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**DOCTORAL THESIS**

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# Operators related to Fourier transform

Department of Mathematical Analysis

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Název práce: Operátory související s Fourierovou transformací

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Abstrakt: Tato práce nabízí jak přehled známých výsledků v teorii singulárních integrálních operátorů a multilineárních multiplikátorů, tak i nové výsledky v této oblasti. První podrobně popsaný výsledek se týká omezenosti maximálních bilineárních singulárních integrálů s hrubým jádrem na prostorech $L^p$, tento výsledek je součástí prvního přiloženého článku. Stejné vlastnosti zkoumáme také v případě bilineárního operátoru, který je tvořen hladkými funkcemi s kompaktním nosičem. Tento výsledek lze najít ve druhém přiloženém článku. Poslední část práce je věnována multilineárnímu případu pro operátor stejného typu a dosažený výsledek je zde podrobně popsán a bude součástí připravovaného článku.

Klíčová slova: Singulární integrály Fourierovy multiplikátory Maximální operátory Hrubá jádra

Title: Operators related to Fourier transform

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Abstract: This manuscript provides an overview of known results in the theory of singular integral operators and multilinear multipliers as well as new results in this context. The first such result we describe is boundedness of rough maximal bilinear singular integrals on $L^p$ spaces which is a part of the first attached paper. The same problem is discussed for a special kind of bilinear operator consisting of smooth bumps, this result is a part of the attached paper number two. Finally we attack the multilinear case of the same type of operator and present a result which is a part of an upcoming paper.

Keywords: Singular integrals Fourier multipliers Maximal operators Rough kernels

# Contents

# Introduction

In this manuscript we provide three original results in the theory of multilinear singular integral operators and Fourier multipliers. We present a historical background on both topics starting with the linear theory of singular integrals, followed by the bilinear counterpart. Afterwards the linear, bilinear and multilinear multipliers are introduced. Our main concern is the boundedness of these operators on $L^p$ spaces, finding optimal results and stating necessary conditions. We provide a summary of the relevant known results in the linear case. The bilinear and the multilinear case is an active area of research with a long history and there is still quite a few gaps and open problems remaining to be solved.

In chapter two and three we attack some of these problems and discuss our results from the attached research papers. In the last chapter we present our original result from the upcoming paper, we give a detailed proof together with some possible applications.

# 1. Historical background and known results

## 1.1 Linear Singular Integrals

In this chapter we describe the linear singular integrals with smooth and rough kernel as well as the related maximal operators. We will provide historical background to this topic and state the essential results for boundedness.

The singular integrals we are interested in are the singular integrals of convolution type, the operators of the form

$$T_\Omega(f)(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\,dy,$$

where $\Omega$ is defined on the unit sphere in $\mathbb{R}^n$ and is integrable with mean value zero. This can be written using the p.v. notation as

$$T_\Omega(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\,dy.$$

This operator is initially defined for Schwartz functions. The basic properties of this operator were described by Calderón and Zygmund and therefore it is often called Calderón-Zygmund operator. The object $K(y) := \frac{\Omega(y/|y|)}{|y|^n}$ is called the kernel of the operator and whether this kernel does or does not posses certain smoothness we distinguish either singular integrals with smooth kernel or rough singular integrals.

The prototype of such operator is the Hilbert transform which was one of the first objects of this type to be studied and served as a motivation for further research in this area. We now take a closer look at this operator.

The Hilbert transform is defined on $\mathbb{R}$ as the singular integral with the kernel $K(y) = \frac{1}{\pi y}$. This operator plays an important role in complex analysis.

Given a Schwartz function $f$ we know that $f$ admits a unique harmonic extension to the upper half-plane given by convolution of $f$ with a Poisson kernel $P_t(x) = P(x,t) = \frac{1}{\pi}\frac{t}{x^2+t^2}$. Then we know there is a unique (up to a constant) harmonic function $g$ defined on the upper half-plane the so-called harmonic conjugate of $f$ such that $f+ig$ is holomorphic.

This harmonic function is obtained by taking convolution of $f$ with conjugate Poisson kernel $Q_t = \frac{1}{\pi}\frac{x}{t^2+x^2}$. Finally the Hilbert transform of a function $f$ is the limit of $v(x,t)$ as $t$ tends to 0, in other words

$$H(f)(x) = \lim_{t \to 0} Q_t * f.$$

For a wide class of functions this is equivalent to the following definition

$$H(f)(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\mathbb{R} \setminus B(0,\varepsilon)} \frac{f(x-y)}{y}\,dy.$$

It was shown by [18] that for $p$ in $(1,\infty)$ there exists a positive constant $C_p$ such that

$$\|H(f)\|_{L^p} \leq C_p \|f\|_{L^p}$$

for all $f$ Schwartz function on $\mathbb{R}$ and that the Hilbert transform admits an extension to a bounded operator on $L^p(\mathbb{R})$. However this fact does not describe $H(f)$ for all functions in $L^p(\mathbb{R})$. To get a notion of the formula we need to take a closer look at the maximal operator assigned to Hilbert transform the so called maximal Hilbert transform. The definition is as follows

$$H^*(f)(x) = \sup_{\varepsilon>0} \frac{1}{\pi} \left\| \int_{\mathbb{R}\setminus B(0,\varepsilon)} \frac{f(x-y)}{y} dy \right\|.$$

This quantity is defined for $f$ in $L^p(\mathbb{R})$, although for some values of $x$, $H^*(f)(x)$ may be infinite. The important result for this operator is that $H^*$ is $L^p$-bounded for $p$ in $(1,\infty)$ and weak type $(1,1)$. We do not provide a proof of this in this manuscript but we remark that the standard way to prove the $L^p$ boundedness, $p$ is in $(1,\infty)$, is to use the Cotlar's inequality which states that for a Schwartz function $f$ it holds

$$H^*(f)(x) \leq M(H(f))(x) + CM(f)(x),$$

where $M$ is the Hardy-Littlewood maximal function.

Using the boundedness of the maximal operator and the fact that the principal value exists almost everywhere on a dense subset of $L^p(\mathbb{R})$ namely the set of Schwartz functions it is easy to obtain almost everywhere existence of $H(f)(x)$ on $\mathbb{R}$ for a given $f$ in $L^p(\mathbb{R})$

With the help of Hilbert transform which is defined in one dimension we can now go back to general singular integrals in higher dimensions. If $\Omega$ is an odd integrable function on a unit sphere in $\mathbb{R}^n$, then we can obtain boundedness of the operator $T$ on $L^p(\mathbb{R}^n)$, $p$ in $(1,\infty)$. We do that by using the so called *method of rotations* on Hilbert transform. This method uses integration in polar coordinates to get directional operators in its radial parts. For example, given $\Omega$ in $L^1(\mathbb{S}^{n-1})$ we define the 'rough' maximal function

$$M_\Omega f(x) = \sup_{R>0} \frac{1}{|B(0,R)|} \int_{B(0,R)} |\Omega(y')||f(x-y)|\,dy.$$

Then we can rewrite this operator in polar coordinates and obtain

$$M_\Omega f(x) = \sup_{R>0} \frac{1}{|B(0,1)|R^n} \int_{\mathbb{S}^{n-1}} |\Omega(u)| \int_0^R |f(x-ru)| r^{n-1} dr d\sigma(u),$$

which can be estimated with $\frac{1}{|B(0,1)|} \int_{\mathbb{S}^{n-1}} |\Omega(u)| M_u f(x) d\sigma(u)$, where $M_u$ is the directional Hardy-Littlewood maximal function $M_u f(x) = \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(x-tu)|\,dt$, which is bounded on $L^p(\mathbb{R}^n)$.

Similarly if we have a general singular integral $T_\Omega$ still with $\Omega$ odd with zero average, then we can obtain

$$T_\Omega f(x) = \lim_{\varepsilon\to 0} \int_{\mathbb{S}^{n-1}} \Omega(u) \int_\varepsilon^\infty f(x-ru)\frac{dr}{r} d\sigma(u),$$

which can be estimated with $\frac{\pi}{2} \int_{\mathbb{S}^{n-1}} \Omega(u) H_u f(x) d\sigma(u)$ where $H_u$ is the directional Hilbert transform defined as $H_u f(x) = \frac{1}{\pi} \lim_{\varepsilon\to 0} \int_{|t|>\varepsilon} f(x-tu)\frac{dt}{t}$ and the Hilbert transform is strong $(p,p)$ for $p$ in $(1,\infty)$.

However, if the function $\Omega$ is even, then the method of rotations does not apply since we cannot represent the singular integral in terms of the Hilbert transform. But the result can be obtained nevertheless by decomposing the operator into a sum of odd operators of Riesz transforms.

For the sake of completeness we now take a closer look at even more general singular integrals. Let us denote the expression $\frac{\Omega(y/|y|)}{|y|^n}$ as $K_\Omega(y)$. Then the operator $T$ can be realized in terms of convolution with the kernel $K_\Omega$ as $T(f) = f * K_\Omega$.

Let $K$ now denote a general locally integrable function on $\mathbb{R}^n \setminus \{0\}$ such that it satisfies the 'size' condition

$$\|\hat{K}(x)\| \leq A,$$

and the 'smoothness' condition

$$\bigcup_{|x|>2|y|} |K(x-y) - K(x)|\, dx \leq B,\, y \in \mathbb{R}^n,$$

where $A$ and $B$ are constants. Then for $p$ in $(1,\infty)$ the operator $T$ is strong $(p,p)$ and weak $(1,1)$.

The smoothness condition is usually referred to as the Hörmander condition and is sometimes written in a stronger version as the gradient condition

$$|\nabla K(y)| \leq \frac{C}{|y|^{n+1}}.$$

Now if we come back yet again to the homogeneous kernel $K_\Omega$, we can ask what do we have to assume about $\Omega$ for the Hörmander condition to hold? The sufficient condition is of course that $\Omega$ is in $C^1(\mathbb{S}^{n-1})$ but much weaker conditions satisfy this purpose. Let us define

$$\omega_\infty(t) = \sup \}|\Omega(u_1) - \Omega(u_2)| : |u_1 - u_2| \leq t,\, u_1, u_2 \in \mathbb{S}^{n-1}| \; .$$

Then if we also have

$$\bigcup_0^1 \frac{\omega_\infty(t)}{t} dt < \infty$$

then the kernel $K_\Omega$ satisfies the Hörmander condition. Such a condition is referred to as a *Dini-type* condition.

Therefore to give a sufficient conclusion of the above mentioned facts we can say that if $\Omega$ is a function defined on $\mathbb{S}^{n-1}$ with zero integral and satisfies the Dini condition, then the operator $T(f) = f * K_\Omega$ is strong $(p,p)$ for $p$ in $(1,\infty)$ and weak $(1,1)$.

In this manuscript we are mainly interested in the singular integrals with rough kernels. The definition of this operator is similar to the previous but the key difference is in the assumptions on the kernel of the operator. We define

$$T_\Omega(f)(x) = \text{p.v.} \bigcup_{\mathbb{R}^n} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\, dy,$$

where $\Omega$ is in $L^1 \;\; \mathbb{S}^{n-1}\{$ with mean value zero. We are now interested in stating conditions for boundedness of the above mentioned operator without any kind of smoothness assumptions on the kernel such as Hörmander condition. We give an overview of the results on this topic which has been an active area of research since the middle of the twentieth century.

It is not difficult to show that with this setup and an extra assumption that $\Omega$ is odd, the operator $T_\Omega$ is bounded on $L^2(\mathbb{R}^n)$. This follows from the fact that the Fourier transform of the principal value of $\frac{\Omega(y/|y|)}{|y|^n}$ is bounded. Therefore with the help of Plancherel theorem we immediately obtain the estimate.

As already mentioned above, with this assumption we can obtain with the use of the *method of rotations* the boundedness of the operator $T_\Omega$ on $L^p(\mathbb{R}^n)$ for $p$ in $(1,\infty)$.

Now we continue dropping the extra assumption of $\Omega$ being odd. In 1956 Calderón and Zygmund showed in [1] that if $\Omega$ is in $L\log L$ $\mathbb{S}^{n-1}\{$, more precisely

$$\bigcup_{\mathbb{S}^{n-1}} |\Omega(x)|\ln(2+|\Omega(x)|)dx < \infty$$

then both $T_\Omega$ and $T_\Omega^*$ are bounded on $L^p(\mathbb{R}^n)$ for $p$ in $(1,\infty)$. Some years later this condition was independently improved by Connett in [6] and Coifman and Weiss in [5] who showed that $\Omega$ only needs to be in the Hardy space $H^1(\mathbb{S}^{n-1})$ which is also sufficient to imply that $T_\Omega^*$ is bounded on $L^p(\mathbb{R}^n)$ for $p$ in $(1,\infty)$.

Considering the weak type $(1,1)$ estimate, the question whether a condition bearing on the size of $\Omega$ alone sufficed for the weak type boundedness of $T_\Omega$ was a problem and the answer turned out to be positive. At first M. Christ and J.-L. Rubio de Francia proved the weak type $(1,1)$ estimate in one dimension for $\Omega$ in $L\log L(\mathbb{S}^1)$, see [7]. Later the same result was proved in all dimensions by A. Seeger in [20]. Nevertheless several questions remain concerning the endpoint behavior of $T_\Omega$ such as if the condition $\Omega \in L\log L(\mathbb{S}^{n-1})$ can be relaxed to $\Omega \in H^1(\mathbb{S}^{n-1})$, or merely $\Omega \in L(\mathbb{S}^{n-1})$ when $\Omega$ is an odd function.

Another question remains considering the $L^1$ theory of $T_\Omega^*$ for $\Omega$ rough. That is whether $T_\Omega^*$ is weak type $(1,1)$ if $\Omega$ does not posses any smoothness, more precisely if $\Omega$ is in $L^\infty(\mathbb{S}^{n-1})$.

## 1.2 Bilinear Singular Integrals

In this chapter we describe bilinear singular integrals with smooth and rough kernel and their behavior on function spaces in the similar fashion like the linear case in a previous chapter.

The setting is quite similar to the linear case. For $\Omega$ in $L^1(\mathbb{S}^{2n-1})$ with mean value zero, we at first define a kernel

$$K(y,z) = \Omega((y,z)/|(y,z)|)/|(y,z)|^{2n}.$$

Now we can define the bilinear singular integral associated with the kernel $K$ as

$$T_\Omega(f,g)(x) = \text{p.v.}\bigcup_{\mathbb{R}^n}\bigcup_{\mathbb{R}^n} K(x-y,x-z)f(y)g(z)\,dydz,$$

where $f$, $g$ are Schwartz functions on $\mathbb{R}^n$.

This type of integral was introduced by Coifman and Meyer in 1975, see [4]. They were also the first to obtain some results considering boundedness of this operator in one dimension. They showed that if $\Omega$ possesses some smoothness, i.e. if it is a function of bounded variation on the circle, then $T_\Omega$ is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$, where $p_1, p_2, p$ are in $(1,\infty)$ and $1/p = 1/p_1 + 1/p_2$.

In higher dimensions it was shown by Grafakos and Torres in [14] that if $\Omega$ is a Lipschitz function on $\mathbb{S}^{2n-1}$, then $T_\Omega$ is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$, where $p_1, p_2$ is in $(1,\infty)$, $p$ in $(1/2,\infty)$ and $1/p = 1/p_1 + 1/p_2$. Same authors also proved the Cotlar's type inequality for the maximal operator $T^*_\Omega$ with smooth kernel, see [13]. It is important to point out that Cotlar's inequality does not hold for rough kernels.

For $\Omega$ rough the boundedness of general $T_\Omega$ remained unresolved until the recent work of Grafakos, He and Honzík. Until then only certain special cases were known. For example the case in one dimension when $\Omega$ is integrable on $\mathbb{S}^1$ and odd function. Then the operator $T_\Omega$ is intimately connected with the directional bilinear Hilbert transform

$$H_{a,b}(f,g)(x) = \int_{-\infty}^{\infty} f(x-ta)g(x-tb)\frac{dt}{t}.$$

The boundedness of $H_{a,b}$ was proved by Lacey and Thiele, see [16], [17] and later addressed by Thiele [22], Grafakos and Li in [12].

We now state the result of Grafakos, He and Honzík from [10] but we will talk about their work more in detail in a separate chapter since the results of this manuscript use the same base, therefore are strongly connected and more explanation will be needed. The result is the following.

**Theorem 1.** *For all $n \geq 1$, if $\Omega$ is in $L^2(\mathbb{S}^{2n-1})$ with mean value zero, then for $T_\Omega$ it holds*

$$\|T_\Omega\|_{L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\to L^1(\mathbb{R}^n)} < \infty.$$

With the use of interpolation the final result of the paper is the boundedness of $T_\Omega$ on $L^p$ for all $p > 1/2$ in all dimensions. The theorem above was further improved by Grafakos and Slavíková to $\Omega$ in $L^{4/3}(\mathbb{S}^{2n-1})$, see [9], however the question of the best possible integrability of $\Omega$ still remains. This result will be mentioned again in the chapter about multipliers.

Having the boundedness of $T_\Omega$ a natural question of boundedness of $T^*_\Omega$ arise which is the main theme of our work.

## 1.3 Multipliers

In this section we introduce the theory of Fourier multipliers and state the important results since this topic is closely related to singular integrals. We will take a closer look on linear and bilinear cases and briefly on multilinear case as well. The linear Fourier multiplier operator has the form

$$T_m(f)(x) = \hat{f}m\{^{\vee}(x) = \int_{\mathbb{R}^n} \hat{f}(y)m(y)e^{2\pi i x\cdot y}dy,$$

where $m$ is a bounded function on $\mathbb{R}^n$, $\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{2\pi i x\cdot\xi}dx$ denotes the Fourier transform of $f$ and $g^{\vee}(\xi)$ denotes the inverse Fourier transform. The operator is initially defined on Schwartz functions and the question is what other conditions we have to impose on $m$ in order to $T_m$ admit a bounded extension on $L^p(\mathbb{R}^n)$ for $p$ in $[1,\infty]$, which is an old and important problem in harmonic analysis. If that is the case for a given $p$, then $m$ is called an $L^p$ *Fourier multiplier*. A basic example would be the multiplier of the Hilbert transform $m(t) = -i\mathrm{sgn}(t)$ as a multiplier on $L^p$.

By the Plancherel theorem $m$ is a $L^2$ Fourier multiplier if and only if $m$ is a $L^\infty$ function. The well known nontrivial result was provided by Mikhlin who proved that if the condition

$$|\partial^\alpha m(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}, \, \xi \neq 0,$$

holds for all multi-indices $\alpha$ with size $|\alpha| \leq [n/2]+1$, then $T_m$ admits a bounded extension on $L^p(\mathbb{R}^n)$ for $p$ in $(1,\infty)$.

This result is well suited for dealing with multipliers whose derivatives have a singularity at one point, such as functions which are homogeneous of degree zero and indefinitely differentiable on the unit sphere. An extension of Mikhlin's result was obtained by Hörmander who showed the same result replacing the Mikhlin's condition with a condition

$$\sup_{k\in\mathbb{Z}} 2^{-kn+2k|\alpha|} \int_{2^k<|y|<2^{k+1}} |\partial^\alpha m(y)|^2 \, dy < \infty.$$

This result was then further extended to fractional Sobolev space again by Hörmander as follows.

For $s>0$ let $(I-\nabla)^{s/2}$ denote the operator given on the Fourier transform side by multiplication by $(1+4\pi^2|\xi|^2)^{s/2}$ and let $\Psi$ be a Schwartz function whose Fourier transform is supported in the annulus $\{x \in \mathbb{R}^n : \, 1/2 < |x| < 2\}$ and which satisfies $\sum_{j\in\mathbb{Z}} \hat{\Psi}(2^{-j}\xi) = 1$ for all $\xi \neq 0$. If for some $r$ in $[1,2]$ and $s$ in $(n/r,\infty)$, the function $m$ satisfies

$$\sup_{k\in\mathbb{Z}} \left\| (I-\nabla)^{s/2} \left[ \hat{\Psi} m(2^k \cdot) \right] \right\|_{L^r(\mathbb{R}^n)} < \infty,$$

then $T_m$ admits a bounded extension on $L^p(\mathbb{R}^n)$ for $p$ in $(1,\infty)$. In the case that $r=2$ and $s$ positive integer we obtain the Hörmander condition from above.

Now it is natural to ask whether this condition still implies that $m$ is an $L^p$ Fourier multiplier for some $p$ in $(1,\infty)$ if $s<n/2$. The answer is positive, via an interpolation argument Calderon and Torchinsky showed that $T_m$ is bounded on $L^p$ whenever the condition holds for all $p$ satisfying $\left|\frac{1}{p}-\frac{1}{2}\right| < \frac{s}{n}$ and $\left|\frac{1}{p}-\frac{1}{2}\right| = \frac{1}{r}$. It was observed that the last assumption can be replaced with a weaker one $\frac{1}{r} < \frac{s}{n}$, moreover it is known that if $T_m$ is bounded on $L^p$ fore every $m$ satisfying the Hörmander condition, then $\left|\frac{1}{p}-\frac{1}{2}\right| \leq \frac{s}{n}$. In other words, when $rs>n$ then the condition $\left|\frac{1}{p}-\frac{1}{2}\right| < \frac{s}{n}$ is essentially optimal.

Additionally, Slavíková recently constructed an example to show that $L^p$ boundedness does not hold on the line $\left|\frac{1}{p}-\frac{1}{2}\right| = \frac{s}{n}$, see [21]. This means that conditions $\left|\frac{1}{p}-\frac{1}{2}\right| < \frac{s}{n}$ and $rs>n$ are optimal for the assumption of Hörmander condition.

Unlike the Mikhlin multiplier theorem, the Hörmander and Calderon-Torchinsky theorems can treat multipliers whose derivatives have infinitely many singularities.

Now let us move on to the bilinear setting. Bilinear multiplier operators are given by

$$T_m(f,g)(z) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} m(x,y) \hat{f}(x) \hat{g}(y) e^{2\pi z\cdot(x+y)} \, dx dy,$$

where $f,g$ are Schwartz functions and $m$ is a bounded function on $\mathbb{R}^{2n}$.

The question of $L^2 \times L^2 \to L^1$ boundedness (the central problem in this setting just as $L^2 \to L^2$ in the linear) is much more intricate due to the lack of Plancherel's theorem on $L^1$. Therefore no straightforward characterization is known.

The study of such operators was initiated by Coifman and Meyer with a classical sufficient condition for boundedness of $T_m$ which says that if $m$ satisfies

$$\left\|\partial^{\alpha}\partial^{\beta}m(x,y)\right\| \leq C_{\alpha,\beta}\,(|x|+|y|)^{-\alpha-\beta}$$

for sufficiently large multiindices $\alpha,\beta$, then the operator $T_m$ admits a bounded extension from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $1/p_1+1/p_2=1/p$ and $p_1,p_2$ are in $(1,\infty]$ and $p$ is in $(1/2,\infty)$. This was proved by Coifman and Meyer in the case $p>1$, see [2] and was extended to the case $p\leq 1$ by Grafakos and Torres in [14] and independently by Kenig and Stein in [15]. This theorem is essentially saying that linear Mikhlin multipliers on $\mathbb{R}^{2n}$ are bounded bilinear multipliers on $\mathbb{R}^n\times\mathbb{R}^n$.

We see that the previous results are quite restrictive in terms of the dependence of the rate of decay on the order of derivatives of $m$. There are, however, many other multipliers emerging naturally in the study of bilinear operators which do not fall under in the scope of the Coifman-Meyer condition such as the rough singular integrals which are the main interest of our research. Grafakos, He, Slavíková showed in [9] the boundedness for a type of operator as follows. Let $r_k$ be a bounded sequence, then we define

$$T=\sum_{k\in\mathbb{Z}} r_k T_{m_k},$$

where $m_k(x,y)=m(2^k(x,y))$ and $m$ is a function on $\mathbb{R}^{2n}$ which satisfies for some $\delta>0$ arbitrarily small that

$$|m(x,y)|\leq C\min\left(|(x,y)|,|(x,y)|^{-\delta}\right),$$

and

$$|\partial^{\alpha}m(x,y)|\leq C_{\alpha}\min\left(1,|(x,y)|^{-\delta}\right)$$

for all multiindices $\alpha$. Unlike the case of Coifman-Meyer conditions, the rate of decay does not depend on the order of derivatives. We note that such operators $T$ indeed include rough bilinear singular integrals with $m=\widetilde{K}\psi$, where $K$ is the kernel of a singular integral and $\psi$ is a smooth function supported in the unit annulus on $\mathbb{R}^{2n}$ satisfying $\sum_{j\in\mathbb{Z}}\psi(2^{-j}\cdot)=1$.

