\mathrm{loc}}{\hookrightarrow} X'$ may be seen as a condition dictating a certain kind of separation.

In [35], the properties of the product operator, and in extension of the product space and the generalized associate space, are investigated in relation to the so called almost compact embedding.

Let $U \subset X$, where X is a function space satisfying at least ($\mathrm{P1}_C$) and (P2). We say that U is *almost compact* in X if U is bounded and

$$\lim_{n\to\infty} \sup_{f\in U} \|f\chi_{E_n}\|_X = 0,$$

whenever E_n are measurable and $\chi_{E_n} \to 0$ a.e. as $n \to \infty$. If Y is another such function space, we say that the embedding $X \hookrightarrow Y$ is *almost compact*, denoted $X \hookrightarrow_* Y$, if B_X is almost compact in Y.

One of the results in [35, Theorem 6.7] tells us that if $\mu(R) < \infty$ and (R, μ) is non-atomic, then for X, Y Banach function spaces $Y \hookrightarrow_* X'$ is in fact necessary and sufficient for $X \cdot Y \hookrightarrow_* L^1$. The target space for the product operator constructed therein is not necessarily the same as our $X \cdot Y$, but we can use the fact that our space is optimal and then the statement easily follows. We conclude this chapter with an interesting example.

Example 1.5. Given exponents $p_i, q_i \in (0, \infty]$, $i = 0, 1$, it holds that $L^{p_0,q_0} \cdot L^{p_1,q_1} = L^{p,q}$ with equivalent quasi-norms, where $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$, $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1}$. For reference see [8]. This holds whenever the spaces are taken over arbitrary σ-finite non-atomic measure space (R, μ). Assume for now that $(R, \mu) = ((0,1), \lambda)$.

It holds that

$$L^2 \cdot L^{2,1} = L^{1,\frac{2}{3}}. \tag{1.14}$$

It is a well known fact, that the space $L^{1,\frac{2}{3}}$ is not equivalently normable, but the proof is most certainly not immediate and requires some work. We will show that the fact that is is not equivalently normable with an r.i. Banach function norm follows directly from the theory we presented and some other general results.

To that end we observe that for a Fatou-convexifiable lattice Z one has $Z \hookrightarrow_* L^1$ if and only if $Z \hookrightarrow_* L^1$ under any equivalent quasi-norm on Z. Moreover, we recall [35, Theorem 5.3], which states that if Z is an r.i. BFS then either $Z = L^1$ or $Z \hookrightarrow_* L^1$. Take for Z the space $L^{1,\frac{2}{3}}$. We know that now Z is a Fatou-convexifiable lattice and since both $L^{2,1}$ and L^2 have the Fatou property $(P3)$, so does $Z = L^2 \cdot L^{2,1}$. It is also easily seen, that Z satisfies (P6) and that $Z \neq L^1$. Hence, Z satisfies all axioms of an r.i. BFS except perhaps for the triangle inequality (T_1). Assume, for contradiction, that there is some equivalent norm on Z under which Z is an r.i. BFS. As $Z \neq L^1$, this implies $Z \hookrightarrow_* L^1$. However, it is very easily seen, that this is false. Consider the sequence

$$f_n = n\chi_{(0,\frac{1}{n})},$$

which is bounded in $L^{1,\frac{2}{3}}$ but is not almost compact in L^1. Indeed, one has for the sequence $E_m = (0, \frac{1}{m})$

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} \|\chi_{E_m} f_n\|_{L^1} \geq \lim_{m\to\infty} \|\chi_{E_m} f_m\|_{L^1} = \lim_{m\to\infty} \|f_m\|_{L^1} = 1.$$

We have shown that $L^{1,\frac{2}{3}}$ does not satisfy the triangle inequality (T_1) without explicitly constructing any counterexamples. Moreover, we have shown that we may not find any equivalent norm on $L^{1,\frac{2}{3}}$ which is also a rearrangement-invariant Banach function norm.

We have, in particular, shown that for a pair of r.i. Banach function spaces X, Y, the condition $Y \hookrightarrow X'$ is not sufficient for the space $X \cdot Y$ to be normable with an r.i. Banach function norm. Indeed, this is seen from the above argument upon taking $X = L^2$ and $Y = L^{2,1}$. The one property that the product space may lack is normability, i.e. the property (T_1).

An analogous argument using, for example, the spaces L^3, $L^{2,1}$ results in $L^3 \cdot L^{2,1} = L^{\frac{6}{5},\frac{3}{4}}$, which is not equivalent to a Banach function space. Notice that $L^{2,1} \hookrightarrow_* (L^3)'$ (e.g. [35]). Therefore, we have illustrated that not even $Y \hookrightarrow_* X'$ is sufficient for $X \cdot Y$ to be equivalent to a Banach function space. It is however sufficient for $P\colon X \times Y \to Z$ for some BFS $Z \hookrightarrow_* L^1$. We have shown in Proposition 1.4 that the choice $Z = (X \cdot Y)''$ is optimal and indeed $(L^{\frac{6}{5},\frac{3}{5}})'' = L^{\frac{6}{5},1}$ (this is not obvious, but it follows e.g. from [27]). More generally $((L^{p,q})^{\mathrm{asc}(L^r)})^{\mathrm{asc}(L^r)} = L^{p,r}$ whenever $p \geq r$, $q \leq r$.

As we stated at the beginning of this example, the space $L^{1,\frac{2}{3}}$ and more generally any space $L^{p,q}$ with $q < 1$, $p < \infty$, is not equivalently normable by any norm. This can be shown directly, but the proof is somewhat more involved. Hence, it is not particularly useful to use the indirect method, example of which was given above, to obtain non-normability of some instances of spaces. However, it serves well as an illustrative example.

2. Preliminaries - Interpolation Theory

Here we recall some basic notions of interpolation theory. For reference see [3] and [4]. Let X_0 and X_1 be quasi-normed spaces (in full generality, we do not require that they are spaces of functions), which are *compatible* in the sense that they are embedded in some common Hausdorff topological vector space V. By $X_0 + X_1$ we denote the set of all vectors $f \in V$ for which there exists a decomposition $f = g + h$ such that $g \in X_0$ and $h \in X_1$. We equip the space $X_0 + X_1$ with the quasi-norm

$$\|f\|_{X_0+X_1} = \inf_{f=g+h}(\|g\|_{X_0} + \|h\|_{X_1}),$$

where the infimum is taken over all such decompositions. By $X_0 \cap X_1$ we denote the classical intersection and we define on it the quasinorm

$$\|f\|_{X_0 \cap X_1} = \max\{\|f\|_{X_0}, \|f\|_{X_1}\}.$$

Recall that if X_0 and X_1 are normed, then $X_0 + X_1$ and $X_0 \cap X_1$ are also normed and if X_0 and X_1 are Banach spaces, then so are $X_0 + X_1$ and $X_0 \cap X_1$. For $f \in X_0 + X_1$ the *Peetre K-functional* is defined by

$$K(t, f; X_0, X_1) := \inf_{f=g+h} (\|g\|_{X_0} + t\|h\|_{X_1}) \quad \text{for } t > 0.$$

In order to define what an "interpolation method" is, we adopt the language of category theory. A *category* $\mathfrak{C}$ is a class of *objects* $A, B, C, ...$ together with a class of *morphisms* $R, S, T, ...$ between objects, where a three-place relation $T\colon A \twoheadrightarrow B$ is defined. We require that if $T\colon A \twoheadrightarrow B$, $S\colon B \twoheadrightarrow C$, then there is a morphism ST, the *product* of S and T, such that $ST\colon A \twoheadrightarrow C$. This product operation must satisfy the associate law $T(SR) = (TS)R$. Moreover for any object A in $\mathfrak{C}$ there must exist a morphism $I = I_A$ such that for all morphisms $T\colon A \twoheadrightarrow A$ we have $TI = IT = T$.

If $\mathfrak{C}_1$, $\mathfrak{C}_2$ are two categories, then a functor from $\mathfrak{C}_1$ to $\mathfrak{C}_2$ is a rule (a class to class mapping, or a regular mapping if the respective classes of objects and morphisms are sets) which to every object A in $\mathfrak{C}_1$ assigns an object $F(A)$ in $\mathfrak{C}_2$ and to every morphism T in $\mathfrak{C}_1$ assigns a morphism $F(T)$ in $\mathfrak{C}_2$ such that the following conditions are satisfied.

1. If $T\colon A \twoheadrightarrow B$, then $F(T)\colon F(A) \twoheadrightarrow F(B)$,
2. $F(ST) = F(S)F(T)$,
3. $F(I_A) = I_{F(A)}$.

Our concept of a functor is in literature usually called a "covariant functor".

As an example, one may take for $\mathfrak{C}_2$ the category of all topological vector spaces, where the class of morphisms is formed by all continuous linear operators and for $\mathfrak{C}_1$ the category of all Banach spaces, where the class of morphisms is again formed by all continuous linear operators. Then the identical functor which to each X in $\mathfrak{C}_1$ assigns $F(X) = X$ and to each morphism T in $\mathfrak{C}_1$ assigns $F(T) = T$ is a functor from $\mathfrak{C}_1$ to $\mathfrak{C}_2$.

Consider some category $\mathfrak{C}$ of quasi-normed spaces such that for any X_0, X_1 in $\mathfrak{C}$ which are compatible, it holds that $X_0 \cap X_1$, $X_0 + X_1$ are in $\mathfrak{C}$. The class of morphisms is an arbitrary class of mappings $T\colon A \twoheadrightarrow B$ between objects A, B in $\mathfrak{C}$ closed under composition and containing identical mappings. We shall call such categories *interpolation categories.*

Denote by $\overline{\mathfrak{C}}$ the category of compatible couples of spaces (X_0, X_1) in $\mathfrak{C}$. We denote such couples $\overline{X}$. The morphisms $T\colon (A_0, A_1) \twoheadrightarrow (B_0, B_1)$ in $\overline{\mathfrak{C}}$ are morphisms T from $\mathfrak{C}$ which satisfy $T\colon X_0 + X_1 \twoheadrightarrow Y_0 + Y_1$, $T\colon A_0 \twoheadrightarrow B_0$ and $T\colon A_1 \twoheadrightarrow B_1$. In that case we write $T\colon \overline{A} \twoheadrightarrow \overline{B}$.

A space X is called an *intermediate space* for the pair $(X_0, X_1) = \overline{X} \in \overline{\mathfrak{C}}$ if $X_0 \cap X_1 \subset X \subset X_0 + X_1$. An intermediate space X is called an *interpolation space* with respect to (X_0, X_1) if, in addition, for an arbitrary morphism $T\colon \overline{X} \twoheadrightarrow \overline{X}$ in $\overline{\mathfrak{C}}$ it holds that

$$T\colon X \twoheadrightarrow X.$$

More generally, if (A_0, A_1), (B_0, B_1) are couples of spaces in $\overline{\mathfrak{C}}$, we say that quasi-normed spaces A and B are *interpolation spaces with respect to* (A_0, A_1) and (B_0, B_1) if for an arbitrary morphism $T\colon \overline{A} \twoheadrightarrow \overline{B}$, it holds that

$$T\colon A \twoheadrightarrow B.$$

Recall that in such a case it does not necessarily follow that A is an interpolation space for the pair (A_0, A_1) or that B is an interpolation space for the pair (B_0, B_1). Moreover, if the morphisms in $\mathfrak{C}$ are (quasi-)linear bounded operators, we define $\|T\|_{\overline{A} \twoheadrightarrow \overline{B}} = \max\{\|T\|_{A_0 \twoheadrightarrow B_0}, \|T\|_{A_1 \twoheadrightarrow B_1}\}$. In that case, we say that A and B are *uniform* interpolation spaces with respect to $\overline{A}$ and $\overline{B}$ if there is $C > 0$ such that for every morphism $T\colon \overline{A} \twoheadrightarrow \overline{B}$ one has

$$\|T\|_{A \twoheadrightarrow B} \leq C K_T \|T\|_{\overline{A} \twoheadrightarrow \overline{B}}, \tag{2.1}$$

where K_T is the triangle inequality constant of T. If (2.1) holds with $C = 1$, we call such A and B *exact* interpolation spaces with respect to $\overline{A}$ and $\overline{B}$.

Let $\mathfrak{C}$ be an interpolation category. We say that a covariant functor $F\colon \overline{\mathfrak{C}} \twoheadrightarrow \mathfrak{C}$ is an *interpolation functor* if for any $\overline{A}, \overline{B} \in \overline{\mathfrak{C}}$ the spaces $F(\overline{A})$, $F(\overline{B})$ are interpolation spaces with respect to $\overline{A}$ and $\overline{B}$. Explicitly said, this means that whenever T is a morphism in $\mathfrak{C}$ satisfying each of the conditions

1. $T\colon A_0 + A_1 \twoheadrightarrow B_0 + B_1$,
2. $T\colon A_0 \twoheadrightarrow B_0$,
3. $T\colon A_1 \twoheadrightarrow B_1$,

then $T\colon F(\overline{A}) \twoheadrightarrow F(\overline{B})$. Analogously we define a *uniform* and an *exact* interpolation functor.

Note that the definition of the interpolation space and functor depend quintessentially on the category we are working with. It is also useful to realize that the meaning of "$\twoheadrightarrow$" is not uniquely defined here. This is best illustrated on an example.

Consider the category $\mathfrak{C}$ of all Banach spaces, where the class of morphisms is the class of all linear continuous mappings between two Banach spaces. Here X and Y are interpolation spaces for the couples (X_0, X_1), (Y_0, Y_1) of compatible Banach spaces if the following holds. If a mapping $T\colon X_0 + X_1 \twoheadrightarrow Y_0 + Y_1$ is continuous and linear from X_0 to Y_0 and it also is continuous and linear from X_1 to Y_1, then the same mapping T is also continuous and linear from X to Y. The meaning of $T\colon X \twoheadrightarrow Y$ is that T is a mapping defined on X which is linear, satisfies $T(X) \subset Y$ and T is a continuous operator from X to Y.

Now consider the category $\mathfrak{B}$ of all Banach function spaces over σ-finite measure spaces. The class of morphisms is the class of all quasi-linear bounded mappings between two Banach function spaces. Here X and Y are interpolation spaces for the couples (X_0, X_1), (Y_0, Y_1) of compatible Banach function spaces if the following holds. If $T\colon X_0 + X_1 \twoheadrightarrow Y_0 + Y_1$ is bounded with $T\colon X_0 \twoheadrightarrow Y_0$ and $T\colon X_0 \twoheadrightarrow Y_0$ being bounded, then the mapping $T\colon X \twoheadrightarrow Y$ is also bounded. The meaning of $T\colon X \twoheadrightarrow Y$ is that T is a mapping defined on X, taking values in Y, which is quasi-linear and bounded from X to Y.

Note that for quasi-linear mappings, boundedness in general only implies continuity in 0, hence the meaning of the symbol "$\twoheadrightarrow$" is strictly different in the latter example.

We shall modify the definition of compatibility for function spaces having properties ($\mathrm{T1}_C$), (P2), (P5). If X_0 and X_1 are function spaces over some σ-finite measure space (R, μ) satisfying ($\mathrm{T1}_C$), (P2), (P5), we call X_0 and X_1 *strongly compatible*. Recall that local convergence in L^1 implies local convergence in measure and thus X_1, X_2 are embedded in the Hausdorff topological space $\mathcal{M}_0(R, \mu)$. Hence strong compatibility implies compatibility.

This is usually omitted in literature, therefore whenever we work with some subcategory of the category of all function spaces having properties ($\mathrm{T1}_C$) (P2), (P5), we shall henceforth consider compatible couples to be strongly compatible. No confusion can arise here. The only problem is that we wish to define functors that can only be defined on couples of strongly compatible spaces, and so in order for them to be well defined, we have to make this adjustment.

We shall usually omit defining explicitly a category when working with interpolation functors. Instead we use statements such as "F is an interpolation functor for quasi-linear mappings on Banach functions spaces". This means that F is an interpolation functor on the category $\mathfrak{B}$ of all Banach function spaces where the class of morphism is the class of all quasi-linear mappings between such spaces. Similarly for other instances of spaces and operators.

Recall that if X and Y are some quasi-normed spaces and T is a mapping, then the meaning of the symbol $T\colon X \to Y$ is that T maps bounded sets in X into bounded sets in Y.

Let us introduce some concrete interpolation methods. We begin with the most classical results, the Riesz-Thorin and the Marcinkiewicz theorem.

Theorem 2.1 (Riesz-Thorin interpolation theorem)**.** *Let (R,μ), (S,ν) be σ-finite measure spaces and let T be a linear operator such that*

$$T\colon L^{p_0}(R,\mu)\to L^{q_0}(S,\nu),$$
$$T\colon L^{p_1}(R,\mu)\to L^{q_1}(S,\nu),$$

for some $p_i,q_i\in[1,\infty]$. Then for any $\lambda\in[0,1]$ it holds that $T\colon L^p(R,\mu)\to L^q(S,\nu)$, where $\frac{1}{p}=\frac{\lambda}{p_0}+\frac{1-\lambda}{p_1}$, $\frac{1}{q}=\frac{\lambda}{q_0}+\frac{1-\lambda}{q_1}$. Moreover, one has

$$\|T\|_{L^p(R,\mu)\to L^q(S,\nu)}\le 2\|T\|^{\lambda}_{L^{p_0}(R,\mu)\to L^{q_0}(S,\nu)}\|T\|^{1-\lambda}_{L^{p_1}(R,\mu)\to L^{q_1}(S,\nu)}.$$

For the classical proof using Hadamard's three line theorem one may consult [3, Chapter 4, Theorem 2.2]. Notice that the norm of the operator between resulting interpolation spaces is dominated by a logarithmically convex combination of operator norms acting on the endpoint spaces.

Theorem 2.2 (Marcinkiewicz interpolation theorem)**.** *Let (R,μ), (S,ν) be σ-finite measure spaces and let T be a quasi-linear operator such that*

$$T\colon L^{p_0,1}(R,\mu)\to L^{q_0,\infty}(S,\nu),$$
$$T\colon L^{p_1,1}(R,\mu)\to L^{q_1,\infty}(S,\nu),$$

for some $p_i,q_i\in[1,\infty]$ with $p_0<p_1<\infty$ and $q_0\neq q_1$. Then one has for any $\lambda\in(0,1)$ and $r\in[1,\infty]$ that $T\colon L^{p,r}\to L^{q,r}$, where $\frac{1}{p}=\frac{\lambda}{p_0}+\frac{1-\lambda}{p_1}$, $\frac{1}{q}=\frac{\lambda}{q_0}+\frac{1-\lambda}{q_1}$.

For proof see [3, Chapter 4, Theorem 4.13].

These two theorems allow us to define two distinct interpolation functors. We now present generalizations to both of these classical theorems. We start with the Riesz-Thorin theorem. This is usually done using the so called complex method, but we choose a different approach.

We introduce the so called Calderón X^sY^{1-s} space. Let Φ be a Young function, which is also invertible (this can be omitted, but requires more technical approach). We define the mapping $\rho_\Phi\colon\mathbb{R}^2\to[0,\infty]$ by the formula

$$\rho_\Phi(x,y)=\begin{cases}|y|\Phi^{-1}(\frac{|x|}{|y|}) & \text{if } x\in[0,\infty], y\in(0,\infty],\\ 0 & \text{otherwise},\end{cases}\tag{2.2}$$

where the possible $\frac{\infty}{\infty}$ is defined to be ∞. Let X and Y be function spaces possessing at least properties ($\mathrm{P1}_C$) and (P2). We define the *Calderón-Lozanovskiĭ* space $\rho_\Phi(X,Y)$ as the collection of all $f\in\mathcal{M}$, for which there exists $g\in X$ and $h\in Y$ such that

$$|f|\le\rho_\Phi(g,h)\quad\mu\text{-a.e.}$$

We equip this space with the functional

$$\|f\|_{\rho_A(X,Y)} = \inf\{\lambda > 0,\ \exists g \in B_X, h \in B_Y \text{ such that } |f| \le \lambda \rho_A(g,h)\} \quad \text{for } f \in \mathcal{M}. \tag{2.3}$$

Note that this, once again, is a Minkowski functional, more precisely

$$\|\cdot\|_{\rho_A(X,Y)} = \mu_{U_{\rho_A(X,Y)}}(\cdot),$$

where $U_{\rho_A(X,Y)} = \{f \in \mathcal{M},\ \exists g \in B_X, h \in B_Y \text{ such that } |f| \le \rho_A(g,h)\}$. In case $\Phi(t) = t^p$, we write $\rho_\Phi(X,Y) = X^{\frac{1}{p}} Y^{1-\frac{1}{p}}$.

These spaces were first introduced by Calderón [10] for Φ a power function and then studied extensively by Lozanovskiĭ in [23], [24], [25]. In relation to the pointwise product and the generalized dual, these spaces are of interest in papers [20], [19]. More basic properties (such as the interpolation property) can be found in [26].

