OPPONENT'S OPINION ON THE PHD THESIS "OPERATORS RELATED TO FOURIER TRANSFORM" BY EVA BURIÁNKOVÁ

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This thesis focuses on the study of various operators in harmonic analysis, with a particular emphasis on multilinear singular integral operators and Fourier multipliers. The thesis consists of an extensive introduction (around 30 pages) and two original papers. The first one, entitled "Rough maximal bilinear singular integrals", is a joint work of Eva Buriánková with Petr Honzík (her PhD advisor) and was published in the journal Collectanea Mathematica in 2019. The second paper is called "The lattice bump multiplier problem" and Eva wrote it jointly with Loukas Grafakos (University of Missouri, USA), Danqing He (Sun Yat-sen University, China) and Petr Honzík. This paper has been submitted for publication. In what follows I will provide a more detailed overview of each part of the thesis.

The introduction of the thesis begins with an overview of the history of the subject, starting with some classical and also more recent results from the linear theory of singular integrals and Fourier multipliers and continuing with the bilinear and multilinear theory. After that, the author describes in detail her own contribution to this subject as obtained in her two joint papers mentioned above. The introduction finishes with a statement and a detailed proof of an original result on multilinear Fourier multipliers that will be a part of an upcoming paper.

Overall, I think that the introduction of the thesis provides a good overview of what is known in the area, what has been left open and how the author contributed to solving some of these open problems. Nevertheless, I have also found a few inaccuracies and some misprints in the introduction; these are listed at the end of this report.

The first attached paper, entitled "Rough maximal bilinear singular integrals", builds on an earlier work of Grafakos, He and Honzík (Adv. Math. 2018), in which the authors used a novel method based on a wavelet decomposition to prove various $L^{p_1} \times L^{p_2} \to L^p$ bounds for bilinear singular integral operators with rough kernels. The present paper combines this technique with some new arguments to obtain analogous results for the maximal version of these operators.

The second attached paper, entitled "The lattice bump multiplier problem", is focused on the study of linear and (more importantly) bilinear Fourier multipliers whose symbols are obtained as a sum of a certain number N of translations of a given smooth bump. In the linear case, the authors obtain bounds for the L^p norm of the operator ($1 \leq p \leq \infty$) that are sharp as $N \to \infty$. In the substantially more complicated bilinear setting, $L^{p_1} \times L^{p_2} \to L^p$ bounds are established whenever $1 \leq p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$, and these estimates are shown to be essentially sharp whenever $p > 1$.

The extension of the previous result to the multilinear setting is still an open problem, but the first step towards achieving this goal has been made in a recent preprint by Grafakos, He, Honzík and Park, in which the $L^2 \times \cdots \times L^2 \to L^{2/m}$ boundedness of these operators has been studied. In the last section of the present thesis, the author extends this result to a larger range of exponents. In the bilinear setting, such an extension follows directly from the $L^2 \times L^2 \to L^1$ endpoint estimate by duality and interpolation. In the m-linear case, the duality argument cannot be used because the range space is L^q with $q < 1$. The author overcomes this difficulty by employing a new direct proof of the corresponding m-linear estimate.

In addition to the papers described above, Eva Buriánková is also the author of a joint paper with David Edmunds (University of Sussex, England) and Luboš Pick (supervisor of Eva's bachelor and master thesis). This paper is entitled "Optimal function spaces for the Laplace transform" and is based on Eva's research work prior to starting her PhD study. The paper was published in the journal Revista Matemática Complutense in 2017.

During the course of her PhD study, Eva has given talks and poster presentations about her research at seminars and conferences in Germany, Austria, Spain, USA and Russia. Her research was also partially supported by a grant from the Grant Agency of the Charles University.

Overall, I believe that Eva Buriánková has demostrated the ability to conduct independent mathematics research and I recommend that she is awarded the title Ph.D.

Some further comments on the introduction of the thesis:

- page 7, line 24: Danqing He should also be mentioned as a coauthor of the paper [9]. Further, the paper [9] does not address the case when $\Omega \in L^{4/3}(\mathbb{S}^{2n-1})$, it only deals with the situation when $\Omega \in L^r(\mathbb{S}^{2n-1})$ for $r > 4/3$. The same comment applies to page 15, lines 18–20.
- page 7, line -7: The negative sign is missing in the argument of the exponential in the definition of the Fourier transform.
- page 8, line 15: It is more customary to denote the Laplacian by Δ, rather than ∇.
- page 11, line 14: I think the range $p > 0$ is not correct, it should be $p > 1/2$ (in the basic case when $\Omega \in L^\infty(\mathbb{S}^{2n-1})$). Perhaps it should also be mentioned that when $\Omega \in L^q(\mathbb{S}^{2n-1})$ for some $q < \infty$ then the range of boundedness is smaller and depends on the value of q.
- page 12, line 17: For general j, the function is supported in $[2^{j-1}, 2^{j+1}]$, the interval $[1/2, 2]$ is only valid for $j = 0$.

• page 13, lines 9 – 10: I think the terminology is switched. I would expect the diagonal part to refer to the group of wavelets that are close to the axes and the off-diagonal part to those that are away from the axes.
• page 17, line 3: The support of the Fourier transform of T_m should appear here instead of the support of T_m itself.
• pages 24 and 25, statements of Theorem 13 and Theorem 14: The function spaces on the left-hand sides of the estimates are switched. In Theorem 13 it should be $L^{2/m}$ and in Theorem 14 it is $L^{2p/(2+p(m-1))}$.
• page 27, line 3: The power of N on the right-hand side should be $N^{\frac{m-1}{2m}+\varepsilon}$ instead of $N^{1/m+\varepsilon}$.
• page 31, reference [10]: I think the journal and year mentioned here are incorrect and refer to a different paper. The correct journal reference is Adv. Math. **326** (2018), 54–78.