The result is as follows. Let $T$ be as above such that for $q$ in $[1,4)$ and $M_q=\max\left(2n,\left\lfloor\frac{2n}{4-q}\right\rfloor+1\right)$, $m$ is in $L^q(\mathbb{R}^{2n})\cap C^{M_q}(\mathbb{R}^{2n})$ satisfying conditions from above for all $|\alpha|\leq M_q$. Then

$$\|T\|_{L^{p_1}\times L^{p_2}\to L^p}<\infty,$$

whenever $1/p_1+1/p_2=1/p$ and $p_1,p_2$ are in $[2,\infty]$ and $p$ is in $[1,2]$.

Applying this result on rough bilinear singular integral we obtain an improvement of the main result from [10] as follows.

**Theorem 2.** *Let $r>4/3$ and assume that $\Omega$ is in $L^r(\mathbb{S}^{2n-1})$ with vanishing integral. Then $\|T\|_{L^{p_1}\times L^{p_2}\to L^p}<\infty$, whenever $1/p_1+1/p_2=1/p$ and $p_1,p_2$ are in $[2,\infty]$ and $p$ is in $[1,2]$.*

Now let us briefly mention the multilinear theory. The study of multilinear operators is not just a question of generalization of the quite well known area of linear operators but rather by their natural appearance in analysis. Coifman and Meyer were one of the first to adopt a multilinear point of view in their study of certain singular integral operators, such as the Calderón commutators, paraproducts, and pseudodifferential operators. An $n$-dimensional $m$-linear multiplier is a bounded function $\sigma$ on $(\mathbb{R}^n)^m$ associated with a $m$-linear operator $T_\sigma$ on $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ as follows

$$T_\sigma(f_1, \ldots, f_m)(x) = \bigcup_{(\mathbb{R}^n)^m} \hat{f}_1(y_1) \cdots \hat{f}_m(y_m) \sigma(y_1, \ldots, y_m) e^{2\pi i x \cdot (y_1 + \cdots + y_m)} dy_1 \cdots dy_m,$$

where $f_j$ for $j = 1, \ldots, m$ are Schwartz functions in $\mathbb{R}^n$. A classical result of Coifman and Meyer says that if for all sufficiently large multiindices $\alpha_1, \ldots, \alpha_m$ we have

$$\left\| \partial_{y_1}^{\alpha_1} \cdots \partial_{y_m}^{\alpha_m} \sigma(y_1, \ldots, y_m) \right\| \leq C(|y_1| + \cdots |y_m|)^{-(|\alpha_1| + \cdots |\alpha_m|)}$$

for all $(y_1, \ldots, y_m)$ in $(\mathbb{R}^n)^m$ away from the origin, then $T_\sigma$ admits a bounded extension from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^m)$ whenever $p_1, \ldots, p_m$ is in $(1, \infty]$, $1/p = 1/p_1 + 1/p_m$ and $p$ is in $[1, \infty)$. The extension of this theorem to indices $p > 1/m$ was simultaneously obtained by Kenig and Stein in [15] and Grafakos and Torres in [14]. This theorem provides an $m$-linear extension of Mikhlin classical linear multiplier result.

# 2. Singular Integrals

## 2.1 Rough Bilinear Singular Integrals

In this section we describe in detail the results from the paper Rough Bilinear Singular Integrals by L. Grafakos, P. Honzík and D. He, see [10] and we will also show the wavelet technique which is the significant method used in this paper.

The question was how to describe the boundedness of the bilinear singular integral operator associated with a function $\Omega$

$$T_\Omega(f,g)(x) = \text{p.v.} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x-y, x-z) f(y) g(z)\, dy dz,$$

where $f$, $g$ are Schwartz functions on $\mathbb{R}^n$, $K(y,z) = \Omega((y,z)/|(y,z)|)/|(y,z)|^{2n}$ and $\Omega$ is rough ( in other words there is no smoothness assumption ) such that $\Omega$ is in $L^q(\mathbb{S}^{2n-1})$ with $\int_{\mathbb{S}^{2n-1}} \Omega\, d\sigma = 0$, where $q$ is in $(1,\infty]$. This problem remained unresolved until this paper, except in situations when it reduces to the uniform boundedness of bilinear Hilbert transforms. In this work they provided a proof of the boundedness of $T_\Omega$ on $L^p$ for all $p > 0$ in all dimensions. This breakthrough is a consequence of the novel technique employed in this context which is the use of the wavelets. The key idea was to decompose the operator in terms of a tensor-type compactly supported wavelet decomposition and to use combinatorial arguments to group the different pieces together with the help of orthogonality.

Generally speaking the wavelet system is a type of orthonormal basis of $L^2(\mathbb{R}^n)$ generated by translations and dilations of a single function $\Psi$. Based on properties of this function we distinguish different types of wavelet systems such as The Haar wavelets in which the function $\chi_{[0,1/2)} - \chi_{[1/2,1)}$ is used or essential for our purposes The Daubechies wavelets using compactly supported functions. To be precise we need product type smooth wavelets with compact support. The construction of such objects can be found in [23] by Triebel. We will use wavelets formed in a following way.

For any fixed $k$ from $\mathbb{N}$ there exist real-valued compactly supported functions $\psi_F, \psi_M$ in $\mathcal{C}^k(\mathbb{R})$, which satisfy $\|\psi_F\|_{L^2(\mathbb{R})} = \|\psi_M\|_{L^2(\mathbb{R})} = 1$ and also that for $\alpha$ in $[0,k]$ we have that $\int_{\mathbb{R}} x^\alpha \psi_M(x) dx = 0$, and if $\Psi^G$ is defined by

$$\Psi^G(\vec{x}) = \psi_{G_1}(x_1) \cdots \psi_{G_{2n}}(x_{2n})$$

for $G = (G_1, \ldots, G_{2n})$ in the set

$$\mathcal{I} := \{(G_1, \ldots, G_{2n}) :\ G_i \in \{F, M\}\},$$

then the family of functions

$$\bigcup_{\vec{\mu} \in \mathbb{Z}^{2n}} \Big[ \{\Psi^{(F,\ldots,F)}(\vec{x} - \vec{\mu})\} \cup \bigcup_{\lambda=0}^{\infty} \{2^{\lambda n} \Psi^G(2^\lambda \vec{x} - \vec{\mu}) :\ G \in \mathcal{I} \setminus \{(F,\ldots,F)\}\} \Big]$$

forms an orthonormal basis of $L^2(\mathbb{R}^{2n})$, where $\vec{x} = (x_1, \ldots, x_{2n})$. The functions $\psi_F, \psi_M$ are called father and mother wavelets.

Using the notation

$$\Psi^{\lambda,G}_{\vec{\mu}}(\vec{x}) = 2^{\lambda n} \Psi^G(2^\lambda \vec{x} - \vec{\mu}), \qquad \vec{x} \in \mathbb{R}^{2n},$$

we utilize the wavelet transform on the kernel $K$ of the operator. More precisely we take the product wavelets described above, with compact supports and $M$ vanishing moments, where $M$ is a large number determined later on. ( To clarify, a function $f$ has $M$ vanishing moments if $\int_{\mathbb{R}^n} f(x)x^{\beta}\,dx = 0$ for $|\beta| \le M$ ). We have the following crucial estimate.

Assuming $m$ is in $\mathcal{C}^{M+1}(\mathbb{R}^{2n})$ such that

$$\sup_{|\alpha|\le M+1} \|\partial^{\alpha} m\|_{\infty} \le C_0 < \infty,$$

then we have

$$\left|\left\langle \Psi^{\lambda,G}_{\vec{\mu}}, m \right\rangle\right| \le CC_0 2^{-(M+1+n)\lambda},$$

provided that $\psi_M$ has $M$ vanishing moments.

This estimate will be used on decomposed parts of the kernel of the operator $T_{\Omega}$. Now we will describe this decomposition.

We fix a smooth function $\alpha$ in $\mathbb{R}^+$ such that $\alpha(t) = 1$ for $t$ in $(0,1]$, $\alpha(t)$ is in $(0,1)$ for $t$ in $(1,2)$ and $\alpha(t) = 0$ for $t$ in $[2,\infty)$. For $(y,z)$ in $\mathbb{R}^{2n}$ and $j$ in $\mathbb{Z}$ we introduce the function

$$\beta_j(y,z) = \alpha(2^{-j}|(y,z)|) - \alpha(2^{-j+1}|(y,z)|).$$

This is a function supported in $[1/2,2]$. We denote $\Delta_j$ the Littlewood-Paley operator $\Delta_j f = \mathcal{F}^{-1}(\beta_j \widehat{f})$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, which is defined via $\mathcal{F}^{-1}(g)(x) = \int_{\mathbb{R}^n} g(\xi)e^{2\pi i x\cdot\xi}\,d\xi = \widehat{g}(-x)$, where $\widehat{g}$ is the Fourier transform of $g$. Now we decompose the kernel $K$ as follows. We denote $K^i = \beta_i K$ and we set $K^i_j = \Delta_{j-i}K^i$ for $i,j$ in $\mathbb{Z}$. Then we write

$$K = \sum_{j=-\infty}^{\infty} K_j,$$

where

$$K_j = \sum_{i=-\infty}^{\infty} K^i_j.$$

We also denote $m_j = \widehat{K_j}$. Then the operator can be written as

$$T^*(f,g)(x) = \sup_{\varepsilon>0}\left|\sum_j \int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \int_{\mathbb{R}^n\setminus B(0,\varepsilon)} K_j(y,z)f(x-y)g(x-z)\,dydz\right|.$$

Now if we denote $m_{j,0}$ the symbol $\widehat{K^0_j}$ we can apply the above estimate and obtain

$$|\langle \Psi^{\lambda,G}_{\vec{\mu}}, m_{j,0}\rangle| \le C\|\Omega\|_{L^q} 2^{-\delta j} 2^{-(M+1+n)\lambda},$$

assuming $\Omega$ is in $L^q(\mathbb{S}^{2n-1})$, $q$ is in $[2,\infty]$, $\delta$ is in $(0,1/q')$.

We observe that we have the identity $m_{j,i} = m_{j,0}(2^i\cdot)$ from the homogeneity of the symbol. Therefore it is enough to show the estimate for $m_{j,0}$. With this setting we will now give the idea of the proof and how the above setup is used.

At first we organize the wavelet basis into groups so that members of the same group have disjoint supports and are of the same product type, in other words they have the same generation index $G$ in the set $\mathcal{I}$. Let us denote by $D_{\lambda,\kappa}$ one of these groups consisting of wavelets whose supports have diameters about $2^{-\lambda}$. So now we can write $m_{j,0}$ as

$$m_{j,0} = \sum_{\substack{\lambda \geq 0 \\ 1 \leq \kappa \leq C_{n,M,N}}} \sum_{\omega \in D_{\lambda,\kappa}} a_\omega \omega$$

and $\omega$ all have disjoint supports within the group $D_{\lambda,\kappa}$. Clearly, using the above estimate we have

$$\left\| \{a_\omega\}_{\omega \in D_{\lambda,\kappa}} \right\|_{l^\infty} \leq C \|\Omega\|_{L^q} 2^{-\delta j} 2^{-(M+1+n)\lambda} .$$

Now in order to obtain an estimate of the symbol we have to decompose $m_{j,0}$ again, within the group $D_{\lambda,\kappa}$. These groups are called 'off-diagonal' part since the wavelets are close to the axes in some sense and the 'diagonal' part with wavelets away from the axes. To be precise the wavelet is 'close' to the axis if

$$\mathrm{supp}\omega \cap \{(\xi_1,\xi_2) \in \mathbb{R}^n \times \mathbb{R}^n : \ 2^{-j}|\xi_1| \geq |\xi_2|\} \neq \emptyset$$

and is 'away' from the axis if

$$\mathrm{supp}\omega \subset \{(\xi_1,\xi_2) \in \mathbb{R}^n \times \mathbb{R}^n : \ 2^{-j}|\xi_1| \leq |\xi_2| \leq 2^j|\xi_1|\},$$

where $j$ is fixed and these groups are disjoint for large $j$.

First we comment on the diagonal part. This group is again further decomposed as follows. We fix $r$ in $[0,\infty)$ which will determine the 'size' of the wavelet within range approximately $(2^{-r-1}, 2^{-r})$, we split the group of wavelets of this size which we call $U_r$ in sets of 'heavy' columns, 'heavy' rows and the remainder. In other words fixing one coordinate, if the number of wavelets with this coordinate in the set $U_r$ is large enough we put them all in either heavy columns or heavy rows, depending on the coordinate. For the heavy parts this is the final decomposition and the operator can be estimated within these groups.

Using technical combinatorial arguments we can estimate the multiplier form of our operator in the set of large columns which we denote $T_{m_j^{r,1}}$ as follows

$$\left\| T_{m_j^{r,1}}(f,g) \right\|_{L^1} \leq C \|\Omega\|_{L^2} 2^{(n-M/8)\lambda - r/8 - \delta j/8} \|f\|_{L^2} \|g\|_{L^2},$$

which is a sufficient decay in $j$, $r$ and $\lambda$ if $M$ is larger than $16n$. The set of large rows is handled in the same way.

Now for the remainder the final decomposition of the operator is into at most $2^{(r+\delta j+M\lambda)}/2$ disjoint sets ( which is easy to show ) , indexed by $s$ where in each group the wavelet are such that they do not share the same coordinate. With the use of Cauchy-Schwartz inequality and Plancherel's theorem we get

$$\left\| T_{m_j^s}(f,g) \right\|_{L^1} \leq C 2^{-r} \|f\|_{L^2} \|g\|_{L^2}.$$

Then, summing over $s$ and choosing sufficient $\delta$ and then putting together the three estimates of 'heavy' columns, rows and the remainder we have the estimate

$$\left\| T_{m_j^r}(f,g) \right\|_{L^1} \leq C \|\Omega\|_{L^q} 2^{-\delta j/5} 2^{(-M\lambda - r)/16} \|f\|_{L^2} \|g\|_{L^2}.$$

Therefore for operator $m_{j,0}^1 = \sum_{r\geq 0}\sum_{\kappa}\sum_{\lambda} m_j^r$ which is one piece of the diagonal part of the operator we get

$$\|T_{m_{j,0}^1}(f,g)\|_{L^1} \leq C\|\Omega\|_{L^q}2^{-\delta j/5}\|f\|_{L^2}\|g\|_{L^2}.$$

In order to have the estimate for the entire diagonal part we need a similar estimate for $T_{m_{j,k}^1}$ which can be obtained on a restricted set, precisely the annulus $E_{j,k} = \{\xi_1 \in \mathbb{R}^n : c_1 2^{-k} \leq |\xi_1| \leq c_2 2^{j-k}\}$. Then using these estimates and applying Cauchy-Schwartz inequality we obtain for the diagonal part $m_j^1$ the estimate

$$\|T_{m_j^1}(f,g)\|_{L^1} \leq C\|\Omega\|_{L^q}2^{-\delta j/5}\|f\|_{L^2}\|g\|_{L^2},$$

which has a sufficient decay in $j$.

Now we describe the estimate for the off-diagonal part of the operator, namely $T_{m_j^2}$ and $T_{m_j^3}$. In order to control these two operators, we need the following inequality

$$\|(T_{m_j^2}(f,g) + T_{m_j^3})(f,g)\|_{L^1} \leq C\,\|\Omega\|_{L^q}\,2^{-j\delta}\,\|f\|_{L^2}\,\|g\|_{L^2}\,.$$

The first step to show this inequality is to show the square function type estimate

$$\Big\|\Big(\sum_{k\in 5\mathbb{Z}} |T_{m_{j,k}^2}(f,g)|^2\Big)^{1/2}\Big\|_{L^1} \leq C\|\Omega\|_{L^q}2^{-j\delta}\|f\|_{L^2}\|g\|_{L^2},$$

in other words

$$\Big\|\Big(\sum_{k\in 5\mathbb{Z}} |T_{m_{j,k}^2}(f,g)|^2\Big)^{1/2}\Big\|_{L^1} < \infty$$

which is rather technical and uses the tensor properties of the wavelet base splitting the operator in a product of two one dimensional part and estimating one with the Hardy-Littlewood maximal function and the preceding estimates for the other.

The next step is to show that there exists a polynomial $Q_1$ of $n$ variables such that $T_{m_j^2}(f,g) - Q_1$ is in $L^1$ which is a consequence of showing for $r = 0,1,2,3,4$ and $m_j^{(r)} = \sum_{k\in 5\mathbb{Z}+r} m_{j,k}^2$ there is a polynomial $Q_j^r$ such that

$$\|T_{m_j^{(r)}}(f,g) - Q_j^r\|_{L^1} \leq \Big\|\Big(\sum_{k\in 5\mathbb{Z}+r} |T_{m_{j,k}^2}(f,g)|^2\Big)^{1/2}\Big\|_{L^1}.$$

This concludes the rough description of dealing with estimating the off-diagonal part.

## 2.2 Rough Maximal Bilinear Singular Integrals

In this section we describe in detail our first main result of this thesis namely the paper Rough Maximal Bilinear Singular Integrals. The question of boundedness of these operators rose naturally after obtaining boundedness for the rough bilinear singular integrals which was described in the previous section. We remind the results known for this operator, namely

$$T_\Omega(f,g)(x) = \text{ p.v.} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\Omega((y,z)/|(y,z)|)}{|(y,z)|^{2n}} f(x-y)g(x-z)\,dydz$$

where $\Omega$ is a function in $L^q(\mathbb{S}^{2n-1})$ with vanishing integral is bounded for

- $q=\infty$ from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ where $p_1,p_2$ is in $(1,\infty)$ and $1/p = 1/p_1+1/p_2$,
- $q=2$ from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ and these two results can be interpolated.

Now we move our attention to the maximal operator. The setting is as follows. We define

$$T^*_\Omega(f,g)(x) = \sup_{\varepsilon>0}\left\|\int_{\mathbb{R}^n\setminus B(0,\varepsilon)}\int_{\mathbb{R}^n\setminus B(0,\varepsilon)}\frac{\Omega((y,z)/|(y,z)|)}{|(y,z)|^{2n}}f(x-y)g(x-z)dydz\right\|,$$

where $\Omega$ is a function in $L^\infty(\mathbb{S}^{2n-1})$ with vanishing integral. Our goal was to try to obtain similar results as for $T_\Omega$ therefore the problem was attacked either in the same way or the techniques were appropriately modified as far as the operator allowed and later on alternative ways were used to get the desired result.

Let us now state the main results.

**Theorem 3.** *For all $n\geq 1$, if $\Omega$ is in $L^\infty(\mathbb{S}^{2n-1})$, then for $T^*_\Omega$ defined above, we have*

$$\|T^*_\Omega\|_{L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \leq C\|\Omega\|_\infty \tag{2.1}$$

*whenever $1<p_1,\,p_2<\infty$ and $1/p=1/p_1+1/p_2$.*

For more general $\Omega$, we get the following:

**Theorem 4.** *For all $n\geq 1$, if $\Omega$ is in $L^2(\mathbb{S}^{2n-1})$, then for $T^*_\Omega$ defined above, we have*

$$\|T^*_\Omega\|_{L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\to L^1(\mathbb{R}^n)} \leq C\|\Omega\|_2. \tag{2.2}$$

We note that Grafakos and Slavíková recently improved the range of boundedness of $T_\Omega$ in [9]. They showed that in the case of $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ boundedness we can assume $\Omega$ in $L^{\frac{4}{3}}\left(\mathbb{S}^{2n-1}\right)$. It is not currently known whether similar result holds for $T^*_\Omega$ as well.

Now we will describe the essential steps of obtaining this result, point out similarities and mainly differences between $T_\Omega$ and $T^*_\Omega$.

At first we will define the operator $T^\#_\Omega$ which will be the main tool we will be working with in the following. The essential property of this operator is that we can equivalently use it instead of the operator $T^*_\Omega$. For functions in the Schwartz class let us define

$$T^\#(f,g)(x) = \sup_{j\in\mathbb{Z}}\left\|\sum_{i>j}\int_{\mathbb{R}^{2n}}K^i(y,z)\,f(x-y)\,g(x-z)\,dydz\right\|.$$

Let us assume for now that the operator $T^*$ is dominated in some sense by $T^\#$ ( the proof of this step is simple for $\Omega$ in $L^\infty(\mathbb{S}^{2n-1})$ but we will provide a more general result later on ). Then it is obviously enough to demonstrate the relevancy of the above mentioned theorems for the operator $T^\#$.

In order to estimate $T^\#$, we split the operator in two parts. Let us define

$$T^\#_j(f,g)(x) = \sup_{k\in\mathbb{Z}}\left\|\sum_{i>k}\int_{\mathbb{R}^{2n}}K^i_j(y,z)\,f(x-y)\,g(x-z)\,dydz\right\|,$$

and

$$T_{\lfloor}^{\#}(f,g)(x) = \sup_{j\in\mathbb{Z}} \left\| \sum_{i>j} \bigcup_{\mathbb{R}^{2n}} \sum_{\gamma<0} K_{\gamma}^{i}(y,z) f(x-y) g(x-z)\, dydz \right\|.$$

Then we can easily get

$$T^{\#}(f,g)(x) \leq T_{\lfloor}^{\#}(f,g)(x) + \sum_{j\geq 0} T_{j}^{\#}(f,g)(x),$$

therefore it is enough to estimate these two segments. What we need is that for $\Omega$ in $L^2(\mathbb{S}^{2n-1})$ and $j \geq 0$ the operators $T_{\lfloor}^{\#}$ and $T_j^{\#}$ are bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.

The first part is included in Proposition 7. This result is quite straightforward. It is easy to check that the maximal operator with the kernel $\sum_{i\in\mathbb{Z}} \sum_{\gamma<0} K_{\gamma}^{i}$ ( we point out that this kernel includes all of the part indexed by $i$, not only those bigger than index $j$ ) is bounded since this kernel is smooth Calderón-Zygmund convolution kernel. If we denote this operator as $T_{\tilde{K}}^{*}$, then we can write

$$\begin{aligned} T_{\lfloor}^{\#}(f,g)(x) &= \sup_{j\in\mathbb{Z}} \left\| \sum_{i>j} \bigcup_{\mathbb{R}^{2n}} \sum_{\gamma<0} K_{\gamma}^{i}(y,z) f(x-y) g(x-z)\, dydz \right\| \\ &\leq T_{K_{\lfloor}}^{*}(f,g)(x) + \\ &\sup_{j\in\mathbb{Z}} \left\| \bigcup_{\mathbb{R}^{2n}} \right) \chi_{\mathbb{R}^{2n}\setminus B(0,2^j)} K_{\lfloor} - \sum_{i>j} \sum_{\gamma<0} K_{\gamma}^{i} \Big( (y,z) f(x-y) g(x-z)\, dydz \right\|. \end{aligned}$$

The error term $\left\| \chi_{\mathbb{R}^{2n}\setminus B(0,2^j)} K_{\lfloor} - \sum_{i>j} \sum_{\gamma<0} K_{\gamma}^{i} \right\|$ can be estimated by $C_N \frac{2^{-2nj}}{1+|2^{-j}x|^N}$ for every $N > 0$ and therefore

$$T_{\lfloor}^{\#}(f,g)(x) \leq T_{K_{\lfloor}}^{*}(f,g)(x) + CMf(x)Mg(x).$$

In order to estimate the operator $T_j^{\#}$ we need to use the methods from the paper [10]. We want to show the following statement.

**Proposition 5.** *Given $q$ in $[2,\infty]$ and $\delta$ in $(0, 1/8q')$, then for any $j$ from $\mathbb{N}_0$, the operator $T_j^{\#}$ maps $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ with norm at most $C\|\Omega\|_{L^q} 2^{-\delta j}$.*

We follow the proof from the paper and first decompose the kernel into dyadic pieces and the set called 'diagonal' part $D_{\lambda,\kappa}^{1}$ and 'off-diagonal' parts $D_{\lambda,\kappa}^{2}$ and the remainder $D_{\lambda,\kappa}^{3}$. In case of the diagonal part we obtain the estimate easily with just a slight modification of the entire procedure described in [10] and previous chapter. Therefore we have

$$\| \sup_{\gamma\in\mathbb{Z}} \sum_{k>\gamma} T_{m_{j,k}^{1}}(f,g) \|_{L^1} \leq C \|\Omega\|_{L^q} j 2^{-\delta j/5} \|f\|_{L^2} \|g\|_{L^2}.$$

The off-diagonal part is significantly more intricate since the procedure used in [10] cannot be applied. We have to estimate the operator

$$\sup_{\gamma\in\mathbb{Z}} \left\| \sum_{k>\gamma} T_{m_{j,k}^{2}}(f,g) \right\|.$$

To do that we need to have an approximate size of the operator $T_{m_{j,k}^{2}}$. We do that with the help of estimating the support of $m_{j,k}^{2}$ because then we can apply the following.

**Lemma 6.** *Let $L^\infty$ function $m$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ be a symbol of a multiplier operator $T_m$. Suppose $I, J \subset \mathbb{R}^n$ are measurable sets and let* supp $m \subset I \times J$. *Then for functions $f, g$ in $\mathcal{S}(\mathbb{R}^n)$ it holds that* supp $T_m(f,g) \subset I + J$.