We continue by presenting some properties of these spaces which are of interest. These are known and scattered in literature, but we present the proofs of the more basic properties for reader's convenience.

Proposition 2.3. *Let X and Y be quasi-normed spaces of functions satisfying (P1$_C$), (P2) and let M_X, M_Y be constants of the triangle inequalities in X and Y, respectively. Let $s \in (0,1)$ and denote $Z = X^s Y^{1-s}$. Then the following statements hold.*

(i) The space Z satisfies (P1$_M$) and (P2) with $M = M_X^s M_Y^{1-s}$. In particular, Z is normed if X and Y are.

(ii) The space Z is an intermediate space for the couple X and Y. Moreover, the norms of embeddings $X \cap Y \hookrightarrow Z \hookrightarrow X + Y$ are equal to 1.

(iii) If both X and Y satisfy any of the properties (P1), (P3), (P4$_w$), (P4) or (P5), then Z also satisfies that property.

(iv) If X and Y satisfy the Fatou property (P3), then the infimum in the definition of $\|\cdot\|_Z$ is attained.

(v) If $p = \frac{1}{s}$, then $Z = X^p \cdot Y^{p'}$ with equal norms.

Proof. For the entirety of the proof, consider $p = \frac{1}{s}$.

(i) Both the positive definiteness and positive homogeneity are obvious, we show the modified triangle inequality. Choose f_1, $f_2 \in Z$ and $\varepsilon > 0$. Then we find $\lambda_i > 0$, $g_i \in B_X$, $h_i \in B_Y$ for $i = 1, 2$ such that

$$\lambda_i \le \|f_i\|_Z + \frac{1}{2}\varepsilon \quad \text{for } i = 1, 2 \tag{2.4}$$

and

$$|f_i| \le \lambda_i |g_i|^s |h_i|^{1-s} \quad \text{for } i = 1, 2.$$

We realize that if a function $F\colon [0,\infty]^2 \to [0,\infty]$ is concave and positively homogenous, then one has

$$\lambda_1 F(x) + \lambda_2 F(y) = \frac{\lambda_1}{\lambda_1+\lambda_2} F((\lambda_1+\lambda_2)x) + \frac{\lambda_2}{\lambda_1+\lambda_2} F((\lambda_1+\lambda_2)y) \leq F(\lambda_1 x + \lambda_2 y), \quad (2.5)$$

for $x, y \in [0,\infty]^2$. Since the function which to $x \in [0,\infty]^2$ assigns $x_1^s x_2^{1-s}$ is positively homogeneous and concave we obtain

$$\begin{aligned}|f_1| + |f_2| &\leq \lambda_1 |g_1|^s |h_1|^{1-s} + \lambda_2 |g_2|^s |h_2|^{1-s} \leq (\lambda_1|g_1| + \lambda_2|g_2|)^s (\lambda_1|h_1| + \lambda_2|h_2|)^{1-s} \\ &= (\lambda_1+\lambda_2) \left(\frac{\lambda_1|g_1| + \lambda_2|g_2|}{\lambda_1+\lambda_2}\right)^s \left(\frac{\lambda_1|h_1| + \lambda_2|h_2|}{\lambda_1+\lambda_2}\right)^{1-s},\end{aligned}$$

where the second inequality is application of (2.5). We have

$$\|\lambda_1|g_1| + \lambda_2|g_2|\|_X \leq M_X(\lambda_1\|g_1\|_X + \lambda_2\|g_2\|_X) \leq M_X(\lambda_1+\lambda_2)$$

and similarly $\|\lambda_1|h_1| + \lambda_2|h_2|\|_Y \leq M_Y(\lambda_1+\lambda_2)$. This means that for g, h defined by

$$g = \frac{\lambda_1|g_1| + \lambda_2|g_2|}{\lambda_1+\lambda_2}$$

and

$$h = \frac{\lambda_1|h_1| + \lambda_2|h_2|}{\lambda_1+\lambda_2},$$

we have $\frac{g}{M_x} \in B_X$ and $\frac{h}{M_Y} \in B_Y$ and

$$|f_1| + |f_2| \leq (\lambda_1+\lambda_2) M_X^s M_Y^{s-1} \left(\frac{g}{M_X}\right)^s \left(\frac{h}{M_Y}\right)^{1-s}.$$

Therefore the definition of the norm in Z dictates

$$\||f_1| + |f_2|\|_Z \leq M_X^s M_Y^{1-s}(\lambda_1+\lambda_2).$$

Combining this with (2.4) yields

$$\||f_1| + |f_2|\|_Z \leq M_X^s M_Y^{1-s}(\|f_1\|_Z + \|f_2\|_Z + \varepsilon),$$

hence sending $\varepsilon \to 0$ gives the triangle inequality with the said constant. The fact that Z satisfies the lattice property (P2) is clear from the definition.

(ii) If $f \in B_{X\cap Y} = B_X \cap B_Y$, then, clearly, $|f| \leq |f|^s |f|^{1-s}$ and so $f \in B_Z$. On the other hand, if $f \in B_Z$, then, by the Young inequality, there are $g \in B_X$, $h \in B_Y$ such that $|f| \leq \frac{|g|}{p} + \frac{|h|}{p'}$. Since $F = \frac{|g|}{p} + \frac{|h|}{p'}$ may be decomposed into a function of norm $\frac{1}{p}$ in X plus a function of norm $\frac{1}{p'}$ in Y, it follows from the definition of the norm in $X+Y$ that $\|F\|_{X+Y} \leq \frac{1}{p} + \frac{1}{p'} = 1$. We have shown that $B_{X\cap Y} \subset B_Z \subset B_{X+Y}$ and so the asserted embeddings hold.

(iii) For (P1), this is an immediate consequence of *(i)*. For (P4) and (P4_w) the statement is also obvious. Let X and Y satisfy (P5) and let $E \subset R$ be of finite positive measure. Let C_X and C_Y be such that for any $f \in \mathcal{M}$ it holds that

$$\int_E |f| \mathrm{d}\mu \leq C_X \|f\|_X$$

and

$$\int_E |f| \mathrm{d}\mu \leq C_Y \|f\|_Y.$$

Let f be measurable, $g \in B_X$, $h \in B_Y$ and $\lambda > 0$ such that $|f| \leq \lambda |g|^s |h|^{s-1}$. Then one has by the Hölder inequality

$$\int_E |f| \mathrm{d}\mu \leq \int_E \lambda |g|^s |h|^{s-1} \mathrm{d}\mu \leq \lambda \left(\int_E |g| \mathrm{d}\mu \right)^s \left(\int_E |h| \mathrm{d}\mu \right)^{1-s} \leq \lambda C_X^s C_Y^{1-s}.$$

Taking now the infimum over all such λ, g and h gives (P5) for Z.

Now assume that X and Y satisfy the Fatou property (P3) and let $f_n \geq 0$ be a non-decreasing sequence of functions in Z, such that $f_n \nearrow f \in \mathcal{M}_+$. Let $\varepsilon > 0$ and find to each $n \in \mathbb{N}$ number $\lambda_n > 0$ such that there exist $0 \leq g_n \in B_X$ and $0 \leq h_n \in B_Y$ such that

$$\lambda_n \leq \|f_n\|_Z + 2^{-n}\varepsilon \tag{2.6}$$

and

$$|f_n| \leq \lambda_n g_n^s h_n^{1-s}.$$

Due to the lattice property (P2) of Z, it is clear that $\|f_n\|_Z$ forms a non-increasing sequence which satisfies $\|f_n\|_Z \leq \|f\|_Z$. We notice that it is possible to choose all of the sequences λ_n, g_n, h_n so that they are non-decreasing. Now there exist pointwise limits g, h, λ of the sequences g_n, h_n and λ_n respectively. Due to (P3) in X and Y it follows that $g \in B_X$ and $h \in B_Y$. Hence, we obtain

$$|f| \leq \lambda g^s h^{1-s}. \tag{2.7}$$

From (2.6) it follows that $\lambda = \lim_{n\to\infty} \|f_n\|_Z$, hence $\|f\|_Z \leq \lambda \leq \lim_{n\to\infty} \|f_n\|_Z \leq \|f\|_Z$. Therefore (P3) is satisfied by Z.

(iv) In the proof of *(iii)*, when showing that Z has the Fatou property (P3) whenever X and Y have (P3), we have, in particular, shown that for any $f \in Z$ there is a $\lambda \geq 0$ which realizes the norm. More precisely, we have found $g \in B_X$ and $h \in B_Y$ such that (2.7) holds and $\lambda = \|f\|_Z$. This is exactly the statement *(iv)*.

(v) If $f \in B_Z$ and $\varepsilon > 0$ we find $g \in B_X$, $h \in B_Y$ such that $|f| \leq (1+\varepsilon) g^{\frac{1}{p}} h^{\frac{1}{p'}} = \tilde{f}$.

Then $\tilde{f} \in (1+\varepsilon) B_{X^p \cdot Y^{p'}}$ by definition of the norm in $X^p \cdot Y^{p'}$. It is easily seen that the property (P2) is preserved when forming the product space and the p-convexification space. In other words (P2) holds in $X^p \cdot Y^{p'}$. Since $f \leq \tilde{f} \in (1+\varepsilon) B_{X^p \cdot Y^{p'}}$, we have $f \in (1+\varepsilon) B_{X^p \cdot Y^{p'}}$. In other words $\|f\|_{X^p \cdot Y^{p'}} \leq 1 + \varepsilon$. Since ε was arbitrary, it is necessary that $\|f\|_{X^p \cdot Y^{p'}} \leq 1$. We have shown $B_Z \subset B_{X^p \cdot Y^{p'}}$.

The fact that $B_{X^p \cdot Y^{p'}} \subset B_Z$ is obvious. Therefore $B_Z = B_{X^p \cdot Y^{p'}}$ and the equality of spaces and norms easily follows. $\square$

Two particular properties of the Calderón-Lozanovskiĭ space which are of great importance are the interpolation property and the fact that the Fatou property (P3) is preserved (in the case of general Φ).

Proposition 2.4. *Let (R,μ) be a σ-finite measure space and let X_0, X_1 be Banach function spaces over it. Let Φ be an invertible Young function and $X = \rho_\Phi(X_0, X_1)$. Then X is also a Banach function space. Moreover the operation $(X_0, X_1) \mapsto \rho_\Phi(X_0, X_1)$ is a uniform interpolation functor for linear operators on Banach function spaces.*

These are deep results with complicated proofs which can be found in [26, Section 15]. Indeed for the second statement this is [26, Section 15, Corollary 5]. For the first statement the properties (P1)-(P5) then follow from the second statement except for perhaps (P3). But (P3) is shown in [26, Section 15, Corollary 3].

We see from Proposition 2.4 that the Calderón-Lozanovskiĭ construction indeed preserves many important properties, most notably (P2) and (P3). Preservation of the property (P6) in a more general context is connected to the famous theorem of Calderón [3, Chapter 3, Theorem 2.12] (for the original result, which is stated in terms of the majorant property, see [11]) which states that a Banach function space X is an interpolation space for the pair (L^1, L^∞) (for quasi-linear operators) if and only if it is a rearrangement-invariant Banach function space. More generally a normed space X is an interpolation space for the pair (L^1, L^∞) for quasi-linear operators if and only if it is a space of functions with the majorant property. That is, whenever $f \in X$ and g is a locally integrable function with $g \prec f$, then $g \in X$ and $\|g\|_X \le \|f\|_X$. This does not imply that X is a Banach function space. It is, however, immediately clear that such a space needs to satisfy properties (P1), (P2), (P4), (P5), (P6), but the Fatou property (P3) is problematic.

It follows immediately that if X is an interpolation Banach function space for a pair of rearrangement-invariant Banach function spaces (X_0, X_1), then X is rearrangement-invariant. This means that any interpolation functor for quasi-linear operators must preserve rearrangement invariance. This fact will be used without further reference.

In light of Proposition 2.4, given an invertible Young function Φ, we define the *Calderón-Lozanovskiĭ interpolation functor* F_Φ on the category of all Banach function spaces (where the class of morphisms is the class of continuous linear mappings) by

$$F_\Phi(X_0, X_1) = \rho_\Phi(X_0, X_1) \quad \text{for } X_0,\ X_1 \text{ strongly compatible BFS.}$$

If $\Phi(t) = t^p$ we write $F_\Phi = F_p$.

We list some known identities concerning this functor. Let $\theta \in (1, \infty)$. Then

$$F_\theta(L^p, L^q) = L^r \quad \text{where } p, q \in (0, \infty],\ \tfrac{1}{r} = \tfrac{1}{\theta p} + \tfrac{1}{\theta' q}. \tag{2.8}$$

If A, B, Φ are Young functions and C is a Young function satisfying

$$C^{-1}(t) = B^{-1}(t)\Phi^{-1}\left(\tfrac{A^{-1}(t)}{B^{-1}(t)}\right) \quad \text{for } t \in (0, \infty),$$

then

$$F_\Phi(L^A, L^B) = L^C. \tag{2.9}$$

Furthermore,

$$F_\theta(L^{p_0,q_0}, L^{p_1,q_1}) = L^{p,q} \quad \text{where } p_i, q_i \in (0,\infty],\ \tfrac{1}{p} = \tfrac{1}{\theta p_0} + \tfrac{1}{\theta' p_1},\ \tfrac{1}{q} = \tfrac{1}{\theta q_0} + \tfrac{1}{\theta' q_1}. \tag{2.10}$$

The identity for Orlicz spaces (2.9) may be found for example in [26, Chapter 15, Example 3]. It is an easy exercise to show that $L^p \cdot L^q = L^r$, where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ (even when $r < 1$) and that $(L^p)^\theta = L^{p\theta}$. Proposition 2.3 now gives (2.8). Alternatively (2.9) implies (2.8).

Similarly one has $(L^{p,r})^\theta = L^{\theta p, \theta r}$ and $L^{p_0,q_0} \cdot L^{p_1,q_1} = L^{p,q}$, where $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$, $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1}$ (see [8]) and so (2.10) follows.

Note that the space $F_\theta(L^p, L^q)$ is precisely the one obtained in the Riesz-Thorin interpolation theorem with $\lambda = \frac{1}{\theta}$.

We continue by generalizing the theorem of Marcinkiewicz.

Given a compatible pair of Banach spaces X_0, X_1, $\lambda \in (0,1)$ and $r \in [1,\infty]$, define the functional

$$\|f\|_{(X_0,X_1)_{\lambda,r}} = \left(\int_0^\infty (t^{-\lambda} K(f,t,X_0,X_1))^r \tfrac{dt}{t} \right)^{\frac{1}{r}}.$$

We define the space $(X_0, X_1)_{\lambda,r} = \{f \in X_0 + X_1,\ \|f\|_{(X_0,X_1)_{\lambda,r}} < \infty\}$. Recall that $\|\cdot\|_{(X_0,X_1)_{\lambda,r}}$ is a norm on $(X_0,X_1)_{\lambda,r}$ under which the resulting normed space is a Banach space. We define the *real interpolation functor* $M_{\lambda,r}$ as

$$M_{\lambda,r}(X_0, X_1) = (X_0, X_1)_{\lambda,r} \quad \text{for } X_1,\ X_2 \text{ compatible Banach spaces.}$$

Recall that this is an interpolation functor on the category of all Banach spaces with linear operators.

Proposition 2.5. *Let $p_i, q_i \in [1,\infty]$, $i = 0,1$ with $p_0 < p_1$. Assume that either $p_1 < \infty$ or $q_1 = \infty$. Let $\lambda \in (0,1)$ and $r \in [1,\infty]$. Then*

$$(L^{p_0,q_0}, L^{p_1,q_1})_{\lambda,r} = L^{p,r},$$

where $\frac{1}{p} = \frac{1-\lambda}{p_0} + \frac{\lambda}{p_1}$ and the spaces are taken over an arbitrary resonant σ-finite measure space.

Proof. First we realize that it is enough to show this for Lorentz spaces over the real half line $(0,\infty)$ and then recall the Luxemburg representation theorem. We assume that $r < \infty$, the modification for the case $r = \infty$ is obvious and hence omitted. Nearly all cases are known and follow from the following results. For any $f \in L^1 + L^\infty$ it holds that

$$K(f, t, L^1, L^\infty) = t f^{**}(t) \quad \text{for } t \in (0,\infty). \tag{2.11}$$

Furthermore, by Hardy's inequality, whenever $p \in (1,\infty)$ and $q \in [1,\infty]$ or $p = q = \infty$, there is $C > 0$ such that for all $f \in \mathcal{M}$ it holds that

$$\|f\|_{L^{p,q}} \le \|f^{**}\|_{L^{p,q}} \le C \|f\|_{L^{p,q}}. \tag{2.12}$$

If $p' \in (1,\infty)$, we have from the definition of the real functor that

$$(L^1, L^\infty)_{\frac{1}{p'},r} = L^{p,r}. \tag{2.13}$$

Indeed from (2.11) and (2.12) we have

$$\|f\|^r_{(L^1,L^\infty)_{\frac{1}{p'},r}} = \int_0^\infty [t^{-\frac{1}{p'}} t f^{**}(t)]^r \tfrac{dt}{t} \approx \int_0^\infty [t^{\frac{1}{p}} f^*(t)]^r \tfrac{dt}{t} = \|f\|^r_{L^{p,r}},$$

for any $f \in \mathcal{M}$ where "$\approx$" holds up to a constant (C from (2.12)) independent of f.

We have shown that the assertion holds if $p_0 = q_0 = 1$ and $p_1 = q_1 = \infty$. Indeed, one then has $\frac{1}{p} = \frac{1-\lambda}{p_0} + \frac{\lambda}{p_1} = 1 - \lambda$ and so $\frac{1}{p'} = \lambda$, hence $(L^1, L^\infty)_{\lambda,r} = (L^1, L^\infty)_{\frac{1}{p'},r} = L^{p,r}$.

Now we distinguish several cases:

(1) $p_i \in (1,\infty)$, $q_i \in [1,\infty]$,

(2) $p_0 = q_0 = 1$, $p_1 \in (1,\infty)$, $q_1 \in [1,\infty]$,

(3) $p_0 \in (1,\infty)$, $q_0 \in [1,\infty]$, $p_1 = q_1 = \infty$,

(4) $p_0 = 1$, $q_0 \in (1,\infty]$, $p_1 \in (1,\infty)$, $q_1 \in [1,\infty]$,

(5) $p_0 = 1$, $q_0 \in (1,\infty]$, $p_1 = q_1 = \infty$.

Cases (1)-(3) are recovered easily by the reiteration principle [3, Chapter 5, Theorem 2.4]. Indeed, each Lorentz space L^{p_i,q_i} satisfies embeddings

$$L^{p_i,1} \hookrightarrow L^{p_i,q_i} \hookrightarrow L^{p_i,\infty},$$

and from (2.13) we have $L^{p_i,1} = (L^1, L^\infty)_{\frac{1}{p_i'},1}$, $L^{p_i,\infty} = (L^1, L^\infty)_{\frac{1}{p_i'},\infty}$. Therefore in any of the cases (1)-(3) the reiteration principle dictates

$$(L^{p_0,q_0}, L^{p_1,q_1})_{\lambda,r} = ((L^1, L^\infty)_{\frac{1}{p_0'},q_0}, (L^1, L^\infty)_{\frac{1}{p_1'},q_1})_{\lambda,r} = (L^1, L^\infty)_{\theta,r},$$

where $\theta = (1-\lambda)\frac{1}{p_0'} + \lambda\frac{1}{p_1'}$. It is now simply calculated that this is equivalent to $\theta = \frac{1}{p'}$, where $\frac{1}{p} = \frac{1-\lambda}{p_0} + \frac{\lambda}{p_1}$, and the assertion follows.

For cases (4) and (5) the reiteration principle cannot be used, as it is assumed that the spaces in question are Banach, but spaces L^{1,q_0} with $q_0 > 1$ are not even normed. This has to therefore be shown by hand.

