An opponent report on the Doctoral Thesis of Eva Burianńková "Operators related to Fourier Transform"

The subject of this very interesting work of Eva Buriánková finds its origin in the classical theory of singular integrals, nowadays called the theory of Calderón-Zygmund singular integral operators. It deals with operators of the form

$$f \to T_\Omega f(x) := \lim_{\varepsilon \to 0^+} T_\Omega^\varepsilon f := \lim_{\varepsilon \to 0^+} \int\limits_{\{y \in \mathbb{R}^n : |y| > \varepsilon\}} f(x-y)K(y)dy,$$

where the kernel $K(y) = |y|^{-n}\Omega(\frac{y}{|y|})$ is a homogeneous function of degree $-n$ satisfying the cancellation property on the unit sphere $\int\limits_{S^{n-1}} \Omega(\theta)d\theta = 0$, and the limit is understood in the appropriate sense (say, almost everywhere or in the L^p-norm). In their seminal papers in 50's, Calderón and Zygmund showed the L^p-boundedness, $1 < p < \infty$, and the weak-type $(1,1)$-bounds of T_Ω under certain smoothness assumptions on Ω, but pointed out that the results should hold for *rough* kernels (here "rough" stands for "not smooth"). In particular, they showed that under mere integrability of an *odd* Ω, T_Ω is bounded in all L^p, $1 < p < \infty$. Nevertheless, it is still open if the weak-type $(1,1)$-bounds of T_Ω holds without the smoothness of Ω, and if the maximal operator $f \to \sup\limits_{\varepsilon>0} |T_\Omega^\varepsilon f|$ is of weak type $(1,1)$, provided $\Omega \in L^\infty(S^{n-1})$.

The first part of the dissertation deals with the *bilinear rough* singular integrals

$$f, g \to T_\Omega(f,g)(x) := \lim_{\varepsilon \to 0^+} T_\Omega^\varepsilon(f,g)(x) :=$$

$$\lim_{\varepsilon \to 0^+} \int\limits_{\{y \in \mathbb{R}^n : |y| > \varepsilon\}} \int\limits_{\{z \in \mathbb{R}^n : |z| > \varepsilon\}} f(x-y)g(x-z)K(y,z)dydz,$$

where $K(y,z) = |(y,z)|^{-n}\Omega(\frac{(y,z)}{|(y,z)|})$ and $\Omega \in L^q(S^{2n-1})$, $1 \le q \le \infty$, has a vanishing integral. In the line of recent results of Grafakos, He and Honzík, it is shown that

$$\| \sup_{\varepsilon>0} |T_\Omega^\varepsilon(f,g)| \, \|_{L^p(\mathbb{R}^n)} \le C \, \|\Omega\|_{L^\infty(S^{2n-1})} \|f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)},$$

provided $\Omega \in L^\infty(S^{2n-1})$, $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and that

$$\| \sup_{\varepsilon>0} |T_\Omega^\varepsilon(f,g)| \, \|_{L^1(\mathbb{R}^n)} \le C \, \|\Omega\|_{L^2(S^{2n-1})} \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)},$$

provided $\Omega \in L^2(S^{2n-1})$. Then the main result is obtained via interpolation.

The second part of the dissertation is devoted to the lattice bump multiplier problem. More precisely, given a smooth bump ϕ which is supported in the ball of radius $\frac{1}{10}$ centered at the origin of $\mathbb{R}^n$, Buriánková considers the multiplier formed by adding the translations of this bump centered at N distinct lattice points of a set $E \subset \mathbb{Z}^n$,

$$L_{E,a,\phi} := \sum_{k \in E} a_k S_{k,\phi}(f),$$

where $a = \{a_k\}_{k \in \mathbb{Z}}$ is a sequence of complex numbers satisfying $|a_k| \leq 1$,

$$S_{k,\phi}(f)(x) = \int\limits_{\mathbb{R}^n} \widehat{f}(\xi)\phi(\xi - k)e^{2\pi i x \cdot \xi} d\xi,$$

$k \in \mathbb{Z}$, and $\widehat{f}$ stands for the Fourier transform of f in $\mathbb{R}^n$.

She attacks the bilinear version of the following problem: given $p \in [1, \infty]$, what is the smallest value $\alpha(p) \in \mathbb{R}$ such that for all subsets $E \subset \mathbb{Z}^n$ of N points and some constant $C = C_{p,n,\phi}$, one has $\|L_{E,a,\phi}(f)\|_{L^p(\mathbb{R}^n)} \leq C\, N^{\alpha(p)} \|f\|_{L^p(\mathbb{R}^n)}$?

In particular, it is proved that for a fixed smooth bump function Φ, which is supported in the ball of radius $\frac{1}{10}$ centered at the origin of $\mathbb{R}^{2n}$, and for a set $E \subset \mathbb{Z}^{2n}$, the estimate

$$\|B_{E,\Phi}(f,g)\|_{L^p(\mathbb{R}^n)} \leq C\, N^{\alpha(p_1,p_2)+\varepsilon}\, \|f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)},$$

with

$$B_{E,\Phi}(f,g)(x) = \sum_{(k,l) \subset E} S_{(k,l),\Phi}(f \otimes g)(x,x),$$

$1 \leq p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, is sharp up to $\varepsilon > 0$.

To conclude, I find that the results in this Thesis are very strong and I support the idea of going for the defense with my highest enthusiasm!

Sincerely yours,
Dmitry Ryabogin

D. A. Ryabogin