In order to find the size of the support of $m^2_{j,k}$ we use the fact $m^2_{j,k}(x) = m^2_{j,0} \;\; 2^k x \{$ therefore we only need to deal with $m^2_{j,0}$. A simple calculation gives

$$\text{supp } m^2_{j,0} \subseteq A \times B,$$

where

$$A = \}\xi \in \mathbb{R}^n : 2^{j-2} - 1 \le |\xi| \le 2^{j+1} + 1|$$

and

$$B = \}\xi \in \mathbb{R}^n : -3 - 2^{-j} \le |\xi| \le 3 + 2^{-j}| \; .$$

Therefore it follows from the Lemma above that

$$\text{supp } \widetilde{T}_{m^2_{j,0}}(f,g) \subseteq \}\xi \in \mathbb{R}^n : 2^{j-3} \le |\xi| \le 2^{j+2}| \; ,$$

thus

$$\text{supp } \widetilde{T}_{m^2_{j,k}}(f,g) \subseteq \Big\}\xi \in \mathbb{R}^n : 2^{j-3-k} \le |\xi| \le 2^{j+3-k} \; .$$

Now we are ready to do the estimate of $\sup_{\gamma\in\mathbb{Z}} \left\| \sum_{k>\gamma} T_{m^2_{j,k}}(f,g) \right\|$. From the paper [10] we know that for every $f, g$ in $L^2(\mathbb{R}^n)$ the following estimate holds

$$\left\| \Bigg) \sum_{k\in 5\mathbb{Z}} \left\| T_{m^2_{j,k}}(f,g) \right\|^2 \Bigg(^{\frac{1}{2}} \right\|_{L^1} \le C \|\Omega\|_{L^q} 2^{-j\delta} \|f\|_{L^2} \|g\|_{L^2} . \tag{2.3}$$

To make sure the Fourier supports of $T_{m^2_{j,k}}$ do not overlap we write the operator as follows. For $\mu = 0, ..., 10$ we have

$$T^{\#}_{m^2_j}(f,g) = \sup_{\gamma\in\mathbb{Z}} \sum_{\mu} \left\| \sum_{\substack{k>\gamma \\ k\in 10\mathbb{Z}+\mu}} T_{m^2_{j,k}}(f,g) \right\|,$$

For $\mu$ fixed we obtain the following inequality

$$\sup_{\gamma\in\mathbb{Z}} \left\| \sum_{\substack{k>\gamma \\ k\in 10\mathbb{Z}+\mu}} T_{m^2_{j,k}}(f,g) \right\| \le \sup_{\beta>0} \left\| \sum_{k\in 10\mathbb{Z}+\mu} T_{m^2_{j,k}}(f,g) - \check{\psi}_\beta * \Bigg) \sum_{k\in 10\mathbb{Z}+\mu} T_{m^2_{j,k}}(f,g) \Bigg( \right\|,$$

where $\psi_\beta$ is a smooth function such that its Fourier transform is equal to 1 on the ball $B\Big)0, 2^{j-\beta+3}\Big($ and vanishes outside of $B\Big)0, 2^{j-\beta+10}\Big($. The reason why this inequality holds is that if we look at it on the Fourier transform side we get on the right hand side an expression of the type $\Big)\sum_{k\in 10\mathbb{Z}+\mu} T_{m^2_{j,k}}\Big( \;\; 1 - \psi_\beta \{$ which is exactly the expression on the left hand side smoothed out. The estimate of the Fourier support of $T_{m^2_{j,k}}$ was used in the choice of $\beta$.

Then if we consider the maximal function $M_\psi f(x) = \sup_{\beta>0} \left|\psi_\beta * f\right|$, the above inequality gives in $L^1$ norm the following

$$\left\| \sup_{\gamma\in\mathbb{Z}} \left| \sum_{k>\gamma} T_{m^2_{j,k}}(f,g) \right| \right\|_{L^1} \le \left\| \sum_{k\in 10\mathbb{Z}+\mu} T_{m^2_{j,k}}(f,g) \right\|_{L^1} + C \left\| M_\psi \left( \sum_{k\in 10\mathbb{Z}+\mu} T_{m^2_{j,k}}(f,g) \right) \right\|_{L^1}.$$

Now we estimate the first part. Given the estimate of the square function it follows from the Littlewood-Paley theorem that there exists a polynomial $Q^\mu_j$ such that

$$\left\| \sum_{k\in 10\mathbb{Z}+\mu} T_{m^2_{j,k}}(f,g) - Q^\mu_j \right\|_{L^1} \le \left\| \left( \sum_{k\in 10\mathbb{Z}} \left| T_{m^2_{j,k}}(f,g) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^1}.$$

Since $\sum_{k\in 10\mathbb{Z}+\mu} T_{m^2_{j,k}}(f,g)$ is in $L^1(\mathbb{R}^n)$ we have $Q^\mu_j = 0$, therefore

$$\left\| \sum_{k\in 10\mathbb{Z}+\mu} T_{m^2_{j,k}}(f,g) \right\|_{L^1} \le C \|\Omega\|_{L^q} 2^{-j\delta} \|f\|_{L^2} \|g\|_{L^2}.$$

In fact (2.3) also implies that $\sum_{k\in\mathbb{Z}} T_{m^2_{j,k}}(f,g)$ is in $H^1$, therefore we can estimate the second expression as

$$\left\| M_\psi \left( \sum_{k\in 10\mathbb{Z}+\mu} T_{m^2_{j,k}}(f,g) \right) \right\|_{L^1} \le \left\| \sum_{k\in 10\mathbb{Z}+\mu} T_{m^2_{j,k}}(f,g) \right\|_{H^1},$$

which is the desired estimate.

So now we know that the Theorem 4 holds for the operator $T^\#$ but we still have to justify the sufficiency of this with respect to $T^*$. In order to do that let us define the following operator which is another type of maximal function. We have

$$M_\Omega(f,g)(x) = \sup_{R>0} \left| \int_{\mathbb{R}^{2n}} \Omega\left( \frac{(y,z)}{|(y,z)|} \right) R^{-2n} \chi_{B(0,R)}(y,z) f(x-y) g(x-z)\, dydz \right|,$$

where $\Omega$ is in $L^1(\mathbb{S}^{2n-1})$ and $\Omega \ge 0$. For $p,q,r$ in $(1,\infty)$ we have $\|M_\Omega(f,g)\|_r \le \|\Omega\|_1 \|f\|_p \|g\|_q$. This result holds due to the same type of boundedness of directional maximal function. It gives us boundedness of the operator $M_\Omega$, but only when $r>1$. In order to do the transition from $T^*$ to $T^\#$, we need to do this precisely when $r=1$. We give this statement in full generality extended also to the region $r \le 1$ for the sake of completeness. Firstly we state the result giving the boundedness of $M_\Omega$.

**Lemma 7.** *Let $r$ be in $(1/2,\infty)$ and $u,v$ in $(1,\infty)$ such that $\frac{1}{u}+\frac{1}{v}=\frac{1}{r}$. Let $\Omega$ be in $L^s(\mathbb{S}^{2n-1})$, $\Omega \ge 0$ where $s$ is from $(1,\infty)$ and is such that $\frac{s}{2s-1} < r$. Then*

$$M_\Omega : L^u(\mathbb{R}^n) \times L^v(\mathbb{R}^n) \to L^r(\mathbb{R}^n).$$

This result is obtained using almost exlusively the multilinear interpolation (Theorem 7.2.2 in [8]). Now we are ready to give the result providing the estimate of $T^*$ with $T^\#$. We point out that the following result is the most important in the situation where $p=q=2$ and $r=1$, but we give the full version.

**Lemma 8.** *Let $r$ be in $(1/2,\infty)$ and $p,q$ in $(1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. Let $\Omega$ be from $L^s\left(\mathbb{S}^{2n-1}\right)$ where $s$ is in $(1,\infty)$ is such that $\frac{s}{2s-1}<q$. Then*

$$\|T^*(f,g)\|_r \le \left\|T^{\#}(f,g)\right\|_r + C\left\|M_{|\Omega|}(f,g)\right\|_r.$$

This estimate is quite simple. We define functions

$$K_\varepsilon(x,y) = K(x,y)\,\chi_{\{(x,y):|x|,|y|\ge\varepsilon\}}(x,y)$$

and

$$\tilde{K}_\varepsilon(x,y) = K(x,y)\left(1-\alpha\left(\frac{1}{\varepsilon}|(x,y)|\right)\right).$$

Then

$$\begin{aligned} T^*(f,g)(x) \le \sup_{\varepsilon>0}&\left|\int_{\mathbb{R}^{2n}} K_\varepsilon(y,z)f(x-y)g(x-z) - \tilde{K}_{\varepsilon_j}(y,z)f(x-y)g(x-z)\,dydz\right| \\ &+\sup_{i\in\mathbb{Z}}\left|\int \tilde{K}_i(y,z)f(x-y)g(x-z)\,dydz\right|, \end{aligned}$$

where in the first part we take $j$ such that $\varepsilon\approx\varepsilon_j=2^j$ and estimate it with $M_{|\Omega|}(f,g)(x)$ while the second part equals $T^{\#}(f,g)(x)$. Therefore we obtain the desired estimate.

This concludes the main part of the paper. The last step in order to prove the Theorem 4 is to do the interpolation. Our intention is to use again the Theorem 7.2.2 from [8]. Since the kernel $K_j$ is a Calderón-Zygmund kernel with $\varepsilon$-Lipschitz constant which is smaller than $C_\varepsilon\|\Omega\|_\infty 2^{|j|\varepsilon}$, then using the Cotlar inequality from [14], we can get for any combination $p,q$ in $(1,\infty)$, $1/r=1/p+1/q$ and any $\varepsilon$ in $(0,1)$ the bound

$$\|T_j^*\|_{L^p\times L^q\to L^r} \le C_{p,q,\varepsilon}\|\Omega\|_\infty 2^{j\varepsilon}.$$

Here we assumed only $j\ge 0$. Next we observe that

$$\|T_j^{\#}\|_{L^p\times L^q\to L^r} \le \|T_j^*\|_{L^p\times L^q\to L^r} + C\|\Omega\|_\infty\|M\|_{L^p\times L^q\to L^r},$$

the proof of which is omitted since it is similar to the proof of boundedness of $T^{\#}$ which was briefly described above.

Now we fix $(p,q)$ such that $p,q$ are in $(1,\infty)$, we find a pair of points $(p_1,q_1)$ and $(p_2,q_2)$ such that $p_1,q_1,p_2,q_2$ are in $(1,\infty)$ and $(1/p,1/q)$ lies inside the triangle $(1/2,1/2)$, $(1/p_1,1/q_1)$ and $(1/p_2,1/q_2)$. According to the Proposition the operators $T_j^{\#}$ have norm at the point $(2,2)$ at most $C\|\Omega\|_{L^q}2^{-\delta j}$, where $\delta>0$ is fixed, while at the remaining two points the norm is $C_{p_i,q_i,\varepsilon}\|\Omega\|_\infty 2^{j\varepsilon}$, where $i=1,2$ and we may choose $\varepsilon>0$ arbitrarily small. Therefore, for a suitable choice of $\varepsilon>0$, we get from the interpolation theorem that the series of norms of $T_j^{\#}$ is convergent at $(p,q)$. This finishes the proof of Theorem 1.

The two results in Theorem 1 and Theorem 2 give by interpolation the following.

**Corollary 9.** *For all $n\ge 1$, if $\Omega$ is in $L^q(\mathbb{S}^{2n-1})$ with $q$ in $[2,\infty]$, then, for $T_\Omega^*$ defined above, we have*

$$\|T_\Omega^*\|_{L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} < \infty \tag{2.4}$$

*whenever $1/p=1/p_1+1/p_2$ and the point $(1/p_1,1/p_2)$ lies inside the quadrilateral with vertices $(1/q,1/q)$, $(1/q,1-1/q)$, $(1-1/q,1-1/q)$ and $(1-1/q,1/q)$.*

To show this we apply interpolation argument similar to the one used for the boundedness of $M_\Omega$.

# 3. Multipliers

## 3.1 The Lattice Bump Multiplier Problem

In this section we describe in detail the results of the manuscript The Lattice Bump Multiplier Problem. This paper studies the behavior of the $L^p$ norm of linear and bilinear multipliers of a specific type. Namely a multiplier given by a finite sum of translations of a given smooth bump. The setup in the linear case is as follows. We fix a smooth bump $\phi$ supported in the ball in $\mathbb{R}^n$ with the radius less than $1/10$. We have the linear operator defined for $k$ in $\mathbb{Z}^n$ as

$$S_{k,\phi}(f)(x) = \int_{\mathbb{R}^n} \widehat{f(y)}\phi(y-k)e^{2\pi i x\cdot y}dy,$$

where $f$ is a Schwartz function on $\mathbb{R}^n$. Next we assume a finite set $E$ in $\mathbb{Z}^n$ and with $E$, $\phi$ and $a = \{a_k\}_{k\in\mathbb{Z}}$ a sequence of complex numbers satisfying $|a_k| \leq 1$, we define a linear operator

$$L_{E,a,\phi} = \sum_{k\in E} a_k S_{k,\phi}(f)$$

for $f$ Schwartz function.

The question is for a given $p$ in $[1,\infty]$ what is the smallest value $\alpha(p)$ such that for all subsets $E$ of $\mathbb{Z}^n$ with $|E| = N$ it holds

$$\|L_{E,a,\phi}\|_{L^p\to L^p} \leq C_{p,n,\phi}N^{\alpha(p)}?$$

The answer for this problem is $\alpha(p) = \left|\frac{1}{p}-\frac{1}{2}\right|$ in the sense that not only the above inequality holds but also

$$\sup_{\|a\|_{l^\infty}\leq 1}\ \sup_{|E|=N} \|L_{E,a,\phi}\|_{L^p\to L^p} \geq CN^{\alpha(p)}.$$

Next we consider the analogous bilinear problem. We fix a smooth bump $\Phi$ supported in the ball in $\mathbb{R}^{2n}$ with the radius less than $1/20$. We consider the following bilinear operator

$$B_{E,\Phi}(f,g)(x) = \sum_{(k,l)\in E} S_{(k,l),\phi}(f\otimes g)(x,x),$$

where $E$ is a subset of $\mathbb{Z}^{2n}$ and $f$ and $g$ are Schwartz functions, $x$ in $\mathbb{R}^n$.

The problem in this case is given $p_1$, $p_2$ in $[1,\infty]$, what is the smallest value $\alpha(p_1,p_2)$ such that for all subsets $E$ of $\mathbb{Z}^{2n}$ with $|E| = N$ it holds

$$\|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \leq C_{p_1,p_2,n,\Phi}N^{\alpha(p_1,p_2)}?$$

The relation between $p$ and $p_1$, $p_2$ is again $1/p = 1/p_1 + 1/p_2$. The answer to this problem is that we can obtain optimal result up to an arbitrarily small $\varepsilon$ with the following value

$$\alpha(p_1,p_2) = \frac{1}{2}\left[\max\left(\frac{1}{p}-\frac{1}{2},0\right) - \min\left(\frac{1}{p_1}-\frac{1}{2},0\right) - \min\left(\frac{1}{p_2}-\frac{1}{2},0\right)\right].$$

The main result is therefore the following theorem.

**Theorem 10.** *For a given $\varepsilon > 0$ and $p_1,\ p_2$ in $[1,\infty)$ it holds*

1. *If $p \leq 1$, then there is a constant $C_{p_1,p_2,\varepsilon}$ such that*

$$\|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \leq C_{p_1,p_2,\varepsilon} N^{\frac{1}{\min(p_1,p_2)}+\frac{1}{2\max(p_1,p_2)}-\frac{1}{2}+\varepsilon}.$$

2. *If $p > 1$, then there is a constant $C_{p_1,p_2,\varepsilon}$ such that*

$$\|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \leq C_{p_1,p_2,\varepsilon} N^{\alpha(p_1,p_2)+\varepsilon}$$

*Moreover the estimate for $p > 1$ is sharp up to $\varepsilon$.*

The proof of the linear case we discus briefly. It is quite straightforward to show the first part using simple estimates and interpolation. To show the optimality we do that by finding counterexamples for fixed smooth bump $\phi$ and $E$ taking different $p$'s. In other words we find a sequence $a = \{a_k\}_{k\in E}$ and a constant $C_p$ such that for all positive integers $N$ we have

$$\|L_{E,a,\phi}\|_{L^p\to L^p} \geq C_p N^{\alpha(p)}.$$

At first we consider the one-dimensional case and $p$ in $(1,2]$. Let

$$m_t(x) = \sum_{|k|\leq N} a_k(t)\phi(x-k),$$

where $a_k$ are the Rademacher functions. Then the following holds

$$\int_0^1 \|T_m(f)\|_{L^p}^p \, dt \leq \|L_{E,a,\phi}\|_{L^p\to L^p}^p.$$

Next we consider the case when $p = 1$ which can be shown by contradiction. Finally the higher dimensional case is considered, where the idea is to assume products of the one-dimensional example.

We know focus on the bilinear case in more detail since our contribution to the paper is in this area. We apply the Fourier series method of Coifman and Meyer developed in [2] and [3] to express the smooth bump $\Phi$ as a sum of products of bumps in each half of the variables. At first we express $\Phi$ in Fourier series as

$$\Phi(x,y) = \sum_{r,s\in\mathbb{Z}^n} c_{r,s} e^{2\pi i r\cdot x} e^{2\pi i s\cdot y} \phi(x)\phi(y),$$

where $\phi(x)$ is a smooth function on $\mathbb{R}^n$ which is equal to 1 on $|x| \leq 1/20$, and vanishes outside $|x| \leq 1/10$ and $c_{r,s}$ is a constant with a rapid decay. Now letting $\phi_r(\xi) = e^{2\pi i r\cdot\xi}\phi(\xi)$, we have that

$$S_{(k,l),\Phi}(f\otimes g)(x,x) = \sum_{r,s\in\mathbb{Z}^n} c_{r,s} S_{k,\phi_r}(f)(x) S_{l,\phi_s}(g)(x)$$

and it will suffice to study an analogous problem for $S_{k,\phi_r}(f)(x)S_{l,\phi_s}(g)(x)$ in place of $S_{(k,l)}(f\otimes g)(x,x)$ and obtain estimates for the norm that are independent of $r$ and $s$.

In order to show the sufficiency part of Theorem 10 we use two main tools, bilinear duality and interpolation. Let us note that considering $p_1, p_2$ in the range $[1,\infty]$ implies that $p$ is in the interval $[1/2,\infty]$. We solve the problem either from the '$p$ range' point of view or the '$p_1, p_2$'s depending on the situation. The first step is to show that the Theorem 10 holds in the local $L^2$ case with constant $\alpha(p_1,p_2) = 1/4$ in other words the case $p_1$, $p_2$, $p$ in $[2,\infty]$. It is enough to consider the case $p_1 = p_2 = 2$ because then the cases $p_1 = 2, p_2 = \infty$ and $p_2 = 2, p_1 = \infty$ are a consequence of duality and the rest follows from interpolation. Afterwards the results outside the local $L^2$ case will be discussed.

The proof of the case $p_1 = p_2 = 2$ uses the same technique as the Theorem 1.3 in [9]. With the set $E$ fixed, we denote $E'$ the set of all first coordinates of elements of $E$. We think of the set $E$ as a union of columns $Col_k$ indexed by $k \in E'$ and we write

$$E = \bigcup_{k \in E'} Col_k.$$

As mentioned before it suffices to consider the case when $B_{E,\Phi}(f,g)$ is a sum of products of operators of the form

$$T_{\sigma_N}(f,g) := \sum_{k \in E'} S_k(f) \sum_{l\colon (k,l) \in Col_k} S_l(g),$$

where $\sigma_N := \sum_{(k,l)\in E} \phi_r(\xi - k)\phi_s(\eta - l)$. Then we split the columns in 'large' and 'small' so we have $E = E_1 \cup E_2$, where $E_1$ contains all the columns of size bigger than some $K$ which is chosen in the end while $E_2$ contains the rest. Therefore we can split the operator as $T_{\sigma_N}(f,g) = T^1_{\sigma_N}(f,g) + T^2_{\sigma_N}(f,g)$ where the operators are the parts indexed by the first coordinates in $E$, in other words indexed by $k$ in $E_1'$ and $E_2'$ respectively.

The operator $T^1_{\sigma_N}$ can be estimated exploiting orthogonality of $S_k$'s on $L^2$ as follows

$$\left\| T^1_{\sigma_N}(f,g) \right\|_{L^1} \leq \|\phi\|_{L^\infty} \|f\|_{L^2} (\#E_1')^{\frac{1}{2}} \|\phi\|_{L^\infty} \|g\|_{L^2}.$$

Then, as there are $N$ points in $E$ and each column in $E_1'$ has at least $K$ elements, this means that there are at most $N/K$ columns in $E_1'$, therefore

$$\left\| T^1_{\sigma_N}(f,g) \right\|_{L^1} \leq (N/K)^{\frac{1}{2}} \|\phi\|^2_{L^\infty} \|f\|_{L^2} \|g\|_{L^2}.$$

We continue with $T^2_{\sigma_N}$. By a sequence of inequalities using mostly Schwartz inequality we get

$$\left\| T^2_{\sigma_N}(f,g) \right\|_{L^1} \leq \|\phi\|_{L^\infty} \|g\|_{L^2} K^{\frac{1}{2}} \|\phi\|_{L^\infty} \|f\|_{L^2}.$$

Therefore the optimal choice of $K$ is $N^{1/2}$ and the proof is finished.

The rest of the sufficiency part is done in several steps. Firstly it can be shown that considering $p_1, p_2, p$ without any further restriction, then

$$\|B_{E,\Phi}\|_{L^{p_1} \times L^{p_2} \to L^p} \leq CN.$$

Moreover in the case $(p_1,p_2) = (1,1)$ (therefore $p = 1/2$) this result is sharp.

The next estimate is for $p$ in $(1/2,1)$ and it is a consequence of interpolation between the estimate at the point $(p_1,p_2) = (1,1)$ and $(p_1,p_2) = (2,2)$. The claim is that there is a constant $C$ such that

$$\|B_{E,\Phi}\|_{L^{p/2} \times L^{p/2} \to L^p} \leq CN^{\frac{3}{4p} - \frac{1}{2}}.$$

The next result will give us the desired estimates in all the remaining regions of $p_1, p_2, p$ apart from the local $L^2$ case which we already have. For $p_1, p_2$ in $(1,\infty)$ it holds the following.

**Proposition 11.** *Let $\varepsilon > 0$ be given.*
*(i) If $p < 1$, then there is a constant $C = C_{p_1,p_2,\varepsilon}$ such that*

$$\|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \leq CN^{\frac{1}{\min(p_1,p_2)}+\frac{1}{2\max(p_1,p_2)}-\frac{1}{2}+\varepsilon}.$$

*(ii) Fix $i \in \{1,2\}$. If $1 < p_i < 2$, and $1 < p < 2$, then there is a constant $C = C_{p_1,p_2,\varepsilon}$ such that*

$$\|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \leq CN^{\frac{1}{2p_i}+\varepsilon}.$$

*(iii) If $p > 2$, then there is a constant $C = C_{p,\varepsilon}$ such that*

$$\|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \leq CN^{\frac{1}{2}-\frac{1}{2p}+\varepsilon}.$$

In order to show all three parts of the previous proposition we need the following lemma.

**Lemma 12.** *For all $p_1, p_2$ in $[2,\infty)$ and $p$ in $[1,\infty)$ with $1/p = 1/p_1 + 1/p_2$, we have*

$$\|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \leq C(\log N)N^{\frac{1}{2}}.$$

This result gives the rate of growth $N^{1/2}\log N$ for $(p_1, p_2, p)$ close to $(\infty,\infty,\infty)$, then by duality we have the same rate of growth for $(p_1, p_2, p)$ close to $(1,\infty,1)$ or $(\infty,1,1)$. Knowing this we can now prove with the use of interpolation the Proposition 11 as follows.

Estimate $(i)$. It suffices to consider the case $p_1 < p_2$. Then the desired estimate is

$$\|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \leq CN^{\frac{1}{p_1}+\frac{1}{2p_2}-\frac{1}{2}+\varepsilon},$$

which follows from interpolation between $(1,1,\frac{1}{2})$, $(2,2,1)$, and $(p_1, p_2, p)$ close to $(1,\infty,1)$.

Estimate $(ii)$ follows by interpolating between $(2,2,1)$, $(2,\infty,2)$, and $(p_1, p_2, p)$ close to $(1,\infty,1)$ enough when $i = 1$. The case $i = 2$ follows by symmetry.