We begin with (5). The K-functional for the pair $(L^{1,\infty}, L^\infty)$ may be found e.g. in [7], it is given by

$$K(f, t, L^{1,\infty}, L^\infty) \approx \sup_{s\in(0,t]} f^*(s)s \quad \text{for } f \in L^{1,\infty} + L^\infty,\ t \in (0,\infty),$$

up to constants independent of f and t. Since $L^1 \hookrightarrow L^{1,q_0} \hookrightarrow L^{1,\infty}$, one has

$$(L^1, L^\infty)_{\lambda,r} \hookrightarrow (L^{1,q}, L^\infty)_{\lambda,r} \hookrightarrow (L^{1,\infty}, L^\infty)_{\lambda,r}, \tag{2.14}$$

We now simply show that $(L^{1,\infty}, L^\infty)_{\lambda,r} \hookrightarrow (L^1, L^\infty)_{\lambda,r}$. We have for $f \in \mathcal{M}$

$$\|f\|^r_{(L^{1,\infty},L^\infty)_{\lambda,r}} \approx \int_0^\infty [t^{-\lambda} \sup_{s\in(0,t)} f^*(s)s]^r \tfrac{\mathrm{d}t}{t} \geq \int_0^\infty [t^{-\lambda+1} f^*(t)]^r \tfrac{\mathrm{d}t}{t} = \|f\|^r_{L^{p,r}},$$

up to a constant independent of f, where $\frac{1}{p} = 1 - \lambda = \frac{1-\lambda}{p_0} + \frac{\lambda}{p_1}$. Hence $(L^{1,\infty}, L^\infty)_{\lambda,r} \hookrightarrow L^{p,r} = (L^1, L^\infty)_{\lambda,r}$ and by (2.14) we have

$$(L^{1,q}, L^\infty)_{\lambda,r} = L^{p,r}.$$

For the case (4) we have

$$(L^1, L^{p_1,q_1})_{\lambda,r} \hookrightarrow (L^{1,q_0}, L^{p_1,q_1})_{\lambda,r} \hookrightarrow (L^{1,\infty}, L^{p_1,\infty})_{\lambda,r}. \tag{2.15}$$

From [36, Proposition 2.2] we have the inequality

$$\sup_{u\in(0,t^{p_1'}]} f^*(2u)u + t \sup_{u\in(t^{p_1'},\infty)} f^*(2u)u^{\frac{1}{p_1}} \lesssim K(f,t,L^{1,\infty},L^{p_1,\infty})$$

for $f \in L^{1,\infty} + L^{p_1,\infty}$, $t \in (0,\infty)$, up to a constant independent of f and t. In particular, we have

$$f^*(2t^{p_1'})t^{p_1'} + f^*(2t^{p_1'})t^{\frac{p_1'}{p_1}+1} \lesssim K(f,t,L^{1,\infty},L^{p_1,\infty}).$$

Since $\frac{p_1'}{p_1} + 1 = p_1'$, it follows that

$$\|f\|^r_{(L^{1,\infty},L^{p_1,\infty})_{\lambda,r}} \gtrsim \int_0^\infty [t^{-\lambda} t^{p_1'} f^*(2t^{p_1'})]^r \tfrac{\mathrm{d}t}{t} = \frac{1}{p_1'} \int_0^\infty [s^{-\frac{\lambda}{p_1'}+1} f^*(2s)]^r \tfrac{\mathrm{d}s}{s} \approx \|f\|_{L^{p,r}},$$

where $\frac{1}{p} = \frac{-\lambda}{p_1'} + 1$, which is equivalent to $\frac{1}{p} = \frac{\lambda}{p_1} - \lambda + 1$. We now have $(L^{1,\infty}, L^{p_1,\infty})_{\lambda,r} \hookrightarrow L^{p,r} = (L^1, L^{p_1,\infty})_{\lambda,r}$ and so (2.15) gives $(L^{1,\infty}, L^{p_1,\infty})_{\lambda,r} = L^{p,r}$. □

Note that $M_{\lambda,r}(L^{p_0,r_0}, L^{p_1,r_1})$ is precisely the space obtained in the Marcinkiewicz interpolation theorem.

We now present the so called Aronszajn-Gagliardo maximal and minimal constructions. Let $\mathfrak{C}$ be the category of all Banach spaces with linear continuous mappings. Let (A_0, A_1) be a compatible couple of spaces in $\mathfrak{C}$ and let A be an interpolation space for the couple $\overline{A}$. For (X_0, X_1) a compatible couple in $\mathfrak{C}$ we define the spaces

$$F^{\min}_{(A_0,A_1),A}(X_0,X_1) = \{f \in X_0 + X_1,\ f = \sum_{j=1}^\infty T_j a_j,\ T_j \colon \overline{A} \to \overline{X},\ a_j \in A\}$$

and

$$F^{\max}_{(A_0,A_1),A}(X_0,X_1) = \{f \in X_0 + X_1,\ T(f) \in A \text{ for all } T\colon \overline{X} \to \overline{A}\}.$$

We define norms on these spaces by

$$\|f\|_{\min} = \inf \left\{ \sum_{j=1}^{\infty} \|T_j\|_{\overline{X}\to\overline{A}} \|a_j\|_A,\ f = \sum_{j=1}^{\infty} T_j a_j \right\} \tag{2.16}$$

and

$$\|f\|_{\max} = \sup_{T \in B_{\overline{X}\to\overline{A}}} \|Tf\|_A,$$

where the infimum in (2.16) is taken over all such representations and

$$B_{\overline{X}\to\overline{A}} = \{T\colon \overline{X} \to \overline{A},\ \|T\|_{\overline{X}\to\overline{A}} \le 1\}.$$

Recall that both of these spaces are Banach spaces and that $F^{\min}_{(A_0,A_1),A}$, $F^{\max}_{(A_0,A_1),A}$ are uniform interpolation functors on $\mathfrak{C}$ with the property that whenever F is another interpolation functor on $\mathfrak{C}$ with $F(A_0,A_1) = A$, then $F^{\min}_{(A_0,A_1),A}(Y_0,Y_1) \hookrightarrow F(Y_0,Y_1) \hookrightarrow F^{\max}_{(A_0,A_1),A}(Y_0,Y_1)$ for all compatible couples of Banach spaces (Y_0,Y_1). For the first embedding, see [4, Corollary 2.5.2]. For the second one, we realize that inclusion $F(Y_0,Y_1) \subset F^{\max}_{(A_0,A_1),A}(Y_0,Y_1)$ holds by definition and then recall the closed graph theorem.

We shall call $F^{\min}_{(A_0,A_1),A}$, $F^{\max}_{(A_0,A_1),A}$ the *Aronszajn-Gagliardo minimal* (resp. *maximal*) *interpolation functor* with respect to $(A_0,A_1),A$.

We conclude by recalling so called Calderón operators. Informally, one could define the notion of a Calderón operator for two couples of spaces (X_0,X_1), (Y_0,Y_1), as an arbitrary operator T which satisfies $T\colon X \to Y$ if and only if $S\colon X \to Y$ for every $S\colon \overline{X} \to \overline{Y}$. Here the spaces have to be taken from some reasonable scale such as the scale of rearrangement-invariant Banach function spaces.

We suffice with a far less general notion. Let $p_i, q_i \in [1,\infty]$, $i = 0,1$, $p_0 < p_1$, $q_0 \neq q_1$, and let σ denote *the interpolation segment*

$$\sigma = \left[\left(\frac{1}{p_0}, \frac{1}{q_0} \right), \left(\frac{1}{p_1}, \frac{1}{q_1} \right) \right],$$

that is, the line segment in the unit square $\{(x,y),\ 0 \le x,y \le 1\}$ with endpoints $(\frac{1}{p_i}, \frac{1}{q_i})$, $i = 0,1$. We denote its slope by m, that is

$$m = \frac{\frac{1}{q_0} - \frac{1}{q_1}}{\frac{1}{p_0} - \frac{1}{p_1}}.$$

We define the *Calderón operator* associated with the interpolation segment σ by the formula

$$S_\sigma f(t) = t^{-\frac{1}{q_0}} \int_0^{t^m} s^{\frac{1}{p_0}} f(s) \frac{\mathrm{d}s}{s} + t^{-\frac{1}{q_1}} \int_{t^m}^{\infty} s^{\frac{1}{p_1}} f(s) \frac{\mathrm{d}s}{s},\ t \in (0,\infty),$$

for $f \in \mathcal{M}(0,\infty)$ and for which it makes sense. We recall that S_σ is well defined for each $f \in L^{p_0,1} + L^{p_1,1}$, where $p_1 < \infty$. We also have

$$S_\sigma \colon L^{p_0,1} \to L^{q_0,\infty};$$
$$S_\sigma \colon L^{p_1,r} \to L^{q_1,\infty}.$$

For reference see [3, Chapter 3, Lemma 4.10]. We also recall that $S_\sigma(f^*) = (S_\sigma(f^*))^*$ for arbitrary $f \in \mathcal{M}(0,\infty)$.

Let (R,μ) and (S,ν) be non-atomic σ-finite measure spaces and let T be a quasi-linear operator satisfying

$$T \colon L^{p_0,1}(R,\mu) \to L^{q_0,\infty}(S,\nu);$$
$$T \colon L^{p_1,1}(R,\mu) \to L^{q_1,\infty}(S,\nu),$$

for $1 \le p_0 < p_1 < \infty$, $q_i \in [1,\infty]$, $i = 0,1$, $q_0 \neq q_1$. Then T is said to be of *weak types* (p_0,q_0) and (p_1,q_1) and there is a constant $C > 0$ such that for every function $f \in L^{p_0,1}(R,\mu) + L^{p_1,1}(R,\mu)$ it holds that

$$(Tf)^* \le C S_\sigma(f^*).$$

It follows immediately from the Luxemburg representation theorem that whenever $X(R,\mu)$ and $Y(S,\nu)$ are rearrangement-invariant Banach function spaces, $T \colon X(R,\mu) \to Y(S,\nu)$ for every such T if and only if $S_\sigma \colon \tilde{X} \to \tilde{Y}$. This means that S_σ may be seen in some sense as the “worst” operator with the given Lorentz-type endpoints.

Calderón operators for more general endpoints may be found for example in [32], while some particular interesting endpoints are treated thoroughly in [16].

3. Abstract factorization of couples of interpolation functors

Assume (R, μ) and (S, ν) are some σ-finite measure spaces. Let S_1 and S_2 be operators defined on some subspaces of $\mathcal{M}(R, \mu)$ and taking values in $\mathcal{M}(S, \nu)$, let $\theta \in (1, \infty)$, and define

$$T(f)(s) = |S_1(f)(s)|^{\frac{1}{\theta}} |S_2(f)(s)|^{\frac{1}{\theta'}}, \tag{3.1}$$

for those $f \in \mathcal{M}(R, \mu)$ for which it makes sense. We shall remark here a very trivial fact, upon which the rest of this chapter is based. If X, Y_0, Y_1 are some quasi-normed function spaces having (P2) such that

$$S_1 \colon X \to Y_1; \quad S_2 \colon X \to Y_2,$$

then $T \colon X \to Y_1^{\theta} \cdot Y_2^{\theta'}$. This should be absolutely clear from definitions and will often be used without further reference. This is also a very special case of a far more general Theorem 3.2 which we present and prove later.

Given some reasonable endpoint estimates on S_1 and S_2 we wish to construct a canonical target space for T (given a suitable domain space). This ends up being quite technical and very abstract, hence we begin by presenting an example, gist of which is the same as that of the construction which will follow.

Example 3.1. Let $(R, \mu) = (S, \nu) = ((0, 1), \lambda)$ and assume S_1, S_2 are linear and satisfy

$$\begin{aligned} S_1 &\colon L^1 \to L^1, & S_2 &\colon L^2 \to L^2, \\ S_1 &\colon L^3 \to L^\infty, & S_2 &\colon L^\infty \to L^\infty. \end{aligned} \tag{3.2}$$

Choosing any exponent $p \in (2, 3)$ one has by the Riesz-Thorin interpolation theorem that

$$S_1 \colon L^p \to L^{q_1}; \quad S_2 \colon L^p \to L^{q_2},$$

where q_1 is such that a number $\lambda \in [0, 1]$ satisfies

$$\frac{1}{p} = \lambda + \frac{1 - \lambda}{3}$$

if and only if it satisfies

$$\frac{1}{q_1} = \lambda.$$

Similarly q_2 is such that $\lambda \in [0, 1]$ satisfies

$$\frac{1}{p} = \frac{\lambda}{2}$$

if and only if it satisfies

$$\frac{1}{q_2} = \frac{\lambda}{2}.$$

In particular

$$q_1 = \frac{1 - \frac{1}{3}}{\frac{1}{p} - \frac{1}{3}} \quad \text{and} \quad q_2 = p.$$

It is a consequence of the definitions that T now satisfies

$$T \colon L^p \to (L^{q_1})^{\theta} \cdot (L^{q_2})^{\theta'} = L^{\theta q_1} \cdot L^{\theta' q_2} = L^q,$$

where $\frac{1}{q} = \frac{1}{\theta q_1} + \frac{1}{\theta' q_2}$, that is

$$\frac{1}{q} = \frac{1}{\theta} \frac{\frac{1}{p} - \frac{1}{3}}{1 - \frac{1}{3}} + \frac{1}{\theta' p}.$$

One may illustrate this principle with two diagrams. In Figure 1 the red line is the interpolation segment for S_1, while the blue line is the interpolation segment for S_2. Given the specific $p \in (2, 3)$ the resulting target space for T is somewhere on the purple line, depending on θ. This dependence is linear with respect to $\frac{1}{\theta}$.

In Figure 2, the situation is similar, except we have a fixed θ (in this case $\theta = 2$, that is $\frac{1}{\theta} = \frac{1}{2}$) and the purple line is the interpolation segment for T with this given θ. That is, for any $p \in (2, 3)$ the q for which $(\frac{1}{p}, \frac{1}{q})$ is a point of the purple line, one has $T \colon L^p \to L^q$.

We define the set $\mathrm{I} = \mathrm{Int}((L^1, L^3), (L^2, L^\infty)) = \mathrm{Int}(L^1, L^2) \cap \mathrm{Int}(L^2, L^\infty)$ (e.g. in the category of all Banach function spaces with linear mappings). Lebesgue spaces in I are precisely the spaces L^p for $p \in [2, 3]$. Of principal importance is that we get result for T for domain spaces which are interpolation simultaneously for the domain endpoints of both S_1 and S_2. For example, it is impossible to say anything about behaviour of T on Lebesgue spaces L^p with $p \notin [2, 3]$.

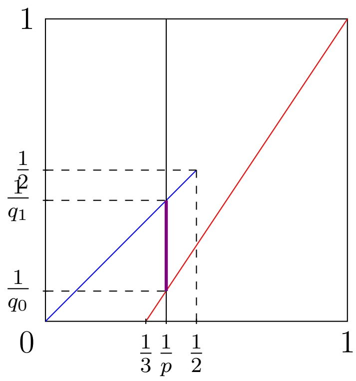

Figure 1: Example of interpolation on Lebesgue spaces with variable θ.

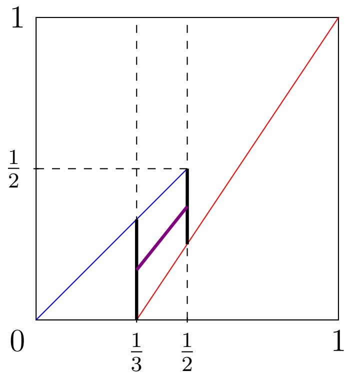

Figure 2: Example of interpolation on Lebesgue spaces with variable p.

We continue with an abstract construction in the spirit of the preceding example. We consider two, possibly distinct, interpolation categories of quasi-normed function spaces having at least the lattice property (P2), we denote these $\mathfrak{C}_1$, $\mathfrak{C}_2$. We shall further assume that classes of objects of both of these categories contain all Banach function spaces $X(R, \mu)$ and $X(S, \nu)$. Note that this means that the classes of objects in $\mathfrak{C}_1$, $\mathfrak{C}_2$ are sets.

We require that morphisms in either of the categories are mappings between objects of the particular category. This implies that the class of morphisms is also a set. We shall have two further requirements upon the sets of morphisms in the particular categories.

(Con) The set of morphisms in $\mathfrak{C}_1$ contains S_1 and the set of morphisms in $\mathfrak{C}_2$ contains S_2.

(Bdd) In both $\mathfrak{C}_1$ and $\mathfrak{C}_2$ the three-place relation $S\colon X \twoheadrightarrow Y$ must imply that S is bounded from X to Y.

We shall further assume that there are some Banach function spaces A_j^i over (R, μ), B_j^i, over (S, ν) $i, j = 1, 2$ such that

$$S_1\colon \overline{A^1} \twoheadrightarrow_{\mathfrak{C}_1} \overline{B^1}; \quad S_2\colon \overline{A^2} \twoheadrightarrow_{\mathfrak{C}_2} \overline{B^2},$$

where we denote $(A_1^i, A_2^i) = \overline{A^i}$, $i = 1, 2$ and similarly for $\overline{B^i}$.

The expression $\mathrm{Int}(\overline{A^1}, \overline{A^2})$ denotes the set of all quasi-normed function spaces X which are interpolation spaces simultaneously for $\overline{A^1}$ in $\mathfrak{C}_1$ and for $\overline{A^2}$ in $\mathfrak{C}_2$.

Let $X \in \mathrm{Int}(\overline{A^1}, \overline{A^2})$ and F_1 be an interpolation functor on $\mathfrak{C}_1$ such that $F_1(\overline{A^1}) = X$. Similarly, let F_2 be an interpolation functor on $\mathfrak{C}_2$ such that $F_2(\overline{A^2}) = X$. Note that such functors only need to exist if X is normed and complete, which is the case if X has (P3), which is equivalent to X being a Banach function space. Indeed by definition of our categories, X has (P2), whereas (P4) and (P5) follow from the fact that the space is intermediate for two Banach function spaces, hence is intermediate for L^1 and L^∞. So X having (P3) is equivalent to X being a Banach function space.

Theorem 3.2. *The logarithmically convex combination T as defined in* (3.1) *is a bounded mapping from X to $F_\theta(F_1(\overline{B^1}), F_2(\overline{B^2}))$. Moreover $W = (F_\theta(F_1(\overline{B^1}), F_2(\overline{B^2})))''$ is a Banach function space and T is bounded from X into W.*

Proof. From (Con), it is immediately clear that we have

$$S_1 \colon X \twoheadrightarrow_{\mathfrak{C}_1} F_1(\overline{B^1}); \quad S_2 \colon X \twoheadrightarrow_{\mathfrak{C}_2} F_2(\overline{B_2}),$$

which must by (Bdd) imply that S_1 is bounded from X into $F_1(\overline{B^1})$ and S_2 is bounded from X into $F_2(\overline{B_2})$.

Let now B be a bounded set in X. Let K be such that $\|f\|_X \leq K$ for all $f \in B$. We then have constants C_1 and C_2 such that $\|S_1 f\|_{F_1(\overline{B^1})} \leq C_1 K$ and $\|S_2 f\|_{F_2(\overline{B_2})} \leq C_2 K$ for each $f \in B$. Recalling Proposition 2.3 *(v)* and the definition of F_θ, we see that

$$F_\theta(F_1(\overline{B^1}), F_2(\overline{B^2})) = (F_1(\overline{B^1}))^\theta \cdot (F_2(\overline{B^2}))^{\theta'}.$$

Note that if Y is a quasi-normed function space having the lattice property (P2), one has for $p > 0$ the identity $\||g|^{\frac{1}{p}}\|_{Y^p} = \|g\|_Y^{\frac{1}{p}}$ for all measurable g. Similarly for the product operator, given another such space X, one has $\|fg\|_{X \cdot Y} \leq \|f\|_X \|g\|_Y$ for all $f \in X$, $g \in Y$. It follows therefore that

$$\||S_1 f|^{\frac{1}{\theta}} |S_2 f|^{\frac{1}{\theta'}}\|_{F_\theta(F_1(\overline{B^1}), F_2(\overline{B^2}))} \leq \||S_1 f|^{\frac{1}{\theta}}\|_{(F_1(\overline{B_1}))^\theta} \||S_2 f|^{\frac{1}{\theta'}}\|_{(F_2(\overline{B_2}))^\theta} \leq C_1^{\frac{1}{\theta}} C_2^{\frac{1}{\theta'}} K$$

for each $f \in B$. This means $T(B)$ is bounded in $F_\theta(F_1(\overline{B^1}), F_2(\overline{B^2}))$, which is precisely what we wanted.

If the space $F_\theta(F_1(\overline{B^1}), F_2(\overline{B^2}))$ is a Fatou-convexifiable lattice, then the space W is a Banach function space by Proposition 1.2. Due to Proposition 2.3 *(iii)* it is enough to show that both $F_1(\overline{B^1}), F_2(\overline{B^2})$ are Fatou-convexifiable lattices. By our assumption on the underlying category, it must be the case that $F_1(\overline{B^1}), F_2(\overline{B^2})$ have ($P1_C$) and (P2) and since they are interpolation spaces for a pair of Banach function spaces, $F_1(\overline{B^1}), F_2(\overline{B^2})$ must satisfy (P4) and (P5). Therefore they are Fatou-convexifiable lattices and the proof is complete. $\square$

On Figure 3 one can see a visualisation of the method described by Theorem 3.2. There the spaces Y_1 and Y_2 of course stand for $F_1(\overline{B^1})$ and $F_2(\overline{B^2})$ respectively, P stands for the product operator and P_θ stands for the mapping which to a measurable function g assigns $|g|^{\frac{1}{\theta}}$.