Estimate $(iii)$ follows from interpolation between $(\infty,2,2)$, $(2,\infty,2)$, and $(p_1, p_2, p)$ close enough to $(\infty,\infty,\infty)$.

# 4. Multilinear Fourier Transform Estimates

In this chapter, we expand the work of Grafakos, He, Honzík and Park [11]. They studied the $L^2 \times \cdots \times L^2 \to L^{\frac{2}{m}}$ boundedness of the operators described below, which is a multilinear extension of the results from the previous chapter in the central point where all $p_i = 2$. Our original result is the extension of this to the point where one of the spaces is $L^p$, $p > 2$. While in the bilinear setting this is rather simple from duality, in the multilinear case the duality argument is no longer valid as the target space is $L^q$ with $q < 1$. We therefore use a novel argument and prove the estimate directly. This work is a part of a larger project, where we hope to settle the precise boundedness of the operator for the full range of spaces.

## 4.1 Multilinear Lattice Bump Result

We have the following setup which is similar to the one from the previous chapter since the result provided is the multilinear continuation of the bilinear case. We again start with a smooth bump $\phi$ supported in the ball in $\mathbb{R}^n$ with a sufficiently small radius.

For $l$ in $\mathbb{Z}^n$ and $y$ in $\mathbb{R}^n$ we put $\omega_l(y) = \phi(y-l)$. For $\vec{k}$ in $(\mathbb{Z}^n)^m$, $\vec{x}$ in $(\mathbb{R}^n)^m$ we define

$$\omega_{\vec{k}}(\vec{x}) := \omega_{k_1}(x_1)\cdots\omega_{k_m}(x_m).$$

We observe that

$$\Big(\sum_{\vec{k}\in\mathbb{Z}^{nm}} \|\omega_k\|^q\Big)^{1/q} = \sum_{\vec{k}\in\mathbb{Z}^{nm}} \|\omega_{\vec{k}}\|$$

for any $q$ in $(0,\infty)$, due to the disjoint compact supports. Let $U$ be a subset of $(\mathbb{Z}^n)^m$ and $\{b_k\}_{k\in U}$ be a sequence of complex numbers. We define

$$\sigma(\vec{x}) := \sum_{k\in U} b_k \omega_k(\vec{x})$$

and let $T_\sigma(f_1 \ldots, f_m)$ be the corresponding $m$-linear multiplier operator defined for Schwartz functions $f_1, \ldots, f_m$ on $\mathbb{R}^n$. The next theorem is the main result of the upcoming paper [11].

**Theorem 13.** *Let $N$ be a positive integer and $U$ be a subset of $(\mathbb{Z}^n)^m$ with $|U| \le N$. Suppose that $\{b_k\}_{k\in(\mathbb{Z}^n)^m}$ is a sequence of complex numbers satisfying $\|\{b_k\}_k\|_{l^\infty} \le A$. Let $\sigma$ be defined as above. Then*

$$\Big\|T_\sigma\ f_1,\ldots,f_m\Big\|_{L^{\frac{2p}{2+p(m-1)}}(\mathbb{R}^n)} \le CAN^{\frac{m-1}{2m}} \|f_1\|_{L^2(\mathbb{R}^n)}\cdots\|f_m\|_{L^2(\mathbb{R}^n)}$$

*for Schwartz functions $f_1, \ldots, f_m$ on $\mathbb{R}^n$.*

This theorem has a range of applications for multilinear singular integral operators with rough kernels and multilinear multipliers. Our goal is now to show that this theorem holds as well in the case $(p,2,2,\ldots,2,q)$, where $p$ is close to infinity and the number of $L^2$ domains is $m-1$ for $m \ge 2$ with a loss of $N^\varepsilon$. We hope to later apply this result to both multilinear multipliers and singular integrals. The result is the following.

**Theorem 14.** *Let $N$ be a positive integer and $U$ be a subset of $(\mathbb{Z}^n)^m$ with $|U| \leq N$. Suppose that $\{b_k\}_{k\in(\mathbb{Z}^n)^m}$ is a sequence of complex numbers satisfying $\|\{b_k\}_k\|_{l^\infty} \leq A$. Let $\sigma$ be defined as above. Then*

$$\left\| T_\sigma(f_1,\dots,f_m)\right\|_{L^{2/m}(\mathbb{R}^n)} \leq CAN^{\varepsilon} N^{\frac{m-1}{2m}} \|f_1\|_{L^p(\mathbb{R}^n)} \|f_2\|_{L^2(\mathbb{R}^n)} \cdots \|f_m\|_{L^2(\mathbb{R}^n)}$$

*for Schwartz functions $f_1,\dots,f_m$ on $\mathbb{R}^n$ and $p$ close to $\infty$. $\varepsilon > 0$ is a constant such that for $p \to \infty$ we have $\varepsilon \to 0$.*

To prove this, we are going to inductively reduce the set $U$ by removing lower dimensional parts with large size. First we take care of long columns. For fixed $k_2,\dots,k_m$ we define $Col_{k_2,\dots,k_m}$ as the collection of points in $U$ with the last $m-1$ coordinates fixed. We denote

$$U^1 = \left\{\vec{k} \in U : \#Col_{k_2,\dots,k_m} \geq N^{\frac{1}{m}}\right\}$$

and

$$PU^1 = \left\{(k_2,\dots,k_m) : \vec{k} \in U^1\right\}.$$

Let $\sigma^1$ be the multiplier related to $U^1$ defined by $\sigma^1 = \sum_{\vec{k}\in U^1} b_{\vec{k}}\omega_{\vec{k}}$ and $T_{\sigma^1}$ the m-linear operator related to $\sigma^1$. We have the following estimates. For simplicity we write $\omega_i := \omega_{k_i}$. Firstly, we have from Cauchy-Schwartz inequality in $k_m$ that

$$T_{\sigma^1}(f_1,\dots,f_m) \leq$$

$$\left(\sum_{k_m\in\mathbb{Z}^n} \left|\left(\omega_m \widehat{f_m}\right)^\vee\right|^2\right)^{\frac{1}{2}}$$

$$\left(\sum_{k_m} \left| \sum_{(k_2,\dots,k_{m-1}):(k_2,\dots,k_m)\in PU^1} \left(\omega_2\widehat{f_2}\right)^\vee \cdots \left(\omega_{m-1}\widehat{f_{m-1}}\right)^\vee \left(\sum_{k_1\in Col_{k_2,\dots,k_m}} b_k\right)\left(\omega_1\widehat{f_1}\right)^\vee\right|^2\right)^{\frac{1}{2}}.$$

The first part can be estimated easily in $L^2$ norm once we take an $L^q$ norm of the operator, the second part needs further estimation. We continue again by Cauchy-Schwartz inequality within the inside sum with the index $k_{m-1}$ and obtain

$$\left(\sum_{k_{m-1}} \left|\left(\omega_{m-1}\widehat{f_{m-1}}\right)^\vee\right|^2\right)^{\frac{1}{2}}$$

$$\left(\sum_{k_m}\sum_{k_{m-1}} \left|\sum_{k_2,\cdots,k_{m-2}} \left(\omega_2\widehat{f_2}\right)^\vee \cdots \left(\omega_{m-2}\widehat{f_{m-2}}\right)^\vee \left(\sum_{k_1} b_k\right)\left(\omega_1\widehat{f_1}\right)^\vee\right|^2\right)^{\frac{1}{2}}.$$

We continue in this fashion using Cauchy-Schwartz inequality until we are in the following situation

$$\left(\sum_{k_m\in\mathbb{Z}^n} \left|\left(\omega_m\widehat{f_m}\right)^\vee\right|^2\right)^{\frac{1}{2}} \cdots \left(\sum_{k_3\in\mathbb{Z}^n} \left|\left(\omega_3\widehat{f_3}\right)^\vee\right|^2\right)^{\frac{1}{2}}$$

$$\Bigg(\sum_{k_m,\dots,k_3\in\mathbb{Z}^n}\Bigg\|\sum_{k_2:(k_2,\dots,k_m)\in PU^1}\big(\omega_2\widehat{f_2}\big)^\vee\Big(\sum_{k_1\in Col_{k_2,\dots,k_m}} b_k\big(\omega_1\widehat{f_1}\big)^\vee\Big)\Bigg\|^2\Bigg)^{\frac{1}{2}}.$$

Now taking the $L^q$ norm of the operator we obtain by Hölder's inequality the terms in the first estimate in $L^2$ norm and $L^{r_1}$ norm respectively, where $\frac{1}{q}=\frac{1}{2}+\frac{1}{r_1}$. The $L^2$ norm of the first term is approximately equal to 1 while the other terms needs to be further estimated. We now apply in each step the Hölder's inequality again on the second terms only until we are in a situation

$$\|T_{\sigma^1}(f_1,\dots,f_m)\|_q \le$$

$$\|\cdot\|_2\cdots\|\cdot\|_2\left\|\Bigg(\sum_{k_m,\dots,k_3\in\mathbb{Z}^n}\Bigg\|\sum_{k_2:(k_2,\dots,k_m)\in PU^1}\big(\omega_2\widehat{f_2}\big)^\vee\Big(\sum_{k_1\in Col_{k_2,\dots,k_m}} b_k\big(\omega_1\widehat{f_1}\big)^\vee\Big)\Bigg\|^2\Bigg)^{\frac{1}{2}}\right\|_{r_{m-2}},$$

where $\frac{1}{r_1}=\frac{1}{2}+\frac{1}{r_2}$, $\frac{1}{r_2}=\frac{1}{2}+\frac{1}{r_3}$ and so on, therefore $\frac{1}{r_{m-2}}=\frac{1}{2}+\frac{1}{p}$ which implies that $r_{m-2}$ is close to 2 when p is close to infinity. Now we can estimate the previous expression with

$$N^{\varepsilon_1}\Bigg(\sum_{k_m,\dots,k_3\in\mathbb{Z}^n}\left\|\sum_{k_2:(k_2,\dots,k_m)\in PU^1}\big(\omega_2\widehat{f_2}\big)^\vee\Big(\sum_{k_1\in Col_{k_2,\dots,k_m}} b_k\big(\omega_1\widehat{f_1}\big)^\vee\Big)\right\|_{r_{m-2}}^2\Bigg)^{\frac{1}{2}}.$$

To achieve this, we first pass to $l^{r_{m-2}}$ norm inside, then switch the norms and finally use Hölder to get back to $l^2$ norm, sacrificing some small power of $N$ depending on $p$. Next with the use of orthogonality we get

$$\left\|\sum_{k_2:(k_2,\dots,k_m)\in PU^1}\big(\omega_2\widehat{f_2}\big)^\vee\Big(\sum_{k_1\in Col_{k_2,\dots,k_m}} b_k\big(\omega_1\widehat{f_1}\big)^\vee\Big)\right\|_{r_{m-2}}\le$$

$$\sum_{k_2:(k_2,\dots,k_m)\in PU^1}\left\|\sum_{k_1\in Col_{k_2,\dots,k_m}} b_k\big(\omega_1\widehat{f_1}\big)^\vee\big(\omega_2\widehat{f_2}\big)^\vee\right\|_{r_{m-2}}.$$

Now we do the critical step of the proof. The functions $(\omega_2\widehat{f_2})^\vee(\omega_1\widehat{f_1})^\vee$, where $k_2$ is fixed and $k_1$ is variable have disjoint Fourier supports. The reason for that is similar as in the proof of Lemma 15 in the attached paper Rough Maximal Bilinear Singular Integrals since the radius of the function $\Phi$ is chosen small enough in order to obtain disjoint supports of the functions $\omega_i$. We therefore interpret the expression as a bump multiplier on $L^{r_{m-2}}$ with length less than $N$. As $r_{m-2}$ is close to 2, we again get

$$\left\|\sum_{k_1\in Col_{k_2,\dots,k_m}} b_k\big(\omega_1\widehat{f_1}\big)^\vee\big(\omega_2\widehat{f_2}\big)^\vee\right\|_{r_{m-2}}\le CN^{\varepsilon_2}\left\|\sum_{k_1\in\mathbb{Z}^n} b_k\big(\omega_1\widehat{f_1}\big)^\vee\big(\omega_2\widehat{f_2}\big)^\vee\right\|_{r_{m-2}},$$

which can be estimated with $CN^{\varepsilon_2}\|(\omega_2\widehat{f_2})^\vee\|_2\|f_1\|_p$. Using orthogonality and Cauchy-Schwartz we get

$$\sum_{k_2:(k_2,\dots,k_m)\in PU^1}\|(\omega_2\widehat{f_2})^\vee\|_2\le C\Big(\sum_{k_2:(k_2,\dots,k_m)\in PU^1}1\Big)^{1/2}\|f_2\|_2.$$

Now, we collect everything to get

$$\begin{aligned}
&\|T_{\sigma^1}(f_1,\dots,f_m)\|_q \\
&\leq CN^{\varepsilon}\Big(\sum_{k_2:(k_2,\dots,k_m)\in PU^1} 1\Big)^{1/2}\|f_1\|_p\cdots\|f_m\|_2 \leq CN^{1/m+\varepsilon}\|f_1\|_p\cdots\|f_m\|_2 .
\end{aligned}$$

Next we will consider a situation when we eliminate the long columns in the $k_1$ coordinate that we estimated above, in other words we continue with the set $U\setminus U^1$ and our goal is to estimate the 'big discs'. We have a set

$$U^2 = \Big\{\vec{k} : \#D_{k_3,\dots,k_m} \geq N^{\frac{2}{m}}\Big\}$$

and

$$PU^2 = \Big\{(k_3,\dots,k_m) : \vec{k}\in U^2\Big\},$$

where $D_{k_3,\dots,k_m}$ is the set of points in $U\setminus U^1$ where the last $m-2$ coordinates are fixed, in this case $(k_3,\dots,k_m)$. We remind that now we have all columns in first coordinate $k_1$ shorter than $N^{\frac{1}{m}}$. We have by Cauchy-Schwartz and Hölder the following

$$\begin{gathered}
\|T_{\sigma^2}(f_1,\dots,f_m)\|_q \leq \\
\Bigg\|\Big(\sum_{k_1\in\mathbb{Z}^n}\Big|\big(\omega_1\widehat{f_1}\big)^{\vee}\Big|^2\Big)^{\frac{1}{2}}\Bigg\|_p \\
\Bigg\|\Big(\sum_{k_1}\Big|\sum_{(k_2,\dots,k_m):(k_3,\dots,k_m)\in PU^2} b_k\big(\omega_2\widehat{f_2}\big)^{\vee}\cdots\big(\omega_m\widehat{f_m}\big)^{\vee}\Big|^2\Big)^{\frac{1}{2}}\Bigg\|_{\frac{2}{m-1}} .
\end{gathered}$$

The first expression can be estimated with $C\|f\|_{L^{p_1}}$ in view of the square function theorem ([19, Theorem 8.1]).So now we focus on the second expression which will be estimated again by Cauchy-Schwartz inequality as

$$\Bigg\|\Big(\sum_{k_1}\sum_{(k_3,\dots,k_m)\in PU^2}\Big|\big(\omega_3\widehat{f_3}\big)^{\vee}\Big|^2\cdots\Big|\big(\omega_m\widehat{f_m}\big)^{\vee}\Big|^2\sum_{(k_3,\dots,k_m)\in PU^2}\Big|\sum_{k_2\in D_{k_3,\dots,k_m}} b_k\big(\omega_2\widehat{f_2}\big)^{\vee}\Big|^2\Big)^{\frac{1}{2}}\Bigg\|_{\frac{2}{m-1}},$$

therefore using Hölder's inequality this can be estimated with

$$\begin{gathered}
\Bigg\|\Big(\sum_{(k_3,\dots,k_m)\in PU^2}\Big|\big(\omega_3\widehat{f_3}\big)^{\vee}\Big|^2\cdots\Big|\big(\omega_m\widehat{f_m}\big)^{\vee}\Big|^2\Big)^{\frac{1}{2}}\Bigg\|_{\frac{2}{m-2}} \cdot \\
\Bigg\|\Big(\sum_{k_1}\sum_{(k_3,\dots,k_m)\in PU^2}\Big|\sum_{k_2\in D_{k_3,\dots,k_m}} b_k\big(\omega_2\widehat{f_2}\big)^{\vee}\Big|^2\Big)^{\frac{1}{2}}\Bigg\|_2 .
\end{gathered}$$

Now the first expression can be estimated as follows. We take the sum over the entire $\mathbb{Z}^n \times \cdots \times \mathbb{Z}^n$ and use Hölder's inequality again to get

$$\left\| \left( \sum_{k_3 \in \mathbb{Z}^n} \left| \left( \omega_3 \widehat{f_3} \right)^{\vee} \right|^2 \right)^{\frac{1}{2}} \right\|_2 \cdots \left\| \left( \sum_{k_m \in \mathbb{Z}^n} \left| \left( \omega_m \widehat{f_m} \right)^{\vee} \right|^2 \right)^{\frac{1}{2}} \right\|_2 ,$$

which can be estimated using orthogonality by $\|f_3\|_2 \cdots \|f_m\|_2$. The estimate for the second expression will be obtained as follows. Assuming $b_k$ is approximately 1, we have using orthogonality that

$$\left\| \left( \sum_{k_1} \sum_{(k_3,\ldots,k_m)\in PU^2} \left| \sum_{k_2 \in D_{k_3,\ldots,k_m}} b_k \left( \omega_2 \widehat{f_2} \right)^{\vee} \right|^2 \right)^{\frac{1}{2}} \right\|_2^2 \leq$$

$$\sum_{k_1} \sum_{k_3} \cdots \sum_{k_m} \left\| \sum_{k_2 \in D_{k_3,\ldots,k_m}} \left| \left( \omega_2 \widehat{f_2} \right)^{\vee} \right|^2 \right\|_2^2 ,$$

where the number of indices $k_1 k_3 \cdots k_m$ is $N^{\frac{m-1}{m}}$ since the number of $k_1$ is less than $N^{\frac{1}{m}}$ and the maximum of $k_1 \cdots k_m$ is $N$ while the number of $k_1 k_2$ is at least $N^{\frac{2}{m}}$ therefore we get the maximum of $k_1 k_3 \cdots k_m$ of $N^{\frac{m-1}{m}}$. Therefore we continue as

$$\sum_{k_1} \sum_{k_3} \cdots \sum_{k_m} \left\| \sum_{k_2 \in D_{k_3,\ldots,k_m}} \left| \left( \omega_2 \widehat{f_2} \right)^{\vee} \right|^2 \right\|_2^2 \leq N^{\frac{m-1}{m}} \left\| \sum_{k_2 \in D_{k_3,\ldots,k_m}} \left| \left( \omega_2 \widehat{f_2} \right)^{\vee} \right|^2 \right\|_2^2 ,$$

which is again with the use of orthogonality equal to

$$\sum_{k_1} \sum_{k_3} \cdots \sum_{k_m} \left\| \sum_{k_2 \in D_{k_3,\ldots,k_m}} \left| \left( \omega_2 \widehat{f_2} \right)^{\vee} \right|^2 \right\|_2^2 \leq N^{\frac{m-1}{m}} \left\| \left( \sum_{k_2 \in D_{k_3,\ldots,k_m}} \omega_2 \widehat{f_2} \right)^{\vee} \right\|_2^2 \leq N^{\frac{m-1}{m}} \|f_2\|_2^2 .$$

Now we want to continue with estimating 'big blocks'. We eliminate long columns and big discs so we continue with the set $U \setminus \left( U^1 \cup U^2 \right)$. We have a set

$$U^3 = \left\{ \vec{k} : \# B_{k_4,\ldots,k_m} \geq N^{\frac{3}{m}} \right\}$$

and

$$PU^3 = \left\{ (k_4, \ldots, k_m) : \vec{k} \in U^3 \right\} ,$$

where $B_{k_4,\ldots,k_m}$ is a block with fixed coordinates $(k_4, \ldots, k_m)$. We have by Cauchy-Schwartz and Hölder the following

$$\|T_{\sigma^3}(f_1, \ldots, f_m)\|_q \leq$$

$$\left\| \left( \sum_{k_1 \in \mathbb{Z}^n} \left| \left( \omega_1 \widehat{f_1} \right)^{\vee} \right|^2 \right)^{\frac{1}{2}} \right\|_p$$

$$\left\| \left( \sum_{k_1} \left\| \sum_{k_2\in\mathbb{Z}^n,\dots,k_{m-1}\in\mathbb{Z}^n,k_m:(k_1,\dots,k_m)\in U^3} b_k \left(\omega_2\widehat{f_2}\right)^{\vee}\cdots\left(\omega_m\widehat{f_m}\right)^{\vee} \right\|^2 \right)^{\frac{1}{2}} \right\|_{\frac{2}{m-1}}.$$

The second expression can be estimated again by Cauchy-Schwartz inequality as

$$\left\| \left( \sum_{k_1}\sum_{k_2} \left\| \left(\omega_2\widehat{f_2}\right)^{\vee} \right\|^2 \cdot \sum_{k_2} \left\| \sum_{k_3,\dots,k_m:(k_1,\dots,k_m)\in U^3} b_k \left(\omega_3\widehat{f_3}\right)^{\vee}\cdots\left(\omega_m\widehat{f_m}\right)^{\vee} \right\|^2 \right)^{\frac{1}{2}} \right\|_{\frac{2}{m-1}},$$

therefore by Hölder's inequality we get

$$\left\| \left( \sum_{k_2\in\mathbb{Z}^n} \left\| \left(\omega_2\widehat{f_2}\right)^{\vee} \right\|^2 \right)^{\frac{1}{2}} \right\|_2$$

$$\left\| \left( \sum_{k_1}\sum_{k_2} \left\| \sum_{k_3,\dots,k_m:(k_1,\dots,k_m)\in U^3} b_k \left(\omega_3\widehat{f_3}\right)^{\vee}\cdots\left(\omega_m\widehat{f_m}\right)^{\vee} \right\|^2 \right)^{\frac{1}{2}} \right\|_{\frac{2}{m-2}}.$$

Now with the second expression we continue the same process of using Cauchy-Schwartz and Hölder's inequality until we have altogether

$$\left\| \left( \sum_{k_1\in\mathbb{Z}^n} \left\| \left(\omega_1\widehat{f_1}\right)^{\vee} \right\|^2 \right)^{\frac{1}{2}} \right\|_p \cdot$$

$$\left\| \left( \sum_{k_2\in\mathbb{Z}^n} \left\| \left(\omega_2\widehat{f_2}\right)^{\vee} \right\|^2 \right)^{\frac{1}{2}} \right\|_2 \left\| \left( \sum_{k_4\in\mathbb{Z}^n} \left\| \left(\omega_4\widehat{f_4}\right)^{\vee} \right\|^2 \right)^{\frac{1}{2}} \right\|_2 \cdots \left\| \left( \sum_{k_m\in\mathbb{Z}^n} \left\| \left(\omega_m\widehat{f_m}\right)^{\vee} \right\|^2 \right)^{\frac{1}{2}} \right\|_2 \cdot$$

$$\left\| \left( \sum_{k_1}\sum_{k_2}\sum_{k_4}\cdots\sum_{k_m} \left\| \sum_{k_3} \left(\omega_3\widehat{f_3}\right)^{\vee} \right\|^2 \right)^{\frac{1}{2}} \right\|_2,$$

where the last expression can be estimated with $N^{\frac{m-1}{2m}}$.

We now proceed inductively and we remove from the set $U$ blocks of increasing dimension, in other words sets $B_{k_l,\dots,k_m}$, where $l$ is in $\{4,\dots,m-1\}$ with size greater than $N^{\frac{l}{m}}$ until we have only the remainder set

$$UR := U \setminus \left\{U^1\cup\cdots\cup U^{m-1}\right\}.$$

We estimate the operator on the set $UR$ in a similar way. We fix the last index $k_m$ and we follow the same procedure until we are left with the expression

$$\left\| \left( \sum_{k_1}\sum_{k_2}\sum_{k_3}\cdots\sum_{k_{m-1}} \left\| \sum_{k_m} \left(\omega_m\widehat{f_m}\right)^{\vee} \right\|^2 \right)^{\frac{1}{2}} \right\|_2.$$

Now for each $k_m$ we have at most $N^{\frac{m-1}{m}}$ indices $k_1\cdots k_{m-1}$ therefore the previous expression can be estimated with $N^{\frac{m-1}{2m}}\|f_m\|_2$ which is the desired estimate and that finishes the proof.

## 4.2 Applications

Using multilinear interpolation, for details see [8], and symmetry we may combine the two theorems from the previous section as follows.

**Theorem 15.** *Let $N$ be a positive integer, $\varepsilon > 0$ and $U$ be a subset of $(\mathbb{Z}^n)^m$ with $|U| \leq N$. Suppose that $\{b_k\}_{k\in(\mathbb{Z}^n)^m}$ is a sequence of complex numbers satisfying $\|\{b_k\}_k\|_{l^\infty} \leq A$. Let $p_1,\dots,p_m$ in $(2,\infty)$ such that the vector $(1/p_1,\dots,1/p_m)$ falls into open simplex with vertices $(1/2,\dots,1/2)$, $(0,1/2,\dots,1/2)$, $(1/2,0,1/2\dots,1/2)$,...,$(1/2,\dots,1/2,0)$. Let $\sigma$ be defined as above. Then*

$$\Big(T_\sigma\ f_1,\dots,f_m\Big\{\Big(_{L^q(\mathbb{R}^n)} \leq CAN^{\varepsilon}N^{\frac{m-1}{2m}}\,\|f_1\|_{L^{p_1}(\mathbb{R}^n)}\,\|f_2\|_{L^{p_2}(\mathbb{R}^n)}\cdots\|f_m\|_{L^{p_m}(\mathbb{R}^n)}$$

*for Schwartz functions $f_1,\dots,f_m$ on $\mathbb{R}^n$, where $1/q = 1/p_1+\cdots+1/p_m$ and $C>0$ is a constant which depends only on $m,n,\varepsilon,p_1,\dots,p_m$.*

In the paper [11] provide a range of applications of the Theorem 13. They proved boundedness of various operators from $L^2\times\cdots\times L^2$ to $L^{\frac{2}{m}}$. In our upcoming paper we plan to obtain similar results for operators from $L^{p_1}\times\cdots\times L^{p_m}$ to $L^q$ using the previous theorem. For example the following multilinear multiplier theorem is proved in [11].

**Theorem 16.** *Let $a > \frac{(m-1)n}{2}$. Suppose that $\sigma$ is in $\mathscr{C}^\infty((\mathbb{R}^n)^m)$ satisfies*

$$\left\|\partial^\beta\sigma(\vec{\xi})\right\| \leq |\xi|^{-a} \tag{4.1}$$

*for all $|\beta| \leq \left]\frac{(m-1)n}{2}\right[ + 1$, where $[x]$ denotes the integer part of $x$. Then the $m$-linear maximal operator $\mathscr{M}_\sigma$ is bounded from $L^2(\mathbb{R}^n)\times\cdots\times L^2(\mathbb{R}^n)$ to $L^{2/m}(\mathbb{R}^n)$.*

We plan to prove this theorem in $L^{p_1}\times\cdots\times L^{p_m}$ to $L^q$ setting.

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# List of publications

# ROUGH MAXIMAL BILINEAR SINGULAR INTEGRALS

EVA BURIÁNKOVÁ AND PETR HONZÍK

ABSTRACT. We study the rough maximal bilinear singular integral

$$T^*_\Omega(f,g)(x) = \sup_{\varepsilon>0}\left|\int_{\mathbb{R}^n\setminus B(0,\varepsilon)}\int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \frac{\Omega((y,z)/|(y,z)|)}{|(y,z)|^{2n}} f(x-y)g(x-z)dydz\right|,$$

where $\Omega$ is a function in $L^\infty(\mathbb{S}^{2n-1})$ with vanishing integral. We prove it is bounded from $L^p \times L^q \to L^r$, where $1 < p,q < \infty$ and $1/r = 1/p + 1/q$. We also discuss results for $\Omega \in L^s(\mathbb{S}^{2n-1})$, $1 < s < \infty$.

CONTENTS



## 1. INTRODUCTION

In this paper, we study the rough maximal bilinear singular integral. Singular integral theory was initiated in the seminal work of Calderón and Zygmund [1]. Bilinear singular operators were introduced by Coifman and Meyer in [4] and the theory was later developed by Grafakos and Torres in [11]. The boundedness of the smooth maximal multilinear singular integrals was obtained also by Grafakos and Torres in [10] via Cotlar type inequality. All these result were obtained for operators with smooth kernels, while the problem of boundedness of the bilinear rough singular integral remained open. Recently, in the paper of Grafakos, He and Honzík [9], the following was proved: For an operator $T_\Omega$ defined as

$$T_\Omega(f,g)(x) = \text{ p.v. } \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |(y,z)|^{-2n}\Omega((y,z)/|(y,z)|)f(x-y)g(x-z)\,dydz$$

where $\Omega$ is a function in $L^q(\mathbb{S}^{2n-1})$ with vanishing integral it holds that for $q=\infty$ we obtain boundedness for $T_\Omega$ from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ where $p_1, p_2 \in$

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$(1,\infty)$ and $1/p = 1/p_1 + 1/p_2$. Also, for $q = 2$ it was proved that $T_\Omega$ is bounded from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ and these two results were interpolated. These results were obtained using a bilinear technique based on tensor-type wavelet decomposition which we will briefly describe later on. In this paper we build on these results and our goal is to describe similar properties of the maximal version of the operator defined above.

## 2. Notation and results

To fix notation, we assume $q \in (1,\infty]$ and we let $\Omega$ in $L^q(\mathbb{S}^{2n-1})$ with $\int_{\mathbb{S}^{2n-1}} \Omega \, d\sigma = 0$, where $\mathbb{S}^{2n-1}$ is the unit sphere in $\mathbb{R}^{2n}$. In this manuscript we will be working with the bilinear singular integral operator associated with $\Omega$ by

$$T^*_\Omega(f,g)(x) = \sup_{\varepsilon>0}\left|\int_{\mathbb{R}^n\setminus B(0,\varepsilon)}\int_{\mathbb{R}^n\setminus B(0,\varepsilon)} K(y,z)f(x-y)g(x-z)\,dydz,\right| \tag{1}$$

where $f,g$ are functions in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$,

$$K(y,z) = \Omega((y,z)')/|(y,z)|^{2n},$$

and $x' = x/|x|$ for $x \in \mathbb{R}^{2n}$.

For $q \in [1,\infty]$ we denote $q'$ the conjugate index. We denote the set of positive integers by $\mathbb{N}$ and we set $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. Finally, we adhere to the standard convention to denote by $C$ a constant that depends only on inessential parameters of the problem.

Let us now state the main results of this paper.

**Theorem 1.** *For all $n \geq 1$, if $\Omega \in L^\infty(\mathbb{S}^{2n-1})$, then, for $T^*_\Omega$ defined in (1), we have*

$$\|T^*_\Omega\|_{L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \leq C\|\Omega\|_\infty \tag{2}$$

*whenever $1 < p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$.*

For more general $\Omega$, we get the following:

**Theorem 2.** *For all $n \geq 1$, if $\Omega \in L^2(\mathbb{S}^{2n-1})$, then, for $T^*_\Omega$ defined in (1), we have*

$$\|T^*_\Omega\|_{L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\to L^1(\mathbb{R}^n)} \leq C\|\Omega\|_2. \tag{3}$$

These two results give by interpolation:

**Corollary 3.** *For all $n \geq 1$, if $\Omega \in L^q(\mathbb{S}^{2n-1})$ with $q \in [2,\infty]$, then, for $T^*_\Omega$ defined in (1), we have*

$$\|T^*_\Omega\|_{L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} < \infty \tag{4}$$

*whenever $1/p = 1/p_1 + 1/p_2$ and the point $(1/p_1, 1/p_2)$ lies inside the quadrilateral with vertices $(1/q,1/q)$, $(1/q,1-1/q)$, $(1-1/q,1-1/q)$ and $(1-1/q,1/q)$.*

Let us now give a brief introduction to wavelets which are the essential tool in the paper [9] and will be used in this manuscript also.

The wavelet system is a form of a complete orthonormal system for $L^2(\mathbb{R}^n)$. For our purposes we need product type smooth wavelets with compact supports, their existence is due to Daubechies [5] and can also be found in Meyer's book [13].

The construction of such objects can be found in Triebel [15] and the existence of a wavelet orthonormal base of $L^2\left(\mathbb{R}^{2n}\right)$ is described by the following statement.

**Lemma 4.** *For any fixed $k \in \mathbb{N}$ there exist real compactly supported functions $\psi_F, \psi_M \in \mathcal{C}^k(\mathbb{R})$, which satisfy $\|\psi_F\|_{L^2(\mathbb{R})} = \|\psi_M\|_{L^2(\mathbb{R})} = 1$, for $0 \le \alpha \le k$ we have $\int_{\mathbb{R}} x^\alpha \psi_M(x)dx = 0$, and, if $\Psi^G$ is defined by*

$$\Psi^G(\vec{x}) = \psi_{G_1}(x_1)\cdots\psi_{G_{2n}}(x_{2n})$$

*for $G = (G_1, \dots, G_{2n})$ in the set*

$$\mathcal{I} := \Big\{(G_1, \dots, G_{2n}) : \ G_i \in \{F, M\}\Big\},$$

*then the family of functions*

$$\bigcup_{\vec{\mu}\in\mathbb{Z}^{2n}} \left[\Big\{\Psi^{(F,\dots,F)}(\vec{x}-\vec{\mu})\Big\} \cup \bigcup_{\lambda=0}^{\infty} \Big\{2^{\lambda n}\Psi^G(2^\lambda\vec{x}-\vec{\mu}) : \ G \in \mathcal{I}\setminus\{(F,\dots,F)\}\Big\}\right]$$

*forms an orthonormal basis of $L^2(\mathbb{R}^{2n})$, where $\vec{x} = (x_1, \dots, x_{2n})$.*

Le us now describe the basic decomposition of the kernel. Now we fix a smooth function $\alpha$ in $\mathbb{R}^+$ such that $\alpha(t) = 1$ for $t \in (0,1]$, $\alpha(t) \in (0,1)$ for $t \in (1,2)$ and $\alpha(t) = 0$ for $t \in [2,\infty)$. For $(y,z) \in \mathbb{R}^{2n}$ and $j \in \mathbb{Z}$ we introduce the function

$$\beta_j(y,z) = \alpha(2^{-j}|(y,z)|) - \alpha(2^{-j+1}|(y,z)|).$$

We write $\beta = \beta_0$ and we note that this is a function supported in $[1/2, 2]$. We denote $\Delta_j$ the Littlewood-Paley operator $\Delta_j f = \mathcal{F}^{-1}(\beta_j \widehat{f})$. Here and throughout this paper $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, which is defined via $\mathcal{F}^{-1}(g)(x) = \int_{\mathbb{R}^n} g(\xi)e^{2\pi i x\cdot\xi}d\xi = \widehat{g}(-x)$, where $\widehat{g}$ is the Fourier transform of $g$.

We decompose the kernel $K$ as follows: we denote $K^i = \beta_i K$ and we set $K^i_j = \Delta_{j-i}K^i$ for $i, j \in \mathbb{Z}$. Then we write

$$K = \sum_{j=-\infty}^{\infty} K_j,$$

where

$$K_j = \sum_{i=-\infty}^{\infty} K^i_j.$$

We also denote $m_j = \widehat{K_j}$.

Then the operator can be written as

$$T^*(f,g)(x) = \sup_{\varepsilon>0}\left|\sum_j \int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \int_{\mathbb{R}^n\setminus B(0,\varepsilon)} K_j(y,z) f(x-y) g(x-z)\, dydz\right|.$$

We have the following lemma whose proof is known (see for instance [6]) and is omitted.

**Lemma 5.** *Given $q \in (1,\infty]$, $\Omega \in L^q(\mathbb{S}^{2n-1})$, $\delta \in (0,1/q')$ and $\vec{\xi} = (\xi_1,\xi_2) \in \mathbb{R}^{2n}$ we have*

$$|\widehat{K^0}(\vec{\xi}\,)| \leq C\|\Omega\|_{L^q} \min(|\vec{\xi}\,|,|\vec{\xi}\,|^{-\delta})$$

*and for all multiindices $\alpha$ in $\mathbb{Z}^{2n}$ with $\alpha \neq 0$ we have*

$$|\partial^\alpha \widehat{K^0}(\vec{\xi}\,)| \leq C_\alpha \|\Omega\|_{L^q} \min(1,|\vec{\xi}\,|^{-\delta}).$$

We recall that $M$ is the Hardy-Littlewood maximal function defined for locally integrable functions $f$ as

$$Mf(x) = \sup_{r>0} \frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x-y)|\,dy,$$

where $x \in \mathbb{R}^n$.

## 3. From $T^*$ to $T^\#$

In this section we will show that to obtain the main results of this paper we can equivalently instead of the operator $T^*$ use the operator $T^\#$ defined as follows. For functions in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ let us define

$$T^\#(f,g)(x) = \sup_{j\in\mathbb{Z}} \left| \sum_{i>j} \int_{\mathbb{R}^{2n}} K^i(y,z)\, f(x-y)\, g(x-z)\, dydz \right|.$$

As already mentioned the above defined operator will be the essential tool that we will be working with in this text.

The following two propositions are the key ingredients to obtain the statement of Theorem 2. The second one is proved immediately, the proof of the first one will be the content of the following sections.

**Proposition 6.** *Let $\Omega \in L^2(\mathbb{S}^{2n-1})$, and for $j \in \mathbb{Z}$ consider the bilinear operator*

$$T_j^\#(f,g)(x) = \sup_{k\in\mathbb{Z}} \left| \sum_{i>k} \int_{\mathbb{R}^{2n}} K_j^i(y,z) f(x-y) g(x-z)\, dydz \right|.$$

*If $j \geq 0$, then $T_j^\#$ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ with norm at most $C\|\Omega\|_{L^2} 2^{-\delta j}$, where $\delta$ is a fixed positive constant.*

Furthermore, we can define an operator

$$\widetilde{T}^\#(f,g)(x) = \sup_{j\in\mathbb{Z}} \left| \sum_{i>j} \int_{\mathbb{R}^{2n}} \sum_{\gamma<0} K_\gamma^i(y,z)\, f(x-y)\, g(x-z)\, dydz \right|.$$

Clearly

$$T^\#(f,g)(x) \leq \widetilde{T}^\#(f,g)(x) + \sum_{j\geq 0} T_j^\#(f,g)(x).$$

We have the following

**Proposition 7.** *For $\Omega \in L^2(\mathbb{S}^{2n-1})$, and for $1 < p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$ the operator $\widetilde{T}^\#$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.*

*Proof.* Let us consider the kernel

$$\widetilde{K} = \sum_{i\in\mathbb{Z}} \sum_{\gamma<0} K^i_\gamma.$$

It is easy to check that this is a smooth Calderón-Zygmund convolution kernel and therefore we immediately get from the Cotlar inequality [10] the boundedness of the maximal operator $T^*_{\widetilde{K}}$. Next, we write

$$\begin{aligned}\widetilde{T}^{\#}(f,g)(x) &= \sup_{j\in\mathbb{Z}} \left| \sum_{i>j} \int_{\mathbb{R}^{2n}} \sum_{\gamma<0} K^i_\gamma(y,z) f(x-y) g(x-z)\, dydz \right| \\ &\leq T^*_{\widetilde{K}}(f,g)(x) + \\ &\sup_{j\in\mathbb{Z}} \left| \int_{\mathbb{R}^{2n}} \left( \chi_{\mathbb{R}^{2n}\setminus B(0,2^j)} \widetilde{K} - \sum_{i>j} \sum_{\gamma<0} K^i_\gamma \right)(y,z) f(x-y) g(x-z)\, dydz \right|.\end{aligned}$$

Standard calculation shows that the error term

$$\left| \chi_{\mathbb{R}^{2n}\setminus B(0,2^j)} \widetilde{K} - \sum_{i>j} \sum_{\gamma<0} K^i_\gamma \right|$$

is dominated by $C_N \frac{2^{-2nj}}{1+|2^{-j}x|^N}$ for every $N>0$ and therefore

$$\widetilde{T}^{\#}(f,g)(x) \leq T^*_{\widetilde{K}}(f,g)(x) + CMf(x)Mg(x)$$

and the proof is finished. □

Next lemmas are important tools for the forgoing considerations. They allow us to work with the operator $T^{\#}$ instead of $T^*$. For $\Omega \in L^\infty(\mathbb{S}^{2n-1})$ the following estimate holds.

**Lemma 8.** *Let $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, where $p,q \in (1,\infty)$ and $r \in [1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. Then*

$$\|T^*(f,g)\|_r \leq \left\|T^{\#}(f,g)\right\|_r + C\|Mf\|_p \|Mg\|_q.$$

The simple proof is omitted, instead we will show that the same results holds for more general $\Omega$. We first state and prove the estimate for a directional bilinear maximal function and then use the method of rotations. We start with a definition of the directional maximal functions.

$$M_\alpha(f)(x) = \sup_{t>0} \frac{1}{t} \int_0^t |f(x-\alpha y)|\, dy,$$

$$M_{\alpha,\beta}(f,g)(x) = \sup_{t>0} \frac{1}{t} \int_0^t |f(x-\alpha y) g(x-\beta y)|\, dy,$$

where $\alpha,\beta \in \mathbb{S}^{n-1}$, $x \in \mathbb{R}^n$ and $f,g$ are locally integrable functions on $\mathbb{R}^n$.

The following lemma describes boundedness of the directional bilinear maximal function.

**Lemma 9.** *Let $p,q \in (1,\infty)$ and $r \in (1,\infty)$ and $\alpha,\beta \in \mathbb{S}^{n-1}$. Then*

$$M_{\alpha,\beta} : L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n).$$

*Proof:*

$$\left\|M_{\alpha,\beta}(f,g)\right\|_r=\left(\int_{\mathbb{R}^n}\left[\sup_{t>0}\int_0^t|f(x-\alpha y)g(x-\beta y)|\frac{dy}{t}\right]^r dx\right)^{\frac{1}{r}}$$

$$\leq\left(\int_{\mathbb{R}^n}\left[\sup_{t>0}\left(\frac{1}{t}\int_0^t|f(x-\alpha y)|^s dy\right)^{\frac{1}{s}}\sup_{t>0}\left(\frac{1}{t}\int_0^t|g(x-\beta y)|^z dy\right)^{\frac{1}{z}}\right]^r dx\right)^{\frac{1}{r}},$$

where $s=\frac{p}{r},z=\frac{q}{r}$ and the inequality follows from Hölder inequality since $\frac{r}{p}+\frac{r}{q}=1$. The last expression can be written as

$$\left(\int_{\mathbb{R}^n}\left[M_\alpha\left(|f|^s\right)(x)\right]^{\frac{r}{s}}\left[M_\beta\left(|g|^z\right)(x)\right]^{\frac{r}{z}}dx\right)^{\frac{1}{r}}$$

and $f\in L^p$ implies $f^s\in L^{\frac{p}{s}}$ which implies $Mf^s\in L^{\frac{p}{s}}$, therefore $(Mf^s)^{\frac{1}{s}}\in L^p$ and $(Mf^s)^{\frac{r}{s}}\in L^{\frac{p}{r}}$. Then again from Hölder inequality and also Hardy-Littlewood maximal theorem we get

$$\begin{aligned}\left(\int_{\mathbb{R}^n}\left[M_\alpha\left(|f|^s\right)(x)\right]^{\frac{r}{s}}\left[M_\beta\left(|g|^z\right)(x)\right]^{\frac{r}{z}}dx\right)^{\frac{1}{r}} &\leq\left(\left\|M_\alpha\left(|f|^s\right)^{\frac{r}{s}}\right\|_{\frac{p}{r}}\left\|M_\beta\left(|g|^z\right)^{\frac{r}{z}}\right\|_{\frac{q}{r}}\right)^{\frac{1}{r}}\\ &=\left(\left\|M_\alpha|f|^s\right\|_{\frac{p}{s}}\right)^{\frac{1}{s}}\left(\left\|M_\beta|g|^z\right\|_{\frac{q}{z}}\right)^{\frac{1}{z}}\\ &\leq C\left(\left\|f^s\right\|_{\frac{p}{s}}\right)^{\frac{1}{s}}\left(\left\|g^z\right\|_{\frac{q}{z}}\right)^{\frac{1}{z}}\\ &=C\|f\|_p\|g\|_q.\end{aligned}$$

□

In the following we show boundedness of another type of maximal function.

**Lemma 10.** *Let $p,q,r$ be like in the previous lemma and define*

$$M_\Omega(f,g)(x)=\sup_{R>0}\left|\int_{\mathbb{R}^{2n}}H_\Omega^R(y,z)f(x-y)g(x-z)\,dydz\right|$$

*where $\Omega\in L^1\left(\mathbb{S}^{2n-1}\right),\Omega\geq 0$ and*

$$H_\Omega^R(y,z)=\left|\Omega\left(\frac{(y,z)}{|(y,z)|}\right)\right|R^{-2n}\chi_{B(0,R)}(y,z),$$

*where $x,y,z\in\mathbb{R}^n$. Then $\|M_\Omega(f,g)\|_r\leq\|\Omega\|_1\|f\|_p\|g\|_q$.*

*Proof:* We can express and estimate the term $M_\Omega(f,g)$ as follows

$$M_\Omega(f,g)(x)=\sup_{R>0}\left|\int_{\mathbb{S}^{2n-1}}\Omega(u')R^{-2n}\int_0^R t^{2n-1}f(x-tu_1)g(x-tu_2)\,dtdu\right|,$$

where $u=(u_1,u_2)$ and we recall $u'=\frac{u}{|u|}$ for $u\in\mathbb{R}^{2n}$. Since $\frac{t^{2n-1}}{R^{2n-1}}\leq 1$ we get

$$M_\Omega(f,g)(x)\leq\sup_{R>0}\left|\frac{1}{R}\int_{\mathbb{S}^{2n-1}}\Omega(u')\int_0^R f(x-tu_1)g(x-tu_2)\,dtdu\right|$$

$$\leq \int_{\mathbb{S}^{2n-1}} \Omega(u') M_u(f,g)(x)\, du$$

where $M_u(f,g)(x) = \sup_{R>0} \frac{1}{R}\int_0^R |f(x-tu_1)\, g(x-tu_2)|\, dt$. According to the previous lemma $M_u : L^p \times L^q \to L^r$ therefore $\|M_\Omega(f,g)\|_r \leq \|\Omega\|_1 \|f\|_p \|g\|_q$. □

The previous lemma gives boundedness of the operator $M_\Omega$, which is a necessary tool for the transition from $T^*$ to $T^\#$, but only when $r > 1$. In order to prove the Theorem 2, we need to do this precisely when $r = 1$. We use interpolation to extend the results on $M_\Omega$ to the region where $r \leq 1$. While in the rest of the paper we mostly use $r = 1$, we include the whole result for general interest. First we state and briefly prove generally known result which will give us the triangle inequality for $L^p$ spaces with $p < 1$ and will be needed in the following.

**Lemma 11.** *Let $p < 1$ and assume that the sequence $\left\{\|a_i\|_p\right\}_i$ has exponential decay. Then $\|\sum_{i=1}^{\infty} a_i\|_p < \infty$.*

*Scetch of a proof.* Since $L^p$ for $p < 1$ is a quasi-Banach space, there exists a constant $C > 1$ such that for every $x, y \in L^p$ it holds $\|x+y\|_p \leq C\left(\|x\|_p + \|y\|_p\right)$. Let $N$ be an integer whose value we specify later.

Since $\|\sum_{i=1}^N a_i\|_p \leq C\left(\left\|\sum_{i=1}^{\left[\frac{N}{2}\right]} a_i\right\|_p + \left\|\sum_{i=\left[\frac{N}{2}\right]+1}^{N} a_i\right\|_p\right)$, then

$$\left\|\sum_{i=1}^{N} a_i\right\|_p \leq C^{\log_2(N+1)} \sum \|a_i\|_p \approx N^L \sum \|a_i\|_p,$$

where$L > 0$ is a constant dependent on $N$.

We assumed $\|a_i\|_p \leq 2^{-i\varepsilon}$ for some $\varepsilon > 0$.