We continue with several remarks. The reason we introduce such an abstract setting with distinct interpolation categories and the conditions (Con), (Bdd) is that we really care very little about the structure of the mappings S_1 and S_2 as long as it is possible to interpolate them in such a way that they are *bounded*. Indeed boundedness is the correct property for the logarithmically convex combination.

There are many possible three-place relations one may assume on the underlying categories that imply boundedness. For example if T is Lipschitz from X into Y (Banach function spaces) then it is also bounded. Hence one may assume the morphisms in the categories to be formed by such operators. Of course the functors F_1 and F_2 have to be interpolation functors for such operators and S_1, S_2 have to be such operators. For interpolation of this kind, one may consult for example [12] or [6, Chapter 4].

$$
\begin{tikzcd}
 & X \arrow[dl, "S_1"'] \arrow[dr, "S_2"] & \\
Y_1 \arrow[d, "P_\theta"'] & & Y_2 \arrow[d, "P_{\theta'}"] \\
Y_1^\theta \arrow[dr] & P & Y_2^{\theta'} \arrow[dl] \\
 & Y_1^\theta \cdot Y_2^{\theta'} &
\end{tikzcd}
$$

Figure 3: A path for the logarithmically convex combination of operators

Note that Theorem 3.2 gives $T\colon X \to F_\theta(F_1(\overline{B^1}), F_2(\overline{B^2}))$, but *not* necessarily $T\colon X \twoheadrightarrow F_\theta(F_1(\overline{B^1}), F_2(\overline{B^2}))$. For example if $\twoheadrightarrow$ means "being a Lipschitz mapping", then it is not necessary that T is Lipschitz, just that it is bounded (between the respective spaces). Quite obviously any sort of sub-additive properties can also be lost.

It would also be possible to forget all other properties but boundedness of S_1 and S_2, such as linearity etc. The problem with that is that to the best of our knowledge no interesting interpolation functors for (only) bounded mappings are known. We are therefore somehow inconveniently forced to assume some structure on S_1 and S_2 which is lost when forming the logarithmically convex combination.

One property which is preserved is the positive homogeneity. That is, if both S_1 and S_2 are positively homogeneous, then T is as well.

In particular if S_1 and S_2 are positively homogeneous and T is bounded between two quasi-normed function spaces X and Y, then T is also continuous at 0 between these spaces.

Example 3.3. Let the following function spaces be over $((0,1), \lambda)$. Let $\theta \in (1, \infty)$ and define the operator P_θ for $f \in \mathcal{M}$ by $P_\theta(f) = |f|^{\frac{1}{\theta}}$. If X is a quasi-normed function space having the property (P2), it holds that $P_\theta\colon X \to X^\theta$ is not Lipschitz. Indeed one has by definition $\|P_\theta(f)\|_{X^\theta} = \|f\|_X^{\frac{1}{\theta}}$. Hence the statement follows from the fact that the function $t \mapsto t^{\frac{1}{\theta}}$ is not Lipschitz on any right neighbourhood of 0.

Let now $S_1 = \mathrm{Id}_{L^2}$ and $S_2(f) = 1$ identically for any $f \in \mathcal{M}$. Then

$$S_1\colon L^2 \to L^2; \quad S_2\colon L^2 \to L^\infty \tag{3.3}$$

are both Lipschitz mappings of constant less than or equal to 1. However, if $\theta = 2$, then the logarithmically convex combination T defined by (3.1) is bounded, but not Lipschitz as a mapping $T\colon L^2 \to F_\theta(L^2, L^\infty) = L^4$. Indeed, the fact that it is bounded is easily seen and the fact that it is not Lipschitz follows from the identity $T(f) = P_2(f)$ for any measurable f, which we have already shown not to be Lipschitz between L^2 and $L^4 = (L^2)^2$.

Remark 3.4. The method introduced by Theorem 3.2 can be easily modified to accommodate for general Banach spaces as domain spaces.

Indeed one may easily consider e.g. Sobolev spaces as domain spaces and use a real interpolation functor in place of F_1 and F_2. The reason our general construction omits this fact is that defining the categories $\mathfrak{C}_1$ and $\mathfrak{C}_2$ in such a case is extremely technical. Suddenly it is hard to make sure that the expression $F_\theta(F_1(\overline{B^1}), F_2(\overline{B^2}))$ is well defined. The main idea should be clear from our construction and modifiable to essentially arbitrary setting, where applying F_θ to resulting target spaces makes sense. We provide an example of this, where only *target* spaces need to be Banach function spaces.

Example 3.5. Let A^i_j, $i, j = 1, 2$ be Banach spaces and B^i_j Banach function spaces over some σ-finite measure space (R, μ). Let S_1 and S_2 be linear mappings satisfying

$$S_1\colon \overline{A^1} \to \overline{B^1}; \quad S_2\colon \overline{A^2} \to \overline{B^2},$$

Assume further that there are $\lambda_j \in (0, 1)$ and $r_j \in [1, \infty]$ such that $A^2_j = (\overline{A})_{\lambda_j, r_j}$, $j = 1, 2$. Then we have

$$S_1\colon \overline{A^2} \to ((\overline{B^1})_{\lambda_1,r_1}, (\overline{B^1})_{\lambda_2,r_2}).$$

Indeed this follows immediately from the fact that the real interpolation functor is an interpolation functor for linear operators. One might call this *normalization* as we are merely interpolating one of the operators twice, in such a way that the resulting estimates have same domain spaces as the other operator.

It now follows from Theorem 3.2 (or rather Remark 3.4) that the logarithmically convex combination T of S_1 and S_2 with exponent $\theta \in (1, \infty)$ satisfies

$$T\colon (\overline{A^2})_{\lambda,r} \to \left[((\overline{B^1})_{\lambda_1,r_1}, (\overline{B^1})_{\lambda_2,r_2})_{\lambda,r}\right]^\theta \cdot \left[(\overline{B^2})_{\lambda,r}\right]^{\theta'}$$

for every $\lambda \in (0, 1)$ and $r \in [1, \infty]$. The spaces on the right possess the property (P2) and so the right hand side is well defined. This can be seen from the definition of the real functor, as $0 \le f \le g$ implies $K(f, t) \le K(g, t)$, $t \in (0, \infty)$ for f, g measurable functions over (R, μ). Here K denotes the Peetre K-functional for an arbitrary pair of Banach function spaces.

Due to the reiteration principle [3, Chapter 5, Theorem 2.4] we may skip the normalization step. Let simply $X = (\overline{A^2})_{\lambda,r}$, then X is certainly an interpolation space for both the couples $\overline{A^1}$ and $\overline{A^2}$. Staying true to the notation from Theorem 3.2, we choose $F_1 = M_{\Lambda,r}$, where $\Lambda = (1 - \lambda)\lambda_1 + \lambda\lambda_2$ and $F_2 = M_{\lambda,r}$. Then from the definition of the real functor we have $X = F_2(\overline{A^2})$ and from the reiteration principle and the definition of Λ we have $X = F_1(\overline{A^1})$. Hence the resulting boundedness for T is

$$\begin{aligned} T\colon X \to F_\theta(F_1(\overline{B^1}), F_2(\overline{B^2})) &= \left[(\overline{B^1})_{\Lambda,r}\right]^\theta \cdot \left[(\overline{B^2})_{\lambda,r}\right]^{\theta'} \\ &= \left[((\overline{B^1})_{\lambda_1,r_1}, (\overline{B^1})_{\lambda_2,r_2})_{\lambda,r}\right]^\theta \cdot \left[(\overline{B^2})_{\lambda,r}\right]^{\theta'}, \end{aligned}$$

where the last equality is the reiteration principle again. This result is the same as the one obtained using the normalization step, which should come as no surprise. We notify the reader that the meaning of the square brackets is the same as that of regular brackets, we use it only to make the expression easier to read.

Let us now assume that the endpoints of S_1 and S_2 have been somehow normalized, that is, we have

$$\begin{aligned} S_1 \colon X_0 \to A, \qquad S_2 \colon X_0 \to D, \\ S_1 \colon X_1 \to B, \qquad S_2 \colon X_1 \to C. \end{aligned} \tag{3.4}$$

Given a suitable interpolation functor F, we can obtain a result of the type

$$T \colon F(X_0, X_1) \to F_\theta(F(A,B), F(D,C)).$$

If one assumes $F = F_\eta$ for some $\eta \in (1, \infty)$ it is not hard to show that in fact

$$F_\theta(F_\eta(A,B), F_\eta(D,C)) = F_\eta(F_\theta(A,D), F_\theta(B,C))$$

(this is further discussed in Example 3.12, see also Figure 4). A natural question arises, does this commutativity-type property hold for wider classes of functors?

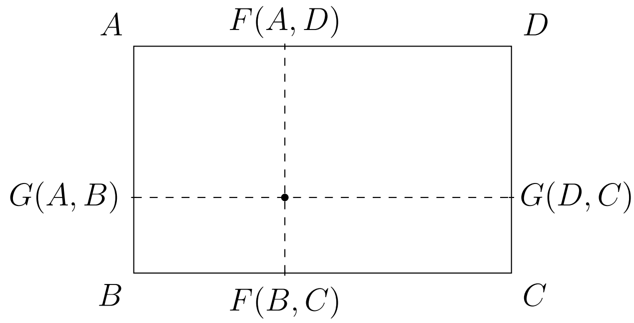

Figure 4: Commutativity on squares

Definition 3.6. Let $\mathfrak{C}$ be some interpolation category of quasi-normed spaces and assume A,B,C,D are mutually compatible spaces in $\mathfrak{C}$. Assume F, G are two interpolation functors on $\mathfrak{C}$. We say that F *commutes* with G on the *square* (A, B, C, D) if $F(G(A,B), G(D,C)) = G(F(A,D), F(B,C))$. We say that F *sub-commutes* with G on the square (A, B, C, D) if $F(G(A,B), G(D,C)) \hookrightarrow G(F(A,D), F(B,C))$. If F commutes with G on all squares of mutually compatible spaces, we say that F and G are *commutative* on $\mathfrak{C}$.

By $\hookrightarrow$ we mean that the natural identity mapping defined on entire $A + B + C + D$ is bounded on the respective spaces. The class of morphisms in $\mathfrak{C}$ is somewhat arbitrary. Indeed one could define this notion of commutativity for general functors.

The definition of commutativity on squares is not symmetric in the sense that the order of the spaces A, B, C, D matters. In particular, the statement that F commutes with G on (A, B, C, D) is equivalent to the statement that G commutes with F on the square (A, D, C, B) but it is not equivalent to the statement that G commutes with F on the square (A, B, C, D). This problem is fixed, however, if F commutes with G on all squares. It then holds that F and G are commutative if and only if G and F are commutative. For a geometric interpretation, see Figure 4. There the intersection of the dashed lines corresponds to both $F(G(A, B), G(D, C))$ and $G(F(A, D), F(B, C))$.

We first present some abstract results and counter-examples. Then we continue with some positive results for concrete functors and examples.

Proposition 3.7. *Let A, B, C, D be mutually compatible Banach spaces and let F, G be interpolation functors on the category $\mathfrak{C}$ of all Banach spaces, with the class of morphisms consisting of all bounded linear mappings. Let $Z_0 = F(A, D)$, $Z_1 = F(B, C)$, $Z = G(F(A, D), F(B, C))$.*

(i) If F sub-commutes with G on the square (A, B, C, D), then so does $F^{min}_{Z_0,Z_1,Z}$.

(ii) If F sub-commutes with $F^{max}_{Z_0,Z_1,Z}$ (A, B, C, D), then F also sub-commutes with G on that square.

Moreover, if $F(G(A, B), G(D, C))$ is not an interpolation space for the pair $(F(A, D), F(B, C))$, then F does not commute with G on the square (A, B, C, D).

Proof. The statements *(i)* and *(ii)* are immediate consequences of the definitions. Indeed, for *(i)* one has $G(F(A, D), F(B, C)) = F^{\min}_{Z_0,Z_1,Z}(F(A, D), F(B, C))$ and so from minimality

$$\begin{aligned} &F(F^{\min}_{Z_0,Z_1,Z}(A, B), F^{\min}_{Z_0,Z_1,Z}(D, C)) \hookrightarrow F(G(A, B), G(D, C)) \\ &\hookrightarrow G(F(A, D), F(B, C)) = F^{\min}_{Z_0,Z_1,Z}(F(A, D), F(B, C)). \end{aligned}$$

The first embedding is merely an interpolation of the identical mapping and the second embedding is the definition of sub-commutativity. For *(ii)* one has $G(F(A, D), F(B, C)) = F^{\max}_{Z_0,Z_1,Z}(F(A, D), F(B, C))$ and so from maximality

$$\begin{aligned} &F(G(A, B), G(D, C)) \hookrightarrow F(F^{\max}_{Z_0,Z_1,Z}(A, B), F^{\max}_{Z_0,Z_1,Z}(D, C)) \\ &\hookrightarrow F^{\max}_{Z_0,Z_1,Z}(F(A, D), F(B, C)) = G(F(A, D), F(B, C)). \end{aligned} \tag{3.5}$$

The last statement is obvious as if F commuted with G on (A, B, C, D), we would have $F(G(A, B), G(D, C)) = G(F(A, D), F(B, C))$ but since G is in interpolation functor, the space on the right must be an interpolation space for the pair $(F(A, D), F(B, C))$. Hence the space on the left must be such. □

We now present an example of functors which do not commute with F_θ on a particular choice of spaces.

Example 3.8. Define $F_\Sigma(X,Y) = X+Y$ and $F_\cap(X,Y) = X\cap Y$ whenever (X,Y) is a compatible couple of Banach spaces. It is then quite obvious that F_Σ and $F_\cap$ are interpolation functors (e.g. in the category of all Banach spaces with bounded mappings). Let all of the following spaces be over $(0,1)$ with the Lebesgue measure. Consider $A = C = L^1$, $B = D = L^2$. It then holds for any $\theta \in (1,\infty)$ that

(i) $F_\theta(F_\cap(A,B), F_\cap(D,C)) \hookrightarrow F_\cap(F_\theta(A,D), F_\theta(B,C))$ with strict inclusion,

(ii) $F_\theta(F_\Sigma(A,B), F_\Sigma(D,C)) \hookleftarrow F_\Sigma(F_\theta(A,D), F_\theta(B,C))$ with strict inclusion and

(iii) Neither $F_\theta(F_\cap(A,B), F_\cap(D,C))$ nor $F_\theta(F_\Sigma(A,B), F_\Sigma(D,C))$ are intermediate spaces for the pair $(F_\theta(A,D), F_\theta(B,C))$.

We continue by proving the assertions.

One has by (2.8) the identity

$$F_\theta(F_\cap(A,B), F_\cap(D,C)) = (L^1\cap L^2)^\theta \cdot (L^2 \cap L^1)^{\theta'} = L^{2\theta}\cdot L^{2\theta'} = L^2,$$

while

$$F_\theta(A,D)\cap F_\theta(B,C) = (L^1)^\theta\cdot (L^2)^{\theta'} \cap (L^2)^\theta \cdot (L^1)^{\theta'} = L^{2\frac{\theta}{\theta+1}} \cap L^{2\frac{\theta}{2\theta-1}}.$$

Since $\frac{\theta}{\theta+1} < 1$ and $\frac{\theta}{2\theta-1} < 1$ we have

$$F_\theta(F_\cap(A,B), F_\cap(D,C)) = L^2 \hookrightarrow L^{2\frac{\theta}{\theta+1}} \cap L^{2\frac{\theta}{2\theta-1}} = F_\theta(A,D)\cap F_\theta(B,C)$$

with strict inclusion and so $F_\theta(F_\cap(A,B), F_\cap(D,C))$ is not an intermediate space for the pair $(F_\theta(A,D), F_\theta(B,C))$. Notice that $F_\theta(A,D)\cap F_\theta(B,C) = F_\cap(F_\theta(A,D), F_\theta(B,C))$ and so we have shown that *(i)* holds. For the sum we have $F_\theta(F_\Sigma(A,B), F_\Sigma(D,C)) = L^1$ and $F_\theta(A,D) + F_\theta(B,C) = L^{2\frac{\theta}{\theta+1}} + L^{2\frac{\theta}{2\theta-1}}$. Since $\frac{\theta}{\theta+1} > \frac{1}{2}$ and $\frac{\theta}{2\theta-1} > \frac{1}{2}$ we have

$$F_\theta(F_\Sigma(A,B), F_\Sigma(D,C)) = L^1 \hookleftarrow L^{2\frac{\theta}{\theta+1}} + L^{2\frac{\theta}{2\theta-1}} = F_\theta(A,D) + F_\theta(B,C)$$

with strict inclusion and so $F_\theta(F_\Sigma(A,B), F_\Sigma(D,C))$ is not an intermediate space for the pair $(F_\theta(A,D), F_\theta(B,C))$. We have shown that *(iii)* holds and since

$$F_\theta(A,D) + F_\theta(B,C) = F_\Sigma(F_\theta(A,D), F_\theta(B,C)),$$

it follows that *(ii)* holds.

Notice that in the previous example, we have shown that the sum and intersection functor do not need to commute with F_θ. Note that by Proposition 3.7, it is necessary for $F_\theta(F_\cap(A,B), F_\cap(D,C))$ to be an interpolation space for the pair $(F_\theta(A,D), F_\theta(B,C))$, in order for F_θ to commute with $F_\cap$ on the square (A,B,C,D). We have proved that not only is this not the case, but the space we require to be an interpolation space is not even an intermediate space. Similarly for the sum functor F_Σ. This means we needed to break the necessary condition from Proposition 3.7 in a very strong fashion to obtain a counterexample.

We are unable to find following examples, hence it may be possible that they do not exist.

(1) A functor F and mutually compatible Banach function spaces (A, B, C, D), such that $F_\theta(F(A, B), F(D, C))$ is an intermediate but not an interpolation space for the pair $(F_\theta(A, D), F_{\theta'}(B, C))$.

(2) A functor F and mutually compatible Banach function spaces (A, B, C, D), such that $F_\theta(F(A, B), F(D, C))$ is an interpolation space for the pair $(F_\theta(A, D), F_\theta(B, C))$, but F_θ does not commute with F on the square (A, B, C, D).

Upon investigating commutativity properties of the Calderón-Lozanovskiĭ functor we arrive at the following chain of results.