Therefore

$$\left\|\sum_{i=1}^{\infty} a_i\right\|_p = \left\|\sum_{k=1}^{N}\left(\sum_{i=1}^{\infty} a_{N(i-1)+k}\right)\right\|_p \leq N^L \sum_{k=1}^{N} \left\|\sum_{i=1}^{\infty} a_{N(i-1)+k}\right\|_p.$$

Also $\sum_{k=1}^N \left\|\sum_{i=1}^{\infty} a_{N(i-1)+k}\right\|_p \leq \sum_{i=1}^{\infty} C^i \left\|a_{N(i-1)+k}\right\|_p \leq \sum_{i=1}^{\infty} C^i 2^{-\varepsilon(N(i-1)+k)}$ so we need $i < \varepsilon(N(i-1)+k)$ for $k = 1, \dots, N$ and every $i > i_0$ for some $i_0$. We obtain $N > \frac{1}{\varepsilon}$. From this we easily get the conclusion. □

**Lemma 12.** *Let $r \in (1/2, \infty)$ and $u, v \in (1, \infty)$ such that $\frac{1}{u} + \frac{1}{v} = \frac{1}{r}$. Let $\Omega \in L^s\left(\mathbb{S}^{2n-1}\right), \Omega \geq 0$ where $s \in (1,\infty)$ is such that $\frac{s}{2s-1} < r$. Then*

$$M_\Omega : L^u(\mathbb{R}^n) \times L^v(\mathbb{R}^n) \to L^r(\mathbb{R}^n).$$

*Proof.* Without loss of generality we can assume $\|\Omega\|_s = 1$. We decompose $\Omega$ as $\Omega = \sum_{i \geq 0} \Omega_i$ where $\Omega_i(x') = \Omega(x')\chi_{E_i}(x)$ and $E_i = \left\{x \in \mathbb{R}^{2n} : \Omega(x') \in \left(2^i, 2^{i+1}\right]\right\}$ for $i > 0$ and $E_0 = \left\{x \in \mathbb{R}^{2n} : \Omega(x') \in [0,2]\right\}$. Then $\Omega_i \in L^1\left(\mathbb{S}^{2n-1}\right)$ and $\|\Omega_i\|_1 \leq 2^{-i\frac{s}{s'}}$ since from Hölder inequality we have

$$\|\Omega_i\|_1 = \int_{\mathbb{S}^{2n-1}} \Omega(x')\chi_{E_i}(x)\, dx \leq \left(\int_{E_i} 1\, dx\right)^{\frac{1}{s'}} \left(\int_{\mathbb{S}^{2n-1}} \Omega^s(x')\, dx\right)^{\frac{1}{s}}$$

$$= |\mathrm{supp}\Omega_i|^{\frac{1}{s'}} K \le K 2^{-i\frac{s}{s'}}$$

where the last inequality follows from the fact that $\int_{\mathrm{supp}\Omega_i} \Omega^s \le 1$ and $\int_{\mathrm{supp}\Omega_i} \Omega^s \cong 2^{is} |\mathrm{supp}\Omega_i|$.

Now let $\varepsilon > 0$. We will use the multilinear interpolation (Theorem 7.2.2 in [8]). Let

- $(p_{11}, p_{12}, q_1) = \left((1+\varepsilon)^2, \frac{(1+\varepsilon)^2}{\varepsilon}, \cdot\right)$
- $(p_{21}, p_{22}, q_2) = \left(\frac{(1+\varepsilon)^2}{\varepsilon}, (1+\varepsilon)^2, \cdot\right)$
- $(p_{31}, p_{32}, q_3) = \left((1+\varepsilon)^2, (1+\varepsilon)^2, \cdot\right)$,

where $q_i$ is such that $\frac{1}{p_{i1}} + \frac{1}{p_{i2}} = \frac{1}{q_i}$ and therefore $q_1 = q_2 = 1+\varepsilon$ and $q_3 = \frac{(1+\varepsilon)^2}{2}$.

Then

- $\|M_{\Omega_i}(f,g)\|_{L^{q_1}} \le 2^{-i\frac{s}{s'}} \|f\|_{p_{11}} \|g\|_{p_{12}}$
- $\|M_{\Omega_i}(f,g)\|_{L^{q_2}} \le 2^{-i\frac{s}{s'}} \|f\|_{p_{21}} \|g\|_{p_{22}}$
- $\|M_{\Omega_i}(f,g)\|_{L^{q_3}} \le 2^{i} \|f\|_{p_{31}} \|g\|_{p_{32}}$

Define

$$\left(\frac{1}{p_1}, \frac{1}{p_2}\right) = \left(\frac{\mu_1}{p_{11}} + \frac{\mu_2}{p_{21}} + \frac{\mu_3}{p_{31}}, \frac{\mu_1}{p_{21}} + \frac{\mu_2}{p_{22}} + \frac{\mu_3}{p_{32}}\right) = \left(\frac{\mu_1+\mu_3+\mu_2\varepsilon}{(1+\varepsilon)^2}, \frac{\mu_2+\mu_3+\mu_1\varepsilon}{(1+\varepsilon)^2}\right)$$

where $\mu_i \in (0,1)$ and $\mu_1+\mu_2+\mu_3 = 1$ and also

$$\frac{1}{q} = \frac{\mu_1}{q_1} + \frac{\mu_2}{q_2} + \frac{\mu_3}{q_3},$$

therefore

$$\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}.$$

We note that $q$ from above is dependent on $\varepsilon$ and $\mu_i$. Then

$$\|M_{\Omega_i}(f,g)\|_{L^q} \le 2^{-i\left(\frac{s}{s'}+\mu_3\left(-\frac{s}{s'}-1\right)\right)} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$$

and we need the exponent to be greater than 0, therefore $s(1-\mu_3) > 1$ which implies $\mu_3 < \frac{1}{s'}$. Then

$$\frac{1}{q} < \frac{1+\varepsilon+\frac{1}{s'}(2-(1+\varepsilon))}{(1+\varepsilon)^2}.$$

Therefore if we choose $(u,v,r)$ such that they satisfy the assumptions and $\frac{1}{r} < \frac{2s-1}{s}$ then we can find $\varepsilon$ small enough so that the above inequality holds. So we get using the previous Lemma that

$$\|M_\Omega(f,g)\|_r \le \sum_i \|M_{\Omega_i}(f,g)\|_r \le C \|f\|_u \|g\|_v.$$

□

Now we can formulate the version of Lemma 8 for $\Omega \in L^s\left(\mathbb{S}^{2n-1}\right)$ where $s$ is like in the previous Lemma. Again, this lemma is most important in the situation where $p = q = 2$ and $r = 1$, but we state it in full generality.

**Lemma 13.** *Let $r \in (1/2,\infty)$ and $p,q \in (1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. Let $\Omega \in L^s\left(\mathbb{S}^{2n-1}\right)$ where $s \in (1,\infty)$ is such that $\frac{s}{2s-1}<q$. Then*

$$\|T^*(f,g)\|_r \leq \left\|T^{\#}(f,g)\right\|_r + C\left\|M_{|\Omega|}(f,g)\right\|_r.$$

*Proof.* Let us define functions $K_\varepsilon(x,y)=K(x,y)\chi_{\{(x,y):|x|,|y|\geq\varepsilon\}}(x,y)$ and $\tilde{K}_\varepsilon(x,y)=K(x,y)\left(1-\alpha\left(\frac{1}{\varepsilon}|(x,y)|\right)\right)$. Then

$$T^*(f,g)(x) \leq \sup_{\varepsilon>0}\left|\int_{\mathbb{R}^{2n}} K_\varepsilon(y,z) f(x-y) g(x-z) - \tilde{K}_{\varepsilon_j}(y,z) f(x-y) g(x-z)\, dydz\right|$$
$$+\sup_{i\in\mathbb{Z}}\left|\int \tilde{K}_i(y,z) f(x-y) g(x-z)\, dydz\right|,$$

where in the first part we take $j$ such that $\varepsilon \approx \varepsilon_j = 2^j$. The second part equals $T^{\#}(f,g)(x)$ since it can be written as

$$\sup_{j\in\mathbb{Z}}\left|\int_{\mathbb{R}^{2n}} K(y,z)\left(1-\alpha\left(2^{-j}|(y,z)|\right)\right) f(x-y) g(x-z)\, dydz\right|$$
$$=\sup_{j\in\mathbb{Z}}\left|\int_{\mathbb{R}^{2n}} \sum_{i>j} K^i(y,z)\left(1-\alpha\left(2^{-j}|(y,z)|\right)\right) f(x-y) g(x-z)\, dydz\right|.$$

Then we can further estimate the first part with

$$\sup_{\varepsilon>0}\left|\int_{\mathbb{R}^{2n}} H^{c_n\varepsilon}_{|\Omega|}(y,z) f(x-y) g(x-z)\, dydz\right|,$$

where $c_n$ is a suitable dimension dependent constant such that the set $\{(x,y):|x|,|y|\geq\varepsilon\}$ is contained in $B(0,c_n\varepsilon)$. □

## 4. Boundedness: a special case

Now we prove the Theorem 2. In view of Proposition 6, Theorem 2 will be a consequence of the following proposition.

**Proposition 14.** *Given $q\in[2,\infty]$ and $\delta\in(0,1/8q')$, then for any $j\in\mathbb{N}_0$, the operator $T^{\#}_j$ maps $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ with norm at most $C\|\Omega\|_{L^q}2^{-\delta j}$.*

To obtain this estimate, we follow the proof from the paper [9] and first decompose the symbol into dyadic pieces, estimate them separately, and then use orthogonality arguments to put them back together which is the point where the arguments will differ from the paper [9]. Let us remind certain properties of the symbol $\widehat{K}^0_j$ which we denote $m_{j,0}$. The classical estimates show that

$$\|m_{j,0}\|_{L^\infty} = \|\widehat{K^0_j}\|_{L^\infty} \leq C\|\Omega\|_{L^q}2^{-\delta j}, \qquad \delta\in\left(0,1/q'\right), \tag{5}$$

while for $q\in[2,\infty]$ it holds

$$\|m_{j,0}\|_{L^2} = \|\beta_j(\widehat{\beta_0 K})\|_{L^2} \leq C\|\widehat{\beta_0 K}\|_{L^2} \leq C\|\Omega\|_{L^2} \leq C\|\Omega\|_{L^q}. \tag{6}$$

We observe that for the case $i\neq 0$ we have the identity $m_{j,i}=\widehat{K}^i_j = m_{j,0}(2^i\cdot)$ from the homogeneity of the symbol, and thus $m_{j,i}$ also lies in $L^2\left(\mathbb{R}^{2n}\right)$.

In this manuscript we will be using the wavelet transform of $m_{j,0}$ taking product wavelets described above in Lemma 4 with compact supports and $M$ vanishing moments, where $M$ is a large number to be determined later. We choose generating functions with support diameter approximately 1. The wavelets with the same dilation factor $2^\lambda$ have some bounded overlap N independent of $\lambda$. With

$$\Psi^{\lambda,G}_{\vec{\mu}}(\vec{x}) = 2^{\lambda n}\Psi^G\left(2^\lambda \vec{x} - \vec{\mu}\right),$$

where $\vec{x} \in \mathbb{R}^{2n}$, we have the next lemma which is another tool to enable us to work with the wavelet technique, the proof can be found in [9].

**Lemma 15.** *Using the preceding notation, for any $j \in \mathbb{Z}$ and $\lambda \in \mathbb{N}_0$ we have*

$$|\langle \Psi^{\lambda,G}_{\vec{\mu}}, m_{j,0}\rangle| \le C\|\Omega\|_{L^q} 2^{-\delta j} 2^{-(M+1+n)\lambda}, \tag{7}$$

*where $M$ is the number of vanishing moments of $\psi_M$ and $\delta$ is as in* (5).

Again as in the paper [9] the wavelets sharing the same generation index $\lambda$ may be organized into $C_{n,M,N}$ groups so that members of the same group have disjoint supports and are of the same product type, i.e., they have the same index $G \in \mathcal{I}$. For $1 \le \kappa \le C_{n,M,N}$ we denote by $D_{\lambda,\kappa}$ one of these groups consisting of wavelets whose supports have diameters about $2^{-\lambda}$. We now have that the wavelet expansion

$$m_{j,0} = \sum_{\substack{\lambda \ge 0 \\ 1 \le \kappa \le C_{n,M,N}}} \sum_{\omega \in D_{\lambda,\kappa}} a_\omega \omega$$

and $\omega$ all have disjoint supports within the group $D_{\lambda,\kappa}$. We recall the following estimates: For the sequence $a = \{a_\omega\}$ we get $\|a\|_{\ell^2} \le C$, if we set $b_\omega = \|a_\omega \omega\|_{L^\infty}$, we have

$$\|\{b_\omega\}_{\omega \in D_{\lambda,\kappa}}\|_{\ell^2} \le C\|\Omega\|_{L^2} 2^{n\lambda}.$$

Then we also have

$$\|\{b_\omega\}_{\omega \in D_{\lambda,\kappa}}\|_{\ell^\infty} \le C\|\Omega\|_{L^q} 2^{-\delta j - (M+1)\lambda}. \tag{8}$$

Now, we split the group $D_{\lambda,\kappa}$ into three parts. Recall the fixed integer $j$ in the statement of Proposition 14. Let us also assume that $j \ge 100\sqrt{n}$ since for $j < 100\sqrt{n}$, Proposition 14 is an easy consequence of Proposition 6. We define sets

$$D^1_{\lambda,\kappa} = \Big\{\omega \in D_{\lambda,\kappa} : a_\omega \ne 0,\ \operatorname{supp}\omega \subset \{(\xi_1,\xi_2) \in \mathbb{R}^n \times \mathbb{R}^n :\ 2^{-j}|\xi_1| \le |\xi_2| \le 2^j|\xi_1|\}\Big\},$$

$$D^2_{\lambda,\kappa} = \Big\{\omega \in D_{\lambda,\kappa} : a_\omega \ne 0,\ \operatorname{supp}\omega \cap \{(\xi_1,\xi_2) \in \mathbb{R}^n \times \mathbb{R}^n :\ 2^{-j}|\xi_1| \ge |\xi_2|\} \ne \emptyset\Big\},$$

and

$$D^3_{\lambda,\kappa} = \Big\{\omega \in D_{\lambda,\kappa} : a_\omega \ne 0,\ \operatorname{supp}\omega \cap \{(\xi_1,\xi_2) \in \mathbb{R}^n \times \mathbb{R}^n :\ 2^{-j}|\xi_2| \ge |\xi_1|\} \ne \emptyset\Big\}.$$

These groups are disjoint for large $j$. Notice that $D^1_{\lambda,\kappa} \cap D^2_{\lambda,\kappa} = \emptyset$ is obvious. For $D^2_{\lambda,\kappa}$ and $D^3_{\lambda,\kappa}$ the worst case is $\lambda = 0$ when we have balls of radius 1 centered at integers, and $D^2_{\lambda,\kappa} \cap D^3_{\lambda,\kappa} = \emptyset$ if $j$ is sufficiently large. We choose $j \ge 100\sqrt{n}$ which works, since if $a_\omega \ne 0$, then $\omega$ is supported in an annulus centered at the origin of size about $2^j$ which is large enough.

We denote, for $\iota = 1,2,3$,

$$m^{\iota}_{j,0} = \sum_{\lambda,\kappa} \sum_{\omega \in D^{\iota}_{\lambda,\kappa}} a_{\omega}\,\omega,$$

and define

$$m^{\iota}_{j} = \sum_{k=-\infty}^{\infty} m^{\iota}_{j,k}$$

with $m^{\iota}_{j,k}(\vec{\xi}\,) = m^{\iota}_{j,0}(2^k\vec{\xi}\,)$. We prove boundedness for each piece $m^1_j, m^2_j, m^3_j$. We call $m^1_j$ the diagonal part of $m_j$ and $m^2_j, m^3_j$ the off-diagonal parts of $m_j = \widehat{K_j}$.

## 5. The diagonal part

We first deal with the first group $D^1_{\lambda,\kappa}$. Using the same arguments like in the paper [9] we will obtain (omitting details) again the similar estimate of $T_{m^1_{j,k}}$, where

$$T_{m^1_{j,k}}(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m^1_{j,k}(y,z)\,\widehat{f}(y)\,\widehat{g}(z)\,e^{2\pi i(y+z)\cdot x}\,dy dz.$$

We denote by $f_k$ the function whose Fourier transform is $\widehat{f}(2^{-k}\xi_1)$ and $E_{j,k} = \{\xi_1 \in \mathbb{R}^n : c_1 2^k \leq |\xi_1| \leq c_2 2^{j+k}\}$, where $c_1, c_2$ are suitable constants such that $\|T_{m^1_{j,0}}(f,g)\|_{L^1} = \|T_{m^1_{j,0}}(\mathcal{F}^{-1}\left(\widehat{f}\chi_{\{c_1 2^j \leq |\xi_1| \leq c_2 2^{j+1}\}}\right), \mathcal{F}^{-1}\left(\widehat{g}\chi_{\{c_1 2^j \leq |\xi_1| \leq c_2 2^{j+1}\}}\right))\|_{L^1}$. Then

$$\begin{aligned}
\|T_{m^1_{j,k}}(f,g)\|_{L^1} &= 2^{-2kn}\|T_{m^1_{j,0}}(f_k,g_k)(2^{-k}\cdot)\|_{L^1} \\
&= 2^{-kn}\|T_{m^1_{j,0}}(f_k,g_k)\|_{L^1} \\
&\leq C\|\Omega\|_{L^q}2^{-kn}2^{-\delta j/5}\|\widehat{f}(2^{-k}\cdot)\chi_{E_{j,0}}\|_{L^2}\|\widehat{g}(2^{-k}\cdot)\chi_{E_{j,0}}\|_{L^2} \\
&= C\|\Omega\|_{L^q}2^{-\delta j/5}\|\widehat{f}\|_{L^2(E_{j,k})}\|\widehat{g}\|_{L^2(E_{j,k})},
\end{aligned}$$

where the inequality is described in detail in [9].

Using this estimate, applying the Cauchy-Schwarz inequality and modifying the first part to get the estimate for our operator we obtain for the diagonal part

$$\begin{aligned}
\|\sup_{\gamma\in\mathbb{Z}}\sum_{k>\gamma} T_{m^1_{j,k}}(f,g)\|_{L^1} &\leq \sum_{k=-\infty}^{\infty}\|T_{m^1_{j,k}}(f,g)\|_{L^1} \\
&\leq C\|\Omega\|_{L^q}\,j2^{-\delta j/5}\|f\|_{L^2}\|g\|_{L^2},
\end{aligned}$$

where again the last inequality was explained in detail in [9]. This completes the estimate of the first piece $m^1_j$.

## 6. The off-diagonal parts

We now estimate the off-diagonal parts of the operator, namely

$$\sup_{\gamma\in\mathbb{Z}}\left|\sum_{k>\gamma} T_{m^2_{j,k}}(f,g)\right|.$$

At first we need to have an approximate size of the Fourier support of

$$T_{m_{j,k}^2}(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m_{j,k}^2(\alpha,\beta)\hat{f}(\alpha)\hat{g}(\beta)e^{2\pi i(\alpha+\beta)\cdot x}d\alpha d\beta.$$

We will estimate the support of $m_{j,k}^2$ since knowing that we can get the required support of $\widehat{T_{m_{j,k}^2}}(f,g)$ from the following lemma.

**Lemma 16.** *Let $L^\infty$ function $m$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ be a symbol of a multiplier operator $T_m$. Suppose $I,J \subset \mathbb{R}^n$ are measurable sets and let* $\operatorname{supp} m \subset I \times J$. *Then for $f,g \in \mathcal{S}(\mathbb{R}^n)$ it holds that* $\operatorname{supp} T_m(f,g) \subset I+J$.

*Proof.* Using change of variables we obtain

$$T_m(f,g)(x) = \int_{\mathbb{R}^n} G(\alpha)e^{2\pi i x\cdot\alpha}d\alpha = \check{G}(x),$$

where

$$G(\alpha) = \int_{\mathbb{R}^n} m(\alpha-\beta,\beta)\hat{f}(\alpha-\beta)\hat{g}(\beta)d\beta.$$

Therefore $\widehat{T_m}(f,g)(x) = G(x)$ and we need to find the support of $G$ which is:

$G(\alpha) \neq 0$ if and only if there exists $\beta \in \mathbb{R}^n$ such that $m(\alpha-\beta,\beta) \neq 0$, therefore if $\operatorname{supp} m \in I \times J$ then $\operatorname{supp} G \in I+J$.

□

Now we want to find the approximate size of the support of $m_{j,k}^2$. We first deal with $m_{j,0}^2$ since the general case will follow from the fact that $m_{j,k}^2(x) = m_{j,0}^2(2^k x)$ and simple modification of the calculation.

It holds that a point $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ is in the support of $m_{j,0}^2$ if there exists $\omega \in D_{\lambda,\kappa}^2$ such that $(x,y) \in \omega$ and

$$\operatorname{supp}\omega \cap \{(u,v) : 2^{-j}|u| \geq |v|\} \cap \{(u,v) : 2^{j-1} \leq |(u,v)| \leq 2^{j+1}\} \neq \emptyset.$$

The size of the wavelet $\omega$ (again in the case $m_{j,0}^2$) can be estimated by 1, therefore by simple calculation we get

$$\operatorname{supp} m_{j,0}^2 \subseteq A \times B,$$

where

$$A = \{\xi \in \mathbb{R}^n : 2^{j-2} - 1 \leq |\xi| \leq 2^{j+1} + 1\}$$

and

$$B = \{\xi \in \mathbb{R}^n : -3 - 2^{-j} \leq |\xi| \leq 3 + 2^{-j}\}.$$

Therefore

$$\operatorname{supp}\widehat{T_{m_{j,0}^2}}(f,g) \subseteq \{\xi \in \mathbb{R}^n : 2^{j-3} \leq |\xi| \leq 2^{j+2}\}.$$

If we now consider $m_{j,k}^2$ we obtain

$$\operatorname{supp}\widehat{T_{m_{j,k}^2}}(f,g) \subseteq \left\{\xi \in \mathbb{R}^n : 2^{j-3-k} \leq |\xi| \leq 2^{j+3-k}\right\}.$$

Now we can proceed to estimating $\sup_{\gamma\in\mathbb{Z}}\left|\sum_{k>\gamma}T_{m_{j,k}^2}(f,g)\right|$. From the paper [9] we know that for every $f,g\in L^2(\mathbb{R}^n)$ the following estimate holds

$$\left\|\left(\sum_{k\in 5\mathbb{Z}}\left|T_{m_{j,k}^2}(f,g)\right|^2\right)^{\frac{1}{2}}\right\|_{L^1}\leq C\|\Omega\|_{L^q}2^{-j\delta}\|f\|_{L^2}\|g\|_{L^2}. \tag{9}$$

For $\mu=0,...,10$ we have

$$T_{m_j^2}^{\#}(f,g)=\sup_{\gamma\in\mathbb{Z}}\sum_{\mu}\left|\sum_{\substack{k>\gamma\\ k\in 10\mathbb{Z}+\mu}}T_{m_{j,k}^2}(f,g)\right|,$$

so we fix $\mu$. Then we have

$$\sup_{\gamma\in\mathbb{Z}}\left|\sum_{\substack{k>\gamma\\ k\in 10\mathbb{Z}+\mu}}T_{m_{j,k}^2}(f,g)\right|\leq\sup_{\beta>0}\left|\sum_{k\in 10\mathbb{Z}+\mu}T_{m_{j,k}^2}(f,g)-\psi_\beta*\left(\sum_{k\in 10\mathbb{Z}+\mu}T_{m_{j,k}^2}(f,g)\right)\right|,$$

where $\psi_\beta$ is a function such that its Fourier transform is equal to 1 on $B\left(0,2^{j-\beta+3}\right)$ and vanishes outside of $B\left(0,2^{j-\beta+10}\right)$, more precisely we have a function $\psi$ such that $\widehat{\psi}\equiv 1$ on $B\left(0,2^{j+3}\right)$, $\widehat{\psi}\equiv 0$ on $B\left(0,2^{j+10}\right)$ and $\widehat{\psi}\in(0,1)$ otherwise. Then we define $\psi_\beta=\widehat{\psi}\left(2^\beta\cdot\right)$.

Then we can further estimate the previous expression with

$$\left|\sum_{k\in 10\mathbb{Z}+\mu}T_{m_{j,k}^2}(f,g)\right|+\sup_{\beta>0}\left|\psi_\beta*\left(\sum_{k\in 10\mathbb{Z}+\mu}T_{m_{j,k}^2}(f,g)\right)\right|.$$

The second part can be estimated with maximal function defined as $M_\psi f(x)=\sup_{\beta>0}\left|\psi_\beta*f\right|$, therefore

$$\sup_{\beta>0}\left|\psi_\beta*\left(\sum_{k\in 10\mathbb{Z}+\mu}T_{m_{j,k}^2}(f,g)\right)(x)\right|\leq CM_\psi\left(\sum_{k\in 10\mathbb{Z}+\mu}T_{m_{j,k}^2}(f,g)\right)(x).$$

If we consider now the $L^1$ norm of the expression $\sup_{\gamma\in\mathbb{Z}}\left|\sum_{k>\beta}T_{m_{j,k}^2}(f,g)\right|$, we get

$$\left\|\sup_{\gamma\in\mathbb{Z}}\left|\sum_{k>\gamma}T_{m_{j,k}^2}(f,g)\right|\right\|_{L^1}\leq\left\|\sum_{k\in 10\mathbb{Z}+\mu}T_{m_{j,k}^2}(f,g)\right\|_{L^1}+C\left\|M_\psi\left(\sum_{k\in 10\mathbb{Z}+\mu}T_{m_{j,k}^2}(f,g)\right)\right\|_{L^1}$$

The first expression can be estimated as follows. Since the Hörmander condition for multilinear multipliers holds for $m_{j,k}^2$ and $m_{j,k}^2$ is smooth, then $\sum_{k\in 10\mathbb{Z}+\mu}T_{m_{j,k}^2}(f,g)$ is in $L^1(\mathbb{R}^n)$. Also since we have the square function estimate in (9), there exists polynomial $Q_j^\mu$ such that

$$\left\|\sum_{k\in 10\mathbb{Z}+\mu}T_{m_{j,k}^2}(f,g)-Q_j^\mu\right\|_{L^1}\leq\left\|\left(\sum_{k\in 10\mathbb{Z}}\left|T_{m_{j,k}^2}(f,g)\right|^2\right)^{\frac{1}{2}}\right\|_{L^1}$$

$$\leq C \|\Omega\|_{L^q} 2^{-j\delta} \|f\|_{L^2} \|g\|_{L^2}.$$

Therefore $Q_j^\mu = 0$ and

$$\left\| \sum_{k \in 10\mathbb{Z}+\mu} T_{m_{j,k}^2}(f,g) \right\|_{L^1} \leq C \|\Omega\|_{L^q} 2^{-j\delta} \|f\|_{L^2} \|g\|_{L^2}.$$

In fact (9) also implies that $\sum_{k\in\mathbb{Z}} T_{m_{j,k}^2}(f,g)$ is in $H^1$, therefore we can estimate the second expression as

$$\left\| M_\psi \left( \sum_{k \in 10\mathbb{Z}+\mu} T_{m_{j,k}^2}(f,g) \right) \right\|_{L^1} \leq \left\| \sum_{k \in 10\mathbb{Z}+\mu} T_{m_{j,k}^2}(f,g) \right\|_{H^1}.$$

This finishes the proof of Proposition 14 and therefore also the proof of the Theorem 2

## 7. Interpolation

Finally, we want to prove the Theorem 1 and the Corollary 3. We recall Lemma 10 from the article [9], which states that the kernel $K_j$ is a Calderón-Zygmund kernel with $\varepsilon$-Lipschitz constant

$$A_\varepsilon \leq C_\varepsilon \|\Omega\|_\infty 2^{|j|\varepsilon},$$

for $\varepsilon \in (0,1)$. We are only using $j \geq 0$ in what follows.