Lemma 3.9. *Let Φ and Ψ be two invertible Young functions. If there exists $K > 0$ such that*

$$\Phi^{-1}\left(\frac{d}{c}\right)\Psi^{-1}\left(\frac{b\Phi^{-1}(\frac{a}{b})}{c\Phi^{-1}(\frac{d}{c})}\right) \leq K\Psi^{-1}\left(\frac{b}{c}\right)\Phi^{-1}\left(\frac{d\Psi^{-1}(\frac{a}{d})}{c\Psi^{-1}(\frac{b}{c})}\right) \quad \text{for all } a, b, c, d > 0, \tag{3.6}$$

then F_Ψ sub-commutes with F_Φ on every square (A, B, C, D) of mutually compatible Banach function spaces. If, in addition, there exists another constant $K' > 0$ such that one obtains the reverse inequality in (3.6) *upon replacing K with K', then F_Ψ and F_Φ are commutative on the category of all Banach function spaces.*

Proof. Let A, B, C, D be arbitrary mutually strongly compatible Banach function spaces and denote $I_0 = F_\Psi(F_\Phi(A, B), F_\Phi(D, C))$ and $I_1 = F_\Phi(F_\Psi(A, D), F_\Psi(B, C))$. Let $\frac{1}{2} > \varepsilon > 0$ and take $x \in (1 - 2\varepsilon)B_{I_0}$. From the definition of the Calderón-Lozanovskiĭ space (2.3) it follows that there exist $x_{AB} \in (1 - \varepsilon)B_{F_\Phi(A,B)}$, $x_{DC} \in (1 - \varepsilon)B_{F_\Phi(D,C)}$ such that

$$|x| \leq x_{DC}\Psi^{-1}\left(\frac{x_{AB}}{x_{DC}}\right). \tag{3.7}$$

For the same reason, we can find $x_A \in B_A$, $x_B \in B_B$, $x_C \in B_C$, $x_D \in B_D$, such that

$$x_{AB} \leq x_B\Phi^{-1}\left(\frac{x_A}{x_B}\right) \quad \text{and} \quad x_{DC} \leq x_C\Phi^{-1}\left(\frac{x_D}{x_C}\right). \tag{3.8}$$

Let Υ be an invertible Young function and $\alpha_i, \beta_i > 0$, $i = 0, 1$. We show that if $\alpha_0 \leq \alpha_1$ and $\beta_0 \leq \beta_1$, then

$$\beta_0\Upsilon^{-1}\left(\frac{\alpha_0}{\beta_0}\right) \leq \beta_1\Upsilon^{-1}\left(\frac{\alpha_1}{\beta_1}\right). \tag{3.9}$$

Carrying β_0 to the right hand side and applying Υ to both sides of the equations yields

$$\frac{\alpha_0}{\beta_0} \leq \Upsilon\left(\frac{\beta_1}{\beta_0}\Upsilon^{-1}\left(\frac{\alpha_1}{\beta_1}\right)\right).$$

The last equation holds since $\frac{\beta_1}{\beta_0} \geq 1$ and so

$$\Upsilon\left(\frac{\beta_1}{\beta_0}\Upsilon^{-1}\left(\frac{\alpha_1}{\beta_1}\right)\right) \geq \frac{\beta_1}{\beta_0}\Upsilon\left(\Upsilon^{-1}\left(\frac{\alpha_1}{\beta_1}\right)\right) = \frac{\alpha_1}{\beta_0} \geq \frac{\alpha_0}{\beta_0}.$$

Now (3.9) allows us to combine (3.7) and (3.8) to obtain

$$|x| \le x_C \Phi^{-1}\left(\frac{x_D}{x_C}\right) \Psi^{-1}\left(\frac{x_B \Phi^{-1}(\frac{x_A}{x_B})}{x_C \Phi^{-1}(\frac{x_D}{x_C})}\right). \tag{3.10}$$

Hence, by (3.10) and (3.6), we have

$$|x| \le K x_C \Psi^{-1}\left(\frac{x_B}{x_C}\right) \Phi^{-1}\left(\frac{x_D \Psi^{-1}(\frac{x_A}{x_D})}{x_C \Psi^{-1}(\frac{x_B}{x_C})}\right). \tag{3.11}$$

We can see that since $x_B \in B_B$ and $x_C \in B_C$, we have $x_{BC} = x_C \Psi^{-1}(\frac{x_B}{x_C}) \in B_{F_\Psi(B,C)}$ and similarly it holds that $x_{AD} = x_D \Psi^{-1}(\frac{x_A}{x_D}) \in B_{F_\Psi(A,D)}$. But then (3.11) reads as

$$|x| \le K x_{BC} \Phi^{-1}\left(\frac{x_{AD}}{x_{BC}}\right),$$

and so $x \in KB_{I_1}$. We have shown that $(1-2\varepsilon)B_{I_0} \subset KB_{I_1}$ holds for every $\frac{1}{2} > \varepsilon > 0$. It follows that

$$\|x\|_{I_1} \le K \tfrac{1}{1-2\varepsilon} \|x\|_{I_0}.$$

Sending ε to zero gives $I_0 \hookrightarrow I_1$ with the constant of embedding being less than or equal to K.

If (3.6) holds in reverse upon replacing K with some $K' > 0$ we get by the preceding that $I_1 \hookrightarrow I_0$ with the constant of embedding being less than or equal to $\frac{1}{K'}$. It follows therefore that $I_0 = I_1$. Since the choice of the Banach function spaces A, B, C, D was arbitrary, F_Φ and F_Ψ are commutative on the category of all Banach function spaces. □

Theorem 3.10. *Assume Φ is an invertible Young function and assume that $\varphi\colon [0,\infty] \to [0,\infty)$ is a non-increasing non-negative function satisfying*

$$\Phi^{-1}(x) = \int_0^x \varphi(t)\mathrm{d}t \quad \text{for } x \in [0,\infty].$$

Assume further that $\Phi^{-1}(1) = 1$. Consider the inequality

$$\Phi^{-1}(x)^{1-\frac{1}{\theta}} \Phi^{-1}(y)^{\frac{1}{\theta}} \le K \Phi^{-1}(x^{1-\frac{1}{\theta}} y^{\frac{1}{\theta}}) \quad \text{for all } x, y \in (0,\infty), \tag{3.12}$$

for some $\theta \in (1,\infty)$, and the following conditions.

(i) The function $\log \circ \Phi^{-1} \circ \exp$ is concave on $\mathbb{R}$.

(ii) The function $x \mapsto \frac{\varphi(x)}{\varphi^{**}(x)}$ is non-increasing on $(0,\infty)$.

Then (ii) $\Rightarrow$ (i) and (i) is equivalent to (3.12) being true for all $\theta \in (1,\infty)$ with $K = 1$. Finally, (3.12) implies that F_θ sub-commutes with F_Φ on all squares (A, B, C, D) of mutually compatible Banach function spaces.

Proof. It is a matter of a simple check that (3.12) is equivalent to (3.6) when one takes $\Psi = t \mapsto t^\theta$ and substitutes $\frac{d}{c}$ with x and $\frac{d}{b}$ with y. Hence (3.12) implies that F_θ subcommutes with F_Φ on all squares of mutually compatible Banach function spaces by virtue of Lemma 3.9.

We notice that applying log from outside and exp from inside on the inequality (provided $K = 1$) in (3.12) results in an equivalent inequality

$$\log(\Phi^{-1}(e^t)^{1-\frac{1}{\theta}}\Phi^{-1}(e^s)^{\frac{1}{\theta}}) \le \log(\Phi^{-1}((e^t)^{1-\frac{1}{\theta}}(e^s)^{\frac{1}{\theta}})) \quad \text{for all } s, t \in \mathbb{R}.$$

But, by virtue of the basic properties of the logarithmic and the exponential function, this is equivalent to

$$\left(1 - \frac{1}{\theta}\right) \log \circ \Phi^{-1} \circ \exp(t) + \frac{1}{\theta} \log \circ \Phi^{-1} \circ \exp(s) \le \log \circ \Phi^{-1} \circ \exp\left(\left(1 - \frac{1}{\theta}\right) t + \frac{1}{\theta} s\right),$$

which is precisely the concavity of $\log \circ \Phi^{-1} \circ \exp$.

It remains to show that *(ii)* implies *(i)*. Denote $f = \log \circ \Phi^{-1} \circ \exp$. Now f is a non-decreasing locally absolutely continuous function, therefore there exists a measurable function $f' \ge 0$ such that $f(x) = \int_0^x f'(t)\mathrm{d}t$, $x \in \mathbb{R}$. Indeed this is the case because $f(0) = 0$, since $\Phi^{-1}(1) = 1$. Moreover, if f' is non-increasing, then f is concave. We have for a.e. $t \in \mathbb{R}$

$$f'(t) = e^t \varphi(e^t) \frac{1}{\Phi^{-1}(e^t)}.$$

Since exp is increasing, it follows that f' is non-increasing if and only if the function $x \mapsto x\varphi(x)\frac{1}{\Phi^{-1}(x)} = \frac{\varphi(x)}{\varphi^{**}(x)}$ is non-increasing, which is precisely *(ii)*. This shows that *(i)* holds if *(ii)* holds.

□

Corollary 3.11. *Assume X_0, X_1 are Banach function spaces over (R, μ) and A, B, C, D are Banach function spaces over (S, ν). Assume further that S_1 and S_2 are linear mappings satisfying the endpoint estimates* (3.4). *Let $\theta \in (1, \infty)$ and Φ be an invertible Young function such that any of the two conditions in Theorem 3.10 is satisfied. Then*

$$T\colon F_\Phi(X_0, X_1) \to F_\Phi(A^\theta \cdot D^{\theta'}, B^\theta \cdot C^{\theta'}).$$

Proof. Theorem 3.2 gives

$$T\colon F_\Phi(X_0, X_1) \to (F_\Phi(A, B))^\theta \cdot (F_\Phi(D, C))^{\theta'} = F_\theta(F_\Phi(A, B), F_\Phi(D, C)).$$

Theorem 3.10 gives

$$F_\theta(F_\Phi(A, B), F_\Phi(D, C)) \hookrightarrow F_\Phi(F_\theta(A, D), F_\theta(B, C)) = F_\Phi(A^\theta \cdot D^{\theta'}, B^\theta \cdot C^{\theta'}).$$

□

Example 3.12. Assume $p, q, \alpha, \beta, \gamma, \delta \in [1, \infty]$ and

$$\begin{aligned} S_1 &\colon L^p(R,\mu) \to L^\alpha(S,\nu), \qquad & S_2 &\colon L^p(R,\mu) \to L^\delta(S,\nu), \\ S_1 &\colon L^q(R,\mu) \to L^\beta(S,\nu), \qquad & S_2 &\colon L^q(R,\mu) \to L^\gamma(S,\nu). \end{aligned}$$

Let $\theta \in (1, \infty)$ and $\eta \in (1, \infty)$. It holds that F_θ and F_η commute on all squares of mutually (strongly) compatible Lebesgue spaces. Denote

$$\begin{aligned} \tfrac{1}{r} &= \tfrac{1}{\eta}\tfrac{1}{p} + (1 - \tfrac{1}{\eta})\tfrac{1}{q} \\ \tfrac{1}{\xi_1} &= \tfrac{1}{\eta}\tfrac{1}{\alpha} + (1 - \tfrac{1}{\eta})\tfrac{1}{\beta} \\ \tfrac{1}{\xi_2} &= \tfrac{1}{\eta}\tfrac{1}{\delta} + (1 - \tfrac{1}{\eta})\tfrac{1}{\gamma} \\ \tfrac{1}{\xi} &= \tfrac{1}{\theta}\tfrac{1}{\xi_1} + (1 - \tfrac{1}{\theta})\tfrac{1}{\xi}. \end{aligned}$$

The following statements hold:

(i) $S_1 \colon L^r \to L^{\xi_1}$, $S_2 \colon L^r \to L^{\xi_2}$

(ii) $T \colon L^r \to L^\xi$

(iii) $F_\eta(F_\theta(L^\alpha, L^\delta), F_\theta(L^\beta, L^\gamma)) = F_\theta(F_\eta(L^\alpha, L^\beta), F_\eta(L^\delta, L^\gamma)) = F_\theta(L^{\xi_1}, L^{\xi_2}) = L^\xi$.

The first statement, *(i)*, is application of the Riesz-Thorin interpolation theorem. Theorem 3.2 implies $T \colon L^r \to F_\theta(F_\eta(L^\alpha, L^\beta), F_\eta(L^\delta, L^\gamma))$ and so *(iii)* gives *(ii)*. The statement *(iii)* may be easily checked by hand. Alternatively, one may realize that if one takes $\Phi = t \mapsto t^\eta$ and $\Psi = t \mapsto t^\theta$, equality is attained in (3.6) for $K = 1$. This implies that F_Φ sub-commutes with F_Ψ on all squares of mutually compatible Banach function spaces and F_Ψ sub-commutes with F_Φ on all squares of mutually compatible Banach function spaces. This means that F_Φ and F_Ψ are commutative on the category of all Banach function spaces. Hence, the first equality in *(iii)* holds. The other equalities are easily computed from (2.8).

In the last example, we have actually observed a far more general statement, which generalizes the first inequality in *(iii)* significantly. We present this result in the form of a proposition.

Proposition 3.13. *If $\theta, \eta \in [1, \infty]$ then F_θ and F_η commute in the category of all Banach function spaces.*

Of course all of these properties on Lebesgue spaces can be calculated by hand with only the knowledge of the Riesz-Thorin interpolation theorem and one does not require the robust theory we built in the first pages of this section. Let us a consider more interesting examples.

Example 3.14. If E, F, G are Young functions, we define the function $G(E, F)$ with the formula

$$(G(E,F))^{-1}(x) = \rho_G(E^{-1}(x), F^{-1}(x)), \quad \text{for } x \in [0, \infty].$$

Assume A, B, C, D, P, Q, Φ are invertible Young functions and S_1, S_2 are operators satisfying

$$\begin{aligned} S_1 &\colon L^P(R,\mu) \to L^A(S,\nu), \qquad & S_2 &\colon L^P(R,\mu) \to L^D(S,\nu), \\ S_1 &\colon L^Q(R,\mu) \to L^B(S,\nu), \qquad & S_2 &\colon L^Q(R,\mu) \to L^C(S,\nu). \end{aligned}$$

Assume further that Φ satisfies any of the conditions *(i)* or *(ii)* in Theorem 3.10. Denote $\Psi_\theta(t) = t^\theta$, $t \in [0, \infty]$. It holds that

$$T\colon L^{\Phi(P,Q)} \to L^{\Psi_\theta(\Phi(A,B),\Phi(D,C))} \hookrightarrow L^{\Phi(\Psi_\theta(A,D),\Psi_\theta(B,C))}.$$

This assertion follows immediately from Theorem 3.2, Theorem 3.10 and the fact that $F_G(L^E, L^F) = L^{G(E,F)}$, which is (2.9).

The last example illustrates somehow the power of sub-commutativity. Of course, not in the sense of obtaining a better target space but in the sense of obtaining a space which may be quite a bit easier to calculate.

What is however the main reason to use the sub-commutativity approach is something we call improvement of endpoints. This is well illustrated in [14], where a particular combination is considered. We shall introduce this in the next example, which will involve Lorentz spaces.

Example 3.15. Let $p, q \in (1, \infty)$, $p < q$, $\theta = p$. Assume further that the operators S_1 and S_2 satisfy the endpoint estimates

$$\begin{aligned} S_1 &\colon L^{p,1}(R,\mu) \to L^{p,1}(S,\nu), \qquad & S_2 &\colon L^{p,1}(R,\mu) \to L^{p,\infty}(S,\nu), \\ S_1 &\colon L^{q,1}(R,\mu) \to L^{\infty}(S,\nu), \qquad & S_2 &\colon L^{q,1}(R,\mu) \to L^{\infty}(S,\nu). \end{aligned}$$

Then $T\colon L^{p,1} \to L^p$ and $T\colon L^{q,1} \to L^\infty$. This is a consequence of (2.10). Furthermore, let Φ be an invertible Young function satisfying any of the conditions *(i)* or *(ii)* in Theorem 3.10. Define $\Phi^p(t) = (\Phi(t))^p$, $t \in [0, \infty]$ as per usual, then we have

$$T\colon F_\Phi(L^{p,1}(\mu), L^{q,1}(\mu)) \to L^{\Phi^p}(R,\mu). \tag{3.13}$$

Indeed from Theorem 3.2 we have

$$T\colon F_\Phi(L^{p,1}(\mu), L^{q,1}(\mu)) \to F_\theta(F_\Phi(L^{p,1}(\nu), L^\infty(\nu)), F_\Phi(L^{p,\infty}(\nu), L^\infty(\nu))). \tag{3.14}$$

But we assumed conditions necessary for F_θ to sub-commute with F_Φ on all squares of mutually strongly compatible Banach function spaces. Hence we have

$$\begin{aligned} F_\theta(F_\Phi(L^{p,1}(\nu), L^\infty(\nu)), &F_\Phi(L^{p,\infty}(\nu), L^\infty(\nu))) \\ &\hookrightarrow F_\Phi(F_\theta(L^{p,1}(\nu), L^{p,\infty}(\nu)), F_\theta(L^\infty(\nu), L^\infty(\nu))). \end{aligned}$$

Using (2.10), (2.9) and the fact that $\theta = p$, the right hand side of the last expression is equal to

$$F_\Phi(L^p(\nu), L^\infty(\nu)) = L^{\Phi^p}(\nu).$$

Furthermore, if $(R,\mu) = ((0,1),\lambda)$, the space on the left of (3.14) is explicitly known. Define

$$w(t) = t^{1-\frac{1}{q}} \frac{1}{\Phi^{-1}(t^{\frac{1}{q}-\frac{1}{p}})} \quad \text{for } t \in (0,1).$$

Then one has $F_\Phi(L^{p,1}, L^{q,1}) = \Lambda_w$, where the space on the right is the classical Lorentz space with the norm given by $\|f\|_{\Lambda_w} = \|f^* w\|_{L^1}$. This is a consequence of the results in [22, Proposition 1] and [21, Theorem 1]. However, a thorough explanation of this would require many definitions, hence we omit it. Note, however, that if we can calculate $F_\Phi(L^{p,1}(R,\mu), L^{q,1}(R,\mu))$ in case $(R,\mu) = ((0,1),\lambda)$ we can recall the Luxemburg representation theorem and calculate said space whenever (R,μ) is a fully non-atomic finite measure space.

Notice that since we only have sub-commutativity of the functors in question, the inclusion

$$F_\theta(F_\Phi(L^{p,1}(\nu), L^\infty(\nu)), F_\Phi(L^{p,\infty}(\nu), L^\infty(\nu))) \hookrightarrow L^{\Phi^p}$$

may be strict. This means that in (3.13), we might have lost some information, as we have increased the target space (with respect to the ordering by inclusion). On the other hand, the space on the right is clearly an Orlicz space built upon an explicitly given Young function.

While the last example illustrates well how one can improve the target spaces for T it does not deal with the problem of the domain spaces all that well. To improve the domain spaces one must consider interpolation functors that deal well with weak-type endpoint estimates. The simplest of these is the real interpolation functor.

Example 3.16. It holds that F_θ commutes with $M_{\lambda,r}$ on every square of mutually strongly compatible Lorentz $L^{p,q}$ spaces (with $p \in [1,\infty), q \in [1,\infty]$ or $p = q = \infty$) for every $\theta \in (1,\infty)$, $\lambda \in (0,1)$, $r \in [1,\infty]$. It is also necessary that first exponents of the spaces in question differ, which we put into more precise terms in the following. The spaces may be taken over an arbitrary non-atomic σ-finite measure space.

Consider four spaces $L^{\alpha,a}, L^{\beta,b}, L^{\gamma,c}, L^{\delta,d}$ with suitable exponents such that $\alpha \neq \beta$, $\delta \neq \gamma$ and

$$\frac{1}{\theta}\frac{1}{\alpha} + \frac{1}{\theta'}\frac{1}{\delta} \neq \frac{1}{\theta}\frac{1}{\beta} + \frac{1}{\theta'}\frac{1}{\gamma}.$$

Then by Proposition 2.5 we have $M_{\lambda,r}(L^{\alpha,a}, L^{\beta,b}) = L^{p_0,r}$, where $\frac{1}{p_0} = \frac{1-\lambda}{\alpha} + \frac{\lambda}{\beta}$. Similarly $M_{\lambda,r}(L^{\delta,d}, L^{\gamma,c}) = L^{p_1,r}$, where $\frac{1}{p_1} = \frac{1-\lambda}{\delta} + \frac{\lambda}{\gamma}$. Hence, by virtue of (2.10), it holds that

$$F_\theta(M_{\lambda,r}(L^{\alpha,a}, L^{\beta,b}), M_{\lambda,r}(L^{\delta,d}, L^{\gamma,c})) = L^{p,r}$$

where $\frac{1}{p} = \frac{1}{\theta}((1-\lambda)\frac{1}{\alpha} + \lambda\frac{1}{\beta}) + \frac{1}{\theta'}((1-\lambda)\frac{1}{\delta} + \lambda\frac{1}{\gamma})$.

On the other hand, we have $F_\theta(L^{\alpha,a}, L^{\delta,d}) = L^{q_0,s}$, $F_\theta(L^{\beta,b}, L^{\gamma,c}) = L^{q_1,s'}$, where $s, s' \in [1,\infty]$ (precise values of s', s are easy to calculate but immaterial) and $\frac{1}{q_0} = \frac{1}{\theta}\frac{1}{\alpha} + \frac{1}{\theta'}\frac{1}{\delta}$ and $\frac{1}{q_1} = \frac{1}{\theta}\frac{1}{\beta} + \frac{1}{\theta'}\frac{1}{\gamma}$. Hence

$$M_{\lambda,r}(F_\theta(L^{\alpha,a}, L^{\delta,d}), F_\theta(L^{\beta,b}, L^{\gamma,c})) = L^{q,r},$$

where $\frac{1}{q} = (1-\lambda)(\frac{1}{\theta}\frac{1}{\alpha} + \frac{1}{\theta'}\frac{1}{\delta}) + \lambda(\frac{1}{\theta}\frac{1}{\beta} + \frac{1}{\theta'}\frac{1}{\gamma})$. One only needs to show $p = q$ but this is easily seen.

This brings us to an interesting question, which we, sadly, do not know an answer to.

Question 3.17. *What are the necessary and sufficient conditions on Banach function spaces A, B, C, D such that F_θ commutes with the real interpolation functor on the square (A, B, C, D)?*

4. A method based on Calderón estimates

Let (R, μ) and (S, ν) be σ-finite non-atomic measure spaces, S_1 and S_2 quasi-linear mappings defined on some subspace $D(T)$ of $\mathcal{M}(\mu)$ and taking values in $\mathcal{M}(\nu)$. By T we denote the logarithmically convex combination of S_1 and S_2 with the exponent $\theta \in (1, \infty)$ as defined in (3.1).

We give a short summary of what our goal is in this chapter. Given some suitable Young function A, we wish to construct another Young function E such that $T\colon L^A(\mu) \to L^E(\nu)$ and E is as sharp as possible. All of this under the assumption that S_1 and S_2 satisfy reasonable weak-type endpoint estimates. We do this in four steps. First, we show that it is enough to show such boundedness when S_1 and S_2 are classical Calderón operators. Then we provide an interpolation technique which has been partially known and used for interpolating quasi-linear operators. Afterwards, we show some inequalities allowing us to apply this interpolation technique on our logarithmically convex combination T. Lastly we use all of this to prove the principal result for T.

Lemma 4.1. *Let $\tilde{S}_1$ and $\tilde{S}_2$ be mappings defined on $\mathcal{M}(0, \infty)$ taking values in $\mathcal{M}(0, \infty)$ with the property*

$$(S_1 f)^* \leq C\tilde{S}_1(f^*), \quad (S_2 f)^* \leq C\tilde{S}_2(f^*) \quad \textit{a.e. and for all } f \in D(T), \tag{4.1}$$

where $C > 0$ is independent of f. Let $F, G\colon [0, \infty] \to [0, \infty]$ be strictly increasing, continuous and onto. It holds that

$$([F \circ (S_1 f)] \cdot [G \circ (S_2 f)])^* \prec [F \circ (C\tilde{S}_1(f^*))] \cdot [G \circ (C\tilde{S}_2(f^*))] \quad \textit{for all } f \in D(T).$$

In particular if T is defined as in (3.1), it holds that

$$(Tf)^* \prec C(\tilde{S}_1(f^*))^{\frac{1}{\theta}}(\tilde{S}_2(f^*))^{\frac{1}{\theta'}} \quad \textit{for all } f \in D(T). \tag{4.2}$$

Proof. If $g \in \mathcal{M}_+(\nu)$ and $t \geq 0$, we have

$$\begin{aligned}(F \circ g)^*(t) &= \inf\{s \geq 0,\ |\{F \circ g > s\}| \leq t\} = \inf\{s \geq 0,\ |\{g > F^{-1}(s)\}| \leq t\} \\ &= \inf\{F(s) \geq 0,\ |\{g > s\}| \leq t\} = F(\inf\{s \geq 0,\ |\{g > s\}| \leq t\}) \\ &= F(g^*(t)).\end{aligned} \tag{4.3}$$

Let $f, g \in \mathcal{M}(\nu)$ and $t \geq 0$ and $\varepsilon > 0$. Since (S, ν) is non-atomic, we find $E \subset S$ measurable with $\nu(E) \leq t$ such that

$$-\varepsilon + \int_0^t (fg)^* \leq \int_E fg \,\mathrm{d}\nu.$$

This follows from [3, Chapter 2, Proposition 3.3]. Furthermore from the Hardy-Littlewood inequality we have

$$\int_E fg \,\mathrm{d}\nu \leq \int_0^{\nu(E)} f^* g^* \leq \int_0^t f^* g^*. \tag{4.4}$$

Sending $\varepsilon \to 0$ gives

$$(fg)^* \prec f^* g^*. \tag{4.5}$$

Hence we have

$$([F \circ (S_1 f)] \cdot [G \circ (S_2 f)])^* \prec (F \circ (S_1 f))^* \cdot (G \circ (S_2 f))^*$$

for any $f \in D(T)$. Moreover, by (4.3), we have

$$(F \circ (S_1 f))^* \cdot (G \circ (S_2 f))^* = (F \circ (S_1 f)^*) \cdot (G \circ (S_2 f)^*) \leq [F \circ (C\tilde{S}_1(f^*))] \cdot [G \circ (C\tilde{S}_2(f^*))],$$

and the assertion follows. The equation (4.2) now follows simply by the fact that $C^{\frac{1}{\theta}} C^{\frac{1}{\theta'}} = C$. □

Application of Lemma 4.1 provides us with the following.

Proposition 4.2. *Let $1 \leq p_1^j < p_2^j < \infty$, $q_i^j \in [1, \infty]$, $q_1^j \neq q_2^j$, $i, j = 1, 2$. Let S_1 be of weak types (p_1^1, q_1^1), (p_2^1, q_2^1) and let S_2 be of weak types (p_1^2, q_1^2), (p_2^2, q_2^2). Let $\tilde{S}_1$ and $\tilde{S}_2$ be Calderón operators associated with the segments given by $(p_1^1, q_1^1; p_2^1, q_2^1)$ and $(p_1^2, q_1^2; p_2^2, q_2^2)$ respectively. Then T given by* (3.1) *satisfies*

$$(Tf)^* \prec C(\tilde{S}_1(f^*))^{\frac{1}{\theta}} (\tilde{S}_2(f^*))^{\frac{1}{\theta'}} \quad \textit{for all } f \in D(T)$$

and some $C > 0$ independent of f. In particular, if $f \in D(T)$, X is a rearrangement-invariant Banach function space over (S, ν) and $\tilde{X}$ is its Luxemburg representation space over $(0, \nu(S))$, it holds that

$$\|Tf\|_X \leq \|(\tilde{S}_1(f^*))^{\frac{1}{\theta}} (\tilde{S}_2(f^*))^{\frac{1}{\theta'}}\|_{\tilde{X}(0,\nu(S))}.$$

In light of Proposition 4.2 we see that it is possible to restrict ourselves to logarithmically convex combinations of classical Calderón operators whenever S_1 and S_2 have classical weak endpoints. Let us therefore continue by studying exactly that. We shall stress that we are mainly interested in some kind of improvement of endpoints in the spirit of Example 3.15. We introduce an interpolation technique used for quasi-linear operators of precisely the same type as T has in that example, namely boundedness $L^{p,1} \to L^p$, $L^{q,1} \to L^\infty$, $p, q \in (1, \infty)$.

If $t \geq 0$ is given, we denote by f_t the function $f_t(s) = \min\{t, |f(s)|\}$ for $s \in (0, 1)$ and we also define $f^t = |f| - f_t$.

We denote by A a Young function which satisfies the following

$$\int_0 \left(\frac{\tau}{A(\tau)} \right)^{\frac{\alpha'}{\alpha}} \mathrm{d}\tau < \infty, \tag{4.6}$$

for some $\alpha \in (1, \infty)$ and

$$\int^\infty \left(\frac{\tau}{A(\tau)} \right)^{\frac{q'}{q}} \mathrm{d}\tau < \infty \tag{4.7}$$

for some $q \in (1, \infty)$. We denote its derivative a.

The main ingredient of our interpolation efforts is the following theorem, strength of which comes from the fact that it may be used for *arbitrary* mappings, for which no (quasi-)linearity or continuity is required. This has been known before at least to some degree. For example with $p = q = r$ this result may be implicitly found in the proof of [14, Theorem 4.14].

Theorem 4.3. *Let T be an arbitrary mapping defined on $B_{L^A(\mu)}(0, \frac{1}{2})$ and taking values in $\mathcal{M}(\nu)$, $p \in (0, \infty)$, $q \in (1, \infty)$, $r \in [1, \infty)$ $C > 0$ and $K > 0$. Let $\sigma\colon (0, \infty) \to (0, \infty)$ be non-decreasing. Assume that* (4.7) *holds and define*

$$F(t) = t^p \left(\int_{\sigma(t)}^{\infty} \left(\frac{\tau}{A(\tau)} \right)^{\frac{q'}{q}} \mathrm{d}\tau \right)^{-\frac{r}{q'}} \quad \text{for } t > 0 \tag{4.8}$$

and

$$E(t) = \int_0^t \frac{F(\tau)}{\tau} \mathrm{d}\tau \quad \text{for } t > 0. \tag{4.9}$$

If $p \geq 1$, then E is a Young function. If for all $f \in B_{L^A(\mu)}(0, \frac{1}{2})$ it holds that

$$\int_t^{\infty} \nu(\{|Tf| > K\tau\})\tau^{-1+p} \mathrm{d}\tau \leq C\|(2f)^{\sigma(t)}\|_{L^{q,1}(\mu)}^r \quad \text{for } t > 0, \tag{4.10}$$

then there exists a constant $C_1 > 0$ such that

$$\int_S E\left(\frac{|Tf|}{K}\right) \mathrm{d}\nu \leq C_1 \quad \text{for all } f \in B_{L^A(\mu)}(0, \tfrac{1}{2}),$$

In particular, if E is a Young function (e.g. if $p \geq 1$) then T maps $B_{L^A(\mu)}(0, \frac{1}{2})$ into a bounded set in the Orlicz space $L^E(\nu)$.

Proof. The fact that if $p \geq 1$, then E is a Young function, follows easily from the definition and the fact that σ is non-decreasing. Indeed in such a case $\frac{F(t)}{t}$ is clearly non-increasing, therefore E is convex and hence a Young function. We shall use freely the fact that

$$\|(2f)^{\sigma(t)}\|_{L^{q,1}(\mu)} \approx \int_{\sigma(t)}^{\infty} \mu(\{|2f| > \tau\})^{\frac{1}{q}} \mathrm{d}\tau$$

holds with a constant independent of f and t. Indeed a change of variables $\tau = s + \sigma(t)$ shows

$$\int_{\sigma(t)}^{\infty} \mu(\{|2f| > \tau\})^{\frac{1}{q}} \mathrm{d}\tau = \int_0^{\infty} \mu(\{|2f| > s + \sigma(t)\})^{\frac{1}{q}} \mathrm{d}s = \int_0^{\infty} \mu(\{|2f|^{\sigma(t)} > s\})^{\frac{1}{q}} \mathrm{d}s,$$

but the last expression is known to be (up to a constant) equal to $\|(2f)^{\sigma(t)}\|_{L^{q,1}(\mu)}$.

Take $f \in L^A$ such that

$$\int_R A(|2f|) \leq 1,$$

which is equivalent to $f \in B_{L^A(\mu)}(0, \frac{1}{2})$. We have by partial integration

$$\begin{aligned}
\int_S E\left(\frac{|Tf|}{K}\right) \mathrm{d}\nu &= \int_0^\infty \frac{F(t)}{t} \nu(\{|Tf| > Kt\}) \mathrm{d}t \\
&= \int_0^\infty \frac{F(t)}{t^p} \frac{\mathrm{d}}{\mathrm{d}t} \left(- \int_t^\infty \nu(\{|Tf| > K\tau\}) \tau^{-1+p} \mathrm{d}\tau \right) \mathrm{d}t \\
&= \left[- \frac{F(t)}{t^p} \int_t^\infty \nu(\{|Tf| > K\tau\}) \tau^{-1+p} \mathrm{d}\tau \right]_{t=0}^{t=\infty} \\
&+ \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{F(t)}{t^p} \right) \int_t^\infty \nu(\{|Tf| > K\tau\}) \tau^{-1+p} \mathrm{d}\tau \mathrm{d}t \\
&\leq \limsup_{t \to 0_+} \frac{F(t)}{t^p} \int_t^\infty \nu(\{|Tf| > K\tau\}) \tau^{-1+p} \mathrm{d}\tau \\
&+ \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{F(t)}{t^p} \right) \int_t^\infty \nu(\{|Tf| > K\tau\}) \tau^{-1+p} \mathrm{d}\tau \mathrm{d}t.
\end{aligned}$$

Let us denote the first term on the right side in the last expression by I and the second one by II. We have, by (4.10), for any $t > 0$

$$\begin{aligned}
\frac{F(t)}{t^p} \int_t^\infty \nu(\{|Tf| > K\tau\}) \tau^{-1+p} \mathrm{d}\tau &\lesssim \frac{F(t)}{t^p} \|(2f)^{\sigma(t)}\|_{L^{q,1}}^r \\
&\approx \frac{F(t)}{t^p} \left(\int_{\sigma(t)}^\infty \mu(\{|2f| > \tau\})^{\frac{1}{q}} \mathrm{d}\tau \right)^r.
\end{aligned} \tag{4.11}$$

Using Hölder's inequality and (4.8), we arrive at

$$\begin{aligned}
\left(\int_{\sigma(t)}^\infty \mu(\{|2f| > \tau\})^{\frac{1}{q}} \mathrm{d}\tau \right)^r &\leq \left(\int_{\sigma(t)}^\infty \frac{A(\tau)}{\tau} \mu(\{|2f| > \tau\}) \mathrm{d}\tau \right)^{\frac{r}{q}} \left(\int_{\sigma(t)}^\infty \left(\frac{\tau}{A(\tau)} \right)^{\frac{q'}{q}} \mathrm{d}\tau \right)^{\frac{r}{q'}} \\
&\leq \left(\int_R A(|2f|) \right)^{\frac{r}{q}} \frac{t^p}{F(t)},
\end{aligned}$$

where the last inequality is a consequence of $\frac{A(\tau)}{\tau} \leq a(\tau)$, $\tau > 0$. Combining this with (4.11) yields

$$\frac{F(t)}{t^p} \int_t^\infty \nu(\{|Tf| > Kt\}) \tau^{-1+p} \mathrm{d}\tau \lesssim \left(\int_R A(|2f|) \right)^{\frac{r}{q}} \leq 1.$$

The relations "$\lesssim$" and "$\approx$" hold up to constants independent of f and t, therefore we can take supremum over $t > 0$ and get $I \lesssim 1$ up to a constant independent of f. On the other hand, to estimate II, let us denote for $t > 0$

$$w(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{F(t)}{t^p} \right)^{\frac{1}{r}},$$

$$h(t) = |\{|2f| > t\}|^{\frac{1}{q}}$$

and

$$v(t) = \left(\frac{A(t)}{t}\right)^{\frac{1}{q}}.$$

Using (4.10) we obtain

$$II \lesssim \int_0^\infty w^r(t) \left(\int_{\sigma(t)}^\infty h(\tau)\mathrm{d}\tau\right)^r \mathrm{d}t = \|w(t)\int_{\sigma(t)}^\infty h(\tau)\mathrm{d}\tau\|_{L^r}^r \lesssim \|vh\|_{L^q}^r,$$

where the last inequality follows from [13, Lemma 1, (ii)]. Indeed, the sufficient (and necessary) condition for the last inequality holds as a consequence of

$$\|w\|_{L^r(0,s)}\|\tfrac{1}{v}\|_{L^{q'}(\sigma(s),\infty)} = \left(\frac{F(s)}{s^p}\right)^{\frac{1}{r}} \left(\int_{\sigma(s)}^\infty \left(\frac{\tau}{A(\tau)}\right)^{\frac{q'}{q}} \mathrm{d}\tau\right)^{\frac{1}{q'}} = 1$$

for all $s > 0$, where the last equality follows form (4.8). But

$$\|vh\|_{L^q}^r = \left(\int_0^\infty \frac{A(t)}{t}|\{|2f| > t\}|\mathrm{d}t\right)^{\frac{r}{q}} \leq \left(\int_R A(|2f|)\mathrm{d}\nu\right)^{\frac{r}{q}} \leq 1.$$

Hence $I + II \lesssim 2$ and therefore also

$$\int_S E\left(\frac{|Tf|}{K}\right)\mathrm{d}\nu \lesssim 2.$$

The relation "$\lesssim$" holds up to a constant independent of f, therefore there exists a constant $C_1 \geq 1$ such that

$$\int_S E\left(\frac{|Tf|}{K}\right)\mathrm{d}\nu \leq C_1 \quad \text{for all } f \in B_{L^A(\mu)}(0, \tfrac{1}{2}),$$

which implies that T maps $B_{L^A(\mu)}(0, \frac{1}{2})$ into $B_{L^E(\nu)}(0, KC_1)$ if E is a Young function. $\square$

We wish to apply Theorem 4.3 to the logarithmically convex combination. We present some technical lemmas which lead to inequalities in the spirit of (4.10). Given a non-decreasing left continuous function $H\colon [0,\infty] \to [0,\infty]$ we define its *left continuous inverse* $H^{-1}\colon [0,\infty] \to [0,\infty]$ by

$$H^{-1}(t) = \sup\{s \in [0,\infty],\ H(s) < t\}.$$

Note that the resulting function is non-decreasing and left continuous with the property that for each $t \in [0,\infty]$ one has $H(H^{-1}(t)) \leq t$.

Lemma 4.4. *Let $\alpha \in (1, \infty)$ and assume (4.6) holds. Let $\sigma = H^{-1}$ be the left continuous inverse of H, where H is the function defined by*

$$H(t) = \left(\int_0^t \left(\frac{\tau}{A(\tau)} \right)^{\frac{\alpha'}{\alpha}} \mathrm{d}\tau \right)^{\frac{1}{\alpha'}} \qquad \textit{for } t \in [0, \infty].$$

Then for any $f \in \mathcal{M}_+(0, \infty)$ and $t \in [0, \infty]$ we have

$$\|f_{\sigma(t)}\|_{L^{\alpha,1}} \leq \alpha t \left(\int_0^\infty A(f(s)) \mathrm{d}s \right)^{\frac{1}{\alpha}}. \tag{4.12}$$

The possible $0 \cdot \infty$ on the right side may be taken as 0. In particular, the mapping $f \mapsto f_{\sigma(t)}$ maps the unit ball in L^A into a ball of diameter αt in $L^{\alpha,1}$ for any $t \geq 0$.

Proof. Note that the left continuous inverse has the following property:

$$H(\sigma(t)) \leq t \quad \text{for } t \geq 0. \tag{4.13}$$

We have, by Hölder's inequality

$$\begin{aligned}
\|f_{\sigma(t)}\|_{L^{\alpha,1}} &= \alpha \int_0^{\sigma(t)} |\{f > \tau\}|^{\frac{1}{\alpha}} \mathrm{d}\tau \\
&\leq \alpha \int_0^{\sigma(t)} \left(\frac{\tau}{A(\tau)} \right)^{\frac{1}{\alpha}} \left(\frac{A(\tau)}{\tau} \right)^{\frac{1}{\alpha}} |\{f > \tau\}|^{\frac{1}{\alpha}} \mathrm{d}\tau \\
&\leq \alpha \left(\int_0^{\sigma(t)} \left(\frac{\tau}{A(\tau)} \right)^{\frac{\alpha'}{\alpha}} \right)^{\frac{1}{\alpha'}} \int_0^\infty \frac{A(\tau)}{\tau} |\{f > \tau\}| \mathrm{d}\tau \\
&\leq \alpha t \left(\int_0^\infty A(f(s)) \mathrm{d}s \right)^{\frac{1}{\alpha}},
\end{aligned}$$

where the last inequality is a consequence of (4.13) and the fact that $\frac{A(t)}{t} \leq a(t)$ for all $t \geq 0$. ☐

Given some (at this point arbitrary) real parameters $\alpha, \beta, \gamma \in (0, \infty]$, we define the operators

$$\begin{aligned}
\mathcal{U}_{\alpha,\beta,\gamma} f(s) &= s^{-\frac{1}{\beta}} \int_0^{s^\gamma} u^{-1+\frac{1}{\alpha}} f(u) \mathrm{d}u \\
\mathcal{D}_{\alpha,\beta\gamma} f(s) &= s^{-\frac{1}{\beta}} \int_{s^\gamma}^\infty u^{-1+\frac{1}{\alpha}} f(u) \mathrm{d}u
\end{aligned} \tag{4.14}$$

for $s \in [0, \infty]$ and $f \in \mathcal{M}(0, \infty)$ for which the right sides are well defined. Recall that we still have non-atomic measure spaces (R, μ) and (S, ν) fixed and our goal is to use Calderón estimates on operators S_1, S_2 acting on them. This means that we shall be working with expressions such as $\mathcal{U}(f^*)$, where $f \in \mathcal{M}(\mu)$. This means, of course, that if $\mu(R) < \infty$, the function f^* may be formally considered to either be defined on $[0, \mu(R)]$ or on $[0, \infty]$ but supported in $[0, \mu(R)]$. This may sometimes lead to some confusion, but, in our case, this does not happen. We can consider the operators $\mathcal{U}$ and $\mathcal{D}$ to operate on appropriate functions defined on the entire half-line and then apply Calderón estimates even in case of finite measure spaces.

Now we can deduce estimates in the spirit of (4.10) for the operators $\mathcal{U}$ and $\mathcal{D}$, defined in (4.14).

Lemma 4.5. *Let $\alpha_i, \beta_i \in [1, \infty]$, $\gamma_i \in [1, \infty)$, $i = 1, 2$, $\theta \in (1, \infty)$. Let $\alpha \in (1, \infty)$ and assume A satisfies* (4.6)*. Let σ be the function defined in Lemma 4.4. Let $q \in [1, \infty]$ and define*

$$B = -\frac{1}{\theta\beta_1} - \frac{1}{\theta'\beta_2} + \frac{\gamma_1}{\theta}\left(\frac{1}{\alpha_1} - \frac{1}{q}\right) + \frac{\gamma_2}{\theta'}\left(\frac{1}{\alpha_2} - \frac{1}{\alpha}\right).$$

Let f be a non-increasing non-negative function in the unit ball of $L^A(0, \infty)$. Then the following statements hold.