Using the Cotlar inequality from [10], we can therefore get for any combination $1 < p,q < \infty$, $1/r = 1/p + 1/q$ and any $\varepsilon \in (0,1)$ the bound

$$\|T_j^*\|_{L^p \times L^q \to L^r} \leq C_{p,q,\varepsilon} \|\Omega\|_\infty 2^{j\varepsilon}.$$

By $T_j^*$ we denote the maximal singular bilinear operator with kernel $K_j$. Next, we need to observe that

$$\|T_j^\#\|_{L^p \times L^q \to L^r} \leq \|T_j^*\|_{L^p \times L^q \to L^r} + C\|\Omega\|_\infty \|M\|_{L^p \times L^q \to L^r}.$$

This is rather trivial, the reasoning is very similar to the proof of the Proposition 7 and so we do not give full detail here. The proof of the Theorem 1 is now simple. If we have a fixed point $(p,q)$, $1 < p,q < \infty$, we find a pair of points $(p_1,q_1)$, $1 < p_1,q_1 < \infty$, and $(p_2,q_2)$, $1 < p_2,q_2 < \infty$ such that $(1/p,1/q)$ lies inside the triangle $(1/2,1/2)$, $(1/p_1,1/q_1)$ and $(1/p_2,1/q_2)$. According to the Proposition 14 the operators $T_j^\#$ have norm at the point $(2,2)$ at most $C\|\Omega\|_{L^q} 2^{-\delta j}$, where $\delta > 0$ is fixed, while at the remaining two points the norm is $C_{p_i,q_i,\varepsilon} \|\Omega\|_\infty 2^{j\varepsilon}$, where $i = 1,2$ and we may choose $\varepsilon > 0$ arbitrarily small. Therefore, for a suitable choice of $\varepsilon > 0$, we get from the interpolation (Theorem 7.2.2 in [8]) that the series of norms of $T_j^\#$ is convergent at $(p,q)$. This finishes the proof of Theorem 2.

To prove the Corollary 3, we apply interpolation argument similar to the proof of Lemma 12. We split $\Omega$ is similar way, but we need to make sure that the integral over sphere vanishes. Let us assume that $\|\Omega\|_q \leq 1$. We decompose $\Omega$ as $\Omega = \sum_{i\geq 0} \widetilde{\Omega}_i$ where we first denote $\Omega_i(x') = \Omega(x') \chi_{E_i}(x)$ and

$$E_i = \left\{ x \in \mathbb{R}^{2n} : |\Omega|(x') \in \left(2^i, 2^{i+1}\right] \right\}$$

for $i > 0$ and $E_0 = \{x \in \mathbb{R}^{2n} : |\Omega|(x') \in [0,2]\}$, and then we set

$$\widetilde{\Omega}_i = \Omega_i - \int_{\mathbb{S}^{2n-1}} \Omega_i(x)dx.$$

We have $\|\Omega_i\|_1 \leq C2^{-i\frac{q}{q}}$ and therefore the sum converges back to $\Omega$. We see that $\|\widetilde{\Omega}_i\|_2 \leq C2^{\frac{-i(q-2)}{2}}$, while $\|\widetilde{\Omega}_i\|_\infty \leq C2^i$. Now, we estimate

$$T^*_\Omega(f,g)(x) \leq \sum_{i\geq 0} T^*_{\widetilde{\Omega}_i}(f,g)(x).$$

We interpolate the norm of each of the operators $T^*_{\widetilde{\Omega}_i}$ in a triangle which contains $(1/p_1, 1/p_2)$, and has one vertex in the point $(1/2, 1/2)$, where the norm $\|T^*_{\widetilde{\Omega}_i}\|_{L^2\times L^2\to L^1} \leq C2^{\frac{-i(q-2)}{2}}$ according to the Theorem 2 and remaining two vertices close to points $(0,0)$, $(0,1)$, $(1,1)$, or $(1,0)$ where the operator has norm less than $C2^i$. The interpolated norms form convergent series precisely when the point $(1/p_1, 1/p_2)$ lies inside the quadrilateral described in the Corollary 3 and the proof is finished.

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# THE LATTICE BUMP MULTIPLIER PROBLEM

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ABSTRACT. We study the lattice bump multiplier problem. Precisely, given a smooth bump supported in a ball centered at the origin, we consider the multiplier formed by adding the translations of this bump centered at $N$ distinct lattice points. We investigate the behavior of the $L^p$ norm of the linear and bilinear multiplier operators associated with this multiplier, as $N \to \infty$. We obtain results in the linear case for all exponents $p$ in $[1, \infty]$ and in the bilinear case for exponents $(p_1, p_2, p)$ satisfying $1/p_1 + 1/p_2 = 1/p$; our results are optimal when $p > 1$.

## 1. Introduction

There has been a lot of work recently done in the theory of bilinear multipliers of Hörmander type; see for instance [16], [7], [13], [11], [3], [4], [12]. The optimal smoothness required of the symbol to have boundedness on a given $L^p$ is closely related to questions about the boundedness of a multiplier given by finite sum of translations of a given bump. In this paper we promote this point of view and we study multiplier operators associated with Fourier multipliers of this type. We focus on the bilinear case, although we briefly discuss the $L^p$ behavior of linear multipliers of this sort.

We fix a smooth bump $\phi$ supported in the ball $|\xi| \le \frac{1}{10}$ in $\mathbb{R}^n$. Consider the linear operator defined for $k \in \mathbb{Z}^n$

$$S_{k,\phi}(f)(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)\phi(\xi - k)e^{2\pi i x\cdot\xi}\, d\xi,$$

where $\widehat{f}(x) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx$ is the Fourier transform of a Schwartz function $f$.

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Suppose we are given a finite subset $E$ of $\mathbb{Z}^n$. Associated with $E$ and $\phi$ we define a linear operator acting on Schwartz functions

$$L_{E,a,\phi} := \sum_{k\in E} a_k S_{k,\phi}(f),$$

where $a = \{a_k\}_{k\in\mathbb{Z}}$ is a sequence of complex numbers satisfying $|a_k| \le 1$ for all $k$. We pose the following problem about $L_{E,a,\phi}$.

**Problem 1:** Given $p \in [1,\infty]$ what is the smallest value $\alpha(p)$ such that for all subsets $E$ of $\mathbb{Z}^n$ with $|E| = N$ we have

$$\|L_{E,a,\phi}\|_{L^p\to L^p} \le C_{p,n,\phi}\, N^{\alpha(p)}\,?$$

The trivial $L^2$ and $L^\infty$ estimates yield by interpolation that

$$\alpha(p) \le 2\Big|\frac{1}{p} - \frac{1}{2}\Big|. \tag{1}$$

But estimate (1) is not sharp as it can be improved by a factor of $1/2$. Results similar to the following are known in different formulations in the literature, e.g. [10, inequality (7)], but for the sake of completeness we include a proof for it in the next section.

**Proposition 1.1.** *For any $p \in [1,\infty]$ and $\alpha(p) = |\frac{1}{p} - \frac{1}{2}|$, we have*

$$\|L_{E,a,\phi}\|_{L^p\to L^p} \le CN^{\alpha(p)}. \tag{2}$$

*Conversely,*

$$\sup_{a:\,\|a\|_{\ell^\infty}\le 1}\ \sup_{E:\,|E|=N} \|L_{E,a,\phi}\|_{L^p\to L^p} \ge CN^{\alpha(p)}. \tag{3}$$

Next, we consider the analogous bilinear problem in $2n$ dimensions. We fix a smooth bump $\Phi$ supported in the ball $|\xi| \le \frac{1}{20}$ in $\mathbb{R}^{2n}$. For a subset $E$ of $\mathbb{Z}^{2n}$ we consider the following bilinear operator

$$B_{E,\Phi}(f,g)(x) := \sum_{(k,l)\in E} S_{(k,l),\Phi}(f\otimes g)(x,x).$$

**Problem 2:** Given $p_1, p_2$ with $1 \le p_1, p_2 \le \infty$, what is the smallest value $\alpha(p_1,p_2)$ such that for all subsets $E$ of $\mathbb{R}^n$ with $|E| = N$ we have

$$\|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \le C_{p_1,p_2,n,\Phi}\, N^{\alpha(p_1,p_2)}\,?$$

Here we consider the situation where $p$, $p_1$, $p_2$ are related as in Hölder's inequality, i.e., $1/p = 1/p_1 + 1/p_2$.

In this paper we mostly focus on Problem 2 and we obtain almost optimal results about it. To state our results we define the index

$$\alpha(p_1,p_2) = \frac{1}{2}\Big[\max\Big(\frac{1}{p}-\frac{1}{2},0\Big) - \min\Big(\frac{1}{p_1}-\frac{1}{2},0\Big) - \min\Big(\frac{1}{p_2}-\frac{1}{2},0\Big)\Big]. \tag{4}$$

**Theorem 1.2.** *Let* $1 \leq p_1, p_2 < \infty$ *and* $1/p = 1/p_1 + 1/p_2$ *and* $\epsilon > 0$.
*(i) If* $p \leq 1$, *then there is a constant* $C = C_{p_1,p_2,\epsilon}$ *such that*

$$\|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \leq CN^{\frac{1}{\min(p_1,p_2)}+\frac{1}{2\max(p_1,p_2)}-\frac{1}{2}+\epsilon}.$$

*(ii) If* $p > 1$, *then there is a constant* $C = C_{p_1,p_2,\epsilon}$ *such that*

$$\|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \leq CN^{\alpha(p_1,p_2)+\epsilon}.$$

*Moreover estimate (ii) is sharp up to* $\epsilon$.

Theorem 1.2 yields that in the local $L^2$ case, i.e., the case where $2 \leq p_1, p_2, p' \leq \infty$, constant $\alpha(p_1, p_2) = 1/4$. We also have results concerning upper and lower bounds for the constant $\alpha(p_1, p_2)$ for indices outside the local $L^2$ case. These are discussed in Sections 5 and 6.

Theorem 1.2 is inspired by the recent solution of the rough bilinear singular integral problem, see [6]. A partial result of this theorem when $p_1 = p_2 = 2$, combined with techniques from harmonic analysis, led us to the solution of the boundedness of rough homogeneous bilinear singular integrals in the largest possible open range of indices.

Throughout this paper, $C$ will denote a constant independent of $N$ and dependent only on auxiliary parameters which may vary in different occurrences, and $\epsilon > 0$ is an arbitrarily small constant.

## 2. The linear case: The proof of Proposition 1.1

The proof of the positive direction of Proposition 1.1, namely (2), is straightforward.

*Proof of* (2). We observe that the multiplier of $L_{E,a,\phi}$ is

$$\sigma(\xi) = \sum_{k\in E} a_k \phi(\xi - k).$$

We have that $\sigma^\vee(x) = \phi^\vee(x) \sum_{k\in E} a_k e^{2\pi i x\cdot k}$ satisfies

$$\left\|\sigma^\vee\right\|_{L^1} \leq \Big\| \sum_{k\in E} a_k e^{e\pi i x\cdot k} \Big\|_{L^1([0,1]^n)}$$

due to the rapid decay of $\phi^\vee$ and the periodicity of $\sum_{k\in E} a_k e^{e\pi i x\cdot k}$. This in turn is bounded by

$$\Big\| \sum_{k\in E} a_k e^{e\pi i x\cdot k} \Big\|_{L^2([0,1]^n)} = \Big( \sum_k |a_k|^2 \Big)^{1/2} \leq N^{1/2}, \tag{5}$$

where in last step we use that $|a_k| \leq 1$. This implies $\|L_{E,a,\phi}\|_{L^1\to L^1} \leq CN^{1/2}$. Interpolating between the trivial $L^2$ estimate

$$\|L_{E,a,\phi}\|_{L^2\to L^2} \leq C$$

and (5) we obtain (2). □

**Remark 2.1.** *We can prove a continuous version of Proposition 1.1. Define $\phi_E(\xi) = \phi * \chi_E$, where the Lebesgue measure of $E$ is $N$. Then it is easy to verify that*

$$\|\phi_E^\vee\|_{L^1} \le \|\phi\|_{L^2}\|\chi_E\|_{L^2} \le C|E|^{1/2},$$

*which implies that the associated operator given by convolution with $\phi_E$ is bounded on $L^p(\mathbf{R}^n)$ with bound $CN^{\alpha(p)}$. We can also replace $\chi_E$ by an arbitrary function $\psi$ and the bound becomes $C\|\psi\|_{L^1}^{\alpha(p)}$. I suggest we remove this. Loukas*

We now turn to the proof of (3). We fix a smooth bump $\phi$ supported in the ball of radius $1/10$. We will need the following lemma.

**Lemma 2.1.** *Let*

$$E = E_N = \{-N, -N+1, \dots, N-1, N\}.$$

*For any fixed $p \in [1,2]$, there exists a constant $C_p > 0$ and a sequence $a = \{a_k\}_{k\in E}$ such that for all positive integers $N$ we have*

$$\|L_{E,a,\phi}\|_{L^p\to L^p} \ge C_p N^{\alpha(p)}. \tag{6}$$

*Proof.* For simplicity we first consider the one-dimensional case.

We first take $p \in (1,2]$. The following counterexample is inspired by an example in [8]. Let

$$m(\xi) = \sum_{|k|\le N} a_k(t)\phi(\xi-k), \quad m_N(\xi) = \sum_{|k|\le N} a_k(t)\phi(N\xi-k),$$

where $a_j$ are the Rademacher functions. Take a smooth function $\varphi$ supported in the support of $\phi$ such that $\varphi\phi \neq 0$ and define $f$, $f_N$ via

$$\widehat{f}(\xi) = N^{-\frac{1}{p'}} \sum_{|k|\le N} \varphi(\xi-k), \qquad \widehat{f_N}(\xi) = \sum_{|k|\le N} \varphi(N\xi-k).$$

Then $\|f\|_{L^p} = \|f_N\|_{L^p} \le C$ as the Dirichlet kernel $D_N$ has $L^p$ norm comparable to $N^{-1/p'}$. Let $T_m$ be the linear operator associated with $m$ in the form $T_m(f) = (\widehat{f}m)^\vee$, then $\|T_m(f)\|_{L^p} = \|T_{m_N}(f_N)\|_{L^p}$, and by applying Khintchine's inequality we have

$$\begin{aligned}\int_0^1 \|T_{m_N}(f_N)\|_{L^p(\mathbb{R})}^p dt &= \int_0^1 \int_{\mathbb{R}} \Big| \sum_{|k|\le N} a_k(t) N^{-1} (\phi\varphi)^\vee (N^{-1}x) e^{2\pi i x \frac{k}{N}} \Big|^p dx dt \\ &\sim \int_{\mathbb{R}} \Big( \sum_{|k|\le N} \Big| N^{-1}(\phi\varphi)^\vee (N^{-1}x) e^{2\pi i x\frac{k}{N}} \Big|^2 \Big)^{\frac{p}{2}} dx \\ &\sim N^{-p} \int_{\mathbb{R}} N^{\frac{p}{2}} |(\phi\varphi)^\vee (N^{-1}x)|^p dx \\ &\sim N^{(\frac{1}{p}-\frac{1}{2})p}. \qquad (7)\end{aligned}$$

Denote $\sup_a \|L_{E,a,\phi}\|_{L^p\to L^p}$ by $C_p(N)$, then

$$\int_0^1 \|T_m(f)\|_{L^p}^p dt \le C_p(N)^p.$$

In summary $C_p(N) \ge C_p' N^{\alpha(p)}$. In particular, we can find a sequence $a$ such that $\|L_{E,a,\phi}\|_{L^p\to L^p} \ge C_p N^{\alpha(p)}$ with $C_p < C_p'$ for $p \in (1,2]$.

Notice that $\|f\|_1 \le C \log N$ by the $L^1$-norm of $D_N$, therefore we can only show that $C_1(N) \ge CN^{1/2}(\log N)^{-1}$ by the same argument. On the other hand, interpolation can help us to remove the logarithmic term below.

We next consider the case when $p = 1$. Suppose that (6) fails for $p = 1$. This is equivalent to saying that for any $C > 0$ there exists a corresponding $N_C$ such that

$$\sup_a \|L_{E,a,\phi}\|_{L^1\to L^1} \le CN_C^{\alpha(1)} = CN_C^{\frac{1}{2}}. \qquad (8)$$

Interpolating between $\sup_a \|L_{E,a,\phi}\|_{L^2\to L^2} \le C'$ and (8) we obtain that, when $p \in (1,2)$, for any $C_p > 0$ there exists a number $N$ (by choosing $C$ in (8) small enough) such that

$$\sup_a \|L_{E,a,\phi}\|_{L^p\to L^p} \le \tfrac{C_p}{2} N^{\alpha(p)};$$

this contradicts (6) for $p \in (1,2)$. In other words (6) holds when $p = 1$.

We now consider the higher dimensional case. The idea is simply to consider products of the one-dimensional example. We briefly describe this example.

Taking

$$m(\xi) = \sum_{|k|\le N} a_k(t) \phi(\xi_1 - k) \prod_{j=2}^{n} \phi(\xi_j - 1),$$

and defining

$$\widehat{f}(\xi) = N^{-1/p'} \sum_{|k|\le N} \varphi(\xi_1 - k) \prod_{j=2}^{n} \varphi(\xi_j - 1),$$

we have

$$T_m(f)(x) = N^{-1/p'} \Big( \sum_{|k|\le N} a_k(t) (\phi\varphi)^\vee (x_1) e^{2\pi i x_1 k} \Big) \prod_{j=2}^{n} (\phi\varphi)^\vee (x_j) e^{2\pi i x_j}.$$

Since the variables are separated, this is essentially the one-dimensional case, and running the same argument as before yields the same necessary condition.

Combining these results we obtain the proof of Proposition 1.1. □

## 3. The bilinear problem

In this section we study the analogous bilinear problem in $2n$ dimensions.

We apply the Fourier series method of Coifman and Meyer [1, 2] to express the smooth bump $\Phi$ as a sum of products of bumps in each half of the variables. As the function $\Phi$ is supported in the ball $B(0, 1/20)$, which is contained in $[-1/2, 1/2]^{2n}$ we can express it in Fourier series as

$$\Phi(\xi, \eta) = \sum_{r,s\in \mathbf{Z}^n} c_{r,s} e^{2\pi i r\cdot\xi} e^{2\pi i s\cdot\eta} \phi(\xi)\phi(\eta)\,,$$

where $\phi(\xi)$ is smooth, is equal to 1 on $|\xi| \le 1/20$, and vanishes outside $|\xi| \le 1/10$. Moreover,

$$c_{r,s} = \int_{B(0,1/20)} \Phi(x, y) e^{-2\pi i (x\cdot r + y\cdot s)} dxdy$$

and an easy integration by parts shows that

$$|c_{r,s}| \le C_M (1 + |r| + |s|)^{-M}$$

for every $M > 0$, where $C_M$ depends on the $L^\infty$ norms of sufficiently many derivatives of $\Phi$. Letting $\phi_r(\xi) = e^{2\pi i r\cdot\xi}\phi(\xi)$, we have that

$$S_{(k,l),\Phi}(f \otimes g)(x, x) = \sum_{r,s\in \mathbf{Z}^n} c_{r,s} S_{k,\phi_r}(f)(x) S_{l,\phi_s}(g)(x)$$

and in view of the rapid decay of $c_{r,s}$, it will suffice to study an analogous problem for $S_{k,\phi_r}(f)(x)S_{l,\phi_s}(g)(x)$ in place of $S_{(k,l)}(f \otimes g)(x, x)$ and obtain estimates for the norm that are independent of $r$ and $s$.

We make the remark that the same approach can handle the two adjoints of $B_E$. Let us look at the first adjoint of $S_{(k,l)}(f \otimes g)$. This is associated with the multiplier

$$\begin{aligned}\Phi(-\xi-\eta-k,\eta-l) =&\Phi(-(\xi+k+l)-(\eta-l),\eta-l)\\ =&\Phi^{*1}(\xi-(-l-k),\eta-l),\end{aligned}$$

where $\Phi^{*1}(\xi,\eta)=\Phi(-\xi-\eta,\eta)$. Now notice that as $(k,l)$ varies over $E$, then $(-k-l,l)$ varies over

$$E^{1*}=\big\{(-k-l,l):\ (k,l)\in E\big\}$$

and $|E^{1*}|=|E|$, while the bump $\Phi^{*1}$ is smooth, has $L^\infty$ norm 1 and is supported in $\{(\xi,\eta):\ |\xi+\eta|^2+|\eta|^2\le\frac{1}{400}\}$ which is contained in $\{(\xi,\eta):\ |(\xi,\eta)|\le\frac{\sqrt{3}}{20}\}$, which is only slightly larger than $B(0,1/20)$. Thus any theorem about $B_{E,\Phi}$ can also be applied to the first adjoint $B^{*1}_{E,\Phi}=B_{E^{*1},\Phi^{*1}}$ of $B_{E,\Phi}$, which has the same characteristics as $B_{E,\Phi}$. This symmetry is one main advantage of $B_{E,\Phi}(f,g)(x)$ compared with $S_{k,\phi_r}(f)(x)S_{l,\phi_s}(g)(x)$.

## 4. The case $p_1=p_2=2$

In this section we prove the sufficiency part of Theorem 1.2. By duality and interpolation it will suffice to consider only the case $p_1=p_2=2$ and $p=1$. The trivial estimate is

$$\|B_{E,\Phi}\|_{L^2\times L^2\to L^1}\le C\,N,$$

but it turns out that the optimal value of the constant $\alpha(2,2)=\frac{1}{4}$. The consideration here is related to the proof of [9, Theorem 1.3], which enhances the combinatorial argument in [6]

*Proof.* We denote by $E'$ the set of all $k\in\mathbb{Z}^n$ with the property that there exists an $l\in\mathbb{Z}^n$ such that the point $(k,l)\in E$. That is $E'$ is the set of all first coordinates of elements of $E$. We think of the set $E$ as a union of columns $Col_k$ indexed by $k\in E'$ and we write

$$E=\bigcup_{k\in E'}Col_k.$$

By the argument in Section 3 it suffices to consider the case when $B_{E,\Phi}(f,g)$ is a sum of products of operators of the form

$$T_{\sigma_N}(f,g):=\sum_{k\in E'}S_k(f)\sum_{l:\,(k,l)\in Col_k}S_l(g),$$

where $\sigma_N:=\sum_{(k,l)\in E}\phi_r(\xi-k)\phi_s(\eta-l)$, and we have dropped the dependence on $\phi_r$ and $\phi_s$ for notational convenience.