(i) If $q \le \alpha_1$ and $\alpha \le \alpha_2$ and $\frac{1}{B} < -p$, then

$$\int_t^\infty |\{(\mathcal{D}_{\alpha_1,\beta_1,\gamma_1} f^{\sigma(t)}(s))^{\frac{1}{\theta}} (\mathcal{D}_{\alpha_2,\beta_2,\gamma_2} f_{\sigma(t)}(s))^{\frac{1}{\theta'}} > \tau\}| \tau^{-1+p} \mathrm{d}\tau \lesssim \|f^{\sigma(t)}\|_{L^{q,1}}^{\frac{-1}{\theta B}} t^{\frac{1}{\theta B}+p}$$

for all $t \in (0, \infty)$.

(ii) If $q \ge \alpha_1$ and $\alpha \ge \alpha_2$ and $\frac{1}{B} < -p$, then

$$\int_t^\infty |\{(\mathcal{U}_{\alpha_1,\beta_1,\gamma_1} f^{\sigma(t)}(s))^{\frac{1}{\theta}} (\mathcal{U}_{\alpha_2,\beta_2,\gamma_2} f_{\sigma(t)}(s))^{\frac{1}{\theta'}} > \tau\}| \tau^{-1+p} \mathrm{d}\tau \lesssim \|f^{\sigma(t)}\|_{L^{q,1}}^{\frac{-1}{\theta B}} t^{\frac{1}{\theta B}+p}$$

for all $t \in (0, \infty)$.

(iii) If $q \ge \alpha_1$ and $\alpha \le \alpha_2$ and $\frac{1}{B} < -p$, then

$$\int_t^\infty |\{(\mathcal{U}_{\alpha_1,\beta_1,\gamma_1} f^{\sigma(t)}(s))^{\frac{1}{\theta}} (\mathcal{D}_{\alpha_2,\beta_2,\gamma_2} f_{\sigma(t)}(s))^{\frac{1}{\theta'}} > \tau\}| \tau^{-1+p} \mathrm{d}\tau \lesssim \|f^{\sigma(t)}\|_{L^{q,1}}^{\frac{-1}{\theta B}} t^{\frac{1}{\theta B}+p}$$

for all $t \in (0, \infty)$.

(iv) If $q \le \alpha_1$ and $\alpha \ge \alpha_2$ and $\frac{1}{B} < -p$, then

$$\int_t^\infty |\{(\mathcal{D}_{\alpha_1,\beta_1,\gamma_1} f^{\sigma(t)}(s))^{\frac{1}{\theta}} (\mathcal{U}_{\alpha_2,\beta_2,\gamma_2} f_{\sigma(t)}(s))^{\frac{1}{\theta'}} > \tau\}| \tau^{-1+p} \mathrm{d}\tau \lesssim \|f^{\sigma(t)}\|_{L^{q,1}}^{\frac{-1}{\theta B}} t^{\frac{1}{\theta B}+p}$$

for all $t \in (0, \infty)$.

In all of the inequalities the relation "$\lesssim$" holds up to a constant independent of f and t.

Proof. We begin by asserting that given B such that $\frac{1}{B} < -p$ and $t \in (0,\infty)$ and $K > 0$ it holds that

$$\int_t^\infty |\{Ks^B t^{\frac{1}{\theta'}} \|f^{\sigma(t)}\|_{L^{q,1}}^{\frac{1}{\theta}} > \tau\}|\tau^{-1+p}\mathrm{d}\tau \lesssim \|f^{\sigma(t)}\|_{L^{q,1}}^{\frac{-1}{\theta B}} t^{\frac{1}{\theta B}+p}. \tag{4.15}$$

Indeed, since $B < 0$, the left hand side is equal to

$$\int_t^\infty |\{s < K^{-\frac{1}{B}} \tau^{\frac{1}{B}} \|f^{\sigma(t)}\|_{L^{q,1}}^{-\frac{1}{\theta B}} t^{-\frac{1}{\theta' B}}\}|\tau^{-1+p}\mathrm{d}\tau = K^{-\frac{1}{B}} \|f^{\sigma(t)}\|_{L^{q,1}}^{-\frac{1}{\theta B}} t^{-\frac{1}{\theta' B}} \int_t^\infty \tau^{\frac{1}{B}-1+p}\mathrm{d}\tau.$$

Since $\frac{1}{B} < -p$, we may integrate and since $\frac{1}{B} + p - \frac{1}{\theta' B} = \frac{1}{\theta B} + p$, we obtain (4.15).

It remains to show that in each of the cases *(i)*-*(iv)* one may obtain the pointwise estimate on the logarithmically convex combination

$$(L_1 f^{\sigma(t)})^{\frac{1}{\theta}} (L_2 f_{\sigma(t)})^{\frac{1}{\theta'}} \leq K s^B t^{\frac{1}{\theta'}} \|f^{\sigma(t)}\|_{L^{q,1}}^{\frac{1}{\theta}} \quad \text{for all } s,t \in (0,\infty),$$

where $K > 0$ is some constant independent of f, s and t, and L_1, L_2 stand for the respective operators $\mathcal{U}$, $\mathcal{D}$. This is a consequence of the various inequalities binding q, α, α_i, $i = 1,2$ and it is done the exactly same way for each of the cases *(i)*-*(iv)*. We show for example (ii). Let $t, s \in [0,\infty]$. One has

$$\begin{aligned}
&(\mathcal{U}_{\alpha_1,\beta_1,\gamma_1} f^{\sigma(t)}(s))^{\frac{1}{\theta}} (\mathcal{U}_{\alpha_2,\beta_2,\gamma_2} f_{\sigma(t)}(s))^{\frac{1}{\theta'}} \\
&= s^{-\frac{1}{\theta\beta_1}-\frac{1}{\theta'\beta_2}} \left(\int_0^{s^{\gamma_1}} u^{-1+\frac{1}{q}+\frac{1}{\alpha_1}-\frac{1}{q}} f^{\sigma(t)}(u)\mathrm{d}u\right)^{\frac{1}{\theta}} \left(\int_0^{s^{\gamma_2}} u^{-1+\frac{1}{\alpha}+\frac{1}{\alpha_2}-\frac{1}{\alpha}} f_{\sigma(t)}(u)\mathrm{d}u\right)^{\frac{1}{\theta'}} \\
&\leq s^{-\frac{1}{\theta\beta_1}-\frac{1}{\theta'\beta_2}+\frac{\gamma_1}{\theta}(\frac{1}{\alpha_1}-\frac{1}{q})+\frac{\gamma_2}{\theta'}(\frac{1}{\alpha_2}-\frac{1}{\alpha})} \left(\int_0^{s^{\gamma_1}} u^{-1+\frac{1}{q}} f^{\sigma(t)}(u)\mathrm{d}u\right)^{\frac{1}{\theta}} \left(\int_0^{s^{\gamma_2}} u^{-1+\frac{1}{\alpha}} f_{\sigma(t)}(u)\mathrm{d}u\right)^{\frac{1}{\theta'}} \\
&\leq s^B \|f^{\sigma(t)}\|_{L^{q,1}}^{\frac{1}{\theta}} \|f_{\sigma(t)}\|_{L^{\alpha,1}}^{\frac{1}{\theta'}} \leq \alpha^{\frac{1}{\theta'}} s^B t^{\frac{1}{\theta'}} \|f^{\sigma(t)}\|_{L^{q,1}}^{\frac{1}{\theta}},
\end{aligned}$$

where the first inequality follows from $q \geq \alpha_1$, $\alpha \geq \alpha_2$ and the last inequality is merely the application of Lemma 4.4 and the fact that f is in the unit ball of $L^A(0,\infty)$. $\square$

Lemma 4.6. *Let $\alpha_i, \beta_i \in [1,\infty]$, $\gamma_i \in [1,\infty)$, $i = 1,2$, $\alpha_2 \neq \infty$, $q \in (1,\alpha_1]$, $q \neq \infty$, $p \in (0,\infty]$ and define*

$$D = -\frac{\theta'}{\gamma_1}\left(\frac{1}{\theta\beta_1} + \frac{1}{\theta'\beta_2}\right) + \frac{\gamma_2}{\gamma_1\alpha_2} + \frac{\theta'}{\theta}\left(\frac{1}{\alpha_1} - \frac{1}{q}\right) + \frac{\theta'}{\gamma_1 p}.$$

Assume further that $0 < D \leq 1$. Let $\sigma \geq 0$ and let f be a non-increasing non-negative function defined a.e. on $[0,\infty]$. Then

$$\int_0^\infty |\{(\mathcal{D}_{\alpha_1,\beta_1,\gamma_1} f^\sigma)^{\frac{1}{\theta}} (\mathcal{U}_{\alpha_2,\beta_2,\gamma_2} f_\sigma)^{\frac{1}{\theta'}} > \tau\}|\tau^{-1+p}\mathrm{d}\tau \lesssim \|(2f)^\sigma\|_{L^{\frac{1}{D},1}}^{\frac{p}{\theta'}} \|(2f)^\sigma\|_{L^{q,1}}^{\frac{p}{\theta}}.$$

Here the relation "$\lesssim$" holds up to a constant independent of f and σ. In particular, if $\frac{1}{q} = D$ it holds that

$$\int_0^\infty |\{(\mathcal{D}_{\alpha_1,\beta_1,\gamma_1} f^\sigma)^{\frac{1}{\theta}} (\mathcal{U}_{\alpha_2,\beta_2,\gamma_2} f_\sigma)^{\frac{1}{\theta'}} > \tau\}|\tau^{-1+p} \mathrm{d}\tau \lesssim \|(2f)^\sigma\|^p_{L^{q,1}}.$$

Proof. Let $s \in (0,\infty)$ and denote $\delta = |\{f > \sigma\}|$. If $\delta = \infty$, then also $\|(2f)^\sigma\|_{L^{q,1}} = \infty$ as $q \neq \infty$ and so there is nothing to prove. Assume therefore that $\delta < \infty$. Then we have

$$\begin{aligned}
&(\mathcal{D}_{\alpha_1,\beta_1,\gamma_1} f^\sigma(s))^{\frac{1}{\theta}} (\mathcal{U}_{\alpha_2,\beta_2,\gamma_2} f_\sigma(s))^{\frac{1}{\theta'}} \\
&= s^{-\frac{1}{\theta\beta_1}} \left(\int_{s^{\gamma_1}}^\infty f^\sigma(u) u^{-1+\frac{1}{\alpha_1}} \mathrm{d}u\right)^{\frac{1}{\theta}} s^{-\frac{1}{\theta'\beta_2}} \left(\int_0^{s^{\gamma_2}} f_\sigma(u) u^{-1+\frac{1}{\alpha_2}} \mathrm{d}u\right)^{\frac{1}{\theta'}} \\
&\lesssim \chi_{(0,\delta^{\frac{1}{\gamma_1}})}(s) s^{-\frac{1}{\theta\beta_1} - \frac{1}{\theta'\beta_2}} \left(\int_{s^{\gamma_1}}^\delta f^\sigma(u) u^{-1+\frac{1}{\alpha_1}} \mathrm{d}u\right)^{\frac{1}{\theta}} \left(\sigma s^{\frac{\gamma_2}{\alpha_2}}\right)^{\frac{1}{\theta'}}.
\end{aligned}$$

Since $q \leq \alpha_1$, we can assert that

$$\int_{s^{\gamma_1}}^\delta f^\sigma(u) u^{-1+\frac{1}{\alpha_1}} \mathrm{d}u = \int_{s^{\gamma_1}}^\delta f^\sigma(u) u^{-1+\frac{1}{q}+\frac{1}{\alpha_1}-\frac{1}{q}} \mathrm{d}u \leq s^{\gamma_1(\frac{1}{\alpha_1}-\frac{1}{q})} \int_{s^{\gamma_1}}^\delta f^\sigma(u) u^{-1+\frac{1}{q}} \mathrm{d}u.$$

Altogether we have

$$(\mathcal{D}_{\alpha_1,\beta_1,\gamma_1} f^\sigma(s))^{\frac{1}{\theta}} (\mathcal{U}_{\alpha_2,\beta_2,\gamma_2} f_\sigma(s))^{\frac{1}{\theta'}} \lesssim \chi_{(0,\delta^{\frac{1}{\gamma_1}})}(s) s^\eta \sigma^{\frac{1}{\theta'}} \|f^\sigma\|^{\frac{1}{\theta}}_{L^{q,1}},$$

where

$$\eta = -\frac{1}{\theta\beta_1} - \frac{1}{\theta'\beta_2} + \frac{\gamma_2}{\theta'\alpha_2} + \frac{\gamma_1}{\theta}\left(\frac{1}{\alpha_1} - \frac{1}{q}\right).$$

Note that $D = \frac{(\eta p+1)\theta'}{\gamma_1 p}$. Since we assumed $D > 0$, $p > 0$, it follows that $\eta p > -1$, therefore

$$\begin{aligned}
&\int_0^\infty |\{(\mathcal{D} f^\sigma)^{\frac{1}{\theta}} (\mathcal{U} f_\sigma)^{\frac{1}{\theta'}} > \tau\}|\tau^{-1+p} \mathrm{d}\tau \lesssim \|(\mathcal{D} f^\sigma)^{\frac{1}{\theta}} (\mathcal{U} f_\sigma)^{\frac{1}{\theta'}}\|^p_{L^p} \lesssim \|f^\sigma\|^{\frac{p}{\theta}}_{L^{q,1}} \sigma^{\frac{p}{\theta'}} \int_0^{\delta^{\frac{1}{\gamma_1}}} s^{\eta p} \mathrm{d}s \\
&\approx \|f^\sigma\|^{\frac{p}{\theta}}_{L^{q,1}} \sigma^{\frac{p}{\theta'}} \delta^{\frac{(\eta p+1)}{\gamma_1}} = \|f^\sigma\|^{\frac{p}{\theta}}_{L^{q,1}} \left(\sigma \delta^{\frac{\theta'(\eta p+1)}{p\gamma_1}}\right)^{\frac{p}{\theta'}} = \|f^\sigma\|^{\frac{p}{\theta}}_{L^{q,1}} \left(\sigma\delta^D\right)^{\frac{p}{\theta'}}.
\end{aligned}$$

Moreover, since for any $\tau \in (\frac{\sigma}{2}, \sigma)$ one has $\delta^D \leq |\{f > \tau\}|^D$, it holds that

$$\|f^\sigma\|^{\frac{p}{\theta}}_{L^{q,1}} \left(\sigma\delta^D\right)^{\frac{p}{\theta'}} \leq 2^{\frac{p}{\theta'}} \|f^\sigma\|^{\frac{p}{\theta}}_{L^{q,1}} \left(\int_{\frac{\sigma}{2}}^\sigma |\{f > \tau\}|^D \mathrm{d}\tau\right)^{\frac{p}{\theta'}} \lesssim \|(2f)^\sigma\|^{\frac{p}{\theta}}_{L^{q,1}} \|(2f)^\sigma\|^{\frac{p}{\theta'}}_{L^{\frac{1}{D},1}},$$

and the assertion of the lemma follows.

∎

Note that in the preceding lemma, σ is invariant under t, it just is an arbitrary non-negative number. Let us give an example of parameters for which the preceding lemma yields nice results. This will be of interest later, namely in Theorem 4.8. Consider $(\alpha, \infty, m) = (\alpha_1, \beta_1, \gamma_1)$, $(p, p, m) = (\alpha_2, \beta_2, \gamma_2)$, where $p \in (1, \infty)$, $\alpha \in (p, \infty)$ and m is the slope of the interpolation segment (p, p, α, ∞), more precisely

$$m = \frac{\frac{1}{p}}{\frac{1}{p} - \frac{1}{\alpha}}.$$

Assume further that $p = q$ and $\theta = p$. It then holds that

$$\begin{aligned} D &= \frac{-p'}{m}\left(\frac{1}{p\infty} + \frac{1}{p'p}\right) + \frac{1}{p} + \frac{p'}{p}\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{p'}{mp} \\ &= \frac{1}{m}\left(-\frac{1}{p} - \frac{1}{(p-1)p} + \frac{1}{p-1}\right) + \frac{1}{p} = \frac{1}{p} = \frac{1}{q}, \end{aligned}$$

and so the preceding lemma provides an estimate in the spirit of (4.10).

We are now prepared to prove the desired interpolation theorem which will allow us to deal efficiently with Example 3.15. But first, let us assert that had T been quasi-linear, our efforts would be much easier. Indeed obtaining estimates necessary to use Theorem 4.3 when T is quasi-linear with appropriate endpoints is rather easy.

Proposition 4.7. *Let $p, q \in [1, \infty)$, $\alpha \in (1, \infty)$, $q < \alpha$. Let S be a quasi-linear operator satisfying*

$$S\colon L^{q,1}(\mu) \to L^p(\nu); \quad S\colon L^{\alpha,1}(\mu) \to L^\infty(\nu).$$

Let the function σ be defined as in Lemma 4.4. Let A and E be as in Theorem 4.3 with $r = q$. Then $S\colon L^A(\mu) \to L^E(\nu)$.

Proof. We show that this holds if T is sub-linear and the norm in the endpoint estimates is less than or equal to 1. The modification for the general case is obvious. It suffices to show that the conditions of Theorem 4.3 are met. Let $K = \alpha$. Then we have for any function $f \in B_{L^A(\mu)}(0, \frac{1}{2})$ and any $t, s \in (0, \infty)$

$$|Sf_{\sigma(t)}(s)| \leq \|f_{\sigma(t)}\|_{L^{\alpha,1}(\mu)} \leq \alpha t = Kt.$$

This follows from the endpoint estimate $S\colon L^{\alpha,1} \to L^\infty$ and Lemma 4.4. Therefore

$$\int_t^\infty \nu(\{|Sf_{\sigma(t)}| > K\tau\})\tau^{-1+p}\mathrm{d}\tau = 0.$$

On the other hand, we have from the other endpoint estimate $S\colon L^{q,1} \to L^p$ that

$$\int_t^\infty \nu(\{|Sf^{\sigma(t)}| > K\tau\})\tau^{-1+p}\mathrm{d}\tau \leq \|Sf^{\sigma(t)}\|^p_{L^p(\nu)} \leq \|S\|^p_{L^{q,1}(\mu)\to L^p(\nu)}\|f^{\sigma(t)}\|^p_{L^{q,1}(\mu)}.$$

Now the result follows from sub-linearity of S as one has

$$\int_0^\infty \nu(\{|Sf| > 2K\tau\})\tau^{-1+p}\mathrm{d}\tau \leq \int_0^\infty \nu(\{|Sf|^{\sigma(t)} > K\tau\})\tau^{-1+p}\mathrm{d}\tau + \int_0^\infty \nu(\{|Sf|_{\sigma(t)} > K\tau\})\tau^{-1+p}\mathrm{d}\tau,$$

and so the estimate (4.10) holds with the constant $2K$. $\square$

We shall remark here that Orlicz-Orlicz type interpolation theory for operators satisfying endpoint estimates

$$S\colon L^{q,1}(\mu) \to L^{p,\infty}(\nu); \quad S\colon L^{\alpha,1}(\mu) \to L^\infty(\nu),$$

is well known. In fact, this problem is essentially completely solved by virtue of [13]. However, the improvement we gained by assuming the stronger condition $S\colon L^{q,1}(\mu) \to L^p(\nu)$ is rather non-trivial. In some sense, we are able to get much closer to the endpoint space $L^{q,1}$, which should not be surprising. To see the difference precisely, we recommend thoroughly consulting [13]. We now show that this improvement is still present even when dealing with the (non-linear) logarithmically convex combination T under a somewhat implicit assumption of a particular extrapolation property of the operators S_i. This property is hidden in the assumptions *(i)-(iii)* below and discussed later.

Theorem 4.8. *Assume that S_1 is of weak types (ξ, η) and (α, ∞) and that S_2 is of weak types (p, p) and (α, ∞), where $\xi, \eta, p, \alpha \in [1, \infty)$ and the following relations hold:*

(i) $p < \alpha$,

(ii) $\xi < p$,

(iii) a number $\lambda \in [0, 1]$ satisfies $\frac{\lambda}{\xi} + \frac{1-\lambda}{\alpha} = \frac{1}{p}$ if and only if it satisfies $\frac{\lambda}{\eta} + \frac{1-\lambda}{\infty} = \frac{1}{p}$.

Let $q = r = p$ and let A be a Young function satisfying (4.7) and (4.6), let E be the Young function defined by (4.9) and let T be defined by (3.1). Finally, let $\theta = p$. It holds that

$$T\colon L^A(\mu) \to L^E(\nu).$$

Proof. Throughout the proof, we shall omit writing out the measure space $(0, \infty)$ when dealing with function spaces over it, this means we write e.g. $L^{q,1}$ instead of $L^{q,1}(0, \infty)$. The proof is rather technical, though it relies only on basic observations and equalities between exponents. We shall start by listing these observations and equalities.

We shall be using the fact that T is positively homogenous, that is, for any $c > 0$ and $f \in D(T)$, one has $T(cf) = cT(f)$. We will also use the fact that for any non-negative numbers a, b one has $(a + b)^{\frac{1}{\theta}} \leq a^{\frac{1}{\theta}} + b^{\frac{1}{\theta}}$. Now we assert important identities for our exponents. Define m to be the slope of the interpolation segment associated with $(p, p; \alpha, \infty)$, that is

$$m = \frac{\frac{1}{p}}{\frac{1}{p} - \frac{1}{\alpha}}.$$

It holds that

$$m = \frac{\frac{1}{\eta}}{\frac{1}{\xi} - \frac{1}{\alpha}} \tag{4.16}$$

and

$$\tfrac{1}{\eta} - m\left(\tfrac{1}{\xi} - \tfrac{1}{p}\right) = \tfrac{1}{p}. \tag{4.17}$$

In fact, if *(i)* and *(ii)* hold, then *(iii)*, (4.16) and (4.17) are all mutually equivalent. This can be seen rather easily from the interpolation diagram (Figure 5) which we present after the proof, proving it by calculation is also easy. For (4.16) one has

$$\frac{\frac{1}{p}}{\frac{1}{p} - \frac{1}{\alpha}} = \frac{\frac{1}{p}}{\frac{\lambda}{\xi} + \frac{1-\lambda}{\alpha} - \frac{1}{\alpha}} = \frac{\frac{1}{p}}{\frac{\lambda}{\xi} - \frac{\lambda}{\alpha}} = \frac{\frac{1}{p}}{\frac{\eta}{p\xi} - \frac{\eta}{p\alpha}} = \frac{\frac{1}{\eta}}{\frac{1}{\xi} - \frac{1}{\alpha}}$$

and for (4.17) we express η from (4.16) and obtain that (4.17) is equivalent to

$$m\left(\frac{1}{\xi} - \frac{1}{\alpha} - \frac{1}{\xi} + \frac{1}{p}\right) = \frac{1}{p},$$

but this holds by the definition of m. We are now ready to prove the main statement. We have by the weak type estimates that

$$(S_1 f)^*(s) \lesssim s^{-\frac{1}{\eta}} \int_0^{s^m} u^{-1+\frac{1}{\xi}} f^*(u)\mathrm{d}u + s^{-\frac{1}{\infty}} \int_{s^m}^{\infty} u^{-1+\frac{1}{\alpha}} f^*(u)\mathrm{d}u \quad \text{for } f \in L^{\xi,1}(\mu) + L^{\alpha,1}(\mu)$$

and

$$(S_2 f)^*(s) \lesssim s^{-\frac{1}{p}} \int_0^{s^m} u^{-1+\frac{1}{p}} f^*(u)\mathrm{d}u + s^{-\frac{1}{\infty}} \int_{s^m}^{\infty} u^{-1+\frac{1}{\alpha}} f^*(u)\mathrm{d}u \quad \text{for } f \in L^{p,1}(\mu) + L^{\alpha,1}(\mu).$$

In our notation this is the same as

$$(S_1 f^*)(s) \lesssim (\mathcal{U}_{\xi,\eta,m} + \mathcal{D}_{\alpha,\infty,m})(f^*)(s)$$

and

$$(S_2 f^*)(s) \lesssim (\mathcal{U}_{p,p,m} + \mathcal{D}_{\alpha,\infty,m})(f^*)(s),$$

respectively. Recalling Lemma 4.1, we have a $C > 0$, independent of f and s, such that

$$\begin{aligned}
(Tf)^*(s) &\prec C\left[(\mathcal{U}_{\xi,\eta,m} + \mathcal{D}_{\alpha,\infty,m})(f^*)(s)\right]^{\frac{1}{\theta}} \left[(\mathcal{U}_{p,p,m} + \mathcal{D}_{\alpha,\infty,m})(f^*)(s)\right]^{\frac{1}{\theta'}} \\
&\lesssim \left[\mathcal{U}_{\xi,\eta,m}(f^*)(s)\right]^{\frac{1}{\theta}} \left[\mathcal{U}_{p,p,m}(f^*)(s)\right]^{\frac{1}{\theta'}} + \left[\mathcal{U}_{\xi,\eta,m}(f^*)(s)\right]^{\frac{1}{\theta}} \left[\mathcal{D}_{\alpha,\infty,m}(f^*)(s)\right]^{\frac{1}{\theta'}} \\
&\quad + \left[\mathcal{D}_{\alpha,\infty,m}(f^*)(s)\right]^{\frac{1}{\theta}} \left[\mathcal{U}_{p,p,m}(f^*)(s)\right]^{\frac{1}{\theta'}} + \left[\mathcal{D}_{\alpha,\infty,m}(f^*)(s)\right]^{\frac{1}{\theta}} \left[\mathcal{D}_{\alpha,\infty,m}(f^*)(s)\right]^{\frac{1}{\theta'}}
\end{aligned}$$

for each $f \in L^{p,1}(\mu) + L^{\alpha,1}(\mu)$ and a.e. $s \in [0, \infty]$. We deal with all of the operators on the right hand side separately. To that end, we denote

$$\begin{aligned}
T_1(f)(s) &= \left[\mathcal{U}_{\xi,\eta,m}(f)(s)\right]^{\frac{1}{\theta}} \left[\mathcal{U}_{p,p,m}(f)(s)\right]^{\frac{1}{\theta'}}; \\
T_2(f)(s) &= \left[\mathcal{U}_{\xi,\eta,m}(f)(s)\right]^{\frac{1}{\theta}} \left[\mathcal{D}_{\alpha,\infty,m}(f)(s)\right]^{\frac{1}{\theta'}}; \\
T_3(f)(s) &= \left[D_{\alpha,\infty,m}(f)(s)\right]^{\frac{1}{\theta}} \left[\mathcal{U}_{p,p,m}(f)(s)\right]^{\frac{1}{\theta'}}; \\
T_4(f)(s) &= \left[\mathcal{D}_{\alpha,\infty,m}(f)(s)\right]^{\frac{1}{\theta}} \left[\mathcal{D}_{\alpha,\infty,m}(f)(s)\right]^{\frac{1}{\theta'}},
\end{aligned}$$

for $f \in L^{p,1} + L^{L^{\alpha},1}$ and $s \in (0, \infty)$. It is important to notice that we consider the operators T_i, $i = 1 \dots 4$ to operate on measurable functions on the real half-line. It is our goal now to show that $T_i \colon L^A \to L^E$ for each $i = 1 \dots 4$. Then it is immediately clear (e.g. by Proposition 4.2) that $T \colon L^A(\mu) \to L^E(\nu)$.

Note that $\mathcal{D}_{\alpha,\infty,m}, \mathcal{U}_{\xi,\eta,m}$ are parts of the Calderón operator for the segment $(\eta, \xi; \alpha, \infty)$ and so they both satisfy boundedness

$$L^{\eta,1} \to L^{\xi,\infty}; \quad L^{\alpha,1} \to L^{\infty}$$

Therefore we have by Marcinkiewicz interpolation theorem and *(iii)* the boundedness

$$L^{p,1} \to L^{p,1} \hookrightarrow L^p; \quad L^{\alpha,1} \to L^{\infty}.$$

and since they are both linear we have by Proposition 4.7 that

$$\begin{aligned} &\mathcal{U}_{\xi,\eta,m} \colon L^A \to L^E, \\ &\mathcal{D}_{\alpha,\infty,m} \colon L^A \to L^E. \end{aligned}$$

Now, since $T_4(f) = \mathcal{D}_{\alpha,\infty,m}(f)$ for any $f \in L^{p,1}(\mu) + L^{\alpha,1}(\mu)$, we have also $T_4 \colon L^A \to L^E$. For T_2, we may use, for example, the Young inequality, to show

$$T_2(f)(s) \le \theta \mathcal{U}_{\xi,\eta,m}(f)(s) + \theta' \mathcal{D}_{\alpha,\infty,m}(f)(s),$$

for any $f \in L^{p,1}(\mu) + L^{\alpha,1}(\mu)$ and a.e. $s \in (0, \infty)$. Hence $T_2 \colon L^A \to L^E$.

Now we shall deal with the other two operators. This is somewhat more complicated. Let now $f \in L^{p,1} + L^{\alpha,1}$ and $t \in [0, \infty]$ be fixed. Assume also that $f \ge 0$ is non-increasing, that is $f = f^*$ a.e. For simplicity we write $\sigma = \sigma(t)$

$$\begin{aligned} T_1(f)(s) &= \left[\mathcal{U}_{\xi,\eta,m}(f^\sigma)(s) + \mathcal{U}_{\xi,\eta,m}(f_\sigma)(s)\right]^{\frac{1}{\theta}} \left[\mathcal{U}_{p,p,m}(f^\sigma)(s) + \mathcal{U}_{p,p,m}(f_\sigma)(s)\right]^{\frac{1}{\theta'}} \\ &\le \left[\mathcal{U}_{\xi,\eta,m}(f^\sigma)(s)\right]^{\frac{1}{\theta}} \left[\mathcal{U}_{p,p,m}(f^\sigma)(s)\right]^{\frac{1}{\theta'}} + \left[\mathcal{U}_{\xi,\eta,m}(f^\sigma)(s)\right]^{\frac{1}{\theta}} \left[\mathcal{U}_{p,p,m}(f_\sigma)(s)\right]^{\frac{1}{\theta'}} \\ &+ \left[\mathcal{U}_{\xi,\eta,m}(f_\sigma)(s)\right]^{\frac{1}{\theta}} \left[\mathcal{U}_{p,p,m}(f^\sigma)(s)\right]^{\frac{1}{\theta'}} + \left[\mathcal{U}_{\xi,\eta,m}(f_\sigma)(s)\right]^{\frac{1}{\theta}} \left[\mathcal{U}_{p,p,m}(f_\sigma)(s)\right]^{\frac{1}{\theta'}} \\ &= T_1(f^\sigma)(s) + T_1(f_\sigma)(s) + \left[\mathcal{U}_{\xi,\eta,m}(f^\sigma)(s)\right]^{\frac{1}{\theta}} \left[\mathcal{U}_{p,p,m}(f_\sigma)(s)\right]^{\frac{1}{\theta'}} \\ &+ \left[\mathcal{U}_{\xi,\eta,m}(f_\sigma)(s)\right]^{\frac{1}{\theta}} \left[\mathcal{U}_{p,p,m}(f^\sigma)(s)\right]^{\frac{1}{\theta'}}, \end{aligned}$$

for $s \in (0, \infty)$. Similarly it holds that

$$\begin{aligned} T_3(f)(s) \le T_3(f^\sigma)(s) + T_3(f_\sigma)(s) &+ \left[\mathcal{D}_{\alpha,\infty,m}(f^\sigma)(s)\right]^{\frac{1}{\theta}} \left[\mathcal{U}_{p,p,m}(f_\sigma)(s)\right]^{\frac{1}{\theta'}} \\ &+ \left[\mathcal{D}_{\alpha,\infty,m}(f_\sigma)(s)\right]^{\frac{1}{\theta}} \left[\mathcal{U}_{p,p,m}(f^\sigma)(s)\right]^{\frac{1}{\theta'}}, \end{aligned}$$

for $s \in (0, \infty)$. Note that each T_i satisfies $T_i \colon L^{p,1} \to L^p$. This is the case because

$$(L^{p,1})^\theta \cdot (L^{p,\infty})^{\theta'} = (L^{p,1})^p \cdot (L^{p,\infty})^{p'} = L^p$$

and each T_i is a logarithmically convex combination of operators out of which one is bounded from $L^{p,1}$ to $L^{p,1}$ and the other from $L^{p,1}$ to $L^{p,\infty}$. Similarly $T_i\colon L^{\alpha,1} \to L^{\infty}$. We continue with an argument nearly identical to that in the proof of Proposition 4.7. It follows from Lemma 4.4 that there is a constant $k > 0$ independent of f such that $|T_i f_\sigma(s)| \le k\|f_\sigma\|_{L^{\alpha,1}} \le k\alpha t$ for a.e. $s \in (0,\infty)$. Therefore one has for $K = \alpha k$ that

$$\int_t^\infty |\{T_i(f_\sigma)(s) > K\tau\}|\tau^{-1+p}\mathrm{d}\tau = 0. \tag{4.18}$$

On the other hand,

$$\int_t^\infty |\{T_i(f^\sigma) > \tau\}|\tau^{-1+p}\mathrm{d}\tau \lesssim \|T_i f^\sigma\|_{L^p}^p \lesssim \|f^\sigma\|_{L^{p,1}}^p. \tag{4.19}$$

Let $(\alpha_1,\beta_1,\gamma_1) = (\xi,\eta,m)$ and $(\alpha_2,\beta_2,\gamma_2) = (p,p,m)$. Then B from Lemma 4.5 satisfies

$$-\theta B = -pB = \frac{1}{\eta} - m\left(\frac{1}{\xi} - \frac{1}{p}\right) = \frac{1}{p},$$

since (4.17) holds. Thus *(ii)* of said lemma provides us with the estimate

$$\int_t^\infty |\{[\mathcal{U}_{\xi,\eta,m}(f^\sigma)(s)]^{\frac{1}{\theta}}\,[\mathcal{U}_{p,p,m}(f_\sigma)(s)]^{\frac{1}{\theta'}} > \tau\}|\tau^{-1+p}\mathrm{d}\tau \lesssim \|f^\sigma\|_{L^{q,1}}^p = \|f^\sigma\|_{L^{p,1}}^p. \tag{4.20}$$

On the other hand, choose $(\alpha_1,\beta_1,\gamma_1) = (p,p,m)$, $(\alpha_2,\beta_2,\gamma_2) = (\xi,\eta,m)$, denote for now $\Theta = \theta'$ and apply *(ii)* interchanging Θ and θ. We obtain

$$-\Theta B = -p'B = \frac{1}{p} + \frac{p'}{p\eta} - \frac{p'}{p}m\left(\frac{1}{\xi} - \frac{1}{\alpha}\right) = \frac{1}{p},$$

by (4.16). Thus we obtain

$$\int_t^\infty |\{[\mathcal{U}_{\xi,\eta,m}(f_\sigma)(s)]^{\frac{1}{\theta}}\,[\mathcal{U}_{p,p,m}(f^\sigma)(s)]^{\frac{1}{\theta'}} > \tau\}|\tau^{-1+p}\mathrm{d}\tau \lesssim \|f^\sigma\|_{L^{q,1}}^p = \|f^\sigma\|_{L^{p,1}}^p. \tag{4.21}$$

Combining (4.18), (4.19), (4.20), (4.21) together with Theorem 4.3 yields $T_1\colon L^A \to L^E$.

Finally, we deal with the operator T_3 in a similar way. Choosing $(\alpha_1,\beta_1,\gamma_1) = (\alpha,\infty,m)$ and $(\alpha_2,\beta_2,\gamma_2) = (p,p,m)$ gives for D from Lemma 4.6

$$\begin{aligned} D &= \frac{-p'}{m}\left(\frac{1}{p\infty} + \frac{1}{p'p}\right) + \frac{1}{p} + \frac{p'}{p}\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{p'}{mp} \\ &= \frac{1}{m}\left(-\frac{1}{p} - \frac{1}{(p-1)p} + \frac{1}{p-1}\right) + \frac{1}{p} = \frac{1}{p} = \frac{1}{q}, \end{aligned}$$

by definition of m, therefore the said lemma gives

$$\int_t^\infty |\{[\mathcal{D}_{\alpha,\infty,m}(f_\sigma)(s)]^{\frac{1}{\theta}}\,[\mathcal{U}_{p,p,m}(f^\sigma)(s)]^{\frac{1}{\theta'}} > \tau\}|\tau^{-1+p}\mathrm{d}\tau \lesssim \|(2f)^\sigma\|_{L^{p,1}}^p. \tag{4.22}$$

This even holds for any σ invariant under t, but for us, that is not necessary. Lastly choose $(\alpha_1, \beta_1, \gamma_1) = (p, p, m)$ and $(\alpha_2, \beta_2, \gamma_2) = (\alpha, \infty, m)$ and interchange once again θ and $\Theta = \theta'$. Then by definition

$$B = -\tfrac{1}{pp'}$$

and so $-\Theta B = -p'B = \frac{1}{p}$. Lemma 4.5 *(iii)* gives

$$\int_t^\infty |\{[\mathcal{D}_{\alpha,\infty,m}(f^\sigma)(s)]^{\frac{1}{\theta}} [\mathcal{U}_{p,p,m}(f_\sigma)(s)]^{\frac{1}{\theta'}} > \tau\}|\tau^{-1+p}\mathrm{d}\tau \lesssim \|f^\sigma\|^p_{L^{q,1}} = \|f^\sigma\|^p_{L^{p,1}}. \tag{4.23}$$

Combining (4.18), (4.19), (4.22), (4.23) together with Theorem 4.3 yields $T_3\colon L^A \to L^E$. We have shown that for each $i = 1 \dots 4$ it holds that $T_i\colon L^A \to L^E$, which concludes the proof. □

We continue with several remarks concerning the preceding theorem. Firstly, the conditions *(i)-(iii)* mean that the interpolation segment for S_2 is a part of the interpolation segment for S_1, but that the segment for S_1 is strictly longer. Hence one may interpolate onto the endpoint of the segment of S_2 using the Marcinkiewicz theorem and obtain $S_1\colon L^{p,1} \to L^{p,1}$ which ensures that $T\colon L^{p,1} \to L^p$. This is better seen on Figure 5.

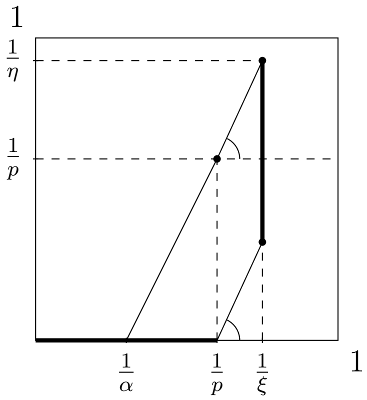

Figure 5: Extrapolation property of S_1

Here the line connecting the points $(\frac{1}{\alpha}, 0)$ and $(\frac{1}{\xi}, \frac{1}{\eta})$ is the interpolation segment for S_1. The part of this line from $(\frac{1}{\alpha}, 0)$ to $(\frac{1}{p}, \frac{1}{p})$ is the interpolation segment for S_2. We can see that the segment of S_1 is merely an extrapolation of the segment for S_2. Our approach unfortunately fails if the segment for S_1 *does not* continue past the point $(\frac{1}{p}, \frac{1}{p})$ for at least some distance. The only two angles that are noted are equal and their tangens is equal to m. We can see from this that (4.16) and (4.17) hold. Indeed the former is equality of the angles and the latter is the equality of lengths of the two thick lines. All of this follows from the conditions *(i)-(iii)*.

Note that we have, in essence, proved that the interpolation method given by Proposition 4.7 and Theorem 4.3 for quasi-linear operators extends in some sense to logarithmically convex combinations. One does not however assume that the combination T itself satisfies any endpoints, but instead that S_1 and S_2 satisfy endpoints which upon applying the F^θ functor from the third chapter give the correct endpoints for T. This can be seen as a more vague sort of commutativity on squares, which is discussed in the third chapter.

We shall remark that a Calderón operator for the endpoints

$$S_1 \colon L^{p,1} \to L^{p,1}; \quad S_1 \colon L^{\alpha,1} \to L^\infty$$

is, in general, not known. Some advancements have, however, been made, namely in [16] and [28]. We should also note that in these papers the Calderón operator does not dominate every operator of given types in the sense of (4.1), but in a weaker sense, description of which an intrigued reader may find in the respective papers. Hence, it is not entirely clear that a result analogous to Lemma 4.1 may be obtained. In other words, we do not know whether a general operator of given types may be replaced by the specific Calderón operator, when forming a logarithmically convex combination.

An interesting question connected to this arises. Given four measurable functions f_i, g_i, $i = 1, 2$ satisfying $f_i \prec g_i$, does it hold that

$$|f_1|^{\frac{1}{\theta}} |f_2|^{1-\frac{1}{\theta}} \prec |g_1|^{\frac{1}{\theta}} |g_2|^{1-\frac{1}{\theta}}$$

for any $\theta \in (1, \infty)$? Note that a positive answer to the question would yield a significant generalization of Lemma 4.1.

As our knowledge stands, we are forced to assume the extrapolation property given by *(i)-(iii)*, so that we may use the classical Calderón operator to replace the general operator S_1.

The method we used is very robust. Various interpolation techniques used for quasi-linear operators abuse the f^t, f_t decomposition and many of them work with endpoints that allow for Calderón type estimates. If one desires to somehow use such a technique for logarithmically convex combinations of operators, approach similar to that of ours is likely to yield positive results.

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