We split the columns in large and small. Precisely, we write

$$E = E_1 \cup E_2,$$

where $E_1$ contains all columns of size $\geq K$ and $E_2$ contains all columns of size $< K$, for some $K$ to be chosen later. Analogously we split

$$E' = E_1' \cup E_2',$$

where $E_1'$ and $E_2'$ is the set of all first coordinates of columns in $E_1$ and $E_2$, respectively. Correspondingly we define:

$$T^1_{\sigma_N}(f,g) = \sum_{k \in E_1'} S_k(f) \sum_{l:\,(k,l)\in Col_k} S_l(g)$$

and

$$\begin{aligned} T^2_{\sigma_N}(f,g) &= \sum_{k \in E_2'} S_k(f) \sum_{l:\,(k,l)\in Col_k} S_l(g) \\ &= \sum_{l:\,\exists k\,(k,l)\in E_2} S_l(g) \sum_{k:\,(k,l)\in E_2} S_k(f) \end{aligned}$$

so that

$$T_{\sigma_N}(f,g) = T^1_{\sigma_N}(f,g) + T^2_{\sigma_N}(f,g).$$

We start with $T^1_{\sigma_N}$. We have

$$\begin{aligned} \left\|T^1_{\sigma_N}(f,g)\right\|_{L^1} &\leq \sum_{k\in E_1'} \Big\|S_k(f) \sum_{l:\,(k,l)\in Col_k} S_l(g)\Big\|_{L^1} \\ &\leq \sum_{k\in E_1'} \left\|S_k(f)\right\|_{L^2} \Big\| \sum_{l:\,(k,l)\in Col_k} S_l(g)\Big\|_{L^2} \\ &\leq \Big(\sum_{k\in E_1'} \left\|S_k(f)\right\|^2_{L^2}\Big)^{\frac{1}{2}} \Big(\sum_{k\in E_1'} \Big\| \sum_{l:\,(k,l)\in Col_k} S_l(g)\Big\|^2_{L^2}\Big)^{\frac{1}{2}} \\ &\leq \|\phi\|_{L^\infty}\|f\|_{L^2}(\#E_1')^{\frac{1}{2}}\|\phi\|_{L^\infty}\|g\|_{L^2}, \end{aligned}$$

exploiting the orthogonality of $S_k$'s on $L^2$.

Notice that as there are $N$ points in $E$ and each column in $E_1'$ has least $K$ elements, this means that there are at most $N/K$ columns in $E_1'$. We conclude that

$$\left\|T^1_{\sigma_N}(f,g)\right\|_{L^1} \leq (N/K)^{\frac{1}{2}}\|\phi\|^2_{L^\infty}\|f\|_{L^2}\|g\|_{L^2}. \tag{9}$$

We continue with $T^2_{\sigma_N}$. We have

$$\begin{aligned} &\left\|T^2_{\sigma_N}(f,g)\right\|_{L^1} \\ &\quad= \Big\| \sum_{l:\,\exists k\,(k,l)\in E_2} S_l(g) \sum_{k:\,(k,l)\in E_2} S_k(f)\Big\|_{L^1} \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{l:\,\exists k\,(k,l)\in E_2} \Big\| S_l(g) \sum_{k:\,(k,l)\in E_2} S_k(f)\Big\|_{L^1}\\
&\le \sum_{l:\,\exists k\,(k,l)\in E_2} \big\| S_l(g)\big\|_{L^2}\Big\| \sum_{k:\,(k,l)\in E_2} S_k(f)\Big\|_{L^2}\\
&\le\Big[\sum_{l:\,\exists k\,(k,l)\in E_2} \big\| S_l(g)\big\|_{L^2}^2\Big]^{\frac12}\Big[\sum_{l:\,\exists k\,(k,l)\in E_2}\Big\| \sum_{k:\,(k,l)\in E_2} S_k(f)\Big\|_{L^2}^2\Big]^{\frac12}\\
&\le\|\phi\|_{L^\infty}\|g\|_{L^2}\Big[\sum_{l:\,\exists k\,(k,l)\in E_2}\sum_{k:\,(k,l)\in E_2}\big\|S_k(f)\big\|_{L^2}^2\Big]^{\frac12}\\
&=\|\phi\|_{L^\infty}\|g\|_{L^2}\Big[\sum_{k\in E_2'}\sum_{l:\,(k,l)\in Col_k}\big\|S_k(f)\big\|_{L^2}^2\Big]^{\frac12}\\
&\le\|\phi\|_{L^\infty}\|g\|_{L^2}K^{\frac12}\Big[\sum_{k\in E_2'}\big\|S_k(f)\big\|_{L^2}^2\Big]^{\frac12}\\
&\le\|\phi\|_{L^\infty}\|g\|_{L^2}K^{\frac12}\|\phi\|_{L^\infty}\|f\|_{L^2}.
\end{aligned}
$$

This yields

$$\big\|T^2_{\sigma_N}(f,g)\big\|_{L^1} \le K^{\frac12}\|\phi\|^2_{L^\infty}\|f\|_{L^2}\|g\|_{L^2}. \tag{10}$$

In view of (9) and (10), the optimal choice of $K = N^{1/2}$. This proves

$$\big\|T_{\sigma_N}(f,g)\big\|_{L^1} \le N^{\frac14}\|\phi\|^2_{L^\infty}\|f\|_{L^2}\|g\|_{L^2}. \tag{11}$$

We have now proved the sufficiency direction in Theorem 1.2. □

## 5. Bilinear case: Sufficiency

We recall that

$$\alpha(p_1,p_2)=\frac12\Big[\max\Big(\frac1p-\frac12,0\Big)-\min\Big(\frac1{p_1}-\frac12,0\Big)-\min\Big(\frac1{p_2}-\frac12,0\Big)\Big].$$

Notice that $\alpha(1,1)=\frac34$ and $\alpha(1,2)=\alpha(2,1)=\frac12$.

We begin with a simple result which is nontrivial only when $p<1$.

**Proposition 5.1.** *If $E\subset\mathbb{Z}^{2n}$ has cardinality $N$, then*

$$\|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p}\le CN$$

*for $1\le p_1,p_2\le\infty$ with $1/p=1/p_1+1/p_2$.*

*Proof.* Recall that

$$B_{E,\Phi}(f,g)(x) = \iint \widehat{f}(\xi)\widehat{g}(\eta) \sum_{(k,l)\in E} \Phi((\xi,\eta)-(k,l))e^{2\pi i x\cdot(\xi+\eta)} d\xi d\eta$$
$$= \iint K(y,z) f(x-y) g(x-z) dydz,$$

where $K(y,z) = \Phi^\vee(y,z)\sum_{(k,l)\in E} e^{2\pi i (y,z)\cdot(k,l)}$.

Setting $\psi(y) = (1+|y|)^{-2n}$, as

$$|\Phi^\vee(y,z)| \le C\psi(y)\psi(z)$$

we have $|K(y,z)| \le CN\psi(y)\psi(z)$, which implies that

$$|B_{E,\Phi}(f,g)(x)| \le CN(|f|*\psi)(x)\,(|g|*\psi)(x).$$

As a result we obtain

$$\|B_{E,\Phi}(f,g)\|_{L^p} \le CN\||f|*\psi\|_{L^{p_1}}\||g|*\psi\|_{L^{p_2}} \le CN\|f\|_{L^{p_1}}\|g\|_{L^{p_2}},$$

hence the conclusion follows. □

**Remark 5.1.** *This result is sharp for $(p_1,p_2) = (1,1)$ by Proposition 6.1, discussed in the next section.*

**Corollary 5.2.** *Fix $\frac{1}{2} < p < 1$. There is a constant $C$ such that*

$$\|B_{E,\Phi}\|_{L^{p/2}\times L^{p/2}\to L^p} \le CN^{\frac{3}{4p}-\frac{1}{2}}.$$

*Proof.* Interpolating using [5, Theorem 7.2.9] between the estimate at the point (1,1,1/2) [Proposition 5.1] and at the point $(2,2,1)$ obtained in Section 4, we deduce the conclusion. □

For the most general case $1 < p_1, p_2 < \infty$, we have the following nontrivial estimate.

**Proposition 5.3.** *Let $\epsilon > 0$ be given.*
*(i) If $p < 1$, then there is a constant $C = C_{p_1,p_2,\varepsilon}$ such that*

$$\|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \le CN^{\frac{1}{\min(p_1,p_2)}+\frac{1}{2\max(p_1,p_2)}-\frac{1}{2}+\epsilon}.$$

*(ii) Fix $i \in \{1,2\}$. If $1 < p_i < 2$, and $1 < p < 2$, then there is a constant $C = C_{p_1,p_2,\varepsilon}$ such that*

$$\|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \le CN^{\frac{1}{2p_i}+\epsilon}.$$

*(iii) If $p > 2$, then there is a constant $C = C_{p,\varepsilon}$ such that*

$$\|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \le CN^{\frac{1}{2}-\frac{1}{2p}+\epsilon}.$$

To prove this proposition, we make use of the following result.

**Lemma 5.4.** *For all* $2 \le p_1, p_2 < \infty$ *and* $1 \le p < \infty$ *with* $1/p = 1/p_1 + 1/p_2$*, we have*

$$\|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \le C\,(\log N)N^{\frac{1}{2}}.$$

*Proof.* Recalling the notation in Section 4, we write

$$T_{\sigma_N}(f,g) = \sum_{k\in E'} S_k(f) \sum_{l\colon (k,l)\in Col_k} S_l(g).$$

It is enough to prove that

$$\|T_{\sigma_N}\|_{L^{p_1}\times L^{p_2}\to L^p} \le C\,(\log N)N^{\frac{1}{2}}.$$

We split the operator $T_{\sigma_N}$ as

$$T_{\sigma_N} = \sum T^i,$$

where $i$ ranges from 1 to $\log_2 N$. Each operator $T^i$ has columns of the size between $N/2^i$ and $N/2^{i-1}$ and therefore there will be at most $2^i$ such columns.

For each $T^i$, we have the estimate

$$\begin{aligned}
&\Big\|\sum_k \sum_{l\colon (k,l)\in Col_k} S_k(f)S_l(g)\Big\|_{L^p}\\
&\le \Big\|\Big(\sum_k |S_k(f)|^2\Big)^{\frac{1}{2}}\Big(\sum_k \Big|\sum_{l\colon (k,l)\in Col_k} S_l(g)\Big|^2\Big)^{\frac{1}{2}}\Big\|_{L^p}\\
&\le \Big\|\Big(\sum_k |S_k(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^{p_1}}\Big\|\Big(\sum_k \Big|\sum_{l\colon (k,l)\in Col_k} S_l(g)\Big|^2\Big)^{\frac{1}{2}}\Big\|_{L^{p_2}}.
\end{aligned}$$

Moreover, we note that

$$\Big\|\Big(\sum_k |S_k(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^{p_1}} \le C\|f\|_{L^{p_1}},$$

in view of the square function theorem ([15, Theorem 8.1]). We remark that the constant $C$ here is independent of $r$ when $\phi_r(\xi) = e^{2\pi i r\cdot\xi}\phi(\xi)$. Actually in this case $\sum_k |S_{k,\phi_r}(f)(x)|^2 = \sum_k |(\phi(\xi-k)\widehat{f_r}(\xi))^\vee(x)|^2$ with $\widehat{f_r}(\xi) = \widehat{f}(\xi)e^{2\pi i r\cdot\xi}$, then the square function estimate gives that

$$\Big\|\Big(\sum_k |S_{k,\phi_r}(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^{p_1}} = \Big\|\Big(\sum_k |S_{k,\phi}(f_r)|^2\Big)^{\frac{1}{2}}\Big\|_{L^{p_1}} \lesssim \|f_r\|_{L^{p_1}} = \|f\|_{L^{p_1}}.$$

To estimate the second part, we first use the Cauchy-Schwarz inequality and that each column under consideration has length about

$N/2^i$, and then the fact that $k$ ranges over set of size at most $2^i$; finally we employ the square function theorem again. We deduce

$$\begin{aligned}
\Big\| \Big( \sum_k \Big| \sum_{l:\,(k,l)\in Col_k} S_l(g) \Big|^2 \Big)^{\frac12} \Big\|_{L^{p_2}} &\le C \Big\| \Big( \sum_k (N/2^i) \sum_{l:\,(k,l)\in Col_k} |S_l(g)|^2 \Big)^{\frac12} \Big\|_{L^{p_2}} \\
&\le C \Big\| \Big( 2^i (N/2^i) \sum_{l\in\mathbb{Z}^n} |S_l(g)|^2 \Big)^{\frac12} \Big\|_{L^{p_2}} \\
&\le C N^{1/2} \|g\|_{L^{p_2}}.
\end{aligned}$$

Summing over $i$ we complete the proof with an extra logarithm. □

We can now prove Proposition 5.3 using multilinear interpolation.

*Proof of Proposition 5.3.* It follows from Lemma 5.4 that we have the rate of growth $N^{1/2}\log N$ for $(p_1,p_2,p)$ close to $(\infty,\infty,\infty)$, then by duality we have the same rate of growth for $(p_1,p_2,p)$ close to $(1,\infty,1)$ or $(\infty,1,1)$.

Estimate $(i)$. It suffices to consider the case $1<p_1<p_2$, and $p<1$, when the desired estimate is

$$\|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \le CN^{\frac{1}{p_1}+\frac{1}{2p_2}-\frac12+\epsilon}.$$

It follows from interpolation between $(1,1,\frac12)$, $(2,2,1)$, and $(p_1,p_2,p)$ close to $(1,\infty,1)$.

Estimate $(ii)$ follows by interpolating between $(2,2,1)$, $(2,\infty,2)$, and $(p_1,p_2,p)$ close to $(1,\infty,1)$ enough when $i=1$. The case $i=2$ follows by symmetry.

Estimate $(iii)$ follows from interpolation between $(\infty,2,2)$, $(2,\infty,2)$, and $(p_1,p_2,p)$ close to $(\infty,\infty,\infty)$ enough. □

## 6. Bilinear case: necessity

Our main result in this section, stated below, includes the necessity direction in Theorem 1.2.

**Proposition 6.1.** *Fix a smooth bump $\Phi$ supported in the ball $|\xi|\le\frac{1}{20}$ in $\mathbb{R}^{2n}$. Then for all $p_1,p_2$ with $1\le p_1,p_2<\infty$ satisfying $1/p_1+1/p_2=1/p$ we have*

$$\sup_E \|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \ge CN^{\max(\alpha(p_1,p_2),\frac1p-1)}.$$

*In particular,* $\|B_{E,\Phi}\|_{L^1\times L^1\to L^{1/2}} \geq CN$ *and estimate (ii) in Theorem 1.2 is sharp (up to* $\epsilon$*) when* $p \geq 1$.

Via an argument similar to that used in the proof of Lemma 2.1, it suffices to consider the case $n=1$, which we discuss below.

Let

$$\alpha'(p_1,p_2) = \frac{1}{2}\Big[\frac{1}{p} - \frac{1}{2} - \min\Big(\frac{1}{p_1} - \frac{1}{2}, 0\Big) - \min\Big(\frac{1}{p_2} - \frac{1}{2}, 0\Big)\Big].$$

Note that $\alpha'(p_1,p_2) = \alpha(p_1,p_2)$ when $p \leq 2$. We need two lemmas to prove Proposition 6.1.

**Lemma 6.2.** *For all* $1 \leq p_1, p_2 < \infty$ *with* $1/p = 1/p_1 + 1/p_2$ *we have*

$$\sup_E \|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \geq CN^{\alpha'(p_1,p_2)}. \tag{12}$$

*Proof.* It suffices to prove the conclusion for $\sum_{(k,l)\in E} a_{k,l}S_k(f)S_l(g)$ with $a_{k,l}\in\{1,-1\}$. Actually if we verify that

$$\Big\|\sum_{(k,l)\in E} a_{k,l}S_k(f)S_l(g)\Big\|_{L^{p_1}\times L^{p_2}\to L^p} \geq 3CN^{\alpha'(p_1,p_2)}$$

with $a_{k,l}\in\{-1,1\}$, then we must have (12); otherwise we obtain that

$$\Big\|\sum_{(k,l)\in E} a_{k,l}S_k(f)S_l(g)\Big\|_{L^{p_1}\times L^{p_2}\to L^p} \leq 2CN^{\alpha'(p_1,p_2)}$$

since we can write

$$\sum_{(k,l)\in E} a_{k,l}S_k(f)S_l(g) = B_{E_1,\Phi}(f,g) - B_{E_2,\Phi}(f,g)$$

for appropriate sets $E_1$ and $E_2$.

Inspired by the examples in [7] for $n=1$, we define[1]

$$m(\xi,\eta) = \sum_{k=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}} a_k(t_1)a_l(t_2)a_{k+l}(t_3)c_{j+k}\phi(\xi-j)\phi(\eta-l),$$

where $a_k(t)$ are Rademacher functions, and $c_l = 1$ when $9\sqrt{N}/10 \leq l \leq 11\sqrt{N}/10$ and 0 elsewhere. We define

$$\widehat{f_N}(\xi) = N^{-\frac{1}{2p_1'}}\sum_{k=1}^{\sqrt{N}} a_k(t_1)\widehat{\varphi}(\xi-k)\,,\ \widehat{g_N}(\eta) = N^{-\frac{1}{2p_2'}}\sum_{l=1}^{\sqrt{N}} a_l(t_2)\widehat{\varphi}(\eta-l).$$

[1] $\sum_{k=1}^{\sqrt{N}} a_k$ means $\sum_{k=1}^{[\sqrt{N}]} a_k$, where $[\sqrt{N}]$ is the integer part of $\sqrt{N}$.

By a calculation analogous to that in (7) we obtain

$$\left(\int_0^1 \left\|T_m(f_N,g_N)\right\|_{L^p}^p dt_3\right)^{\frac{1}{p}} \sim N^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})}.$$

On the other hand

$$\left(\int_0^1 \|f_N\|_{L^{p_1}}^{p_1}\, dt_1\right)^{\frac{1}{p_1}} \sim N^{\frac{1}{2}(\frac{1}{p_1}-\frac{1}{2})}.$$

Let $C_0(N) = \sup_E \|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p}$, where the supremum is taken over all $E$ with $|E| = N$, then

$$N^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})} \sim \Big\|\left\|T_m(f_N,g_N)\right\|_{L^p}\Big\|_{L^p(dt_3)} \le C_0(N)\|f_N\|_{L^{p_1}}\|g_N\|_{L^{p_2}}.$$

Taking $L^{p_1}(dt_1)$ and $L^{p_2}(dt_2)$ norms on both sides, we obtain that

$$C_0(N) \ge C\frac{N^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})}}{N^{\frac{1}{2}(\frac{1}{p_1}-\frac{1}{2})+\frac{1}{2}(\frac{1}{p_2}-\frac{1}{2})}} = CN^{\frac{1}{4}},$$

using the estimates for $f_N$ and $g_N$. This estimate works for all choices of indices $p_1, p_2, p$ with $1/p_1 + 1/p_2 = 1/p$ but it is sharp only in the local $L^2$ case, i.e. in the case where $2 \le p_1, p_2, p' \le \infty$.

Now if all the coefficients $a_k(t)$ are equal to 1 in the definition of $f_N$, then $\|f_N\|_{p_1} \le C$ for $p_1 \in (1,2]$, which is smaller than $N^{\frac{1}{2}(\frac{1}{p_1}-\frac{1}{2})}$ if $p_1 < 2$. So in the case $p_1 \le 2 \le p_2$, we modify the multiplier $m$ by

$$m(\xi,\eta) = \sum_{k=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}} a_l(t_2)a_{k+l}(t_3)c_{j+k}\phi(\xi-j)\phi(\eta-l)$$

correspondingly, which gives then

$$C_0(N) \ge CN^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2}-\frac{1}{p_2}+\frac{1}{2})} = CN^{\frac{1}{2p_1}}.$$

By symmetry we have $C_0(N) \ge CN^{\frac{1}{2p_2}}$ when $p_2 \le 2 \le p_1$. In analogous way, when $1 \le p_1, p_2 \le 2$ we set $a_k(t_1) = a_l(t_2) = 1$ to obtain the lower bound $C_0(N) \ge CN^{\frac{1}{2}(\frac{1}{p}-\frac{1}{2})}$. Combining these estimates in one form, we obtain the lower bound $C\,N^{\alpha'(p_1,p_2)}$. □

**Lemma 6.3.** *There exists a set $E \subset \mathbb{Z}^2$ with cardinality $N$ such that*

$$\|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \ge CN^{\frac{1}{p}-1}. \tag{13}$$

*In particular $\|B_{E,\Phi}\|_{L^1\times L^1\to L^{1/2}} \ge CN$.*

This estimate is stronger than (12) when $\frac{1}{2} \le p < \frac{2}{3}$.

*Proof.* We consider the multiplier

$$m(\xi,\eta) := \sum_{j=-N}^{N} \phi(\xi-j)\phi(\eta+j),$$

whose inverse Fourier transform is

$$K(y,z) = \phi^{\vee}(y)\phi^{\vee}(z)\sum_{j=-N}^{N} e^{2\pi i j(y-z)}.$$

We remark that $\sum_{j=-N}^{N} e^{2\pi i j s}$ is real by symmetry. Moreover we have $|(2N+1) - \sum_{j=-N}^{N} e^{2\pi i j s}| \leq N$ for $s \leq 1/(50N)$. We now take

$$f(y) = g(y) = 100N\chi_{[0,(100N)^{-1}]}(y),$$

which satisfy $\|f\|_{L^1} = \|g\|_{L^1} = 1$. Then

$$T_m(f,g)(x) = \iint K(x-y, x-z)f(y)g(z)dydz$$

satisfies that $|T_m(f,g)(x)| \geq CN$ for $|x| \leq (100)^{-1}$ if we choose $\phi$ appropriately. This yields that $\|T_m(f,g)\|_{L^{1/2}} \geq CN$. In summary $\|T_m\|_{L^1\times L^1\to L^{1/2}} \geq CN$. From this example we also obtain that

$$\|T_m\|_{L^{p_1}\times L^{p_2}\to L^p} \geq C\frac{N}{N^{1/p_1'}N^{1/p_2'}} \geq CN^{\frac{1}{p}-1}.$$

This concludes the proof of the lemma. □

We provide the following intuitive understanding of the proof of (13). As $m$ is supported in a tube with dimensions $N\times 1$ along the antidiagonal, the kernel $K = m^{\vee}$ is essentially equal to the constant $N$ in a tube of dimensions $1\times N^{-1}$ along the diagonal, in view of the uncertainty principle. If $f\otimes g$ is supported in a square of length $N^{-1}$ of height $N^2$, then $K*(f\otimes g)(x,x)$ is essentially $K*(f\otimes g)(0,0) \sim N^3N^{-2} \sim N$ for $|x| \leq C$. This gives the claimed lower bound of $B_{E,\Phi}$.

**Remark 6.1.** *Suppose that* $1 \leq p_1, p_2 \leq 2$. *We have* $\alpha'(p_1,p_2) = \frac{1}{2p} - \frac{1}{4}$. *Note that* $\alpha'(p_1,p_2) \geq \frac{1}{p} - 1$ *if and only if* $p \geq \frac{2}{3}$. *In other words, the example in Lemma 6.3 provides a larger lower bound for* $\|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p}$ *when* $p < \frac{2}{3}$.

We now provide the proof of Proposition 6.1.

*Proof of Proposition 6.1.* It follows from (12) and the discussion in Section 3 that

$$\sup_E \|B^{*1}_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \geq CN^{\alpha'(p_1,p_2)}.$$

More precisely if $1 < p_1 \le 2$ and $1 \le p \le 2$, we have

$$\sup_E \|B^{*1}_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \ge CN^{1/(2p_1)},$$

which implies by duality that

$$\sup_E \|B_{E,\Phi}\|_{L^{p'}\times L^{p_2}\to L^{p_1'}} \ge CN^{1/(2p_1)}.$$

We can rephrase this estimate as

$$\sup_E \|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \ge CN^{1/(2p')},$$

which matches $N^{\alpha(p_1,p_2)}$ and is greater than $N^{1/4}$ when $p \ge 2$, which happens exactly when $\alpha(p_1,p_2) \ge \alpha'(p_1,p_2)$.

In summary, we obtain that

$$\sup_E \|B_{E,\Phi}\|_{L^{p_1}\times L^{p_2}\to L^p} \ge CN^{\alpha(p_1,p_2)},$$

which combined with (13) finishes the proof. □

We finish this section by giving the formal proof of Theorem 1.2; this was essentially done in last three sections.

*Proof of Theorem 1.2.* We refer where we discussed the sufficient part first. The local $L^2$ case is proved in Section 4. The case when $p > 1$ but the local $L^2$ case is given by Proposition 5.3 (ii) and (iii). The case when $p \le 1$ is Proposition 5.3 (i). The necessity is provided by Proposition 6.1. □

**Remark 6.2.** *One notices that* $\frac{3}{4p} - \frac{1}{2} > \frac{1}{p} - 1$ *when* $p > \frac{1}{2}$*, and* $\frac{3}{4p} - \frac{1}{2} > \frac{1}{2p} - \frac{1}{4}$ *for* $p < 1$*, therefore, as of this writing, there is a gap between the positive result from* $L^p \times L^p \to L^{p/2}$ *for* $p < 1$ *in Corollary 5.2 and our two counterexamples.*

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